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REVIEW OF FUZZY NUMBERS AND ITS APPLICATION IN CAPITAL BUDGETING

ABDUL MALEK BIN YAAKOB

A dissertation submitted in fulfilment of the
requirements for the award of the degree of
Master of Science (Mathemtics)

Faculty of Science
Universiti Teknologi Malaysia

MAY 2011

I declare that this thesis titled "*Review of Fuzzy Numbers and Its Application on Capital Budgeting*" is the result of my own research except as cited in the references. This thesis has not been acceptd for any degree and is not concurrently submitted in candidature of any other degree.

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To my beloved mum, Mek Mat Ali;

my late father, Yaakob Ghani;

To my beloved *queenmyheart*, Siti Fatimah Abdul Rahman;

my newborn son, Muhammad Ihsan Wafiuddin;

my sisters and brother.

I'll Loving You and We are Always Connected.

ACKNOWLEDGEMENTS

First and foremost, praise be to Allah s.w.t for giving me His blessings throughout my studies. I would like to express my gratitude to those who had extended their help and support throughout my studies, towards the completion of this dissertation in particular.

In preparing this thesis, I was in contact with many people, academicians, and practitioners. They have contributed towards my understanding and thoughts. In particular, I wish to express my sincere appreciation to my supervisor, Dr. Normah Maan, for her encouragement, guidance, advices, motivation, critics and friendship. Without her continued support and interest, this dissertation would not have been the same as presented here.

I am grateful to all my family members especially queenmyheart Siti Fatimah Abdul Rahman, my parents Yaakob Ghani and Mek Mat Ali and through all my siblings for their numerous support. I am also indebted to Universiti Utara Malaysia(UUM) for granting me the study leave and Kementerian Pengajian Tinggi (KPT) for their funding of my masters study.

My fellow postgraduate students should also be recognised for their support. My sincere appreciation also extends to all my colleagues and others who have provided assistance at various occasions. Their views and tips are useful indeed. Unfortunately, it is not possible to list all of them in this limited space.

ABSTRACT

A fuzzy number is an extension of a regular number in the sense that it does not refer to one single value but rather to a connected set of possible values. Whereas, capital budgeting is the planning process used to determine whether an organisation's long term investments such as new machinery, replacement machinery, new plants, new products, and research development projects are worth pursuing.

The purpose of this study is to review the concept of one, two and n-dimensional fuzzy numbers and investigate the application of fuzzy numbers in financial field specifically in capital budgeting problem. There are five capital budgeting techniques usually used in practices, such as revenue per one dollar, payback period, net present value(NPV), net future value(NFV), and the modified of internal rate of return(MIRR) methods. In this research both, classical and fuzzy approach are used in order to evaluate the project as mentioned above.

The research will also focus on the application of special case of trapezoidal fuzzy numbers, triangular fuzzy number. At the end of this research the comparison of classical and fuzzy methods are discussed according to five capital budgeting methods. The result of this study provides the alternative way in financial field to evaluate the profitable project to invest.

ABSTRAK

Suatu nombor kabur adalah satu perluasan daripada nombor biasa, dalam erti kata, ia tidak merujuk kepada satu nilai sahaja tetapi ia merujuk kepada suatu set yang berhubung dengan nilai mungkin. Manakala, bajet modal adalah proses perancangan yang digunakan untuk menentukan sama ada pelaburan jangka panjang organisasi seperti mesin baru, pengantian mesin, kilang baru, produk baru dan projek pembangunan penyelidikan adalah suatu yang menguntungkan.

Tujuan kajian ini adalah untuk meninjau konsep nombor kabur dimensi satu, dimensi dua, dan dimensi-n dan menyiasat penggunaan nombor kabur dalam bidang kewangan khususnya dalam masalah bajet modal. Terdapat lima teknik bajet modal yang biasa dipraktiskan seperti kaedah pendapatan pada satu dolar, tempoh pengambalian, nilai terkini bersih, nilai hadapan bersih dan kadar pulangan dalaman. Dalam kajian ini, kedua-dua pendekatan klasik dan kabur digunakan untuk menilai projek seperti disebut diatas.

Kajian ini juga akan menumpukan pada pelaksanaan kes khas trapesium nombor kabur, segitiga nombor kabur. Pada akhir kajian ini perbandingan kaedah klasik dan kabur dibincangkan mengikut lima kaedah bajet modal. Hasil kajian ini menberikan cara alternatif dalam bidang kewangan untuk menilai projek yang menguntungkan untuk dilabur.

TABLE OF CONTENTS

CHAPTER	TITLE	PAGE
	DECLARATION	ii
	DEDICATION	iii
	ACKNOWLEDGEMENTS	iv
	ABSTRACT	v
	ABSTRAK	vi
	TABLE OF CONTENTS	vii
	LIST OF TABLES	x
	LIST OF FIGURES	xi
1	**INTRODUCTION**	**1**
	1.1 Background of the Problem	1
	1.2 Statement of the Problem	2
	1.3 Objective of Study	3
	1.4 Scope of Study	3
	1.5 Significant of Study	3
	1.6 Summary for Each Chapter	4

2	**REVIEW ON FUZZY NUMBERS**		**5**
	2.1	Introduction	5
	2.2	Interval Numbers and Some Operations	5
	2.3	One Dimensional Fuzzy Numbers	7
		2.3.1 Addition of Fuzzy Numbers	9
	2.4	Two Dimensional Fuzzy Numbers	12
		2.4.1 Domain of Confidence	14
		2.4.2 Addition of Two Domain of Confidence	15
		2.4.3 Addition of Fuzzy Numbers of Dimension Two	17
	2.5	n-Dimensional Fuzzy Numbers	19
	2.6	Summary	19
3	**THE CONCEPT OF CAPITAL BUDGETING TECHNIQUE**		**20**
	3.1	Introduction	20
	3.2	Capital Budgeting Techniques	20
	3.3	Classical Capital Budgeting Methods [16]	21
	3.4	Fuzzy Capital Budgeting with Crisp Project Duration [16]	25
	3.5	Summary	27
4	**IMPLEMENTATION**		**28**
	4.1	Shenandoah Furniture Capital Budgeting Problem	28
	4.2	Classical Capital budgeting Method	29
		4.2.1 Revenue per One Dollar Method	29
		4.2.2 Payback Period Method	30
		4.2.3 Net Present Value Method	30
		4.2.4 Net Future Value Method	31
		4.2.5 Modified Internal Rate of Return Method	31
		4.2.6 Summary of Classical Capital budgeting Methods	32
	4.3	Fuzzy Capital Budgeting with Crisp Project Duration	32
		4.3.1 Fuzzy Revenue per One Dollar Method	37

4.3.2 Fuzzy Payback Period Method 37
4.3.3 Fuzzy Net Present Value Method 39
4.3.4 Fuzzy Net Future Value Method 43
4.3.5 Fuzzy Modified Internal Rate of Return Method 48
4.3.6 Summary of Fuzzy Capital Budgeting Methods 53

5 CONCLUSION AND RECOMMENDATION 54

5.1 Introduction 54
5.2 Result and Discussion 54
5.3 Conclusion 56
5.4 Further Research 57

REFERENCES 58

LIST OF TABLES

TABLE NO.	TITLE	PAGE
4.1	Summary of Classical Capital Budgeting Methods	32
4.2	Summary of Fuzzy Capital Budgeting Measure	53
5.1	The Comparison of Classical and Fuzzy Capital Budgeting Methods	55

LIST OF FIGURES

FIGURE NO.	TITLE	PAGE
2.1	An ordinary subset in $\Re$	7
2.2	An fuzzy subset in $\Re$	8
2.3	A convex fuzzy subset(nonnormal)	8
2.4	A normal fuzzy subset(nonconvex)	9
2.5	A normal and convex fuzzy subset	9
2.6	A Fuzzy Number is 6-tuple Form	11
2.7	Definition of convex surface	15
2.8	Definition of non-convex surface	15
2.9	Sum of two fuzzy rectangular domains, $D = D_1 + D_2$	17
2.10	Sum of two pyramidal fuzzy numbers	18
4.1	The Fuzzified Cash Outflows $(C\tilde{O}F)$	34

CHAPTER 1

INTRODUCTION

1.1 Background of the Problem

The theory of fuzzy set was introduce by Lutfi A. Zadeh [1] in United States, and has been studied by many researchers. In the first decade, the application-oriented papers in the fuzzy field had mainly concerned with theoretical studies. In 1978, the concept of fuzzy numbers has been introduced by S.Nahmias [2] in United States and H.Dubois and D. Prade [3] in France. The concept of fuzziness can be represented by several different approaches [4]. In this research we will represent a fuzzy numbers based on the concept of the interval of confidence, which will be discuss further in the literature. A fuzzy numbers is an extension of a regular numbers in the sense that it does not refer to one single value but rather to connected set of possible values, which is a special case of a convex and normal fuzzy set.

In 1986, Goetschel and Vaxman [5] was proposed the representation theorem of one dimensional fuzzy numbers. This theorem express a fuzzy numbers as two real valued function and it has been proved very useful in solving many problems concerning fuzzy numbers. However this concept is meaningful only in one dimensional case. Facing the real problems which are mostly of multidimensional systems inspires the work of the discussion on the concept of n-dimensional fuzzy numbers. Therefore, in 2002 Zhang and Wu [6], present the representation theorems of n-dimensional fuzzy numbers. The theorems are very useful in solving problems of fuzzy numbers space. The fuzzy approach also widely used in various field engineering, management and financial [7, 8, 12], etc.

There are variety of methods used in capital budgeting application.The details will be discussed in Chapter 3. In capital budgeting, the classical method used do not take into account the uncertainty which may be inherent in the information used. Here the information are cash inflow (CIF), cash outflow (COF), the required rate of return (r), and the duration of project (n).

In the literature, there exist several ways of incorporating uncertainty in capital budgeting decision making such as intuitive method, risk adjustment, deterministic data and probabilistic approach. Intuitive methods are widely used in practices, but they have the disadvantage of depending too much on the intuition of decision maker. Probabilistic approaches is much less flexible then fuzzy set theory. The fuzzy set can be applied even with rather scanty information [16], that is the reason why several researchers have used fuzzy approach. In this research the concept of fuzzy number have been implemented in capital budgeting method. In general capital budgeting is the process of selecting capital investment such as equipment replacement, advertising, new product etc. Here, it is not easy to make a decision which investment is profitable to company. This research will provide a guideline to the manager to make a profitable decision by using fuzzy approach.

1.2 Statement of the Problem

Below are the statement of the problems:

(i) What is the definition and properties of one, two and n-dimensional fuzzy numbers?

(ii) What is budgeting problem in financial point of view?

(iii) How to implement the concept of fuzzy numbers in capital budgeting problem?

1.3 Objective of Study

Objective of this research:

(i) To review the definition, properties, and application of one, two and n-dimensional fuzzy numbers.

(ii) To understand the budgeting problem in financial point of view.

(iii) To implement the concept of fuzzy numbers in capital budgeting problem.

1.4 Scope of Study

This research will focus on the application of special case of trapezoidal fuzzy numbers in capital budgeting. The duration of project is assume in n years.

1.5 Significant of Study

The concept of fuzzy numbers is very important in solving vague problem. In this research, we apply the concept of one dimensional fuzzy numbers in the capital budgeting problem. The result of the research will give benefits in mathematical, decision making process and finance field. In mathematics, we want to highlight the new application of fuzzy numbers in financial problems. On the other hand, the result of this research can also be used by decision maker as a guideline to determine the best project to invest. This research will also provide the alternative way in financial field to evaluate the profitable project to invest.

1.6 Summary for Each Chapter

This dissertation is organized into five chapters. Chapter 1 is the introduction of study. Background of the research discussed in this chapter. Then, the objectives and the scopes of this research are explained briefly. Finally, the significances of the study is discussed to elaborate the contributions of this research in mathematics and financial field.

In Chapter 2, an overview of the study has been discussed. This chapter will focus on details about one, two and n-dimensional fuzzy numbers. Some basic concepts and properties of interval number and fuzzy numbers that will be used throughout the dissertation are presented.

In Chapter 3, the details discussion about capital budgeting technique will be presented. Then, the classical method and fuzzy method of capital budgeting are presented. These information about capital budgeting technique will be used in our analysis.

In Chapter 4, the real case application of capital budgeting is stated. Then, the calculation of classical method and fuzzy method are shown. At the end of this chapter, the summary of each methods are presented.

The last chapter of this dissertation is Chapter 5. In this chapter, the conclusion of this research have been make and the outline of some further research which are worthwhile investigating in the future are listed.

CHAPTER 2

REVIEW ON FUZZY NUMBERS

2.1 Introduction

Fuzzy set theory proposed by Zadeh [1] allows us to process and transform imprecise information effectively and flexibly. Fuzzy numbers plays a significant role among all fuzzy sets since the predominant representation of information is numerics [13]. Triangular fuzzy numbers have been extensively applied in many application of real world such as engineering, science, management, and stock selection [7]. In this research, we will give another application of fuzzy number in financial. Therefore the following sections will give the details of interval numbers and fuzzy numbers.

2.2 Interval Numbers and Some Operations

Let $A = [a_1; a_2]$; $B = [b_1; b_2]$, etc., denote real, closed intervals, which we will call interval numbers. Let us start with the following definition:

Definition 2.1 [10] Let s be an arbitrary real number. Then

(i) $[a_1, a_2] > s \iff a_1 > s$ and $[a_1, a_2] < s \iff a_2 < s$.

(ii)

$$s[a_1, a_2] = \begin{cases} [sa_1, sa_2], & \text{if } s > 0, \\ [0, 0], & \text{if } s = 0, \\ [sa_2, sa_1], & \text{if } s < 0. \end{cases}$$

Moreover, the interval exponential function will be defined in the following way:

(iii)

$$e^{[a_1, a_2]} = [e^{a_1}, e^{a_2}]$$

Let us now go over to operations on two interval numbers. The basic idea of these operations is contained in the following formula, where * denotes an arbitrary operation defined for real numbers:

$$\bar{A} * \bar{B} = \{a * b | a \in \bar{A}, b \in \bar{B}\}.$$

The following definition is a consequence of the above formula.

Definition 2.2 [10]

(i) $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$;

(ii) $[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$:

(iii) $[a_1, a_2] \cdot [b_1, b_2] = [\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \max\{a_1b_1, a_1b_2; a_2b_1, a_2b_2\}]$.

(iv) Let $0 \notin [b_1, b_2]$. Then $[a_1, a_2]/[b_1, b_2] = [a_1, a_2] \cdot [1/b_2, 1/b_1]$.

The following lemma is proved in [10].

Lemma 2.1 [10] Let $\bar{A} = [a_1, a_2]$, $\bar{B} = [b_1, b_2]$, $\bar{C} = [c_1, c_2]$ be arbitrary intervals such that $\bar{B} \cdot \bar{C} > 0$. Then $\bar{A} \cdot (\bar{B} + \bar{C}) = \bar{A} \cdot \bar{B} + \bar{A} \cdot \bar{C}$ holds.

Now let us look at the definition and some properties of fuzzy numbers.

