GEOMETRIC REPRESENTATIONS OF DISTINCT HAMILTONIAN CIRCUITS IN COMPLETE GRAPH DECOMPOSITION

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Abstrak

Penggambaran bagi perwakilan geometri untuk litar Hamiltonan berbeza dalam graf lengkap diperlukan untuk mengelakkan kemiripan struktur dalam aplikasi sebenar. Walau bagaimanapun, terdapat hanya sebilangan kecil kajian yang mempertimbangkan penggambaran graf sedangkan ramai penyelidik memberi tumpuan terhadap masa pengiraan. Oleh itu, kajian ini bertujuan untuk membina satu kaedah gambaran baharu iaitu Kaedah Rama-rama Separuh (HBM) untuk menyelesaikan senario tersebut. Bagi membina HBM, satu konsep Strategi Sayap baharu diperkenalkan untuk mendapatkan arah dari satu bucu ke bucu yang lain. Seterusnya, arah ini digunakan untuk memetakan bucu-bucu yang berbeza. Bagi mendapatkan litar Hamiltonan berbeza, konsep matriks transposisi digunakan untuk mengesan imej cermin bagi litar tersebut. Beberapa teorem dan lema baharu dibuktikan dalam penghuraian graf lengkap kepada litar Hamiltonan berbeza. Selanjutnya, hasil HBM ini digunakan untuk menyenaraikan semua pilih atur $n!$ dan beberapa teorem berkaitan dibentuk. Kesimpulannya, kajian ini berjaya menghasilkan satu kaedah baharu untuk menggambarkan litar Hamiltonan berbeza dalam penghuraian graf lengkap.

Kata kunci: Graf lengkap, Kaedah Rama-rama Separuh, Litar Hamilton, Penghuraian graf

Abstract

Visualization of geometric representations of distinct Hamiltonian circuits in complete graphs is needed to avoid structures resemblance in real application. However, there are only a few studies that consider graph visualization, whereas most researchers focus on computation time. Thus, this study aims to construct a novel picturing method called Half Butterfly Method (HBM) to address the aforementioned scenario. Towards developing HBM, the concept of Wing Strategy is introduced to create directions from one vertex to another vertex. Then, these directions are used to map distinct vertices. In order to obtain the distinct Hamiltonian circuits, the concept of matrix transpose is used to capture the mirror image of that circuit. Several new theorems and lemmas are proved in the decomposition of complete graphs into distinct Hamiltonian circuits. Furthermore, the result of HBM is applied to list $n!$ permutations and some related theorems are established. In conclusion, this study successfully produced a novel method to visualize distinct Hamiltonian circuit in complete graph decomposition.

Keywords: Complete graph, Hamiltonian circuit, Graph decomposition

Acknowledgement

In the name of Allah, the Most Gracious and the Most Merciful. Thank you Allah for Your will and blessings that I am able to complete this thesis.

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This thesis is dedicated to them.

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Declaration Associated with this Thesis

Darus, M. M., Ibrahim, H. & Karim, S. (2015). The construction of distinct circuits of length six for complete graph K_6. *International Journal of Applied Mathematics and Statistics, 53*(1), 17-31.

Darus, M. M., Ibrahim, H., Karim, S., Omar, Z., & Ahmad, N. (2014). Complete graph K_4 decomposition into circuits of length 4. *Proceedings of the 21st National Symposium on Mathematical Sciences (SKSM 21), 1605,* 605-610.

Darus, M. M., Ibrahim, H., & Karim, S. (2013). On the generation of distinct circuits via complete graph decomposition. *The First Innovation and Analytics Conference and Exhibition,* 26-32.

Table of Contents

List of Tables

List of Figures

List of Appendices

List of Symbols

$\boldsymbol{G}$	A graph consists of vertices and edges
$\boldsymbol{V}(\boldsymbol{G})$	The vertices of a graph G
$\boldsymbol{E}(\boldsymbol{G})$	The edges of a graph G
$\boldsymbol{G}(\boldsymbol{V},\boldsymbol{E})$	A graph G with the set of vertices $V(G)$ and edges $E(G)$
$\boldsymbol{K_n}$	A complete graph with n vertices
$\boldsymbol{C_n}$	A cycle with n vertices
$\boldsymbol{W_n}$	A wheel with n vertices
$\boldsymbol{K_n}^*$	A complete directed graph with n vertices
$\boldsymbol{C_n}^*$	A circuit with n vertices
$\boldsymbol{P_n}$	A path with n vertices
$\boldsymbol{K_{m,n}}$	A bipartite graph with two disjoint sets, m and n
$\boldsymbol{K_{m,n}^c}$	The complement set of bipartite graph $K_{m,n}$
$\boldsymbol{CTn}$	A complete-transposition graph with n vertices
$\boldsymbol{S_k}$	A star graph
$\boldsymbol{D_{r\times c}}$	A Cartesian product with r rows and c columns
$\mathbf{deg}\,(\boldsymbol{u})$	The degree of vertex u where u is a vertex in any graph
$\boldsymbol{x_i}$	A vertex in any graph, where $1 \leq i \leq n$
$\boldsymbol{n} \in \mathbb{Z}^+$	The total number of vertices of any graph, where n is positive
$\boldsymbol{\lambda}$	The block for each direction in WS
$\boldsymbol{\gamma}$	Total permutations of n elements

List of Abbreviations

WS	Wing Strategy
BS	Butterfly Strategy
S-R	Shift-and-Rotate Strategy
HBM	Half Butterfly Method

CHAPTER ONE
INTRODUCTION

Graph decomposition is an important research in graph theory because it can model several networks in our daily life such as social network, railway network and internet network (Kante, 2008). In the literature, several studies regarding graph decomposition have been focused such as the studies done by Granville, Moisiadis and Rees (1989); Adams, Bryant, Forbes and Griggs, (2012); and Yuan and Kuang, (2012); among others.

1.1 General Reviews on Graph

Let say, we are planning to visit five historic sites in Malacca without visiting each site more than once. Figure 1.1 shows the routes connecting each site. From the map, we need to manage possible route which will make sure we will never visit any site twice.

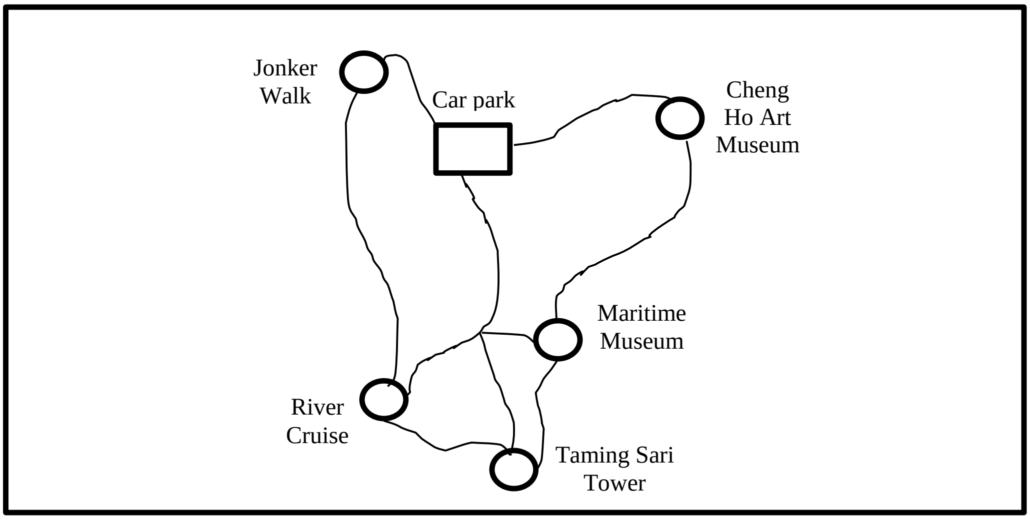

Figure 1.1. Routes for five historical sites in Malacca (this map has been modified from Google maps)

To be specific, we need to find a route (a circuit) from the car park to visit each site before returning to the car park, without visiting any site twice. This kind of circuit is known as a Hamiltonian circuit (HC) as it starts and ends at the same point (car park). Searching the HC in complete graph is one of the problem in graph theory.

Broadly speaking, this study is concerned on searching distinct HC in complete graphs. That is, we are interested in finding geometric representation for such circuits. Specifically, we shall focus on extracting the HC with different path which will determine the distinct circuits among them. Hamiltonian graphs are further discussed in Section 1.1.1.

As known, graph theory arises from the problem of Seven Bridges of Königsberg in Russia in 1735 (West, 2001). We shall follow standard graph theory notations. A graph G consists of a set of vertices $V(G)$ together with a set of edges $E(G)$. A vertex (plural is called vertices) is also known as node or point. A vertex with zero degree is called an isolated vertex. Otherwise, it is a non-isolated vertex. The degree of a vertex is the number of edges incident on it. The vertices that belong to an edge are called ends, endpoints or end-vertices of the edge. An edge is also known as line or link. Two vertices are also adjacent if they share a common edge.

A graph is called trivial when it has only one vertex and no edges. A graph is called a null graph when it has zero vertices and edges. An edgeless graph (or empty graph) is a graph when it has zero or more vertices but contains no edges (West, 2001; Skiena, 1990).

Graphs can be classified into many properties such as Petersen graph, perfect graph, cographs, chordal graphs, cycle, wheel, n-cubes, simple graph, subgraph, tree, directed graph (also known as digraph), undirected graph, partite graph, regular graph, path, circuit and complete graph. Further details on these graphs can be found in West (2001) and Rosen (2013).

A complete graph with n vertices (K_n) is a simple undirected graph where each vertex is adjacent to all the other vertices. A complete directed graph with n vertices (K_n^*) is a complete graph in which each edge is bidirected. A cycle with n vertices (C_n) is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A subgraph of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in $G.$ A path in graph G with n vertices (P_n) is a walk in which all the vertices $(v_1, v_2, \dots, v_n)$ are all distinct. A circuit (C_n^*) is a closed walk that starts and ends at the same vertex, and has no repeated edges or vertices except the start and end vertices. A Hamiltonian circuit is in the class of Hamiltonian graph. The following section discusses further on Hamiltonian graph.

1.1.1 Hamiltonian Graph

The Icosian game (or Hamilton's puzzle) was invented back in 1850s by Sir William Rowan Hamilton, about eight years before he died. He gave his name to the mathematical field of "Hamiltonian Graphs" (Wilson, 1988).

A Hamiltonian path (or traceable path) is any simple path that passes through every vertex in a connected graph exactly once. A Hamiltonian circuit (or Hamiltonian cycle) is a Hamiltonian path that starts at one vertex, visits every other vertex exactly once, and then returns to its starting vertex. This means that not all edges need to be traversed. In other words, a HC is a cycle that passes through every vertex exactly once (West, 2001).

A graph $G = (V, E)$ is a Hamiltonian graph if it possesses a HC. Obviously, if a graph has a HC, it has a Hamiltonian path. On the contrary, a graph with a Hamiltonian path may not contain a Hamiltonian circuit. Removing one edge of any Hamiltonian circuit one gets a Hamiltonian path. However, a Hamiltonian path can be extended to a HC only if its endpoints are adjacent. All Hamiltonian graphs are biconnected, but a biconnected graph is not always a Hamiltonian graph. A biconnected graph is a "nonseparable" graph, that is, the graph remains connected if any vertex is removed. For instance, a Petersen graph is a biconnected graph but it does not have any HC (West, 2001).

Since this study will decompose complete graph into HC, several related decompositions are discuss in the following section including Hamiltonian decomposition.

1.2 Graph Decomposition

Graph decomposition has been applied in various fields such as architecture (Raney, Cahill, Patterson & Bussey, 2012), biology (Botton, Fortz, Gouveia & Poss, 2013), chemical physics (Studer, Blosch, Friedli & Burg, 2007) and computer networking (Rapanotti, Hall, Jackson & Nuseibeh, 2004). Generally, graph decomposition involves partitioning the edge set of a graph into other graphs or subgraphs. Specifically, complete graph decomposition into HC is the partitioning of the edge set of the complete graph into HC with length n (Rao, 2006).

From the literature, several studies related to complete graph decomposition have been found e.g. complete graph decomposition into Hamiltonian graph (Kumar, 2003; Akbari & Herman, 2007; Cranston, 2008; Brualdi & Schroeder, 2011; and Shi & Niu, 2009), cycles (Froncek, Kovar & Kubesa, 2010; and Gyarfas, Ruszinko, Sarkozy & Szeremedi, 2011), one-factors (Kaski & Ostergard, 2009), n-suns (Anitha & Lekshmi, 2008), and Cartesian product (Fu, Hwang, Jimbo, Mutoh & Shiue, 2004).

Among these, complete graph decomposition into HC is attention-grabbing due to its vast advantages such as analyzing interconnection network in multicomputer (Chung, 2000; Choi, Lee & Chung, 2008), privacy data mining (Dong & Kresman, 2010), butterfly network (Hwang & Chen, 2000), and DNA physical mapping (Grebinski, 1998).

Decomposing a graph into HC is a challenging process due to the fact that the number of HC for any graph with n vertices is $n!$ (Riaz & Khiyal, 2006). This fact can resulted in high numbers of HC as n increases. Therefore, numerous studies have been concentrated on developing fast algorithms for finding HC (Dharwadker, 2004; Riaz & Khiyal, 2006; Babar, Khiyal & Saeed, 2006; Chalaturnyk, 2008). However, these studies do not presented the geometric representation of the HC.

1.3 Motivation of the Study

Geometric representation of a graph has played a significant role in dealing with fundamental problems of combinatorial and computational geometry since the approaches done by Avital and Hanani (1966), and Alon and Perles (1986). In addition, graph representation has also been used in solving mathematical problem for a visualization purpose (Toth & Valtr, 1999; and Lovasz, 2009).

Up until now, there are many advantages in solving mathematical problems when using graph representation. Among them, the representation of graphs can help researcher explore or explain data especially in handling huge data (Munzner, 2000; Samee & Rahman, 2007; Shai & Preiss, 1999; Jeron & Jard, 1995), to do data comparison and interpretation in regression model (Hurley &

Oldford, 2008), solve fingerprints classification which is mainly used in criminal investigation (Marcialis, Roli & Serrau, 2007), and solve various problems in our daily life such as file hierarchy on a computer system, browsing history and document management system (Herman, Melancon & Marshall, 2000).

In light of the importance mentioned above, graph representation has strong relationship with computational geometry and algorithms as well as connection to daily life problems. Indeed, Euler himself depended on the graph representation to solve his Königsberg bridge problem (Simonetto, 2011).

Current research to date tends to focus on theoretical works in computation time for decomposing complete graph into HC without classified the distinct circuits (Cranston, 2008; and Brualdi & Schroeder, 2011). Even though several fast algorithms have been discovered (Dharwadker, 2004; Riaz & Khiyal, 2006; Babar *et al.*, 2006; and Chalaturnyk, 2008), the geometric results of both HC and distinct HC have not been presented. This representation is vital to present the circuits directly. Therefore, the similarities or differences among the circuits can be perceived.

To the best of our knowledge, there is no study focusing on geometric representation of distinct HC in K_n. Hence, a study needs to be conducted to fill the existing gap. To ensure our proposed method plays a key role, the research questions below must be refined clearly.

1.4 Research Questions

1. How to determine the geometric solution for HC with different path from K_n?
2. How to develop a new method to visualize HC with different path from K_n?
3. What is the related application to the proposed method?

1.5 Research Objectives

The main objective of this study is to propose a new method in finding geometric representation of K_n decomposition into distinct HC with different path. In order to achieve this objective, the following sub-objectives need to be accomplished:

1. To investigate the structure and related techniques of K_n decomposition into HC.
2. To formulate a new geometric representation method to decompose K_n into distinct Hamiltonian circuits with different path.
3. To apply the proposed method in the context of listing permutations of n elements.

1.6 Thesis Outline

Chapter One briefly explains the general concept and some backgrounds of graphs and graph decompositions. Several K_n decompositions are discussed in this chapter such as K_n decomposition into Hamiltonian graphs. Then, the Hamiltonian graphs and Hamiltonian decompositions are also discussed.

Chapter Two presents the existing studies and related definitions. The existing Butterfly Strategy and Shift-and-Rotate Strategy are simplified clearly as a basis to our proposed method.

Chapter Three provides numerical examples of K_n decomposition into distinct Hamiltonian circuits with different path for cases $n = 3$, $n = 4$, $n = 5$ and $n = 6$ as basis to the general solution.

Chapter Four discusses the main contribution of this study: The Half Butterfly Method. Then, the general method to decompose K_n into distinct HC with different path for all n is also discussed. This chapter starts with some definitions that are needed to be understood before the general method is presented.

Chapter Five gives an attention to the application of our proposed method which is the listing permutation of n elements. Some examples for listing permutations of three, four and five elements are presented before the generalization is developed.

Chapter Six presents the overall conclusion regarding this study. The first section concludes the whole chapters of this study. Next, the second section presents the contributions of this study. The third section discusses some open problems and future researches, while the last section discusses the application of this study that may be applied in the real world problems.

CHAPTER TWO
RELATED MATERIALS AND DEFINITIONS

We start our investigation by discussing previous studies on the existing method of graph decomposition that relates to our proposed method. Then, the related definitions, several existing theorems, and some background of HC in a complete graph are provided.

We modified the Shift-and-Rotate Strategy and the Butterfly Strategy introduced by Gopal, Kothapalli, Venkaiah and Subramaniam (2007) in developing our new method. Therefore, the following section reviewed on both strategies that serves as a basis to our proposed method.

2.1 The Shift-and-Rotate Strategy and the Butterfly Strategy

The Shift-and-Rotate Strategy (S-R) and the Butterfly Strategy (BS) have been introduced to solve complete bipartite graph $K_{n,n}$ decomposition into one-factor (Gopal *et al.*, 2007). The one-factor of S-R, denoted by (a, b) where $a = k$ and $b = k + 1$ for $k = 1,2, \ldots ,2n$ have been considered. For each one-factor (a, b), a is the first endpoint and b is the second endpoint. The general one-factor is given as $\{(1, n + 1), (2, n + 2), \ldots , (n, 2n)\}$. The second endpoints shifted to the left by i places to gain the one-factor $\{(1, n + 1 + i), (2, n + 2 + i), \ldots , (n, 2n + i - n)\}$.

In a similar case, the BS also involved first endpoints and second endpoints. The general one-factor is given as $\{(1, n), (2, n+1), (3, n+2), \dots, (n, 2n)\}$ while the standard one-factor simply appeared as $(1, v_1), (2, v_2), \dots, (n, v_n)$ where $n + 1 \leq v_1, v_2, \dots, v_n \leq 2n$.

Our proposed method is motivated from these S-R and BS which will be discussed further in Chapter Four. The S-R and the BS decomposed bipartite graph into one-factor. However, we will modify these methods to decompose K_n into distinct HC with different path. In addition, the results of S-R and the BS are in one-factorization form whereas our results focused on geometric representation of the HC.

2.2 Related Materials and Definitions

This section presents several materials and definitions related to this study. Standard notations are used. It is assumed that the reader is familiar with the material summarized below. A thorough scan of this section is therefore strongly suggested for every reader.

The term *mapping* is usually shortened to map. A map is a way of associating objects to every element in a given set. So, a mapping $f: A \mapsto B$ is a function f that maps every element in a set A to every element in B.

Definition 2.1 Let $\begin{pmatrix} 1 & 2 & 3 \cdots n \\ a & b & c \cdots z \end{pmatrix}$ denotes the function of vertices set $\{1,2,3, \dots, n\}$, which maps $1 \longmapsto a$, $2 \longmapsto b$, …, $n \longmapsto z$, for $n \in \mathbb{Z}^+$ and a, b, c, z be the images.

Example 2.2 The function of vertices set $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ maps $1 \longmapsto 2$, $2 \longmapsto 3$, $3 \longmapsto 4$, and $4 \longmapsto 1$.

Definition 2.3 Let A and B be two circuits with n vertices. If the mapping of A and B is $(1, a)(2, b)(3, c) \dots (n, z)$ and $(z, n) \dots (c, 3)(b, 2)(a, 1)$ respectively, where $1 \leq a, b, c, z \leq n$, then A and B has an opposite mapping.

Example 2.4 Circuit A with mapping $(1,2)(2,3)(3,4)(4,5)(5,1)$ and circuit B with mapping $(1,5)(5,4)(4,3)(3,2)(2,1)$ have an opposite mapping.

A mapping can be obtained by using the idea of *product of transposition*. Any permutation set is a product of transposition.

Definition 2.5 Let $(x_1, x_2, x_3, \dots, x_{n-1}, x_n)$ be the vertices that form a cycle with n vertices. The product of transposition for $(x_1, x_2, x_3, \dots, x_{n-1}, x_n)$ is written as $(x_1, x_2)(x_2, x_3)(x_3, x_4) \dots (x_{n-1}, x_n)$.

Example 2.6 The product of transposition for $(1,4,2,6,3,5)$ is $(1,4)(4,2)(2,6)(6,3)(3,5)$.

The product of transposition can be used to determine whether a given graph has a HC. As known, a complete graph with three or more vertices is already known to be in a class of Hamiltonian graph (Rosen, 2013). Thus, K_n with $n \geq 3$ vertices possesses some HC.

Corollary 2.7 (Babar *et al.*, 2006) There exists a subgraph H of K_n with the following properties:

1. H contains every vertex of K_n,
2. H is connected,
3. H has the same number of edges as vertices, and
4. every vertex of H has degree 2.

Theorem 2.8 (West, 2001) A complete graph K_n with $n \geq 3$ vertices has $(n-1)!$ Hamiltonian circuits.

Theorem 2.9 (West, 2001) A complete graph K_n with $n \geq 3$ vertices has $\frac{(n-1)!}{2}$ distinct Hamiltonian circuits.

There are several conditions for two graphs to be isomorphic. We need to determine the isomorphic graph because we want to know when two graphs are essentially the same and when they are essentially different. The following paragraph discusses the *isomorphism classes* for Hamiltonian circuits.

Definition 2.10 Let $C_1^* = (V_1, E_1)$ and $C_2^* = (V_2, E_2)$ be two Hamiltonian circuits. $C_1^* \cong C_2^*$ if there is a one-to-one function $f: V(C_1^*) \to V(C_2^*)$ such that $uv \in E(C_1^*)$ if and only if $f(u)f(v) \in (C_2^*)$.

To prove C_1^* and C_2^* are isomorphic, a formula (picture) must be given, and both have the same number of vertices, the same number of edges, the same degrees for corresponding vertices, and the existence of one-to-one function.

Example 2.11 Let C_1^* and C_2^* be two Hamiltonian circuits as shown in Figure 2.1.

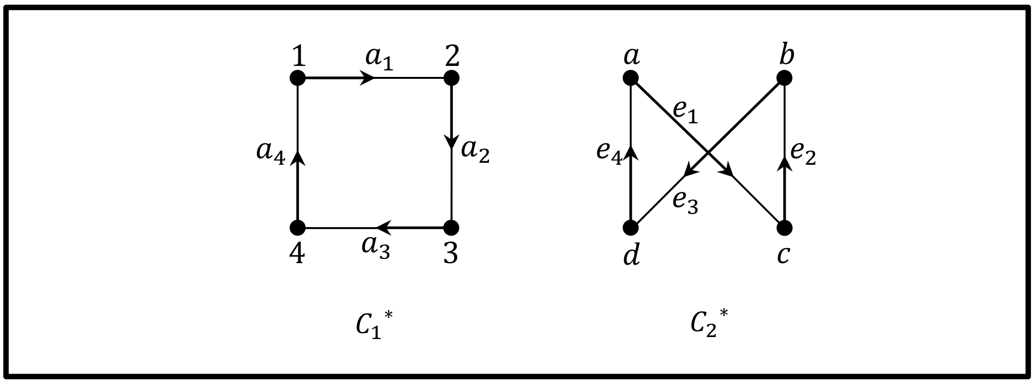

Figure 2.1. Isomorphic graphs

Both C_1^* and C_2^* have four vertices, four edges, and vertices of degree two. Since these invariants all agree, we relabel the circuits by the following mappings to investigate the one-to-one function.

$$f_1 : 1 \to a \qquad\qquad\qquad f_2 : a_1 \to e_1$$
$$2 \to c \qquad\qquad\qquad a_2 \to e_2$$
$$3 \to b \quad \text{and} \quad a_3 \to e_3$$
$$4 \to d \qquad\qquad\qquad a_4 \to e_4$$

Since there are one-to-one function, then ${C_1}^*$ and ${C_2}^*$ are isomorphic.

When dealing with graph isomorphism, the adjacency of vertices can be used to represent the isomorphic graphs.

Definition 2.11 Suppose $G = (V, E)$ where $v_1, v_2, v_3, \dots, v_n \in V$. The *adjacency matrix* $\mathbf{A}$ of G (or $\mathbf{A}_G$), with respect to this listing of vertices, is the $n \times n$ matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, 0 as its (i, j)th entry when they are not adjacent.

In other words, the adjacency matrix $\mathbf{A} = [a_{i,j}]$ with

$$a_{i,j} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.12 If $\mathbf{M}$ is a $m \times n$ matrix, then the *transpose matrix* of $\mathbf{M}$ denoted by $\mathbf{M}^T$ is an $n \times m$ matrix, where the columns of $\mathbf{M}$ be the rows of $\mathbf{M}^T$ and the rows of $\mathbf{M}$ be the columns of $\mathbf{M}^T$.

In this study, we aim to present a new method to extract geometric representation of distinct HC in the decomposition of the complete graph K_n. The new method of K_n decomposition into distinct HC with different path for all n is discussed in the following chapters. Some examples will be discussed in Chapter Three before the general method is presented in Chapter Four.

CHAPTER THREE
SOME NUMERICAL EXAMPLES

This chapter presents some examples of K_n decomposition into distinct HC with different paths. We begin with several definitions and terminologies used for this method. After that, the proposed method will be presented for several cases such as $n = 3$, $n = 4$, $n = 5$ and $n = 6$.

3.1 Definitions and Terminologies

This section presents the definitions needed in this study. Some examples are disclosed for better understanding.

Definition 3.1 Let G and H be two complete graphs. Suppose the sets of vertices $\{x_1, x_2, x_3, \dots, x_n\} \in G$ and $\{x_1f, x_2f, x_3f, \dots, x_nf\} \in H$, where f maps the adjacent vertex. A function $g = \begin{pmatrix} x_1 & x_2 & x_3 \cdots & x_n \\ x_1f & x_2f & x_3f \cdots & x_nf \end{pmatrix}$ maps the vertices $\{x_1, x_2, x_3, \dots, x_n\}$ of G to other vertices $\{x_1f, x_2f, x_3f, \dots, x_nf\}$ of H where $\{x_1f, x_2f, x_3f, \dots, x_nf\}$ are the images for every element in G. That is, $x_1 \longmapsto x_1f$, $x_2 \longmapsto x_2f$, …, $x_n \longmapsto x_nf$ for $n \in \mathbb{Z}^+$. Then, the mapping is written as a product of transposition $(x_1, x_a)(x_2, x_b) \dots (x_n, x_z)$.

Example 3.2 Let $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$.

Then, $1 \longmapsto 2$, $2 \longmapsto 3$, $3 \longmapsto 4$, and $4 \longmapsto 1$. Thus, we write the mapping for g as $(1,2)(2,3)(3,4)(4,1)$.

