UNRESTRICTED SOLUTIONS OF ARBITRARY LINEAR FUZZY SYSTEMS

GHASSAN O. MALKAWI

DOCTOR OF PHILOSOPHY
UNIVERSITI UTARA MALAYSIA
2015

UNRESTRICTED SOLUTIONS OF ARBITRARY LINEAR FUZZY SYSTEMS

Awang Had Salleh
Graduate School
of Arts And Sciences

Universiti Utara Malaysia

PERAKUAN KERJA TESIS / DISERTASI
(Certification of thesis / dissertation)

Kami, yang bertandatangan, memperakukan bahawa
(We, the undersigned, certify that)

GHASSAN OMAR MAHMOUD MALKAWI

calon untuk Ijazah PhD
(candidate for the degree of)

telah mengemukakan tesis / disertasi yang bertajuk:
(has presented his/her thesis / dissertation of the following title):

"UNRESTRICTED SOLUTIONS OF ARBITRARY LINEAR FUZZY SYSTEMS"

seperti yang tercatat di muka surat tajuk dan kulit tesis / disertasi.
(as it appears on the title page and front cover of the thesis / dissertation).

Bahawa tesis/disertasi tersebut boleh diterima dari segi bentuk serta kandungan dan meliputi bidang ilmu dengan memuaskan, sebagaimana yang ditunjukkan oleh calon dalam ujian lisan yang diadakan pada : *22 Disember 2015.*
That the said thesis/dissertation is acceptable in form and content and displays a satisfactory knowledge of the field of study as demonstrated by the candidate through an oral examination held on: December 22, 2015.

Pengerusi Viva: *(Chairman for VIVA)*	Assoc. Prof. Dr. Wan Rozaini Sheik Osman	Tandatangan *(Signature)*
Pemeriksa Luar: *(External Examiner)*	Prof. Dr. Mohd Lazim Abdullah	Tandatangan *(Signature)*
Pemeriksa Luar: *(External Examiner)*	Prof. Dr. Mohd Salmi Md Noorani	Tandatangan *(Signature)*
Pemeriksa Dalam: *(Internal Examiner)*	Dr. Sharmila Karim	Tandatangan *(Signature)*
Nama Penyelia/Penyelia-penyelia: *(Name of Supervisor/Supervisors)*	Dr. Nazihah Ahmad	Tandatangan *(Signature)*
Nama Penyelia/Penyelia-penyelia: *(Name of Supervisor/Supervisors)*	Assoc. Prof. Dr. Haslinda Ibrahim	Tandatangan *(Signature)*

Tarikh:
(Date) *December 22, 2015*

Permission to Use

In presenting this thesis in fulfilment of the requirements for a postgraduate degree from Universiti Utara Malaysia, I agree that the Universiti Library may make it freely available for inspection. I further agree that permission for the copying of this thesis in any manner, in whole or in part, for scholarly purpose may be granted by my supervisor(s) or, in their absence, by the Dean of Awang Had Salleh Graduate School of Arts and Sciences. It is understood that any copying or publication or use of this thesis or parts thereof for financial gain shall not be allowed without my written permission. It is also understood that due recognition shall be given to me and to Universiti Utara Malaysia for any scholarly use which may be made of any material from my thesis.

Requests for permission to copy or to make other use of materials in this thesis, in whole or in part, should be addressed to :

Dean of Awang Had Salleh Graduate School of Arts and Sciences
UUM College of Arts and Sciences
Universiti Utara Malaysia
06010 UUM Sintok

Abstrak

Penyelesaian sistem kabur linear telah menarik perhatian ramai penyelidik disebabkan keupayaannya dalam menangani ketidaktepatan maklumat bagi masalah sebenar. Walau bagaimanapun, terdapat beberapa kelemahan dalam kaedah sedia ada. Antaranya kebersandaran yang tinggi kepada pengaturcaraan linear, pengelakan penggunaan hampir sifar bagi nombor kabur, penyelesaian tidak jitu, tumpuan pada saiz sistem yang terhad, dan pembatasan pada pekali matriks serta penyelesaian. Oleh itu, kajian ini bertujuan untuk membina kaedah baharu iaitu sistem linear bersekutu, sistem min-maks dan sistem mutlak dalam teori matriks dengan nombor kabur segi tiga dalam menyelesaikan sistem kabur linear berasaskan kelemahan yang dinyatakan. Sistem linear bersekutu yang dibina terbukti setara dengan sistem kabur linear tanpa melibatkan sebarang operasi kabur. Seterusnya, sistem linear bersekutu yang dibangunkan adalah efektif dalam memberi penyelesaian tepat berbanding dengan pengaturcaraan linear, yang tertakluk kepada sebilangan kekangan. Kaedah ini juga berupaya menyediakan penyelesaian yang jitu bagi sistem yang besar. Selanjutnya, sistem linear bersekutu ini boleh menyemak kewujudan penyelesaian kabur dan pengkelasan penyelesaian yang mungkin. Bagi kes sistem linear kabur penuh hampir sifar, operasi kabur diperlukan untuk menentukan sifat penyelesian bagi sistem kabur dan untuk memastikan kekaburan penyelesaian. Penyelesaian terhingga yang merupakan konsep baru bagi ketekalan sistem linear diperoleh dengan sistem min-maks terbina dan sistem mutlak terbina. Kaedah-kaedah terbina boleh juga diubah suai bagi menyelesaikan sistem kabur lanjutan seperti persamaan matriks kabur penuh dan persamaan Sylvester kabur penuh, dan kaedah ini juga boleh dimanfaatkan untuk jenis nombor kabur yang lain seperti nombor kabur trapezoidal. Kajian ini menyumbang kepada kaedah menyelesaikan sebarang sistem kabur linear tanpa batasan pada sistem tersebut.

Kata kunci: Sebarang sistem kabur linear penuh, Sistem kabur bersekutu, Teori matriks, Nombor kabur segitiga.

Abstract

Solving linear fuzzy system has intrigued many researchers due to its ability to handle imprecise information of real problems. However, there are several weaknesses of the existing methods. Among the drawbacks are heavy dependence on linear programing, avoidance of near zero fuzzy numbers, lack of accurate solutions, focus on limited size of the systems, and restriction to the matrix coefficients and solutions. Therefore, this study aims to construct new methods which are associated linear systems, min-max system and absolute systems in matrix theory with triangular fuzzy numbers to solve linear fuzzy systems with respect to the aforementioned drawbacks. It is proven that the new constructed associated linear systems are equivalent to linear fuzzy systems without involving any fuzzy operation.

Furthermore, the new constructed associated linear systems are effective in providing exact solution as compared to linear programing, which is subjected to a number of constraints. These methods are also able to provide accurate solutions for large systems. Moreover, the existence of fuzzy solutions and classification of possible solutions are being checked by these associated linear systems. In case of near zero fully fuzzy linear system, fuzzy operations are required to determine the nature of solution of fuzzy system and to ensure the fuzziness of the solution. Finite solutions which are new concept of consistency in linear systems are obtained by the constructed min-max and absolute systems. These developed methods can also be modified to solve advanced fuzzy systems such as fully fuzzy matrix equation and fully fuzzy Sylvester equation, and can be employed for other types of fuzzy numbers such as trapezoidal fuzzy number. The study contributes to the methods to solve arbitrary linear fuzzy systems without any restriction on the system.

Keywords: Arbitrary fully fuzzy linear system, Associated linear system, Matrix theory, Triangular fuzzy number.

Acknowledgement

I am grateful to the Almighty Allah for giving me the opportunity to complete my PhD thesis. May peace and blessing of Allah be upon His beloved Prophet Muhammad (SAW), his family, and his companions.

In completing this thesis, I owe a debt of gratitude and thanks to many persons and institutions that have supported me throughout this difficult and challenging journey. While being thankful to all of them, I must register my gratitude to some in particular. First, I would like to express my deepest appreciation to my principal supervisors, Dr. Nazihah Binti Ahmad and Associate Professor Dr. Haslinda Ibrahim, who has been very patient in guiding and supporting me from the very beginning of my first arrival here in Malaysia and throughout the completion of this thesis. They assisted me immensely in developing a correct focus for my study and has given me their valuable ideas, insights, comments and suggestions towards understanding the empirical predicaments I was encountering. Honestly, I considered both of them not only as my supervisors, but also my friends and my sisters in Malaysia. I would like to also thank my dear friends in Malaysia for the friendship rendered and assistance provided during my stay here in Malaysia and UUM in particular. Also, I would like to thank all academic and administrative staff in the College of Arts of Sciences.

I would like to express my ending appreciation and gratitude to people in Jordan. First and foremost, I would like to thank my father who has been a great and wise teacher in my life and my lovely mother for her infinite patience especially during my absence, and her sincere flow of love has accompanied me all the way in my long struggle and has pushed me to pursue my dreams. Last but not least, to my family, friends, teachers, brothers and sisters, I thank you so much for continuously giving me the undivided support and eternal prayers. To all of you, I have this to say: I love you, respect you, pray for you, and May Allah bless you.

Declaration Associated with This Thesis

1- Malkawi, G., Ahmad, N., & Ibrahim, H. (2013). Revisiting Fuzzy Approach for Solving System Of Linear Equations, [CD -Rom], *ICDeM* 2012, 13 - 16 March, Kedah, Malaysia.

2- Malkawi, G., Ahmad, N., & Ibrahim, H. (2014). A Note On The Nearest Symmetric Fuzzy Solution For A Symmetric Fuzzy Linear System. *An. St. Univ. Ovidius Constanta*.

3- Malkawi, G., Ahmad, N., & Ibrahim, H. (2014). Solving Fully Fuzzy Linear System With The Necessary And Sufficient Condition To Have A Positive Solution. *Appl. Math*, *8*(3), 1003-1019.

4- Malkawi, G., Ahmad, N., & Ibrahim, H. (2014). On The Weakness Of Linear Programming To Interpret The Nature Of Solution Of Fully Fuzzy Linear System.*Journal of Uncertainty Analysis and Applications*, *2*(1), 1-23.

5- Malkawi, G., Ahmad, N., Ibrahim, H., & Alshmari, B. (2014). Row Reduced Echelon Form For Solving Fully Fuzzy System With Unknown Coefficients. *Journal of Fuzzy Set Valued Analysis*, *2014*, 1-18.

6- Malkawi, G., Ahmad, N., & Ibrahim, H. (2014, December). Finite Solutions Of Fully Fuzzy Linear System. In *International Conference On Quantitative Sciences And Its Applications (ICOQSIA 2014): Proceedings of the 3rd International Conference on Quantitative Sciences and Its Applications* (Vol. 1635, pp. 447-454). AIP Publishing.

7- Malkawi, G., Ahmad, N., Ibrahim, H., & Albayari, D. J. (2015). A Note On Solving Fully Fuzzy Linear Systems By Using Implicit Gauss---Cholesky Algorithm.*Computational Mathematics and Modeling*, *26*(4), 585-592.

8- Malkawi, G., Ahmad, N., & Ibrahim, H. (2015). An Algorithm For A Positive Solution Of Arbitrary Fully Fuzzy Linear System. *Computational Mathematics and Modeling*, , *26*(3), 436-465.

9- Malkawi, G., Ahmad, N., & Ibrahim, H. (2015). Solving The Fully Fuzzy Sylvester Matrix Equation With Triangular Fuzzy Number. Far East Journal of Mathematical Sciences, 89(1), 73-55..

10- Ahmad, N., Malkawi, G., & Ibrahim, H. (2015, August). Solution Of LR-Fuzzy Linear System With Trapezoidal Fuzzy Number Using Matrix Theory. In Malaysian Technical Universities Conference on Engineering and Technology (Acepted).

Table of Contents

List of Tables

List of Figures

List of Appendices

CHAPTER ONE
INTRODUCTION

1.1 Linear system

In applied mathematics, some fields consist problems of several parts which interact and affect each other, such as economics, finance, engineering and physics. These parts can be represented as a set of linear equations. A system of linear equations or linear system is a collection of linear equations involving the same set of the equation that deals with all variables at once. A linear system of equations is considered to be the simplest and the most helpful method to solve these equations. A general system of linear equations can be written as follows.

Definition 1.1.1. (Hogben, 2006) A linear system is a set of one or more linear equations with the same set of variables. A general system of m linear equations with n unknowns can be written as,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots \ldots . + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots \ldots . + a_{2n}x_n = b_2, \\ \vdots \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots \ldots . + a_{mn}x_n = b_m. \end{cases} \tag{1.1}$$

This system can be written in matrix form as,

$$AX = B, \tag{1.2}$$

where A is a $m \times n$ matrix, while X and B are column vectors with n and m entries, respectively.

However, the classical linear system is not well equipped to handle uncertain ties of information in real life problems, because some values of the coefficients may be vague and imprecise due to incomplete data. In practice, the data of the mathematical method are not always exactly known. Moore (1979) declared that exact numerical data might be unrealistic, but vague data can consider more features of a real life problem. A natural way to describe vague data is using fuzzy data. Thus, in this case, fuzzy numbers is a better usage than crisp numbers for modeling uncertain problems. The concept of fuzzy numbers and arithmetic operations with these numbers were introduced and investigated by Zadeh (1965), Dubois and Prade (1980), and Kaufmann and Gupta (1991).

The next section presents background of fuzzy systems and insight of existing methods for solving them.

1.2 Linear Fuzzy Systems

In the application of fuzzy numbers in a linear system, some crisp entries of the linear system should be replaced by fuzzy numbers such as particular form of fuzzy numbers, triangular fuzzy numbers and trapezoidal fuzzy numbers. This results new categories of linear systems called linear fuzzy system. For instance, if the elements of the matrix A are crisp numbers and the elements of vector B are fuzzy numbers, this leads to new systems called fuzzy linear system (FLS), Left- Right fuzzy linear system ($LR-FLS$) and Left- Right trapezoidal fuzzy linear system ($LR-TFLS$). On the other hand, it is called fully fuzzy linear system ($FFLS$), if all elements in A

and B are fuzzy numbers. Moreover, if the fuzzy vectors in $FFLS$ are extended to fuzzy matrices, it is called fully fuzzy matrix equation ($FFME$).

1.2.1 Fuzzy linear system

FLS is obtained by replacing crisp numbers in B with fuzzy numbers to create $\tilde{B}$ in Equation (1.1). The most achievable approach of FLS $A\tilde{X} = \tilde{B}$ was obtained by Friedman, Ming and Kandel (1998). They proposed a generic method for solving an $n \times n$ FLS by employing the embedding approach. In this method, they used fuzzy numbers in a particular form to construct the fuzzy system FLS. The FLS was replaced by a $2n \times 2n$ crisp linear system to solve the fuzzy system.

In the last few years, some authors have discussed the deficiencies of the Friedman's method. The main disadvantage was in the definition of fuzzy solution, in which Friedman's method classified the fuzzy solutions to strong or weak fuzzy solution. Allahviranloo, Ghanbari, Hosseinzadeh, Haghi and Nuraei (2011a), by a counter example, showed that the definition of weak fuzzy solution is not always correct i.e. it does not always yield a fuzzy number vector. Additionally, Mansouri and Asady (2011) showed that sometimes the proposed $2n \times 2n$ linear system in Friedman's method (1998) is singular even if the original coefficient matrix A is non-singular.

Also, Friedman's method deals with particular form in fuzzy numbers as FLS was not represented as triangular or trapezoidal in fuzzy vector $\tilde{B}$. In general, it is easier to solve the systems using constants instead of variables. For example, Ghanbari, Mahdavi-Amiri and Yousefpour (2015), Ghanbari and Mahdavi-Amiri (2015), and Allahviranloo, Lotfi, Kiasari and Khezerloo (2013) have attempted to extend FLS to $LR-FLS$ by replacing the entries of vector $\tilde{B}$ with triangular fuzzy numbers. Additionally, Nasseri and Gholami (2011), Allahviranloo, Haghi and Ghanbari (2012a), Allahviranloo, Nuraei, Ghanbari, Haghi and Hosseinzadeh (2012b) have attempted to extend FLS to $LR-TFLS$ by replacing the entries of vector with trapezoidal fuzzy numbers.

In the next section, $LR-FLS$ is discussed as an alternative of FSL by changing the type of fuzzy numbers.

1.2.2 Left- Right Fuzzy Linear System

The $LR-FLS$ is a fuzzy system $A\tilde{X}=\tilde{B}$ where the A is crisp matrix, $\tilde{X}$ and $\tilde{B}$ are fuzzy vectors including triangular fuzzy numbers, thus, $LR-FLS$ is considered as an expansion of FLS.

Ghanbari et al. (2010), and Ghanbari and Mahdavi-Amiri (2015) found the solutions of $LR-FLS$. They transformed the fuzzy system $LR-FLS$ into a corresponding crisp linear system and a constrained least squares problem, and they proved that the $LR-FLS$ has a solution if and only if the corresponding crisp system has a solution and the corresponding least squares value equaled zero. Thus, to determine the sufficient and necessary condition to have a solution, it is required to solve the corresponding crisp linear system and the constrained least squares problem. Allahviranloo et al. (2013) provided a method that is able to obtain infinitely many solutions, however this method cannot provide all solution sets for infinitely many solutions because they used convex set. Moreover, their method cannot determine if the solution is infinite or finite before they get the final solution, this is because the sufficient and necessary condition to have a solution were not investigated. Nikuie and Ahmad (2014) found the solution for $LR - FLS$ using a corresponding linear system. Next, the necessary and sufficient conditions for solving a $LR - FLS$ was investigated by ranking of corresponding linear system, as a result, they could not determine the singularity of $LR - FLS$ before constructing or solving the system. Later, Ghanbari and Mahdavi-Amiri (2015) established some necessary and sufficient conditions for solving of $LR - FLS$, an minimization problem was used to obtain the solution.

Thus, the exact and approximation the solution of $LR - FLS$ can be obtained by linear system or minimization problem in existing methods. In addition, the existing methods cannot investigate the necessary and sufficient conditions before obtaining the solution.

In the next section, the entries for A and B in Equation (1.1) are replaced with fuzzy numbers in order to construct $FFLS$.

1.2.3 Fully Fuzzy Linear Systems

In the fuzzy system $\tilde{A}\tilde{X} = \tilde{B}$, all entries in $\tilde{A}, \tilde{X}$ and $\tilde{B}$ are fuzzy numbers. The most common usage of fuzzy numbers in $FFLS$ is triangular fuzzy number. There are many scenarios of $FFLS$ that depend on the sign of triangular fuzzy numbers (positive or negative or near zero). On the contrary of FLS, $LR - FLS$ and $LR - TFLS$ which have only one scenario because the multiplication in FLS, $LR - FLS$ and $LR - TFLS$ between matrix coefficient A and fuzzy vector $\tilde{X}$ does not depend on the sign, while in $FFLS$ the multiplication between matrix coefficient $\tilde{A}$ and fuzzy vector $\tilde{X}$ depends on the sign for both. Dehghan, Hashemi and Ghatee (2006), Dehghan and Hashemi (2006a), and Dehghan, Hashemi and Ghatee (2007) were the first researchers developed FLS by using particular form of fuzzy numbers in left hand side to $FFLS$ by using triangular fuzzy numbers in the left and right hand sides.

In establishing $FFLS$, Dehghan and his colleagues declared that there are many scenarios that can be derived from a $FFLS$. The methods of finding a solution for these scenarios of $FFLS$ create new scenarios of fuzzy systems (Kumar, Bansal and Neetu, 2010). Moreover, Kumar, Neetu and Bansal (2012), mentioned that there is infinite number of scenarios that can be derived form a $FFLS$. Abbasbandy and Hashemi (2012) questioned as to what occurs in a FLS if all parameters are replaced by fuzzy numbers, and also what is the solution of this type of FLS.

Dehghan and his colleagues in Dehghan et al. (2006), Dehghan and Hashemi (2006a), and Dehghan et al. (2007) found positive solution $\tilde{X}$ when all coefficients of $\tilde{A}$ and $\tilde{B}$ are positive triangular fuzzy numbers. Scholars like Nasseri, Sohrabi and Ardil (2008), Nasseri and Zahmatkesh (2010), Nasseri, Behmanesh and Sohrabi (2012),

Gao and Zhang (2009), Kumar et al. (2012), Abbasbandy and Hashemi (2012), Ezzati, Khezerloo, Mahdavi-Amiri and Valizadeh (2014), Abdolmaleki and Edalatpanah (2014) and Radhakrishnan, Gajivaradhan and Govindarajan (2014); proposed methods for solving $FFLS$ in a similar case to Dehghan and his colleagues, in which the sign of triangular fuzzy number was restricted to be positive for all entries in $\tilde{A}, \tilde{X}$ and $\tilde{B}$.

However, to date, there is no computational method for solving $FFLS$ without any constraint on the coefficients (Kumar, Bansal and Neetu (2011a), Kumar, Neetu and Bansal (2011b) and Babbar, Kumar and Bansal (2013)). Besides, the existing methods cannot check if the achieved solution is unique, trivial, infinite many solutions or no solution (Babbar et al. (2013)).

On other hand, several methods were developed in order to solve $FFLS$ which include $\propto$-cuts or particular form of fuzzy number instead of triangular fuzzy numbers, such as in Allahviranloo and Mikaeilvand (2006), Muzziolia and Reynaerts (2006), Allahviranloo, Mikaeilvand, Kiani and Shabestari (2008) and Allahviranloo, Salahshourb and Khezerloo (2011b). In these studies $\propto$-cuts were included in methods, although, some of them included the triangular fuzzy number in the coefficient of matrix $\tilde{A}$. In general, it is easier to solve the systems using constants instead of variables. As such, in the case of $FFLS$, the computational time can be reduced by employing the triangular fuzzy numbers, thus allowing researchers in other fields to use. This is because the triangular fuzzy numbers consist order of three crisp numbers which is similar to classical linear system.

After reviewing the literature, we can classify the following scenarios of $FFLS$ and its solutions based on the sign of triangular fuzzy numbers:

- If all entries of $\tilde{A}$ are positives, regardless of the sign of entries in $\tilde{B}$, it is called Positive $FFLS$, abbreviated $P - FFLS$.

- If $\tilde{A}$ has at least two entries that are negative and positive, regardless of the sign of entries in $\tilde{B}$, it is called General $FFLS$, abbreviated $G - FFLS$.
- If at least one entry of $\tilde{A}$ is near zero, regardless of the sign of entries in $\tilde{B}$, it is called Near Zero (Arbitrary or Unrestricted) $FFLS$, abbreviated $NZ - FFLS$.
- If all entries of $\tilde{X}$ are positives it is called Positive solution, abbreviated $P - \tilde{X}$.
- If $\tilde{X}$ has at least two entries that are negative and positive it is called General solution, abbreviated $G - \tilde{X}$.
- If at least one entry of $\tilde{X}$ is near zero, it is called Near Zero (Arbitrary or Unrestricted) solution, abbreviated $NZ - \tilde{X}$.

The researchers tried to generalize methods for dealing with the scenarios of $FFLS$, since almost existing methods did not clearly distinguish these scenarios. Many restrictions were added to $FFLS$ which can be noted by the limitations in the illustrated numerical examples in their studies. Liu (2010) and Kumar et al. (2011a) tried to generalize the previous methods $P - \tilde{X}$ for $NZ - FFLS$, but actually they restricted the signs of coefficients positive or negative as an attempt to avoid the near zero triangular fuzzy numbers, such that they found $G - FFLS$ instead of $NZ - FFLS$. Using their methods, they obtained exact and infinite many solutions of $P - \tilde{X}$and $G - \tilde{X}$ for $G - FFLS$. However, the proposed solutions by Liu (2010) is not accurate since the left hand side is not equal to the right hand side, moreover, his unique example has a non fuzzy solution. Ezzati, Khezerloo and Yousefzadeh (2012) found $P - \tilde{X}$ for $NZ - FFLS$ using the corresponding linear systems for exact solution and linear programming (LP) for getting approximate solution, some examples are not accurate since the left hand side is not equal to right hand side.

The exact and infinite $P - \tilde{X}$ for $NZ - FFLS$ was studied by Kumar et al. (2011a) through a method similar to the technique employed in Dehghan et al. (2006). In a way this means a strong restriction to employ this method in real life situations. Otadi, Mosleh and Abbasbandy (2011) found an approximate $G - \tilde{X}$ for $G - FFLS$, in which the near zero fuzzy numbers were not included. They proposed a numerical method by fuzzy neural network denoted by FNN. This method was restricted by many constraints, where $G - FFLS$ can be solved only when the system has a unique fuzzy solution and the matrix $\tilde{A}$ has to be squares which prevents an extension of the method to solve a rectangle matrix. Moreover, Babbar et al. (2013) proposed another method that depends on Cramer's rule, which also expected the possibility of ‘‘near zero’’ fuzzy numbers in the solution vector.

In order to extend the methods for dealing with $G - FFLS$ or $NZ - FFLS,$ and overcoming the shortcomings in previous methods, the researchers relied heavily on LP and nonlinear programing (NLP) to solve $G - FFLS$ or $NZ - FFLS$. Unfortunately, LP and NLP can give answers to the linear system but not to fuzzy systems. For instance, the $FFLS$ may yield two unique solutions or many infinite solutions despite of the fact that it is constructed by only one equation which contradicts with linear system, such as the methods in Kumar et al. (2011b) and Babbar et al. (2013).

Also, $FFLS$ may have two unique solutions which contradict the uniqueness of optimal solution that can be obtained through LP and NLP. Moreover, LP and NLP

methods add more restrictions on the final exact solution to obtain the fuzzy solution. These restrictions cannot detect all solution sets of infinite many solutions.

The exact and infinite of $P-\tilde{X}$ or $NZ-\tilde{X}$ for $NZ-FFLS$ has been studied by Otadi and Mosleh (2012), Kumar et al. (2011a), and Babbar et al. (2013) using LP and NLP. They illustrated some examples that have a unique fuzzy solution, while the systems have two unique fuzzy solutions or infinite many solutions. They added many restrictions to the systems which required many steps, thus requiring longer time to reach the ultimate solution. For example, Babbar et al. (2013) method leads to complicated LP since it has $3n+2n^2$ of constraints in some cases for $n\times n$ $FFLS$, so their examples does not exceed fuzzy matrix $\tilde{A}$ of size $n=2$. However almost all examples in literature does not exceed fuzzy matrix $\tilde{A}$ of size $n=3$ or 4 if at least one entry is near zero triangular fuzzy number. Allahviranloo, Hosseinzadeh, Ghanbari, Haghi and Nuraei (2014) transformed $FFLS$ for two fully interval linear systems, then the interval systems transformed to three system, $2n$ linear equations, $4n$ nonlinear equations and n nonlinear equations, then, the all the equations are transformed for NLP to compute the optimal solution. Therefore, their method required a construction more than six different systems. This method cannot provide more than one solution because the final solution is provided by NLP.

In the next fuzzy system, the $FFLS$ is extended to linear matrix equation where the fuzzy vector replaced by fuzzy matrix to construct the $FFME$.

1.2.4 Fully Fuzzy Matrix Equation

In this fuzzy system, the fuzzy vectors in $FFLS$ is extend to fuzzy matrix to introduce $FFME$.