2.3 One Dimensional Fuzzy Numbers

Here we will define one dimensional fuzzy numbers in a mathematically strict manner and discuss properties of them. Let H be a universal set(for example, $\Re$ or $\mathbb{Z}$). An ordinary subset, A, of this universal set is defined by its characteristic function,$\forall x \in H$:

$$\mu_A(x) \in \{0,1\}$$

which shows that an element of H belongs to, or does not belong to A according to the value of the characteristic function(1 or 0). For the same universal set H a fuzzy subset A will be defined by its characteristic function called the membership function, which takes the value in the interval $[0,1]$ instead of in the binary set {0,1}. $\forall x \in H$:

$$\mu_A(x) \in [0,1]$$

That is the elements of E belong to A with level(α-level) located in $[0,1]$. Figure 2.1 and Figure 2.2 shows an ordinary subset and a fuzzy subset in $\Re$.

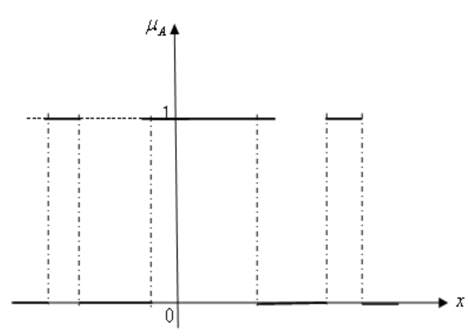

Figure 2.1: An ordinary subset in $\Re$

Definition 2.3 [14] By $\Re$ we denote the set of all real numbers. A fuzzy numbers in $\Re$ is a fuzzy subset of R that is convex and normal.

The following theorem 2.1 is proved in [14].

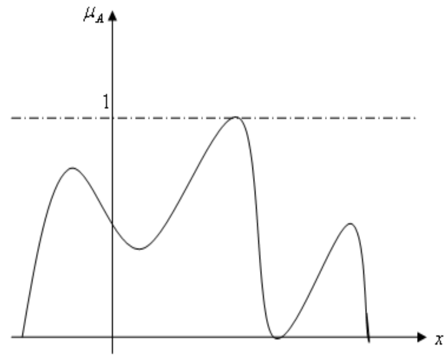

Figure 2.2: An fuzzy subset in $\Re$

Theorem 2.1 [14] A fuzzy subset $A \subset \Re$ is convex if and only if every ordinary

$$A_\alpha = \{x|\mu_A(x) \geqslant \alpha\}, \alpha \in [0,1] \tag{2.1}$$

subset is convex;

in other word, if it is a closed interval of $\Re$. Figure 2.3 shows a convex fuzzy subset.

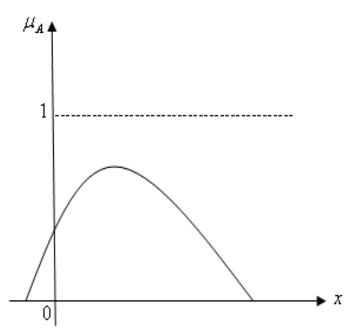

Figure 2.3: A convex fuzzy subset(nonnormal)

We now define normality.

Definition 2.4 [14] A fuzzy subset $A \subset \Re$ is normal if and only if $\forall x \in \Re$ such that

$$\bigvee_x \mu_A(x) = 1 \tag{2.2}$$

this means that the highest(max) values of $\mu_A(x)$is equal to 1. Figure 2.4 a normal fuzzy subset(non-convex). Figure 2.5 shows a normal and convex fuzzy set.

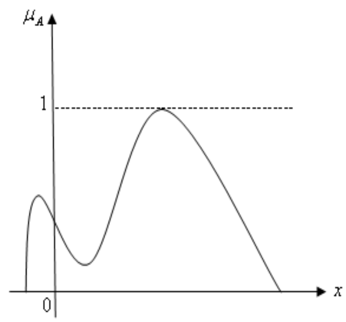

Figure 2.4: A normal fuzzy subset(nonconvex)

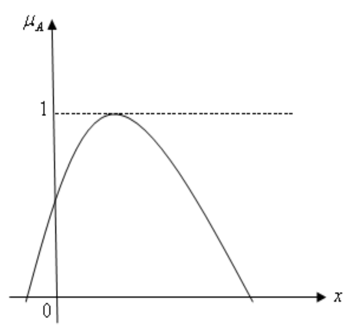

Figure 2.5: A normal and convex fuzzy subset

2.3.1 Addition of Fuzzy Numbers

Let A and B be two fuzzy numbers and A_α and B_α are their intervals of confidence for the alpha level, $\alpha \in [0, 1]$. We can write [13]:

$$\begin{aligned} A_\alpha(+)B_\alpha &= [a_1^{(\alpha)}, a_2^{(\alpha)}](+)[b_1^{(\alpha)}, b_2^{(\alpha)}] \\ &= [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] \end{aligned} \tag{2.3}$$

If $A, B \subset \Re$, then for the interval of confidence at the level α, we may define the ordinary subset A_α and B_α :

$$A_\alpha = \{x|\mu_A(x) \geqslant \alpha\} \tag{2.4}$$

$$B_\alpha = \{x|\mu_B(x) \geqslant \alpha\} \tag{2.5}$$

the following theorems is proved in [13].

Theorem 2.2 If A and B are fuzzy numbers in $\Re$, then $A(+)B$ is also subset in $\Re$ that is convex and normal.

In this research we will used a notation similiar to that used in [11] and will adopt the following definition of fuzzy number :

Definition 2.5 [11] A fuzzy number is a 6-tuple of the following form :

$$\tilde{r} = (r_1, r_2, r_3, r_4, f_1^r(\lambda), f_2^r(\lambda))$$

or shorter as shown in Figure 2.6,

$$\tilde{r} = (r_1, r_2, r_3, r_4, f_1^r, f_2^r)$$

where

- r_1, r_2, r_3, r_4 are real numbers s.t. $r_1 \leq r_2 \leq r_3 \leq r_4$.
- f_1^r is a continuous non-decreasing real function defined on the interval $[0, 1]$, such that $f_1^r(0) = r_1, f_1^r(1) = r_2$.

  f_2^r is a continuous non-increasing real function defined on the interval $[0, 1]$, such that $f_2^r(1) = r_3,\ f_2^r(0) = r_4$.

Let us recall now the following definition:

Definition 2.6 [16] Let $\tilde{r} = (r_1, r_2, r_3, r_4, f_1^r, f_2^r)$ be an arbitrary fuzzy number and λ an arbitrary number such that $0 \leq \lambda \leq 1$. The symbol r^λ stands for the λ -level of the fuzzy number $\tilde{r}$, defined as

$$r^\lambda = [f_1^r(\lambda), f_2^r(\lambda)]$$

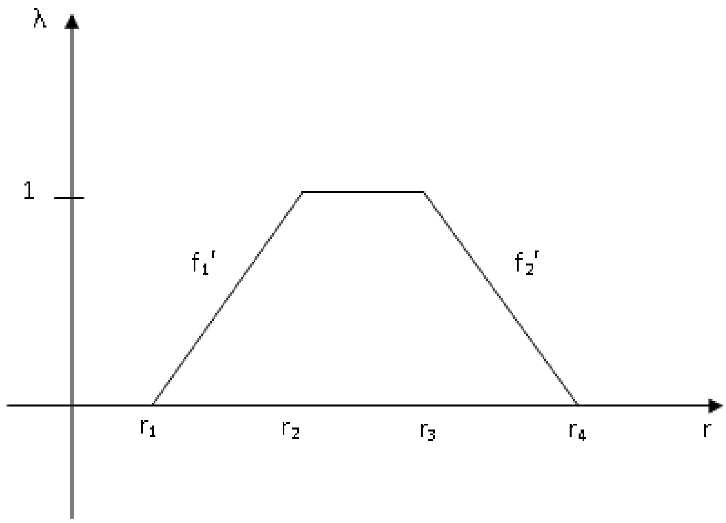

Figure 2.6: A Fuzzy Number is 6-tuple Form

The symbols $(r^\lambda)_1, (r^\lambda)_2$ or r_1^λ, r_2^λ stand for, respectively, the lower and upper end of the interval r^λ. The following well-known lemma is obvious.

Lemma 2.2 [16] A fuzzy number is unequivocally determined by the set of its λ-levels $(0 \leq \lambda \leq 1)$. Thus, we can write $\tilde{r} = r^\lambda, 0 \leq \lambda \leq 1$.

Definition 2.7 [16]

(i) An arbitrary fuzzy number $\tilde{r}$ fulfils a relation R iff for each $0 \leq \lambda \leq 1$, λ -level r^λ fulfils this relation.

(ii) If $*$ is an arbitrary operation and $\tilde{r}$ and $\tilde{p}$ any two fuzzy numbers, then $\tilde{r} * \tilde{p} = r^\lambda * s^\lambda$, for such λ that $0 \leq \lambda \leq 1$ and operation $*$ is defined.

Definition 2.8 [16] Let s be an arbitrary real number. Then

(i) $(r_1, r_2, r_3, r_4, f_1^r, f_2^r) > s \Leftrightarrow r_1 > s, (r_1, r_2, r_3, r_4, f_1^r, f_2^r) < s \Leftrightarrow r_4 > s.$

(ii) $$s(r_1, r_2, r_3, r_4, f_1^r, f_2^r) = \begin{cases} (sr_1, sr_2, sr_3, sr_4, sf_1^r, sf_2^r) & \text{if } s > 0, \\ (0,0,0,0,0,0) & \text{if } s = 0, \\ (sr_4, sr_3, sr_2, sr_1, sf_2^r, sf_1^r) & \text{if } s < 0. \end{cases}$$

Moreover, the fuzzy exponential function will be defined in the following way:

(iii) $e^{(r_1, r_2, r_3, r_4, f_1^r, f_2^r)} = (e^{r_1}, e^{r_2}, e^{r_3}, e^{r_4}, e^{f_1^r}, e^{f_2^r})$

Definition 2.9 [16]

(i) $(r_1, r_2, r_3, r_4, f_1^r, f_2^r) + (p_1, p_2, p_3, p_4, f_1^p, f_2^p)$
$= (r_1 + p_1, r_2 + p_2, r_3 + p_3, r_4 + p_4, f_1^r + f_1^p + f_r^2 + f_2^p)$.

(ii) $(r_1, r_2, r_3, r_4, f_1^r, f_2^r) - (p_1, p_2, p_3, p_4, f_1^p, f_2^p)$

$= (r_1 - p_4, r_2 - p_3, r_3 - p_2, r_4 - p_1, f_1^r - f_2^p, f_2^p - f_1^p)$.

(iii) $(r_1, r_2, r_3, r_4, f_1^r, f_2^r) \cdot (p_1, p_2, p_3, p_4, f_1^p, f_2^p)$

$$= \left\{ \begin{array}{l} \min(r_1p_1, r_1p_4, r_4p_1, r_4p_4), \min(r_2p_2, r_2p_3, r_3p_2, r_3p_3), \\ \max(r_2p_2, r_2p_3, r_3p_2, r_3p_3), \\ \max(r_1p_1, r_1p_4, r_4p_1, r_4p_4), \\ \min\{f_1^r f_1^p, f_1^r f_2^p, f_2^r f_1^p f_2^r f_2^p\}, \max\{f_1^r f_1^p, f_1^r f_2^p, f_2^r f_1^p, f_1^r f_2^p\} \end{array} \right\}$$

(iv) If $0 \notin [p_1, p_4]$ then

$(r_1, r_2, r_3, r_4, f_1^r, f_2^r)/(p_1, p_2, p_3, p_4, f_1^p, f_2^p)$

$= (r_1, r_2, r_3, r_4, f_1^r, f_2^r) \cdot (1/p_4, 1/p_3, 1/p_2, 1/p_1, 1/f_2^p, 1/f_1^p)$.

It is easy to show that the results of the operations from Definitions 2.8 and 2.9 are fuzzy numbers.

The following lemma is a direct consequence of Lemma 2.1 and Definition 2.7.

Lemma 2.3 [16] Let $\tilde{r}$, $\tilde{p}$, $\tilde{t}$ be arbitrary fuzzy numbers such that $\tilde{p} \cdot \tilde{t} = 0$. Then $\tilde{r} \cdot (\tilde{p} + \tilde{t}) = \tilde{r} \cdot \tilde{p} + \tilde{r} \cdot \tilde{t}$ holds.

2.4 Two Dimensional Fuzzy Numbers

The basic concept of one dimensional fuzzy numbers is used to derive the fuzzy numbers of dimension two [14].

Definition 2.10 [15] Let $C_{F_{\alpha_i}}$ and $D_{F_{\alpha_i}}$ be the set of all fuzzy numbers with membership value of $\alpha_i \in [0,1]$. The fuzzy numbers of dimension two is a subset of the cartesian product of two fuzzy numbers i.e.

$$C_{F_{\alpha_i}} \times D_{F_{\alpha_i}} = \{(x,y)|x \in C_{F_{\alpha_i}}, y \in D_{F_{\alpha_i}}\} \subset C_F \times D_F \tag{2.6}$$

The fuzzy numbers of dimension two must satisfy the following properties

(i) $\forall x_0 \in C_F$:

$$\mu_R(x_0, y) \in [0,1] \tag{2.7}$$

is a convex membership function

(ii) $\forall y_0 \in D_F$:

$$\mu_R(x, y_0) \in [0,1] \tag{2.8}$$

is a convex membership function

(iii) $\forall \alpha \in [0,1]$ and for all α-level:

$$\{R\}_\alpha = \{(x,y)|(x,y) \in C_F \times D_F, \mu_R(x,y) \geqslant \alpha\} \tag{2.9}$$

is a convex surface.

(iv) $\exists (x_n, y_n) \in C_F \times D_F$:

$$\mu(x_n, y_n) = 1 \tag{2.10}$$

is a normal surface.

Now, let us discuss some examples of Cartesian product of two one dimensional fuzzy numbers.

Definition 2.11 [15] Let $C_{F_{\alpha_i}}$ and $D_{F_{\alpha_i}}$ be the set of all triangular fuzzy numbers with membership value of α_i, where $i \in [0,1]$. The pyramidal fuzzy numbers is a subset of the Cartesian product of two fuzzy sets

$$C_{F_{\alpha_i}} \times D_{F_{\alpha_i}} = \{(x,y)|x \in C_{F_{\alpha_i}}, y \in D_{F_{\alpha_i}}\} \subset C_F \times D_F \tag{2.11}$$

Theorem 2.3 The pyramidal fuzzy numbers is fuzzy numbers of dimension two.

Theorem 2.4 The flatten top pyramidal fuzzy numbers is fuzzy numbers of dimension two.

Definition 2.12 [15] Let $A_{F_{\alpha_i}}$ and $B_{F_{\alpha_i}}$ be the set of all parabolic triangular fuzzy numbers with membership value of α_i, where $i \in [0, 1]$. The two dimensional fuzzy numbers is a subset of the cartesian product of two fuzzy sets

$$A_{F_{\alpha_i}} \times B_{F_{\alpha_i}} = \{(x, y) | x \in A_{F_{\alpha_i}}, y \in B_{F_{\alpha_i}}\} \subset A_F \times B_F \tag{2.12}$$

Theorem 2.5 The paraboloid fuzzy numbers is fuzzy numbers of dimension two.