Definition 3.3 Let A be a circuit with direction $(x_1, x_2, x_3, \dots, x_{n-1}, x_n, x_1)$. Then, a circuit B is a mirror image to circuit A if the direction of B is $(x_1, x_n, x_{n-1}, \dots, x_3, x_2, x_1)$.

Definition 3.4 Let A and B be two circuits with n vertices. A is isomorphic to B when B is the mirror image of A.

Example 3.5 Figure 3.1 shows two circuits A and B. Suppose the direction of A is $(1,2,3,4,1)$, then B is the mirror image of A with direction $(1,4,3,2,1)$. Then, A is isomorphic to B since there is a one-to-one mapping from A to B.

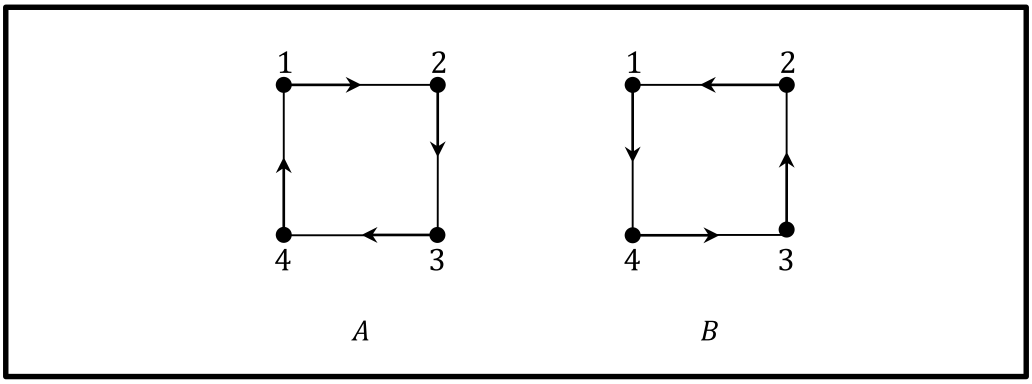

Figure 3.1. Circuit A is isomorphic to circuit B

3.2 Some Examples of the Proposed Method

In this section, several examples for K_3, K_4, K_5 and K_6 decomposition into distinct HC with different paths will be investigated.

3.2.1 Case K_3

For a graph with n vertices, the total number of circuits is $n!$ at most (Riaz & Khiyal, 2006). Consider K_3 as shown in Figure 3.2. Since there are three vertices, then K_3 can be decomposed into $3! = 6$ circuits as presented in Figure 3.3.

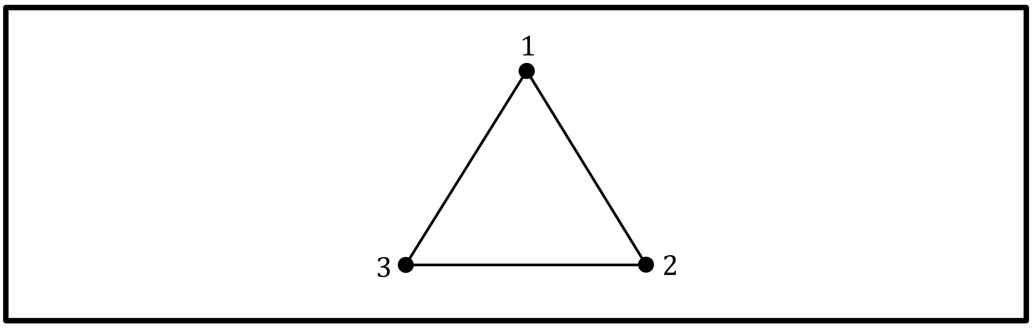

Figure 3.2. A complete graph K_3

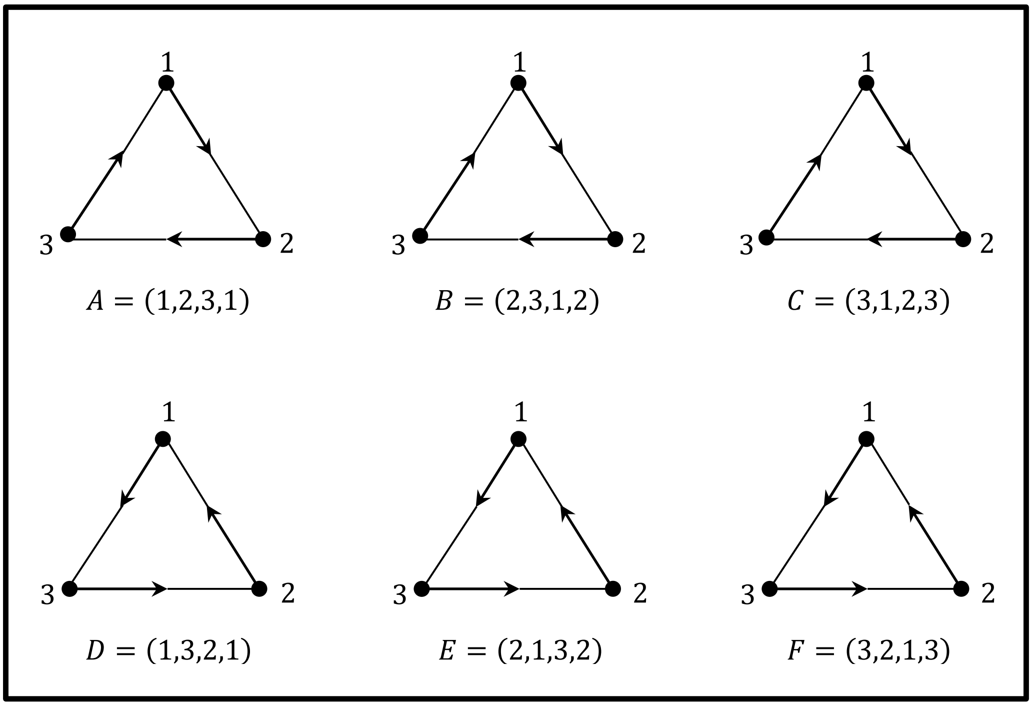

Figure 3.3. The circuits from K_3

The six decomposed circuits in Figure 3.3 have similar paths. Circuits A, B and C have opposite direction compared to circuits D, E and F. Following Definition 3.3, we can see that circuit D is the mirror image of A, E is the mirror image of B, and F is the mirror image of C.

Since these six circuits have similar paths, then K_3 is decomposable into a distinct HC with different path. We conclude this solution as trivial. Figure 3.4 presents the geometric representation of the decomposed HC in K_3, and we discuss the case for K_4 in the following section.

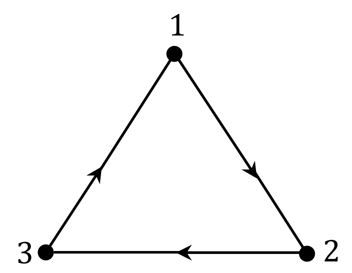

Figure 3.4. The distinct Hamiltonian circuit with different paths from K_3

Remark 3.6 A complete graph K_3 is decomposable into a distinct Hamiltonian circuit with different path.

3.2.2 Case K_4

Let us consider K_4, which can be decomposed into $4! = 24$ circuits. Figure 3.5 shows several circuits in K_4.

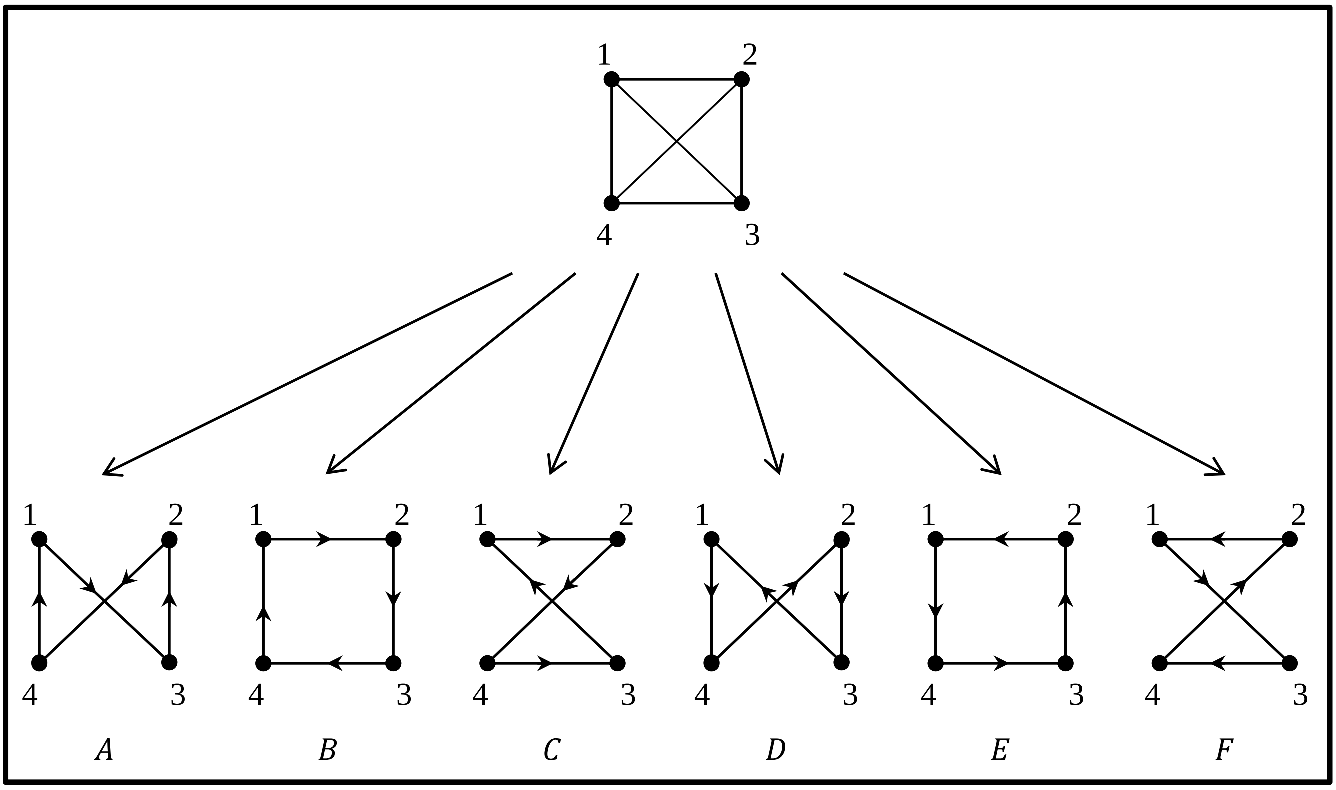

Figure 3.5. Several circuits from K_4

The direction of the circuits are $A = (1,3,2,4,1)$, $B = (1,2,3,4,1)$, $C = (1,2,4,3,1)$, $D = (1,4,2,3,1)$, $E = (1,4,3,2,1)$ and $F = (1,3,4,2,1)$.

Consider circuit $B = (1,2,3,4,1)$; By changing the starting point of B, circuits $(2,3,4,1,2)$, $(3,4,1,2,3)$, and $(4,1,2,3,4)$ are the same circuits with similar paths to B as presented in Figure 3.6.

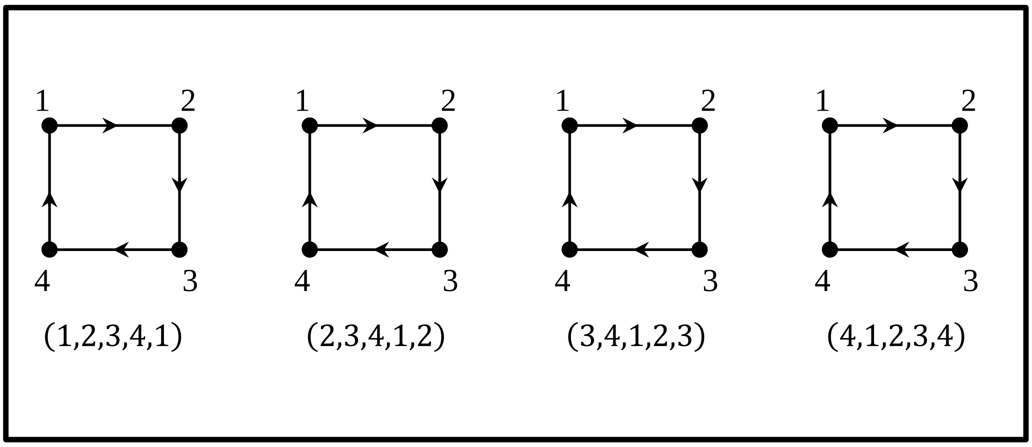

Figure 3.6. The same circuits from B with different starting point

Similar to other circuits in Figure 3.5, each circuit can have any other three similar circuits with similar path by changing the starting point.

Next, we consider circuits C and F as shown below.

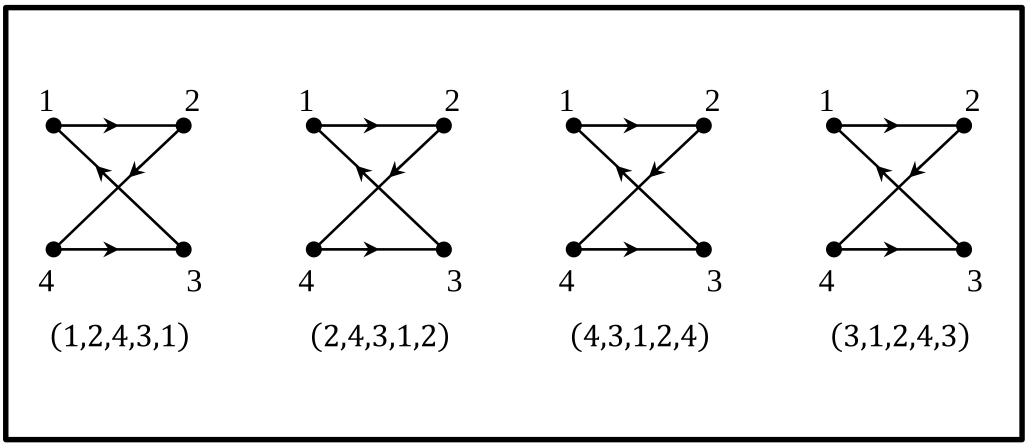

Figure 3.7. The same circuits from C with different starting point

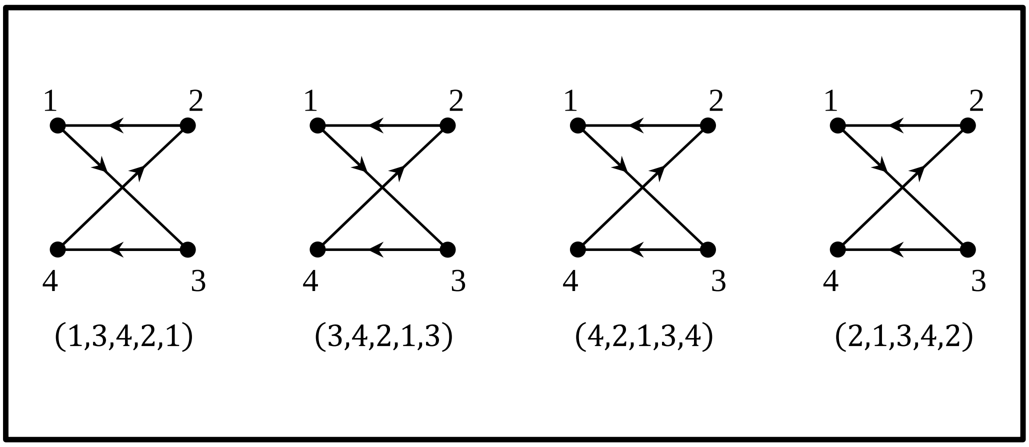

Figure 3.8. The same circuits from F with different starting point

Similar to circuit B, circuits C and F produced the other three circuits by changing the starting point, as shown in Figures 3.7 and 3.8. However, the circuits are similar since they traverse the same path. Then, by referring to Figure 3.5, we can see that circuits A and D traverse the same path. Similarly, circuits B and C have similar path with circuits E and F, respectively. This situation can be strengthened by considering the mapping for the circuits.

The direction of each circuit with different paths is used to obtained the mapping, before the mirror image is determined. By Definition 3.1, the mapping for each circuit in Figure 3.5 is obtained as shown in the next page.

$$A = (1,3,2,4,1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = (1,3)(3,2)(2,4)(4,1),$$

$$B = (1,2,3,4,1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1,2)(2,3)(3,4)(4,1),$$

$$C = (1,2,4,3,1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = (1,2)(2,4)(4,3)(3,1),$$

$$D = (1,4,2,3,1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} = (1,4)(4,2)(2,3)(3,1),$$

$$E = (1,4,3,2,1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1,4)(4,3)(3,2)(2,1), \text{ and}$$

$$F = (1,3,4,2,1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = (1,3)(3,4)(4,2)(2,1).$$

Following Definition 3.3, circuit A is the mirror image of D. Likewise, B is the mirror image of E, and C is the mirror image of F. Then, by Definition 3.4, we have three distinct HC with different paths from K_4. The complete geometric representation of distinct HC with different paths in K_4 are shown in Figure 3.9.

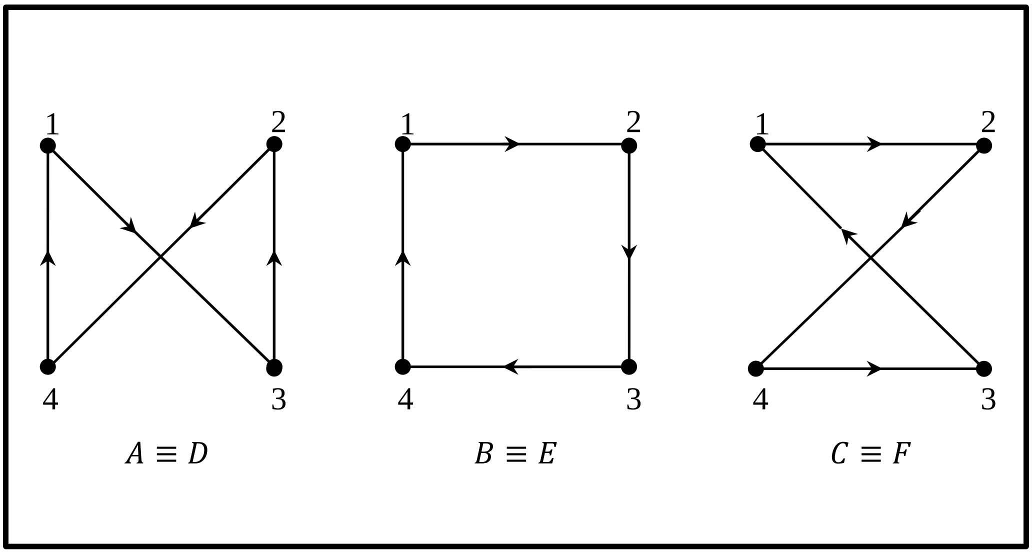

Figure 3.9. Three distinct Hamiltonian circuits with different paths from K_4

Now, we would like to present how the above result is enumerated. The following discussion provides the method to decompose K_4 into three distinct HC with different paths.

Step 1: Creating direction

In this step, we created directions to obtain the circuits of K_4. Since we fixed vertex 1 as a starting location in all blocks, then there are three vertices left. Thus, there are three blocks for K_4, i.e. block 1, block 2, and block 3. In block 1, we fixed vertex 1 as the starting location, and vertex 2 as the second location. When we mentioned "vertex 2 is the second location", it means the arrow will move from vertex 1 (starting location) to vertex 2. The third and fourth locations for directions in block 1 are substituted by the remaining vertices 3 and 4. The directions obtained in block 1 are shown below.

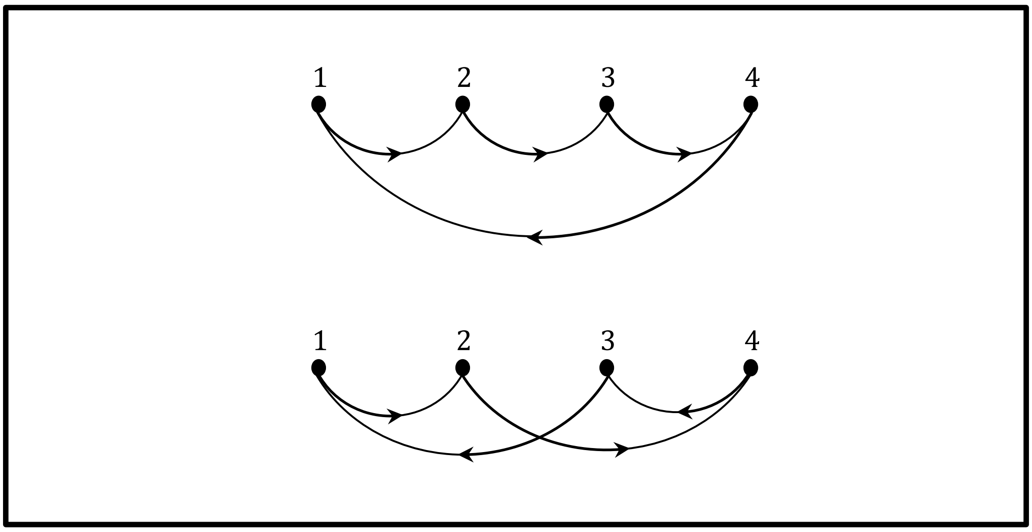

Figure 3.10. Creating directions in block 1 for K_4

The direction $(1,2,3,4,1)$ in Figure 3.10 is obtained by moving the arrow from vertex 1 to vertex 2, from vertex 2 to vertex 3 (third location), from vertex 3 to vertex 4 (fourth location) and from vertex 4 back to vertex 1. Since the third location for the direction $(1,2,3,4,1)$ is placed by vertex 3, then vertex 4 will be the third location for the direction $(1,2,4,3,1)$. Thus, the arrow is moving from one vertex to another vertex to get the direction as shown in Figure 3.10.

Next, in block 2, we fixed vertex 1 as the starting location and vertex 3 as the second location. The direction $(1,3,2,4,1)$ is obtained by moving the arrow from vertex 1 to vertex 3, from vertex 3 to vertex 2 (third location), from vertex 2 to vertex 4 (fourth location), and from vertex 4 back to the starting location. Then, the direction $(1,3,4,2,1)$ is constructed similarly with vertex 4 as the third location and vertex 2 as the fourth location. The directions obtained are presented in Figure 3.11.

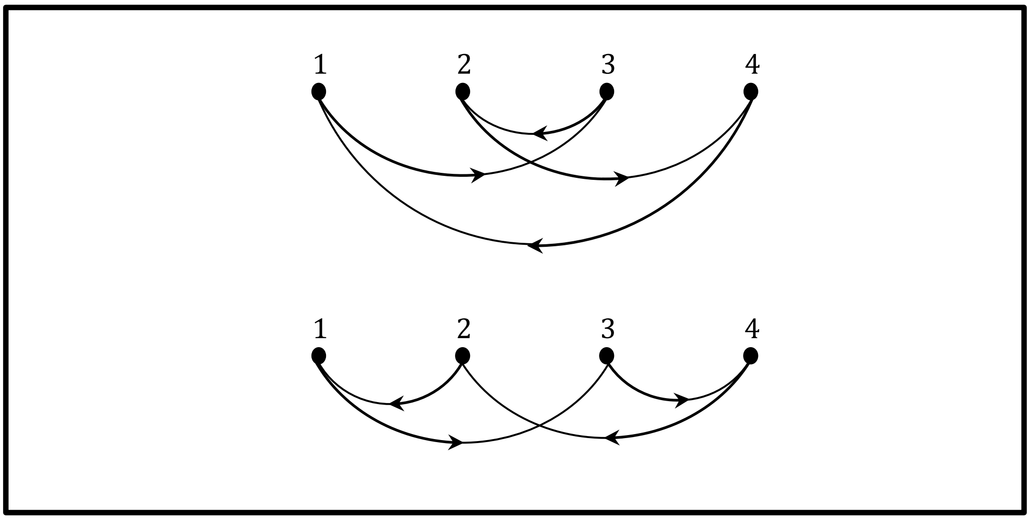

Figure 3.11. Creating directions in block 2 for K_4

Then, in block 3, we fixed vertex 1 and vertex 4 as the starting location and second location, respectively. To get the first direction in block 3, vertex 2 be the third location and the remaining vertex (vertex 3) be the fourth location. The second direction in block 3 is developed with vertex 3 as the third location and the remaining vertex (vertex 2) as the fourth location. Figure 3.12 provides the directions; they are, $(1,4,2,3,1)$ and $(1,4,3,2,1)$.

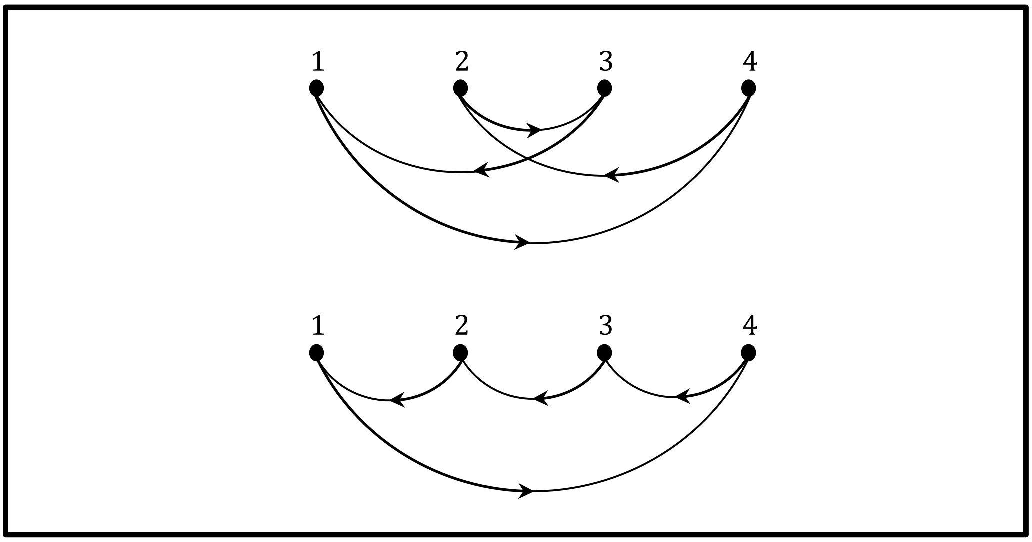

Figure 3.12. Creating directions in block 3 for K_4

Step 2: Fixing and shifting

All directions obtained in Step 1 will be used in Step 2. For example, we used the path $(1,2,3,4)$ in the direction $(1,2,3,4,1)$. The path is used because vertex 1 is not considered more than once since it is the same vertex for the circuit to start with. We fixed one vertex (vertex 1), and shifted the remaining vertices to the left as shown below.