The most recent study on extending $FFLS$ to $FFME$ was done by Otadi and Mosleh (2012), in which they extend the LP technique to obtain positive solution for arbitrary $FFME$ which is $\tilde{A}\tilde{X}_m = \tilde{B}_m$, where the entries of $\tilde{A}, \tilde{X}_m$ and $\tilde{B}_m$ are *TFNs*. This method cannot detect all possible cases of infinitely many solutions since the ultimate solution for $FFME$ is proposed as equivalent to optimal solution in LP, they provided the unique solution for an example, where as the solution has infinite many solutions.

The fully fuzzy Sylvester matrix equation ($FFSE$) is $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$, where $\tilde{A}$,$\tilde{B}$ and $\tilde{C}$ are given fuzzy matrices and the problem is to find the fuzzy matrix $\tilde{X}$. The classical form of this matrix equation plays an important role in control theory, signal processing, filtering, method reduction, image restoration, and decoupling techniques for ordinary and partial differential equations (Benner (2004) and Darouach (2006)). The fuzzy form of Sylvester matrix equation $FFSE$ is considered as an extension for most fuzzy systems involving fuzzy matrix equation. To date, there is no method to solve $FFSE$ where the entries of fuzzy matrices $\tilde{A}$,$\tilde{B}, \tilde{X}$ and $\tilde{C}$ are *TFNs*.

In the next section, a new fuzzy system is discussed with a new type of fuzzy number. The triangular fuzzy number is replaced in $LR - FLS$ by trapezoidal fuzzy number to show the ability for developing all previous fuzzy systems to other type of fuzzy numbers.

1.2.5 Left- Right Trapezoidal Fuzzy Linear System

In the linear system, the $LR-TFLS$ is a fuzzy system $A\tilde{X}=\tilde{B}$ where the A is a crisp matrix $\tilde{X}$ and $\tilde{B}$ are fuzzy vectors constructing of trapezoidal fuzzy numbers.

The first development of $LR-FLS$ to $LR-TFLS$ was obtained by Nasseri et al. (2011), where they proposed a solution of $LR-TFLS$. This method requires an associated triangular fuzzy numbers and a particular form of fuzzy numbers. Allahviranloo et al. (2012a, b) introduced a metric function to provide an exact fuzzy solution and nearest approximation solution. The fuzzy system was transformed into the minimization problem, where the constraints of *NLP* guarantee that the solution is a fuzzy number. Unfortunately, the proposed solution from an example was not corresponding to the fuzzy system, because the left hand side was not equal to the right hand side after substituting the proposed solution in $A\tilde{X}=\tilde{B}$. Moreover, there are many nearer (symmetric or non symmetric) approximate solutions as compared to the proposed solution in Allahviranloo et al. (2012a, b).

Table 1.1 provides a summary for fuzzy systems.

Table 1.1

Summary for Fuzzy Systems.

Name of system	**Abbreviation**	**Fuzziness**			**Matrix**	**The first achievable work**
		The fuzzy number	**Left hand side**	**Right hand side**	**Form**	
Fuzzy Linear System	FLS	Particular Form	A:crisp matrix $\tilde{X}$:fuzzy vector	$\tilde{B}$:fuzzy vector	$A\tilde{X} = \tilde{B}$	1998
Left Right - Fuzzy linear system	$LR - FLS$	Triangular Fuzzy number	A:crisp matrix $\tilde{X}$:fuzzy vector	$\tilde{B}$:fuzzy vector	$A\tilde{X} = \tilde{B}$	2010
Positive Fully Fuzzy Linear System	$P - FFLS$	Triangular Fuzzy number	$\tilde{A}$: fuzzy matrix $\tilde{X}$:fuzzy vector	$\tilde{B}$:fuzzy vector	$\tilde{A}\tilde{X} = \tilde{B}$	2006
Near Zero Fully Fuzzy Linear System	$NZ - FFLS$	Triangular Fuzzy number	$\tilde{A}$: fuzzy matrix $\tilde{X}$:fuzzy Vector	$\tilde{B}$:fuzzy vector	$\tilde{A}\tilde{X} = \tilde{B}$	2011
Fully Fuzzy Matrix Equation	$FFME$	Triangular Fuzzy number	$\tilde{A}$,$\tilde{X}_m$: fuzzy matrices	$\tilde{B}_m$:fuzzy matrix	$\tilde{A}\tilde{X}_m = \tilde{B}_m$	2012
Fully Fuzzy Sylvester Matrix Equation	$FFSE$	Triangular Fuzzy number	$\tilde{A}$,$\tilde{X}_m$: fuzzy matrices	$\tilde{B}_m$:fuzzy matrix	$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$	2015
Left Right - Trapezoidal Fuzzy linear system	$LR - TFLS$	Trapezoidal Fuzzy number	A:crisp matrix $\tilde{X}$:fuzzy vector	$\tilde{B}$:fuzzy matrix	$A\tilde{X} = \tilde{B}$	2011

1.3 Problem Statement

Based on the vast amount of discussion on methods for solving fuzzy systems ($LR - FLS, FFLS, FFME, FFSE, LR - TFLS$), it is apparent that these methods have three main drawbacks: first, the dependence on *LP* and *NLP* to obtain exact fuzzy solution; second, the restriction on the signs of *TFNs* when dealing with near zero TFN; third, fuzzy solutions are incompatible with the fuzzy system.

For the first drawback, some existing methods relied heavily on *LP* and *NLP* where both of these can only give answers to the linear system but not for fuzzy systems such as presented by Kumar et al. (2011b), Babbar et al.(2013), Otadi and Mosleh (2012), and Allahviranloo et al. (2014). The existence of the $FFLS$ solutions relies on the existence of the solutions of the *LP* and *NLP*. In these studies, the researchers computed all fuzzy operations to produce linear equations. Next, the artificial variables were added to each equation. All artificial values were dropped to zero to get the optimal values. The optimal solution was restricted by more constrains, as a result the fuzzy solution was preserved and other non fuzzy solution was omitted. Hence, some solutions for fuzzy systems may be omitted through this approach and cannot be obtained. Moreover, the possibilities of solutions for fuzzy systems do not follow the classical linear system i.e. no solution, a unique solution, or infinite many solutions; however, this possibility can not be applied to linear equations in the constrains part of optimization problem. In conclusion, these studies could not detect all possible fuzzy or non fuzzy exact solutions. For instance, Babbar et al. (2013) and Kumar et al. (2011b) provided a unique solution where the fuzzy system has two unique solutions (finite solutions and not infinite solutions); whereas, two unique solutions are not acceptable for linear systems. Otadi and Mosleh (2012) provided a unique solution where the system has infinite many solutions. Allahviranloo et al. (2014) provided a unique solution where the example provided no fuzzy solution for the system.

Moreover, because of the complicated of optimization problem, the studies only illustrated limited fuzzy matrices that did not exceed $n = 3$ or 4 of fuzzy matrices. This is because when the size increases many steps are required to reach the final solution, which leads to more computation time. For instance, Babbar et al. (2013) proposed that the optimization problem may have $3n + 2n^2$ constraints to solve arbitrates $n \times n$ fuzzy system. Meanwhile, Allahviranloo et al. (2014) needed to solve more than six different linear systems or nonlinear, which comprised of two fully interval linear systems, three $2n \times 2n$ linear systems, a $4n \times 4n$ nonlinear equations, and an $n \times n$ nonlinear equations, before constructing the optimization problem.

The second drawback is that many researchers tried to generalize methods for solving the fuzzy system by including any signs of $TFNs$ while dealing with constricting the signs of $TFNs$ (positive, negative, near zero). However, by restricting the signs for positive and negative, the complicated arithmetic fuzzy operation can be avoided. This is because near zero TFN required more arithmetic fuzzy operation of positive or negative TFN. For example, Liu (2010) and Kumar et al. (2011a) restricted the two signs that are, positive or negative fuzzy number of coefficient.Similarly, Kumar et al. (2012), Kumar et al. (2011b), and Otadi and Mosleh (2012) restricted the positive solution only, or restricted the negative solution only Kumar, Babbar and Bansal, (2013). Therefore, these methods are unable to solve arbitrary systems that have near zero TFN in the coefficient.

Lastly, the proposed fuzzy solution in the existing methods is not compatible with the fuzzy system. Some existing methods are incomplete and have many flaws. Since the proposed solution is incorrect, this flaw can be easily identified by substituting the proposed fuzzy solution in the system and concluding that the right hand side is not equal to the left hand side; its evidence can be found in the studies by Liu, (2010), Allahviranloo et al. (2012a), Abbasbandy and Hashemi (2012) and Allahviranloo et al. (2014).

Thus, new methods are deemed necessary to be constructed to resolve aforementioned drawbacks of solving $FFLS$, $FFME$, $FFSE$ and $LR - TFLS$ focusing on positive and unrestricted parameters for scenarios for $FFLS$.

1.4 Research Objectives

The crux objective of this study is to propose new algorithms for solving all scenarios of fuzzy systems. In order to accomplish this main objective, the following sub-objectives must be considered:

i- To construct new algorithms for solving $FFLS$, $FFME$ and $LR - TFLS$.

ii- To develop theoretical background of the new constructed algorithms which involves the following possibilities of solution:

- Case of no solution, non fuzzy solutions.

- Unique fuzzy solution, finite fuzzy solution and infinite number of fuzzy solutions.

iii- To apply the new constructed algorithms in the following application:

- The simpler fuzzy systems such as $LR - FLS$.
- Fully Fuzzy Sylvester Equation in control theory.
- The nearest approximated fuzzy solution in the case of non fuzzy solution.
- Fuzzy systems with other type of fuzzy numbers such as trapezoidal fuzzy numbers.

1.5 Scope of the Study

This study develops computational methods of linear systems and matrix equations where the coefficients are triangular fuzzy numbers and trapezoidal fuzzy numbers.

1.6 Significance of Findings

The findings of this study will have the following contributions:

i- New constructed methods for solving fuzzy systems without fuzzy operations.

ii- The theoretical development on the existence of fuzzy solution, possibilities of solution of fuzzy systems.

iii- The methods can be used to solve uncertain problems in the following field:

- In optimization problem: To find the nearest approximation fuzzy solution.

- In fuzzy system with other type of fuzzy numbers as trapezoidal fuzzy numbers.
- In control theory as Sylvester equation.

1.7 Overview of the Thesis

This thesis has eight chapters. Chapter One provides the introduction of the research. This chapter discusses the research background and a brief survey of fuzzy systems, the problem statement, the research objectives, the scope of study and the significance of the thesis.

Chapter Two presents the selected reviews of matrix theory and fuzzy numbers, definitions, basic concepts and established results of fuzzy systems.

In Chapter Three, the unique solution of $P-\tilde{X}$ for $P-FFLS$ without fuzzy operation is obtained. The existence of $P-\tilde{X}$ for $P-FFLS$ is checked and proved before solving the system. The possibilities of solution are classified. Also, the general form solution for arbitrary fuzzy vector $\tilde{B}$ is formulated.

In Chapter Four, a new method is formulated to obtain the $P-\tilde{X}$ of $G-FFLS$ or $NZ-FFLS$. The coefficients of fuzzy matrix $\tilde{A}$ and the entries of fuzzy vectors $\tilde{B}$ are represented in a linear system, without either fuzzy operation or min-max system. The existence of solution for $P-\tilde{X}$ is checked and proved before solving the system.

Chapter Three and Four provide the positive solution for $FFLS$. Meanwhile, in Chapter Five, unrestricted solution for arbitraries $FFLS$ is provided. The near zero is included

($NZ - \tilde{X}$ for $NZ - FFLS$) by using only arithmetic operations of fuzzy numbers in order to avoid adding any restrictions to the system. As a result, all possible solutions for systems are attained. Hence, it is concluded that the nature of solution of classical linear system (no solution, unique, infinitely many solutions) is not sufficient to provide all possibility of solutions for $FFLS$, where the $FFLS$ have more than two solutions, and not infinite solutions.

In Chapter Six, it is shown that these methods of solving $FFLS$ can be developed to solve $FFME$ and $FFSE$. The solution is obtained without any fuzzy operation, similar to previous chapters.

Chapter Seven, provides a new approach of fuzzy system by applying other types of fuzzy number such as trapezoidal fuzzy number. It shows that our approach in previous chapters can be extended to any type of fuzzy number. In addition, the nearest approximation fuzzy solution using a minimization problem is provided, when the exact solution is non fuzzy. It shows that our approach in previous chapters can be extended to provide approximation solution when the exact solution is non fuzzy.

Finally, Chapter Eight concludes the whole thesis with a summary of this study and discuses some insights of the possibilities for further research conducted in this area of study.

CHAPTER TWO

LITERATURE REVIEW

This chapter provides some basic concepts of matrix theory, and types of fuzzy numbers. These concepts will be used to discuss the methods for solving fuzzy systems at the last section of this chapter.

2.1 Fundamental Concepts of Matrix Theory

The following are basic definitions in matrix theory, will be used in solving fuzzy systems.

Definition 2.1.1. *A matrix A is called non−negative inverse if $A > 0$ and $A^{-1} > 0$.*

Definition 2.1.2. *A matrix A is called a generalized permutation matrix (or monomial matrix) if each column and each row has exactly one non−zero entry.*

Theorem 2.1.1. (Minc, 1988) The inverse of a non negative matrix A is non negative if and only if A is a generalized permutation matrix.

Remark 2.1.1. (Friedman et al. 1998) The odds of A^{-1} to be non negative is very small.

Definition 2.1.3. *Block or partitioned matrix is a matrix that has been created from other smaller matrices.*

Definition 2.1.4. *An upper triangular block matrix is a block matrix, if the diagonal elements have square matrices of any size (possibly even 1×1), with zero matrices below the main diagonal.*

Definition 2.1.5. *A block diagonal matrix is a block matrix, if the diagonal elements have square matrices of any size (possibly even 1×1), and the other elements are zeroes.*

Definition 2.1.6. *Let A and B be $n \times p$ and $m \times q$ matrices respectively. The $nm \times pq$ matrix,*

$$A \otimes B = (a_{ij}B) = \begin{bmatrix} a_{11}B & \cdots & a_{1p}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{np}B \end{bmatrix}, \tag{2.1}$$

is called the Kronecker product of A and B, it is also called the direct product or tensor product.

Definition 2.1.7. *The vec operator generates a column vector from a matrix A by stacking the column vectors of $A = [c_1 \quad c_2 \quad \dots \quad c_n]$ as,*

$$vec(A) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}. \tag{2.2}$$

Theorem 2.1.2. (Zhang, 2011) Assuming $A, B, C,$ and D are matrices with a common size in block matrix H:

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

If A and $A - BD^{-1}C$ are invertible, then the inverse of matrix H is,

$$H^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}. \quad (2.3)$$

2.1.1 Elementary Operation of Block or Partitioned Matrices

Since the techniques used for manipulating block matrices are similar to those for ordinary matrices, as an application, the elementary row or column operations for ordinary matrices can be generalized to block or partitioned matrices as follows:

1. *By interchanging two block rows or columns.*
2. *By multiplying a block row or columns from the left or right by a non−singular matrix of appropriate size.*
3. *By multiplying a block row or column by a non−zero matrix from the left or right, then add it to another row or column* (Zhang, 2011).

2.1.2 Rules for Operation with Kronecker Product and Vec Operator

The formal rules for operation with Kronecker product are as follows, where the matrices $A, B,$ and C are approprite size

1. $rank(A \otimes B) = rank(A)\, rank(B)$.
2. $det(A \otimes B) = (det(A))^n\,(det(B))^m$, *if A and B are m and n squre matrices, respectively.*
3. $vec(A + B) = vec(A) + vec(B)$.
4. $vec(ABC) = (C^T \otimes A) vec\, B$. (Abadir and Magnus, 2005)

2.2 Types of Fuzzy Numbers

The following are basic definitions and results related to fuzzy numbers. Here we have discussed two types of fuzzy numbers: triangular fuzzy numbers, and trapezoidal fuzzy numbers (Dubois and Prade (1980),Kaufmann and Gupta(1991)).

Definition 2.2.1. *Let Xbe a universal set.Then, we define the fuzzy subset $\tilde{A}$ of X by its membership function $\mu_{\tilde{A}} : X \to [0, 1]$ which assigns to each element $x \in X$ a real number $\mu_{\tilde{A}}(x)$ in the interval $[0, 1]$, where the function value of $\mu_{\tilde{A}}(x)$ represents the grade of membership of x in $\tilde{A}$. A fuzzy set $\tilde{A}$ is written as*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), x \in X, \mu_{\tilde{A}}(x) \in [0, 1]\}.$$

Definition 2.2.2. *A fuzzy subset $\tilde{A}$ of the real line $\mathbb{R}$ with membership function $\mu_{\tilde{A}}(x): \mathbb{R} \to [0,1]$ is called a fuzzy number if,*

1. $\mu_{\tilde{A}}$ *is upper semi−continuous.*

2. $\mu_{\tilde{A}}(x) = 0$ *is outside some interval* $[c, d]$.

3. *There are real numbers a and b such that $c \leq a \leq b \leq d$ and:*

   i. $\mu_{\tilde{A}}(x)$ *is monotonic increasing on* $[c, a]$,

   ii. $\mu_{\tilde{A}}(x)$ *is monotonic decreasing on* $[b, d]$,

   iii. $\mu_{\tilde{A}}(x) = 1,$ *for* $a \leq x \leq b$.

Definition 2.2.3. *A matrix* $\tilde{A} = (\tilde{a}_{ij})$ *is called a fuzzy number matrix, or shortly fuzzy matrix, if each element of* $\tilde{A}$ *is a fuzzy number. A matrix* $\tilde{A}$ *is a positive fuzzy matrix* $\left(denoted\ by\ \tilde{A} \geq 0\right)$, *if each element of* $\tilde{A}$ *is positive. A vector* $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ *is called a fuzzy vector, if elements of* $\tilde{X}$ *are a fuzzy numbers.*

Definition 2.2.4. *Let* $\tilde{A} = (\tilde{a}_{ij})$ *and* $\tilde{B} = (\tilde{b}_{ij})$ *be two* $m \times n$ *and* $n \times p$ *respectively. We define* $\tilde{A} \otimes \tilde{B} = \tilde{C} = (\tilde{c}_{ij})$ *which is the* $m \times p$ *matrix where,*

$$\tilde{c}_{ij} = \sum_{k=1,\dots,n}^{\oplus} \tilde{a}_{ik} \otimes \tilde{b}_{kj}. \tag{2.4}$$

The next fuzzy number is a parametric form of fuzzy number which is used in FLS.

Definition 2.2.5. *A fuzzy number* $\tilde{u}$ *is represented in parametric form by an ordered pair of functions* $\tilde{u} = \left(\underline{u}(r), \bar{u}(r)\right),\ for\ 0 \leq r \leq 1$ *which will satisfy the following conditions:*

1. $\underline{u}(r) \leq \bar{u}(r),\ for\ 0 \leq r \leq 1.$

2. $\bar{u}(r)$ *is a bounded left continuous non−increasing function on [0, 1].*

3. $\underline{u}(r)$ *is a bounded left continuous non−decreasing function on [0, 1].*

Among the several fuzzy numbers, the most common one used is triangular fuzzy number, where particular form of fuzzy number can be represented by triangular fuzzy number. In the next section, triangular fuzzy number is discussed.

2.2.1 Triangular Fuzzy Number

Definition 2.2.6. *A fuzzy number* $\widetilde{m} = (m, \alpha, \beta)$ *is said to be a triangular fuzzy number (TFN), if its membership function is given by,*

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{m - x}{\alpha}, & m - \alpha \le x \le m \ , \ \alpha > 0, \\ 1 - \dfrac{x - m}{\beta}, & m \le x \le m + \beta \ , \ \beta > 0, \\ 0, & otherwise. \end{cases} \tag{2.5}$$

We denote the set of triangular fuzzy numbers as $G(R)$. $\widetilde{m}$ *is called symmetric if* $\alpha = \beta$. $\widetilde{m}$ is called non fuzzy number if α or β are negative. Figure 2.1. displays the TFN.

The sign of $\widetilde{m} = (m, \alpha, \beta)$ *is classified as follows:*

- $\widetilde{m}$ *is called Positive (Negative) iff* $m - \alpha \ge 0$ $(\beta + m \le 0)$.
- $\widetilde{m}$ *is called Zero if* $(m = 0, \alpha, \beta = 0)$.
- $\widetilde{m}$ *is called Near Zero iff* $m - \alpha \le 0 \le \beta + m$.

Remark 2.2.1 (Dubois and Prade, 1980) If the spreads α and β increase in $\widetilde{m} = (m, \alpha, \beta)$, $\widetilde{m}$ becomes fuzzier and fuzzier. Moreover, it is considered as non fuzzy (crisp number) when $\alpha, \beta = 0$.

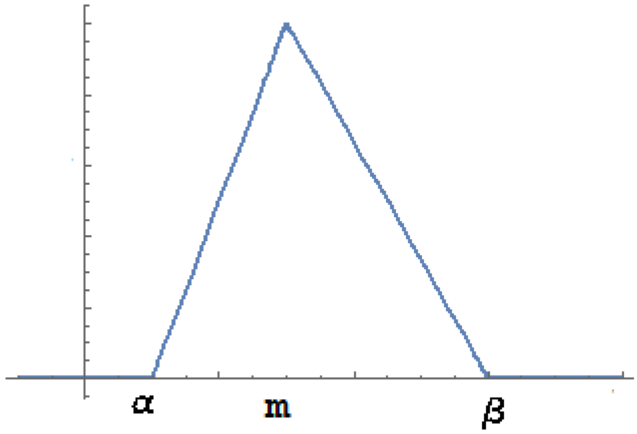

Figure 2.1. Representation of triangular fuzzy number (m, α, β)

TFN can be represented by Definition 2.2.2. as follows:

$$\underline{u}(r) = r\alpha + m - \alpha \text{ and } \bar{u}(r) = m + \beta - r\beta.$$

The triangular fuzzy number can be represented in another form; it is derived if we suppose:

$$a_1 = m - \alpha, \qquad a_2 = m, \qquad a_3 = m + \beta. \tag{2.6}$$

In this case, it is symbolically written as $\tilde{a} = (a_1, a_2, a_3)$ or (a, b, c).

Then the membership function for this form is,

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2, \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3, \\ 0, & otherwise. \end{cases} \tag{2.7}$$

Definition 2.2.7. *Two TFNs* $\widetilde{m}_1 = (m_1, \alpha_1, \beta_1)$ *and* $\widetilde{m}_2 = (m_2, \alpha_2, \beta_2)$ *are called equal, iff* $m_1 = m_2$, $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$.

For choosing the nearest approximate and more accurate solution, the following metric distance function which proposed in Allahviranloo et al. (2012a, b) for trapezoidal fuzzy numbers. This metric function is modified to for triangular fuzzy number, by supposing the mean value interval in trapezoidal fuzzy number is a unique value to produce triangular fuzzy number.

Definition 2.2.8. (Allahviranloo et al. 2012a, b) *For two fuzzy number* $\tilde{A} = (a, \alpha_1, \beta_1)$, $\tilde{B} = (b, \alpha_2, \beta_2)$, *we define the distance between* $\tilde{A}$ *and* $\tilde{B}$ *as follows:*

$$d^2(\tilde{A}, \tilde{B}) = \frac{[(a-b)-(\alpha_1-\alpha_2)]^2 + [(a-b)+(\beta_1-\beta_2)]^2 + 2\,(a-b)^2}{4}, \tag{2.7}$$

for two fuzzy vectors $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$, $\tilde{Y} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ *we define the distance between* $\tilde{A}$ *and* $\tilde{B}$ *as follows:*

$$D_p(\tilde{X}, \tilde{Y}) = \left(\sum_{i=1}^{n} d^p(\tilde{A}, \tilde{B})\right)^{\frac{1}{p}}. \tag{2.8}$$

The idea of constructing the concept *TFNs* is referred to Dubois and Prade (1980). It aims at simplifying operations of fuzzy numbers to make the computational formulas easier and quick. The next definition presents the arithmetic operation of *TFNs*.

Definition 2.2.9. (Dubois and Prade, 1980) *The arithmetic operations for two TFN Fuzzy numbers* $\tilde{m} = (m, \alpha, \beta)$ *and* $\tilde{n} = (n, \gamma, \delta)$ *are represented as follows:*

Addition:

$$\tilde{m} \oplus \tilde{n} = (m + n, \alpha + \gamma, \beta + \delta). \qquad (2.9a)$$

Opposite:

$$-\tilde{m} = -(m, \alpha, \beta) = (-m, \beta, \alpha). \qquad (2.9b)$$

Subtraction:

$$\tilde{m} \ominus \tilde{n} = (m - n, \alpha + \delta, \beta + \gamma). \qquad (2.9c)$$

Approximated multiplication operation of two triangular fuzzy numbers:

1. *Let* $\tilde{m} > 0$ *and* $\tilde{n} > 0$. *Then,*

$$\tilde{m} \otimes \tilde{n} \cong (mn, m\gamma + n\alpha, m\delta + n\beta). \qquad (2.10a)$$

2. *Let* $\tilde{m} < 0$ *and* $\tilde{n} > 0$. *Then,*

$$\tilde{m} \otimes \tilde{n} \cong (mn, n\alpha - m\delta, n\beta - m\gamma). \qquad (2.10b)$$

3. *Let* $\tilde{m} < 0$ *and* $\tilde{n} < 0$. *Then,*

$$\tilde{m} \otimes \tilde{n} \cong (mn, -n\beta - m\delta, -n\alpha - m\gamma). \qquad (2.10c)$$

4. *Let* $\tilde{m} = 0$ *or* $\tilde{n} = 0$. *Then,*

$$\tilde{m} \otimes \tilde{n} \cong 0. \qquad (2.10d)$$

Scalar multiplication:

Let $\lambda \in \mathbb{R}$. *Then,*

$$\lambda \otimes (m, \alpha, \beta) = \begin{cases} (\lambda m, \lambda\alpha, \lambda\beta), & \lambda \geq 0, \\ (\lambda m, -\lambda\beta, -\lambda\alpha), & \lambda < 0. \end{cases} \qquad (2.11)$$

Some authors discussed the limitations of Dubois and Prade's (1980) approximation for multiplication on *TFNs*. Babbar et al. (2013) claimed that this approximation suitable when the spreads (right and left) of the *TFN* to be negligible if compared to the mean. According to Dubois and Prade (1980), when spreads are not small compared with mean values, other approximation formulas can be used to give the rough shape. Fortin, Dubois and Fargier, (2008) mentioned that this method for multiplication is very suitable for a positive *TFN* only; it can give a closed form result for the basic arithmetic multiplication of positive numbers.

In fuzzy systems, the mean value m may be too remote of spreads to the right α or left β. Furthermore, the sign is not required to be positive all time. Thus, an Approximation for Multiplication was introduced by Kaufmann and Gupta (1991) is used which depends on the min-max function.