Theorem 2.3,2.4,2.5 is proven in [14, 15]. Next we will discuss domain of confidence and some properties of fuzzy numbers of dimension two.

2.4.1 Domain of Confidence

The concept of domain of confident in $\Re^2$ is a generalization of that of the interval of confidence of one dimensional fuzzy numbers ($\Re$). It is a convex surface in $\Re^2$. Recalled that the conditions under which surface is convex in $\Re^2$. An ordinary subset in $\Re^2$ is convex if only if, considering two points of this subset, $(x_1) = (x_1^{(1)}, {x_1}^{(2)})$ and $(x_2) = (x_2^{(1)}, {x_2}^{(2)})$. Then any point $(x_3) = (x_3^{(1)}, {x_3}^{(2)})$ located in between (x_1) and (x_2) belong to the subset [13]. For this reason, the subset or surface S in Figure2.7 is convex and that in Figure 2.8 is not convex, since (x_3) does not belong to S.

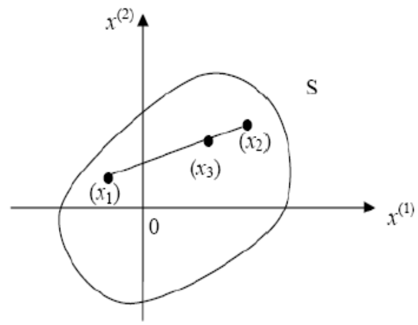

Figure 2.7: Definition of convex surface

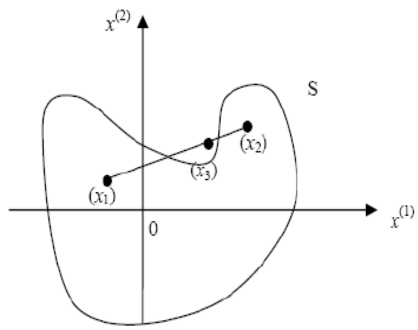

Figure 2.8: Definition of non-convex surface

2.4.2 Addition of Two Domain of Confidence

Let D_1 be a domain or a subset of $\Re^2$ and consider a point of (x_1) of D_1, which we denote as follows:

$$(x_1) = (x_1^{(1)}, x_1^{(2)}) \tag{2.13}$$

Similarly, we let D_2 be a domain or subset of $\Re^2$ and consider a point of (x_2) of D_2 as follows:

$$(x_2) = (x_2^{(1)}, x_2^{(2)}) \tag{2.14}$$

The addition of D_1 and D_2 is $D = D_1 + D_2$ and is defined by

$$D = \{(x^{(1)}, x^{(2)}) | (x^{(1)}, x^{(2)}) = (x_1^{(1)} + x_2^{(1)}, x_1^{(2)} + x_2^{(2)}), (x_1^{(1)}, x_1^{(2)}) \in D_1, (x_2^{(1)}, x_2^{(2)}) \in D_2\} \tag{2.15}$$

or in short,

$$D = \{(x)|(x) = (x_1) + (x_2), (x_1) \in D_1, (x_2) \in D_2\} \tag{2.16}$$

Theorem 2.6 [13] Let $D_1 \subset \Re^2$ and $D_2 \subset \Re^2$ be convex domains, $(x_1) \in D_1$, $(x_2) \in D_2$. If we build a domain D using addition, we will have

$$(x) = (x_1) + (x_2) \tag{2.17}$$

then the D is a convex domain in $\Re^2$.

Example 2.1 Consider rectangular domains D_1 and D_2 shown in Figure 2.9 defined by their respective vertices.

$$\begin{aligned} D_1 &= ((a_1), (b_1), (c_1), (d_1)) \\ &= ((2, -4), (7, -4), (7, -2), (2, -2)) \end{aligned}$$

and

$$\begin{aligned} D_2 &= ((a_2), (b_2), (c_2), (d_2)) \\ &= ((9, 2), (12, 2), (12, 6), (9, 6)) \end{aligned}$$

It is trivial to show that the addition of these two rectangular domains gives a rectangular domain that can immediately be obtained by adding of points that are vertices of the rectangles. Of course, if the edges are parallel to the axis we have $0x_1$ and $0x_2$, respectively. We then obtain

$$\begin{aligned} D = D_1 + D_2 &= ((a_1) + (a_2), (b_1) + (b_2), (c_1) + (c_2), (d_1) + (d_2)) \\ &= ((2, -4) + (9, 2), (7, -4) + (12, 2), (7, -2) + (12, 6), (2, -2) + (9, 6)) \\ &= ((11, -2), (19, -2), (19, 4)(11, 4)) \end{aligned}$$

this result shown in Figure 2.9 □

The following Theorem 2.7 and Theorem 2.8 are proven in [14]

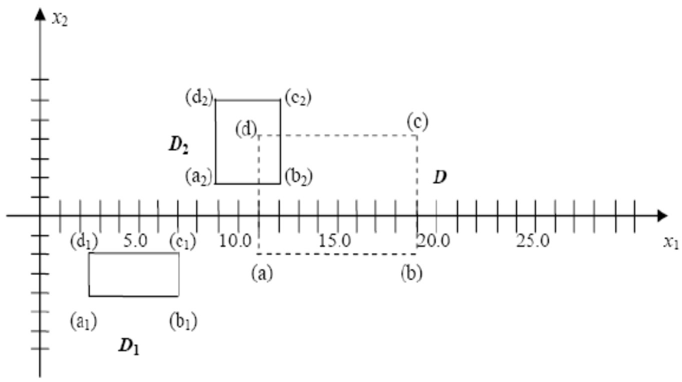

Figure 2.9: Sum of two fuzzy rectangular domains, $D = D_1 + D_2$

Theorem 2.7 Let $D_1 \subset \Re^2$ be convex domains and $D_2 \subset \Re^2$ be nonconvex domain, $(x_1) \in D_1$, and $(x_2) \in D_2$. Suppose $\lambda, \lambda' \in [0,1]$ and $\lambda \leqslant \lambda'$, if we build a domain D using addition, we have

$$(x) = (x_1) + (x_2)$$

then the domain is a convex domain in $\Re^2$.

Theorem 2.8 Let $D_1 \subset \Re^2$ and $D_2 \subset \Re^2$ be a nonconvex domain, $(x_1) \in D_1$, and $(x_2) \in D_2$. , if we build a domain D using addition, we have

$$(x) = (x_1) + (x_2)$$

then the domain is not necessary a convex domain in $\Re^2$.

2.4.3 Addition of Fuzzy Numbers of Dimension Two

Now, we will concentrate on the addition of fuzzy numbers of dimension two. For every $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \Re^2$:

$$\mu_{D_c}(z_1, z_2) = \bigvee_{(z_1,z_2)=(x_1,x_2)+(y_1,y_2)} \{\mu D_a(x_1, x_2) \bigwedge \mu D_b(y_1, y_2)\}; \tag{2.18}$$

where $D_a, D_b, D_c \subset \Re^2$ and $\mu D_a(x_1, x_2), \mu D_b(y_1, y_2), \mu_{D_c}(z_1, z_2) \in [0, 1]$. For each α-level, $\alpha \in [0, 1]$. Therefore, we have

$$D_{c,\alpha} = D_{a,\alpha} + D_{b,\alpha}$$

Example 2.2 Consider the pyramidal fuzzy numbers in $\Re^2$ with edges having rectangular bases parallel to the axis $0x_1$ and $0x_2$. Three coordinates are required for descriptions of triangular fuzzy numbers in $\Re$ and five coordinates are required for the descriptions of the pyramidal fuzzy numbers in $\Re$, as shown in Figure 2.10

$$\begin{aligned} D_a &= ((a_1), (a_2), (a_3), (a_4), (a_5)) \\ &= ((4, 5), (4, 9), (2, 9), (2, 5), (3, 7)) \\ D_b &= ((b_1), (b_2), (b_3), (b_4), (b_5)) \\ &= ((1, 11), (1, 14), (-2, 14), (-2, 11), (0, 12)) \end{aligned}$$

$$\begin{aligned} D_c &= ((a_1), (a_2), (a_3), (a_4), (a_5)) + ((b_1), (b_2), (b_3), (b_4), (b_5)) \\ &= ((a_1) + (b_1), (a_2) + (b_2), (a_3) + (b_3), (a_4) + (b_4), (a_5) + (b_5)) \\ &= ((4 + 1, 5 + 11), (4 + 1, 9 + 14), (2 - 2, 9 + 4), (2 - 2, 5 + 11), (3 + 0, 7 + 12)) \\ &= ((5, 16), (5, 23), (0, 23), (0, 16), (3, 19)) \end{aligned}$$

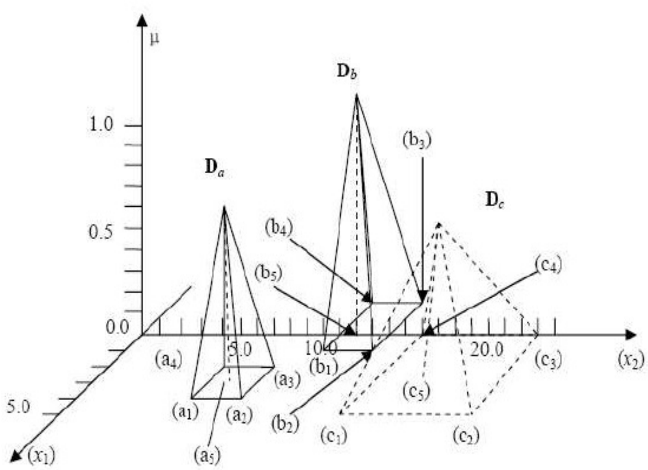

Figure 2.10: Sum of two pyramidal fuzzy numbers

□

2.5 n-Dimensional Fuzzy Numbers

Definition 2.13 Let $F(\Re^n)$ denoted the set of all fuzzy subset on $\Re^n$. If $u \in \Re^n$,$r \in (0,1]$, then we write $[u]^r = \{x|u(x) \geqslant r\}$. Suppose $u \in F(\Re^n)$, satisfies the following conditions.

(i) u is a normal fuzzy set, i.e. there exists $x_0\Re$ such that $u(x_0) = 1$;

(ii) u is a convex fuzzy set, i.e. $u(tx + (1-t)y \geqslant \min\{u(x), u(y)\}$, for $x, y \in \Re^n$ and $t \in (0,1]$;

(iii) u is upper semicontinuous;

(iv) $[u]^0 = \bigcup_{r\in(0,1]} [u]^r$ is bounded

then u is call a fuzzy numbers. We use D^n to denote the fuzzy number space [6].

Theorem 2.9 If $u \in D^n$, then

(i) $[u]^r$ is a nonempty bounded closed convex subset of $\Re^n, r \in (0,1]$;

(ii) $[u]^{r_1} \subseteq [u]^{r_2}$, whenever $0 \leqslant r_2 \leqslant r_1 \leqslant 1$;

(iii) if $r_n > 0$ and r_n converging to $r \in [0,1]$ is nondecreasing, then $\bigcap_{n=1}^{\infty} [u]^{r_n} = [u]^r$.

2.6 Summary

In this chapter, we devided the review into several sections. The first section discussed about the interval numbers and operations perform on them in general. The following 3 sections discussed on one, two, and n-dimensional fuzzy number and it properties. In the next chapter, the concept of capital budgeting problem in financial point of view will be stated.

CHAPTER 3

THE CONCEPT OF CAPITAL BUDGETING TECHNIQUE

3.1 Introduction

As mentioned before, a capital budgeting is the process of selecting the best investment, that is expected to result in benefits over a period more then one year. The aim of capital budgeting is to determine whether an organisation's long term investments such as new machinery, replacement machinery, new plants, new products, and research development projects are worth pursuing. It is budget for major capital, or investment and expenditures. The following sections will give the details of the important terminology involved in capital budgeting.

3.2 Capital Budgeting Techniques

There are many techniques available to evaluate project. We start by defining the problem and listing the capital investment methods used in practise in their classical form. Further in the section all of them will be given a "fuzzified" form. Here we consider the following problem: There is a project or another type of investment which might be interesting for company to under take. We want to get a numerical estimation of this investment which will allow us to say whether to accept it or not and to compare this project with alternative ones. The following data for the project must be given.

(i) n- the duration of the project in years.

(ii) COF_i- cash outflow at the end of the ith year, $i = 0, 1, ..., n$

(iii) CIF_i-cash inflow at the end of the ith year, $i = 0, 1, ..., n$

(iv) CF_i- cash flow, both yearly cash flows will be often summarized as a single cash flow CF_i, occurring at the end of the ith year, $i = 0, 1, ..., n$. where

$$CF_0 = -COF_0 \qquad\qquad CF_i = CIF_i - COF_i$$

(v) r- the required rate of return of the investment or cost of capital.

for the way the r and various assumption that are made see scope of our research. Now, we will review more about capital budgeting methods as given in the following section.

3.3 Classical Capital Budgeting Methods [16]

Here are several methods used in practise to evaluate a project given by the data described above. E will denote the output of each method. Several estimations are possible:

- E may be the difference between all the revenue and all the expenditure related with the project.
- E may be the time it is going to take for the project to pay off the money which were initially invested in it.

E will serve as a basis for the rejection or acceptance decision. The methods used are:

(i) **Revenues per One Dollar Method**:

Revenues per one dollar method is a measure of the revenue generated per one dollar. This method allows for the analysis of a company's revenue generation and growth at the per dollar level, which can help investors to identify which products are high or low revenue-generators.

The advantages and disadvantages of this method are listed as follows :

Advantages

(a) Easy to compute and understand

Disadvantages

(a) Ignore time value of money

(b) Assumes net investment is written off at constant rate.

(c) No objective criterion for decision making.

The revenue per one dollar can be calculate as

$$E = \sum_{i=1}^{n} CIF_i / \sum_{i=1}^{n} COF_i. \tag{3.1}$$

The acceptance-rejection criteria are:

if E > Minimum Acceptable return, then accept.

if E <Minimum Acceptable return, then reject.

(ii) **Payback Period Method:**

Payback period is the number of years required to recover initial investment. It measures how quickly the project will return its original investment. The advantages and disadvantages of this method as the following :

Advantages

(a) Easy to compute and interpret.

(b) Provides a measure of liquidity(measure the time required to recover initial expenditure).

(c) Provide a crude measure of risk (project with longer payback periods are more risky).

Disadvantages

(a) Does not consider time value of money.

(b) Does not measure profitability of an investment as it ignore cash flow after payback period.

(c) No strong decision indicator to indicate whether project will increase firm's value.

(d) Does not consider the riskiness of future cash flows.

The payback period can be calculate as

$$E = \min_{k=1,\ldots,n} \left\{ k : \sum_{i=0}^{k} CF_i \geqslant 0, \infty \right\} \tag{3.2}$$

The acceptance-rejection criteria are:

if E < Maximum Acceptable period, then accept.

if E >Maximum Acceptable period, then reject.