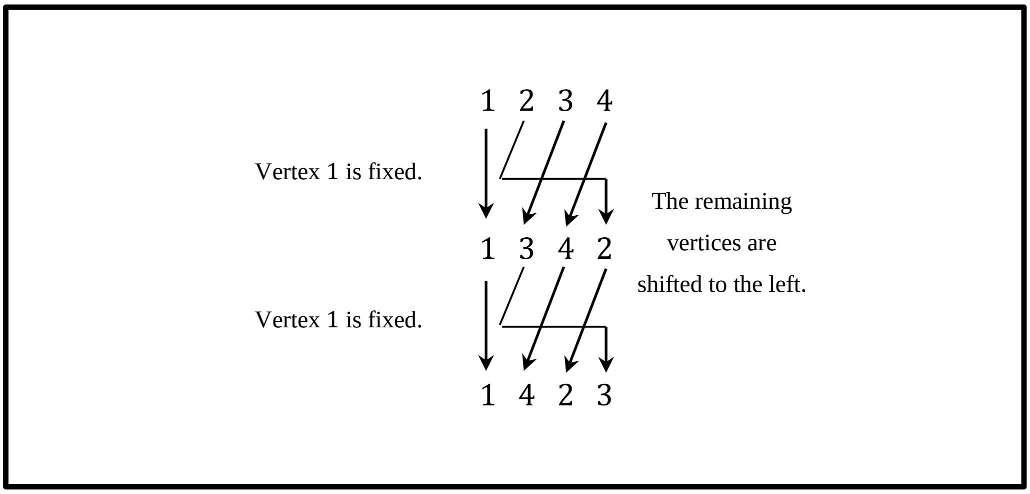

Figure 3.13. Fix-and-shift for K_4 where vertex 1 is fixed

Then we have $(1,2,3,4)$, $(1,3,4,2)$, and $(1,4,2,3)$. Now, we fixed vertex 2 and shifted the remaining vertices to the left as shown below.

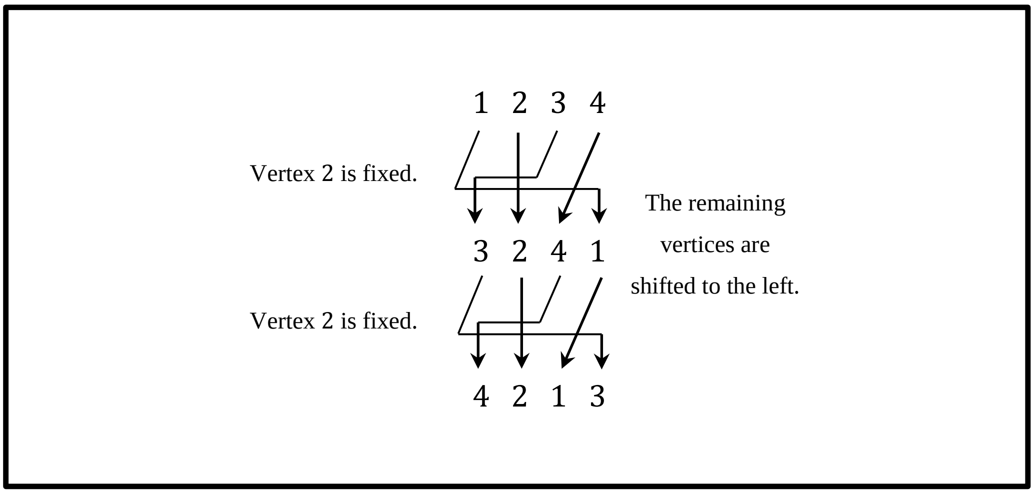

Figure 3.14. Fix-and-shift for K_4 where vertex 2 is fixed

Then, we fixed vertex 3 and shifted the remaining vertices to the left.

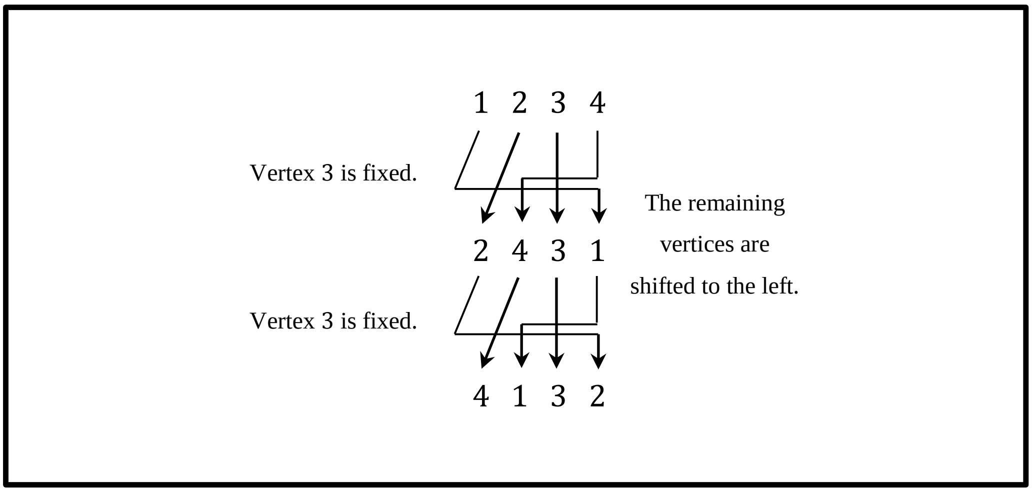

Figure 3.15. Fix-and-shift for K_4 where vertex 3 is fixed

Next, we fixed vertex 4 and shifted the remaining vertices to the left.

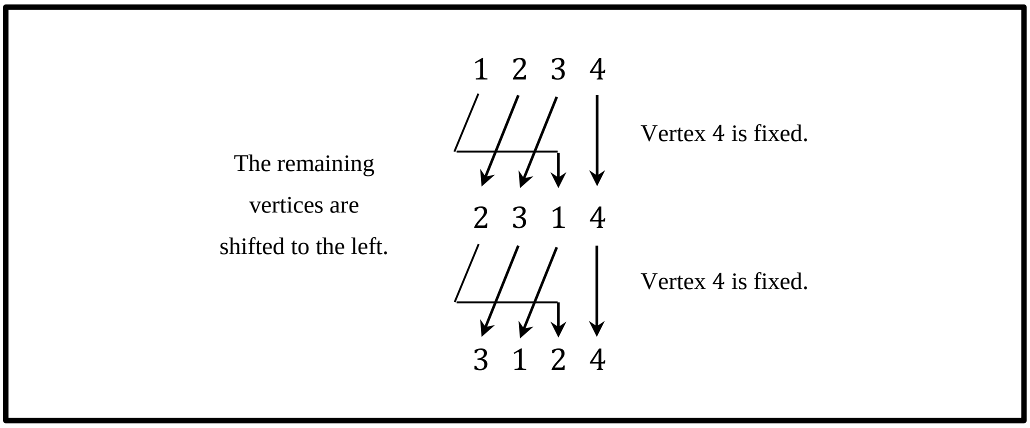

Figure 3.16. Fix-and-shift for K_4 where vertex 4 is fixed

Figures 3.13, 3.14, 3.15, and 3.16 show the fix-and-shift carried out for the first direction in block 1, that is, $(1,2,3,4,1)$. This fix-and-shift is repeated for all directions obtained in Step 1; these are $(1,2,4,3,1)$, $(1,3,2,4,1)$, $(1,3,4,2,1)$, $(1,4,2,3,1)$, and $(1,4,3,2,1)$, as presented in Appendix A.

Step 3: Finding the mapping

In this step, we followed Definition 3.1 to get the mapping for all the paths obtained in Step 2 as shown below.

$$(1,2,3,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1,2)(2,3)(3,4)(4,1),$$

$$(1,3,4,2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = (1,3)(3,4)(4,2)(2,1),$$

$$(1,4,2,3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} = (1,4)(4,2)(2,3)(3,1),$$

.

.

.

$$(2,4,1,3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = (1,3)(3,2)(2,4)(4,1).$$

The maping for the rest of the path is continued until block 3. By following Definitions 2.3, 3.3 and 3.4, the similar mappings are eliminated as well as the opposite mappings to get the distinct mappings $(1,2)(2,3)(3,4)(4,1)$, $(1,3)(3,4)(4,2)(2,1)$ and $(1,4)(4,2)(2,3)(3,1)$.

Step 4: Drawing the circuits

Using the mapping obtained in Step 3, we developed the following geometric representation of three distinct HC with different paths from K_4. For example, the mapping of $C_1{}^* = (1,2)(2,3)(3,4)(4,1)$ is the direction from one vertex to another vertex, that is, vertex 1 maps to 2, 2 maps to 3, 3 maps to 4, and 4 maps to 1.

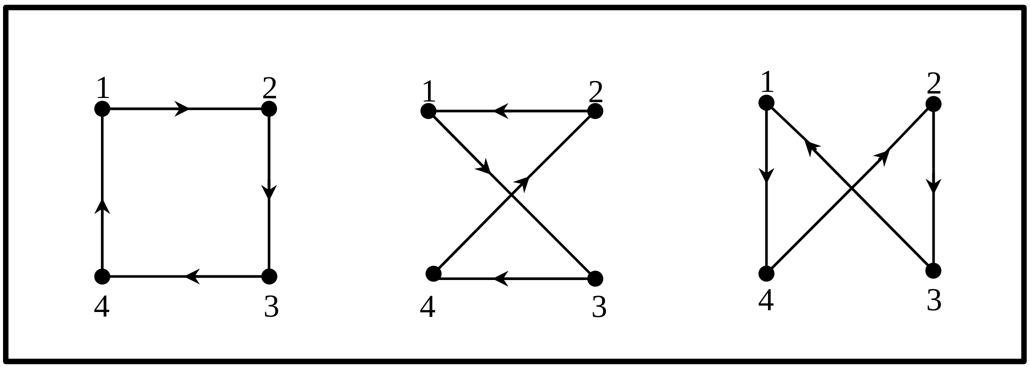

Figure 3.17. The three distinct Hamiltonian circuits with different paths from K_4

Remark. A complete graph K_4 is decomposable into three distinct Hamiltonian circuits with different path.

The following section discusses the case $n = 5$.

3.2.3 Case K_5

Consider K_5 as shown in Figure 3.18, where K_5 can be decomposed into $5! = 120$ circuits. Since there are five similar circuits with similar paths, we have $\frac{120}{5} = 24$ circuits. Considering the mirror image based on Definitions 3.3 and 3.4, then K_5 can be decomposed into $\frac{24}{2} = 12$ distinct circuits.

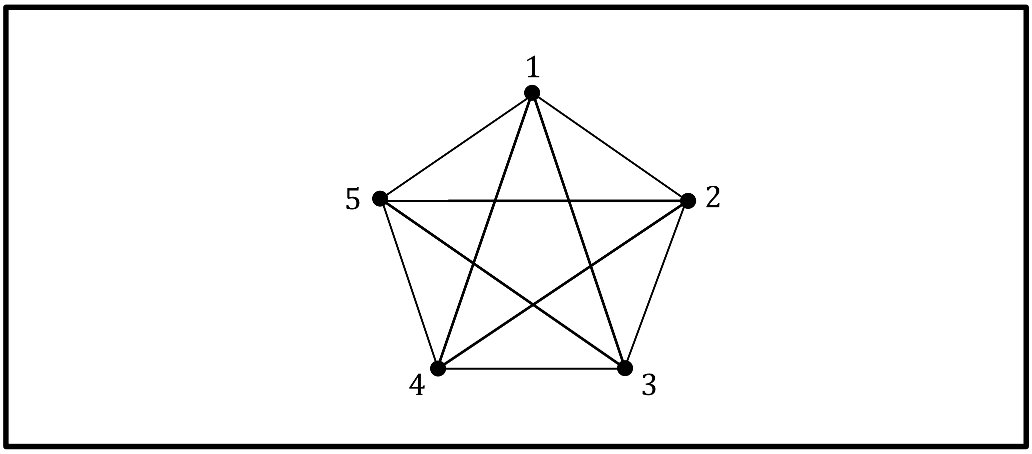

Figure 3.18. A complete graph K_5

The following paragraphs discuss the method to decompose K_5 into twelve distinct HC with different paths.

Step 1: Creating direction

In step 1, we fixed vertex 1 as the starting location for all directions, then there are four vertices left. Thus, K_5 has four blocks, i.e. block 1, block 2, block 3, and block 4. Similar to K_4, in block 1, we fixed vertex 1 as the starting location and vertex 2 as the second location. The third, fourth and fifth locations are substituted consecutively by the remaining vertices in ascending order. Figure 3.19 shows the direction obtained for K_5.

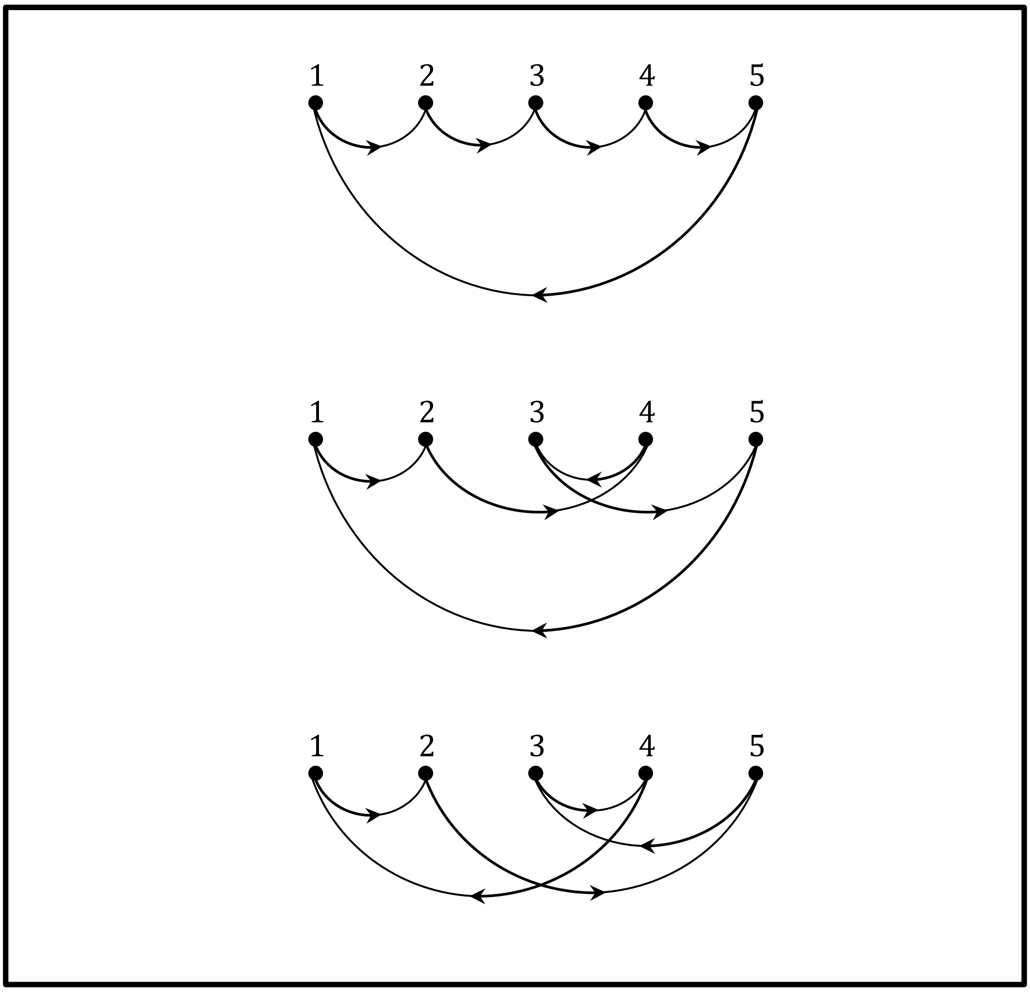

Figure 3.19. Creating directions in block 1 for K_5

Figure 3.19 gives directions $(1,2,3,4,5,1)$, $(1,2,4,3,5,1)$ and $(1,2,5,3,4,1)$. In details, for example, the direction $(1,2,3,4,5,1)$ is obtained by moving forward the arrow from vertex 1 (starting location) to vertex 2 (second location), from vertex 2 to vertex 3 (third location), from vertex 3 to vertex 4 (fourth location), from vertex 4 to vertex 5 (fifth location), and from vertex 5 back to vertex 1 (starting location). This process is stopped after vertex 5 be the third location since there are no vertices can be substituted as the third location after vertex 5.

In block two, we fixed vertex 1 as the starting location and vertex 3 as the second location. The remaining vertices in K_5 are substituted consecutively in ascending order to be the third, fourth and fifth locations. Figure 3.20 presents the directions obtained in block 2.

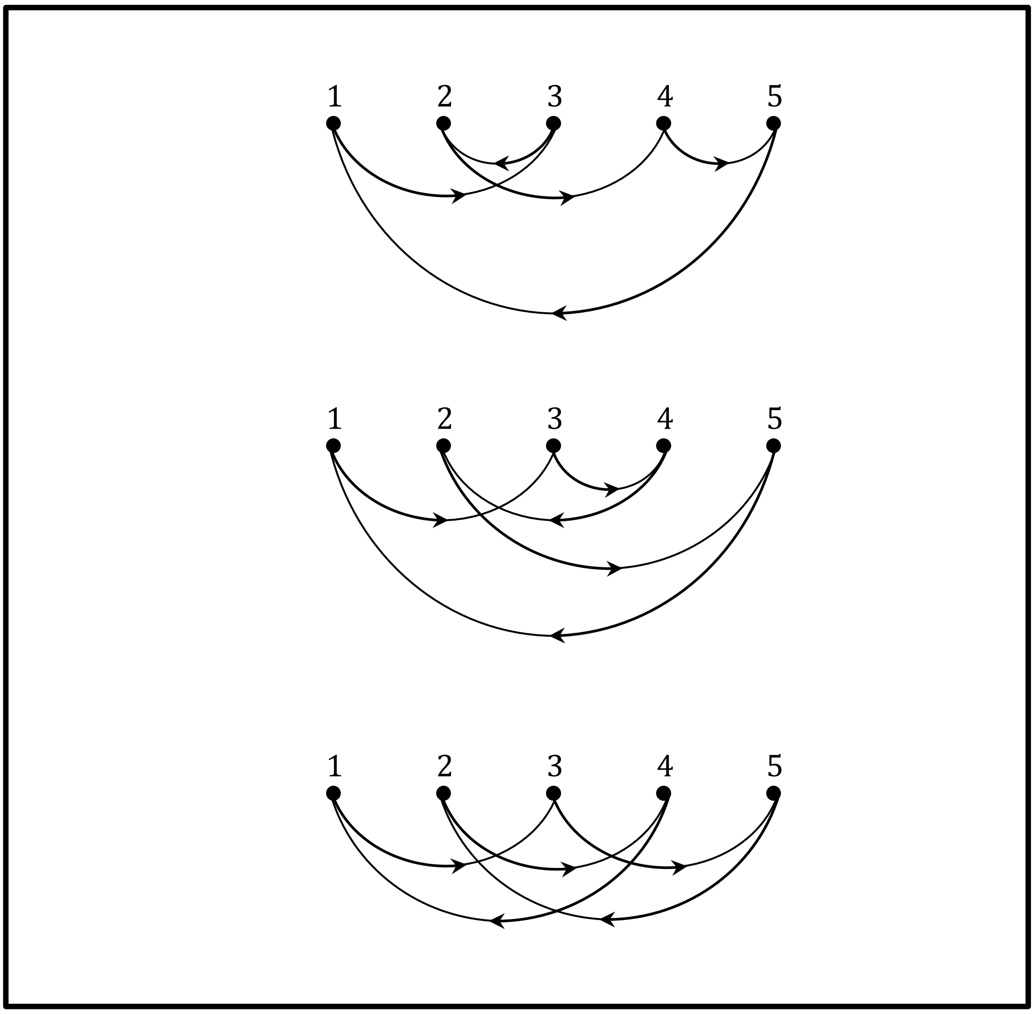

Figure 3.20. Creating directions in block 2 for K_5

The directions obtained from Figure 3.20 are $(1,3,2,4,5,1)$, $(1,3,4,2,5,1)$ and $(1,3,5,2,4,1)$. The third, fourth and fifth locations are substituted consecutively in ascending order, i.e. vertex 2, vertex 4 and vertex 5.

Next, we fixed vertex 1 as the starting location and vertex 4 as the second location. Similarly, the remaining vertices in K_5 are substituted consecutively to be the third, fourth and fifth locations. Figure 3.21 provides the directions obtained in block 3 for K_5.

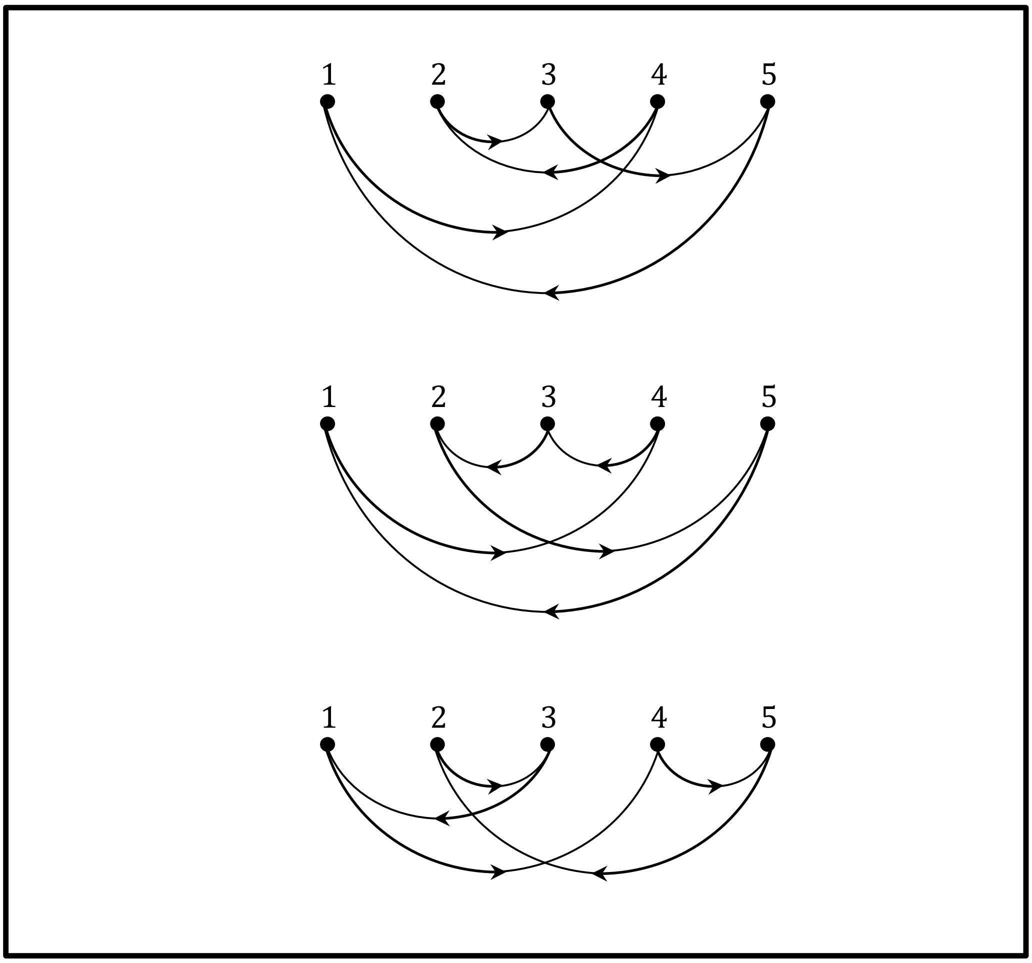

Figure 3.21. Creating directions in block 3 for K_5

Since vertex 1 and vertex 4 are fixed to be the starting location and second location, the third location is substituted with vertex 2 to produce $(1,4,2,3,5,1)$, vertex 3 to produce $(1,4,3,2,5,1)$, and vertex 5 to produce $(1,4,5,2,3,1)$. The fourth and fifth locations are substituted consecutively.

In block 4, we fixed vertex 1 as the starting location and vertex 5 as the second location. The third location is substituted with vertex 2 to produce $(1,5,2,3,4,1)$, vertex 3 to produce $(1,5,3,2,4,1)$, and vertex 4 to produce $(1,5,4,2,3,1)$. Figure 3.22 shows the directions in block 4.

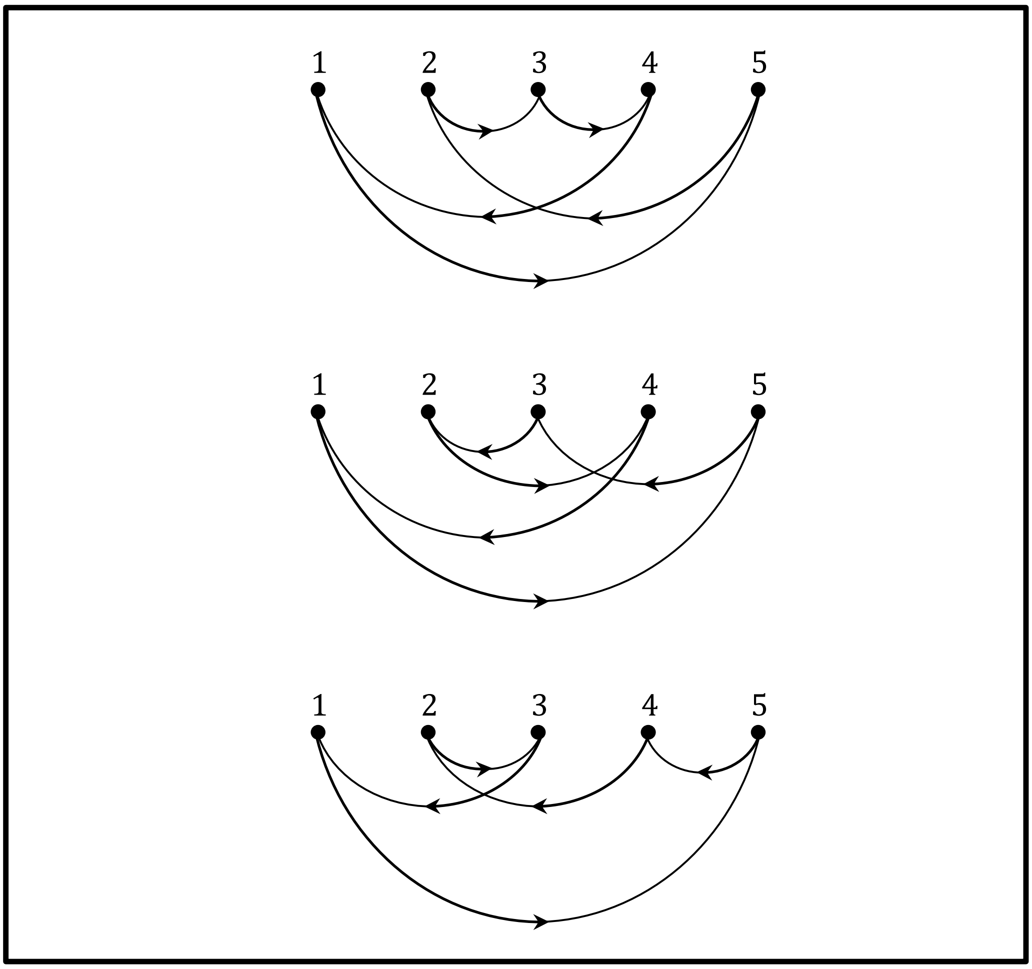

Figure 3.22. Creating directions in block 4 for K_5

Step 2: Fixing and shifting

All directions obtained in Step 1 will be used in Step 2. From the direction $(1,2,3,4,5,1)$, we use the path $(1,2,3,4,5)$. The path is used because vertex 1 is not considered more than once since the circuit starts and ends at vertex 1. We fixed vertex 1 and shifted the remaining vertices to the left.