Definition 2.2.10. (Kaufmann and Gupta,1991) *Let* $\tilde{m} = (m, \alpha, \beta)$ *and* $\tilde{n} = (n, \gamma, \delta)$, *be two unrestricted TFNs. Then,*

$$\tilde{m} \otimes \tilde{n} = (mn, f_1, f_2), \tag{2.12}$$

where,

$$f_1 = mn - Min\{(m-\alpha)(n-\gamma), (m+\beta)(n-\gamma), (m+\beta)(n+\delta), (m-\alpha)(n+\delta)\},$$

and,

$$f_2 = Max\{(m-\alpha)(n-\gamma), (m+\beta)(n-\gamma), (m+\beta)(n+\delta), (m-\alpha)(n+\delta)\} - mn.$$

2.2.2 Trapezoidal Fuzzy Number

Trapezoidal fuzzy number (*TZFN*) is a generalization to triangular fuzzy number by extended mean value m in *TFN* to produce interval $[m, n]$ which is presented in the next definition.

Definition 2.2.11. *A fuzzy number* $\tilde{A} = (m, n, \alpha, \beta)$ *is said to be a trapezoidal fuzzy number (TZFN) if its membership function is given by,*

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{m-x}{\alpha}, & m-\alpha \leq x \leq m \;\;, \;\; \alpha > 0, \\ 1, & m < x < n, \\ 1 - \dfrac{x-n}{\beta}, & n \leq x \leq n+\beta \;\;, \; \beta > 0, \\ 0, & otherwise. \end{cases} \quad (2.13)$$

A set of all trapezoidal fuzzy numbers is denoted by $T(R)$. $\widetilde{m}$ *is called symmetric if* $\alpha = \beta$. $\tilde{A}$ *is called non trapezoidal fuzzy number* if α, β *is negative or* $m > n$. *Figure 2.2. Displays the* $TZFN$.

The sign of $\widetilde{m} = (m, n, \alpha, \beta)$ *is classified as follows:*

- $\widetilde{m}$ *is called Positive (Negative) iff* $m - \alpha \geq 0$, $(\beta + n \leq 0)$.
- $\widetilde{m}$ *is called Zero if* $(m, n = 0, \alpha, \beta = 0)$.
- $\widetilde{m}$ *is called Near Zero iff* $m - \alpha \leq 0 \leq \beta + n$.

Similarly for *TFN*, there is another very advantageous form for *TZFN*. This form is derived if we suppose,

$$a_1 = m - \alpha, \qquad a_2 = m, \qquad a_3 = n, \qquad a_4 = n + \beta, \qquad (2.14)$$

which can be symbolically written as $\tilde{a} = (a_1, a_2, a_3, a_4)$ or (a, b, c, d).

Then, the membership function for this form is as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ 1, & a_2 < x < a_3, \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_3 \leq x \leq a_4, \\ 0, & otherwise. \end{cases} \tag{2.15}$$

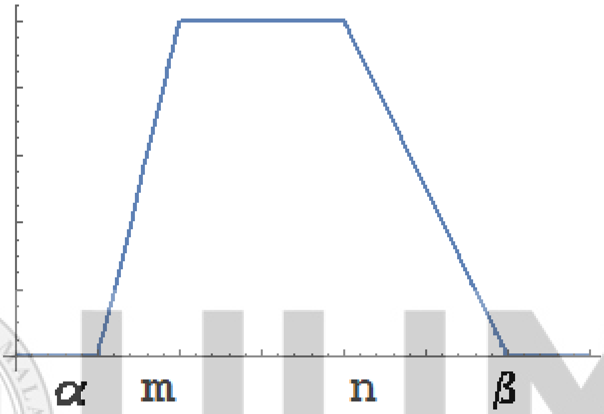

Figure 2.2. Representation of trapezoidal fuzzy number (m, n, α, β)).

Definition 2.2.12. *Two TZFNs* $\tilde{m}_1 = (m_1, n_1, \alpha_1, \beta_1)$ *and* $\tilde{m}_2 = (m_2, n_2, \alpha_2, \beta_2)$ *are called equal, iff* $m_1 = m_2$, $n_1 = n_2$, $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$.

Definition 2.2.13. (Dubois and Prade, 1980) *The arithmetic operations for two fuzzy numbers* $\tilde{m} = (m_1, n_1, \alpha_1, \beta_1)$ *and* $\tilde{n} = (m_2, n_2, \alpha_2, \beta_2)$ *is as follows:*

Addition:

$$\tilde{m} + \tilde{n} = (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2). \tag{2.16}$$

Scalar multiplication:

Let $\lambda \in \mathbb{R}$. Then,

$$\lambda \otimes (m,n,\alpha,\beta) = \begin{cases} (\lambda m, \lambda n, \lambda\alpha, \lambda\beta) & \lambda \geq 0, \\ (\lambda n, \lambda m, -\lambda\beta, -\lambda\alpha) & \lambda < 0. \end{cases} \tag{2.17}$$

Definition 2.2.14. (Allahviranloo et al. 2012a, b) *For two fuzzy number* $\tilde{A} = (m_1, n_1, \alpha_1, \beta_1)$, $\tilde{B} = (m_2, n_2, \alpha_2, \beta_2)$, *we define the distance between* $\tilde{A}$ *and* $\tilde{B}$ *as follows:*

$$d^2(\tilde{A},\tilde{B}) = \frac{[(m_1-m_2)-(\alpha_1-\alpha_2)]^2+[(n_1-n_2)+(\beta_1-\beta_2)]^2+(m_1-m_2)^2+(n_1-n_2)^2}{4}, \tag{2.18}$$

for two fuzzy vectors $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$, $\tilde{Y} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ *we define the distance between* $\tilde{A}$ *and* $\tilde{B}$ *as follows:*

$$D_p(\tilde{X},\tilde{Y}) = \left(\sum_{i=1}^{n} d^p(\tilde{A},\tilde{B})\right)^{\frac{1}{p}}. \tag{2.19}$$

2.3 Linear Fuzzy Systems

In this section a survey of existing methods for solving linear fuzzy systems are reviewed. Some examples are illustrated to show the problems in the existing method.

2.3.1 Fuzzy Linear System

We will start with fuzzy linear system in particular form of fuzzy numbers which considers the simplest fuzzy system, in the definition below.

Definition 2.3.1. *Consider the $n \times n$ linear system,*

$$\begin{cases} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \ldots\ldots + a_{1n}\tilde{x}_n = \tilde{b}_1, \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \ldots\ldots + a_{2n}\tilde{x}_n = \tilde{b}_2 \\ \qquad\qquad\qquad \vdots \\ \qquad\qquad\qquad \vdots \\ a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \ldots\ldots + a_{nn}\tilde{x}_n = \tilde{b}_n, \end{cases} \tag{2.20}$$

where the entries of matrix $A = (a_{ij})_{i,j,=1}^{n}$ are crisp numbers and $\tilde{X} = (\tilde{x}_j)$ and $\tilde{B} = (\tilde{b}_j)$ is a column vector with entries of fuzzy numbers in particular form $\tilde{b}_j = \left((\underline{b}_j(r), \overline{b}_j(r)\right), j = 1, \ldots, n$. This system is called Fuzzy Linear System (FLS). In matrix form the system can be represented as,

$$A \otimes \tilde{X} = \tilde{B}, \tag{2.21}$$

In the next section Friedman's Model (1998) for solving FLS is discussed to indicate the problem in using parametric form.

Friedman Model

Friedman et al. (1998) created a generic model in order to obtain the solution to $n \times n$ FLS. They wrote the FLS in the following function linear system*:*

$$\begin{aligned} \underline{\sum_{j=1}^{n} a_{ij}x_j} &= \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y}_j, \\ \overline{\sum_{j=1}^{n} a_{ij}x_j} &= \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y}_j, \quad j = 1, 2, \ldots, m. \end{aligned} \tag{2.22}$$

They constructed a crisp $2n \times 2n$ linear system,

$$SX = Y, \tag{2.23}$$

instead of the original $n \times n$ FLS, where matrix S is contained in a non−negative coefficients matrix represented as follows:

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix} \ , \ X = \begin{pmatrix} X \\ -\underline{X} \end{pmatrix} \ , \ Y = \begin{pmatrix} Y \\ -\underline{Y} \end{pmatrix}.$$

In the above block matrix $S,$ the matrix B comprises the positive coefficients of the original matrix A along with the absolute values of its negative coefficients C.

Definition 2.3.2. (Friedman et al. 1998) *Let vector* $\tilde{X} = \left\{\left(\underline{x}_j(r), -\bar{x}_j(r)\right), 1 \le i \le n\right\}$ *indicated the unique solution to* FLS. *The fuzzy number vector* $\widetilde{U} = \left\{\left(\underline{u}_j(r), \bar{u}_j(r)\right), 1 \le i \le n\right\}$ *is located by*

$$\begin{aligned} \underline{u}_j(r) &= Min\{\underline{x}_j(r), \bar{x}_j(r), \underline{x}_j(1), \bar{x}_j(1)\}, \\ \bar{u}_j(r) &= Max\{\underline{x}_j(r), \bar{x}_j(r), \bar{x}_j(1), \bar{x}_j(1)\}, \end{aligned} \tag{2.24}$$

as X *is the solution of* $SX = Y$, $\widetilde{U}$ *is the fuzzy solution of original* $A \otimes \tilde{X} = \tilde{Y}$.

The following type of fuzzy solution,

$$\begin{cases} \underline{u}_j(r) = \underline{x}_j(r), \\ \bar{u}_j(r) = \overline{x}_j(r), \end{cases} \qquad for\ 1 \le i \le n,$$

is named a strong fuzzy solution. Otherwise, a fuzzy solution is named a weak fuzzy solution.

Allahviranloo (2004, 2005) and Allahviranloo and Kermani (2006) utilized Friedman's model, where he proposed a solution of FLS by using well−known iterative methods like Gauss Seidel and Jacobi. They illustrated $SX = Y$ by a numerical example where the unique solution is either a weak or strong fuzzy number. Allahviranloo (2005), argued that by using the Gauss–Sidel method, where the systems are not convergent, he proposed the solution of FLS by using another iterative method which is called Successive Over Relaxation (SOR). In the same year, Allahviranloo (2005) completed his work in iterative methods by comparing the previous result with Adomian decomposition method; in order to enhance his result in Allahviranloo (2004), he proved that the Jacobi iterative method and Adomian are equivalent.

As for Dehghan and Hashemi (2006b), in order to solve FLS, they extend several numerical algorithms for solving linear systems, such as Extrapolated Richardson, Richardson, Gaus–Seidel, JOR, Jacobi, SOR, EGS, ESOR, SSOR, AOR, USSOR, MSOR and EMA. These methods were followed by convergence theorems, and proved some results that were studied by Allahviranloo (2004). On the other hand, based on Friedman's model, many studies used several numerical methods to solve FLS (Matinfar, Nasseri and Sohrabi, (2008), Dafchahi, (2008), Jafari, Saeidy and Vahidi, (2009), Lazim, and Hakimah, (2010), Ezzati, (2011), Najafi and Edalatpanah, (2012) and Amirfakhrian, (2012)).

However, Allahviranloo et al. (2011a) said that "fuzzy linear systems" of Friedman model (1998) almost never have a genuine solution and hence this research line is totally barren. Allahviranloo et al. (2011a) have indicated that by a counterexample, the definition for weak fuzzy solution does not always yield a fuzzy number vector. It is therefore clear that the definition of weak fuzzy solution by Friedman et al.

(1998) cannot be used any more. Consequently, Friedman's model for solving fuzzy linear systems almost never had a genuine solution.

In the next section, a new fuzzy system $(LR - FLS)$ is considered as generalization of FLS, with triangular fuzzy number, to avoid the limitations in FLS.

2.3.2 Left−Right Fuzzy Linear System

LR fuzzy linear system $(LR - FLS)$ is a linear system whose right hand side $\tilde{B}$ is a triangular fuzzy number vector. This fuzzy system is simplest linear system including triangular fuzzy number as definition below.

Definition 2.3.3. *(Left−Right fuzzy linear system) Consider the $n \times n$ linear system,*

$$\begin{cases} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \ldots\ldots + a_{1n}\tilde{x}_n = \tilde{b}_1, \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \ldots\ldots + a_{2n}\tilde{x}_n = \tilde{b}_2 \\ \vdots \\ \vdots \\ a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \ldots\ldots + a_{nn}\tilde{x}_n = \tilde{b}_n, \end{cases} \tag{2.25}$$

it can be written in matrix form,

$$A \otimes \tilde{X} = \tilde{B}, \tag{2.26}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{pmatrix},$$

this system is called a Left−Right fuzzy linear system $(LR - FLS)$ where the coefficient matrix $A = (a_{ij})$, $1 \le i, j \le n$ is a crisp matrix, the entries of fuzzy vectors $\tilde{X}, \tilde{B}$ are TFNs, and $\tilde{X}$ is unknown to be found.

The interest on $LR - FLS$ has been increasing in the last five years for three main reasons, namely:

1. In applications, $LR - FLS$ is easier to understand and applied since triangular fuzzy number does not have parameter $r, 0 \leq r \leq 1$ as particular form fuzzy number in FLS, because *TFN* is a collection of three ordered crisp numbers.
2. Allahviranloo's (2011a) argument on the weak solution of Friedman'model (1998) has been showed that many cases and examples were wrongly solved $LR - FLS$.
3. $LR - FLS$ is considered as particular scenarios of $FFLS$, where all spreads are zero in left hand side.

For these reasons, $LR - FLS$ is considered as an alternative of FLS.

In next section $FFLS$ is presented as expanding of FLS and $LR - FLS$, where all the entries are *TFNs*.

2.3.3 Fully Fuzzy Linear System

The interest in $FFLS$ is due to the needs to widen the scope of FLS and $LR - FLS$, as it is found in many applications there is no control to fix coefficients of matrix A similar to FLS or $LR - FLS$.

Definition 2.3.4. *Consider the $n \times n$ linear system,*

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \ldots\ldots + \tilde{a}_{1n}\tilde{x}_n = \tilde{b}_1, \\ \tilde{a}_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \ldots\ldots + \tilde{a}_{2n}\tilde{x}_n = \tilde{b}_{2,} \\ \qquad\qquad\vdots \\ \qquad\qquad\vdots \\ \tilde{a}_{n1}\tilde{x}_1 + \tilde{a}_{n2}\tilde{x}_2 + \ldots\ldots + \tilde{a}_{nn}\tilde{x}_n = \tilde{b}_n, \end{cases} \tag{2.27}$$

where $\forall\ \tilde{a}_{ij}, \tilde{b}_j \in G(R)$. *This system is called a Fully fuzzy linear system* ($FFLS$).

The matrix $\tilde{A} = (\tilde{a}_{ij})_{i,j=1}^{n}$ *and the vector* $\tilde{B} = (\tilde{b}_j)_{j=1}^{n}$ *can be represented as,*

$$\tilde{A} \otimes \tilde{X} = \tilde{B}. \tag{2.28}$$

The vector $\tilde{X} = (\tilde{x}_j)_{j=1}^{n}$ *is called exact fuzzy solution if* $\forall \tilde{x}_j \in G(R)$, $j = 1, \dots, n$, *otherwise it is called exact non fuzzy solution.*

In the literature, many methods were obtained to solve the scenarios of $FFLS$, it is noted the methods for solving $FFLS$ applied the approximations multiplication operators numbers (Dubois and Prade's or Kaufmann's approximation for multiplication).

These methods are classified according approximation formula for multiplication, direct methods which used Dubois and Prade's approximate and indirect methods which used Kaufmann's approximation for multiplication.

2.3.3.1 Direct methods for solving $FFLS$

Through direct methods, the solution directly obtained by applying a multiplication operators from Dubois and Prade's approximate arithmetic and then solving the associated linear system for $FFLS$.

For example, if all entries of fuzzy matrices $\tilde{A}$ and $\tilde{X}$ are positives $TFNs$, we need only Equations (2.9a) and (2.10a) to obtain the associated linear system. Although the scenarios of $FFLS$ which can be solved by direct methods are very restricted, these methods have a wide scientific applications in real life situations, such as supposing the time as a parameter, then $P-\tilde{X}$ is required for $P-FFLS$ or $G-FFLS$.

Next theorem provides the conditions of $P-FFLS$ to have $P-\tilde{X}$.

Theorem 2.3.1. (Dehghan et al. 2006) *Let* $\tilde{A}=(A,M,N)$, $\tilde{B}=(b,h,g)\geq 0$ *in* $FFLS$

$\tilde{A}\otimes\tilde{X}=\tilde{B}$ *, and:*

i- Centric matrix A *be a non negative−inverse, i.e.* A^{-1} *exist and* $A^{-1}>0$.

ii- $h\geq MA^{-1}b$ *and* $g\geq NA^{-1}b$.

iii- $(MA^{-1}+I)b\geq h$.

Then, the FLLS has $P-\tilde{X}$.

Dehghan et al. (2006) used the direct methods in solving $P-\tilde{X}$ for $P-FFLS$ to obtain a technique similar to the classic methods derived in linear algebra, such as the

LU decomposition and Cramer's rule (with its explanation to find the approximated solution of a system). Apart from that, they also proposed a new method using *LP* in order to obtain the solution of non−square, square and matrix (over−determined) fuzzy systems. The iterative techniques like Gauss–Seidel and Jacobi Adomian decomposition method are also expanded by Dehghan and Hashemi (2006b), Dehghan, et al. (2007). Similarly, Nasseri, Sohrabi and Ardil (2008), Nasseri and Zahmatkesh (2010), Nasseri, Behmanesh and Sohrabi (2012), Gao and Zhang

(2009), and Kumar, Neetu and Bansal (2012), Abbasbandy and Hashemi (2012). Ezzati, Khezerloo, Mahdavi−Amiri and Valizadeh (2014) and Abdolmaleki and Edalatpanah (2014) in all their studies proposed new methods for solving $P - FFLS$ are similar to Dehghan's method, all the previous studies depended on Theorem 2.3.1. In addition, Kumar et al. (2012) added a contribution in these methods by investigating the unique and infinite many solutions.

Liu (2010) found $P - \tilde{X}$ for $G - FFLS$ by Homotopy Perturbation Methods (HPM), in his study, followed the three multiplication operators in Dubois and Prade's approximate arithmetic, since it is easier to apply in numerical method. However, he used a small fuzziness α, β comparing with m. He avoided dealing with near zero triangular fuzzy number.

The problem in direct methods for solving $FFLS$

The following limitations are noted in the direct methods:

1. All direct methods depend only on Dubois and Prade's approximate arithmetic, which means that the direct fuzzy approximation solution will be obtained without taking into consideration the distance of spreads α and β about the mean m. The evidence for this, can be seen in most examples in previous studies that takes the spreads α and β to be very small, as compared with mean m. For example the fuzziness in fuzzy numbers $\tilde{8} = (8, 0.1, 0.2)$ and $\widetilde{50} = (50, 1, 3)$ are very small, in fact, these fuzzy numbers are very close to crisp numbers 8 and 50, respectively, based on Remark 2.2.1. if the spreads α and β increase in $\widetilde{m} = (m, \alpha, \beta)$, $\widetilde{m}$ becomes fuzzier and fuzzier.

2. The guarantee for existing solution by these methods is very small. For example, the discussion in Chapter Three indicates that Theorem 2.3.1. is limited to $A^{-1} > 0$. Until now, there is no numerical example investigated by Theorem 2.3.1.
3. The guarantee of fuzziness solution must depend on $\tilde{A}$ as well as $\tilde{B}$. The evidence for that, until now, there is no numerical example that can guarantee $P - \tilde{X}$ for an $P - FFLS$. Moreover some methods provided an $P - \tilde{X}$ where the system doesn't have $P - \tilde{X}$ as for $P - FFLS$ such as examples in Abbasbandy and Hashemi (2012).
4. Some existing methods are incomplete and have many flaws, as shown in the below examples. Many of these methods provide incorrect examples because the right hand is not equal to the left hand which is discussed in Chapter Three such as examples in Abbasbandy and Hashemi (2012) and example in $P - \tilde{X}$ for an $G - FFLS$ (Liu, 2010).

2.3.3.2 Indirect methods for solving $FFLS$

Indirect methods can obtain a solution by applying Kaufmann's approximation for multiplication, but there is no guarantee that the associated system will be a corresponding linear system for $FFLS$. Such that, Kumar et al. (2011a), Otadi and Mosleh (2012) and Babbar et al. (2013) their methods depend on transferring the fuzzy system to LP or NLP.

Kumar et al. (2011a) found the $P - \tilde{X}$ for $G - FFLS$. The $n \times n$ $FFLS$ is transferred to a $3n \times 3n$ classical linear system by applying Kaufmann's approximation, they calculated all multiplication by fuzzy operation between fuzzy matrices, thus requiring more computational times. Then, they studied the consistency of the system. However, this method is unable to treat near zero TFN (neither positive nor negative). It is noted that in the two examples illustrated in his paper, each system is neither fully positive nor fully negative TFN. Moreover, all size of $FFLS$ examples do not exceed $n = 2$.

Otadi et al. (2011) obtained $G - \tilde{X}$ for $G - FFLS$ using numerical method that depends on neural network. However, their method is restricted by many constraints, the method solves $FFLS$ when the system has a unique fuzzy solution only and the fuzzy matrix $\tilde{A}$ has to be square. These constraints prevent the extension of the method to solve singular or rectangle matrix to provide infinity many solution when it is available.

Later, Ezzati et al. (2012) proposed method to find $P - \tilde{X}$ for $NZ - FFLS$ by transferring the $FFLS$ into two $n \times n$ and $2n \times 2n$ crisp linear systems, they employed a least squares problem to find an approximated fuzzy solution. Actually, his method produced inaccurate solution as shown in the example in Chapter Four, since the left hand side is not equal to the right hand side.

In addition, Otadi and Mosleh (2012) investigated $P - \tilde{X}$ for $NZ - FFLS$ by employing LP, which leads to more computational times, their method involves much arithmetic fuzzy operations. Because of that, they only provided the size of $FFLS$ that do not exceed $n = 2$. This method cannot check the possibilities of solution since it provides optimal solution only using LP. Moreover this method is constructed to solve fully fuzzy matrix equation ($FFME$),

Kumar et al. (2011b) and Babbar et al. (2013) found the $NZ-\tilde{X}$ for $NZ-FFLS$ by LP, the methods are valid for square and rectangle systems such as $n=3$ $FFLS$. The solution is obtained in six steps. In the fifth step, where the $FFLS$ was transferred to equivalent LP in order to solve, the required constraints added to guarantee the fuzzy solution. This produces complicated LP since it needed $3n+2n^2$ constraints in for $n \times n$ $FFLS$, with conditions of obtained solution to be a fuzzy solution. So their examples do not exceed $n=3$. Also, they found $G-\tilde{X}$ for $G-FFLS$ by fuzzy Cramer's. However they proposed the same consistency in the solution of $FFLS$ and the equivalent LP.

They introduced Remark 2.3.1, but the contradiction for this remark is located in Chapter Five by counter example.

Allahviranloo, et al. (2014) transformed $FFLS$ for two fully interval linear systems, then to LP, this method required to construct more than 6 different systems, the final solution provided by *NLP*, moreover their method cannot provide more than one solution.

The problems in Indirect Methods:

1- It is apparent that the previous methods treating possible ways of solutions for $FFLS$ is similar to possible ways of solution for linear systems or LP. The concept of uniqueness of solution for systems produced by applying Kaufmann's approximation for multiplication. But this approximation can provide two different unique solutions (finite solution), unlike the uniqueness concept in linear system where the uniqueness happens by one solution. Similarly, the infinite many solutions case may happen through min or max conditions, unlike the infinite many solutions in linear systems which is based on the consistency of the system. Moreover, the $FFLS$ may have two solutions only and not infinity many solutions. The discussion in Chapter Five is contrary with their following Remark 2.3.1. provided in Kumar et al. (2011b) and Babbar et al. (2013).

Remark 2.3.1. The existence of the $FFLS$ solutions relies on the existence of the solutions of the LP which may be no solution, trivial, unique, or infinite many solutions (Babbar et al. 2012, Kumar et al. 2011b).

2- Many of the methods in Kumar et al. (2011b), Babbar et al. (2013) and Otadi and Mosleh (2012), Allahviranloo, et al. (2014) relied heavily on LP and NLP to solve the $FFLS$ without separation in algorithms between exact ($\tilde{X}$) and approximate ($\tilde{X}'$) fuzzy solution. To overcome this mixture in algorithms, they added further constraints on the systems. As stated before, to solve

$n \times n$ $FFLS$, for example in Babbar et al. (2013) the associated LP must have $3n + 2n^2$ constraints. All provided examples in the previous methods do not exceeded $n = 4$, which indicate the insufficiency of these methods to provide solution for $FFLS$ in more than three parameters, because any example more than $n = 4$ needs high constraints as $(3)(5) + (2)(5^2) = 65$ for 5×5 $FFLS$, these constrains are computed by fuzzy operations. In addition, Allahviranloo, et al. (2014)'s method also has many fuzzy operation since the final solution gained by solving 6 systems, moreover the proposed solution is not satisfied the system as it is showed in Chapter Five.

3- Some methods, such as Otadi et al. (2011), required information about the solution to choose suitable initial data before solving the system. This means the method is insufficient to solve arbitrary systems, when these date are not available.

The next remark will be used in indirect methods to solve $NZ - FFLS$.

Remark 2.3.2. For any variables x and y. $Min[x,y]$ and $Max[x,y]$ denote the minimum and maximum of x and y, respectively as follows,

$$Min[x,y] = \left(\frac{x+y}{2}\right) - \left|\frac{x-y}{2}\right|, \quad Max[x,y] = \left(\frac{x+y}{2}\right) + \left|\frac{x-y}{2}\right|. \tag{2.29}$$

2.3.4 Fully Fuzzy Matrix Systems

This section is considered as an extension to $FFLS$ scenarios, since all coefficients are fuzzy numbers. There are many linear matrix equations that can be formed in this section. The most important two linear matrix equations are fully fuzzy matrix system as well as fully fuzzy Sylvester equation, where the other fuzzy linear matrices can be formed by them.

Definition 2.3.5. *(Fully Fuzzy Matrix System) Let* $\tilde{A} = (\tilde{a}_{ij})_{i,j=1}^{n} = (m_{ij}^{a}, \alpha_{ij}^{a}, \beta_{ij}^{a})_{i,j=1}^{n}$ *and* $\tilde{B}_m = (\tilde{b}_{ij})_{i,j=1}^{n} = (m_{ij}^{b}, \alpha_{ij}^{b}, \beta_{ij}^{b})_{i,j=1}^{n}$ *are known positive fuzzy matrices,* $\tilde{X}_m = (\tilde{x}_{ij})_{i,j=1}^{n} = (m_{ij}^{x}, \alpha_{ij}^{x}, \beta_{ij}^{x})_{i,j=1}^{n}$ *is an unknown positive fuzzy matrix,*

$$\tilde{A}_m \otimes \tilde{X}_m = \tilde{B}_m, \tag{2.30}$$

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nn} \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \dots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \dots & \tilde{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{n1} & \tilde{x}_{n2} & \cdots & \tilde{x}_{nn} \end{pmatrix} = \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \dots & \tilde{b}_{1n} \\ \tilde{b}_{21} & \tilde{b}_{22} & \dots & \tilde{b}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{b}_{n1} & \tilde{b}_{n2} & \cdots & \tilde{b}_{nn} \end{pmatrix},$$

is called fully fuzzy matrix Equation ($FFME$).

The next equation is most important linear matrix equation in Control theory is named Sylvester equation.