(iii) **Net Present Value(NPV)Method**:

Net present value method will give a requirement of absolute value of project in term of today's dollars. The advantages and disadvantages of this method as the following :

Advantages

(a) Consider time value of money.

(b) Consider all cash flows.

(c) Considers the riskiness of future cash flows as risk is reflected in the discount rate.

(d) Is an indicator of whether project increases firm's value.

Disadvantages

(a) Required estimate of cost of capital.

(b) Is express of ringgit and not percentage.

The NPV can be calculate as

$$E = CF_0 + \sum_{i=0}^{n} \frac{CF_i}{(1+r)^i} \tag{3.3}$$

E represents the value of the project at the beginning of the year it is started.

The acceptance-rejection criteria are:

if E ≥ 0 , then accept.

if E ≤ 0 , then reject.

(iv) **Net Future Value (NFV)Method**:

Future Value is the amount of money that an investment made today (the present value) will grow to by some future date. Since money has time value, we naturally expect the future value to be greater than the present value.

Here we calculate the value of the project at the end of mth year.

$$E(m) = \sum_{i=0}^{n} CF_i(1+r)^m - 1 \tag{3.4}$$

for any $m \geqslant n$.

The acceptance-rejection criteria are:

if E ≥ 0 , then accept.

if E ≤ 0 , then reject.

(v) **The Modified Internal Rate of Return (MIRR) Method**:

While the IRR assumes the cash flows from a project are reinvested at the IRR, the MIRR assumes that postive cash flows are reinvested at the firm's cost of capital, and the intial outlays are financed at the firm's financing cost. Therefore, MIRR more accurately reflects the cost and profitability of a project. In this case no additional assumption about cash flows CF_i are necessary. E is the solution of the following equation:

$$\sum_{i=0}^{n} \frac{COF_i}{1+r}^i = \frac{\sum_{i=0}^{n} CIF_i(1+r)^{n-i}}{(1+E)^n} \tag{3.5}$$

On the left hand side of above equations we have the present value of all the outflows, and on the right hand side the future value of all the inflows reported afterwards to the present moment through the unknown modified internal rate of return.

The acceptance-rejection criteria are:

if E $\geq$ required rate of return, then accept.

if E $\leq$ required rate of return, then reject.

The advantage and disadvantages of this method as the following :

Advantages

(i) Consider time value of money.

(ii) Consider all cash flows.

(iii) Considers riskiness of future cash flows.

(iv) Is an indicator of whether project increase firm's value.

Disadvantages

(i) Not easy to calculate.

(ii) Selecting the highest MIRR may not result in value-maximizing decision in capital rationing and mutually exclusive project.

The following section describes the fuzzy capital budgeting technique where all the input data are transformed into fuzzy numbers.

3.4 Fuzzy Capital Budgeting with Crisp Project Duration [16]

For each of the methods described before, we will give a formula or procedure permitting to calculate the fuzzy evaluation $\widetilde{E}^*$ or $\widetilde{E}^{*\lambda}$ for $\lambda \in [0,1]$ of the given project.

(i) Fuzzy Revenue per One Dollar Method:

$$\widetilde{E}^* = \sum_{i=1}^{n} C\widetilde{I}F_i / \sum_{i=0}^{n} C\widetilde{O}F_i. \tag{3.6}$$

where $\widetilde{E}^*$ is a fuzzy numbers.

(ii) Fuzzy Payback Period Method: For each $\lambda \in [0,1]$, $E^{*\lambda}$ will calculated according to the following formula :

$$E^{*\lambda} = \left[\min_{k=1,\ldots,n} \left\{ k : \sum_{i=0}^{k} (CF_i^{\lambda})_2 \geqslant 0, \infty \right\}, \min_{k=1,\ldots,n} \left\{ k : \sum_{i=0}^{k} (CF_i^{\lambda})_1 \geqslant 0, \infty \right\} \right] \tag{3.7}$$

(iii) Fuzzy Net Present Value (NPV) Method:

$$\widetilde{E}^* = N\widetilde{P}V = \sum_{i=0}^{n} \frac{\widetilde{C}F_i}{(1+\widetilde{r})^i} \tag{3.8}$$

where $\widetilde{E}^*$ is a fuzzy number determined by a natural power of a fuzzy number is defined as a repeated multiplication.

(iv) Fuzzy Net Future Value (NFV) Method:

$$\widetilde{E}(m)^* = \sum_{i=0}^{n} \widetilde{C}F_i(1+\widetilde{r})^{m-i} \tag{3.9}$$

for any integer $m \geqslant n$ where $\widetilde{E}(m)^*$ is a fuzzy number determined by a natural power of a fuzzy number is defined as a repeated multiplication.

(v) The Fuzzy Modified Internal Rate of Return (MIRR) Method:

Analogously to the classical method, we assume that $\widetilde{C}F_i \geqslant 0$ for $i = 0, ..., n$. for each $\lambda \in [0, 1]$, $E^{*\lambda}$ will be calculated according to the following formula:

$$E^{*\lambda} = \left[(E^{*\lambda})_1, (E^{*\lambda})_2\right], \tag{3.10}$$

where $(E^{*\lambda})_1$, $(E^{*\lambda})_2$ are respectively solutions of the following equations:

$$\sum_{i=0}^{n} \frac{(COF_i^{\lambda})_2}{(1+r_1^{\lambda})^i} = \frac{\sum_{i=0}^{n}(CIF_i^{\lambda})_1(1+r_1^{\lambda})^{n-1}}{(1+(E^{*\lambda})_1)^n}$$

$$\sum_{i=0}^{n} \frac{(COF_i^{\lambda})_1}{(1+r_2^{\lambda})^i} = \frac{\sum_{i=0}^{n}(CIF_i^{\lambda})_2(1+r_2^{\lambda})^{n-1}}{(1+(E^{*\lambda})_2)^n}$$

note that $(CF_i^{\lambda})_1$, $(CF_i^{\lambda})_2$ denote the lower and upper end of the interval CF_i^{λ}

3.5 Summary

In this chapter, the review is devided into four sections. The first section discussed on capital budgeting. The following two sections discussed on data for the project and the concept of capital budgeting technique. Finally, in the last section the application of fuzzy numbers on capital budgeting technique is discussed. In the next chapter, the real application of capital budgeting problem with fuzzy method are shown. Then, for the purpose of comparison, the classical method will also be presented.

CHAPTER 4

IMPLEMENTATION

4.1 Shenandoah Furniture Capital Budgeting Problem

Shenandoah Furniture, Incorporation is a private company categorized under wood-household furn-upholstered manufacturers. Companies like Shenandoah Furniture, Inc usually offer: lane furniture, chairs massage, garden furniture, painted furniture and stanley furniture.

Shenandoah Furniture is considering replacing one of the machines in its manufacturing facility. The cost of new machine will be RM67370. The company expect to dispose of the machine in 6 years. The cash inflows are as below:

Year	Cash Inflow (RM)
1	20371
2	22657
3	16352
4	12314
5	10994
6	6448

The rate of return for the company is 12%. The uncertainties forecasted by the company's administration are $\pm$10% for all variables. All variables and its uncertainties are fuzzified and used in the calculation. By using the various evaluation methods stated in the last chapter. The result from this chapter can assist the Company's Chief Financial Officer to decide whether or not the firm should purchase the new asset.

For the purpose of comparison, both methods will be used in this chapter. Hence, the next two sections will discussed the computation of capital budgeting in classical and fuzzy methods.

4.2 Classical Capital budgeting Method

In this section, we want to solve the Shenandoah Furniture capital budgeting problem by using classical method. All the methods mentioned in Chapter 3 will be used.

4.2.1 Revenue per One Dollar Method

According to the Equation (3.1), we have

$$\begin{aligned} E &= \frac{RM20371 + RM22657 + RM16352 + RM12314 + RM10994 + RM6448}{RM67370} \\ &= \frac{RM89136}{RM67370} \\ &= RM1.32/\text{one dollar.} \end{aligned}$$

Therefore the revenue per one dollar is RM1.32. It means that if the company invest in this machine, they will received RM1.32 per each dollar invested.

4.2.2 Payback Period Method

According to the Equation (3.2), we have

$$
\begin{aligned}
i &= 0\ k = 0 \Rightarrow CF_0 = -RM67370 \\
i &= 1\ k = 1 \Rightarrow CF_0 + CF_1 = -RM67370 + RM20371 = -RM46999 \\
i &= 2\ k = 2 \Rightarrow CF_0 + \cdots + CF_2 = -RM46999 + RM22657 = -RM24342 \\
i &= 3\ k = 3 \Rightarrow CF_0 + \cdots + CF_3 = -RM24342 + RM16352 = -RM7990 \\
i &= 4\ k = 4 \Rightarrow CF_0 + \cdots + CF_4 = -RM7990 + RM12314 = RM4324
\end{aligned}
$$

Therefore the payback period is 4 years, that is if company invest in this machine,it will take 4 years to recover it initial investment.

4.2.3 Net Present Value Method

According to the Equation (3.3), we have

$$
\begin{aligned}
E &= -RM67370 + \frac{RM20371}{(1+0.12)} + \frac{RM22657}{(1+0.12)^2} + \frac{RM16352}{(1+0.12)^3} \\
&\quad + \frac{12314}{(1+0.12)^4} + \frac{RM10994}{(1+0.12)^5} + \frac{RM2918}{(1+0.12)^6} \\
&= -RM2149.75
\end{aligned}
$$

Therefore the NPV is -RM2149.75. When NPV is less than zero (a negative number), it means the discounted value of future cash flows is less than the initial investment. The company are receive a lower return than they desire. The company must pay RM2149.75 less for the property to achieve their desired return.

4.2.4 Net Future Value Method

According to the Equation (3.4), and let m = 7 (1 year after n) we have

$$
\begin{aligned}
E(7) &= -RM67370(1+0.12)^7 + RM20371(1+0.12)^6 + RM22657(1+0.12)^5 \\
&\quad + RM16352(1+0.12)^4 + RM12314(1+0.12)^3 + RM10994(1+0.12)^2 \\
&\quad + RM6448(1+0.12) \\
E(7) &= -RM4752.38
\end{aligned}
$$

Therefore, at 7^{th} year the NFV is -RM4752.38. It means that the value of this project at 7^{th} year are -RM4752.38.

4.2.5 Modified Internal Rate of Return Method

According to the Equation (3.5), as follows

$$
\begin{aligned}
\sum_{i=0}^{n} \frac{COF_i}{1+r}^{i} &= RM67370 \\
\frac{\sum_{i=0}^{n} CIF_i(1+r)^{n-i}}{(1+E)^n} &= \frac{RM20371(1+E)^5 + RM22657(1+E)^4 + RM16352(1+E)^3}{(1+E)^6} \\
&\quad + \frac{RM12314(1+E)^2 + RM10994(1+E) + RM6448}{(1+E)^6} \\
&= \frac{RM128734.23}{(1+E)^6}
\end{aligned}
$$

by equating the equations, we have

$$
\begin{aligned}
RM67370 &= \frac{RM128734.23}{(1+E)^6} \\
(1+E)^6 &= \frac{RM128734.23}{RM67370} \\
E &= (1.91)^{1/6} - 1 \\
&= 0.114 * 100\% \\
E &= 11.4\%.
\end{aligned}
$$

The value of modified internal rate of return is 11.4%, it means that 11.4% is the compound annual rate of return that the company will earn if it invests in the project and the given cash flows.

4.2.6 Summary of Classical Capital budgeting Methods

The following Table 4.1 show about the summary of classical capital budgeting method:

Table 4.1: Summary of Classical Capital Budgeting Methods

Methods	*Value*
Revenue per One Dollar	$RM1.32$
Payback Period	4years
Net present Value	$-RM2149.74$
Net Future Value	$-RM4752.38$
Modified Internal Rate of Return	11.4%

Next, the calculation of capital budgeting in the case of fuzzified cash flow, fuzzified rate of return and crisp project duration will be discussed by applying the concept of fuzzy numbers from Chapter 2.

4.3 Fuzzy Capital Budgeting with Crisp Project Duration

The Shenandoah Furniture problem has been fuzzified according to $\pm 10\%$ uncertainties.The project data will be presented in the form of trapezoidal fuzzy numbers.

From Definition **??** and Figure 2.6, the calculation of fuzzy cash outflows $(C\tilde{O}F)$,fuzzy cash inflows$(C\tilde{I}F)$and fuzzy rate of return $(\tilde{r})$ will be discussed.

The following calculation is based on the special case of trapezoidal fuzzy number (TrFN) when $r_2 = r_3$. The value 60633 is obtained from

$$67370 - (67370 \times 10\%) = 60633$$

next,

$$67370 + (67370 \times 10\%) = 74107$$

by applying the straight line equation, the non-decreasing equation is obtained as in Figure 4.1.