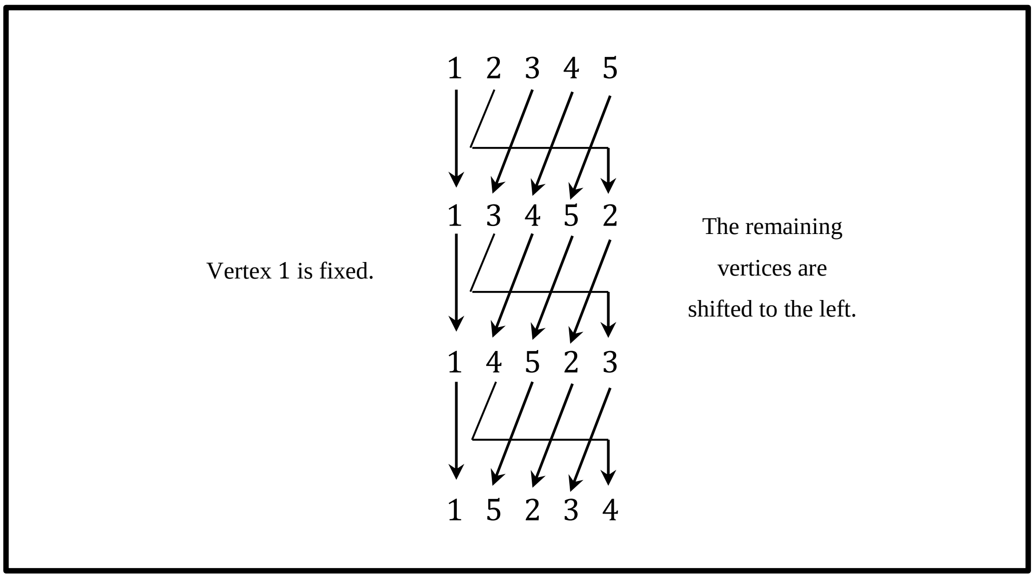

Figure 3.23. Fix-and-shift for K_5 where vertex 1 is fixed

We stopped at $(1,5,2,3,4)$ since fix-and-shift $(1,5,2,3,4)$ will get $(1,2,3,4,5)$ once again. Next, we fixed vertex 2 and shifted the remaining vertices to the left.

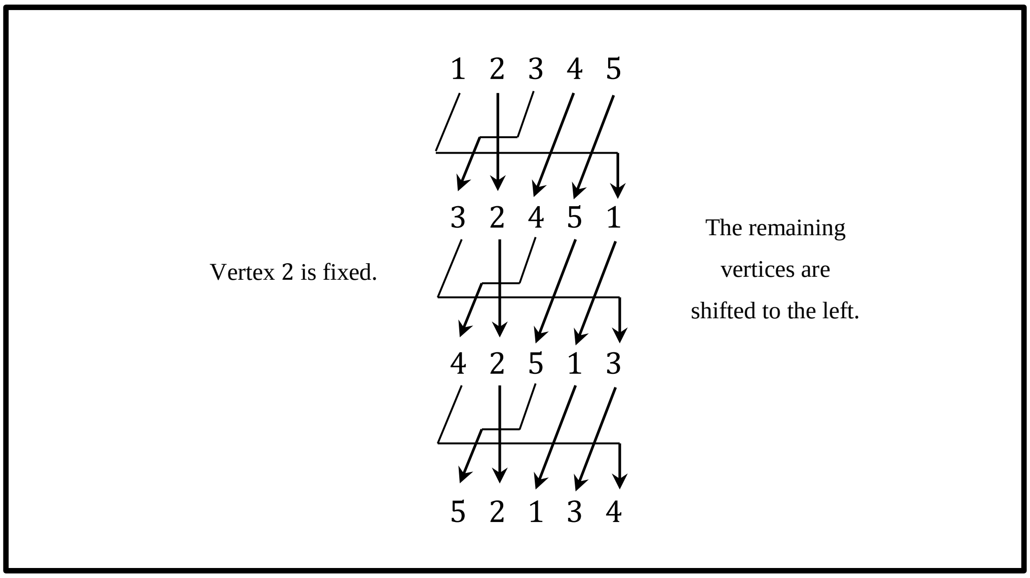

Figure 3.24. Fix-and-shift for K_5 where vertex 2 is fixed

We stopped at $(5,2,1,3,4)$ since fix-and-shift $(5,2,1,3,4)$ will have $(1,2,3,4,5)$ once again. The same step is applied to the remaining vertices 3, 4 and 5 as shown in Figure 3.25.

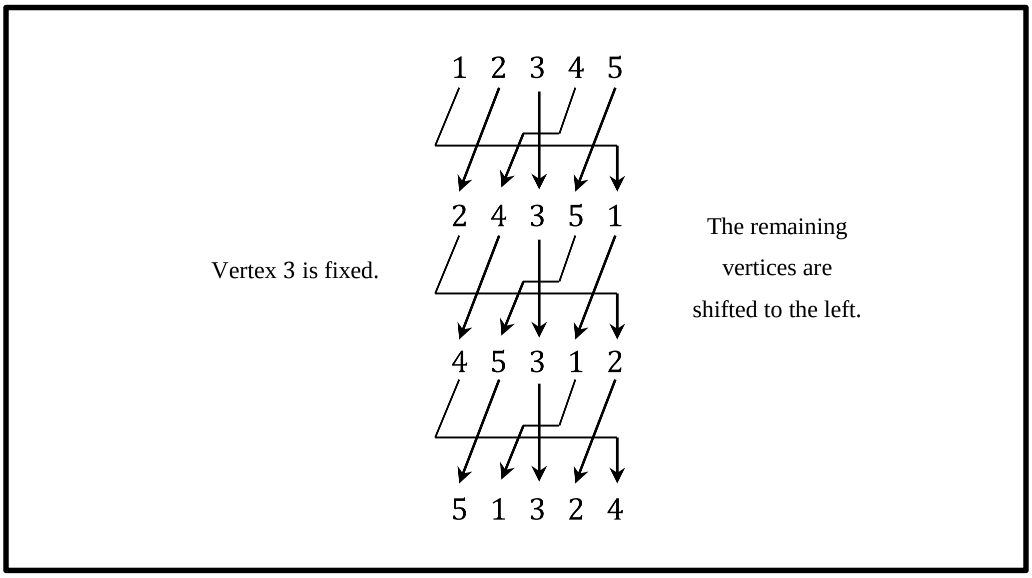

Figure 3.25. Fix-and-shift for K_5 where vertex 3 is fixed

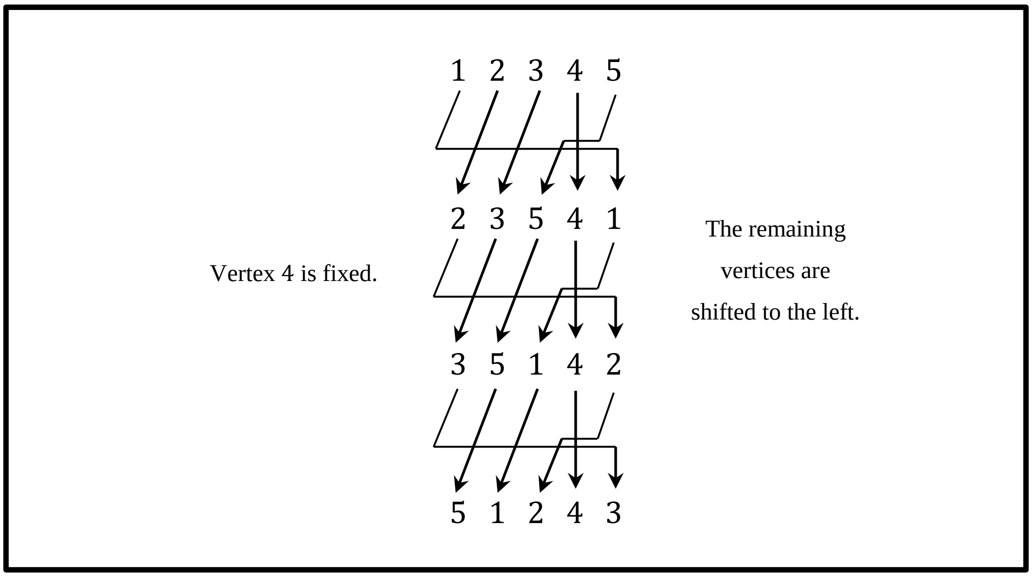

Figure 3.26. Fix-and-shift for K_5 where vertex 4 is fixed

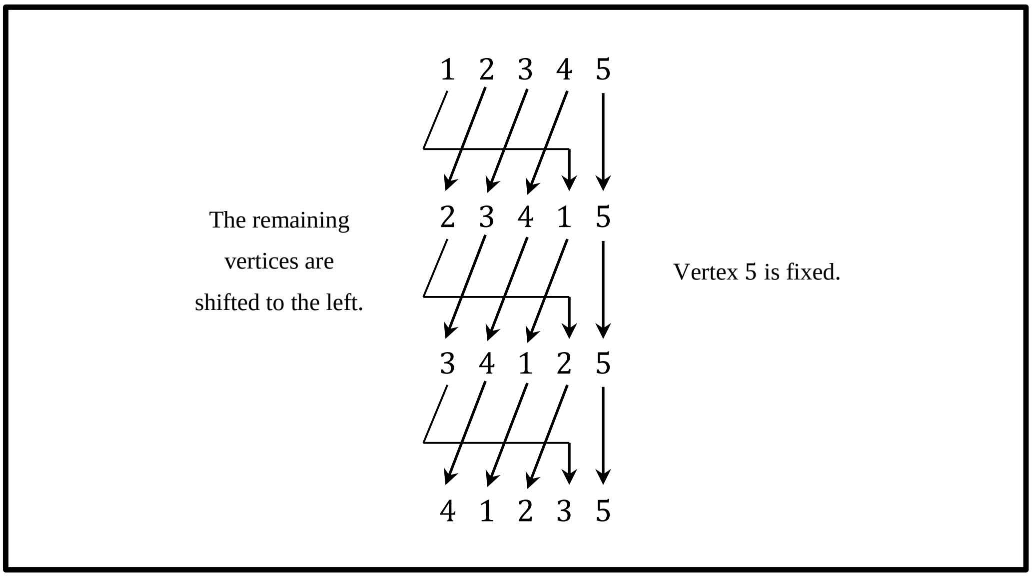

Figure 3.27. Fix-and-shift for K_5 where vertex 5 is fixed

Figures 3.23, 3.24, 3.25, 3.26 and 3.27 are the fix-and-shift done for direction (1,2,3,4,5,1). The fix-and-shift step is applied for all directions developed in Step 1 as presented in Appendix B.

Step 3: Finding the mapping

In this step, we followed Definition 3.1 to get the mapping for each path obtained in Step 2.

$$(1,2,3,4,5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} = (1,2)(2,3)(3,4)(4,5)(5,1),$$

$$(1,3,4,5,2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix} = (1,3)(3,4)(4,5)(5,2)(2,1),$$

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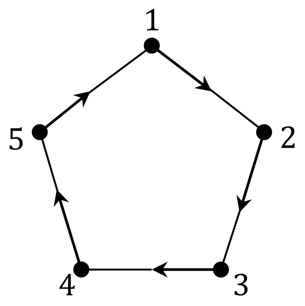
$$(3,5,1,4,2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 2 & 1 \end{pmatrix} = (1,4)(4,2)(2,3)(3,5)(5,1),$$

As previously stated, two important procedures are needed to get the distinct circuits. First, the similar mapping is eliminated. Then, the mapping with opposite direction is also eliminated followed by Definitions 2.3, 3.3 and 3.4. After the elimination, there are twelve mappings which are distinct. The result of distinct mappings for K_5 is presented in Appendix C. Next, in Step 4, the circuits are drawn.

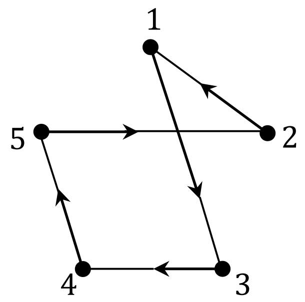
Step 4: Drawing the circuits

We used each of the distinct mapping in Step 3 to obtain the following geometric representation of distinct HC with different path from K_5. For instance, the mapping of ${C_1}^* = (1,2)(2,3)(3,4)(4,5)(5,1)$ is the direction from one vertex to another vertex, that is, vertex 1 maps to 2, 2 maps to 3, 3 maps to 4, 4 maps to 5, and 5 maps to 1.

$C_1^* = (1,2)(2,3)(3,4)(4,5)(5,1) \equiv$



$C_2^* = (1,3)(3,4)(4,5)(5,2)(2,1) \equiv$



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$C_{12}^* = (1,3)(3,5)(5,2)(2,4)(4,1) \equiv$

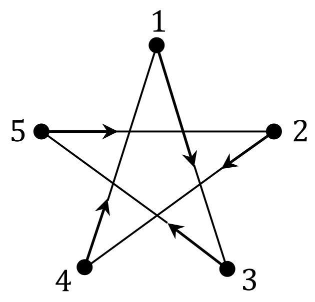


Figure 3.28. Drawing the circuits for K_5

The complete geometric representation of twelve distinct HC with different paths from K_5 is presented in Appendix D.

Remark 3.7 A complete graph K_5 is decomposable into twelve distinct Hamiltonian circuits with different paths.

The following section discusses the case $n = 6$.

3.2.4 Case $\boldsymbol{K_6}$

This section discusses K_6 decomposition into distinct HC with different path. Since K_6 has six vertices, then K_6 can be decomposed into $6! = 720$ circuits. Since there are six similar circuits which traverse the same path, then we have $\frac{6!}{6} = 120$ circuits. By Definition 3.4, K_6 can be decomposed into $\frac{120}{2} = 60$ circuits.

A complete graph K_6 and several circuits from K_6 is presented in Figure 3.29 as examples before we presented the method to decompose K_6 into sixty distinct HC with different paths.

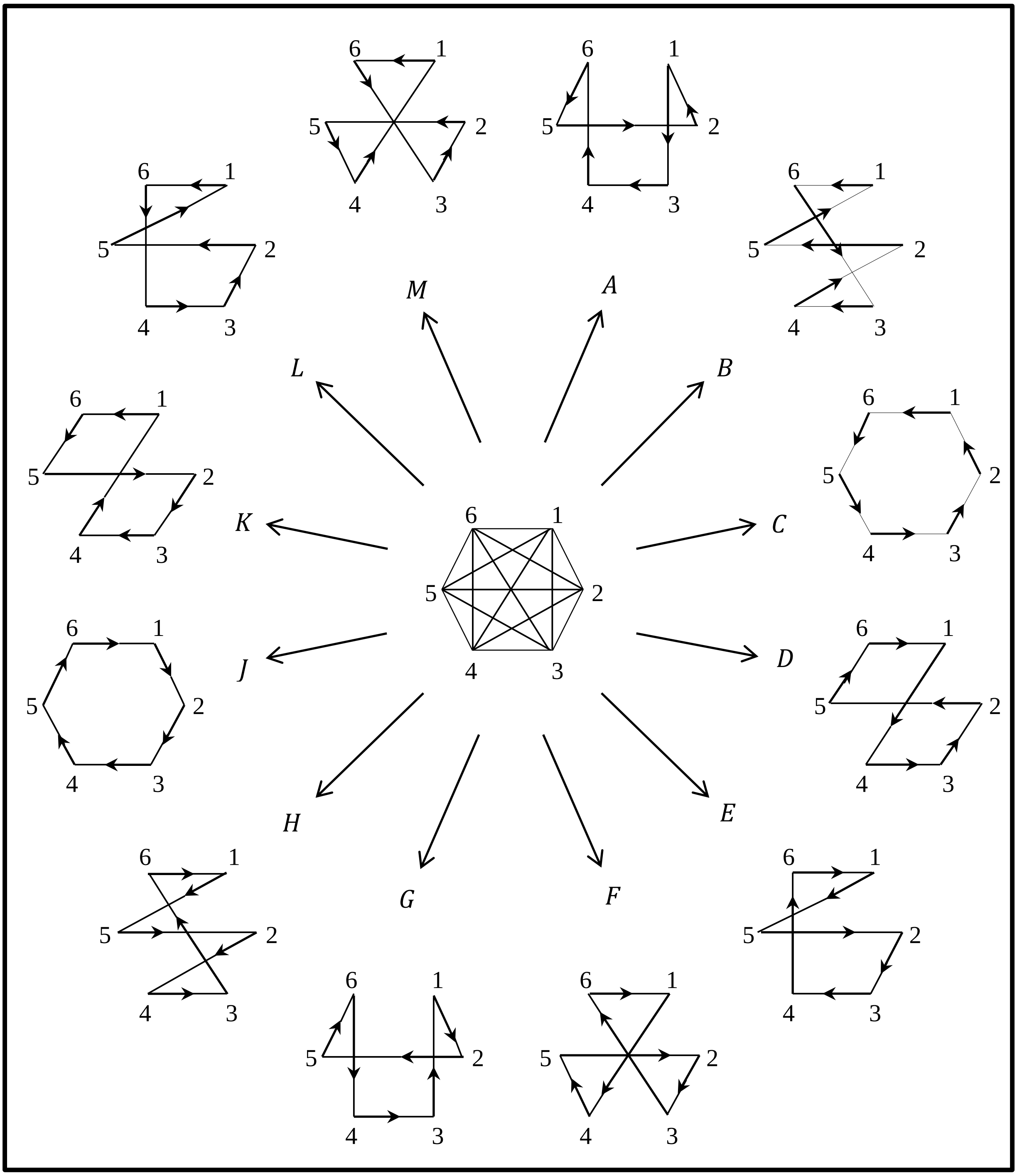


Figure 3.29. Several circuits from K_6

Step 1: Creating direction

Similar to K_5, we obtained the directions for K_6 by moving forward the arrow from one vertex to other vertices consecutively as shown in Figures 3.30, 3.31, 3.32, 3.33 and 3.34. Since vertex 1 is fixed as a starting location in all blocks, then there are five vertices left. Thus, there are five blocks for K_6, i.e. block 1, block 2, block 3, block 4, and block 5.

In block 1, we fixed vertex 1 as the starting location and vertex 2 as the second location. The third location is substituted by the remaining vertices consecutively, as well as the fourth, fifth and sixth locations. Figure 3.30 presents the directions obtained in block 1 for K_6.

In block 2, we fixed vertex 1 as the starting location and vertex 3 as the second location. In block 3, vertex 1 and vertex 4 are fixed as the starting location and second location respectively. While in block 4, vertex 1 is the starting location and vertex 5 is the second location. Finally, in block 5, vertex 1 is fixed as the starting location and vertex 6 is fixed to be the second location. The third, fourth, fifth and sixth location of each block are substituted by the remaining vertices consecutively. The directions obtained are presented in Figures 3.30, 3.31, 3.32, 3.33 and 3.34.