Definition 2.3.6. *(Fully Fuzzy Sylvester Equation) Consider the following fuzzy matrix equation $\tilde{A}, \tilde{B}, \tilde{X}$ and $\tilde{C}$ where $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, $\tilde{X} = (\tilde{x}_{ij})_{n\times m}$, $\tilde{B} = (\tilde{b}_{ij})_{m\times m}$ and*

$\tilde{C} = (\tilde{c}_{ij})_{n\times m}$. The equation below is called fully fuzzy Sylvester equation ($FFSE$),

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}. \tag{2.31}$$

The methods of solving fully fuzzy matrix systems is an extension form $FFLS$, for that all stated problem in direct and indirect methods for solving $FFLS$ are effective for solving fuzzy matrix equation such as methods in (Otadi and Mosleh (2012) and Kargar, Allahviranloo, Rostami-Malkhalifeh and Jahanshaloo (2014). In addition, there are two further problems:

1. The arithmetic fuzzy matrices multiplication are computed between fuzzy matrices to produce $FFLS$, such as, Guo and Shang (2013a,b) which required many more computational time due to applying fuzzy operation.

2. Basic definitions and operation in crisp matrices such as identity and transpose are not developed to fuzzy matrices, where these definitions are essential to establish required theorems. Such as Kronecker product cannot be applied between fuzzy and crisp matrices. Because of that, in Guo and Jin (2014) the fuzzy matrix equations are separated to solve each values m, α and β, then collected again in one system.

CHAPTER THREE

POSITIVE SOLUTION FOR POSITIVE FULLY FUZZY LINEAR SYSTEM

This chapter develops associated linear system to obtain $P-\tilde{X}$ of $P-FFLS$. In order to develop this linear system, block matrix and block vectors will be used to include all entries of $P-FFLS$ into the associated linear system. A necessary conditions to ensure $P-\tilde{X}$ are derived. The consistency of the fully fuzzy linear system is provided.

3.1 Fundamental Concepts for Associated Linear System

This section presents the fundamental definition and main theorem to develop the new method for solving $P-\tilde{X}$ of $P-FFLS$, the next definition is used in associated linear system for $P-FFLS$.

Definition 3.1.1. *A $3n\times 3n$ crisp matrix S consists of $n\times n$ zero matrix $Z=(0)_{n\times n}$ and three $n\times n$ crisp matrices $A=\left(m^{a}_{i,j}\right)_{n\times n}$,$M=\left(\alpha^{a}_{i,j}\right)_{n\times n}$ and $N=\left(\alpha^{a}_{i,j}\right)_{n\times n}$ of fuzzy matrix $\tilde{A}=\left(\tilde{a}_{ij}\right)_{n\times n}=\left(m^{a}_{i,j},\alpha^{a}_{i,j},\beta^{a}_{i,j}\right)_{n\times n}$ is defined as,*

$$S=\begin{pmatrix} A & Z & Z \\ N & A & Z \\ M & Z & A \end{pmatrix}=\begin{pmatrix} \begin{pmatrix} m^{a}_{11} & \dots & m^{a}_{1n} \\ \vdots & \ddots & \vdots \\ m^{a}_{n1} & \cdots & m^{a}_{nn} \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} \alpha^{a}_{11} & \dots & \alpha^{a}_{1n} \\ \vdots & \ddots & \vdots \\ \alpha^{a}_{n1} & \cdots & \alpha^{a}_{nn} \end{pmatrix} & \begin{pmatrix} m^{a}_{11} & \dots & m^{a}_{1n} \\ \vdots & \ddots & \vdots \\ m^{a}_{n1} & \cdots & m^{a}_{nn} \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} \beta^{a}_{11} & \dots & \beta^{a}_{1m} \\ \vdots & \ddots & \vdots \\ \beta^{a}_{m1} & \cdots & \beta^{a}_{mm} \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} m^{a}_{11} & \dots & m^{a}_{1n} \\ \vdots & \ddots & \vdots \\ m^{a}_{n1} & \cdots & m^{a}_{nn} \end{pmatrix} \end{pmatrix}.$$

The matrix S is called the associated matrix of fuzzy matrix $\tilde{A}$.

The next theorem shows that the singularity of matrix S depends on the singularity of matrix A.

Theorem 3.1.1. Block matrix S is non-singular if and only if the matrix A in is non-singular.

Proof. The crisp matrices A, M and N are square matrices in common order $n,$ so we can easily make the block matrix S becomes a diagonal matrix, using elementary operation of partitioned matrices in Section 2.1.1, by subtracting the first row multiplied by MA^{-1} from the second row, and subtracting the first row multiplied by NA^{-1} from the third row, is as follows:

$$S = \begin{pmatrix} A & 0 & 0 \\ M & A & 0 \\ N & 0 & A \end{pmatrix} \rightarrow \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix} = S_1.$$

Now, we will expand S_1 through 3 block matrices of order $3n \times 3n,$ which are, $\{E_1,\ E_2,\ E_3\}$ where,

$$E_1 = \begin{pmatrix} A & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix}, E_2 = \begin{pmatrix} I_n & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_n \end{pmatrix}, E_3 = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & A \end{pmatrix}.$$

Hence,

$$S_1 = E_1 E_2 E_3.$$

Clearly,

$$|S| = |S_1| = |E_1||E_2||E_3| = |A|^3.$$

Therefore,

$$|S| \neq 0 \text{ if and only if } |A| \neq 0. \tag{3.1}$$

Moreover,

$$|S| = |A|^3 \,. \tag{3.2}$$

□

Next section provides method to obtain the $P-\tilde{X}$ for $P-FFLS$, using the block matrix S in Definition 3.1.1.

3.2 Positive Solution For Positive Fully Fuzzy Linear System

The $P-\tilde{X}$ for $P-FFLS$ is obtained in this section. The $P-FFLS$ is transferred to linear system then to matrix form. The solution of this linear system provides a vector X is equivalent to $P-\tilde{X}$.

The solution is obtained in following two steps:

Step 1 Transferring the fuzzy system $P-FFLS$ to crisp matrices and vectors.

Consider $\tilde{A} \otimes \tilde{X} = \tilde{B}$, where $\tilde{A} = (A, M, N) \geq 0$, and $\tilde{B} = (m^b, \alpha^b, \beta^b)$, $\tilde{X} = (m^x, \alpha^x, \beta^x) \geq 0$, using Equations $(2.9a)$ and $(2.10a)$, we have,

$$\tilde{A} \otimes \tilde{X} = \begin{cases} A \otimes m^x = m^b, & \Rightarrow \quad Am^x = m^b, \\ M \otimes \alpha^x = \alpha^b, & \Rightarrow \quad A\alpha^x + Mm^x = \alpha^b, \\ N \otimes \beta^x = \beta^b, & \Rightarrow \quad A\beta^x + Nm^x = \beta^b. \end{cases} \tag{3.3}$$

The system in Equation (3.3) is written through its coefficients as follow:

The crisp matrices are,

$$A = \left(m_{i,j}^{a}\right)_{n\times n}, M = \left(\alpha_{i,j}^{a}\right)_{n\times n}, N = \left(\alpha_{i,j}^{a}\right)_{n\times n}.$$

The crisp vectors are,

$m^{x} = \left(m_{j}^{x}\right)_{n\times 1}$, $\alpha^{x} = \left(\alpha_{j}^{x}\right)_{n\times 1}$, $\beta^{x} = \left(\beta_{j}^{x}\right)_{n\times 1}$ and $m^{b} = \left(m_{j}^{b}\right)_{n\times 1}$, $\alpha^{b} = \left(\alpha_{j}^{b}\right)_{n\times 1}$, $\beta^{b} = \left(\beta_{j}^{b}\right)_{n\times 1}$.

Hence, Equation (3.3) is equivalent to following $3n$ linear equations,

$$\begin{cases} (m_{i1}^{a}m_{1}^{x} + m_{i2}^{a}m_{2}^{x} + \cdots + m_{in}^{a}m_{n}^{x}) = m_{i}^{b}, \\ (m_{i1}^{a}\alpha_{1}^{x} + m_{i2}^{a}\alpha_{2}^{x} + \cdots + m_{in}^{a}\alpha_{n}^{x}) + (\alpha_{i1}^{a}m_{1}^{x} + \alpha_{i2}^{a}m_{2}^{x} + \cdots + \alpha_{in}^{a}m_{n}^{x}) = \alpha_{i}^{b}, \\ (m_{i1}^{a}\beta_{1}^{x} + m_{i2}^{a}\beta_{2}^{x} + \cdots + m_{in}^{a}\beta_{n}^{x}) + (\beta_{i1}^{a}m_{1}^{x} + \beta_{i2}^{a}m_{2}^{x} + \cdots + \beta_{in}^{a}m_{n}^{x}) = \beta_{i}^{b}. \end{cases} \tag{3.4}$$

Rearranging Equation (3.4) and adding zero terms to make it $3n \times 3n$ linear system,

$$\begin{cases} (m_{i1}^{a}m_{1}^{x} + m_{i2}^{a}m_{2}^{x} + \cdots + m_{in}^{a}m_{n}^{x}) + 0\cdots 0 + 0\cdots 0 = m_{i}^{b}, \\ \\ (m_{i1}^{a}\alpha_{1}^{x} + m_{i2}^{a}\alpha_{2}^{x} + \cdots + m_{in}^{a}\alpha_{n}^{x}) + 0\cdots 0 \\ +(\alpha_{i1}^{a}m_{1}^{x} + \alpha_{i2}^{a}m_{2}^{x} + \cdots + \alpha_{in}^{a}m_{n}^{x}) = \alpha_{i}^{b}, \\ \\ (m_{i1}^{a}\beta_{1}^{x} + m_{i2}^{a}\beta_{2}^{x} + \cdots + m_{in}^{a}\beta_{n}^{x}) + 0\cdots 0 \\ +(\beta_{i1}^{a}m_{1}^{x} + \beta_{i2}^{a}m_{2}^{x} + \cdots + \beta_{in}^{a}m_{n}^{x}) = \beta_{i}^{b}. \end{cases} \tag{3.5}$$

Step 2 Assigning $3n \times 3n$ linear system which equivalent to fuzzy system.

- Assign the coefficients of block matrix $S = \left(s_{ij}\right)_{3n \times 3n}$:

$$\begin{cases} s_{i,j} = s_{i+n,j+n} = s_{i+2n,j+2n} = m_{i,j}^{a}, \\ s_{i+n,j} = \alpha_{i,j}^{a}, \\ s_{i+2n,j} = \beta_{i,j}^{a}, \end{cases} \tag{3.6}$$

The others s_{ij} are not determined in Equation (3.6) are zero.

- Assign the vectors $X = \left(x_j\right)_{3n \times 1}$ and $B = (b_i)_{3n \times 1}$:

$$\begin{cases} x_j = m_j^{x} \\ x_{j+n} = \alpha_j^{x}, \\ x_{j+2n} = \beta_j^{x}. \end{cases} \tag{3.7a}$$

$$\begin{cases} b_i = m_i^{b} \\ b_{i+n} = \alpha_i^{b}, \\ b_{i+2n} = \beta_i^{b}. \end{cases} \tag{3.7b}$$

Using Equations (3.6), (3.7a) and (3.7b), we have the following $3n \times 3n$ linear system,

$$SX = B,$$

$$\begin{pmatrix}
\begin{pmatrix} m_{11}^{a} & m_{12}^{a} & \dots & m_{1n}^{a} \\ m_{21}^{a} & m_{22}^{a} & \dots & m_{2n}^{a} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1}^{a} & m_{n2}^{a} & \cdots & m_{nn}^{a} \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\
\begin{pmatrix} \alpha_{11}^{a} & \alpha_{12}^{a} & \dots & \alpha_{1n}^{a} \\ \alpha_{21}^{a} & \alpha_{22}^{a} & \dots & \alpha_{2n}^{a} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}^{a} & \alpha_{n2}^{a} & \cdots & \alpha_{nn}^{a} \end{pmatrix} & \begin{pmatrix} m_{11}^{a} & m_{12}^{a} & \dots & m_{1n}^{a} \\ m_{21}^{a} & m_{22}^{a} & \dots & m_{2n}^{a} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1}^{a} & m_{n2}^{a} & \cdots & m_{nn}^{a} \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\
\begin{pmatrix} \beta_{11}^{a} & \beta_{12}^{a} & \dots & \beta_{1n}^{a} \\ \beta_{21}^{a} & \beta_{22}^{a} & \dots & \beta_{2n}^{a} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1}^{a} & \beta_{n2}^{a} & \cdots & \beta_{nn}^{a} \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} m_{11}^{a} & m_{12}^{a} & \dots & m_{1n}^{a} \\ m_{21}^{a} & m_{22}^{a} & \dots & m_{2n}^{a} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1}^{a} & m_{n2}^{a} & \cdots & m_{nn}^{a} \end{pmatrix}
\end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ \vdots \\ m_n^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \vdots \\ \alpha_n^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \vdots \\ \beta_n^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} m_1^b \\ m_2^b \\ \vdots \\ m_n^b \end{pmatrix} \\ \begin{pmatrix} \alpha_1^b \\ \alpha_2^b \\ \vdots \\ \alpha_n^b \end{pmatrix} \\ \begin{pmatrix} \beta_1^b \\ \beta_2^b \\ \vdots \\ \beta_n^b \end{pmatrix} \end{pmatrix}.$$

$$\begin{pmatrix} A & 0 & 0 \\ N & A & 0 \\ M & 0 & A \end{pmatrix} \begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} m^b \\ \alpha^b \\ \beta^b \end{pmatrix}, \tag{3.8}$$

where X and B are called the associated crisp vectors of fuzzy vectors $\tilde{X}$ and $\tilde{B},$ respectively. The linear system in Equation (3.8) is called positive associated linear system ($P - ALS$), where $P - ALS$ and linear systems in (3.4) are equivalent. Hence, $P - ALS$ is equivalent to $\tilde{A} \otimes \tilde{X} = \tilde{B}$. Using Theorem 3.1.1. the singularity of matrix A is examined, if $|A| \neq 0$ then $|S| \neq 0$. Hence the linear system $SX = B$ has a unique solution if, $|A| \neq 0$.

By solving the $P - ALS$ in Equation (3.8) we can find the crisp parameters m_j^x, α_j^x and β_j^x for $j = 1, \dots, n,$ which are equivalent to $P - \tilde{X}$ of $P - FFLS$, $\tilde{x}_j = (m_j^x, \alpha_j^x, \beta_j^x)$.

Moreover, in Section 3.4, this method is developed to obtain infinitely many solutions whenever it exist, where $|A| = 0$ or A is rectangle. Also, to provide interval of solution when some coefficients are unknown.

Remark 3.1.1. The Crisp vector X and fuzzy vector $\tilde{X}$ are equivalent if components m_j^x, α_j^x and β_j^x belong in X and $\tilde{X}$ are equal for all $j = 1, \dots, n$.

Theorem 3.2.1. The unique solution of crisp system $SX = B$ and $P - \tilde{X}$ for $P - FFLS$ is equivalent.

Proof. The proof is straightforward from by Theorem 3.1.1 and Equation (3.8)

Two examples were in literature are solved, the first example has a unique fuzzy solution, while the second has a non fuzzy unique solution; which shows that *P-ALS* can provide the exact fuzzy or non fuzzy solution. The next example is the same example in Dehghan et al. (2006) where the fuzzy system has a fuzzy unique solution. The new method proposed a similar solution for Dehghan et al. (2006).

Examples 3.2.1. Dehghan et al. (2006) consider following $P - FFLS$,

$$\begin{cases} (6,1,4) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (5,2,2) \otimes (m_2^x, \alpha_2^x, \beta_2^x) \oplus (3,2,1) \\ \otimes (m_3^x, \alpha_3^x, \beta_3^x) = (58,30,60), \\ (12,8,20) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (14,12,15) \otimes (m_2^x, \alpha_2^x, \beta_2^x) \oplus (8,8,10) \\ \otimes (m_3^x, \alpha_3^x, \beta_3^x) = (142,139,257), \\ (24,10,34) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (32,30,30) \otimes (m_2^x, \alpha_2^x, \beta_2^x) \oplus (20,19,24) \\ \otimes (m_3^x, \alpha_3^x, \beta_3^x) = (316,297,514). \end{cases}$$

The $P - \tilde{X}$ is obtained using *P-ALS*.

In matrix form, this become,

$$\begin{pmatrix} (6,1,4) & (5,2,2) & (3,2,1) \\ (12,8,20) & (14,12,15) & (8,8,10) \\ (24,10,34) & (32,30,30) & (20,19,24) \end{pmatrix} \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (58,30,60) \\ (142,139,257) \\ (316,297,514) \end{pmatrix}.$$

According to Step 1, the crisp matrices A, M and N are,

$$A=\begin{pmatrix}6&5&3\\12&14&8\\24&32&20\end{pmatrix},M=\begin{pmatrix}1&2&2\\8&12&8\\10&30&19\end{pmatrix},N=\begin{pmatrix}4&2&1\\20&15&10\\34&30&24\end{pmatrix}.$$

The crisp vectors $m^x, m^b, \alpha^x, \alpha^b, \beta^x$ and β^b are,

$$m^x=\begin{pmatrix}m_1^x\\m_2^x\\m_3^x\end{pmatrix},\alpha^x=\begin{pmatrix}\alpha_1^x\\\alpha_2^x\\\alpha_3^x\end{pmatrix},\beta^x=\begin{pmatrix}\beta_1^x\\\beta_2^x\\\beta_3^x\end{pmatrix},m^b=\begin{pmatrix}58\\142\\316\end{pmatrix},\alpha^b=\begin{pmatrix}30\\139\\297\end{pmatrix},\beta^b=\begin{pmatrix}60\\257\\514\end{pmatrix},$$

Since $|A|=48,\ |A|\neq 0$, according to Theorem 3.1.1, $|S|\neq 0$. Hence, the $FFLS$ has a unique solution.

According to Step 2, the *P-ALS* can be constructed by $SX=B$ in (3.8), where S is,

$$S=\begin{pmatrix}A&0&0\\M&A&0\\N&0&A\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}6&5&3\\12&14&8\\24&32&20\end{pmatrix}&\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\\\begin{pmatrix}1&2&2\\8&12&8\\10&30&19\end{pmatrix}&\begin{pmatrix}6&5&3\\12&14&8\\24&32&20\end{pmatrix}&\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\\\begin{pmatrix}4&2&1\\20&15&10\\34&30&24\end{pmatrix}&\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}6&5&3\\12&14&8\\24&32&20\end{pmatrix}\end{pmatrix}.$$

Also X and B are,

$$X=\begin{pmatrix}m^x\\\alpha^x\\\beta^x\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}m_1^x\\m_2^x\\m_3^x\end{pmatrix}\\\begin{pmatrix}\alpha_1^x\\\alpha_2^x\\\alpha_3^x\end{pmatrix}\\\begin{pmatrix}\beta_1^x\\\beta_2^x\\\beta_3^x\end{pmatrix}\end{pmatrix},\quad B=\begin{pmatrix}m^b\\\alpha^b\\\beta^b\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}58\\142\\316\end{pmatrix}\\\begin{pmatrix}30\\139\\297\end{pmatrix}\\\begin{pmatrix}60\\257\\514\end{pmatrix}\end{pmatrix}.$$

However, using Equation (3.2) our approach in Theorem 3.1.1, can be verified by computing $|S|$.

$$|S| = |A|^3 = (48)^3 = 110592.$$

The crisp solution can be easily obtained by inversion matrix method, $X = S^{-1}B$:

$$X = \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0.5 \\ 0.5 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \end{pmatrix}.$$

Then, the fuzzy solution is,

$$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (4, 1, 3) \\ (5, 0.5, 2) \\ (3, 0.5, 1) \end{pmatrix}.$$

The proposed method obtained similar solution as Dehghan et al. (2006). A comparison between Dehghan et al. (2006) approach and the proposed method will be presented in Section 3.3 using Table 3.2a,b.

The next example illustrates our method where the fuzzy system has a unique non fuzzy solution. This example is the same example as in Abbasbandy et al. (2012). However, the new method proposes a different solution for Abbasbandy et al. (2012), because the solutions are different; the verification of solutions are provided.

Example 3.2.2. Abbasbandy et al. (2012) consider the following $P-FFLS$,

$$\begin{cases} (4,3,2)\otimes(m_1^x,\alpha_1^x,\beta_1^x)\oplus(5,2,1)\otimes(m_2^x,\alpha_2^x,\beta_2^x)\oplus(3,0,3)\otimes(m_3^x,\alpha_3^x,\beta_3^x) \\ \qquad =(71,54,76), \\ (7,4,3)\otimes(m_1^x,\alpha_1^x,\beta_1^x)\oplus(10,6,3)\otimes(m_2^x,\alpha_2^x,\beta_2^x)\oplus(2,1,1)\otimes(m_3^x,\alpha_3^x,\beta_3^x) \\ \qquad =(118,115,129), \\ (6,2,2)\otimes(m_1^x,\alpha_1^x,\beta_1^x)\oplus(7,1,2)\otimes(m_2^x,\alpha_2^x,\beta_2^x)\oplus(15,5,4)\otimes(m_3^x,\alpha_3^x,\beta_3^x) \\ \qquad =(155,89,151). \end{cases}$$

The $P-\tilde{X}$ is obtained using *P-ALS*.

In matrix form, this becomes,

$$\begin{pmatrix} (4,3,2) & (5,2,1) & (3,0,3) \\ (7,4,3) & (10,6,3) & (2,1,1) \\ (6,2,2) & (7,1,2) & (15,5,4) \end{pmatrix}\otimes\begin{pmatrix} (m_1^x,\alpha_1^x,\beta_1^x) \\ (m_2^x,\alpha_2^x,\beta_2^x) \\ (m_3^x,\alpha_3^x,\beta_3^x) \end{pmatrix}=\begin{pmatrix} (71,54,76) \\ (118,115,129) \\ (155,89,151) \end{pmatrix},$$

$$A=\begin{pmatrix} 4 & 5 & 3 \\ 7 & 10 & 2 \\ 6 & 7 & 15 \end{pmatrix}, |A|=46,$$

using Theorem 3.1.1, $|A|\neq 0$, then $|S|\neq 0$, thus the *P-ALS*, $SX=B$, has a unique solution.

$$\begin{pmatrix} \begin{pmatrix} 4 & 5 & 3 \\ 7 & 10 & 2 \\ 6 & 7 & 15 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 3 & 2 & 0 \\ 4 & 6 & 1 \\ 2 & 1 & 5 \end{pmatrix} & \begin{pmatrix} 4 & 5 & 3 \\ 7 & 10 & 2 \\ 6 & 7 & 15 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 1 & 3 \\ 3 & 3 & 1 \\ 2 & 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 4 & 5 & 3 \\ 7 & 10 & 2 \\ 6 & 7 & 15 \end{pmatrix} \end{pmatrix}\begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} \end{pmatrix}=\begin{pmatrix} \begin{pmatrix} 71 \\ 118 \\ 155 \end{pmatrix} \\ \begin{pmatrix} 54 \\ 115 \\ 89 \end{pmatrix} \\ \begin{pmatrix} 76 \\ 129 \\ 151 \end{pmatrix} \end{pmatrix},$$

by computing $X = S^{-1}B$,

$$X = \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 4 \\ 8 \\ 5 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \\ \begin{pmatrix} -16.782 \\ 19.608 \\ 4.695 \end{pmatrix} \end{pmatrix},$$

since $\beta_1^x \leq 0$, then the exact unique solution of this system is non fuzzy vector, is as follows,

$$\tilde{X}_g = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} \left(4,2,-\frac{386}{23}\right) \\ \left(8,3,\frac{451}{23}\right) \\ \left(5,1,\frac{108}{23}\right) \end{pmatrix} = \begin{pmatrix} (4, 2, -16.782) \\ (8, 3, 19.608) \\ (5, 1, 4.695) \end{pmatrix}.$$

The verification of solution shows that, the multiplication of two fuzzy matrices $\tilde{A}, \tilde{X}_g$ satisfies $\tilde{B}$. Also, using Definition 2.2.8. the distance metric function is equal zero, $D_3(\tilde{A} \otimes \tilde{X}_g, \tilde{B}) = 0$. Hence, the solution $\tilde{X}_g$ satisfies the fuzzy system.

While, the original work of Abbasbandy et al. (2012) provided $\tilde{X}_a$ as exact unique fuzzy solution for $FFLS$,

$$\tilde{X}_a = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (4, 2, 2) \\ (8, 3, 5) \\ (5, 1, 4) \end{pmatrix}.$$

However, $\tilde{X}_a$ is not satisfied the system, since $\tilde{A} \otimes \tilde{X}_a \neq \tilde{B}$,

$$\tilde{A} \otimes \tilde{X}_a = \begin{pmatrix} (71,54,76) \\ (118,115,113) \\ (155,89,151) \end{pmatrix}, \text{while } \tilde{B} = \begin{pmatrix} (71,54,76) \\ (118,115,129) \\ (155,89,151) \end{pmatrix}.$$

Moreover, the distance metric function is not equal zero,

$$D_2\left(\tilde{A} \otimes \tilde{X}_a, \tilde{B}\right) = \sqrt{(\tfrac{1}{4})(16)^2} = 8.$$

As shown, the new method proposed solution $\tilde{X}_g$ which is a different solution for Abbasbandy et al. (2012)'solution $\tilde{X}_a$. Thus, the verification of solutions are provided for both methods.

Verification of solution of $\tilde{X}_g$

$$\begin{cases} (4,3,2) \otimes \left(4,2,-\frac{386}{23}\right) \oplus (5,2,1) \otimes \left(8,3,\frac{451}{23}\right) \oplus (3,0,3) \otimes \left(5,1,\frac{108}{23}\right) \\ = \left(16,20,-\frac{1360}{23}\right) \oplus \left(40,31,\frac{2439}{23}\right) \oplus \left(15,3,\frac{669}{23}\right) = (71,54,76), \\ \\ (7,4,3) \otimes \left(4,2,-\frac{386}{23}\right) \oplus (10,6,3) \otimes \left(8,3,\frac{451}{23}\right) \oplus (2,1,1) \otimes \left(5,1,\frac{108}{23}\right) \\ = \left(28,30,-\frac{2426}{23}\right) \oplus \left(80,78,\frac{5062}{23}\right) \oplus \left(10,7,\frac{331}{23}\right) = (118,115,129), \\ \\ (6,2,2) \otimes \left(4,2,-\frac{386}{23}\right) \oplus (7,1,2) \otimes \left(8,3,\frac{451}{23}\right) \oplus (15,5,4) \otimes \left(5,1,\frac{108}{23}\right) \\ = \left(24,20,-\frac{2132}{23}\right) \oplus \left(56,29,\frac{3525}{23}\right) \oplus \left(75,40,\frac{2080}{23}\right) = (155,89,151). \end{cases}$$

Verification of solution of $\tilde{X}_a$

$$\begin{cases} (4,3,2) \otimes (4,2,2) \oplus (5,2,1) \otimes (8,3,5) \oplus (3,0,3) \otimes (5,1,4) \\ = (16,20,16) \oplus (40,31,33) \oplus (15,3,27) = (71,54,76), \\ \\ (7,4,3) \otimes (4,2,2) \oplus (10,6,3) \otimes (8,3,5) \oplus (2,1,1) \otimes (5,1,4) \\ = (28,30,26) \oplus (80,78,74) \oplus (10,7,13) = (118,115,113), \\ \\ (6,2,2) \otimes (4,2,2) \oplus (7,1,2) \otimes (8,3,5) \oplus (15,5,4) \otimes (5,1,4) \\ = (24,20,20) \oplus (56,29,51) \oplus (75,40,80) = (155,89,151). \end{cases}$$

Table 3.1 compares both solutions $\tilde{X}_g$ and $\tilde{X}_a$ in terms of accuracy of solution $\tilde{X}$, the distance for right hand side vector $\tilde{B}$ and the possibility of unique solution.