First, find the slope of the straight line, M. By considering $\lambda_1 = 1, \lambda_2 = 0$ and $x_1 = 60633, x_2 = 67370$.

$$\begin{aligned}
\frac{\lambda_2 - \lambda_1}{x_2 - x_1} &= M \\
\frac{1-0}{67370 - 60633} &= M \\
M &= \frac{1}{6737}
\end{aligned}$$

then, the straight line equation is

$$\begin{aligned}
\frac{\lambda - \lambda_2}{x - x_2} &= M \\
\frac{\lambda - 0}{x - 60633} &= \frac{1}{6737} \\
\lambda &= \frac{1}{6737}(x - 60633) \\
6737\lambda &= x - 60633 \\
x &= 6737\lambda + 60633
\end{aligned}$$

Now, considering $\lambda_1 = 1, \lambda_2 = 0$ and $x_1 = 67370, x_2 = 74107$. The non-increasing function as in Figure 4.1 is calculated as follows:

Similarly, find the slope of the line

$$\begin{aligned}
\frac{\lambda_2 - \lambda_1}{x_2 - x_1} &= M \\
\frac{0-1}{74107 - 67370} &= M \\
M &= -\frac{1}{6737}
\end{aligned}$$

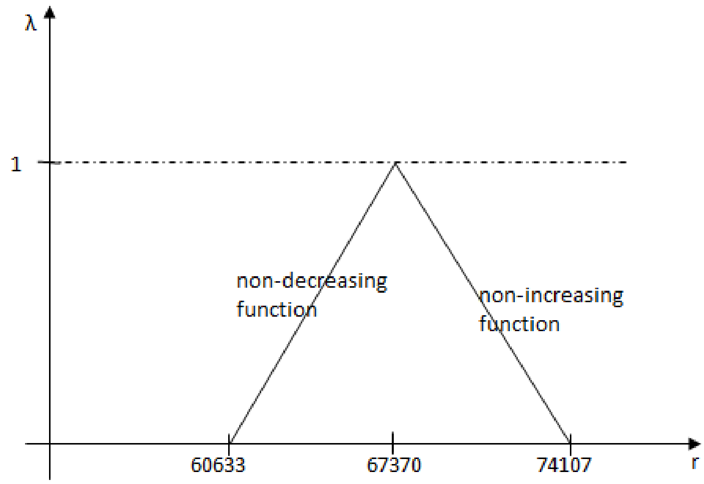

Figure 4.1: The Fuzzified Cash Outflows $(C\tilde{O}F)$

then, the equation is

$$\begin{aligned}
\frac{\lambda - \lambda_2}{x - x_2} &= M \\
\frac{\lambda - 0}{x - 74107} &= -\frac{1}{6737} \\
\lambda &= -\frac{1}{6737}(x - 74107) \\
-6737\lambda &= x - 74107 \\
x &= -6737\lambda + 74107
\end{aligned}$$

Finally, we have the fuzzified cash outflow $(C\tilde{O}F)$ as

$$C\tilde{O}F_0 = (60633, 67370, 67370, 74107, 6737\lambda + 60633, -6737\lambda + 74107)$$

The method of calculation for the case of fuzzy cash inflows and fuzzy rate of return are the same as the method of calculation for fuzzy cash outflows. The fuzzy cash outflows $(C\tilde{O}F)$,fuzzy cash inflows$(C\tilde{I}F)$and fuzzy rate of return $(\tilde{r})$

for the Shenandoah Furniture are as below:

$$\begin{aligned} C\tilde{O}F_0 &= (60633, 67370, 67370, 74107, 6737\lambda + 60633, -6737\lambda + 74107) \quad (4.1) \\ C\tilde{I}F_1 &= (18334, 20371, 20371, 22408, 2037\lambda + 18334, -2037\lambda + 22408) \\ C\tilde{I}F_2 &= (20391, 22657, 22657, 24923, 2266\lambda + 20391, -2266\lambda + 24923) \\ C\tilde{I}F_3 &= (14717, 16352, 16352, 17987, 1635\lambda + 14717, -1635\lambda + 17987) \\ C\tilde{I}F_4 &= (11083, 12314, 12314, 13545, 1231\lambda + 11083, -1231\lambda + 13545) \\ C\tilde{I}F_5 &= (9895, 10994, 10994, 12093, 1099\lambda + 9895, -1099\lambda + 12093) \\ C\tilde{I}F_6 &= (5803, 6448, 6448, 7093, 645\lambda + 5803, -645\lambda + 7093) \\ \tilde{r} &= (0.11, 0.12, 0.12, 0.13, 0.01\lambda + 0.11, -0.01\lambda + 0.13) \\ n &= 6. \end{aligned}$$

The fuzzy cash flows ($\tilde{CF}$) is calculated according to the information given in Section 3.2. The calculation for fuzzy cash flows are the as the above calculations. The fuzzy cash flow are stated as below :

$$\begin{aligned} \tilde{CF}_0 &= (-74107, -67370, -67370, -60633, -6737\lambda - 74107, -6737\lambda - 60633) \\ \tilde{CF}_1 &= (18334, 20371, 20371, 22408, 2037\lambda + 18334, -2037\lambda + 22408) \\ \tilde{CF}_2 &= (20391, 22657, 22657, 24923, 2266\lambda + 20391, -2266\lambda + 24923) \\ \tilde{CF}_3 &= (14717, 16352, 16352, 17987, 1635\lambda + 14717, -1635\lambda + 17987) \\ \tilde{CF}_4 &= (11083, 12314, 12314, 13545, 1231\lambda + 11083, -1231\lambda + 13545) \\ \tilde{CF}_5 &= (9895, 10994, 10994, 12093, 1099\lambda + 9895, -1099\lambda + 12093) \\ \tilde{CF}_6 &= (5803, 6448, 6448, 7093, 645\lambda + 5803, -645\lambda + 7093) \end{aligned}$$

The rate of return ($\tilde{r}$) and the power of fuzzy number $(1+\tilde{r})^i$ for $i = 1, \cdots, 7$ which is defined as repeated multiplication are calculated . Then $(1+\tilde{r})^i$ for $i = 1, \cdots, 7$ will be writen in the following ways:

$$\begin{aligned} (1+\tilde{r})^2 &= (1.11, 1.12, 1.12, 1.13, 0.01\lambda + 1.11, -0.01\lambda + 1.13) \\ &\quad \cdot (1.11, 1.12, 1.12, 1.13, 0.01\lambda + 1.11, -0.01\lambda + 1.13) \end{aligned}$$

by applying the multiplications as defined in Definition 2.9,

$$= \begin{pmatrix} \min\{1.23, 1.25, 1.25, 1.28\}, \min\{1.25, 1.25, 1.25, 1.25\} \\ \max\{1.25, 1.25, 1.25, 1.25\}, \max\{1.23, 1.25, 1.25, 1.28\} \\ \min\{(0.01\lambda + 1.11)^2, (0.01\lambda + 1.11)(-0.01\lambda + 1.13), \\ -0.01\lambda + 1.13)(0.01\lambda + 1.11), (-0.01\lambda + 1.13)^2\} \\ \max\{(0.01\lambda + 1.11)^2, (0.01\lambda + 1.11)(-0.01\lambda + 1.13), \\ -0.01\lambda + 1.13)(0.01\lambda + 1.11), (-0.01\lambda + 1.13)^2\} \end{pmatrix}$$

$$(1 + \tilde{r})^2 = (1.23, 1.25, 1.25, 1.28, (0.01\lambda + 1.11)^2, (-0.01\lambda + 1.13)^2)$$

The next calculations are also same as the above calculations depend on the value of i. For example for $i = 3$ then the multiplication will be repeated by three times. For $i = 4$, the multiplication will be repeated four times and others.

$$\begin{aligned} \tilde{r} &= (0.11, 0.12, 0.12, 0.13, 0.01\lambda + 0.11, -0.01\lambda + 0.13) \\ (1 + \tilde{r}) &= (1.11, 1.12, 1.12, 1.13, 0.01\lambda + 1.11, -0.01\lambda + 1.13) \\ (1 + \tilde{r})^2 &= (1.23, 1.25, 1.25, 1.28, (0.01\lambda + 1.11)^2, (-0.01\lambda + 1.13)^2) \\ (1 + \tilde{r})^3 &= (1.37, 1.40, 1.40, 1.44, (0.01\lambda + 1.11)^3, (-0.01\lambda + 1.13)^3) \\ (1 + \tilde{r})^4 &= (1.52, 1.57, 1.57, 1.63, (0.01\lambda + 1.11)^4, (-0.01\lambda + 1.13)^4) \\ (1 + \tilde{r})^5 &= (1.69, 1.76, 1.76, 1.84, (0.01\lambda + 1.11)^5, (-0.01\lambda + 1.13)^5) \\ (1 + \tilde{r})^6 &= (1.87, 1.97, 1.97, 2.08, (0.01\lambda + 1.11)^6, (-0.01\lambda + 1.13)^6) \\ (1 + \tilde{r})^7 &= (2.08, 2.21, 2.21, 2.35, (0.01\lambda + 1.11)^7, (-0.01\lambda + 1.13)^7) \end{aligned}$$

Now we will find the value of fuzzy capital budgeting method for each methods defined in Chapter 3.

4.3.1 Fuzzy Revenue per One Dollar Method

According to the Equation (3.6) and Definition 2.9, we have

$$\begin{aligned}
\tilde{E}^* &= (80223, 89136, 89136, 98049, 8914\lambda + 80223, -8914\lambda + 98049) \\
&\cdot(\frac{1}{74107}, \frac{1}{67370}, \frac{1}{67370}, \frac{1}{60633}, \frac{1}{6737\lambda + 74107}, \frac{1}{-6737\lambda + 60633}) \\
&= \begin{pmatrix} \min(1.08, 1.32, 1.32, 1.62), \min(1.32, 1.32, 1.32, 1.32), \\ \max(1.32, 1.32, 1.32, 1.32), \max(1.08, 1.32, 1.32, 1.62), \\ \\ \min(\frac{8913\lambda+80223}{-6737\lambda+74107}, \frac{8913\lambda+80223}{6737\lambda+60633}, \frac{-8913\lambda+98049}{-6737\lambda+74107}, \frac{-8913\lambda+98049}{6737\lambda+60633}), \\ \\ \max(\frac{8913\lambda+80223}{-6737\lambda+74107}, \frac{8913\lambda+80223}{6737\lambda+60633}, \frac{-8913\lambda+98049}{-6737\lambda+74107}, \frac{-8913\lambda+98049}{6737\lambda+60633}) \end{pmatrix} \\
&= (1.08, 1.32, 1.32, 1.62, \frac{8913\lambda + 80223}{-6737\lambda + 74107}, \frac{-8913\lambda + 98049}{6737\lambda + 60633}) \\
\tilde{E}^* &= [1.08, 1.62]
\end{aligned}$$

Since the value of $\tilde{E}^*$ is $[1.08, 1.62]$, therefore the company have an oppurtunity to recieve the revenue atlease 1.08 and atmost 1.62 per one dollar invested.

4.3.2 Fuzzy Payback Period Method

According to the Equation (3.7), the value of $\tilde{E}^{*\lambda}$ were calculated for some value of λ. For $\lambda = 0$ then $\tilde{E}^{*\lambda}$ are calculated as below:

$$\begin{aligned}
k &= 1 : \tilde{CF}_0 + \tilde{CF}_1 = [8774\lambda - 55773, -8774\lambda - 38225] \Rightarrow [-55773, -38225]. \\
k &= 2 : \tilde{CF}_0 + \cdots + \tilde{CF}_2 = [11040\lambda - 35382, -11040\lambda - 13302] \Rightarrow [-35382, -13302]. \\
k &= 3 : \tilde{CF}_0 + \cdots + \tilde{CF}_3 = [12675\lambda - 20665, -12675\lambda + 4685] \Rightarrow [-20665, 4685]. \\
k &= 4 : \tilde{CF}_0 + \cdots + \tilde{CF}_4 = [13906\lambda - 9582, -13906\lambda + 18230] \Rightarrow [-9582, 18230]. \\
k &= 5 : \tilde{CF}_0 + \cdots + \tilde{CF}_5 = [15005\lambda + 313, -15005\lambda + 30323] \Rightarrow [313, 30323]. \\
k &= 6 : \tilde{CF}_0 + \cdots + \tilde{CF}_6 = [15650\lambda + 6116, -15650\lambda + 37416] \Rightarrow [6116, 37416]. \\
\tilde{E}^{*0} &= [3, 5]
\end{aligned}$$

Since at $\lambda = 0$, the value $\tilde{E}^{*0}$ is $[3, 5]$. The payback period of this project is [3,5]. It means that if the company invest in this machine, it will take minimum 3 years and maximum 5years to recover it initial investment.

For $\lambda = 0.5$

$$
\begin{aligned}
k &= 1 \Rightarrow [-51385, -42612] \\
k &= 2 \Rightarrow [-29862, -18822] \\
k &= 3 \Rightarrow [-14328, -1653] \\
k &= 4 \Rightarrow [-2629, 11277] \\
k &= 5 \Rightarrow [7815.5, 22821] \\
k &= 6 \Rightarrow [13941, 29591]
\end{aligned}
$$

$$E^{*0.5} = [4, 5]$$

Since at $\lambda = 0.5$, the value $\tilde{E}^{*0.5}$ is $[4, 5]$. The payback period of this project is [4,5]. It means that if the company invest in this machine, it will take minimum 4 years and maximum 5years to recover it initial investment.

For $\lambda = 1$

$$
\begin{aligned}
k &= 1 \Rightarrow [-46999, -46999] \\
k &= 2 \Rightarrow [-24342, -24342] \\
k &= 3 \Rightarrow [-7990, -7990] \\
k &= 4 \Rightarrow [4324, 4324] \\
k &= 5 \Rightarrow [15318, 15318] \\
k &= 6 \Rightarrow [21766, 21766]
\end{aligned}
$$

$$E^{*1} = [4, 4]$$

Since at $\lambda = 1$, the value $\tilde{E}^{*1}$ is $[4, 4]$. The payback period of this project is 4years. It means that if the company invest in this machine, it will take 4 years to recover it initial investment.

4.3.3 Fuzzy Net Present Value Method

According to the Equation (3.8),the formula can be expand it into the following form. The multiplication and the division of fuzzy number are calculated as Definition 2.9.

$$\tilde{E}^* = \frac{\tilde{CF}_0}{(1+r)^0} + \frac{\tilde{CF}_1}{(1+r)^1} + \frac{\tilde{CF}_2}{(1+r)^2} + \frac{\tilde{CF}_3}{(1+r)^3} + \frac{\tilde{CF}_4}{(1+r)^4} + \frac{\tilde{CF}_5}{(1+r)^5} + \frac{\tilde{CF}_6}{(1+r)^6}$$

for $i = 0$, we have the fuzzy cash outflow as below:

$$\frac{\tilde{CF}_0}{(1+r)^0} = (-74107, -67370, -67370, -60633, 6737\lambda - 74107, -6737\lambda - 60633)$$

for $i = 1$, then we have $\tilde{CF}_1 = (18334, 20371, 20371, 22408, 2037\lambda + 18334, -2037\lambda + 22408)$ and $(1+r)^1 = (1.11, 1.12, 1.12, 1.13, 0.01\lambda + 1.11, -0.01\lambda + 1.13)$. The multiplication and division of fuzzy number are calculated as Definition 2.9.

$$\begin{aligned}
\frac{\tilde{CF}_1}{(1+r)^1} &= (18334, 20371, 20371, 22408, 2037\lambda + 18334, -2037\lambda + 22408) \\
&\quad \cdot (\frac{1}{1.13}, \frac{1}{1.12}, \frac{1}{1.12}, \frac{1}{1.11}, \frac{1}{-0.01\lambda + 1.13}, \frac{1}{0.01\lambda + 1.11}) \\
&= \begin{pmatrix}
\min(16224.78, 16517.12, 19830.09, 20187.39), \\
\min(18188.39, 18188.39, 18188.39, 18188.39), \\
\max(18188.39, 18188.39, 18188.39, 18188.39), \\
\max(16224.78, 16517.12, 19830.09, 20187.39), \\
\min(\frac{2037\lambda+18334}{-0.01\lambda+1.13}, \frac{2037\lambda+18334}{0.01\lambda+1.11}, \frac{-2037\lambda+22408}{-0.01\lambda+1.13}, \frac{-2037\lambda+22408}{0.01\lambda+1.11}), \\
\max(\frac{2037\lambda+18334}{-0.01\lambda+1.13}, \frac{2037\lambda+18334}{0.01\lambda+1.11}, \frac{-2037\lambda+22408}{-0.01\lambda+1.13}, \frac{-2037\lambda+22408}{0.01\lambda+1.11})
\end{pmatrix} \\
&= \left(16224.78, 18188.39, 18188.39, 20187.39, \frac{2037\lambda + 18334}{-0.01\lambda + 1.13}, \frac{-2037\lambda + 22408}{0.01\lambda + 1.11}\right)
\end{aligned}$$

for $i = 2$, then we have $\tilde{CF}_2 = (20391, 22657, 22657, 24923, 2266\lambda + 20391, -2266\lambda + 24923)$ and $(1 + r)^2 = (1.23, 1.25, 1.25, 1.28, (0.01\lambda + 1.11)^2, (-0.01\lambda + 1.13)^2)$. The multiplication and division of fuzzy number are calculated as Definition 2.9.