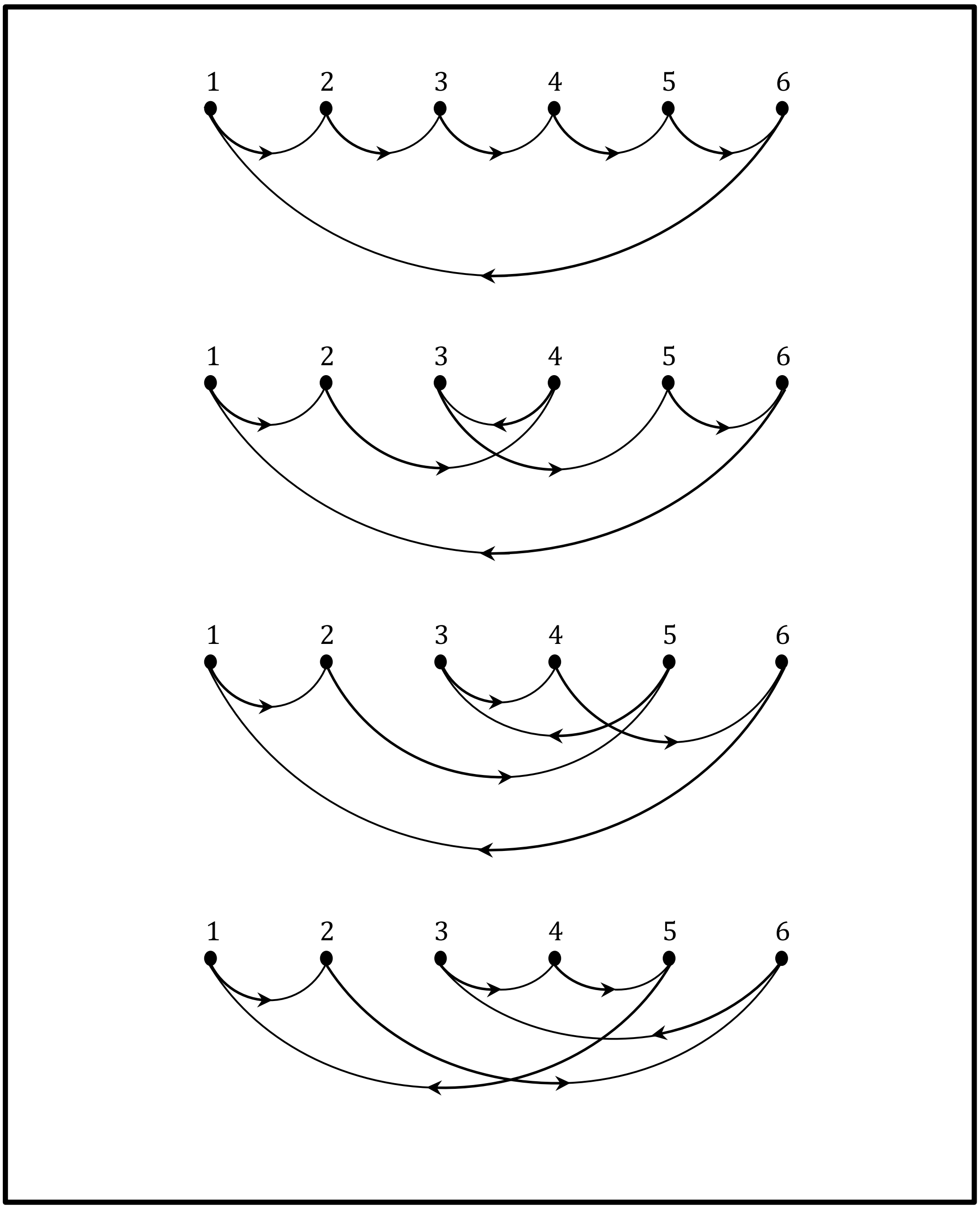


Figure 3.30. Creating directions in block 1 for K_6

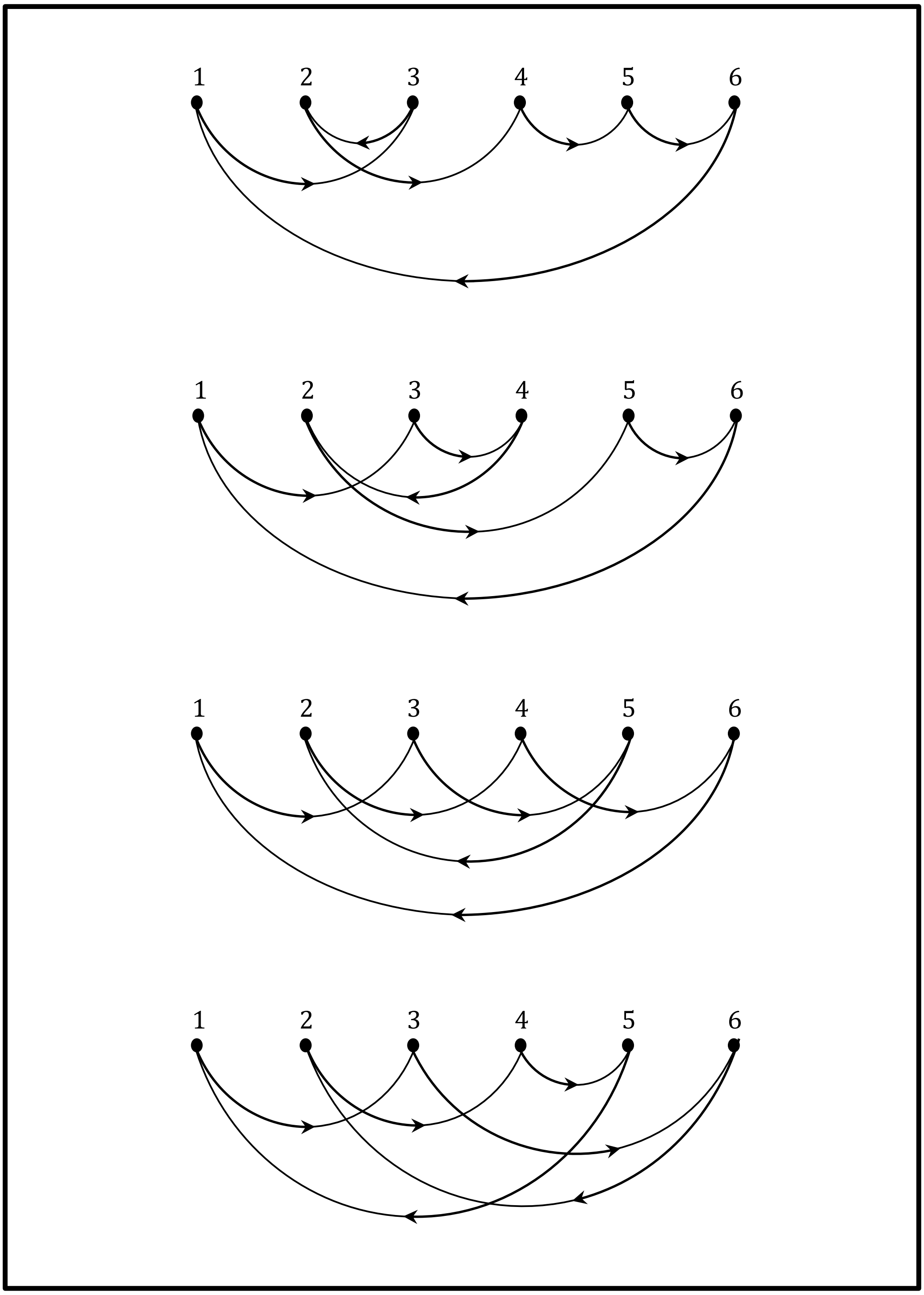


Figure 3.31. Creating directions in block 2 for K_6

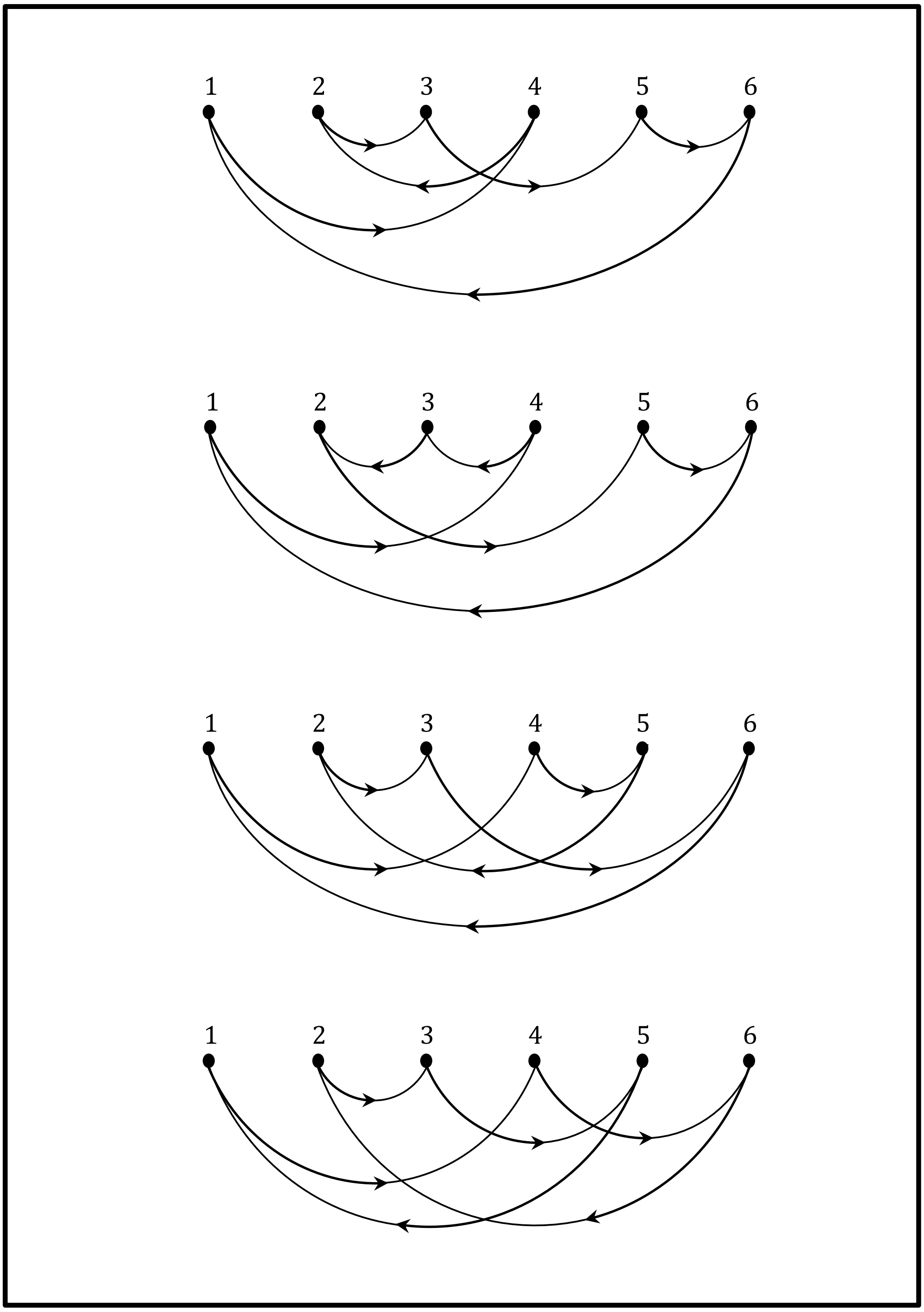


Figure 3.32. Creating directions in block 3 for K_6

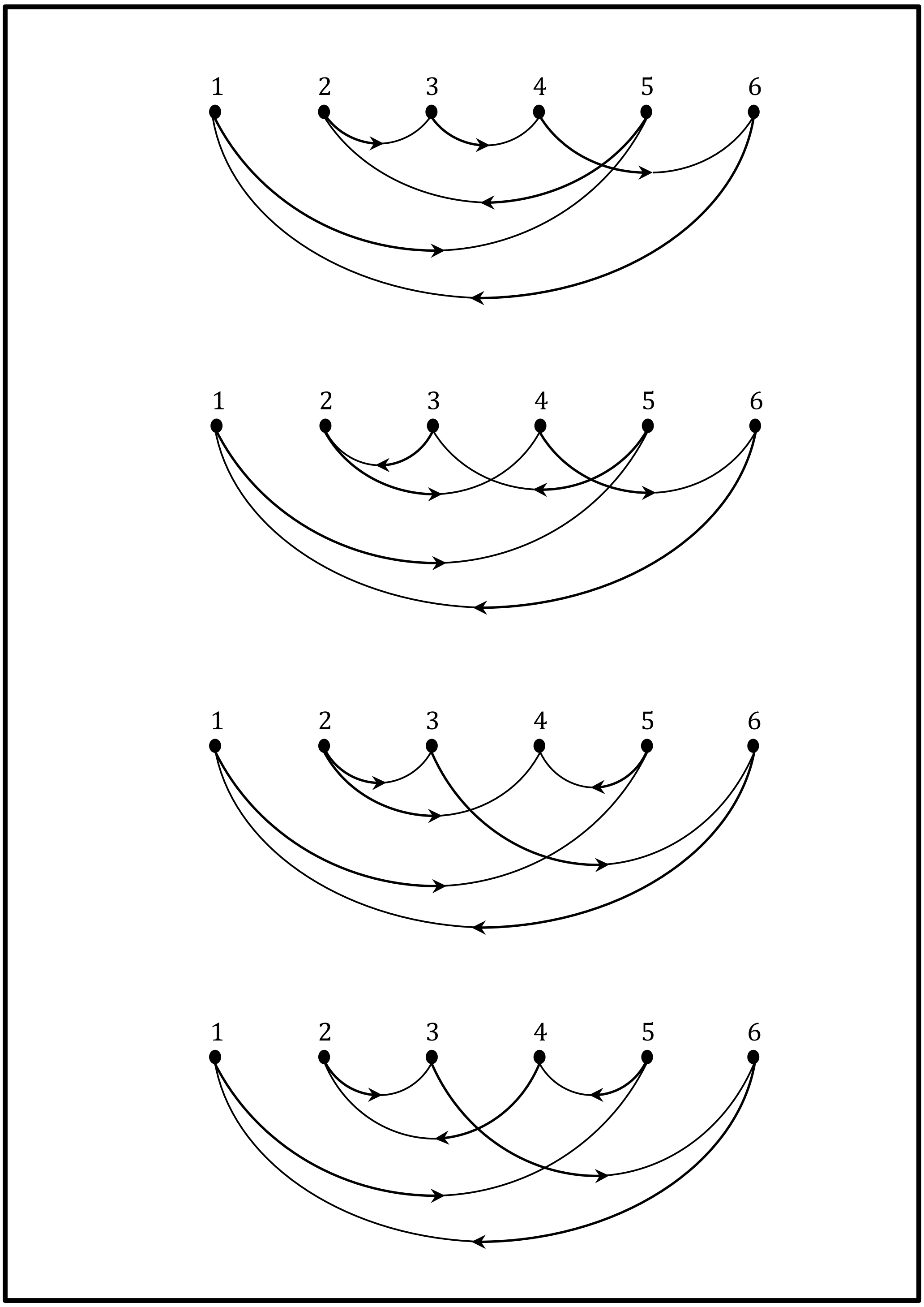


Figure 3.33. Creating directions in block 4 for K_6

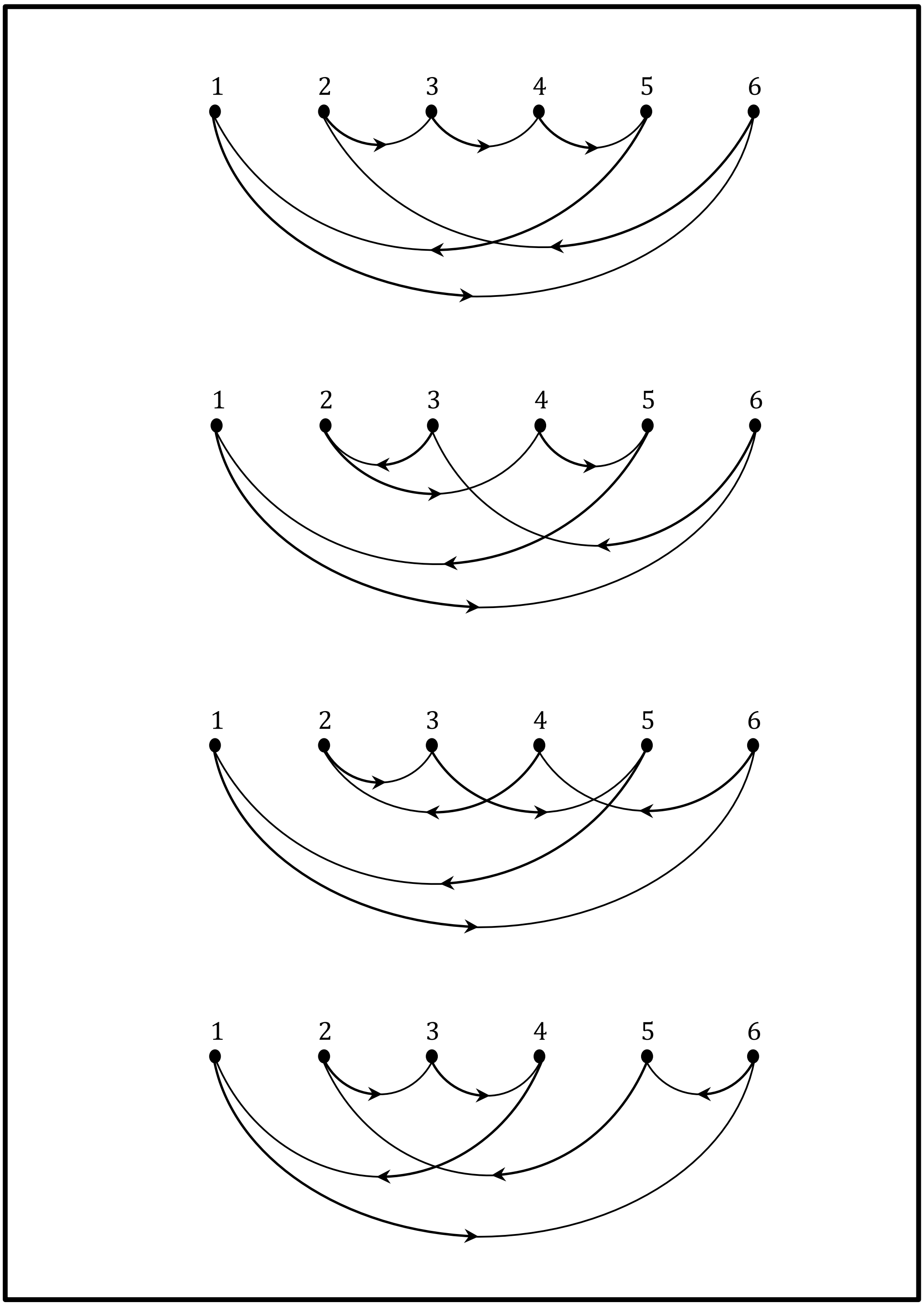


Figure 3.34. Creating directions in block 5 for K_6

Step 2: Fixing and shifting

All directions obtained in Step 1 will be used in Step 2. Let us considered the direction $(1,2,3,4,5,6,1)$. Vertex 1 is not considered more than once since the direction starts and ends at vertex 1. Thus, in this step, we used the path $(1,2,3,4,5,6)$ in the direction $(1,2,3,4,5,6,1)$. Figure 3.35 shows the results obtained when vertex 1 is fixed and the remaining vertices are shifted to the left.

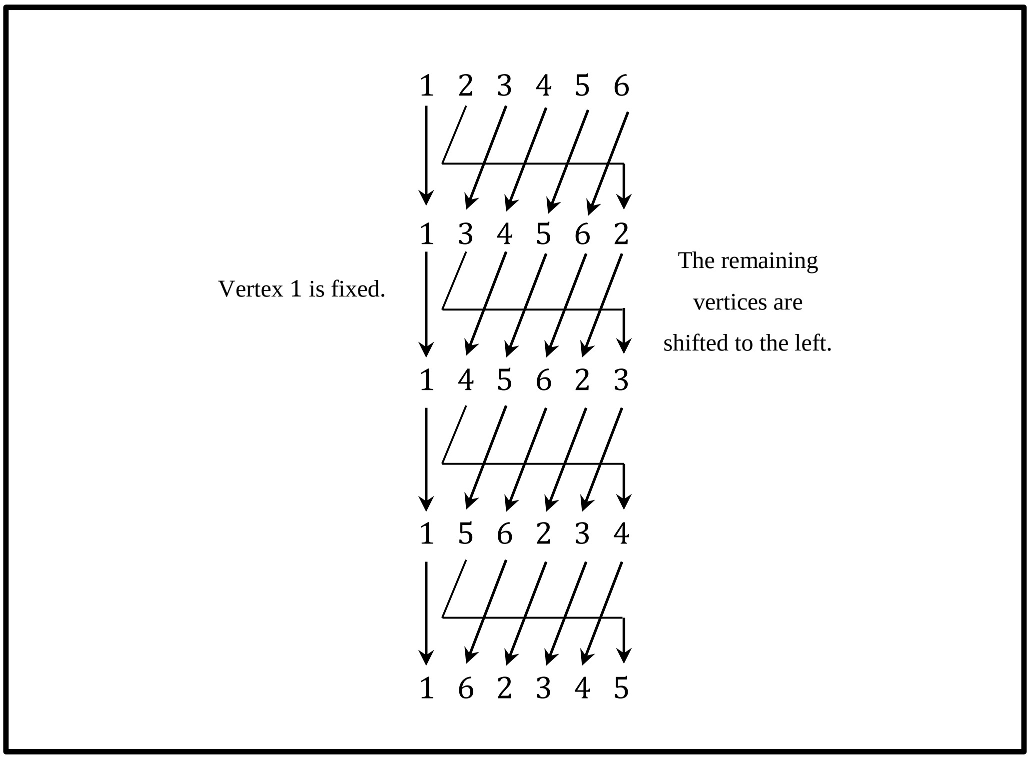


Figure 3.35. Fix-and-shift for K_6 where vertex 1 is fixed

If we fixed-and-shifted $(1,6,2,3,4,5)$, we obtained $(1,2,3,4,5,6)$ once again. Then, we have $(1,2,3,4,5,6)$, $(1,3,4,5,6,2)$, $(1,4,5,6,2,3)$, $(1,5,6,2,3,4)$ and $(1,6,2,3,4,5)$.

Figures 3.36, 3.37, 3.38, 3.39 and 3.40 present the results when vertex 2, vertex 3, vertex 4, vertex 5 and vertex 6 are fixed.

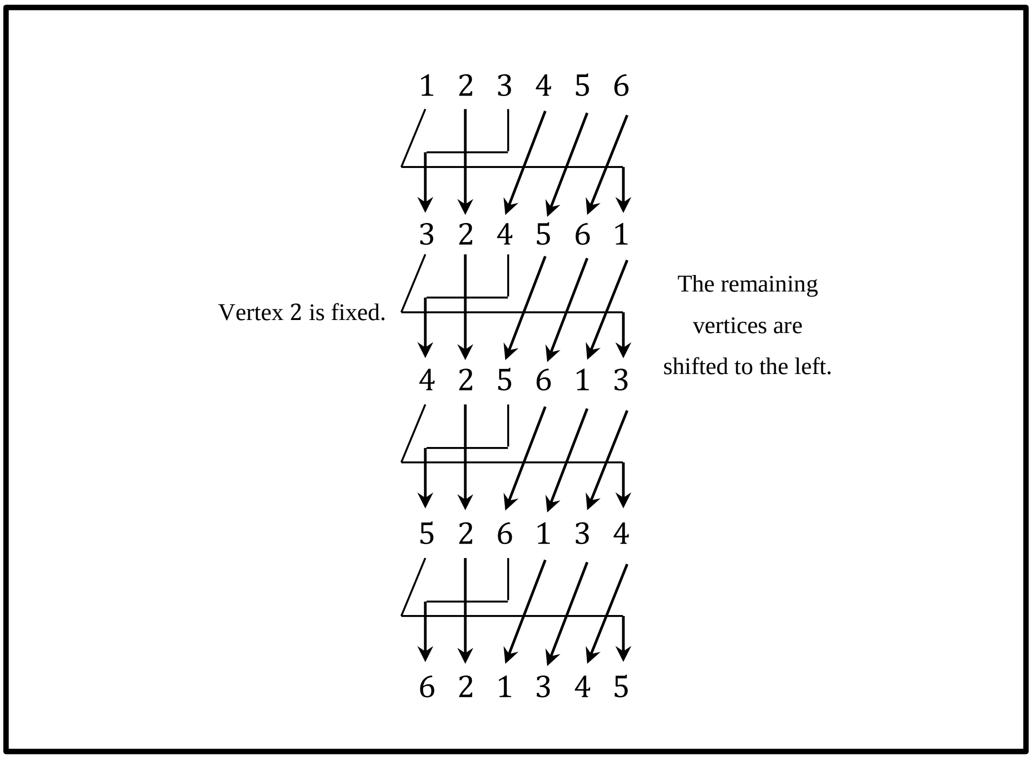


Figure 3.36. Fix-and-shift for K_6 where vertex 2 is fixed

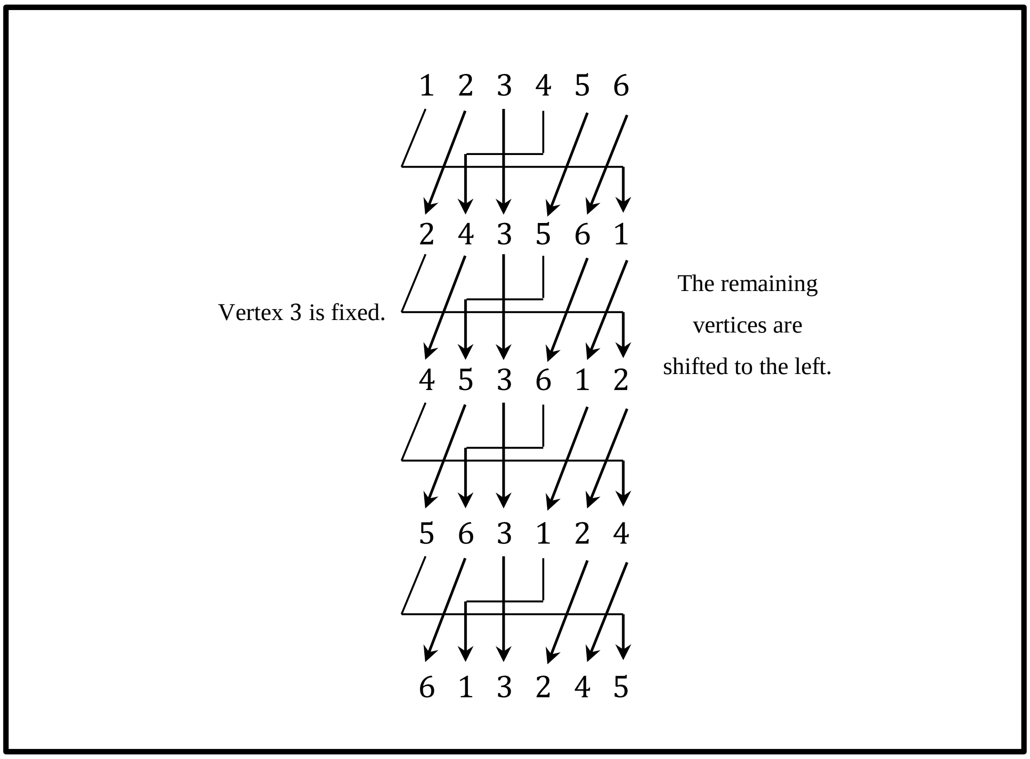


Figure 3.37. Fix-and-shift for K_6 where vertex 3 is fixed

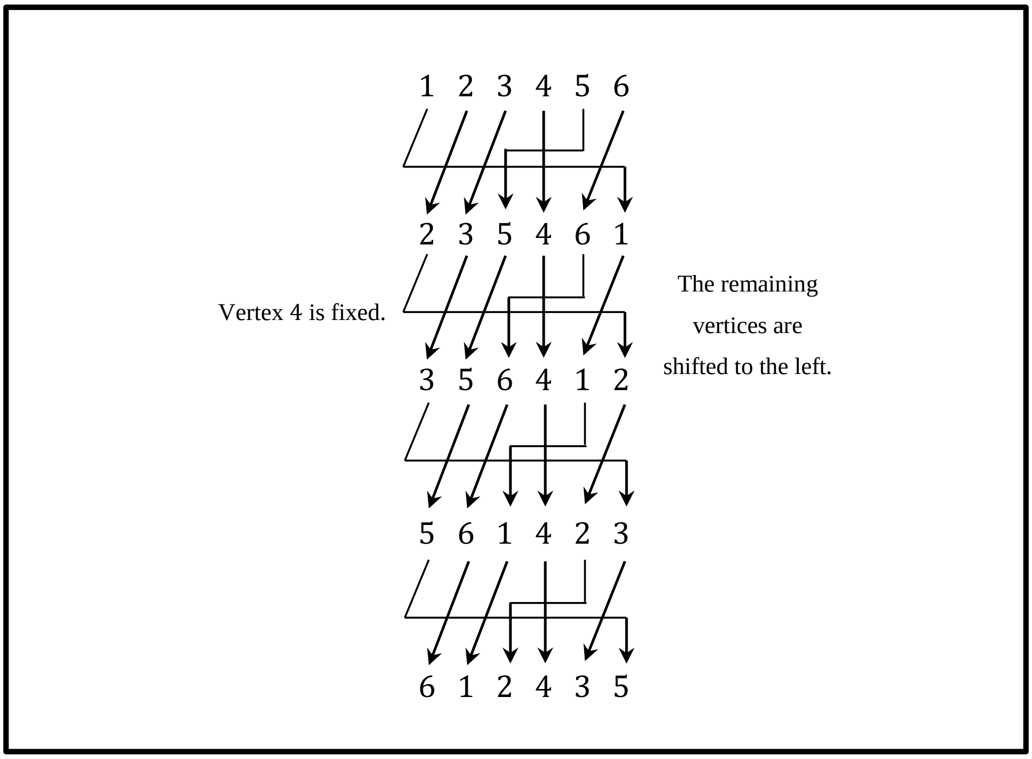


Figure 3.38. Fix-and-shift for K_6 where vertex 4 is fixed

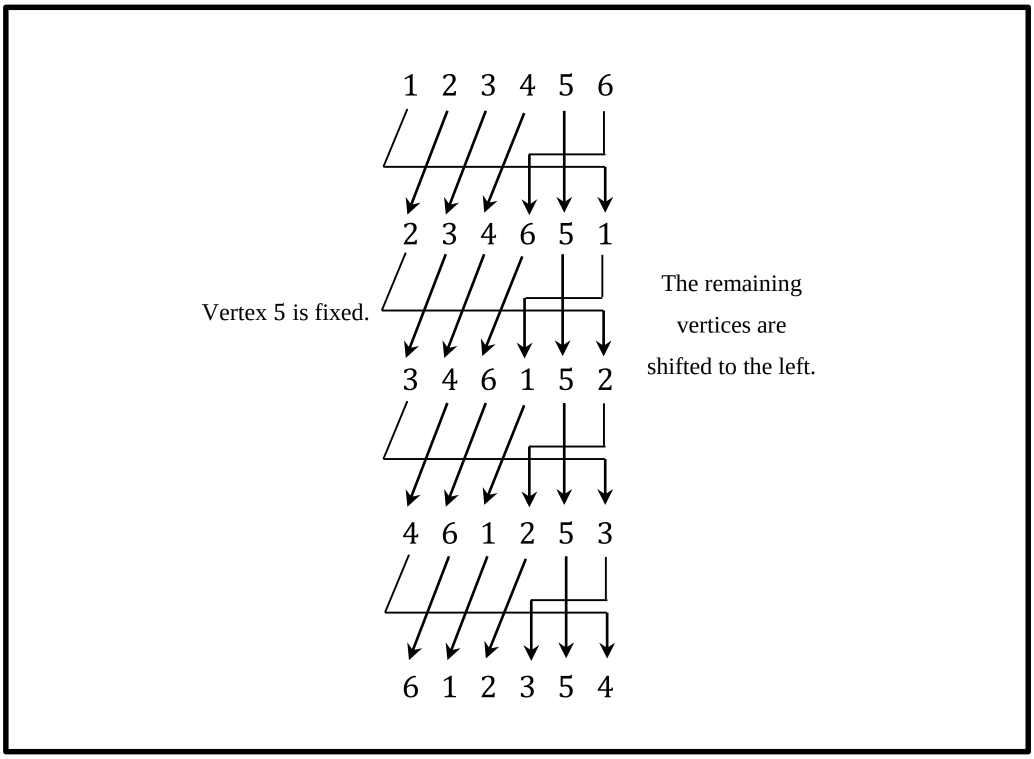


Figure 3.39. Fix-and-shift for K_6 where vertex 5 is fixed.

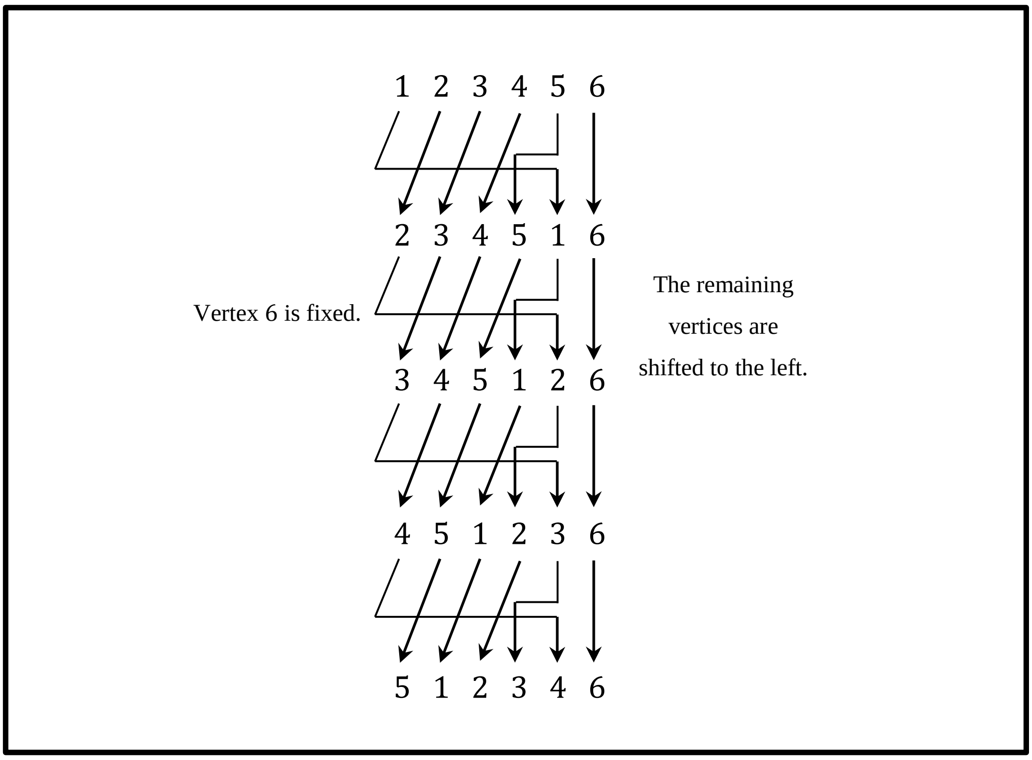


Figure 3.40. Fix-and-shift for K_6 where vertex 6 is fixed

Figures 3.35, 3.36, 3.37, 3.38, 3.39 and 3.40 show fix-and-shift for direction $(1,2,3,4,5,6,1)$ obtained in block 1. Next, we applied fix-and-shift for all directions obtained in Step 1 as presented in Appendix E.