Table 3.1

A comparison between $\tilde{X}_g$ and $\tilde{X}_a$.

	$\tilde{X}_g$	$\tilde{X}_a$
Accuracy of solution $\tilde{X}$	$\tilde{A} \otimes \tilde{X}_g = \tilde{B}$.	$\tilde{A} \otimes \tilde{X}_a \neq \tilde{B}$.
Distance metric function	0	8
The possibilities of solution	A unique solution is determined.	Is not determined unique or non-unique solution.

As shown in Table 3.1, the $\tilde{X}_g$ is non fuzzy solution. This means that our method is able to provide an exact unique solution even it is not $P - \tilde{X}$. This motivates us to determine the conditions of $P - FFLS$ to have a $P - \tilde{X}$ in Section 3.3.

This method is able to solve large scale of system as illustrates in Examples 3.2.3, which indicates the efficiency of the proposed method. The details and verification of solution are provided in Appendix A.

Example 3.2.3. Consider the following 10×10 $FFLS$,

$$\begin{pmatrix} (6,3,6) & (3,1,3) & (5,4,9) & (5,1,6) & (6,2,1) & (3,1,2) & (5,2,3) & (6,5,7) & (7,5,2) & (7,1,1) \\ (4,2,1) & (3,2,0) & (8,7,1) & (7,3,3) & (3,3,3) & (7,2,6) & (3,1,1) & (7,5,3) & (2,1,6) & (8,0,1) \\ (4,3,2) & (7,6,4) & (1,1,4) & (1,1,9) & (2,1,0) & (3,3,5) & (4,1,5) & (3,3,0) & (4,4,8) & (8,1,9) \\ (2,2,8) & (3,0,4) & (5,3,3) & (6,3,8) & (2,1,1) & (4,0,3) & (4,2,9) & (6,3,4) & (2,0,6) & (6,1,9) \\ (7,6,9) & (3,1,6) & (1,1,5) & (3,2,4) & (5,4,2) & (5,4,6) & (5,2,7) & (5,4,9) & (7,4,9) & (6,1,6) \\ (8,5,6) & (4,0,3) & (8,5,5) & (7,1,2) & (4,1,1) & (4,2,9) & (6,2,3) & (6,6,8) & (7,3,9) & (1,0,4) \\ (8,7,3) & (4,1,9) & (7,2,1) & (8,1,5) & (8,4,7) & (5,3,2) & (7,4,4) & (3,1,8) & (7,4,5) & (3,1,7) \\ (4,3,9) & (5,1,6) & (6,2,6) & (1,0,4) & (5,5,2) & (2,1,3) & (1,1,7) & (7,4,3) & (6,0,2) & (8,5,3) \\ (5,1,9) & (1,1,0) & (8,3,5) & (4,1,6) & (7,0,0) & (3,1,8) & (7,1,0) & (6,2,3) & (7,6,1) & (2,2,3) \\ (1,0,4) & (8,3,9) & (2,0,1) & (7,3,9) & (6,2,1) & (8,2,5) & (4,3,3) & (8,6,9) & (3,3,2) & (5,4,7) \end{pmatrix}$$

$$\otimes \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \\ (m_4^x, \alpha_4^x, \beta_4^x) \\ (m_5^x, \alpha_5^x, \beta_5^x) \\ (m_6^x, \alpha_6^x, \beta_6^x) \\ (m_7^x, \alpha_7^x, \beta_7^x) \\ (m_8^x, \alpha_8^x, \beta_8^x) \\ (m_9^x, \alpha_9^x, \beta_9^x) \\ (m_{10}^x, \alpha_{10}^x, \beta_{10}^x) \end{pmatrix} = \begin{pmatrix} (499,355,783) \\ (591,449,509) \\ (277,270,669) \\ (414,230,644) \\ (334,289,784) \\ (593,381,729) \\ (609,324,671) \\ (493,278,704) \\ (558,277,601) \\ (431,273,671) \end{pmatrix}.$$

The $P - \tilde{X}$ using the proposed method is

$$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \tilde{x}_8 \\ \tilde{x}_9 \\ \tilde{x}_{10} \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \\ (m_4^x, \alpha_4^x, \beta_4^x) \\ (m_5^x, \alpha_5^x, \beta_5^x) \\ (m_6^x, \alpha_6^x, \beta_6^x) \\ (m_7^x, \alpha_7^x, \beta_7^x) \\ (m_8^x, \alpha_8^x, \beta_8^x) \\ (m_9^x, \alpha_9^x, \beta_9^x) \\ (m_{10}^x, \alpha_{10}^x, \beta_{10}^x) \end{pmatrix} = \begin{pmatrix} (4,1,4) \\ (9,6,9) \\ (37,0,6) \\ (8,2,2) \\ (9,2,5) \\ (8,1,9) \\ (2,0,4) \\ (5,1,4) \\ (8,3,1) \\ (7,0,9) \end{pmatrix}.$$

Next section provides necessary conditions of solution to be positive fuzzy solution.

3.3 The Necessary Conditions of Positive Fuzzy Solution

A necessary conditions for $P - FFLS$ to has $P - \tilde{X}$ are checked for fuzzy matrix $\tilde{A}$ and fuzzy vector $\tilde{B}$. The conditions are provided for both left and right hand side for ($\tilde{A}$ and $\tilde{B}$), then for left hand side $\tilde{A}$ only where $\tilde{B}$ is an arbitrary.

3.3.1 The Necessary Conditions of Left and Right Hand Side ($\tilde{A}$ and $\tilde{B}$)

Dehghan et al. (2006) studied the conditions for $P - FFLS$ to have $P - \tilde{X}$ as stated in Theorem 2.3.1. This theorem determines the fuzzy solution or non fuzzy solution only when matrix $A^{-1} > 0$. To date, no numerical example is checked whether the $P - FFLS$ have $P - \tilde{X}$ or not, even in Dehghan et al. (2006). According to Remark 2.1.1 the odds of the matrices which satisfies the condition $A^{-1} > 0$ is very small.

For that, Theorem 2.3.1. is incompatible to apply in most of $FFLS$. Thus, the restriction

$A^{-1} > 0$ is demonstrated in $P-FFLS$ by the following corollary, and it is omitted in our approach.

Corollary 3.3.1. Consider $\tilde{A}=(A,M,N)$ is positive fuzzy matrix and A is generalized permutation matrix. Then the right spread matrix M is also a generalized permutation matrix, and must have the same structure of A. In other word,

$$\text{If } a_{i,j}=0, \text{ then } m_{i,j}=0 \text{ in } \tilde{a}_{i,j}=\left(a_{i,j},m_{i,j},n_{i,j}\right)\ \forall i,j=1,\dots,n.$$

Proof. Suppose $m_{i,j}\neq 0$, and $a_{i,j}=0$. Then, $a_{i,j}-m_{i,j}\ngtr 0$. This leads to negative TFN, which contradicts with the hypothesis of positive fuzzy matrix $\forall\tilde{a}_{i,j}\geq 0$.

□

According to Corollary 3.3.1, $A^{-1}>0$ is only satisfied in $P-FFLS$ when the entries of the matrices $A=(a_{i,j})$ and $M=(m_{i,j})$ are all zero, except for a single positive entry in each row and column.

In order to point out numerically the structure of fuzzy matrices which satisfies $A^{-1}>0$, the next fuzzy matrix is presented,

$$\tilde{A}=\begin{pmatrix}(0,0,0.1) & (0,0,0.6) & (0,0,0.2) & (4,2,0.2)\\ (11,0,0) & (0,0,0.2) & (0,0,0.11) & (0,0,0.13)\\ (0,0,0.5) & (5,3,0.6) & (0,0,0.9) & (0,0,0.17)\\ (0,0,0.9) & (0,0,0.9) & (2,0.5,0.7) & (0,0,0.3)\end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 11 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \text{ then } A^{-1} = \begin{pmatrix} 0 & \frac{1}{11} & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & 0 & 0 \end{pmatrix}.$$

So, in this section, the conditions for $P - FFLS$ to has $P - \tilde{X}$ is proposed without the condition $A^{-1} > 0$. The inverse of matrix S and sub-vectors for the solution m^x, α^x and β^x are required to provide the conditions for $P - FFLS$ to has $P - \tilde{X}$. For that, the following lemma provides the inverse of matrix S.

Lemma 3.3.1. If S^{-1} exist it must have the same structure as S, which is

$$S^{-1} = \begin{pmatrix} A' & 0 & 0 \\ M' & A' & 0 \\ N' & 0 & A' \end{pmatrix},$$

where,

$$A' = A^{-1},$$

$$M' = -A^{-1}MA^{-1},$$

$$N' = -A^{-1}MA^{-1}.$$

Proof. Let $(S \vdots I)$ be a rectangle block matrix $3n \times 6n$,

$$(S \vdots I) = \begin{pmatrix} A & 0 & 0 & \vdots & I & 0 & 0 \\ M & A & 0 & \vdots & 0 & I & 0 \\ N & 0 & A & \vdots & 0 & 0 & I \end{pmatrix}.$$

Using Theorem 3.1.1, S^{-1} is exist then A^{-1} is exist. By multiplying each rows by A^{-1},

$$\left(\begin{array}{ccc:ccc} I & 0 & 0 & A^{-1} & 0 & 0 \\ A^{-1}M & I & 0 & 0 & A^{-1} & 0 \\ A^{-1}N & 0 & I & 0 & 0 & A^{-1} \end{array}\right).$$

After that, subtracting the first row multiplied by $A^{-1}M$ and $A^{-1}N$ from the second row and third row, respectively,

$$\left(\begin{array}{ccc:ccc} I & 0 & 0 & A^{-1} & 0 & 0 \\ 0 & I & 0 & -A^{-1}MA^{-1} & A^{-1} & 0 \\ 0 & 0 & I & -A^{-1}NA^{-1} & 0 & A^{-1} \end{array}\right),$$

then,

$$S^{-1} = \begin{pmatrix} A' & 0 & 0 \\ M' & A' & 0 \\ N' & 0 & A' \end{pmatrix},$$

where A', M' and N' are,

$$\begin{aligned} A' &= A^{-1}, \\ M' &= -A^{-1}MA^{-1}, \\ N' &= -A^{-1}NA^{-1}. \end{aligned} \tag{3.9}$$

□

The vectors m^x, α^x and β^x are computed in next remark using S^{-1}.

Remark 3.3.1. The sub-vectors m^x, α^x and β^x are computed by $X = S^{-1}B$,

$$\begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} A' & 0 & 0 \\ M' & A' & 0 \\ N' & 0 & A' \end{pmatrix} \begin{pmatrix} m^b \\ \alpha^b \\ \beta^b \end{pmatrix},$$

then,

$$m^x = (A')m^b = (A^{-1})m^b,$$

$$\alpha^x = (M')m^b + (A')\alpha^b = (-A^{-1}MA^{-1})m^b + (A^{-1})\alpha^b,$$

$$\beta^x = (N')m^b + (A')\beta^b = (-A^{-1}MA^{-1})m^b + (A^{-1})\beta^b.$$

Now, in next theorem, a necessary and sufficient conditions for $P - FFLS$ to have $P - \tilde{X}$ are proposed, without the condition $A^{-1} > 0$.

Theorem 3.3.1 The $P - FFLS$ has $P - \tilde{X}$ when matrix A is invertible, that is if,

i- $A^{-1}\alpha^b \geq A^{-1}(MA^{-1})m^b$.

ii- $A^{-1}\beta^b \geq A^{-1}NA^{-1}m^b$.

iii- $A^{-1}(I + MA^{-1})m^b \geq A^{-1}\alpha^b$.

Proof. The spreads $\alpha^x, \beta^x \geq 0$ can be easily verified using Remark 3.3.1, is as follows,

$\alpha^x \geq 0$ is obtained from $A^{-1}\alpha^b \geq A^{-1}(MA^{-1})m^b$.

Similarly, $\beta^x \geq 0$ is obtained from $A^{-1}\beta^b \geq A^{-1}NA^{-1}$.

Now, we have to proof $m^x - \alpha^x \geq 0$ without involving $A^{-1} > 0$.

$$m^x - \alpha^x = [A^{-1}m^b] - [A^{-1}\alpha^b - (A^{-1})M(A^{-1})m^b]$$
$$= [A^{-1}m^b + (A^{-1})M(A^{-1})m^b] - [A^{-1}\alpha^b],$$

but $[A^{-1}m^b + (A^{-1})M(A^{-1})m^b] = A^{-1}[m^b + M(A^{-1})m^b] = A^{-1}[I + MA^{-1}]m^b$.

Then,

$$m^x - \alpha^x \geq 0 \text{ if and only if } A^{-1}[I + MA^{-1}]m^b - A^{-1}\alpha^b \geq 0. \qquad \square$$

As shown, the condition $A^{-1} \ngtr 0$ is omitted in Theorem 3.3.1, so it can apply for generalized or arbitrary crisp matrix A to check the solution whether fuzzy or non fuzzy. The previous Example 3.2.2. is checked below by the proposed theorem, while Theorem 2.3.1. cannot check the example since $A^{-1} \ngtr 0$. However, Theorem 3.3.1. can be determined whether the system has fuzzy or non fuzzy solution. In addition, the theorem is able to check the fuzzy solution whenever non positive fuzzy number as Example 3.3.1.

Now, using the proposed method in last section we obtain non fuzzy solution $\tilde{X}_g$ for Example 3.2.2, while Abbasbandy et al. (2012) proposed solution $\tilde{X}_a$. Theorem 3.3.1. verifies that Example 3.2.2 originally has non fuzzy solution.

By applying (iii) in Theorem 3.3.1. we get,

$$A^{-1}\beta^b - A^{-1}(NA^{-1})m^b = \begin{pmatrix} \frac{175}{23} \\ \frac{313}{46} \\ \frac{177}{46} \end{pmatrix} - \begin{pmatrix} \frac{561}{23} \\ -\frac{589}{46} \\ -\frac{39}{46} \end{pmatrix} = \begin{pmatrix} -\frac{386}{23} \\ \frac{451}{23} \\ \frac{108}{23} \end{pmatrix}.$$

Since $A^{-1}\beta^b - A^{-1}(NA^{-1})m^b$ has negative entry $-\frac{386}{23}$, then $A^{-1}\beta^b \ngeq A^{-1}NA^{-1}m^b$, which mean the fuzzy system has non fuzzy solution. This result, verifies the proposed solution $\tilde{X}_g$ which was non fuzzy solution.

As stated before, Theorem 3.3.1. also able to check the fuzzy solution whenever it is negative as Example 3.3.1.

Example 3.3.1 Given the following $FFLS$,

$$\begin{pmatrix} (3.1, 0.4, 0.1) & (2, 0.6, 0.4) \\ (8, 0.6, 0.7) & (6.9, 0.2, 0.1) \end{pmatrix} \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (17.3, 14.75, 7.86) \\ (51.6, 32.91, 22.72) \end{pmatrix}.$$

Since $A^{-1} \ngeq 0$, Theorem 2.3.1. fails to check this system. The inverse of A is

$$A^{-1} = \begin{pmatrix} 3.1 & 2 \\ 8 & 6.9 \end{pmatrix}^{-1} = \begin{pmatrix} 1.28015 & -0.371058 \\ -1.48423 & -0.371058 \end{pmatrix}.$$

By applying (i) in Theorem 3.3.1. we get,

$$A^{-1}(I + MA^{-1})m^b = \begin{pmatrix} 6.64378 \\ 0.152134 \end{pmatrix} \text{ and } A^{-1}\alpha^b = \begin{pmatrix} 6.67069 \\ -2.96456 \end{pmatrix}.$$

$A^{-1}\,(I + MA^{-1})\,m^b \ngeq A^{-1}\alpha^b$, because $6.64378 \ngeq 6.67069$. Consequently, the $P - FLLS$ doesn't have $P - \tilde{X}$.

Furthermore, by solving the example using *P-ALS* in Equation (3.8), we confirm the exact and unique solution is negative fuzzy is as follows, $SX = B$,

$$\begin{pmatrix} \begin{pmatrix} 3.1 & 2 \\ 8 & 6.9 \end{pmatrix} & \begin{pmatrix} 0. & 0 \\ 0 & 0. \end{pmatrix} & \begin{pmatrix} 0. & 0 \\ 0 & 0. \end{pmatrix} \\ \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.2 \end{pmatrix} & \begin{pmatrix} 3.1 & 2 \\ 8 & 6.9 \end{pmatrix} & \begin{pmatrix} 0. & 0 \\ 0 & 0. \end{pmatrix} \\ \begin{pmatrix} 0.1 & 0.4 \\ 0.7 & 0.1 \end{pmatrix} & \begin{pmatrix} 0. & 0 \\ 0 & 0. \end{pmatrix} & \begin{pmatrix} 3.1 & 2 \\ 8 & 6.9 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 17.3 \\ 51.6 \end{pmatrix} \\ \begin{pmatrix} 14.75 \\ 32.91 \end{pmatrix} \\ \begin{pmatrix} 7.86 \\ 22.72 \end{pmatrix} \end{pmatrix},$$

By computing, $X = S^{-1}B$,

$$X = \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \end{pmatrix} \\ \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \end{pmatrix} \\ \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ \\ \begin{pmatrix} 3.0269 \\ 0.883302 \end{pmatrix} \\ \\ \begin{pmatrix} 0.126902 \\ 2.7833 \end{pmatrix} \end{pmatrix},$$

$$\tilde{X} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (3, 3.0269, 0.126902) \\ \\ (4, 0.883302, 2.7833) \end{pmatrix}.$$

Since $m_1^x - \alpha_1^x \ngeq 0$, based on Definition 2.2.6, $(m_1^x, \alpha_1^x, \beta_1^x)$ is a negative TFN.

Finally, the sub-vectors solution m^x, α^x and β^x in Remark 3.3.1. show the incompatibility of Theorem 2.3.1. which is not only for the condition $A^{-1} > 0$, but also for the further condition $(MA^{-1} + I)m^b \geq \alpha^b$ which is used to proof $m^x - \alpha^x \geq 0$ in $\tilde{x} = (m^x, \alpha^x, \beta^x)$.

The following alternative proof of $m^x - \alpha^x \geq 0$ shows that $(MA^{-1} + I)m^b \geq \alpha^b$ is unnecessary to proof $m^x - \alpha^x \geq 0$, and must be omitted, because $m^x - \alpha^x \geq 0$ can be obtained without $(MA^{-1} + I)m^b \geq \alpha^b$ as follows,

$$m^b \geq 0 \text{ and } \mathrm{A}^{-1} > 0 \Rightarrow m^x = \mathrm{A}^{-1} m^b \geq 0,$$

$$m^x - \alpha^x = (A^{-1} m^b) - (A^{-1}\alpha^b - A^{-1} M A^{-1} m^b)$$

$$= A^{-1} m^b - A^{-1}\alpha^b + A^{-1} M A^{-1} m^b,$$

hence,

$$m^x - \alpha^x = A^{-1}[(m^b - \alpha^b) + (MA^{-1}b)] = A^{-1}[(m^b - \alpha^b) + (Mm^x)].$$

$A^{-1} > 0$. $(m^b - \alpha^b) \geq 0$ since $\tilde{B} = (m^b, \alpha^b, \beta^b) \geq 0$. $(M), (m^x) \geq 0$ then $(Mm^x) \geq 0$. Hence,

$$m^x - \alpha^x \geq 0. \qquad \square$$

However, $(MA^{-1} + I)m^b \geq \alpha^b$ is already satisfied by the hypothesis $A^{-1} > 0$,

$$(MA^{-1} + I)m^b - \alpha^b = (MA^{-1}m^b + m^b) - \alpha^b$$

$$= (MA^{-1}m^b) + (m^b - \alpha^b),$$

$$A^{-1} > 0, \qquad M, m^b \geq 0, \qquad (m^b - \alpha^b) \geq 0.$$

Hence,

$$(MA^{-1} + I)m^b - \alpha^b \geq 0. \qquad \square$$

The next section provides the necessary conditions to have $P - \tilde{X}$ using only left hand side $\tilde{A}$. This let the left hand side $\tilde{B}_G$ be arbitrary.

3.3.2 The Necessary Conditions of Right Hand Side $\tilde{A}$ with an Arbitrary $\tilde{B}$

The necessary and sufficient conditions in $\tilde{A}$ to have $P - \tilde{X}$ are provided. This let us provide general solution $\tilde{X}_G$ for a given $\tilde{A}$ only, where $\tilde{B}_G$ is an arbitrary positive fuzzy vector.

Next lemma provides the conditions on crisp matrices A, M and N to warrant matrix S is non negative inverse. Because the condition S is non negative, inverse is required in Theorem 3.3.2. to provide general solution.

Lemma 3.3.2. The matrix S is an non negative, inverse, if and only if the center matrix A be an inverse-non negative and the spreads matrices M and N are zero matrices, in other words,

$$S^{-1} > 0 \text{ if and only if } A^{-1} > 0 \text{ and } \tilde{A} = (A, 0, 0).$$

Proof. By Lemma 3.3.1,

$$S^{-1} = \begin{pmatrix} A' & 0 & 0 \\ M' & A' & 0 \\ N' & 0 & A' \end{pmatrix} = \begin{pmatrix} A^{-1} & 0 & 0 \\ -A^{-1}MA^{-1} & A^{-1} & 0 \\ -A^{-1}NA^{-1} & 0 & A^{-1} \end{pmatrix},$$

$A' > 0$ if and only if $A^{-1} > 0$.

$M' \geq 0$ if and only if $-A^{-1}MA^{-1} \geq 0$ implies that $M = 0$.

$N' \geq 0$ if and only if $-A^{-1}NA^{-1} \geq 0$ implies that $N = 0$. □

Now, next theorem provides the conditions on crisp matrices A, M and N to obtain general form solution for $P - FFLS$ when $\tilde{B}_G$ is an arbitrary.

Theorem 3.3.2. If $A^{-1} > 0$ and the spreads matrices M and N are zeros, then the unique solution X of $SX = B$ represents a general form solution $P - \tilde{X}_G$ for an arbitrary positive vector $\tilde{B}_G$ in $\tilde{A} \otimes \tilde{X}_G = \tilde{B}_G$.

Proof. $\tilde{B}$ is fuzzy vector implies,

$$\alpha^b \text{ and } \beta^b \geq 0.$$

$\tilde{B}$ is positive fuzzy vector implies,

$$m^b - \alpha^b \geq 0 \Rightarrow m^b \geq \alpha^b \geq 0 \Rightarrow \; m^b \geq 0.$$

Thus, m^b, α^b and $\beta^b \geq 0$, then $B = \begin{pmatrix} m^b \\ \alpha^b \\ \beta^b \end{pmatrix} \geq 0$.

By Lemma 3.2.2. if $A^{-1} > 0$ and $M = N = 0$, then $S^{-1} > 0$.

$$SX = B \Rightarrow X = S^{-1}B \geq 0,$$

hence,

$$\alpha^x, \beta^x \geq 0.$$

Since the spreads α^x and β^x are non negative, the entries of vector $\tilde{X}$ are fuzzy numbers based on Definition 2.2.6.

The positivity of $\tilde{X}$ is proved as follows,

$$m^x - \alpha^x = A^{-1}m^b - A^{-1}\alpha^b + A^{-1}MA^{-1}m^b,$$

$$m^x - \alpha^x = A^{-1}(m^b - \alpha^b), \text{since } m^b \geq \alpha^b,$$

$$m^x - \alpha^x \geq 0,$$

since $m^x - \alpha^x, \alpha^x, \beta^x \geq 0$, then $\tilde{X}$ is a positive fuzzy vector. □

Unfortunately, the odds of A^{-1} to be non negative is very small according to Remark 2.3.1. Now, we illustrate an example satisfies the previous conditions in Theorem 3.3.2. This conditions let a fuzzy system have a general solution $\tilde{X}$ for an arbitrary $\tilde{B}_G$ in right hand side, where only $\tilde{A}$ is given in left hand side.

Example 3.3.2. Consider the fuzzy matrix,

$$\tilde{A} = \begin{pmatrix} (0,0,0) & (0,0,0) & (3,0,0) \\ (4,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (7,0,0) & (0,0,0) \end{pmatrix}.$$

The general positive solution $\tilde{X}_G$ for an arbitrary positive fuzzy vector $\tilde{B}_G$ is obtained as follows:

Let $\tilde{B}_G = (\tilde{b}_i) = (b_i, h_i, g_i),\ i = 1, \dots, n$ is an arbitrary fuzzy vector,

$$\tilde{B}_G = \begin{pmatrix} (b_1, h_1, g_1) \\ (b_2, h_2, g_2) \\ (b_3, h_3, g_3) \end{pmatrix}.$$

Then, the $FFLS\ \tilde{A} \otimes \tilde{X}_G = \tilde{B}_G$ can be written as,

$$\begin{cases} (3,0,0)(m_3^x, \alpha_3^x, \beta_3^x) = (b_1, h_1, g_1), \\ (4,0,0)(m_1^x, \alpha_1^x, \beta_1^x) = (b_2, h_2, g_2), \\ (7,0,0)(m_2^x, \alpha_2^x, \beta_2^x) = (b_3, h_3, g_3). \end{cases}$$

In matrix form,

$$\begin{pmatrix} (0,0,0) & (0,0,0) & (3,0,0) \\ (4,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (7,0,0) & (0,0,0) \end{pmatrix} \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (b_1, h_1, g_1) \\ (b_2, h_2, g_2) \\ (b_3, h_3, g_3) \end{pmatrix}.$$

By solving the system using *P-ALS* in Equation (3.8). The general solution for an arbitrary fuzzy vector $\tilde{B}_G$ is,

$$\tilde{X}_G = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} \left(\frac{b_2}{4}, \frac{h_2}{4}, \frac{g_2}{4}\right) \\ \\ \left(\frac{b_3}{7}, \frac{h_3}{7}, \frac{g_3}{7}\right) \\ \\ \left(\frac{b_1}{3}, \frac{h_1}{3}, \frac{g_1}{3}\right) \end{pmatrix}.$$

As an application for Example 3.3.2, consider the particular vector $\tilde{B}_p$,

$$\tilde{B}_p = \begin{pmatrix} (b_1, h_1, g_1) \\ \\ (b_2, h_2, g_2) \\ \\ (b_3, h_3, g_3) \end{pmatrix} = \begin{pmatrix} (5, 3, 8) \\ \\ (3, 2, 1) \\ \\ (4, 3, 1) \end{pmatrix},$$

using the general solution $\tilde{X}_G$, the particular solution $\tilde{X}_p$ is,

$$\tilde{X}_p = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} \left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right) \\ \\ \left(\frac{4}{7}, \frac{3}{7}, \frac{1}{7}\right) \\ \\ \left(\frac{5}{3}, 1, \frac{8}{3}\right) \end{pmatrix}.$$

Lastly, a comparison between Dehghan et al. (2006) approach and this study for the necessary conditions of left and right hand side ($\tilde{A}$ and $\tilde{B}$) is presented in Table 3.2a. While, Table 3.2b compares the necessary conditions of right hand side $\tilde{A}$ where left hand side $\tilde{B}$ is arbitrary.