$$\begin{aligned}
\frac{\tilde{CF}_2}{(1+r)^2} &= (20391, 22657, 22657, 24923, 2266\lambda + 20391, -2266\lambda + 24923) \\
&\cdot \left(\frac{1}{1.28}, \frac{1}{1.25}, \frac{1}{1.25}, \frac{1}{1.23}, \frac{1}{(-0.01\lambda + 1.13)^2}, \frac{1}{(0.01\lambda + 1.11)^2}\right) \\
&= \begin{pmatrix}
\min(15930.47, 16578.05, 16578.05, 20262.60), \\
\min(18125.60, 18125.60, 18125.60, 18125.60), \\
\max(18125.60, 18125.60, 18125.60, 18125.60), \\
\max(15930.47, 16578.05, 16578.05, 20262.60), \\
\min(\frac{2266\lambda+20391}{-0.01\lambda+1.13)^2}, \frac{2266\lambda+20391}{(0.01\lambda+1.11)^2}, \frac{-2266\lambda+24923}{-0.01\lambda+1.13)^2}, \frac{-2266\lambda+24923}{(0.01\lambda+1.11)^2}), \\
\max(\frac{2266\lambda+20391}{-0.01\lambda+1.13)^2}, \frac{2266\lambda+20391}{(0.01\lambda+1.11)^2}, \frac{-2266\lambda+24923}{-0.01\lambda+1.13)^2}, \frac{-2266\lambda+24923}{(0.01\lambda+1.11)^2}).
\end{pmatrix} \\
&= \left(15930.47, 18125.60, 18125.60, 20262.60, \frac{2266\lambda + 20391}{-0.01\lambda + 1.13)^2}, \frac{-2266\lambda + 24923}{(0.01\lambda + 1.11)^2}\right)
\end{aligned}$$

for $i = 3$, then we have $\tilde{CF}_3 = (14717, 16352, 16352, 17987, 1635\lambda + 14717, -1635\lambda + 17987)$ and $(1 + r)^3 = (1.37, 1.40, 1.40, 1.45, (0.01\lambda + 1.11)^3, (-0.01\lambda + 1.13)^3)$. The multiplication and division of fuzzy number are calculated as Definition 2.9.

$$\begin{aligned}
\frac{\tilde{CF}_3}{(1+r)^3} &= (14717, 16352, 16352, 17987, 1635\lambda + 14717, -1635\lambda + 17987) \\
&\cdot \left(\frac{1}{1.45}, \frac{1}{1.40}, \frac{1}{1.40}, \frac{1}{1.37}, \frac{1}{(-0.01\lambda + 1.13)^3}, \frac{1}{(0.01\lambda + 1.11)^3}\right) \\
&= \begin{pmatrix}
\min(10149.66, 10742.34, 10742.34, 13129.20), \\
\min(11680, 11680, 11680, 11680), \\
\max(11680, 11680, 11680, 11680), \\
\max(10149.66, 10742.34, 10742.34, 13129.20), \\
\min(\frac{1635\lambda+14717}{(-0.01\lambda+1.13)^3}, \frac{1635\lambda+14717}{(0.01\lambda+1.11)^3}, \frac{-1635\lambda+17987}{(-0.01\lambda+1.13)^3}, \frac{-1635\lambda+17987}{(0.01\lambda+1.11)^3}), \\
\max(\frac{1635\lambda+14717}{(-0.01\lambda+1.13)^3}, \frac{1635\lambda+14717}{(0.01\lambda+1.11)^3}, \frac{-1635\lambda+17987}{(-0.01\lambda+1.13)^3}, \frac{-1635\lambda+17987}{(0.01\lambda+1.11)^3}).
\end{pmatrix} \\
&= \left(10149.66, 11680, 11680, 13129.20, \frac{1635\lambda + 14717}{(-0.01\lambda + 1.13)^3}, \frac{-1635\lambda + 17987}{(0.01\lambda + 1.11)^3}\right)
\end{aligned}$$

for $i = 4$, then we have $\tilde{CF}_4 = (11083, 12314, 12314, 13545, 1231\lambda + 11083, -1231\lambda + 13545)$ and $(1 + r)^4 = (1.52, 1.57, 1.57, 1.63, (0.01\lambda + 1.11)^4, (-0.01\lambda + 1.13)^4)$. The multiplication and division of fuzzy number are calculated as Definition 2.9.

$$
\begin{aligned}
\frac{\tilde{CF}_4}{(1+r)^4} &= (11083, 12314, 12314, 13545, 1231\lambda + 11083, -1231\lambda + 13545) \\
&\quad \cdot \left(\frac{1}{1.63}, \frac{1}{1.57}, \frac{1}{1.57}, \frac{1}{1.52}, \frac{1}{(-0.01\lambda + 1.13)^4}, \frac{1}{(0.01\lambda + 1.11)^4}\right) \\
&= \begin{pmatrix}
\min(6799.39, 7291.45, 7291.45, 8911.18), \\
\min(7843.31, 7843.31, 7843.31, 7843.31), \\
\max(7843.31, 7843.31, 7843.31, 7843.31), \\
\max(6799.39, 7291.45, 7291.45, 8911.18), \\
\min(\frac{1231\lambda+11083}{(-0.01\lambda+1.13)^4}, \frac{1231\lambda+11083}{(0.01\lambda+1.11)^4}, \frac{-1231\lambda+13545}{(-0.01\lambda+1.13)^4}, \frac{-1231\lambda+13545}{(0.01\lambda+1.11)^4}), \\
\max(\frac{1231\lambda+11083}{(-0.01\lambda+1.13)^4}, \frac{1231\lambda+11083}{(0.01\lambda+1.11)^4}, \frac{-1231\lambda+13545}{(-0.01\lambda+1.13)^4}, \frac{-1231\lambda+13545}{(0.01\lambda+1.11)^4}).
\end{pmatrix} \\
&= \left(6799.39, 7843.31, 7843.31, 8911.18, \frac{1231\lambda + 11083}{(-0.01\lambda + 1.13)^4}, \frac{-1231\lambda + 13545}{(0.01\lambda + 1.11)^4}\right)
\end{aligned}
$$

for $i = 5$, then we have $\tilde{CF}_5 = (9895, 10994, 10994, 12093, 1099\lambda + 9895, -1099\lambda + 12093)$ and $(1+r)^5 = (1.69, 1.76, 1.76, 1.84, (0.01\lambda + 1.11)^5, (-0.01\lambda + 1.13)^5)$. The multiplication and division of fuzzy number are calculated as Definition 2.9.

$$
\begin{aligned}
\frac{\tilde{CF}_5}{(1+r)^5} &= (9895, 10994, 10994, 12093, 1099\lambda + 9895, -1099\lambda + 12093) \\
&\quad \cdot \left(\frac{1}{1.84}, \frac{1}{1.76}, \frac{1}{1.76}, \frac{1}{1.69}, \frac{1}{(-0.01\lambda + 1.13)^5}, \frac{1}{(0.01\lambda + 1.11)^5}\right) \\
&= \begin{pmatrix}
\min(5377.72, 5855.03, 6572.28, 7464.81), \\
\min(6246.59, 6246.59, 6246.59, 6246.59), \\
\max(6246.59, 6246.59, 6246.59, 6246.59), \\
\max(5377.72, 5855.03, 6572.28, 7464.81), \\
\min(\frac{1099\lambda+9895}{(-0.01\lambda+1.13)^5}, \frac{1099\lambda+9895}{(0.01\lambda+1.11)^5}, \frac{-1099\lambda+12093}{(-0.01\lambda+1.13)^5}, \frac{-1099\lambda+12093}{(0.01\lambda+1.11)^5}), \\
\max(\frac{1099\lambda+9895}{(-0.01\lambda+1.13)^5}, \frac{1099\lambda+9895}{(0.01\lambda+1.11)^5}, \frac{-1099\lambda+12093}{(-0.01\lambda+1.13)^5}, \frac{-1099\lambda+12093}{(0.01\lambda+1.11)^5}).
\end{pmatrix} \\
&= \left(5377.72, 6246.59, 6246.59, 7464.81, \frac{1099\lambda + 9895}{(-0.01\lambda + 1.13)^5}, \frac{-1099\lambda + 12093}{(0.01\lambda + 1.11)^5}\right)
\end{aligned}
$$

for $i = 6$, then we have $\tilde{CF}_6 = (5803, 6448, 6448, 7093, 645\lambda + 5803, -645\lambda + 7093)$ and $(1+r)^6 = (1.87, 1.97, 1.97, 2.08, (0.01\lambda + 1.11)^6, (-0.01\lambda + 1.13)^6)$. The multiplication and division of fuzzy number are calculated as Definition 2.9.

$$\begin{aligned}
\frac{\tilde{CF}_6}{(1+r)^6} &= (5803, 6448, 6448, 7093, 645\lambda+5803, -645\lambda+7093) \\
&\cdot \left(\frac{1}{2.08}, \frac{1}{1.97}, \frac{1}{1.97}, \frac{1}{1.87}, \frac{1}{(-0.01\lambda+1.13)^6}, \frac{1}{(0.01\lambda+1.11)^6}\right) \\
&= \left(\begin{array}{l}
\min(2789.9, 3103.21, 3103.21, 3793.05), \\
\min(3273.1, 3273.1, 3273.1, 3273.1), \\
\max(3273.1, 3273.1, 3273.1, 3273.1), \\
\max(2789.9, 3103.21, 3103.21, 3793.05), \\
\min(\frac{645\lambda+5803}{(-0.01\lambda+1.13)^6}, \frac{645\lambda+5803}{(0.01\lambda+1.11)^6}, \frac{-645\lambda+7093}{(-0.01\lambda+1.13)^6}, \frac{-645\lambda+7093}{(0.01\lambda+1.11)^6}), \\
\max(\frac{645\lambda+5803}{(-0.01\lambda+1.13)^6}, \frac{645\lambda+5803}{(0.01\lambda+1.11)^6}, \frac{-645\lambda+7093}{(-0.01\lambda+1.13)^6}, \frac{-645\lambda+7093}{(0.01\lambda+1.11)^6}).
\end{array}\right) \\
&= \left(2789.9, 3273.1, 3273.1, 3793.05, \frac{645\lambda+5803}{(-0.01\lambda+1.13)^6}, \frac{-645\lambda+7093}{(0.01\lambda+1.11)^6}\right)
\end{aligned}$$

Finally, we have the value of net present value as follows:

$$\begin{aligned}
\text{NPV} &= (-74107, -67370, -67370, -60633, 6737\lambda-74107, -6737\lambda-60633) \\
&+ \left(16224.78, 18188.39, 18188.39, 20187.39, \frac{2037\lambda+18334}{-0.01\lambda+1.13}, \frac{-2037\lambda+22408}{0.01\lambda+1.11}\right) \\
&+ \left(15930.47, 18125.60, 18125.60, 20262.60, \frac{2266\lambda+20391}{-0.01\lambda+1.13)^2}, \frac{-2266\lambda+24923}{(0.01\lambda+1.11)^2}\right) \\
&+ \left(10149.66, 11680, 11680, 13129.20, \frac{1635\lambda+14717}{(-0.01\lambda+1.13)^3}, \frac{-1635\lambda+17987}{(0.01\lambda+1.11)^3}\right) \\
&+ \left(6799.39, 7843.31, 7843.31, 8911.18, \frac{1231\lambda+11083}{(-0.01\lambda+1.13)^4}, \frac{-1231\lambda+13545}{(0.01\lambda+1.11)^4}\right) \\
&+ \left(5377.72, 6246.59, 6246.59, 7464.81, \frac{1099\lambda+9895}{(-0.01\lambda+1.13)^5}, \frac{-1099\lambda+12093}{(0.01\lambda+1.11)^5}\right) \\
&+ \left(2789.9, 3273.1, 3273.1, 3793.05, \frac{645\lambda+5803}{(-0.01\lambda+1.13)^6}, \frac{-645\lambda+7093}{(0.01\lambda+1.11)^6}\right) \\
&= \left(\begin{array}{l}
-16835.08, -2013.01, -2013.01, 13115.23, \\
6737\lambda - 74107 + \frac{2037\lambda+18334}{-0.01\lambda+1.13} + \frac{2266\lambda+20391}{-0.01\lambda+1.13)^2} + \frac{1635\lambda+14717}{(-0.01\lambda+1.13)^3} \\
+\frac{1231\lambda+11083}{(-0.01\lambda+1.13)^4} + \frac{1099\lambda+9895}{(-0.01\lambda+1.13)^5} + \frac{645\lambda+5803}{(-0.01\lambda+1.13)^6}, \\
-6737\lambda - 60633 + \frac{-2037\lambda+22408}{0.01\lambda+1.11} + \frac{-2266\lambda+24923}{(0.01\lambda+1.11)^2} + \frac{-1635\lambda+17987}{(0.01\lambda+1.11)^3} \\
+\frac{-1231\lambda+13545}{(0.01\lambda+1.11)^4} + \frac{-1099\lambda+12093}{(0.01\lambda+1.11)^5} + \frac{-645\lambda+7093}{(0.01\lambda+1.11)^6}
\end{array}\right) \\
\text{NPV} &= [-16835.08, 13115.23]
\end{aligned}$$

The possible value of net present value is [-16835.08,13115.23], since it is negative which means that the company is possible to get a lower return then they desire. The company must pay RM16835.08 less for the machine to achieve their desire return.