Step 3: Finding the mapping

Using each path in Step 2, we followed Definition 3.1 to get the mapping.

$$(1,2,3,4,5,6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix} = (1,2)(2,3)(3,4)(4,5)(5,6)(6,1),$$

$$(1,3,4,5,6,2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} = (1,3)(3,4)(4,5)(5,6)(6,2)(2,1),$$

.

.

.

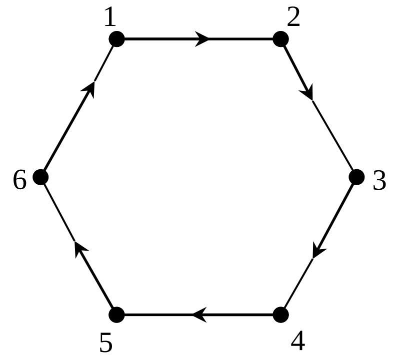
$$(4,6,1,5,2,3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 4 & 6 & 2 & 1 \end{pmatrix} = (1,5)(5,2)(2,3)(3,4)(4,6)(6,1).$$

The mappings obtained in this step are investigated, in other words, the similar mappings and the opposite mappings are eliminated to get the distinct mappings as shown in Appendix F.

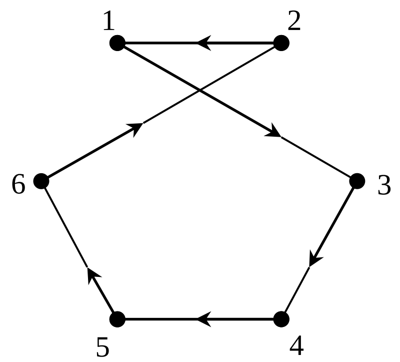
Step 4: Drawing the circuits

We used the distinct mapping obtained in Step 3 to draw the circuits as shown in Figure 3.41. As an example, the mapping of ${C_1}^* = (1,2)(2,3)(3,4)(4,5)(5,6)(6,1)$ is the direction from one vertex to another vertex, that is, vertex 1 maps to 2, 2 maps to 3, 3 maps to 4, 4 maps to 5, 5 maps to 6, and 6 maps to 1. All sixty circuits are presented in Appendix G.

$C_1^* = (1,2)(2,3)(3,4)(4,5)(5,6)(6,1) \equiv$



$C_2^* = (1,3)(3,4)(4,5)(5,6)(6,2)(2,1) \equiv$



$\vdots$

$C_{60}^* = (1,4)(4,6)(6,3)(3,2)(2,5)(5,1) \equiv$

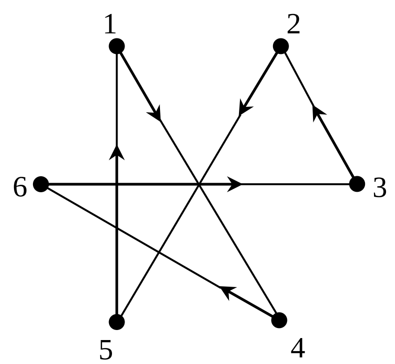


Figure 3.41. Drawing the circuits for K_6

Remark 3.8 A complete graph K_6 is decomposable into sixty distinct Hamiltonian circuits with different path.

We have presented several examples for cases $n = 3$, $n = 4$, $n = 5$ and $n = 6$ by using our new method. By looking the pattern for those cases, the decomposition of K_n into distinct HC with different paths can be generalized. The generalization is presented in Chapter Four.

CHAPTER FOUR
THE HALF BUTTERFLY METHOD

The novel method in this study is named the Half Butterfly Method (HBM). The name of HBM is choosen due to the idea of mirror image. This method is inspired by Gopal *et al.* (2007) as they introduced the BS. However, the BS is totally different to the proposed HBM. The differences are simplified in Table 4.1.

Table 4.1

Differences between BS and HBM.

The BS	**The HBM**
Decompose bipartite graph $K_{n,n}$ into one-factor.	Decompose complete graph K_n into distinct Hamiltonian circuits.
Worked for n where n is a multiple of 4.	Worked for all $n \geq 3$.

In addition, the BS involves the theoretical explanation and examples regardless of any geometric representation. After several efforts have been taken to visualize the BS, the idea of the HBM is developed by adding the construction of direction, the concept of mirror image, and the mapping of each circuit.

The HBM is presented to decompose K_n into distinct HC with different paths. As a preamble, the HBM consists of four steps. The first step is creating direction using Wing Strategy (WS) which will be discussed in detail in Section 4.1. The second step is fix-and-shift as discussed in Chapter Three. The third step is finding the mapping while the final step involves the process of drawing the circuits. The steps are summarized in Table 4.2.

Table 4.2

The steps of HBM.

Step	**Procedure**
1	Create direction using WS.
2	Fix-and-shift every vertex of the direction obtained in Step 1.
3	Determine the mapping for each circuit obtained in Step 2. Similar mappings as well as the opposite mappings are eliminated to get the distinct circuits.
4	Draw the circuits based on the mapping obtained in Step 3.

In this chapter, we introduce the WS that is used to create direction. This chapter is divided into four folds. Section 4.1 discusses WS. Section 4.2 presents the generalization for K_n decomposition into distinct HC with different path using the HBM. Section 4.3 provides the theoretical concepts while Section 4.4 summarized the method of this study.

4.1 Wing Strategy

The WS is used to create direction for every K_n. The following paragraph discusses the steps to create direction using WS. The formula given below is the arrow movement from one vertex to other vertices.

For example, if given

$$x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n, x_1,$$

then, the direction will be as shown in Figure 4.1. In order to implement WS into HBM, the starting point is fixed to be at vertex 1. The arrangement of the vertices must be consecutive and in ascending order to avoid duplicate result. Each arrow indicates the next chosen vertex to obtain the direction. From Figure 4.1, the direction is $(x_1, x_2, x_3, x_4, x_5, x_6, \dots, x_{n-1}, x_n, x_1)$.

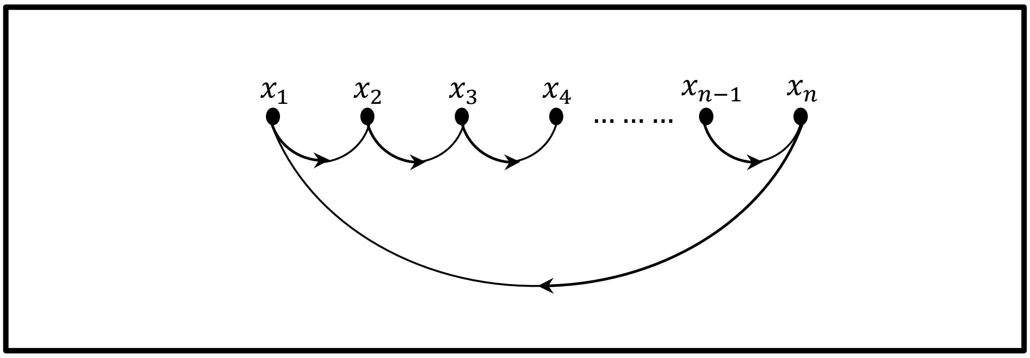


Figure 4.1. Creating direction using WS

The following paragraph discusses the procedures to create direction using WS.

Block 1:

In block 1, fix vertex x_1 as the starting location and vertex x_2 as the second location. Then, the remaining vertices will be substituted consecutively and in ascending order to obtain the direction for K_n in block 1. The procedures are shown below.

The first direction is

$$x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n, x_1, \quad (4.1)$$

where

x_1 is the starting location,

x_2 is the second location,

x_3 is the third location, ...,

x_n is the n^{th} location of (4.1).

To construct the second direction in block 1,

1. fix x_1 and x_2 as the first and second locations,
2. substitute x_4 as the third location, and
3. place the remaining vertices in (4.1) to be the fourth, fifth, ..., nth locations.

Then, from (4.1), the second direction in block 1 is:

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_n,\ x_1$$

$$x_1,\ x_2,\ x_4,\ x_3,\ x_5,\ x_6,\ x_7,\ x_8,\ \ldots,\ x_n,\ x_i.$$

(4.2)

To obtain the third direction in block 1,

1. fix x_1 and x_2 as the first and second locations,
2. substitute x_5 as the third location, and
3. place the remaining vertices in (4.1) to be the fourth, fifth, ..., nth locations.

Then, from (4.1), the third direction is:

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_n,\ x_1$$

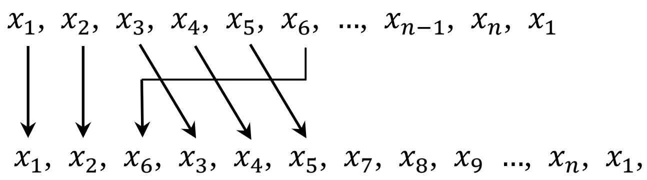
$$x_1,\ x_2,\ x_5,\ x_3,\ x_4,\ x_6,\ x_7,\ x_8,\ \ldots,\ x_n,\ x_1.$$

(4.3)

To develop the remaining directions in block 1,

1. fix x_1 and x_2 as the first and second locations,
2. substitute a vertex as the third location, and
3. place the remaining vertices in (4.1) to be the fourth, fifth, ..., nth locations.

The remaining directions in block 1 are shown below. Then, from (4.1) we have



(4.4)

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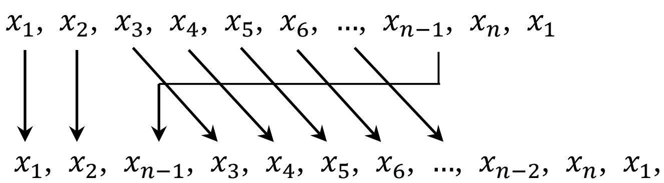
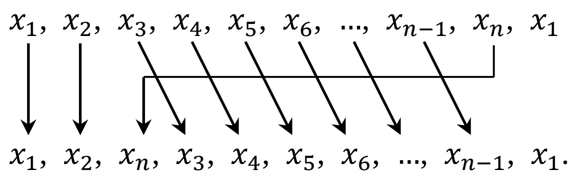


(4.5)



(4.6)

Block 2:

To construct directions in block 2,

1. fix vertex x_1 as the first location and vertex x_3 as the second location,
2. take one vertex as the third location, and
3. place the remaining vertices as the fourth, fifth, ..., nth locations consecutively.
4. Repeat the above procedures until vertex x_n is the third location.

The first direction in block 2 is

$$x_1, x_3, x_2, x_4, x_5, x_6, \ldots, x_{n-1}, x_n, x_1. \tag{4.7}$$

Then, from (4.7), the remaining directions in block 2 are

$$x_1,\ x_3,\ x_2,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_n,\ x_1$$

$$x_1,\ x_3,\ x_4,\ x_2,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_n,\ x_1, \tag{4.8}$$

$$x_1,\ x_3,\ x_2,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_n,\ x_1$$

$$x_1,\ x_3,\ x_5,\ x_2,\ x_4,\ x_6,\ \ldots,\ x_{n-1},\ x_n,\ x_1, \tag{4.9}$$

$x_1,\ x_3,\ x_2,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_n,\ x_1$

$x_1,\ x_3,\ x_6,\ x_2,\ x_4,\ x_5,\ \ldots,\ x_{n-1},\ x_n,\ x_1,$

(4.10)

.

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$x_1,\ x_3,\ x_2,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_n,\ x_1$

$x_1,\ x_3,\ x_{n-1},\ x_2,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-2},\ x_n,\ x_1,$

(4.11)

$x_1,\ x_3,\ x_2,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_n,\ x_1$

$x_1,\ x_3,\ x_n,\ x_2,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_1.$

(4.12)

Block 3:

Directions in block 3 are developed as follows:

1. Fix vertex x_1 as the first location and vertex x_4 as the second location.
2. Take one vertex as the third location.
3. Place the remaining vertices as the fourth, fifth, ..., nth locations consecutively.
4. Repeat the above procedures until vertex x_n is the third location.

The first direction in block 3 is

$$x_1, x_4, x_2, x_3, x_5, x_6, x_7, \ldots, x_{n-1}, x_n, x_1.$$

$$(4.13)$$

Then, from (4.13), the remaining directions in block 3 are

$$x_1,\ x_4,\ x_2,\ x_3,\ x_5,\ x_6,\ x_7,\ \ldots,\ x_{n-1},\ x_n,\ x_1$$

$$x_1,\ x_4,\ x_3,\ x_2,\ x_5,\ x_6,\ x_7,\ \ldots,\ x_{n-1},\ x_n,\ x_1,$$

$$(4.14)$$

$$x_1,\ x_4,\ x_2,\ x_3,\ x_5,\ x_6,\ x_7,\ \ldots,\ x_{n-1},\ x_n,\ x_1$$

$$x_1,\ x_4,\ x_5,\ x_2,\ x_3,\ x_6,\ x_7,\ \ldots,\ x_{n-1},\ x_n,\ x_1,$$

$$(4.15)$$

$$x_1,\ x_4,\ x_2,\ x_3,\ x_5,\ x_6,\ x_7,\ \ldots,\ x_{n-1},\ x_n,\ x_1$$

$$x_1,\ x_4,\ x_6,\ x_2,\ x_3,\ x_5,\ x_7,\ \ldots,\ x_{n-1},\ x_n,\ x_1,$$

$$(4.16)$$

.

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.

$$x_1,\ x_4,\ x_2,\ x_3,\ x_5,\ x_6,\ x_7,\ \ldots,\ x_{n-1},\ x_n,\ x_1$$

$$x_1,\ x_4,\ x_{n-1},\ x_2,\ x_3,\ x_5,\ x_6,\ x_7,\ \ldots,\ x_n,\ x_1,$$

(4.17)

$$x_1,\ x_4,\ x_2,\ x_3,\ x_5,\ x_6,\ x_7,\ \ldots,\ x_{n-1},\ x_n,\ x_1$$

$$x_1,\ x_4,\ x_n,\ x_2,\ x_3,\ x_5,\ x_6,\ x_7,\ \ldots,\ x_{n-1},\ x_1.$$

(4.18)

Repeat the same procedures as in block 1, block 2, and block 3 to construct all directions for K_n until x_n in (4.1) be the second location. The directions constructed in the last block are discussed below.

Last block:

To develop the directions in the last block,

1. fix vertex x_1 as the first location and vertex x_n as the second location,
2. take one vertex as the third location, and
3. place the remaining vertices as the fourth, fifth, ..., nth locations.
4. Repeat the above procedures until vertex x_{n-1} be the third location.

The first direction obtained in the last block is

$$x_1, x_n, x_2, x_3, x_4, x_5, x_6, \ldots, x_{n-1}, x_1.$$

(4.19)

Then, from (4.19), the remaining directions constructed in the last block are

$$\begin{array}{l} x_1,\ x_n,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_1 \\ \\ x_1,\ x_n,\ x_3,\ x_2,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_1, \end{array}$$

(4.20)

$$\begin{array}{l} x_1,\ x_n,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_1 \\ \\ x_1,\ x_n,\ x_4,\ x_2,\ x_3,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_1, \end{array}$$

(4.21)

$$\begin{array}{l} x_1,\ x_n,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-1},\ x_1 \\ \\ x_1,\ x_n,\ x_5,\ x_2,\ x_3,\ x_4,\ x_6,\ \ldots,\ x_{n-1},\ x_1, \end{array}$$

(4.22)

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$$\begin{array}{l} x_1,\ x_n,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-2},\ x_{n-1},\ x_1 \\ \\ x_1,\ x_n,\ x_{n-1},\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ \ldots,\ x_{n-2},\ x_1. \end{array}$$

(4.23)

Now, we have presented the steps to generate the directions. We summarize the results of the directions obtained in block 1, block 2, block 3 and last block in Tables 4.3, 4.4, 4.5, and 4.6 respectively. The sequences of the directions are presented in Appendix L.

Table 4.3

Creating directions in block 1.

Wing Strategy
$x_1, x_2, x_3, x_4, \ldots, x_{n-1}, x_n, x_1$
$x_1, x_2, x_4, x_3, x_5, x_6, x_7, x_8, \ldots, x_n, x_1$
$x_1, x_2, x_5, x_3, x_4, x_6, x_7, x_8, \ldots, x_n, x_1$
$x_1, x_2, x_6, x_3, x_4, x_5, x_7, x_8, \ldots, x_n, x_1$
.
.
.
$x_1, x_2, x_{n-1}, x_3, x_4, x_5, \ldots, x_n, x_1$
$x_1, x_2, x_n, x_3, x_4, x_5, x_6, \ldots, x_{n-1}, x_1$

Table 4.4

Creating directions in block 2.

Wing Strategy
$x_1, x_3, x_2, x_4, x_5, x_6, x_7, \ldots, x_{n-1}, x_n, x_1$
$x_1, x_3, x_4, x_2, x_5, x_6, x_7, \ldots, x_{n-1}, x_n, x_1$
$x_1, x_3, x_5, x_2, x_4, x_6, x_7, \ldots, x_{n-1}, x_n, x_1$
$x_1, x_3, x_6, x_2, x_4, x_5, x_7, \ldots, x_{n-1}, x_n, x_1$
.
.
.
$x_1, x_3, x_{n-1}, x_2, x_4, x_5, x_6, \ldots, x_{n-2}, x_n, x_1$
$x_1, x_3, x_n, x_2, x_4, x_5, x_6, \ldots, x_{n-1}, x_1$

Table 4.5

Creating directions in block 3.

Wing Strategy
$x_1, x_4, x_2, x_3, x_5, x_6, x_7, x_8, x_9, \ldots, x_{n-1}, x_n, x_1$
$x_1, x_4, x_3, x_2, x_5, x_6, x_7, x_8, x_9, \ldots, x_{n-1}, x_n, x_1$
$x_1, x_4, x_5, x_2, x_3, x_6, x_7, x_8, x_9, \ldots, x_{n-1}, x_n, x_1$
$x_1, x_4, x_6, x_2, x_3, x_5, x_7, x_8, x_9, \ldots, x_{n-1}, x_n, x_1$
.
.
.
$x_1, x_4, x_{n-1}, x_2, x_3, x_5, x_6, x_7, x_8, \ldots, x_{n-2}, x_n, x_1$
$x_1, x_4, x_n, x_2, x_3, x_5, x_6, x_7, x_8, \ldots, x_{n-1}, x_1$

Table 4.6

Creating directions in the last block.

Wing Strategy
$x_1, x_n, x_2, x_3, x_4, x_5, x_6, x_7, x_8, ..., x_{n-1}, x_1$
$x_1, x_n, x_3, x_2, x_4, x_5, x_6, x_7, x_8, ..., x_{n-1}, x_1$
$x_1, x_n, x_4, x_2, x_3, x_5, x_6, x_7, x_8, ..., x_{n-1}, x_1$
$x_1, x_n, x_5, x_2, x_3, x_4, x_6, x_7, x_8, ..., x_{n-1}, x_1$
.
.
.
$x_1, x_n, x_{n-1}, x_2, x_3, x_4, x_5, x_6, ..., x_{n-2}, x_1$

The following section discusses the general method to decompose K_n into distinct Hamiltonian circuits with different paths using the HBM.

4.2 K_n Decomposition into Distinct Hamiltonian Circuits with Different Path

The general method for extracting geometric representations of distinct HC with different paths in K_n is discussed below.

Step 1: Creating direction

In this step, WS as discussed in Section 4.1 is used to get the direction. From Table 4.3, the directions are shown in Figure 4.2. The complete directions from Tables 4.3, 4.4, 4.5 and 4.6 are presented in Appendices H, I, J and K respectively.

As previously stated, we follow the vertices in Table 4.3 to move the arrow to construct the directions in block 1. These directions are shown in Figure 4.2.

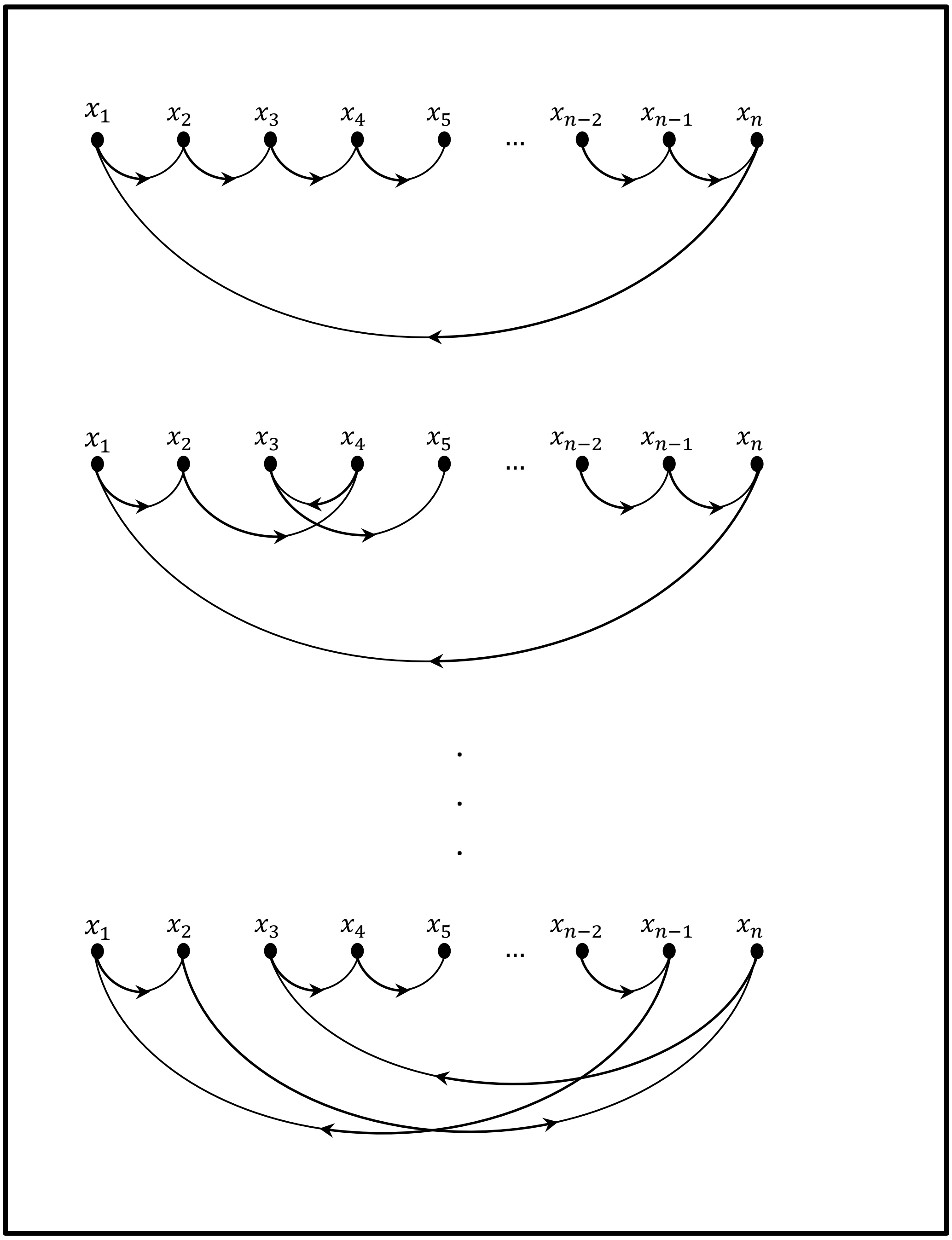

Figure 4.2. Creating directions in block 1 for K_n

Step 2: Fix-And-Shift

Fix vertex x_1, and shift the remaining vertices to the left. Vertex x_1 is not considered twice in this step since it is the starting and ending location.

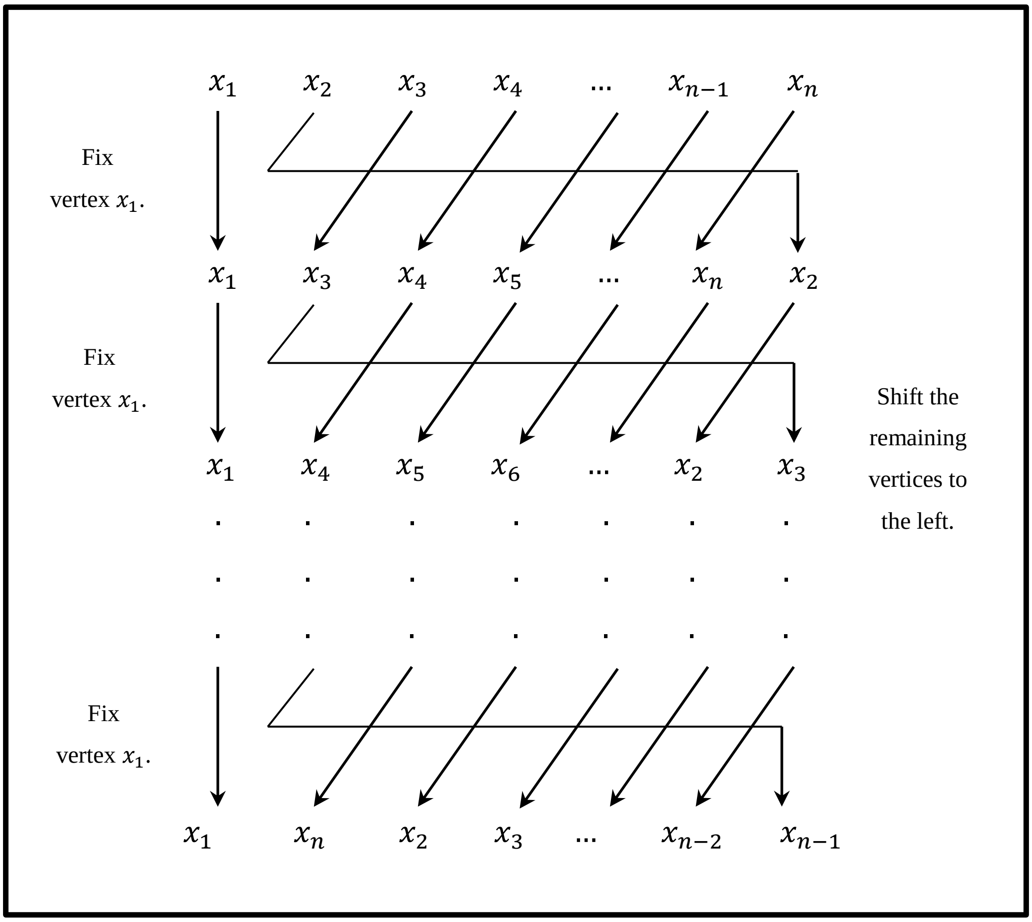

Figure 4.3. Step of fix-and-shift, vertex x_1 is fixed

Next, fix vertex x_2, and shift the remaining vertices to the left. Repeat this step until vertex x_n is fixed.