Table3.2a

Comparison the conditions on both hand sides between P-ALS method and Dehghan et al. (2006)'method.

	***P-ALS* method**	**Dehghan et al. (2006)'method**
$\boldsymbol{A}$	Arbitrary.	Generalized permutation matrix.
$\widetilde{\boldsymbol{B}}$	Arbitrary.	Arbitrary.

It is noted in Table 3.2a, the matrix A in $\tilde{A} = (A, M, N) \geq 0$ can be chosen arbitrary without $A^{-1} > 0$ in this study. While the matrix A should be generalized permutation matrix in Dehghan et al. (2006).

Table 3.2b

Comparison the conditions on left hand side between P-ALS method and Dehghan et al. (2006)'method.

	***P-ALS* method**	**Dehghan et al. (2006)'method**
$\boldsymbol{A}$	Generalized permutation matrix.	Not investigated.
$\widetilde{\boldsymbol{B}}$	Arbitrary.	Not investigated.

It is noted in Table 3.2a, the use of restriction $A^{-1} > 0$ in this approach for providing positive fuzzy using only left hand side. While this case is not investigated in Dehghan et al. (2006).

In current section, the condition of system to have positive fuzzy solution are provided, in next section, consistency of general solution is proposed, to classify the possibilities for the solution (unique solution, infinite number of solution, no solution).

3.4 The Consistency of Fully Fuzzy Linear System

In this section, the nature of solution of the fully fuzzy linear system are studied. The consistency of fuzziness of positive solution is checked and the possibilities for the solution for $P - FFLS$ are classified. Moreover, the proposed method are modified using row reduced echelon method to find the solution whenever its not unique.

Using P-ALS there are three cases for the solution for $n \times n\ P - FFLS$, according to the possibilities for solution of classical linear system (unique solution, infinite many solutions, no solution):

Case 1: Unique solution.

If $|A| \neq 0$. Then $P - FFLS$ has unique solution. Through Theorem 3.1.1, If $|A| \neq 0$, then S is invertible. Hence, $SX = B$ has a unique solution, which is $X = S^{-1}B$.

Case 2: Infinite number of solutions.

If $|A| = 0$ then $|S| = 0$, using Theorem 3.1.1. and if $rank(S) = rank\ (S{:}B) = m$, $m < 3n$, then $SX = B$ has infinite number of solutions. Is this case, row reduced method echelon for classical linear system is modified to provide infinite number of solutions for fuzzy system.

Case 3: No solution.

If $|A| = 0$ then $|S| = 0$ and $rank(S) < rank\ (S{:}B)$, the $SX = B$ has no solution. Thus the fuzzy system doesn't have a unique solution.

Therefore, using the linear system $SX = B$, the solution of $\tilde{A} \otimes \tilde{X} = \tilde{B}$ can be obtained whenever it is exist. The possibility for fuzzy unique solution is illustrated in Examples 3.2.1, while the possibility for non fuzzy unique solution is illustrated in Examples 3.2.2.

The next section provides the fuzzy row reduced echelon method. This method is obtained the infinity many solutions whenever its exist which happens if $|A| = 0$ or A is rectangle. In addition, the interval of solution when some coefficients are unknown can be obtained using this method.

Fuzzy Row Reduced Echelon Method

In this section, the fuzzy row reduced echelon method is proposed to solve $FFLS$ where $|A| = 0$ or A is rectangle. So, infinitely many solutions can be provided whenever it exist, and interval of solution when some coefficients are unknown. The method is obtained by three steps, where, $|A| = 0$:

Step 1 Computing the $rank\,(S)$ and $rank\,(S{:}B)$. Based on Case 2, if $rank(S) = rank\,(S{:}B) = m,$ then the system has infinitely many solutions.

Step 2 Transforming the *P-ALS* in Equation (3.8) to $3n$ linear equations. Then, reducing it to m linear equation.

Step 3 Solving linear equations with positive fuzzy inequalities,

$$\begin{cases} m_i^x \geq \alpha_i^x, \\ \alpha_i^x \geq 0, \\ \beta_i^x \geq 0, \qquad \forall i = 1, 2, \dots, n. \end{cases} \tag{3.10}$$

The next example illustrates the fuzzy row reduced echelon method where the fuzzy system has a infinitely many solutions. This example is the same example as in Kumar et al. (2012). However, using the proposed method, the infinitely many solutions are provided by a different solution sets for Kumar et al. (2012)'solution set. Finally, Kumar et al. (2012)'solution set is showed that as subset of a solution set using the proposed method.

Example 3.4.1. Kumar et al. (2012) consider the following $FFLS,$

$$\begin{cases} (2,1,1) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (4,1,1) \otimes (m_2^x, \alpha_2^x, \beta_2^x) = (10,5,5), \\ (4,2,2) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (8,2,2) \otimes (m_2^x, \alpha_2^x, \beta_2^x) = (20,10,10). \end{cases}$$

The $P - \tilde{X}$ is obtained as follows,

In matrix form,

$$\begin{pmatrix} (2,1,1) & (4,1,1) \\ (4,2,2) & (8,2,2) \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (10,5,5) \\ (20,10,10) \end{pmatrix},$$

Since $|A| = 0$, then the fuzzy system don't have a unique solution.

Using Step 1, $rank(S)$ and $rank(S : B)$ are computed as follows,

$$S = \begin{pmatrix} 2 & 4 & 0 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 4 & 0 & 0 \\ 2 & 2 & 4 & 8 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 4 \\ 2 & 2 & 0 & 0 & 4 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 4 & 8 \\ 0 & 1 & 0 & 0 & -2 & -4 \\ 0 & 0 & 1 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, rank(S) = 3,$$

$$(S : B) = \begin{pmatrix} 2 & 4 & 0 & 0 & 0 & 0 & 10 \\ 4 & 8 & 0 & 0 & 0 & 0 & 20 \\ 1 & 1 & 2 & 4 & 0 & 0 & 5 \\ 2 & 2 & 4 & 8 & 0 & 0 & 10 \\ 1 & 1 & 0 & 0 & 2 & 4 & 5 \\ 2 & 2 & 0 & 0 & 4 & 8 & 10 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 4 & 8 & 5 \\ 0 & 1 & 0 & 0 & -2 & -4 & 0 \\ 0 & 0 & 1 & 2 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$rank(S : B) = 3.$

Since $rank(S) = rank(S : B) = 3$, hence $m = 3$, $m < 3n$, then, the system has infinity many solutions.

Using Step 2 the six linear equations of *P-ALS* is,

$$
\begin{cases}
2m_1^x + 4m_2^x = 10, \\
4m_1^x + 8m_2^x = 20, \\
m_1^x + m_2^x + 2\alpha_1^x + 4\alpha_2^x = 5, \\
2m_1^x + 2m_2^x + 4\alpha_1^x + 8\alpha_2^x = 10, \\
m_1^x + m_2^x + 2\beta_1^x + 4\beta_2^x = 5, \\
2m_1^x + 2m_2^x + 4\beta_1^x + 8\beta_2^x = 10,
\end{cases}
$$

which can be reduced to three linear equations,

$$
\begin{cases}
\alpha_1^x = -2\alpha_2^x + \beta_1^x + 2\beta_2^x, \\
m_2^x = 2\beta_1^x + 4\beta_2^x, \\
m_1^x = 5 - 4\beta_1^x - 8\beta_2^x.
\end{cases}
$$

Using Step 3, the previous linear equations should be solved with positive fuzzy inequalities in Equation (3.10) is as follows,

$$
\begin{cases}
\alpha_1^x = -2\alpha_2^x + \beta_1^x + 2\beta_2^x, \\
m_2^x = 2\beta_1^x + 4\beta_2^x, \\
m_1^x = 5 - 4\beta_1^x - 8\beta_2^x. \\
\alpha_1^x \geq 0, \alpha_2^x \geq 0, \\
\beta_1^x \geq 0, \beta_2^x \geq 0, \\
m_1^x \geq \alpha_1^x, m_2^x \geq \alpha_2^x.
\end{cases}
$$

Hence the general form solution $\tilde{X}_G$ for infinitely many solutions is:

$$\tilde{X}_G = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} \left(5 - 2(2\beta_1^x + 4\beta_2^x), \frac{1}{2}\left(-4\alpha_2^x + (2\beta_1^x + 4\beta_2^x)\right), \beta_1^x\right) \\ (2\beta_1^x + 4\beta_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix}.$$

The entries β_1^x, β_2^x and α_2^x provide three different solution set $\tilde{X}_{Set} = \{\tilde{X}_{g_1}, \tilde{X}_{g_2}, \tilde{X}_{g_3}\}$ presented in Table 3.3.

While general solution set $\tilde{X}_k$ in Kumar et al. (2012) is,

$$\tilde{X}_k = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} \left(5 - 2u, \frac{u}{2}, \frac{u}{2}\right) \\ (u, 0,0) \end{pmatrix}, u \in [0,2];$$

As shown, $\tilde{X}_k$ represented the infinitely many solutions set by $u = m_1^x \in [0,2]$. The general solution set $\tilde{X}_k$ is presented in Table 3.3 to compare the sets of infinitely many solutions using the proposed method and Kumar et al. (2012)'method.

Table 3.3

Comparison the entries for three solution sets of $\tilde{X}_{Set} = \{\tilde{X}_{g_1}, \tilde{X}_{g_2}, \tilde{X}_{g_3}\}$ *and a solution set of* $\tilde{X}_k$.

Solution set	m_1^x	α_1^x	β_1^x	m_2^x	α_2^x	β_2^x
$\tilde{X}_{g_1}$	$5 - 2(2\beta_1^x + 4\beta_2^x)$	$-2\alpha_2^x + \beta_1^x + 2\beta_2^x$	$[0,1]$	$2\beta_1^x + 4\beta_2^x$	$\left[0, \frac{\beta_1^x}{2}\right]$	$\left[0, \frac{1}{10}(5 + 2\alpha_2^x - 5\beta_1^x\right]$
$\tilde{X}_{g_2}$	$5 - 2(2\beta_1^x + 4\beta_2^x)$	$-2\alpha_2^x + \beta_1^x + 2\beta_2^x$	$[0,1]$	$2\beta_1^x + 4\beta_2^x$	$\left(\frac{\beta_1^x}{2}, \frac{5}{8}\right)$	$\left[\frac{1}{2}(2\alpha_2^x - \beta_1^x), \frac{1}{10}(5 + 2\alpha_2^x - 5\beta_1^x)\right]$
$\tilde{X}_{g_3}$	$5 - 2(2\beta_1^x + 4\beta_2^x)$	$-2\alpha_2^x + \beta_1^x + 2\beta_2^x$	$\left[1, \frac{5}{4}\right)$	$2\beta_1^x + 4\beta_2^x$	$\left(\frac{1}{2}(-5 + 5\beta_1^x), \frac{\beta_1^x}{2}\right]$	$\left[0, \frac{1}{10}(5 + 2\alpha_2^x - 5\beta_1^x)\right]$
$\tilde{X}_k$	$5 - 2\, m_2^x$	$\left[0, \frac{m_2^x}{2}\right]$	$\left[0, \frac{m_2^x}{2}\right]$	$[0,2]$	0	0

However, the solution set of $\tilde{X}_k$ can be considered as a subset of $\tilde{X}_{g_1}$, because $\alpha_2^x, \beta_2^x = 0$ in $\tilde{X}_k$.

By supposing $\alpha_2^x = \beta_2^x = 0$ in $\tilde{X}_G$, we get,

$$\begin{pmatrix} \left(5 - 2(2\beta_1^x + 0), \frac{1}{2}\left(0 + (2\beta_1^x + 0)\right), \beta_1^x\right) \\ (2\beta_1^x + 0,0,0) \end{pmatrix} = \begin{pmatrix} \left(5 - 2(2\beta_1^x), \frac{1}{2}(2\beta_1^x), \beta_1^x\right) \\ (2\beta_1^x, 0,0) \end{pmatrix},$$

but $\beta_1^x \in [0,1]$ in solution set $\tilde{X}_{g_1}$, then $2\beta_1^x \in [0,2]$. Suppose $h = 2\beta_1^x$, then $\tilde{X}_G$ becomes,

$$\begin{pmatrix} \left(5 - 2(h), \frac{1}{2}(h), \frac{h}{2}\right) \\ (h, 0,0) \end{pmatrix} = \begin{pmatrix} \left(5 - 2h, \frac{h}{2}, \frac{h}{2}\right) \\ (h, 0,0) \end{pmatrix} = \tilde{X}_k, \text{ where } h \in [0,2].$$

The values of m_1^x are represented by Figure 4.1, 4.2, 4.3 and 4.4, respectively, in order to compare the vales of m_1^x using the proposed method and Kumar et al. (2012)'method.

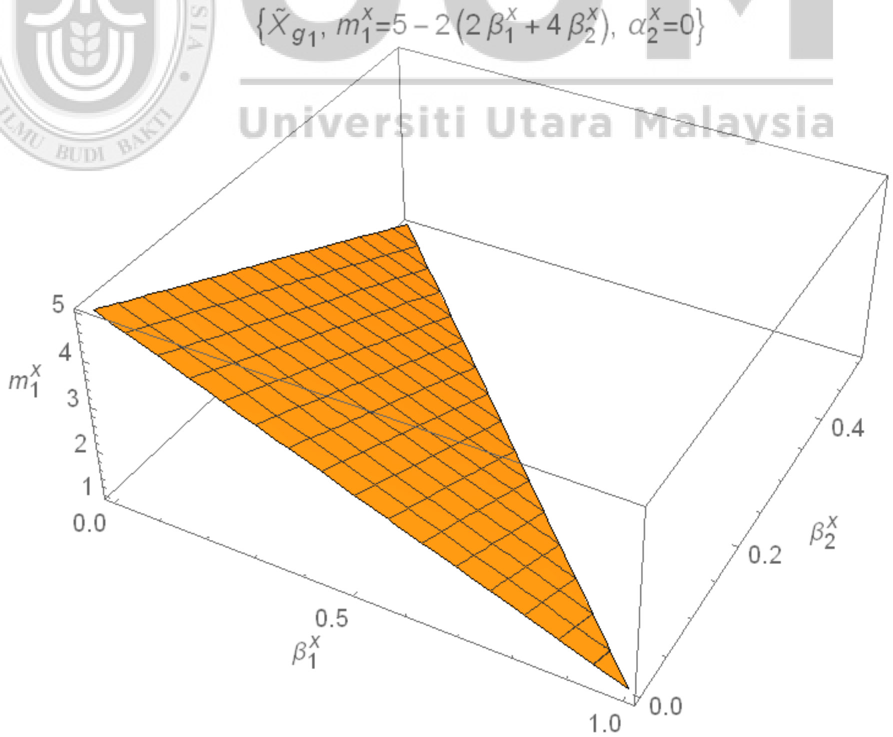

Figure 3.1. The values of m_1^x in solution set $\tilde{X}_{g_1}$.

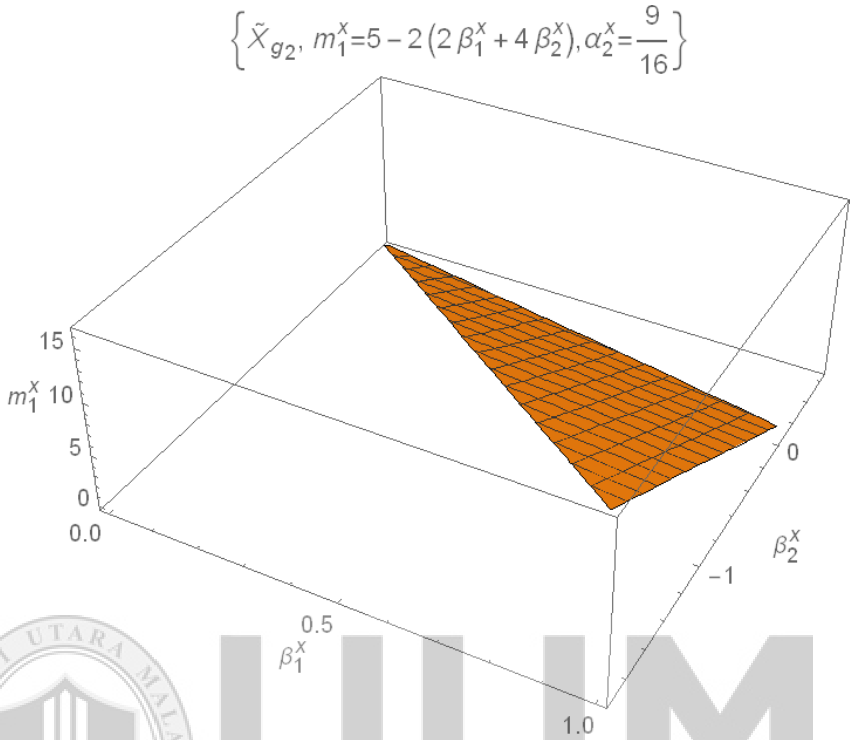

Figure 3.2. The values of m_1^x in solution set $\tilde{X}_{g_2}$.

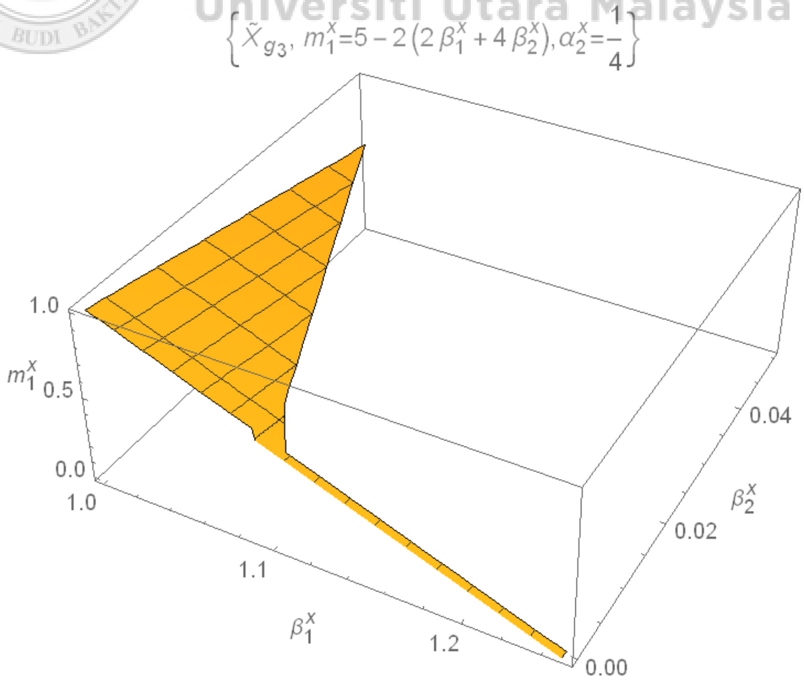

Figure 3.3. The values of m_1^x in solution set $\tilde{X}_{g_3}$.

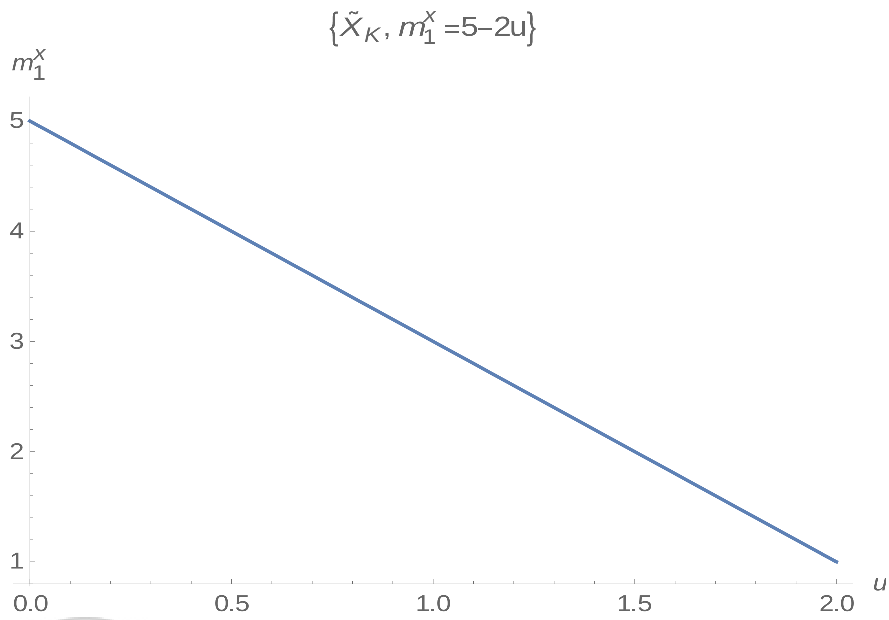

Figure 3.4. The values of m_1^x in solution set $\tilde{X}_k$.

The proposed method is compared with Kumar et al. (2012)'method in Table 3.4 in terms of solution sets, symmetry, fuzziness and independence between $\tilde{x}_1$ and $\tilde{x}_2$.

Table 3.4

Comparison of the solutions set $\tilde{X}_G$ and $\tilde{X}_k$.

	$\tilde{X}_G$	$\tilde{X}_k$
Solution set	Three solution sets.	One solution set.
Symmetry	Symmetric or non symmetric.	Symmetric.
fuzziness	$\tilde{x}_1$ and $\tilde{x}_2$ can be chosen fuzzy.	$\tilde{x}_2$ represents only by it mean value based on a crisp number m_2^x with no spreads.
Independence between $\tilde{x}_1$ and $\tilde{x}_2$	β_1^x α_1^x and β_2^x can be chosen independently.	$\tilde{x}_1$ and $\tilde{x}_2$ depend on chosen values u

As shown in Table 3.4, the general form solution shows that the new method provides three sets $\tilde{X}_{Set} = \{\tilde{X}_{g_1},\ \tilde{X}_{g_2},\ \tilde{X}_{g_3}\}$, while the original work of Kumar et al. (2012) has provided one solution set $\tilde{X}_k$. In addition, the particular solution of $\tilde{X}_G$ can be considered symmetric or non symmetric, while the particular solution $\tilde{X}_k$ should be symmetric because in $\tilde{x}_1$ the both spreads equal $\frac{u}{2}$ and in $\tilde{x}_2$ both spreads equal 0. Also, all fuzzy numbers of particular solution in $\tilde{X}_G$ are nontrivial since the spreads in $\tilde{x}_1$ and $\tilde{x}_2$ are nonzero, while $\tilde{x}_2$ in $\tilde{X}_k$ is represented by non fuzzy number (crisp number) based on Remark 2.2.1, because the both spreads are zero, $\alpha_2^x = \beta_2^x = 0$. Lastly, the entries of $\tilde{x}_1$ and $\tilde{x}_2$ have independent intervals to provide infinite many solutions in $\tilde{X}_G$, while all entries of $\tilde{x}_1$, $\tilde{x}_2$ depend on one interval $u \in [0,2]$ in $\tilde{X}_k$.

The next example is solved by the proposed method in Section 3.2 and Step 3 in this section to provide the interval solution, when some coefficients are unknown in $FFLS$.

Examples 3.4.2. Consider δ is an arbitrary real valued number in the following $FFLS$,

$$\begin{cases}(\delta,2,1)\otimes(m_1^x,\alpha_1^x,\beta_1^x)\oplus(7,3,1)\otimes(m_2^x,\alpha_2^x,\beta_2^x)=(54,46,54),\\ (6,1,2)\otimes(m_1^x,\alpha_1^x,\beta_1^x)\oplus(4,1,2)\otimes(m_2^x,\alpha_2^x,\beta_2^x)=(52,27,56).\end{cases}$$

The $P-\tilde{X}$ is obtained based on values of δ for the $P-FFLS$ as follows.

The system may be written in matrix form, is as follows,

$$\begin{pmatrix}(\delta,2,1) & (7,3,1)\\ (6,1,2) & (4,1,2)\end{pmatrix}\otimes\begin{pmatrix}(m_1^x,\alpha_1^x,\beta_1^x)\\ (m_2^x,\alpha_2^x,\beta_2^x)\end{pmatrix}=\begin{pmatrix}(54,46,54)\\ (52,27,56)\end{pmatrix}.$$

Hence, the *P-ALS* in (3.8) is given by $SX=B$,

$$\begin{pmatrix}\delta & 7 & 0 & 0 & 0 & 0\\ 6 & 4 & 0 & 0 & 0 & 0\\ 2 & 3 & \delta & 7 & 0 & 0\\ 1 & 1 & 6 & 4 & 0 & 0\\ 1 & 1 & 0 & 0 & \delta & 7\\ 2 & 2 & 0 & 0 & 6 & 4\end{pmatrix}\begin{pmatrix}m_1^x\\ m_2^x\\ \alpha_1^x\\ \alpha_2^x\\ \beta_1^x\\ \beta_2^x\end{pmatrix}=\begin{pmatrix}54\\ 52\\ 46\\ 27\\ 54\\ 56\end{pmatrix}.$$

The crisp solution X in term of δ is,

$$X = \begin{pmatrix} m_1^x \\ m_2^x \\ \alpha_1^x \\ \alpha_2^x \\ \beta_1^x \\ \beta_2^x \end{pmatrix} = \begin{pmatrix} \frac{74}{21-2\delta} \\ \frac{162-26\delta}{21-2\delta} \\ \frac{989-140\delta}{2(21-2\delta)^2} \\ \frac{1992-415\delta+28\delta^2}{2(21-2\delta)^2} \\ \frac{668-46\delta}{(21-2\delta)^2} \\ \frac{2694-598\delta+30\delta^2}{(21-2\delta)^2} \end{pmatrix}.$$

Using Step 3,

- $\tilde{x}_1$ is a positive fuzzy number if and only if $m_1^x \geq \alpha_1^x, \alpha_1^x \geq 0$ and $\beta_1^x \geq 0$, then,

$$\begin{cases} \frac{74}{21-2\delta} \geq \frac{989-140\delta}{2(21-2\delta)^2}, \\ \frac{989-140\delta}{2(21-2\delta)^2} \geq 0, \\ \frac{668-46\delta}{(21-2\delta)^2} \geq 0, \end{cases} \longrightarrow \begin{cases} \frac{163-12\delta}{(21-2\delta)^2} \geq 0, \\ \frac{989-140\delta}{(21-2\delta)^2} \geq 0, \\ \frac{334-23\delta}{(21-2\delta)^2} \geq 0, \end{cases} \longrightarrow \left\{ \delta \leq \frac{989}{140}. \right.$$

- $\tilde{x}_2$ is a positive fuzzy number if and only if $m_2^x \geq \alpha_2^x, \alpha_2^x \geq 0$ and $\beta_2^x \geq 0$, then,

$$\begin{cases} \dfrac{162-26\delta}{21-2\delta} \geq \dfrac{1992-415\delta+28\delta^2}{2(21-2\delta)^2}, \\ \dfrac{1992-415\delta+28\delta^2}{2(21-2\delta)^2} \geq 0, \\ \dfrac{2694-598\delta+30\delta^2}{(21-2\delta)^2} \geq 0, \end{cases} \longrightarrow \begin{cases} \dfrac{4812-1325\delta+76\delta^2}{(21-2\delta)^2} \geq 0, \\ \dfrac{1992-415\delta+28\delta^2}{(21-2\delta)^2} \geq 0, \\ \dfrac{1347-299\delta+15\delta^2}{(21-2\delta)^2} \geq 0, \end{cases}$$

$$\longrightarrow \begin{cases} \delta \leq \dfrac{1}{152}\left(1325-\sqrt{292777}\right), \\ \delta \geq \dfrac{1}{30}(299+\sqrt{8581}. \end{cases}$$

Hence, the interval solution of δ using $\tilde{x}_1$, $\tilde{x}_2$ is,

$$\begin{cases} \delta \leq \dfrac{989}{140}, \\ \delta \leq \dfrac{1}{152}\left(1325-\sqrt{292777}\right), \\ \text{and } \delta \geq \dfrac{1}{30}(299+\sqrt{8581}, \end{cases} \longrightarrow \begin{cases} \delta \leq 7.0642, \\ \delta \leq 5.157 \quad \text{and} \quad \delta \geq 13.054, \end{cases}$$

$$\longrightarrow \{\delta \leq 5.157.$$

Thus, $P-\tilde{X}$,

$$\tilde{X}_G = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{74}{21-2\delta}, \frac{989-140\delta}{2(21-2\delta)^2}, \frac{668-46\delta}{(21-2\delta)^2}\right) \\ \left(\frac{162-26\delta}{21-2\delta}, \frac{1992-415\delta+28\delta^2}{2(21-2\delta)^2}, \frac{2694-598\delta+30\delta^2}{(21-2\delta)^2}\right) \end{pmatrix},$$

where, $\delta \in [2, 5.157]$.