4.3.4 Fuzzy Net Future Value Method

According to the Equation (3.9)and we let $m = 7$ then we have

$$\begin{aligned}\tilde{E}^*(m) &= \tilde{CF}_0(1+\tilde{r})^7 + \tilde{CF}_1(1+\tilde{r})^6 + \tilde{CF}_2(1+\tilde{r})^5 + \tilde{CF}_3(1+\tilde{r})^4 \\ &\quad + \tilde{CF}_4(1+\tilde{r})^3 + \tilde{CF}_5(1+\tilde{r})^2 + \tilde{CF}_6(1+\tilde{r})^1 + \tilde{CF}_7(1+\tilde{r})^0\end{aligned}$$

$$\begin{aligned}\tilde{CF}_0(1+\tilde{r})^7 &= (-74107, -67370, -67370, -60633, 6737\lambda - 74107, -6737\lambda - 60633) \\ &\quad \cdot \big(2.08, 2.21, 2.21, 2.35, (0.01\lambda + 1.11)^7, (-0.01\lambda + 1.13)^7\big)\end{aligned}$$

$$= \begin{pmatrix} \min(-154142.56, -174151.45, -126116.64, -142487.55), \\ \min(-148887.7, -148887.7, -148887.7, -148887.7), \\ \max(-148887.7, -148887.7, -148887.7, -148887.7), \\ \max(-154142.56, -174151.45, -126116.64, -142487.55), \\ \min((6737\lambda - 74107)(0.01\lambda + 1.11)^7, (6737\lambda - 74107)(-0.01\lambda + 1.13)^7, \\ (-6737\lambda - 60633)(0.01\lambda + 1.11)^7, (-6737\lambda - 60633)(-0.01\lambda + 1.13)^7, \\ \\ \max((6737\lambda - 74107)(0.01\lambda + 1.11)^7, (6737\lambda - 74107)(-0.01\lambda + 1.13)^7, \\ (-6737\lambda - 60633)(0.01\lambda + 1.11)^7, (-(6737\lambda - 60633)(-0.01\lambda + 1.13)^7 \end{pmatrix}$$

$$= \begin{pmatrix} -174151.45, -148887.7, -148887.7, -126116.64, \\ (6737\lambda - 74107)(-0.01\lambda + 1.13)^7, (-6737\lambda - 60633)(0.01\lambda + 1.11)^7 \end{pmatrix}$$

$$\begin{aligned}\tilde{CF}_1(1+\tilde{r})^6 &= (18334, 20371, 20371, 22408, 2037\lambda + 18334, -2037\lambda + 22408) \\ &\quad \cdot \big(1.87, 1.97, 1.97, 2.08, (0.01\lambda + 1.11)^6, (-0.01\lambda + 1.13)^6\big)\end{aligned}$$

$$= \begin{pmatrix} \min(34284.58, 38134.72, 41902.96, 46608.64), \\ \min(40130.87, 40130.87, 40130.87, 40130.87), \\ \max(40130.87, 40130.87, 40130.87, 40130.87), \\ \max(34284.58, 38134.72, 41902.96, 46608.64), \\ \min((2037\lambda + 18334)(0.01\lambda + 1.11)^6, (2037\lambda + 18334)(-0.01\lambda + 1.13)^6, \\ (-2037\lambda + 22408)(0.01\lambda + 1.11)^6, (-2037\lambda + 22408)(-0.01\lambda + 1.13)^6, \\ \max((2037\lambda + 18334)(0.01\lambda + 1.11)^6, (2037\lambda + 18334)(-0.01\lambda + 1.13)^6, \\ (-2037\lambda + 22408)(0.01\lambda + 1.11)^6, (-2037\lambda + 22408)(-0.01\lambda + 1.13)^6 \end{pmatrix}$$

$$= \begin{pmatrix} 34284.58, 40130.87, 40130.87, 46608.64, \\ (2037\lambda + 18334)(0.01\lambda + 1.11)^6, (-2037\lambda + 22408)(-0.01\lambda + 1.13)^6 \end{pmatrix}$$

$$\begin{aligned} \tilde{CF}_2(1+\tilde{r})^5 &= (20391, 22657, 22657, 24923, 2266\lambda + 20391, -2266\lambda + 24923) \\ &\quad \cdot \left(1.69, 1.76, 1.76, 1.84, (0.01\lambda + 1.11)^5, (-0.01\lambda + 1.13)^5\right) \end{aligned}$$

$$= \begin{pmatrix} \min(34460.79, 37519.44, 37519.44, 45858.32), \\ \min(39876.32, 39876.32, 39876.32, 39876.32), \\ \max(39876.32, 39876.32, 39876.32, 39876.32), \\ \max(34460.79, 37519.44, 37519.44, 45858.32), \\ \min((2266\lambda + 20391)(0.01\lambda + 1.11)^5, (2266\lambda + 20391)(-0.01\lambda + 1.13)^5, \\ (-2266\lambda + 24923)(0.01\lambda + 1.11)^5, (-2266\lambda + 24923)(-0.01\lambda + 1.13)^5, \\ \max((2266\lambda + 20391)(0.01\lambda + 1.11)^5, (2266\lambda + 20391)(-0.01\lambda + 1.13)^5, \\ (-2266\lambda + 24923)(0.01\lambda + 1.11)^5, (-2266\lambda + 24923)(-0.01\lambda + 1.13)^5 \end{pmatrix}$$

$$= \begin{pmatrix} 34460.79, 39876.32, 39876.32, 45858.32, \\ (2266\lambda + 20391)(0.01\lambda + 1.11)^5, (-2266\lambda + 24923)(-0.01\lambda + 1.13)^5 \end{pmatrix}$$

$$\begin{aligned} \tilde{CF}_3(1+\tilde{r})^4 &= (14717, 16352, 16352, 17987, 1635\lambda + 14717, -1635\lambda + 17987) \\ &\quad \cdot \left(1.52, 1.57, 1.57, 1.63, (0.01\lambda + 1.11)^4, (-0.01\lambda + 1.13)^4\right) \end{aligned}$$

$$= \begin{pmatrix} \min(22369.84, 23988.71, 23988.71, 29318.81), \\ \min(25672.64, 25672.64, 25672.64, 25672.64), \\ \max(25672.64, 25672.64, 25672.64, 25672.64), \\ \max(22369.84, 23988.71, 23988.71, 29318.81), \\ \min((1635\lambda + 14717)(0.01\lambda + 1.11)^4, (1635\lambda + 14717)(-0.01\lambda + 1.13)^4, \\ (-1635\lambda + 17987)(0.01\lambda + 1.11)^4, (-1635\lambda + 17987)(-0.01\lambda + 1.13)^4, \\ \max((1635\lambda + 14717)(0.01\lambda + 1.11)^4, (1635\lambda + 14717)(-0.01\lambda + 1.13)^4, \\ (-1635\lambda + 17987)(0.01\lambda + 1.11)^4, (-1635\lambda + 17987)(-0.01\lambda + 1.13)^4 \end{pmatrix}$$

$$= \begin{pmatrix} 22369.84, 25672.64, 25672.64, 29318.81, \\ (1635\lambda + 14717)(0.01\lambda + 1.11)^4, (-1635\lambda + 17987)(-0.01\lambda + 1.13)^4 \end{pmatrix}$$

$$\begin{aligned} \tilde{CF}_4(1+\tilde{r})^3 &= (11083, 12314, 12314, 13545, 1231\lambda + 11083, -1231\lambda + 13545) \\ &\quad \cdot \big(1.37, 1.40, 1.40, 1.44, (0.01\lambda + 1.11)^3, (-0.01\lambda + 1.13)^3\big) \end{aligned}$$

$$= \begin{pmatrix} \min(15183.71, 15959.52, 15959.52, 19504.80), \\ \min(17239.60, 17239.60, 17239.60, 17239.60), \\ \max(17239.60, 17239.60, 17239.60, 17239.60), \\ \max(15183.71, 15959.52, 15959.52, 19504.80), \\ \min((1231\lambda + 11083)(0.01\lambda + 1.11)^3, (1231\lambda + 11083)(0.01\lambda + 1.13)^3, \\ (-1231\lambda + 13545)(0.01\lambda + 1.11)^3, (-1635\lambda + 17987)(-0.01\lambda + 1.13)^3, \\ \max((1231\lambda + 11083)(0.01\lambda + 1.11)^3, (1231\lambda + 11083)(0.01\lambda + 1.13)^3, \\ (-1231\lambda + 13545)(0.01\lambda + 1.11)^3, (-1635\lambda + 17987)(-0.01\lambda + 1.13)^3 \end{pmatrix}$$

$$= \begin{pmatrix} 15183.71, 17239.60, 17239.60, 19504.80, \\ (1231\lambda + 11083)(0.01\lambda + 1.11)^3, (-1635\lambda + 17987)(-0.01\lambda + 1.13)^3 \end{pmatrix}$$

$$\begin{aligned} \tilde{CF}_5(1+\tilde{r})^2 &= (9895, 10994, 10994, 12093, 1099\lambda + 9895, -1099\lambda + 12093) \\ &\quad \cdot \big(1.23, 1.25, 1.25, 1.28, (0.01\lambda + 1.11)^2, (-0.01\lambda + 1.13)^2\big) \end{aligned}$$

$$= \begin{pmatrix} \min(12170.85, 12665.6, 14874.39, 15479.04), \\ \min(13742.5, 13742.5, 13742.5, 13742.5), \\ \max(13742.5, 13742.5, 13742.5, 13742.5), \\ \max(12170.85, 12665.6, 14874.39, 15479.04), \\ \min((1099\lambda + 9895)(0.01\lambda + 1.11)^2, (1099\lambda + 9895)(0.01\lambda + 1.13)^2, \\ (-1099\lambda + 12093)(0.01\lambda + 1.11)^2, (-1099\lambda + 12093)(-0.01\lambda + 1.13)^2, \\ \max((1099\lambda + 9895)(0.01\lambda + 1.11)^2, (1099\lambda + 9895)(0.01\lambda + 1.13)^2, \\ (-1099\lambda + 12093)(0.01\lambda + 1.11)^2, (-1099\lambda + 12093)(-0.01\lambda + 1.13)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 12170.85, 13742.5, 13742.5, 15479.04, \\ (1099\lambda + 9895)(0.01\lambda + 1.11)^2, (-1099\lambda + 12093)(-0.01\lambda + 1.13)^2 \end{pmatrix}$$

$$\begin{aligned} \tilde{CF}_6(1 + \tilde{r})^1 &= (5803, 6448, 6448, 7093, 645\lambda + 5803, -645\lambda + 7093) \\ &\quad \cdot \big(1.11, 1.12, 1.12, 1.13, (0.01\lambda + 1.11)^1, (-0.01\lambda + 1.13)^1\big) \end{aligned}$$

$$= \begin{pmatrix} \min(6441.33, 6557.39, 7873.23, 8013.09), \\ \min(7221.76, 7221.76, 7221.76, 7221.76), \\ \max(7221.76, 7221.76, 7221.76, 7221.76), \\ \max(6441.33, 6557.39, 7873.23, 8013.09), \\ \min((645\lambda + 5803)(0.01\lambda + 1.11)^1, (645\lambda + 5803)(0.01\lambda + 1.13)^1, \\ (-645\lambda + 7093)(0.01\lambda + 1.11)^1, (-645\lambda + 7093)(-0.01\lambda + 1.13)^1, \\ \max((645\lambda + 5803)(0.01\lambda + 1.11)^1, (645\lambda + 5803)(0.01\lambda + 1.13)^1, \\ (-645\lambda + 7093)(0.01\lambda + 1.11)^1, (-645\lambda + 7093)(-0.01\lambda + 1.13)^1 \end{pmatrix}$$

$$= \begin{pmatrix} 6441.33, 7221.76, 7221.76, 8013.09, \\ (645\lambda + 5803)(0.01\lambda + 1.11)^1, (-645\lambda + 7093)(-0.01\lambda + 1.13)^1 \end{pmatrix}$$

Therefore, we have NFV as

$$
\begin{aligned}
&= \left(\begin{array}{l} -174151.45, -148887.7, -148887.7, -126116.64, \\ (6737\lambda - 74107)(-0.01\lambda + 1.13)^7, (-6737\lambda - 60633)(0.01\lambda + 1.11)^7 \end{array}\right) \\
&+ \left(\begin{array}{l} 34284.58, 40130.87, 40130.87, 46608.64, \\ (2037\lambda + 18334)(0.01\lambda + 1.11)^6, (-2037\lambda + 22408)(-0.01\lambda + 1.13)^6 \end{array}\right) \\
&+ \left(\begin{array}{l} 34460.79, 39876.32, 39876.32, 45858.32, \\ (2266\lambda + 20391)(0.01\lambda + 1.11)^5, (-2266\lambda + 24923)(-0.01\lambda + 1.13)^5 \end{array}\right) \\
&+ \left(\begin{array}{l} 22369.84, 25672.64, 25672.64, 29318.81, \\ (1635\lambda + 14717)(0.01\lambda + 1.11)^4, (-1635\lambda + 17987)(-0.01\lambda + 1.13)^4 \end{array}\right) \\
&+ \left(\begin{array}{l} 15183.71, 17239.60, 17239.60, 19504.80, \\ (1231\lambda + 11083)(0.01\lambda + 1.11)^3, (-1635\lambda + 17987)(-0.01\lambda + 1.13)^3 \end{array}\right) \\
&+ \left(\begin{array}{l} 12170.85, 13742.5, 13742.5, 15479.04, \\ (1099\lambda + 9895)(0.01\lambda + 1.11)^2, (-1099\lambda + 12093)(-0.01\lambda + 1.13)^2 \end{array}\right) \\
&+ \left(\begin{array}{l} 6441.33, 7221.76, 7221.76, 8013.09, \\ (645\lambda + 5803)(0.01\lambda + 1.11)^1, (-645\lambda + 7093)(-0.01\lambda + 1.13)^1 \end{array}\right) \\
&= \left(\begin{array}{l} -49246.35, -5004.01, -5004.01, 38666.06, \\ (6737\lambda - 74107)(-0.01\lambda + 1.13)^7 + (2037\lambda + 18334)(0.01\lambda + 1.11)^6 \\ +(2266\lambda + 20391)(0.01\lambda + 1.11)^5 + (1635\lambda + 14717)(0.01\lambda + 1.11)^4 \\ +(1231\lambda + 11083)(0.01\lambda + 1.11)^3 + (1099\lambda + 9895)(0.01\lambda + 1.11)^2 \\ +(645\lambda + 5803)(0.01\lambda + 1.11)^1, \\ (-6737\lambda - 60633)(0.01\lambda + 1.11)^7 + (-2037\lambda + 22408)(-0.01\lambda + 1.13)^6 \\ +(-2266\lambda + 24923)(-0.01\lambda + 1.13)^5 + (-1635\lambda + 17987)(-0.01\lambda + 1.13)^4 \\ +(-1635\lambda + 17987)(-0.01\lambda + 1.13)^3 + (-1099\lambda + 12093)(-0.01\lambda + 1.13)^2 \\ +(-645\lambda + 7093)(-0.01\lambda + 1.13)^1. \end{array}\right) \\
&= [-49246.35, 38666.06]
\end{aligned}
$$

Hence, these value of interval is the amount of money that an investment made today will grow to by some future date.