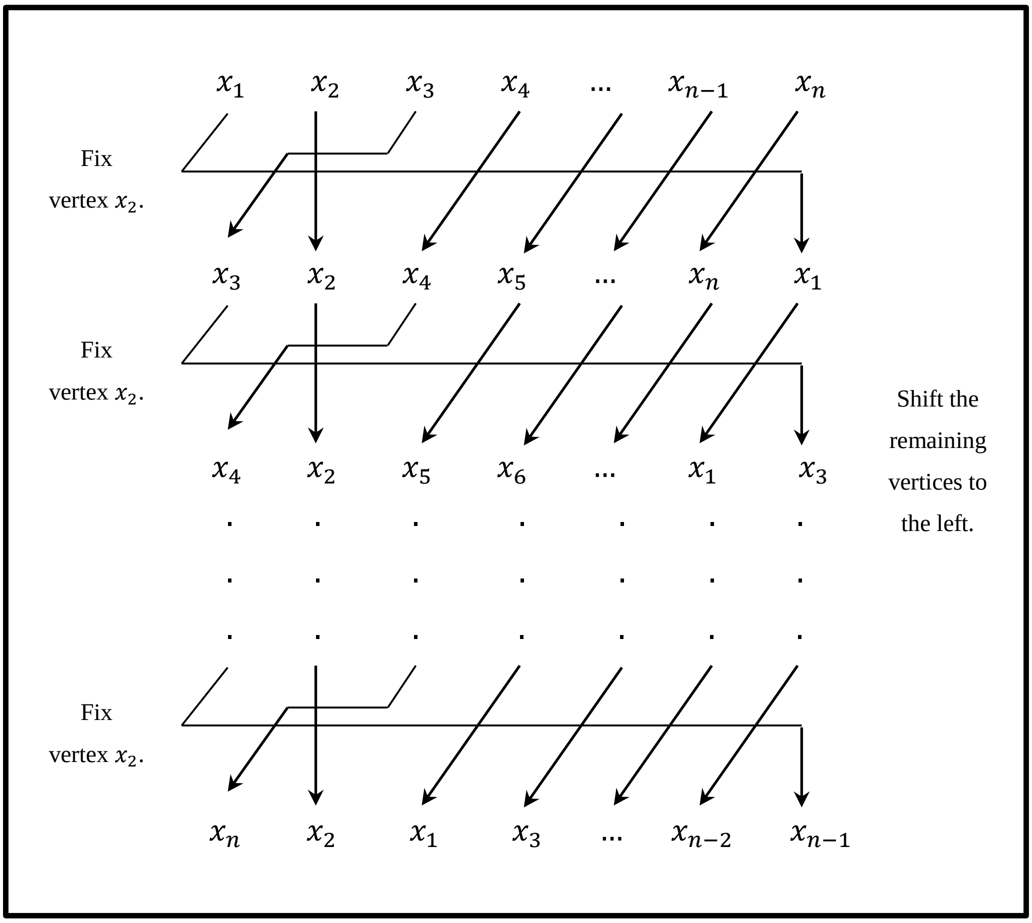

Figure 4.4. Step of fix-and-shift, vertex x_2 is fixed

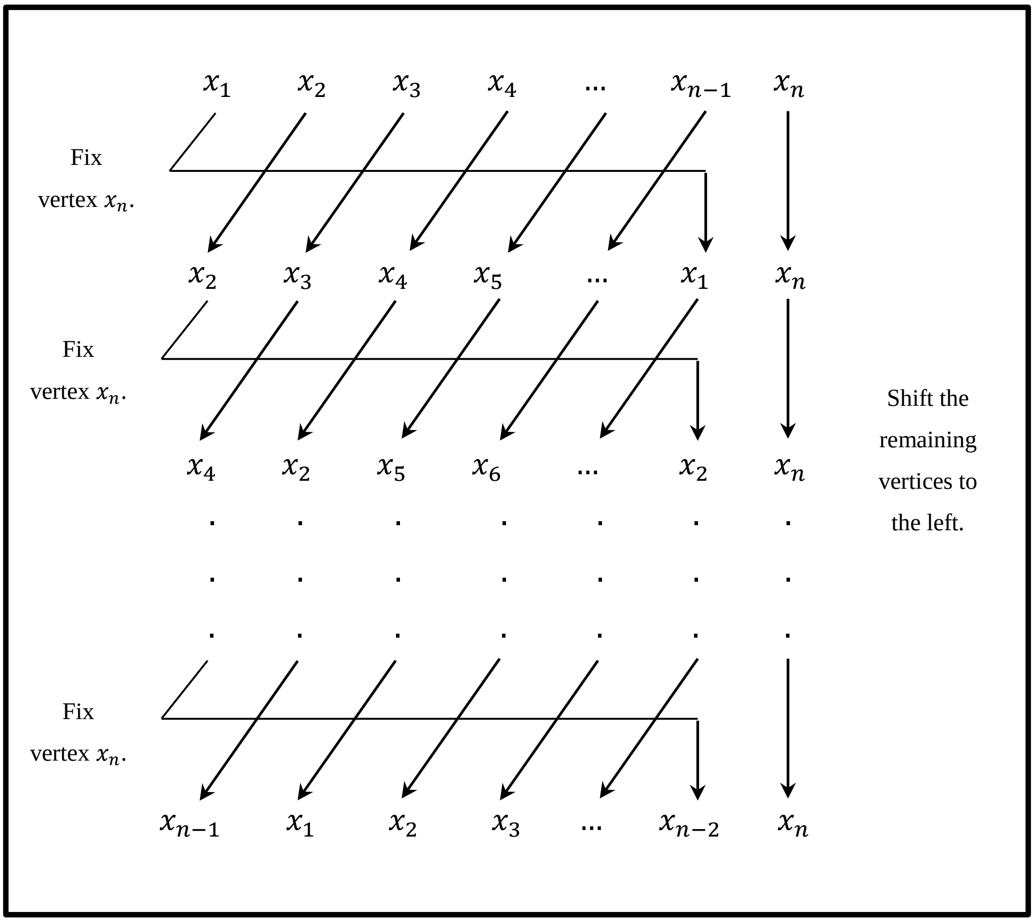

Figure 4.5. Step of fix-and-shift, vertex x_n is fixed

Figures 4.3, 4.4 and 4.5 fixed-and-shifted the first direction obtained in block 1. Repeat this step for all directions developed in Step 1. Then, find the mapping.

Step 3: Finding the mapping

Used each of the path obtained in Step 2 to get the mapping.

From (4.1) we have

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \cdots & x_{n-1} & x_n \\ x_2 & x_3 & x_4 & x_5 & \cdots & x_n & x_1 \end{pmatrix}.$$

Then, by referring Definition 3.1 the mapping is

$$(x_1, x_2)(x_2, x_3)(x_3, x_4)(x_4, x_5) \ldots (x_{n-1}, x_n)(x_n, x_1),$$

(4.24)

which maps

$x_1 \mapsto x_2$, $x_2 \mapsto x_3$, $x_3 \mapsto x_4$, $x_4 \mapsto x_5$, ..., $x_{n-1} \mapsto x_n$, and $x_n \mapsto x_1$.

In Step 3, there are two important procedures in obtaining the distinct mapping. First, eliminate the similar mapping. Next, eliminate the opposite mapping using Definitions 2.3, 3.3 and 3.4.

Find the mapping for each of the path obtained in Step 2. Next, draw the circuits.

Step 4: Drawing the circuits

Used each of the mapping in Step 3 to draw the circuits. The arrow movement is based on the mapping obtained in Step 3. To be clear, for a mapping ${C_1}^* = (x_1, x_2)(x_2, x_3)(x_3, x_4) \ldots (x_{n-1}, x_n)(x_n, x_1)$, the direction is x_1 maps to x_2, x_2 maps to x_3, x_3 maps to x_4, until x_n maps to x_1.

From (4.24), we have the circuit as shown in Figure 4.6.

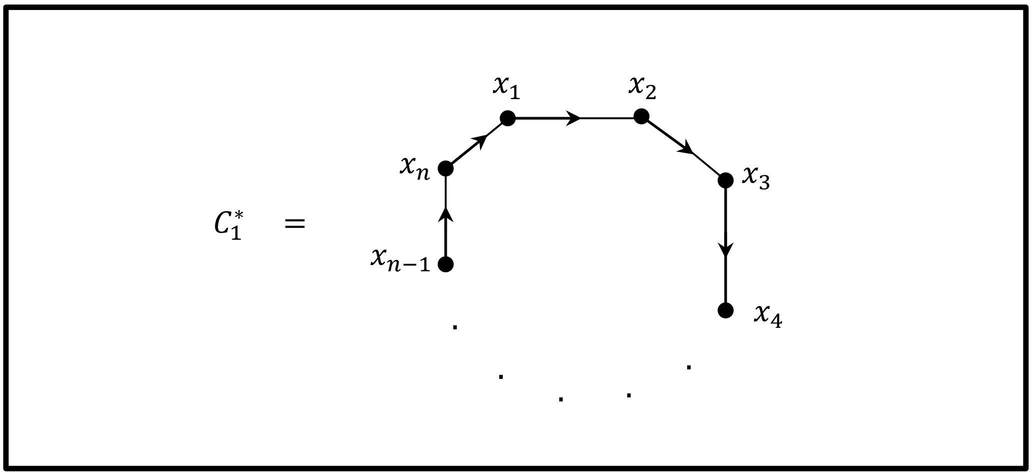

Figure 4.6. Drawing the circuits for all K_n

Now, we have completed the generalization of decomposing complete graph K_n into distinct Hamiltonian circuits with different paths. The above discussions yield the following lemmas and theorems as discussed in Section 4.3.

4.3 Conceptual Results on the HBM

In this chapter, several lemmas and theorems are discussed based on the methods presented in previous sections. We define λ as the block for each direction in K_n.

Lemma 4.1 There are $(n-1)$ blocks to create directions for all K_n, $n \geq 3$.

Proof. Suppose $\{x_1, x_2, x_3, x_4, \ldots, x_n\}$ are vertices of K_n. Let λ_1 be block 1, λ_2 be block 2, λ_3 be block 3, and λ_m be the last block. In generating the directions, in each block, we have the following situation:

λ_1 : Vertex x_1 is fixed as the starting location and x_2 is fixed to be the second location for the direction.

λ_2 : Vertex x_1 is fixed as the starting location and x_3 is fixed to be the second location for the direction.

λ_3 : Vertex x_1 is fixed as the starting location and x_4 is fixed to be the second location for the direction.

.

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λ_m : Vertex x_1 is fixed as the starting location and x_n is fixed to be the second location for the direction.

For each block, a vertex (vertex x_1) is fixed to be the starting location. Then, there are $(n-1)$ vertices left to be chosen as the second location, consecutively, in each block. Since there are $(n-1)$ vertices to be the second location, then there are $(n-1)$ blocks to create direction for all $K_n, n \geq 3$. □

Lemma 4.2 There are $(n-2)$ directions in each block.

Proof. By Lemma 4.1, there are $(n-1)$ blocks to create the directions. In each block, two vertices are fixed to be the starting location and the second location. Since there are n vertices, and two vertices are fixed such that vertex x_i and vertex x_j for $j > i$, then there are $(n-2)$ vertices left to be the third location for each direction. Thus, $(n-2)$ directions are obtained in each block. □

Lemma 4.1 and Lemma 4.2 provide the total number of directions and blocks for a complete graph. The following propositions and theorems are provided based on Lemma 4.1 and Lemma 4.2.

Proposition 4.3 The total number of directions that could be created in K_n, $n \geq 3$ is $(n-2)(n-1)$.

Proof. From Lemma 4.1, let λ_k be the blocks in creating the directions, for $1 \leq k \leq n-1$. Since we have $(n-1)$ blocks and each block has $(n-2)$ directions, thus we have

$$\sum_{k=1}^{n-1} \lambda_k = (n-2)(n-1)$$

directions in K_n.

In other way, we can prove this theorem using the idea of arithmetic sequences as discussed below. From Lemmas 4.1 and 4.2, we have the total number of directions for each block as shown below.

$$\lambda_1 = (n-2) \text{ directions}$$

$$\begin{aligned} \lambda_1 + \lambda_2 &= (n-2) + (n-2) \\ &= 2(n-2) \\ &= (2n-4) \text{ directions} \end{aligned}$$

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= (n-2) + (n-2) + (n-2) \\ &= 3(n-2) \\ &= (3n-6) \text{ directions} \end{aligned}$$

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= (n-2) + (n-2) + (n-2) + (n-2) \\ &= 4(n-2) \\ &= (4n-8) \text{ directions} \end{aligned}$$

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 &= (n-2) + (n-2) + (n-2) + (n-2) \\ &\quad +(n-2) \\ &= 5(n-2) \\ &= (5n-10) \text{ directions} \end{aligned}$$

We transform the above results into arithmetic sequence as presented below. Let A_k be the arithmetic sequence and a_k denotes the elements in A_k for $1 \leq k \leq n$.

$$A_k = \{n-2,\ 2n-4,\ 3n-6,\ 4n-8,\ 5n-10, \dots, a_k\},$$

(4.25)

where $a_1 = n-2$, $a_2 = 2n-4$, $a_3 = 3n-6$, $\dots a_k$. To find the last term of A_k, we consider the formula

$$a_k = a_1 + (k-1)d,$$

(4.26)

where a_1 is the first element of A_k, d is the common difference and k is the number of the element to find. To determine the common difference, d we use

$$d = a_i - a_j,$$

(4.27)

where $i = j+1$ and $1 \leq i < j \leq k$. Then, by taking any element of A_k in (4.25), we have

$$d = n-2.$$

(4.28)

Here, we take $k = n - 1$ due to Lemma 4.1. Therefore, from (4.26), (4.27) and (4.28), we have

$$\begin{aligned} a_k &= a_1 + (k-1)d \\ a_{n-1} &= a_1 + \big((n-1)-1\big)d \\ &= (n-2) + \big((n-1)-1\big)(n-2) \\ &= (n-2) + (n-2)(n-2) \\ &= n - 2 + n^2 - 4n + 4 \\ &= n^2 - 3n + 2 \\ &= (n-2)(n-1). \end{aligned}$$

Thus, the last element of A_k is $[(n-2)(n-1)]^{\text{th}}$ element, which gives the total number of directions that can be produced for every K_n for $n \geq 3$. □

Proposition 4.4 There are $n(n-1)$ Hamiltonian circuits from each direction in K_n, $n \geq 3$.

Proof. From Proposition 4.3, there are $(n-2)(n-1)$ directions in K_n. The vertices $x_1, x_2, x_3, \dots, x_n$ in each direction are fixed-and-shifted, where one vertex is fixed and the remaining vertices are shifted to the left.

Suppose the first direction of K_n is $\alpha_1 = \{x_1, x_2, x_3, x_4, \dots, x_n\}$. When each vertex of α_1 is fixed, then we have $(n-1)$ vertices left to be shifted to the left. Since there are n vertices to be fixed, then we have

$$\sum_{i=1}^{n}(n-1) = \underbrace{(n-1)+(n-1)+(n-1)+\cdots+(n-1)}_{n \text{ times}}$$

$$= n(n-1)$$

Hamiltonian circuits with similar paths from each direction in K_n. $\square$

Proposition 4.5 There are $\frac{(n-1)(n!)}{(n-3)!}$ Hamiltonian circuits from all directions in K_n, $n \geq 3$.

Proof. For every K_n, $(n-2)(n-1)$ directions are obtained (Proposition 4.3). For each direction, when one vertex is fixed, there are $(n-1)$ vertices left to be shifted to the left. Since there are n vertices to fix, then we have

$$\sum_{j=1}^{n}\sum_{i=1}^{n-1}(n-2)(n-1)$$

$$= \sum_{i=1}^{n-1}\underbrace{[(n-2)(n-1)+(n-2)(n-1)+\cdots+(n-2)(n-1)]}_{n \text{ times}}$$

$$= \sum_{i=1}^{n-1}(n)(n-2)(n-1)$$

$$= \underbrace{(n)(n-2)(n-1) + (n)(n-2)(n-1) + \cdots + (n)(n-2)(n-1)}_{(n-1)\text{ times}}$$

$$= (n-1)(n)(n-2)(n-1)$$

$$= (n-1)[(n-2)(n-1)(n)]$$

$$= (n-1)\left[\frac{n!}{(n-3)!}\right]$$

$$= \left[\frac{(n-1)(n!)}{(n-3)!}\right]$$

circuits from all directions in K_n. □

Remark 4.6 The Hamiltonian circuits produced from all directions in K_n (Proposition 4.5) are inclusive of duplicate circuits, i.e. circuits with similar mapping and circuits with opposite mapping. By referring Definitions 2.3 and 3.1, the similar and opposite mapping can be found and eliminated.

Theorem 4.7 Let G be a complete graph. Then G is decomposable into $\frac{(n-1)!}{2}$ distinct Hamiltonian circuits with different paths for all $n \geq 3$.

Proof. Suppose G is a K_n, then there exist n vertices. The total number of Hamiltonian circuits of K_n is known to be $n!$. Since there are n vertices, then there are n vertices available as starting locations. Thus, K_n has $\frac{n!}{n}$ Hamiltonian circuits with similar paths. Since there are $(n-1)$ blocks, $(n-2)$ directions in each block, and $(n)(n-1)$ circuits from each direction, by considering the mirror image of each circuit, then we have

$$\left[(n-1)(n-2)(n)(n-1)-\frac{n!}{n}\right]\times\frac{1}{2}$$

(4.29)

circuits that must be eliminated. Then, from Proposition 4.5 and (4.29), for every K_n where $n \geq 3$, there are

$$\frac{(n-1)(n)!}{(n-3)!}-\left[\left((n)(n-1)^2(n-2)-\frac{n!}{n}\right)\times\frac{1}{2}\right]$$

$$=\frac{(n-1)(n)!}{(n-3)!}-\left[\frac{(n)(n-1)^2(n-2)}{2}-\frac{n!}{2n}\right]$$

$$=\frac{(n-1)(n)!}{(n-3)!}-\left[\frac{(n)(n-1)^2(n-2)}{2}-\frac{(n)(n-1)!}{2n}\right]$$

$$=\frac{(n-1)(n)!}{(n-3)!}-\left[\frac{(2n^2)(n-1)^2(n-2)-(n)(n-1)!}{2n}\right]$$

$$=\frac{(n-1)(n)!}{(n-3)!}-\left[\frac{(2n)(n-1)^2(n-2)-(n-1)!}{2}\right]$$

$$=\frac{2(n-1)(n)!-(2n)(n-1)^2(n-2)(n-3)!+(n-1)!\,(n-3)!}{2(n-3)!}$$

$$=\frac{\begin{array}{c}(2n)(n-1)^2(n-2)(n-3)!-(2n)(n-1)^2(n-2)(n-3)!\\+(n-1)!\,(n-3)!\end{array}}{2(n-3)!}$$

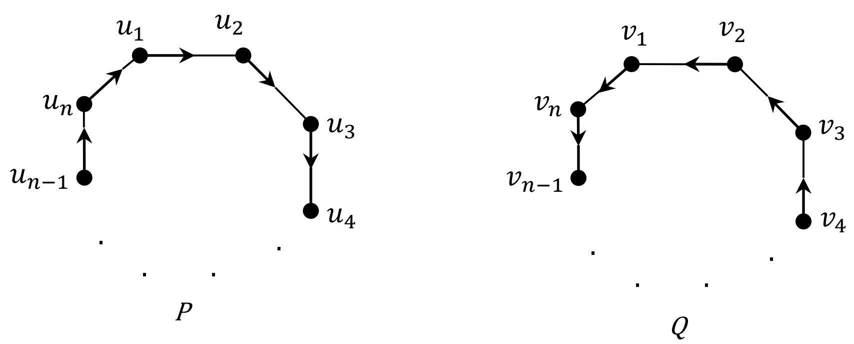
$$=\frac{(2n)(n-1)^2(n-2)-(2n)(n-1)^2(n-2)+(n-1)!}{2}$$

$$=\frac{(n-1)!}{2}$$

distinct Hamiltonian circuits with different paths from K_n. □

Theorem 4.8 Let P and Q be two Hamiltonian circuits that traverse the same path, but in the opposite direction. If the adjacency matrix of P is equal to the transpose of the adjacency matrix of Q, i.e. $\mathbf{A}_P = \mathbf{A}_Q{}^T$, then $P \cong Q$.

Proof. Suppose P and Q are two Hamiltonian circuits with n vertices as shown below.



Both P and Q have n vertices, n edges, and vertices of degree two. Because P and Q agree with respect to these invariants, we define a function f to investigate the one-to-one function. Since all vertices in both P and Q have degree two, then we have $f(u_n) = v_3$, $f(u_1) = v_2$, $f(u_2) = v_1$, $f(u_3) = v_n$, $f(u_4) = v_{n-1}$, ..., $f(u_{n-1}) = v_4$. To examine whether f preserves edges, we examine the adjacency matrices of P and Q as well as their transpose, with the rows and columns labeled by the images of their corresponding vertices.

$$\mathbf{A}_P = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-1} \\ u_n \end{array} \begin{array}{c} \begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & \cdots & u_{n-1} & u_n \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{A}_P{}^T = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-1} \\ u_n \end{array} \begin{array}{c} \begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & \cdots & u_{n-1} & u_n \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & & 1 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{A}_Q = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-1} \\ v_n \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & \cdots & v_{n-1} & v_n \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & & 1 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{A}_Q{}^T = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-1} \\ v_n \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & \cdots & v_{n-1} & v_n \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 0 \end{bmatrix} \end{array}$$

From the above matrices, we have $\mathbf{A}_P = \mathbf{A}_Q{}^T$ and $\mathbf{A}_Q = \mathbf{A}_P{}^T$ which shows that f preserves the edges. Thus, we conclude that P and Q are isomorphic. □

4.4 Conclusion

This chapter discussed the general method to decompose K_n into distinct HC with different paths. In summary, we have achieved the first and second objectives. In the next chapter, we will continue our discussion on the application of the HBM in order to accomplish the third objective.

CHAPTER FIVE
AN APPLICATION OF THE HALF BUTTERFLY METHOD

To date, a new approach to list all permutations of n elements has been explored where several starter sets are needed (Ibrahim, Omar & Rohni, 2010). In this chapter, we will find an alternative approach by proving that our results of distinct HC from K_n are enough to list all permutations of n elements.

Section 5.1 briefly explains the concept of listing permutation. Section 5.2 presents the permutation of three elements. Section 5.3 discusses the permutation of four elements. Section 5.4 provides the permutation of five elements. Finally, section 5.5 explains the construction of permutations of n elements before a new theorem derived from this application is disclosed.

5.1 The Permutation of n Elements

A permutation of a set of numbers is an ordering of all its elements. For example, a set of n elements is given as $\{1,2,3,4,\ldots,n\}$. Then, this set of n elements has $n!$ permutations. The following sections discuss how we can obtain all $n!$ permutations by using the HBM.

We provided examples of three, four and five elements before presenting the generalization of n elements.

5.2 The Permutation of Three Elements using the HBM

Let $\{1,2,3\}$ be a set of three elements. The total permutations of these three elements is six, that is, $3! = 6$.

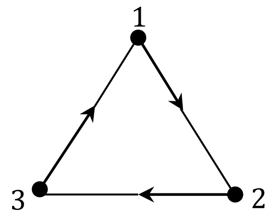
In Section 3.2.1, we have discussed that K_3 is decomposable into a distinct HC with different paths. Now, we will present how the circuit is sufficient in obtaining all the permutations of three elements.

From Figure 3.3, circuit A has direction $(1,2,3,1)$. However, there is no repeatition of numbers in listing permutation. Therefore, from circuit A, we consider the path $(1,2,3)$ as the permutation set $\{1,2,3\}$. This process is applied to all circuits in obtaining the permutations of n elements.

In listing $n!$ permutations, the distinct circuits and its mirror image are considered. Figure 3.3 shows circuits A, B and C that give the permutations $\{1,2,3\}$, $\{2,3,1\}$ and $\{3,1,2\}$ respectively. The mirror image of A, B and C, which are circuits D, E and F, give the permutations $\{1,3,2\}$, $\{2,1,3\}$ and $\{3,2,1\}$ respectively. These six circuits are the same circuits since traverse the same path. Therefore, a distinct HC as shown in Figure 3.4 is sufficient to generate all the permutations of three elements. The results are presented in Table 5.1.

Table 5.1

The permutations of three elements

No.	The distinct HC from K_3	Permutations obtained from the circuits with similar paths	Permutations obtained from the mirror images of the circuits with similar paths
1.	 (1,2,3,1)	{1,2,3} {2,3,1} {3,1,2}	{3,2,1} {2,1,3} {1,3,2}

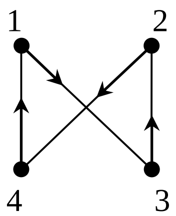
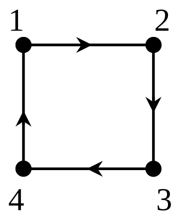
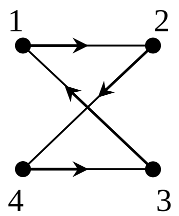
5.3 The Permutation of Four Elements using the HBM

Let {1,2,3,4} be a set of four elements. The total permutations of this four elements are 24, that is, $4! = 24$. Now, we will prove that the three distinct HC in Figure 3.9 are enough as a basis to list all permutations of four elements.

We denote the circuit $A \equiv D$ as $C_1{}^*$, the circuit $B \equiv E$ as $C_2{}^*$, and circuit $C \equiv F$ as $C_3{}^*$. The path in $C_1{}^*$ gives the permutation {1,3,2,4}. By changing the starting point of $C_1{}^*$, the permutations {3,2,4,1}, {2,4,1,3} and {4,1,3,2} are obtained. The mirror image of $C_1{}^*$ gives the permutation {4,2,3,1}. Then, the permutations {2,3,1,4}, {3,1,4,2} and {1,4,2,3} are obtained by changing the starting point of the mirror image of $C_1{}^*$. Thus, it is proven that the three distinct HC with different paths from K_4 are enough to list all permutations of four elements. The results are presented in Table 5.2.

Table 5.2

The permutations of four elements.

No.	The distinct HC from K_4	Permutations obtained from the circuits with different paths	Permutations obtained from the mirror images of the circuits with different paths
1.	 (1,3,2,4,1)	{1,3,2,4} {3,2,4,1} {2,4,1,3} {4,1,3,2}	{4,2,3,1} {1,4,2,3} {3,1,4,2} {2,3,1,4}
2.	 (1,2,3,4,1)	{1,2,3,4} {2,3,4,1} {3,4,1,2} {4,1,2,3}	{4,3,2,1} {1,4,3,2} {2,1,4,3} {3,2,1,4}
3.	 (1,2,4,3,1)	{1,2,4,3} {2,4,3,1} {4,3,1,2} {3,1,2,4}	{3,4,2,1} {1,3,4,2} {2,1,3,4} {4,2,1,3}

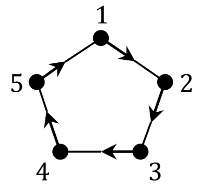
5.4 The Permutation of Five Elements using the HBM

Let $\{1,2,3,4,5\}$ be a set of five elements. The total permutations of these five elements is 120, that is, $5! = 120$. Similar to the previous cases, the twelve circuits in Appendix D can be used to list all permutations of five elements.