As a particular solution $\tilde{X}_p$, using $\tilde{X}_G$ when $\delta = 5.1$ is

$$\tilde{X}_p = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (6.851, 1.178, 3.715) \\ (2.722, 2.588, 3.639) \end{pmatrix}.$$

In last section the solution of positive $LR - FLS$ are proposed. To show that the proposed methods not only for solving positive $FFLS$, but also positive $LR - FLS$.

3.5 Positive Solution for Positive Left Right-Fuzzy Linear System

In this section, we will show the $P - \tilde{X}$ for positive $LR - FLS$ can be obtained by supposing the spreads matrices are zero, i.e. $M = N = 0$.

Next corollary shows that the relation between the solution of $FFLS$ and the solution of $LR - FLS$.

Corollary 3.5.1. The unique solution X of $SX = B$ represents a positive fuzzy vector $\tilde{X}$ for an arbitrary positive fuzzy vector $\tilde{B}$ if:

i- $A^{-1} > 0$.

ii- The FFLS is a LR-FLS.

Proof. According to Theorem 3.3.2, $M = N = 0$ in $\tilde{A} = (A, 0, 0)$. Then, based on Remark 2.2.1, the entries of left hand side in $FFLS$ are represented by only non fuzzy number (crisp numbers), hence the $FFLS$ is $LR - FLS$. □

In the next example $LR - FLS$ is solved by proposed method for solving $FFLS$ in Section 3.2.

Examples 3.5.1. Consider following $LR - FLS$,

$$\begin{cases} 10m_1^x + 9m_2^x = (120, 4, 9), \\ m_1^x + 8m_2^x = (80, 1, 5). \end{cases}$$

We can rewrite the $LR - FLS$ as $FFLS$ to obtain $P - \tilde{X}$,

$$\begin{cases} (10, 0, 0) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (9, 0, 0) \otimes (m_2^x, \alpha_2^x, \beta_2^x) = (120, 4, 9), \\ (1, 0, 0) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (8, , 0, 0) \otimes (m_2^x, \alpha_2^x, \beta_2^x) = (80, 1, 5). \end{cases}$$

Using *P-ALS*, $SX = B$ is,

$$\begin{pmatrix} 10 & 9 & 0 & 0 & 0 & 0 \\ 1 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 9 & 0 & 0 \\ 0 & 0 & 1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 9 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ \alpha_1^x \\ \alpha_2^x \\ \beta_1^x \\ \beta_2^x \end{pmatrix} = \begin{pmatrix} 120 \\ 80 \\ 4 \\ 1 \\ 9 \\ 5 \end{pmatrix}.$$

Hence, the positive solution is,

$$\tilde{X} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} \left(\frac{240}{71}, \frac{23}{71}, \frac{27}{71}\right) \\ \left(\frac{680}{71}, \frac{6}{71}, \frac{41}{71}\right) \end{pmatrix}.$$

3.6 Conclusion and Contribution

In this chapter, we proposed new methods to obtain the $P - FFLS$ for $P - \tilde{X}$. The coefficients of $\tilde{A}$ and the entries of fuzzy numbers $\tilde{B}$ were represented in a linear system.

We can summarize the findings in this chapter by the following contributions:

1- Obtain the $P - \tilde{X}$ to $P - FFLS$ without fuzzy operation.

2- The nature of the solution of $P - FFLS$ is distinguished for fuzziness of $P - \tilde{X}$ and possibilities of the solution $\tilde{X}$.

3- Provide the sufficient and necessary conditions for $P - FFLS$ to have a $P - \tilde{X}$ based on the fuzziness of the solution, where the $P - FFLS$ is examined before solving the system.

4- Classify the possibilities of the solution (i.e., unique solution, infinite number of solutions, no solution), in which the existence of for $\tilde{X}$ is examined before solving the $P - FFLS$.

5- Formulate fuzzy row reduced echelon method to provide infinite many solutions $P - \tilde{X}$ to $P - FFLS$.

6- Formulate the general form solution for $P - FFLS$ for arbitrary fuzzy vector $\tilde{B}_G$.

CHAPTER FOUR

POSITIVE SOLUTION FOR NEAR ZERO FULLY FUZZY LINEAR SYSTEM

In the previous Chapter Three, the $P-\tilde{X}$ for $P-FFLS$ is proposed using associated linear system *P-ALS*. While in this chapter the $P-\tilde{X}$ is proposed for $NZ-FFLS$ using a new associated linear system, because the arithmetic operations for near fuzzy numbers system depend of min-max function. In order to develop this linear system, min-max system is used to transfer the fuzzy system to linear equations, then a block matrix and block vectors will be used to include all entries of linear equations in the associated linear system. The consistency of the fully fuzzy linear system is provided.

4.1 Fundamental Concepts for Associated Linear System

This section presents the fundamental definitions and the main theorems to develop the new method for solving $P-\tilde{X}$ of $NZ-FFLS$. The next definitions are used in associated linear system for $NZ-FFLS$.

Definition 4.1.1. *Let $\tilde{A}$ be a fuzzy matrix $\tilde{A}=(A,M,N)$, where $A=\left(m_{i,j}^{a}\right)_{n\times n}$ and $M=\left(\alpha_{i,j}^{a}\right)_{n\times n}$, define following the matrices:*

i- *The matrix $C=\left({c_{i,j}}^{a}\right)_{n\times n}=A-M.$*

ii- *The matrix* $P_c=\left(c_{i,j}^{+a}\right)_{n\times n}=\begin{cases}c_{i,j}, & c_{i,j}^{a}\geq 0,\\ 0 & Otherwise.\end{cases}$

iii- *The matrix* $N_c = \left(c_{i,j}^{-a}\right)_{n\times n} = \begin{cases} c_{i,j}, & c_{i,j}^{a} < 0, \\ 0, & Otherwise. \end{cases}$

Definition 4.1.2. *Let* $\tilde{A}$ *be a fuzzy matrix* $\tilde{A} = (A, M, N)$, $A = \left(m_{i,j}^{a}\right)_{n\times n}$ *and* $N = \left(\beta_{i,j}^{a}\right)_{n\times n}$, *define the following matrices:*

i- *The matrix* $D = \left({d_{i,j}}^{a}\right)_{n\times n} = A + N.$

ii- *The matrix* $P_d = \left(d_{i,j}^{+a}\right)_{n\times n} = \begin{cases} d_{i,j}, & d_{i,j}^{a} \geq 0, \\ 0, & Otherwise. \end{cases}$

iii- *The matrix* $N_d = \left(d_{i,j}^{-a}\right)_{n\times n} = \begin{cases} d_{i,j} & d_{i,j}^{a} < 0, \\ 0 & Otherwise. \end{cases}$

The next definition and theorem are used in transforming the $NZ - FLLS$ to linear equations.

Definition 4.1.3. *A min-max system (or a system of min-max equation) is a collection of equations such that at least one of them has min or max equations.*

Theorem 4.1.1. Consider two *TFNs*, $\tilde{a} = (m^{a}, \alpha^{a}, \beta^{a})$, $\tilde{x} = (m^{x}, \alpha^{x}, \beta^{x})$, $\tilde{a}$ is an arbitrary fuzzy number while $\tilde{x}$ is a positive *TFN*.

i- If $\tilde{a}$ is positive, then the following inequalities are satisfied for all $\tilde{x}$.

$$0 \leq (m^{x} - \alpha^{x})(m^{a} - \alpha^{a}) \leq (m^{x} + \beta^{x})(m^{a} - \alpha^{a}), \qquad (4.1a)$$

$$0 \leq (m^{x} - \alpha^{x})(m^{a} + \beta^{a}) \leq (m^{x} + \beta^{x})(m^{a} + \beta^{a}). \qquad (4.1b)$$

ii- If $\tilde{a}$ is negative, then the following inequalities are satisfied for all $\tilde{x}$.

$$0 \geq (m^x - \alpha^x)(m^a - \alpha^a) \geq (m^x + \beta^x)(m^a - \alpha^a), \qquad (4.2a)$$

$$0 \geq (m^x - \alpha^x)(m^a + \beta^a) \geq (m^x + \beta^x)(m^a + \beta^a). \qquad (4.2b)$$

iii- If $\tilde{a}$ *TFN* is near zero, then the inequalities in Equations (4.1b) and (4.2a) are satisfied for all $\tilde{x}$.

Proof. Since $\tilde{x}$ is positive then,

$$0 \leq (m^x - \alpha^x) \leq (m^x + \beta^x). \qquad (4.3)$$

i- If $\tilde{a}$ is positive, then $(m^a + \alpha^a)$ and $(m^a - \beta^a)$ are positive.

Since $0 \leq (m^a - \beta^a)$, the inequality in Equation (4.1a) can be obtained by multiplying $(m^a - \beta^a)$ at the inequality in Equation (4.3), to get,

$$0 \leq (m^x - \alpha^x)(m^a - \beta^a) \leq (m^x + \beta^x)(m^a - \beta^a).$$

Similarly, since $0 \leq (m^a + \beta^a)$, the inequality in Equation $(4.1b)$ can be obtained by multiplying $(m^a + \beta^a)$ at the inequality in Equation (4.3),

$$0 \leq (m^x - \alpha^x)(m^a + \beta^a) \leq (m^x + \beta^x)(m^a + \beta^a).$$

ii- If $\tilde{a}$ is negative then $(m^a - \beta^a)$ and $(m^a + \alpha^a)$ are negative.

Since $0 \geq (m^a - \beta^a)$, the inequality in Equation (4.2a) can be obtained by multiplying $(m^a - \beta^a)$ at the inequality in Equation (4.3),

$$0 \geq (m^x - \alpha^x)(m^a - \alpha^a) \geq (m^x + \beta^x)(m^a - \alpha^a).$$

Similarly, since $0 \geq (m^a + \beta^a)$, the inequality in Equation (4.2b) can be obtained by multiplying $(m^a + \beta^a)$ at the inequality in Equation (4.3),

$$0 \geq (m^x - \alpha^x)(m^a + \beta^a) \geq (m^x + \beta^x)(m^a + \beta^a).$$

iii- $\tilde{a}$ is near zero, then $(m^a + \beta^a)$ is positive and $(m^a - \alpha^a)$ is negative.

Because of $0 \leq (m^a + \beta^a)$, the inequality in Equation (4.1b) can be obtained by multiplying $(m^a + \beta^a)$ at the inequality in Equation (4.3),

$$0 \leq (m^x - \alpha^x)(m^a + \beta^a) \leq (m^x + \beta^x)(m^a + \beta^a).$$

Similarly, since $0 \geq (m^a - \alpha^a)$ the inequality in Equation (4.2a) can be obtained by multiplying $(m^a - \alpha^a)$ at the inequality in Equation (4.3),

$$0 \geq (m^x - \alpha^x)(m^a - \alpha^a) \geq (m^x + \beta^x)(m^a - \alpha^a). \qquad \square$$

The previous theorem is used in next section to transfer the $NZ - FFL$ to associated linear system without fuzzy operation.

4.2 Positive Solution for Near Zero Fully Fuzzy Linear System

The $P - \tilde{X}$ for $NZ - FFLS$ is obtained in this section, then $NZ - FFLS$ is transformed to min-max system. Subsequently using Theorem 4.1.1. the min-max system is transformed to linear system. The solution of this linear system provides a vector X which is an equivalent to $P - \tilde{X}$.

The solution is obtained in following three steps:

Step 1 Transferring the fuzzy system $NZ - FFLS$ to min-max system.

Applying Equation (2.12) for $\tilde{A} \otimes \tilde{X} = \tilde{B}$, where $\tilde{A} = (A, M, N), \tilde{B}(m^b, \alpha^b, \beta^b)$ and $\tilde{X} = (m^x, \alpha^x, \beta^x) \geq 0$, then we have,

$$\left(m_i^b, \alpha_i^b, \beta_i^b\right) = \left(m_{i,j}^a m_j^x, m_{i,j}^a m_j^x - Min\left[\left(m_{i,j}^a - \alpha_{i,j}^a\right)\left(m_j^x - \alpha_j^x\right), \left(m_{i,j}^a - \alpha_{i,j}^a\right)\left(m_j^x + \beta_j^x\right)\right], -m_{i,j}^a m_j^x + Max\left[\left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x - \alpha_j^x\right), \left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x + \beta_j^x\right)\right]\right),$$

thus,

$$m_i^b = m_{i,j}^a m_j^x, \tag{4.4a}$$

$$\alpha_i^b = m_{i,j}^a m_j^x - Min\left[\left(m_{i,j}^a - \alpha_{i,j}^a\right)\left(m_j^x - \alpha_j^x\right), \left(m_{i,j}^a - \alpha_{i,j}^a\right)\left(m_j^x + \beta_j^x\right)\right], \tag{4.4b}$$

$$\beta_i^b = -m_{i,j}^a m_j^x + Max\left[\left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x - \alpha_j^x\right), \left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x + \beta_j^x\right)\right]. \tag{4.4c}$$

Hence, $\forall i = 1, .., n$, we have three equations m_i^b, α_i^b and β_i^b, where α_i^b and β_i^b are functions of fuzzy number $\tilde{a}_{ij} = (m_{i,j}^a, \alpha_{i,j}^a, \beta_{i,j}^a)$ including min and max functions, respectively. Then we have $3n$ min-max system.

Step 2 Transferring the min equations α_i^b in Equation (4.4b) and max equations β_i^b in Equation (4.4c) to linear equations.

For α_i^b:

Suppose

$$f_i^\alpha = Min\left[\left(m_j^x - \alpha_j^x\right)\left(m_{i,j}^a - \alpha_{i,j}^a\right), \left(m_{i,j}^a - \alpha_{i,j}^a\right)\left(m_j^x + \beta_j^x\right)\right]. \tag{4.5}$$

According to the sign of $\left(m_j^a, \alpha_j^a, \beta_j^a\right)$, we have two possibilities for f_i^α:

i- If $\tilde{a}_{ij}$ is positive, then using Equation (4.1a) in Theorem 4.1.1,

$$f_i^\alpha = Min\left[\left(m_j^x - \alpha_j^x\right)\left(m_{i,j}^a - \alpha_{i,j}^a\right), \left(m_{i,j}^a - \alpha_{i,j}^a\right)\left(m_j^x + \beta_j^x\right)\right]$$

$$= \left(m_j^x - \alpha_j^x\right)\left(m_{i,j}^a - \alpha_{i,j}^a\right). \tag{4.6a}$$

ii- If $\tilde{a}_{ij}$ is negative or near zero then, using Equation (4.2a),

$$f_i^\alpha = Min\left[\left(m_j^x - \alpha_j^x\right)\left(m_{i,j}^a - \alpha_{i,j}^a\right), \left(m_{i,j}^a - \alpha_{i,j}^a\right)\left(m_j^x + \beta_j^x\right)\right]$$

$$= \left(m_j^x + \beta_j^x\right)\left(m_{i,j}^a - \alpha_{i,j}^a\right). \tag{4.6b}$$

Hence, Equations (4.6a) and (4.6b) can be written in a piecewise function as follows,

$$f_i^\alpha = \begin{cases} \left(m_{i,j}^a - \alpha_{i,j}^a\right)\left(m_j^x - \alpha_j^x\right), & \left(m_{i,j}^a - \alpha_{i,j}^a\right) \geq 0, \\ \left(m_{i,j}^a - \alpha_{i,j}^a\right)\left(m_j^x + \beta_j^x\right), & \left(m_{i,j}^a - \alpha_{i,j}^a\right) < 0. \end{cases} \tag{4.7}$$

Using Definition 4.1.1. $c_{i,j}^a = m_{i,j}^a - \alpha_{i,j}^a$, then,

$$f_i^\alpha = \begin{cases} c_{i,j}^{+a}\left(m_j^x - \alpha_j^x\right), & c_{i,j}^a \geq 0, \\ c_{i,j}^{-a}\left(m_j^x + \beta_j^x\right), & c_{i,j}^a < 0. \end{cases} \tag{4.8}$$

Since, either $c_{i,j}^{+a} = 0$ or $c_{i,j}^{-a} = 0$ (or both),

$$f_i^\alpha = c_{i,j}^{+a}\left(m_j^x - \alpha_j^x\right) + c_{i,j}^{-a}\left(m_j^x + \beta_j^x\right)$$

$$= c_{i,j}^{+a} m_j^x - c_{i,j}^{+a}\alpha_j^x + c_{i,j}^{-a} m_j^x + c_{i,j}^{-a}\beta_j^x$$

$$= \left(c_{i,j}^{+a} + c_{i,j}^{-a}\right)m_j^x + c_{i,j}^{-a}\beta_j^x - c_{i,j}^{+a}\alpha_j^x$$

$$= \left(c_{i,j}^a\, m_j^x\right) + \left(c_{i,j}^{-a}\beta_j^x - c_{i,j}^{+a}\alpha_j^x\right),$$

hence,

$$f_i^{\alpha} = \left(m_{i,j}^a - \alpha_{i,j}^a\right)m_j^x + \left(c_{i,j}^{-a}\beta_j^x - c_{i,j}^{+a}\alpha_j^x\right). \quad (4.9)$$

Thus, Equation (4.4b) can be written as,

$$\alpha_i^b = m_{i,j}^a m_j^x - f_i^{\alpha}$$

$$= m_{i,j}^a m_j^x - \left\{m_{i,j}^a m_j^x - \alpha_{i,j}^a m_j^x + c_{i,j}^{-a}\beta_j^x - c_{i,j}^{+a}\alpha_j^x\right\}$$

$$= \left(m_{i,j}^a m_j^x - m_{i,j}^a m_j^x\right) + \alpha_{i,j}^a m_j^x - c_{i,j}^{-a}\beta_j^x + c_{i,j}^{+a}\alpha_j^x$$

$$= \alpha_{i,j}^a m_j^x - c_{i,j}^{-a}\beta_j^x + c_{i,j}^{+a}\alpha_j^x, \quad (4.10)$$

Hence, the min equation in Equation (4.4b) is equal to following linear equation for all $i = 1, \ldots, n$.

$$\alpha_i^b = \alpha_{i,j}^a m_j^x - c_{i,j}^{-a}\beta_j^x + c_{i,j}^{+a}\alpha_j^x, \quad (4.11)$$

For $\boldsymbol{\beta_i^b}$:

Suppose

$$\boldsymbol{f_i^{\beta} = Max\left[\left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x - \alpha_j^x\right), \left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x + \beta_j^x\right)\right],} \quad (\mathbf{4.12})$$

according to the sign of $\left(m_j^a, \alpha_j^a, \beta_j^a\right)$, we have two possibilities for f_i^{β}:

i- If $\tilde{a}_{ij}$ is positive or near zero, then using Equation (4.1b),

$$f_i^{\beta} = Max\left[\left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x - \alpha_j^x\right), \left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x + \beta_j^x\right)\right]$$

$$= \left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x + \beta_j^x\right). \quad (4.13a)$$

ii- If $\tilde{a}_{ij}$ is negative, then using Equation (4.2b),

$$f_i^{\beta} = Max\left[\left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x - \alpha_j^x\right), \left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x + \beta_j^x\right)\right]$$

$$= \left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x - \alpha_j^x\right). \tag{4.13b}$$

Hence, Equations (4.13a) and (4.13b) can be written in a piecewise function is as follows,

$$f_i^{\beta} = \begin{cases} \left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x + \beta_j^x\right), & m_{i,j}^a + \beta_{i,j}^a \geq 0, \\ \left(m_{i,j}^a + \beta_{i,j}^a\right)\left(m_j^x - \alpha_j^x\right), & m_{i,j}^a + \beta_{i,j}^a \leq 0. \end{cases} \tag{4.14}$$

Using Definition 4.1.2. $d_{i,j}^a = m_{i,j}^a + \beta_{i,j}^a$, then

$$f_i^{\beta} = \begin{cases} d_{i,j}^{+a}\left(m_j^x + \beta_j^x\right), & d_{i,j}^a \geq 0, \\ d_{i,j}^{-a}\left(m_j^x - \alpha_j^x\right), & d_{i,j}^a < 0, \end{cases} \tag{4.15}$$

Since, either $d_{i,j}^{+a} = 0$ or $d_{i,j}^{-a} = 0$ (or both),

$$f_i^{\beta} = d_{i,j}^{+a}\left(m_j^x + \beta_j^x\right) + d_{i,j}^{-a}\left(m_j^x - \alpha_j^x\right)$$

$$= d_{i,j}^{+a} m_j^x + d_{i,j}^{+a}\beta_j^x + d_{i,j}^{-a} m_j^x - d_{i,j}^{-a}\alpha_j^x$$

$$= \left(d_{i,j}^{+a} m_j^x + d_{i,j}^{-a} m_j^x\right) + \left(d_{i,j}^{+a}\beta_j^x - d_{i,j}^{-a}\alpha_j^x\right)$$

$$= \left(d_{i,j}^{+a} + d_{i,j}^{-a}\right) m_j^x + \left(d_{i,j}^{+a}\beta_j^x - d_{i,j}^{-a}\alpha_j^x\right)$$

$$= \left(d_{i,j}^{a} m_j^x\right) + \left(d_{i,j}^{+a}\beta_j^x - d_{i,j}^{-a}\alpha_j^x\right) \tag{4.16}$$

hence,

$$f_i^{\beta} = \left(m_{i,j}^{a} + \beta_{i,j}^{a}\right)m_j^{x} + \left(d_{i,j}^{+a}\beta_j^{x} - d_{i,j}^{-a}\alpha_j^{x}\right),$$

Equation (4.5c) can be written,

$$\beta_i^{b} = -m_{i,j}^{a}m_j^{x} + f_i^{\beta}$$

$$= -m_{i,j}^{a}m_j^{x} + \left\{\left(m_{i,j}^{a} + \beta_{i,j}^{a}\right)m_j^{x} + \left(d_{i,j}^{+a}\beta_j^{x} - d_{i,j}^{-a}\alpha_j^{x}\right)\right\},$$

$$= \left(-m_{i,j}^{a}m_j^{x} + m_{i,j}^{a}m_j^{x}\right) + \beta_{i,j}^{a}m_j^{x} + \left(d_{i,j}^{+a}\beta_j^{x} - d_{i,j}^{-a}\alpha_j^{x}\right), = 0 + \beta_{i,j}^{a}m_j^{x} + \left(d_{i,j}^{+a}\beta_j^{x} - d_{i,j}^{-a}\alpha_j^{x}\right).$$

Hence, the max equation in Equation (4.4c) is equal to following linear equation for all $i = 1, \ldots, n$.

$$\beta_i^{b} = \beta_{i,j}^{a}m_j^{x} + d_{i,j}^{+a}\beta_j^{x} - d_{i,j}^{-a}\alpha_j^{x}. \tag{4.17}$$

Step 3 Collecting the linear equations in Equations (4.4a), (4.11) and (4.17) in a linear system.

The fuzzy system can be written as follows,

$$\tilde{A} \otimes \tilde{X} = \tilde{B} = \sum_{j=1}^{\oplus}\left(m_{i,j}^{a}, \alpha_{i,j}^{a}, \beta_{i,j}^{a}\right) \otimes \left(m_j^{x}, \alpha_j^{x}, \beta_j^{x}\right)$$

$$= \left(\sum_{j=1}^{n} m_{i,j}^{a}m_j^{x}, \sum_{j=1}^{n}\left(\alpha_{i,j}^{a}m_j^{x} - c_{i,j}^{-a}\beta_j^{x} + c_{i,j}^{+a}\alpha_j^{x}\right), \sum_{j=1}^{n}\left(\beta_{i,j}^{a}m_j^{x} + d_{i,j}^{+a}\beta_j^{x} - d_{i,j}^{-a}\alpha_j^{x}\right)\right),$$

$$\forall i = 1, \ldots, n, \tag{4.18}$$

then, the parameters m_j^x, α_j^x and β_j^x can be obtained as follows:

- The mean values m_j^x are obtained separately using the following $n \times n$ linear system,

$$\sum_{j=1}^{n} m_{i,j}^{a} m_j^x = m_i^b \text{ }, \forall \; i = 1, .., n. \tag{4.19a}$$

By solving the $n \times n$ linear system in Equation (4.19a), we can find the mean values.

- The spreads values α_j^x and β_j^x are obtained jointly using the following $2n \times 2n$ linear system, where the values m_j^x are given from Equation (4.19a),

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \left(\alpha_{i,j}^{a} m_j^x - c_{i,j}^{-a} \beta_j^x + c_{i,j}^{+a} \alpha_j^x \right) = \alpha_i^b \text{ }, \forall \; i = 1, .., n, \\ \displaystyle\sum_{j=1}^{n} \left(\beta_{i,j}^{a} m_j^x + d_{i,j}^{+a} \beta_j^x - \mathrm{d}_{i,j}^{-a} \alpha_j^x \right) = \beta_j^b \text{ }, \forall \; i = 1, .., n. \end{cases} \tag{4.19b}$$

By solving the $2n \times 2n$ linear system in Equation (4.19b), we can find spreads values α_j^x, β_j^x.

The next definition and theorem are used to write the linear systems in Equations (4.19a) and (4.19b) in matrix form.