4.3.5 Fuzzy Modified Internal Rate of Return Method

According to the Equation (3.10), we have

for $\lambda = 0$ and to find the value of $(E^{*\lambda})_1$

$$\begin{aligned}\sum_{i=0}^{n}\frac{(COF_i^0)_2}{(1+r_1^0)^i} &= \frac{(COF_0^0)_2}{(1+r_1^0)^0}+\frac{(COF_1^0)_2}{(1+r_1^0)^1}+\cdots+\frac{(COF_6^0)_2}{(1+r_1^0)^6}\\ &= 74107\end{aligned}$$

$$\begin{aligned}\frac{\sum_{i=0}^{n}(CIF_i^0)_1(1+r_1^0)^{n-i}}{(1+(E^{*0})_1)} &= \frac{CIF_0^0(1+r_1^0)^6+\cdots+CIF_6^0(1+r_1^0)^0}{(1+(E^{*0})_1)^6}\\ &= \frac{18334(1.11)^5+20391(1.11)^4+14717(1.11)^3}{(1+(E^{*0})_1)^6}\\ &\quad+\frac{11083(1.11)^2+9895(1.11)+5803}{(1+(E^{*0})_1)^6}\\ &= \frac{112418.07}{(1+(E^{*0})_1)^6}\end{aligned}$$

$$\begin{aligned}74107 &= \frac{112418.07}{(1+(E^{*0})_1)^6}\\ (1+(E^{*0})_1)^6 &= \frac{112418.07}{74107}\\ (E^{*0})_1 &= (1.517)^{1/6}-1\\ &= 0.072\times 100\\ &= 7.2\%\end{aligned}$$

Next, we find $(E^0)_2$

$$\begin{aligned}\sum_{i=0}^{n}\frac{(COF_i^0)_1}{(1+r_2^0)^i} &= \frac{(COF_0^0)_1}{(1+r_2^0)^0}+\frac{(COF_1^0)_1}{(1+r_2^0)^1}+\cdots+\frac{(COF_6^0)_1}{(1+r_2^0)^6}\\ &= 60633\end{aligned}$$

$$\begin{aligned}
\frac{\sum_{i=0}^{n}(CIF_i^0)_2(1+r_2^0)^{n-i}}{(1+(E^{*0})_2)} &= \frac{CIF_0^0(1+r_2^0)^6+\cdots+CIF_6^0(1+r_2^0)^0}{(1+(E^{*0})_2)^6} \\
&= \frac{22408(1.13)^5+24923(1.13)^4+17987(1.13)^3}{(1+(E^{*0})_2)^6} \\
&\quad + \frac{13545(1.13)^2+12093(1.13)+7093}{(1+(E^{*0})_2)^6} \\
&= \frac{145928.67}{(1+(E^{*0})_2)^6}
\end{aligned}$$

$$\begin{aligned}
60633 &= \frac{145928.67}{(1+(E^{*0})_2)^6} \\
(1+(E^{*0})_2)^6 &= \frac{145928.67}{60633} \\
(E^{*0})_2 &= (2.407)^{1/6}-1 \\
&= 0.157\times 100 \\
&= 15.7\%
\end{aligned}$$

Hence $E^{*0} = [7.2\%, 15.76\%]$ For $\lambda = 0$ the value of modified internal rate of return is [7.2%,15.76%], it means that [7.2%,15.76%] is the compound annual rate of return that the company will earn if it invests in the project and the given cash flows.

For $\lambda = 0.5$, we have

$$\begin{aligned}
COF_0^{0.5} &= [64001.5, 70738.5] \\
CIF_1^{0.5} &= [19352, 21389.5] \\
CIF_2^{0.5} &= [21524, 23790] \\
CIF_3^{0.5} &= [15534.5, 17169.5] \\
CIF_4^{0.5} &= [11698.5, 12929.5] \\
CIF_5^{0.5} &= [10444.5, 11543.5] \\
CIF_6^{0.5} &= [6125.5, 6770.5] \\
r^{0.5} &= [0.115, 0.125]
\end{aligned}$$

for $\lambda = 0.5$ and the value of $(E^{*\lambda})_1$

$$\sum_{i=0}^{n} \frac{(COF^0.5_i)_2}{(1+r^0.5_1)^i} = \frac{(COF^0.5_0)_2}{(1+r^0.5_1)^0} + \frac{(COF^0.5_1)_2}{(1+r^0.5_1)^1} + \cdots + \frac{(COF^0.5_6)_2}{(1+r^0.5_1)^6}$$
$$= 70738.5$$

$$\frac{\sum_{i=0}^{n}(CIF_i^0)_1(1+r_1^0)^{n-i}}{(1+(E^{*0.5})_1)} = \frac{CIF_0^0(1+r_1^0)^6 + \cdots + CIF_6^0(1+r_1^0)^0}{(1+(E^{*0.5})_1)^6}$$
$$= \frac{19352(1.115)^5 + 21524(1.115)^4 + 15534.5(1.115)^3}{(1+(E^{*0.5})_1)^6}$$
$$+ \frac{11698.5(1.115)^2 + 10444.5(1.115) + 6125.5}{(1+(E^{*0.5})_1)^6}$$
$$= \frac{120466.85}{(1+(E^{*0.5})_1)^6}$$

$$70738.5 = \frac{120466.85}{(1+(E^{*0.5})_1)^6}$$
$$(1+(E^{*0.5})_1)^6 = \frac{120466.85}{70738.5}$$
$$(E^{*0.5})_1 = (1.703)^{1/6} - 1$$
$$= 0.093 \times 100$$
$$= 9.3\%$$

Next, we find $(E^{0.5})_2$

$$\sum_{i=0}^{n} \frac{(COF_i^{0.5})_1}{(1+r_2^{0.5})^i} = \frac{(COF_0^{0.5})_1}{(1+r_2^{0.5})^0} + \frac{(COF_1^{0.5})_1}{(1+r_2^{0.5})^1} + \cdots + \frac{(COF_6^{0.5})_1}{(1+r_2^{0.5})^6}$$
$$= 64001.5$$

$$\frac{\sum_{i=0}^{n}(CIF_i^{0.5})_2(1+r_2^{0.5})^{n-i}}{(1+(E^{*0.5})_2)} = \frac{CIF_0^{0.5}(1+r_2^{0.5})^6 + \cdots + CIF_6^{0.5}(1+r_2^{0.5})^0}{(1+(E^{*0.5})_2)^6}$$
$$= \frac{21389.5(1.125)^5 + 23790(1.125)^4 + 17169.5(1.125)^3}{(1+(E^{*0.5})_2)^6}$$
$$+ \frac{12929.5(1.125)^2 + 11543.5(1.125) + 6770.5}{(1+(E^{*0.5})_2)^6}$$
$$= \frac{137218.81}{(1+(E^{*0.5})_2)^6}$$

$$
\begin{aligned}
64001.5 &= \frac{137218.81}{(1+(E^{*0.5})_2)^6} \\
(1+(E^{*0.5})_2)^6 &= \frac{137218.81}{64001.5} \\
(E^{*0.5})_2 &= (2.144)^{1/6}-1 \\
&= 0.136\times 100 \\
&= 13.6\%
\end{aligned}
$$

Hence $E^{*0.5}=[9.3\%,13.6\%]$

For $\lambda=0.5$ the value of modified internal rate of return is [9.3%,13.6%], it means that [9.3%,13.6%] is the compound annual rate of return that the company will earn if it invests in the project and the given cash flows.

For $\lambda=1$, we have

$$
\begin{aligned}
COF_0^1 &= [67370,67370] \\
CIF_1^1 &= [20371,20371] \\
CIF_2^1 &= [22657,22657] \\
CIF_3^1 &= [16352,16352] \\
CIF_4^1 &= [12314,12314] \\
CIF_5^1 &= [10994,10994] \\
CIF_6^1 &= [6448,6448] \\
r^1 &= [0.12,0.12]
\end{aligned}
$$

The value of $(E^{*\lambda})_1$

$$
\begin{aligned}
\sum_{i=0}^{n}\frac{(COF_i^1)_2}{(1+r_1^1)^i} &= \frac{(COF_0^1)_2}{(1+r_1^1)^0}+\frac{(COF_1^1)_2}{(1+r_1^1)^1}+\cdots+\frac{(COF_6^1)_2}{(1+r_1^1)^6} \\
&= 67370
\end{aligned}
$$

$$\begin{aligned}
\frac{\sum_{i=0}^{n}(CIF_i^1)_1(1+r_1^1)^{n-i}}{(1+(E^{*1})_1)} &= \frac{CIF_0^1(1+r_1^1)^6+\cdots+CIF_6^1(1+r_1^1)^0}{(1+(E^{*1})_1)^6} \\
&= \frac{20371(1.12)^5+22657(1.12)^4+16352(1.12)^3}{(1+(E^{*1})_1)^6} \\
&\quad + \frac{12314(1.12)^2+10994(1.12)+6448}{(1+(E^{*1})_1)^6} \\
&= \frac{128733.23}{(1+(E^{*1})_1)^6}
\end{aligned}$$

$$\begin{aligned}
67370 &= \frac{128733.23}{(1+(E^{*1})_1)^6} \\
(1+(E^{*1})_1)^6 &= \frac{128733.23}{67370} \\
(E^{*0.5})_1 &= (1.911)^{1/6}-1 \\
&= 0.114\times 100 \\
&= 11.4\%
\end{aligned}$$

Next, we find $(E^1)_2$

$$\begin{aligned}
\sum_{i=0}^{n}\frac{(COF_i^1)_1}{(1+r_2^1)^i} &= \frac{(COF_0^1)_1}{(1+r_2^1)^0}+\frac{(COF_1^1)_1}{(1+r_2^1)^1}+\cdots+\frac{(COF_6^1)_1}{(1+r_2^1)^6} \\
&= 67370
\end{aligned}$$

$$\begin{aligned}
\frac{\sum_{i=0}^{n}(CIF_i^1)_2(1+r_2^1)^{n-i}}{(1+(E^{*1})_2)} &= \frac{CIF_0^1(1+r_2^1)^6+\cdots+CIF_6^1(1+r_2^1)^0}{(1+(E^{*1})_2)^6} \\
&= \frac{20371(1.12)^5+22657(1.12)^4+16352(1.12)^3}{(1+(E^{*1})_1)^6} \\
&\quad + \frac{12314(1.12)^2+10994(1.12)+6448}{(1+(E^{*1})_1)^6} \\
&= \frac{128733.23}{(1+(E^{*1})_2)^6}
\end{aligned}$$

$$
\begin{aligned}
67370 &= \frac{128733.23}{(1+(E^{*1})_2)^6} \\
(1+(E^{*1})_2)^6 &= \frac{128733.23}{67370} \\
(E^{*1})_2 &= (1.911)^{1/6} - 1 \\
&= 0.114 \times 100 \\
&= 11.4\%
\end{aligned}
$$

Hence $E^{*1} = [11.4\%, 11.4\%]$

For $\lambda = 1$ the value of modified internal rate of return is [11.4%,11.4%], it means that 11.4% is the compound annual rate of return that the company will earn if it invests in the project and the given cash flows.

4.3.6 Summary of Fuzzy Capital Budgeting Methods

Table4.2 the summary of fuzzy capital budgeting methods:

Table 4.2: Summary of Fuzzy Capital Budgeting Measure

Methods	*Value*	
Revenue per One Dollar		[1.06,1.62]
Payback Period	$\lambda = 0$	[3,5]
	$\lambda = 0.5$	[4,5]
	$\lambda = 1$	[4,4]
Net present Value		[-16835.08,13115.23]
Net Future Value		[-49246.35,38666.06]
Modified Internal Rate of Return	$\lambda = 0$	[7.2%,15.76%]
	$\lambda = 0.5$	[9.5%,13.6%]
	$\lambda = 1$	[11.4%,11.4%]

In the next chapter will be discussed on details about the above result. The comparison of result for both classical and fuzzy methods are presented.

CHAPTER 5

CONCLUSION AND RECOMMENDATION

5.1 Introduction

This chapter provides a summary and an overall conclusion of the findings presented in this research. The first section of this chapter will be discussed about the result and discussion that obtain in this research. It also give the advantages of using fuzzy method rather than using classical one. Finally this chapter will give an outline of some further research which are worthwhile investigating in future.

5.2 Result and Discussion

Table 5.1 shows the comparison of result by classical and fuzzy methods. There are five methods of capital budgeting which are revenue per one dollar, payback period, net present value, net future value and modified internal rate of return. The result of the calculation are tabulated in Table 5.1.

From the table, for revenue per one dollar, by classical method the project was projected to earn 1.32 per one dollar invested. Meanwhile, by fuzzy methods the result show that the company has a potential to receive maximum 1.62 per one dollar. Therefore, the company should reject their proposal to purchase new

Table 5.1: The Comparison of Classical and Fuzzy Capital Budgeting Methods

Methods	*Classical*	*Fuzzy*	
Revenue per One Dollar	RM1.32		[1.06,1.62]
Payback Period	4 years	$\lambda = 0$	[3,5]
		$\lambda = 0.5$	[4,5]
		$\lambda = 1$	[4,4]
Net present Value	-RM2149.74		[-16835.08,13115.23]
Net Future Value	-RM4752.38		[-49246.35,38666.06]
Modified Internal Rate of Return	11.4%	$\lambda = 0$	[7.2%,15.76%]
		$\lambda = 0.5$	[9.5%,13.6%]
		$\lambda = 1$	[11.4%,11.4%]

machine, if the company required rate is greater than 1.32 per one dollar for classical method and 1.62 per one dollar for fuzzy method.

For the payback period method, the company required 4 years by classical methods to recover the initial investment and 3 to 5 years by fuzzy method. For the accept-reject decision, the project should be rejected if the required payback is less then 4 years for classical method and less then 3 years by fuzzy method.

For the net present value method, the present value is -RM2149.74 for the classical method. Meanwhile, by fuzzy method the present value are estimated from -RM16835.08 to RM13115.23. Therefore the company should reject the project, since the present value from both classical and fuzzy methods are involved negetive value.

Next, by classical in the net future value method , the future value is -RM4752.38 whereas by fuzzy methods the future value are from -RM49246.35 to RM38666.06. By acceptance and rejection critaria, the company should reject the project.

For the last method, by classical method the MIRR is 11.4%, while the rate are from 7.2% to 15.76% using fuzzy method. The acceptance and rejection criteria of MIRR, stated that the company should reject the project if the rate is less than the rate of return.

Actually, the value of result that we get from the classical methods is likely the same as fuzzy methods at the value $\lambda = 1$. But by fuzzy methods, we got the result in interval form, which present the minimum and maximum value for each measurement in capital budgeting. As well, by using fuzzy methods we can trace, the range of interval at any value of $\lambda \in [0, 1]$, which represent the degree of intuition of decision maker.

5.3 Conclusion

The initial task of the research was to review the definition, properties of one, two and n-dimensional fuzzy numbers. The applications of one dimensional fuzzy number on capital budgeting problem was presented.

In conclusion, we found that by classical method the result is in single value in order to make a decision,that is, it is not considering uncertainty. Meanwhile by fuzzy methods, the result that we obtained is more flexible than classical method. It is because, by fuzzy methods we can trace, the range of interval at any value of $\lambda \in [0, 1]$, which represent the degree of intuition of decision maker. Therefore the classical methods was enhanced by fuzzy method in capital budgeting problems.

From the result and discussion in Table 5.1, we conclude that all methods of capital budgeting used will give the same suggestion to the chief financial officer of Shenandoah Furniture, that is to reject their proposal in purchasing new machine for company production activity.

5.4 Further Research

From this research, the definition and some properties of one, two and n-dimensional fuzzy number have been review, then the application of one-dimensional fuzzy numbers on capital budgeting problem have been solved. Whereas the application of two dimensional fuzzy number is open for further research. Here, we have implement the fuzzy arithmatics into classical method to find the better result. It is also as an alternative way to evaluate the project. All the objetives of the research are achieved successfully. Based on this success one would think that there are many other areas in financials that are waiting extension to fuzzy set. The following research suggestions, in our opinion are worthwhile to explored:

(i) Further investigation and development of one dimensional fuzzy number on capital budgeting problem need to be carried out.

(ii) Implementation of two dimensional fuzzy number in financial problem.

(iii) The application of other type of fuzzy number of dimension one can also be investigated.

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