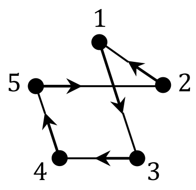
For example, ${C_1}^*$ in Appendix D gives permutation $\{1,2,3,4,5\}$. By changing the starting point of ${C_1}^*$, we have the circuits that traverse the same path as ${C_1}^*$. Then, the permutations $\{2,3,4,5,1\}$, $\{3,4,5,1,2\}$, $\{4,5,1,2,3\}$ and $\{5,1,2,3,4\}$ can be obtained. The mirror images of these circuits with similar paths gives the permutations $\{5,4,3,2,1\}$, $\{1,5,4,3,2\}$, $\{2,1,5,4,3\}$, $\{3,2,1,5,4\}$ and $\{4,3,2,1,5\}$. Table 5.3 provides the complete results for the permutations of five elements that are obtained from the distinct HC with different paths in K_5.

Table 5.3

The permutations of five elements.

No.	**The distinct HC from K_5**	**Permutations from the circuits with different paths**	**Permutations from the mirror images of the circuits with different paths**
1.	 $(1,2,3,4,5,1)$	$\{1,2,3,4,5\}$ $\{2,3,4,5,1\}$ $\{3,4,5,1,2\}$ $\{4,5,1,2,3\}$ $\{5,1,2,3,4\}$	$\{5,4,3,2,1\}$ $\{1,5,4,3,2\}$ $\{2,1,5,4,3\}$ $\{3,2,1,5,4\}$ $\{4,3,2,1,5\}$

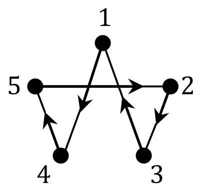
2.



(1,3,4,5,2,1)

{1,3,4,5,2}	{2,5,4,3,1}
{3,4,5,2,1}	{1,2,5,4,3}
{4,5,2,1,3}	{3,1,2,5,4}
{5,2,1,3,4}	{4,3,1,2,5}
{2,1,3,4,5}	{5,4,3,1,2}

3.



(1,4,5,2,3,1)

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{4,5,2,3,1}	{2,5,4,1,3}
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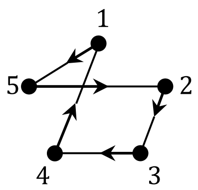
{3,1,4,5,2}	{1,3,2,5,4}

4.



(1,5,2,3,4,1)

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{2,3,4,1,5}	{5,1,4,3,2}
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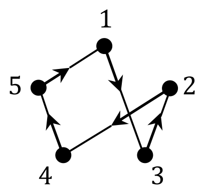
{4,1,5,2,3}	{3,2,5,1,4}

5.



(1,3,2,4,5,1)

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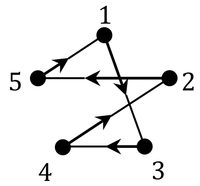
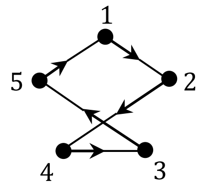
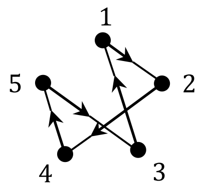
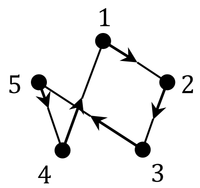
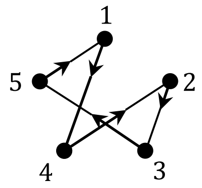
{3,2,4,5,1}	{4,2,3,1,5}
{2,4,5,1,3}	{2,3,1,5,4}
{4,5,1,3,2}	{3,1,5,4,2}
{5,1,3,2,4}	{1,5,4,2,3}

6.	 (1,3,4,2,5,1)	{1,3,4,2,5} {3,4,2,5,1} {4,2,5,1,3} {2,5,1,3,4} {5,1,3,4,2}	{5,2,4,3,1} {2,4,3,1,5} {4,3,1,5,2} {3,1,5,2,4} {1,5,2,4,3}
7.	 (1,2,4,3,5,1)	{1,2,4,3,5} {2,4,3,5,1} {4,3,5,1,2} {3,5,1,2,4} {5,1,2,4,3}	{5,3,4,2,1} {3,4,2,1,5} {4,2,1,5,3} {2,1,5,3,4} {1,5,3,4,2}
8.	 (1,2,4,5,3,1)	{1,2,4,5,3} {2,4,5,3,1} {4,5,3,1,2} {5,3,1,2,4} {3,1,2,4,5}	{3,5,4,2,1} {5,4,2,1,3} {4,2,1,3,5} {2,1,3,5,4} {1,3,5,4,2}
9.	 (1,2,3,5,4,1)	{1,2,3,5,4} {2,3,5,4,1} {3,5,4,1,2} {5,4,1,2,3} {4,1,2,3,5}	{4,5,3,2,1} {5,3,2,1,4} {3,2,1,4,5} {2,1,4,5,3} {1,4,5,3,2}

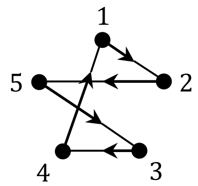
10.



(1,4,2,3,5,1)

{1,4,2,3,5}	{5,3,2,4,1}
{4,2,3,5,1}	{3,2,4,1,5}
{2,3,5,1,4}	{2,4,1,5,3}
{3,5,1,4,2}	{4,1,5,3,2}
{5,1,4,2,3}	{1,5,3,2,4}

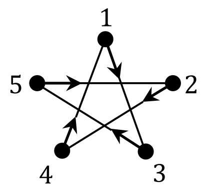
11.



(1,2,5,3,4,1)

{1,2,5,3,4}	{4,3,5,2,1}
{2,5,3,4,1}	{3,5,2,1,4}
{5,3,4,1,2}	{5,2,1,4,3}
{3,4,1,2,5}	{2,1,4,3,5}
{4,1,2,5,3}	{1,4,3,5,2}

12.



(1,3,5,2,4,1)

{1,3,5,2,4}	{4,2,5,3,1}
{3,5,2,4,1}	{2,5,3,1,4}
{5,2,4,1,3}	{5,3,1,4,2}
{2,4,1,3,5}	{3,1,4,2,5}
{4,1,3,5,2}	{1,4,2,5,3}

5.5 The Permutation of n Elements using the HBM

Let $S = \{x_1, x_2, x_3, x_4, \dots, x_{n-1}, x_n\}$ be the set of n elements. To develop the permutations of n elements, consider the circuit below.

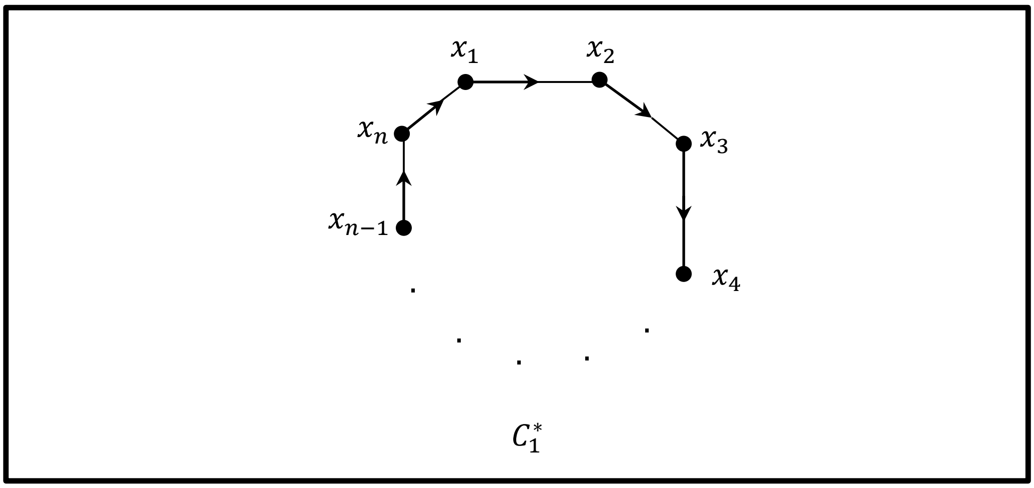


Figure 5.1. Hamiltonian circuit with n vertices.

The above circuit produces permutation

$$\{x_1, x_2, x_3, x_4, \dots, x_{n-1}, x_n\}.$$

When changing the starting point of C_1^*, the circuits with similar paths are obtained, which then lead to the development of permutations

$$\{x_2, x_3, x_4, \dots, x_{n-1}, x_n, x_1\},$$

$$\{x_3, x_4, \dots, x_{n-1}, x_n, x_1, x_2\},$$

$$\{x_4, \dots, x_{n-1}, x_n, x_1, x_2, x_3\}, \dots,$$

$$\{x_n, x_1, x_2, x_3, \dots, x_{n-1}\}.$$

The mirror image of C_1^* as well as the mirror images of the circuits that traverse the same path as C_1^* produce the permutations

$$\{x_n, x_{n-1}, \dots, x_4, x_3, x_2, x_1\},$$
$$\{x_1, x_n, x_{n-1}, \dots, x_4, x_3, x_2\},$$
$$\{x_2, x_1, x_n, x_{n-1}, \dots, x_4, x_3\},$$
$$\{x_3, x_2, x_1, x_n, x_{n-1}, \dots, x_4\}, \dots,$$
$$\{x_{n-1}, \dots, x_3, x_2, x_1, x_n\}.$$

The remaining permutations can be obtained from each distinct circuit of K_n by following the same procedure as discussed before.

In summary, there are n permutations from each circuit and n permutations from its mirror images that can be produced. This leads us to the following results.

Theorem 5.1 The $\frac{(n-1)!}{2}$ distinct Hamiltonian circuits with different paths in K_n will produce a total of $2n$ permutations which include the n permutations from the circuit and the other n permutations from their mirror images.

Proof. Let G be a K_n and K_n is decomposable into $\frac{(n-1)!}{2}$ distinct HC with different paths (Theorem 4.7). Since K_n possesses $n!$ permutations, then for every K_n, each of $\frac{(n-1)!}{2}$ decomposed circuit will have

$$\frac{n!}{\frac{(n-1)!}{2}} = n! \times \frac{2}{(n-1)!}$$

$$= n(n-1)! \times \frac{2}{(n-1)!}$$

$$= 2n$$

permutations. □

Theorem 5.2 There are $n!$ permutations from $\frac{(n-1)!}{2}$ distinct Hamiltonian circuits with different paths in K_n.

Proof. Let γ indicates the total permutations of n elements. For every K_n, there are $\frac{(n-1)!}{2}$ distinct Hamiltonian circuits with different paths (Theorem 4.8). These $\frac{(n-1)!}{2}$ distinct circuits are exclusive of the circuits with similar paths and mirror images. The circuits with similar paths are obtained by changing the starting point, x_1. Thus, there are n circuits with similar path for every $\frac{(n-1)!}{2}$ distinct circuits.

Then, by reincluding the circuits with similar paths, we have

$$\gamma = \frac{(n-1)!}{2} \times n$$

$$= \frac{n(n-1)!}{2} .$$

Considering the mirror image of each circuit obtained above, we have

$$\gamma = \frac{n(n-1)!}{2} \times 2$$

$$= n(n-1)!$$

$$= n!\,. \qquad \square$$

Therefore, it is proven that the decomposed $\frac{(n-1)!}{2}$ distinct Hamiltonian circuits with different paths in K_n are enough to obtain all the permutations of n elements. As a summary, generally, geometric representation is capable in listing permutations of n elements. To be specific, it is proven that the HBM can be used in listing all permutations of n elements.

CHAPTER SIX
CONCLUSION

The study presented in this thesis explored how to solve the geometric representation of distinct HC with different path in complete graph K_n. Section 6.1 is the summary of each chapter. The contribution of this study is presented in Section 6.2. Then, Section 6.3 points out some possibilities for future researches related to this study. Finally, Section 6.4 provides some discussions and potential applications of the HBM to some real world problems.

6.1 Summary of Each Chapter

Chapter One is started with a brief description of HC. The background of the graphs is also presented such as several types of graphs as well as their definitions. Related historical surveys are also discussed there. This chapter closed with research questions and the objectives.

Chapter Two began with a summary of the BS introduced by Gopal *et al.* (2007) as an initial relation to our novel method. Then, some definitions, existing corollary and related theorems that are associated with our study are also discussed.

Chapter Three commenced with several definitions and terminologies that are used in the method development, followed by several examples for better understanding. The method is explained using numerical examples for K_3, K_4, K_5 and K_6 decomposition where the steps involved are clearly explained. The results of each example are pointed out as remarks.

Chapter Four is the real journey of our study where the novel method named the HBM is presented. We began with an explanation on how our method was produced. Then, we showed the differences between the existing BS and our new HBM. This chapter is divided into four folds. The WS is introduced in Section 4.1 as a procedure to find the directions. Section 4.2 presented the generalization of our method while Section 4.3 discussed the theoretical results obtained from the HBM. Then, we conclude our method in Section 4.4.

We presented the application of our new HBM in Chapter Five. The basic knowledge of permutation is briefly discussed. Next, we gave the examples for permutations of three, four, and five elements, before presenting the listing permutation of n elements. Each geometric representation of the HC is presented together with the permutation lists as shown in Tables 5.1, 5.2 and 5.3. This chapter is ended with Theorems 5.1 and 5.2, both derived from the application of the HBM.

The details of our results are explored in the following section.

6.2 Contribution of the Study

Motivated mainly by graph decomposition, this study focuses in constructing a novel method to decompose complete graph K_n into distinct Hamiltonian circuits with different paths. The contributions of this study are discussed below.

- **Constructing a novel HBM**

This study constructed a new method in finding the decomposition of distinct HC with different path from K_n. The HBM is presented in Chapter Four (refer to Section 4.2). In the HBM, we introduced the WS, used to generate direction, which is then used to produce the distinct HC (refer to Section 4.1). Using the HBM, we are able to produce the result of the HC in geometric representation for visualization purpose. In addition, the visualization of the HC can reveal a deeper layer of the structure.

- **Producing new theoretical concepts from the HBM**

Based on the proposed HBM as presented in Chapter Four, two lemmas, three propositions and four theorems are produced. The development of directions in the HBM is divided into several blocks, which bring us to Lemma 4.1. In each block, there is a total number of direction produced for every K_n which brings us to Lemma 4.2. The total directions in all blocks are discussed in Proposition 4.3. Theorem 4.6 proved that the result of our method is equivalent to the existing ones (Theorem 2.3) by considering the total number of HC produced from each direction (Proposition 4.4), and the total number of HC obtained from all directions (Proposition 4.5).

- **Listing the permutations of n elements using the distinct Hamiltonian circuits with different paths in K_n**

We proved that the $\frac{(n-1)!}{2}$ distinct HC with different paths are enough to list all the n! permutations of n elements (Theorem 5.2). We explained the permutation obtained from each distinct HC by providing the permutations of three, four, and five elements, before generalizing it for n elements (refer to Tables 5.1, 5.2 and 5.3).

6.3 Suggestion for Future Research

We believe that the research presented in this thesis opens up a lot of interesting directions to pursue. We may end with a nice problem that is enlighten by this study, which consists in proving the distinct HC with different structures. Table 6.1 shows the results for K_3, K_4, K_5 and K_6.

Table 6.1

The number of different structure of distinct Hamiltonian circuits from K_n.

Complete Graph	**Number of circuits with different structure**
K_3	1
K_4	2
K_5	4
K_6	14

The above results are obtained by eliminating the circuits with similar structure for each complete graph. For example, by ignoring the direction of each circuit of K_4 in Figure 3.9, there are two different structures. From this result, we conclude that K_4 has two distinct HC with different structure.

Next, Appendix D shows the distinct HC which can be decomposed from K_5. The direction of each circuit is ignored to classify the different structure. Thus, K_5 possesses four distinct HC with different structures. Similarly, K_6 has fourteen distinct HC with different structures. Appendix G gives the result when the direction of each circuit is not considered. The details will be discussed in the following paragraph.

At this stage, the total number of distinct HC with different structures from K_n has the sequence of

$$1,2,4,14, \dots . \tag{6.1}$$

We might consider the problem of trigonometry to the $\frac{(n-1)!}{2}$ distinct HC from K_n in order to get the general results of (6.1).

For instance, when the direction of each circuit in Figure 3.9 is ignored, K_4 has two different structures as presented in Figure 6.1.

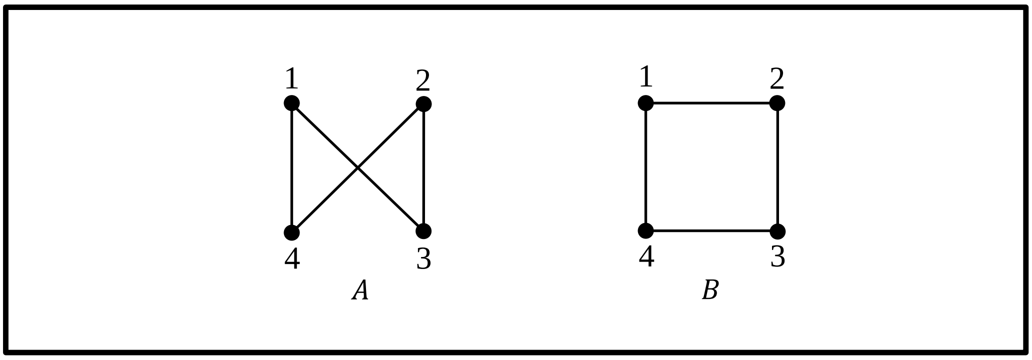


Figure 6.1. Two different structures of distinct Hamiltonian circuits with different paths from K_4

Next, when the direction of each circuit of order five in Appendix D is ignored, the circuit has the following result as presented in Figure 6.2. Table 6.2 analyzed the details of the construction of distinct structures.

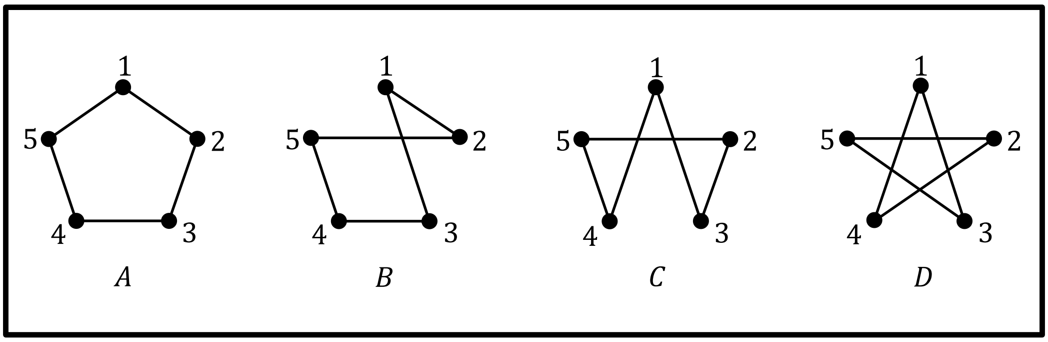


Figure 6.2. Four different structures of distinct Hamiltonian circuits with different paths from K_5

Table 6.2

Analysis of distinct Hamiltonian circuits with different structure for K_5.

The distinct structure	**Analysis**
A	Structure of $C_1{}^*$.
B	Structure of $C_2{}^*$, $C_4{}^*$, $C_5{}^*$, $C_7{}^*$ and $C_9{}^*$.
C	Structure of $C_3{}^*$, $C_6{}^*$, $C_8{}^*$, $C_{10}{}^*$ and $C_{11}{}^*$.
D	Structure of $C_{12}{}^*$.

Since there are five vertices in K_5, the degree between one vertex to another vertex is $\frac{360°}{5} = 72°$. Thus, the circuits that formed structure B has a rotation of $72°$ between one circuit to another circuit. The same situation occurs to the circuits that produced structure C. Now, we consider the circuits of order six in Appendix G. The results of distinct structures for the circuits are presented in Figure 6.3 and Table 6.3.

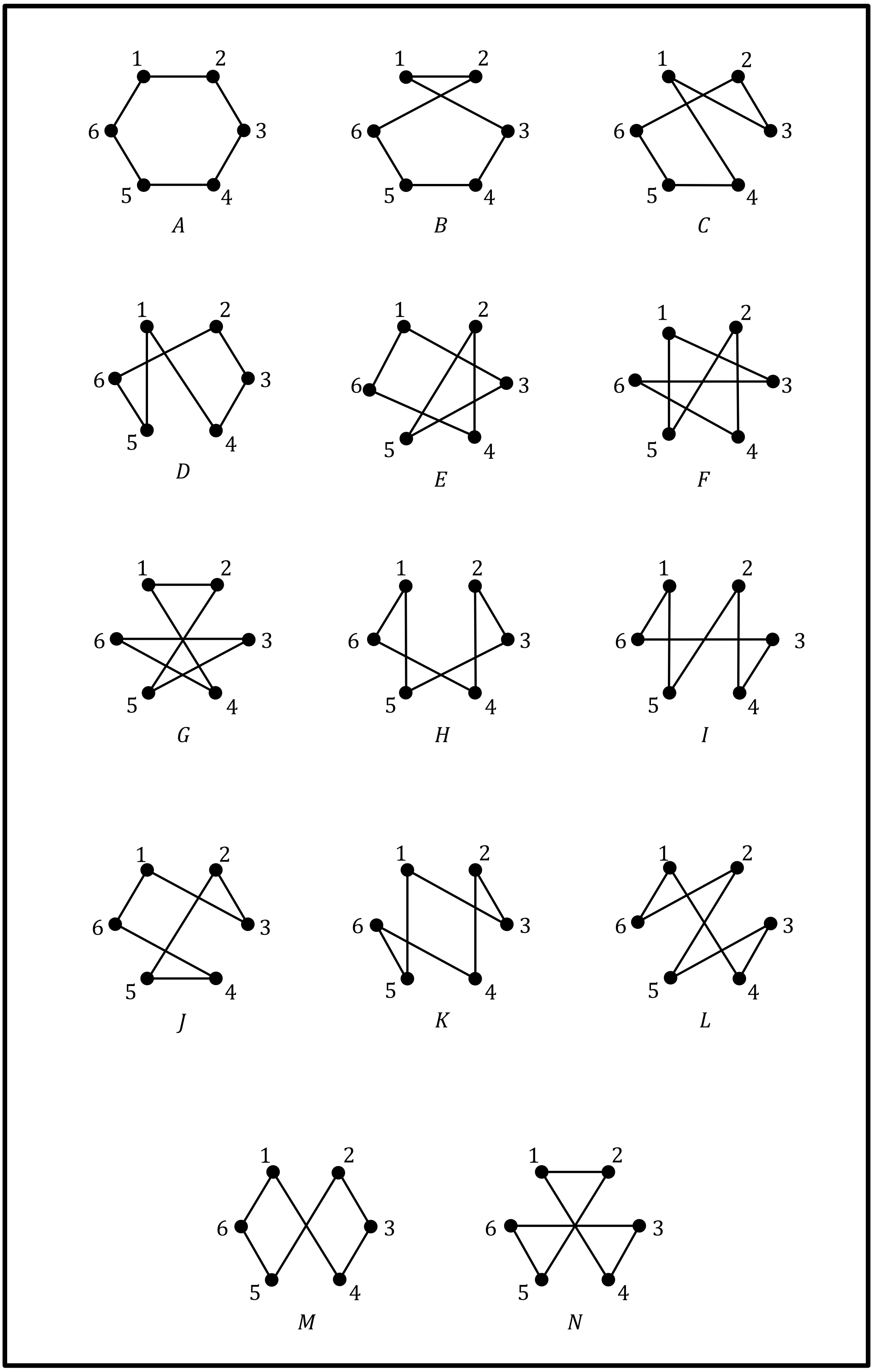


Figure 6.3. Fourteen different structures of distinct Hamiltonian circuits with different paths from K_6.

Table 6.3

Analysis of distinct Hamiltonian circuits with different structures for K_6.

The distinct structure	Analysis
A	Structure of $C_1{}^*$.
B	Structure of $C_2{}^*$, $C_5{}^*$, $C_6{}^*$, $C_9{}^*$, $C_{12}{}^*$ and $C_{15}{}^*$.
C	Structure of $C_3{}^*$, $C_7{}^*$, $C_{10}{}^*$, $C_{13}{}^*$, $C_{16}{}^*$ and $C_{18}{}^*$.
D	Structure of $C_4{}^*$, $C_8{}^*$, $C_{11}{}^*$, $C_{14}{}^*$, $C_{17}{}^*$ and $C_{19}{}^*$.
E	Structure of $C_{21}{}^*$, $C_{30}{}^*$, $C_{49}{}^*$, $C_{50}{}^*$, $C_{54}{}^*$ and $C_{57}{}^*$.
F	Structure of $C_{31}{}^*$, $C_{44}{}^*$ and $C_{53}{}^*$.
G	Structure of $C_{35}{}^*$, $C_{36}{}^*$, $C_{38}{}^*$, $C_{39}{}^*$, $C_{51}{}^*$ and $C_{60}{}^*$.
H	Structure of $C_{22}{}^*$, $C_{29}{}^*$, $C_{32}{}^*$, $C_{42}{}^*$, $C_{52}{}^*$ and $C_{55}{}^*$.
I	Structure of $C_{27}{}^*$, $C_{43}{}^*$ and $C_{47}{}^*$.
J	Structure of $C_{20}{}^*$, $C_{26}{}^*$, $C_{34}{}^*$, $C_{40}{}^*$, $C_{58}{}^*$ and $C_{59}{}^*$.
K	Structure of $C_{23}{}^*$, $C_{41}{}^*$ and $C_{48}{}^*$.
L	Structure of $C_{25}{}^*$, $C_{33}{}^*$ and $C_{46}{}^*$.
M	Structure of $C_{24}{}^*$, $C_{28}{}^*$ and $C_{45}{}^*$.
N	Structure of $C_{37}{}^*$ and $C_{56}{}^*$.

There are six vertices in K_6. The degree between one vertex to another vertex is $\frac{360°}{6} = 60°$. Thus, every six circuits with similar paths that produced structures B, C, D, E, G, H and J each has a rotation of $60°$. Next, every three circuits with similar paths that produced structures F, I, K, L and M each has a rotation of $\frac{360°}{3} = 120°$. However, two circuits with similar paths which produced structure N is the complement to each other.

At this stage, we have the following conjecture.

Conjecture 6.1 There exist a way to identify the distinct Hamiltonian circuits with different structures from K_n for all $n \geq 3$.

As previously discussed, K_3 has a distinct circuit with different structures, K_4 has two, K_5 has four, and K_6 has fourteen. These different structures are obtained by finding the distinct structures of the output, manually. We believe that this sequence might generate an interesting idea to generalize the total number of circuits with different structures from K_n for all n. In addition, we may use the idea of arithmetic sequences and series for proving purposes.

In addition, when dealing with geometric representation of graphs, the major problems that need to be addressed are the size of the graphs and time complexity (Herman *et al.*, 2000). Thus, any visualization system needs to provide near real-time interaction, where updates must be done in very short time intervals. Thus, this study can be extended to the computer source code implementation.

6.4 Discussion

The novel method in this study hopefully can be used in the field of network security and wireless local area networks where the access points are interfering with some other access points in the same region (Shirinivas, Vetrivel & Elango, 2010). Other than that, the result of this study might be able to improve the design of edge-clustering framework for general graphs (Cui *et al.*, 2008). On the contrary, the idea of this study, however, may renewed the structural fingerprint classification to get more general results than the existing one (Marcialis *et al.*, 2007). These improvements are likely, in turn, to be applicable in investigating other states of graph decomposition.

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