Definition 4.2.1. *Let* $\tilde{A} = (A, M, N)$ *be a fuzzy matrix.* $\tilde{B} = (m^b, \alpha^b, \beta^b)$ *and* $\tilde{X} = (m^x, \alpha^x, \beta^x)$ *are fuzzy vectors. Define the* $3n \times 3n$ *linear systems as,*

$$JX = B,$$

$$\begin{pmatrix} A & 0 & 0 \\ M & P_c & -N_c \\ N & -N_d & P_d \end{pmatrix} \begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} m^b \\ \alpha^b \\ \beta^b \end{pmatrix}, \tag{4.20}$$

where $J_{12} = J_{13} = (0)_{n \times n}$.

In this study, the linear system $JX = B$ is called the near zero associated linear system *(*$NZ - ALS$*)* for $NZ - FFLS$.

The following theorem shows the relation between the components $3n$ dimensional crisp solution m^x, α^x and β^x of vector X in Equation (4.20) and the $P - \tilde{X}$ for $NZ - FFLS$ in Equations (4.19a) and (4.19b), according to equivalent concept between X and $P - \tilde{X}$ in Remark 3.1.1.

Theorem 4.2.1. The unique solution of crisp system $JX = B$ and $P - \tilde{X}$ for $NZ - FFLS$ is equivalent.

Proof. The linear system in Equation (4.20) is $JX = B$,

$$\begin{pmatrix} A & 0 & 0 \\ M & P_c & -N_c \\ N & -N_d & P_d \end{pmatrix} \begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} m^b \\ \alpha^b \\ \beta^b \end{pmatrix},$$

$$\begin{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} & \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} & \begin{pmatrix} c_{11}^+ & \cdots & c_{1n}^+ \\ \vdots & \ddots & \vdots \\ c_{n1}^+ & \cdots & c_{nn}^+ \end{pmatrix} & \begin{pmatrix} -c_{11}^- & \cdots & -c_{1n}^- \\ \vdots & \ddots & \vdots \\ -c_{n1}^- & \cdots & -c_{nn}^- \end{pmatrix} \\ \begin{pmatrix} n_{11} & \cdots & n_{1n} \\ \vdots & \ddots & \vdots \\ n_{n1} & \cdots & n_{nn} \end{pmatrix} & \begin{pmatrix} -d_{11}^- & \cdots & -d_{1n}^- \\ \vdots & \ddots & \vdots \\ -d_{n1}^- & \cdots & -d_{nn}^- \end{pmatrix} & \begin{pmatrix} d_{11}^+ & \cdots & d_{1n}^+ \\ \vdots & \ddots & \vdots \\ d_{n1}^+ & \cdots & d_{nn}^+ \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m_1^x \\ \vdots \\ m_n^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \vdots \\ \alpha_1^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \vdots \\ \beta_1^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} m_1^b \\ \vdots \\ m_n^b \end{pmatrix} \\ \begin{pmatrix} \alpha_1^b \\ \vdots \\ \alpha_1^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^b \\ \vdots \\ \beta_1^b \end{pmatrix} \end{pmatrix} \qquad (4.21)$$

$JX = B$ is separated into the following three linear systems.

For $\boldsymbol{i = 1, \ldots, n}$,

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} m_1^x \\ \vdots \\ m_n^x \end{pmatrix} = \begin{pmatrix} m_1^b \\ \vdots \\ m_n^b \end{pmatrix},$$

then,

$$\sum_{j=1}^{n} a_{i,j} m_j^x = m_i^b \text{ }, \forall\, i = 1, .., n,$$

by substituting $a_{i,j} = m_{i,j}^a$, we get $(4.19a)$.

For $\boldsymbol{i = n + 1, \ldots, 2n}$,

$$\begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} \begin{pmatrix} m_1^x \\ \vdots \\ m_n^x \end{pmatrix} + \begin{pmatrix} c_{11}^+ & \cdots & c_{1n}^+ \\ \vdots & \ddots & \vdots \\ c_{n1}^+ & \cdots & c_{nn}^+ \end{pmatrix} \begin{pmatrix} \alpha_1^x \\ \vdots \\ \alpha_1^x \end{pmatrix}$$

$$+\begin{pmatrix} -c_{11}^{-} & \cdots & -c_{1n}^{-} \\ \vdots & \ddots & \vdots \\ -c_{n1}^{-} & \cdots & -c_{nn}^{-} \end{pmatrix}\begin{pmatrix} \beta_1^x \\ \vdots \\ \beta_1^x \end{pmatrix} = \begin{pmatrix} \alpha_1^b \\ \vdots \\ \alpha_1^x \end{pmatrix},$$

then,

$$\sum_{j=1}^{n}\left(m_{i,j}\, m_j^x - c_{i,j}^{-a}\, \beta_j^x + c_{i,j}^{+a}\, \alpha_j^x\right) = \alpha_i^b\,, \forall\, i = 1, \ldots, n,$$

by rearranging the equation,

$$\sum_{j=1}^{n}\left(m_{i,j}\, m_j^x + c_{i,j}^{+a}\, \alpha_j^x - c_{i,j}^{-a}\, \beta_j^x\right) = \alpha_i^b\,, \forall\, i = 1, \ldots, n. \qquad (4.22)$$

For $i = 2n + 1, \ldots, 3n$,

$$\begin{pmatrix} n_{11} & \cdots & n_{1n} \\ \vdots & \ddots & \vdots \\ n_{n1} & \cdots & n_{nn} \end{pmatrix}\begin{pmatrix} m_1^x \\ \vdots \\ m_n^x \end{pmatrix} + \begin{pmatrix} -d_{11}^{-} & \cdots & -d_{1n}^{-} \\ \vdots & \ddots & \vdots \\ -d_{n1}^{-} & \cdots & -d_{nn}^{-} \end{pmatrix}\begin{pmatrix} \alpha_1^x \\ \vdots \\ \alpha_1^x \end{pmatrix}$$

$$+\begin{pmatrix} d_{11}^{+} & \cdots & d_{1n}^{+} \\ \vdots & \ddots & \vdots \\ d_{n1}^{+} & \cdots & d_{nn}^{+} \end{pmatrix}\begin{pmatrix} \beta_1^x \\ \vdots \\ \beta_1^x \end{pmatrix} = \begin{pmatrix} \beta_1^b \\ \vdots \\ \beta_1^b \end{pmatrix},$$

then,

$$\sum_{j=1}^{n}\left(n_{i,j}\, m_j^x + d_{i,j}^{+a}\beta_j^x - d_{i,j}^{-a}\alpha_j^x\right) = \beta_i^b\,, \forall\, i = 1, \ldots, n,$$

by rearranging the equation,

$$\sum_{j=1}^{n}\left(n_{i,j}\, m_j^x - d_{i,j}^{-a}\, \alpha_j^x + d_{i,j}^{+a}\, \beta_j^x\right) = \beta_i^b\,, \forall\, i = 1, \ldots, n. \qquad (4.23)$$

by substituting $m_{i,j} = \alpha_{i,j}^{a}$ and $n_{i,j} = \beta_{i,j}^{a}$, we get Equations (4.22) and (4.23) are equivalent to Equation (4.19b). □

The next theorem investigates the singularity of block matrix which is similar to structure of block matrix in Equation (4.21).

Theorem 4.2.2. Let F be a $3n \times 3n$ matrix and consists of $n \times n$ crisp matrices $J_{i,j}$ for $i, j = 1, .., n$, where $J_{12} = J_{13} = Z = (0)_{n\times n}$ and J_{22} is invertible,

$$F = \begin{pmatrix} J_{11} & Z & Z \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}.$$

Then,

$$|F| = |J_{11}|.|J_{22}|.|H|, \tag{4.24}$$

where $H = (J_{33}) - (J_{32})(J_{22})^{-1}(J_{23})$.

Proof. Consider the matrix F,

$$F = \begin{pmatrix} J_{11} & Z & Z \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} = \begin{pmatrix} J_{11} & 0 & 0 \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix},$$

J_{22} is invertible, then multiplying column two by $-(J_{23})(J_{22})^{-1}$ and adding to column three,

$$\begin{vmatrix} J_{11} & 0 & 0 \\ J_{21} & J_{22} & -(J_{23})(J_{22})^{-1}(J_{22}) + (J_{23}) \\ J_{31} & J_{32} & -(J_{23})(J_{22})^{-1}(J_{32}) + (J_{33}) \end{vmatrix} = \begin{vmatrix} J_{11} & 0 & 0 \\ J_{21} & J_{22} & 0 \\ J_{31} & J_{32} & -(J_{23})(J_{22})^{-1}(J_{32}) + (J_{33}) \end{vmatrix}$$

$$= \begin{vmatrix} J_{11} & 0 & 0 \\ J_{21} & J_{22} & 0 \\ J_{31} & J_{32} & -(J_{23})(J_{22})^{-1}(J_{32}) + (J_{33}) \end{vmatrix} = |K_2|,$$

then,

$$|F| = |K_2| = |J_{11}|.\,|J_{22}|.\,|-(J_{32})(J_{22})^{-1}(J_{23}) + J_{33}|.$$

But,

$$|-(J_{32})(J_{22})^{-1}(J_{23}) + (J_{33})| = |(J_{33}) - (J_{32})(J_{22})^{-1}(J_{23})| = |H|,$$

hence,

$$|F| = |J_{11}|.\,|J_{22}|.\,|H|. \qquad \square$$

Corollary 4.2.1. The unique $P - \tilde{X}$ can be checked for $NZ - FFLS$ by two approaches according to singularity of P_c:

a- If P_c is nonsingular, then $NZ - FFLS$ has a unique solution if $|A|$ and $|H| \neq 0$, where $H = (P_d) - (N_d)(P_c)^{-1}(N_c)$.

b- If P_c is singular, then the $NZ - FFLS$ has a unique solution if $|J| \neq 0$.

Proof.

a- Since the structure of matrix F is as structure of matrix J, Equation (4.24) in Theorem 4.2.2. can be applied by supposing that $J_{11} = A$, $J_{22} = P_c$ and the matrix H can be obtained as follows,

$$H = -(J_{32})(J_{22})^{-1}(J_{23}) + (J_{33}) = -(-N_d)(P_c)^{-1}(-N_c) + (P_d) =$$
$$-(-N_d)(P_c)^{-1}(-N_c) + (P_d) = (P_d) - (N_d)(P_c)^{-1}(N_c).$$

Then, if P_c is nonsingular, the linear system $JX = B$ has a unique solution if $|A|$ and $|H| \neq 0$. Thus, $NZ - FFLS$ has a unique solution.

b- If P_c is singular, then Theorem 4.2.2. can not be applied. Thus we have to check the singularity of matrix J in Equation (4.20). Then, $NZ - FFLS$ has a unique solution if and only if $|J| \neq 0$.

Two examples were in literature are solved. The first example has a unique fuzzy solution while the second one has non fuzzy unique solution, which shows show that *NZ-ALS* can provide the exact fuzzy or non fuzzy solution. Example 4.2.1. illustrates our method where the fuzzy system has a fuzzy unique solution. This example is the same example as in Babbar et al. (2013). The new method proposed a similar solution for it. The uniqueness of system is checked using Corollary 4.2.1a.

Example 4.2.1. Babbar et al. (2013) consider the following $FFLS$,

$$\begin{cases}(4,6,1) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (4,2,4) \otimes (m_2^x, \alpha_2^x, \beta_2^x) = (24,26,31), \\ (3,2,1) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (2,3,1) \otimes (m_2^x, \alpha_2^x, \beta_2^x) = (14,18,13),\end{cases}$$

where $\tilde{x}_i = ({m_i}^x, {\alpha_i}^x, {\beta_i}^x) \geq 0, i = 1,2.$

The system can be written in matrix form,

$$\begin{pmatrix}(4,6,1) & (4,2,4)\\ (3,2,1) & (2,3,1)\end{pmatrix} \otimes \begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix} = \begin{pmatrix}(24,26,31)\\ (14,18,13)\end{pmatrix}.$$

$P-\tilde{X}$ can be obtained by proposed methods as follows,

$$A = \begin{pmatrix}4 & 4\\ 3 & 2\end{pmatrix}, \qquad M = \begin{pmatrix}6 & 2\\ 2 & 3\end{pmatrix}, \qquad N = \begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix},$$

$$A-M = \begin{pmatrix}-2 & 2\\ 1 & -1\end{pmatrix}, \quad P_c = \begin{pmatrix}0 & 2\\ 1 & 0\end{pmatrix}, \quad N_c = \begin{pmatrix}-2 & 0\\ 0 & -1\end{pmatrix},$$

$$A+N = \begin{pmatrix}5 & 8\\ 4 & 3\end{pmatrix}, \qquad N_d = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \qquad P_d = \begin{pmatrix}5 & 8\\ 4 & 3\end{pmatrix}.$$

$|P_c| = -2$, then Corollary 4.2.1a. can be applied, $|A| = -4$, and $|H| = -17$, where H is,

$$H = (P_d) - (N_d)\,(P_c)^{-1}(N_c) = \begin{pmatrix}5 & 8\\ 4 & 3\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 2\\ 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}-2 & 0\\ 0 & -1\end{pmatrix}$$

$$= \begin{pmatrix}5 & 8\\ 4 & 3\end{pmatrix}.$$

Hence, the *NZ-ALS* has a unique solution. Thus, the fuzzy system *NZ-FFLS* has a unique solution.

Also, the determinate of matrix J is also confirms the result of Equation (4.24) in Theorem 4.2.2,

$$|J| = \begin{vmatrix}4 & 4 & 0 & 0 & 0 & 0\\ 3 & 2 & 0 & 0 & 0 & 0\\ 6 & 2 & 0 & 2 & 2 & 0\\ 2 & 3 & 1 & 0 & 0 & 1\\ 1 & 4 & 0 & 0 & 5 & 8\\ 1 & 1 & 0 & 0 & 4 & 3\end{vmatrix} = -136,$$

and,

$$|A|.|P_c|.|H| = (-4)(-2)(-17) = -136.$$

The solution can be obtained using *NZ-ALS* in Equation (4.21),

$$\begin{pmatrix} \begin{pmatrix} 4 & 4 \\ 3 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 5 & 8 \\ 4 & 3 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 24 \\ 14 \end{pmatrix} \\ \begin{pmatrix} 26 \\ 18 \end{pmatrix} \\ \begin{pmatrix} 31 \\ 13 \end{pmatrix} \end{pmatrix},$$

using inversion matrix method, then the crisp solution is,

$$X = J^{-1}\,B = \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix},$$

which is equivalent to fuzzy solution in Babbar et al. (2013),

$$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (2, 1, 1) \\ (4, 2, 1) \end{pmatrix}.$$

Babbar et al. (2013)'method is compared with the proposed method in Table 4.1.

Table 4.1

Comparison between the proposed method and Babbar et al. (2013)'method.

	The proposed method	**Babbar et al. (2013)'method**
Fuzzy operation	No fuzzy operation.	Need to find positive, negative and near zero fuzzy matrices.
Existence of solution	It can be determined before solving the system.	After solving the system.
The size of system	Large system as $n = 10$.	All examples are in size $n = 2$.

As noted in Table 4.1, the proposed method can determine the uniqueness of solution of $NZ - FFLS$ before obtaining the solution, while the method in Babbar et al. (2013) cannot check that before obtaining the fully solution. Moreover, the proposed method can obtain the solution without fuzzy operation while in Babbar et al. (2013)'method needs to find three fuzzy matrices; positive, negative and near zero fuzzy matrices. Additionally, in the proposed method there is no linear or min-max systems that needs to be solved, all systems are transferred to matrix form and find the solution with matrix inversion method, while Babbar et al. (2013)'method needs to construct and solve linear system according to signs of fuzzy matrices. For that, the proposed method can solve any $NZ - FFLS$ regardless the size of matrix $\tilde{A}$ as $n = 10$. In Example 4.2.3, we solve *NZ-FLS* when the fuzzy matrix $n = 10$ using the proposed method, while the size of matrix $\tilde{A}$ is not more $n = 2$ in Babbar et al. (2013)'method.

The next example illustrates our method where the fuzzy system has a unique non fuzzy solution. Note that, this example is negative *FFLS* since all the entries are negative *TFNs*, which indicates the proposed method can provide positive solution for negative *FFLS.* This example uses the same example as in Ezzati et al. (2012). However, the new method proposed a different solution for Ezzati et al. (2012). A verification of solution is provided for both solutions. The uniqueness of the system is checked using Corollary 4.2.1b.

Example 4.2.2. Ezzati et al. (2012) consider the following $FFLS$ $\left(written\ in\ the\ form\ (a,b,c)\right)$,

$$\begin{pmatrix} (-10,-7,-4) & (-7,-5,-3) & (-5,-2,-1) \\ (-6,-4,-2) & (-4,-3,-1) & (-8,-5,-5) \\ (-8,-7,-3) & (-11,-1,-1) & (-3,-2,-1) \end{pmatrix} \otimes \begin{pmatrix} (a_1^x, b_1^x, c_1^x) \\ (a_2^x, b_2^x, c_2^x) \\ (a_3^x, b_3^x, c_3^x) \end{pmatrix}$$

$$= \begin{pmatrix} (-36,-36,-33) \\ (-26,-25,-23) \\ (-44,-21,-19) \end{pmatrix}.$$

Because we follow (m,α,β) form for TFN in this study, this example is converted to form (m,α,β). We will use $\tilde{A}', \tilde{X}'$ and $\tilde{B}'$ to form (a,b,c), while $\tilde{A}, \tilde{X}$ and $\tilde{B}$ to form (m,α,β).

The matrix form $\tilde{A} \otimes \tilde{X} = \tilde{B}$ becomes, $\left(in\ the\ form\ (m,\alpha,\beta)\right)$

$$\begin{pmatrix} (-7,3,3) & (-5,2,2) & (-2,3,1) \\ (-4,2,2) & (-3,1,2) & (-5,3,0) \\ (-7,1,4) & (-1,10,0) & (-2,1,1) \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (-36,0,3) \\ (-25,1,2) \\ (-21,23,2) \end{pmatrix}.$$

The positive solution can be obtained as follows,

$$A = \begin{pmatrix} -7 & -5 & -2 \\ -4 & -3 & -5 \\ -7 & -1 & -2 \end{pmatrix}, M = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 10 & 1 \end{pmatrix}, N = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 4 & 0 & 1 \end{pmatrix},$$

$$A - M = \begin{pmatrix} -10 & -7 & -5 \\ -6 & -4 & -8 \\ -8 & -11 & -3 \end{pmatrix} \text{ then } P_c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ N_c = \begin{pmatrix} -10 & -7 & -5 \\ -6 & -4 & -8 \\ -8 & -11 & -3 \end{pmatrix},$$

$$A + N = \begin{pmatrix} -4 & -3 & -1 \\ -2 & -1 & -5 \\ -3 & -1 & -1 \end{pmatrix} \text{ then } N_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ P_d = \begin{pmatrix} -4 & -3 & -1 \\ -2 & -1 & -5 \\ -3 & -1 & -1 \end{pmatrix}.$$

Since P_c is zero matrix then $|P_c| = 0$, Corollary 4.2.1b. is applied, $|J| = -636768$, then the *NZ-ALS* has a unique solution. Thus, the fuzzy system *FFLS* has a unique solution.

$$\begin{pmatrix} \begin{pmatrix} -7 & -5 & -2 \\ -4 & -3 & -5 \\ -7 & -1 & -2 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 3 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 10 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 10 & 7 & 5 \\ 6 & 4 & 8 \\ 8 & 11 & 3 \end{pmatrix} \\ \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 4 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 4 & 3 & 1 \\ 2 & 1 & 5 \\ 3 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} -36 \\ -25 \\ -21 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \\ 23 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \end{pmatrix}.$$

by computing $X = J^{-1}B$,

$$X = \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \frac{235}{108} \\ \frac{15}{4} \\ \frac{109}{108} \end{pmatrix} \\ \begin{pmatrix} -\frac{197}{108} \\ -\frac{5}{4} \\ -\frac{107}{108} \end{pmatrix} \\ \begin{pmatrix} -\frac{127}{108} \\ -\frac{3}{4} \\ -\frac{1}{108} \end{pmatrix} \end{pmatrix}.$$

Since all spreads in $\tilde{X}_g$ are negative, then the exact unique solution $\tilde{X}_g$ of this system is non fuzzy vector, is as follows,

$$\tilde{X}_g = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} \left(\frac{235}{108}, -\frac{197}{108}, -\frac{127}{108}\right) \\ \left(\frac{15}{4}, -\frac{5}{4}, -\frac{3}{4}\right) \\ \left(\frac{109}{108}, -\frac{107}{108}, -\frac{1}{108}\right) \end{pmatrix} \text{ or } \tilde{X}'_g = \begin{pmatrix} (a_1^x, b_1^x, c_1^x) \\ (a_2^x, b_2^x, c_2^x) \\ (a_3^x, b_3^x, c_3^x) \end{pmatrix} = \begin{pmatrix} \left(4, \frac{235}{108}, 1\right) \\ \left(5, \frac{15}{4}, 3\right) \\ \left(2, \frac{109}{108}, 1\right) \end{pmatrix}.$$

The verification of solution of $\tilde{X}_g$ $\left(\boldsymbol{in\ form}\ (\boldsymbol{m}, \boldsymbol{\alpha}, \boldsymbol{\beta})\right)$

$$\begin{cases} (-7,3,3) \otimes \left(\frac{235}{108}, -\frac{197}{108}, -\frac{127}{108}\right) \oplus (-5,2,2) \otimes \left(\frac{15}{4}, -\frac{5}{4}, -\frac{3}{4}\right) \\ \oplus (-2,3,1) \otimes \left(\frac{109}{108}, -\frac{107}{108}, -\frac{1}{108}\right) = \left(-\frac{1645}{108}, -\frac{565}{108}, -\frac{83}{108}\right) \\ \oplus \left(-\frac{75}{4}, \frac{9}{4}, \frac{15}{4}\right) \oplus \left(-\frac{109}{54}, \frac{161}{54}, \frac{1}{54}\right) = (-36,0,3), \\ \\ (-4,2,2) \otimes \left(\frac{235}{108}, -\frac{197}{108}, -\frac{127}{108}\right) \oplus (-3,1,2) \otimes \left(\frac{15}{4}, -\frac{5}{4}, -\frac{3}{4}\right) \\ \oplus (-5,3,0) \otimes \left(\frac{109}{108}, -\frac{107}{108}, -\frac{1}{108}\right) = \left(-\frac{235}{27}, -\frac{73}{27}, \frac{19}{27}\right) \\ \oplus \left(-\frac{45}{4}, \frac{3}{4}, \frac{25}{4}\right) \oplus \left(-\frac{545}{108}, \frac{319}{108}, -\frac{535}{108}\right) = (-25,1,2), \\ \\ (-7,1,4) \otimes \left(\frac{235}{108}, -\frac{197}{108}, -\frac{127}{108}\right) \oplus (-1,10,0) \otimes \left(\frac{15}{4}, -\frac{5}{4}, -\frac{3}{4}\right) \\ \oplus (-2,1,1) \otimes \left(\frac{109}{108}, -\frac{107}{108}, -\frac{1}{108}\right) = \left(-\frac{1645}{108}, -\frac{781}{108}, \frac{349}{108}\right) \\ \oplus \left(-\frac{15}{4}, \frac{117}{4}, -\frac{5}{4}\right) \oplus \left(-\frac{109}{54}, \frac{53}{54}, \frac{1}{54}\right) = (-21,23,2). \end{cases}$$

The verification of solution shows that $\tilde{A} \otimes \tilde{X}_g$ satisfies $\tilde{B}$. Also, using Definition 2.2.8. The distance metric function is equal zero $D_2\left(\tilde{A} \otimes \tilde{X}_g, \tilde{B}\right) = 0$. Hence, the solution $\tilde{X}_g$ satisfies the fuzzy system.

While the solution $\tilde{X}'_e$ in Ezzati et al. (2012) is

$$\tilde{X}'_e = \begin{pmatrix} (a_1^x, b_1^x, c_1^x) \\ \\ (a_2^x, b_2^x, c_2^x) \\ \\ (a_3^x, b_3^x, c_3^x) \end{pmatrix} = \begin{pmatrix} (1,2,4) \\ \\ (3,4,5) \\ \\ (1,1,2) \end{pmatrix}, \text{ or } \tilde{X}_e = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (2,1,2,) \\ \\ (4,1,1,) \\ \\ (1,0,1,) \end{pmatrix}.$$

The verification of solution of $\tilde{X}_e$ $\left(\boldsymbol{in\ form\ (m, \alpha, \beta)}\right)$

$$\begin{cases} (-7,3,3) \otimes (2,1,2) \oplus (-5,2,2) \otimes (4,1,1) \oplus (-2,3,1) \otimes (1,0,1) = \\ (-14,26,10) \oplus (-20,15,11) \oplus (-2,8,1) = (-36,49,22), \\ (-4,2,2) \otimes (2,1,2) \oplus (-3,1,2) \otimes (4,1,1) \oplus (-5,3,0) \otimes (1,0,1) = \\ (-8,16,6) \oplus (-12,8,9) \oplus (-5,11,0) = (-25,35,15), \\ (-7,1,4) \otimes (2,1,2) \oplus (-1,10,0) \otimes (4,1,1) \oplus (-2,1,1) \otimes (1,0,1) = \\ (-14,18,11) \oplus (-4,51,1) \oplus (-2,4,1) = (-20,73,13), \end{cases}$$

the verification of their solution $\tilde{X}_e$ shows that, $\tilde{A}\tilde{X}_e$ provide $\tilde{B}_e$, thus $\tilde{A} \otimes \tilde{X}_e \neq \tilde{B}$,

$$\tilde{B}_e = \begin{pmatrix} (-36,49,22) \\ (-25,35,15) \\ (-20,73,13) \end{pmatrix} \text{ while } \tilde{B} = \begin{pmatrix} (-36,0,3) \\ (-25,1,2) \\ (-21,23,2) \end{pmatrix}.$$

Also, using Definition 2.2.8. the distance metric function is not equal zero between $\tilde{B}_e$ and $\tilde{B}_e$,

$$D_2\left(\tilde{A} \otimes \tilde{X}_e, \tilde{B}\right) = \sqrt{\frac{1381}{2} + \frac{1325}{4} + \frac{2547}{4}} = 40.724.$$

As shown, the new method proposed solution $\tilde{X}_g$ which is a different solution for Ezzati et al. (2012)'solution $\tilde{X}_e$. Thus, both methods are compared in Table 4.2 in terms of accuracy of solution $\tilde{X}$, the distance for right hand side vector $\tilde{B}$ and the possibilities of unique solution.

Table 4.2

Comparison of both solutions $\tilde{X}_g$ and $\tilde{X}_e$.

	$\tilde{\boldsymbol{X}}_{\boldsymbol{g}}$	$\tilde{\boldsymbol{X}}_{\boldsymbol{e}}$
Accuracy of solution $\tilde{\boldsymbol{X}}$	$\tilde{X}_g$ satisfied the system, since $\tilde{A} \otimes \tilde{X}_g = \tilde{B}$.	$\tilde{X}_a$ did not satisfy the system, since $\tilde{A} \otimes \tilde{X}_e \neq \tilde{B}$.
Distance metric function	0	40.724
The possibilities of solution	A unique solution is determined.	Is not determined.

In Examples 4.2.3, we show the efficiency of the proposed method in obtaining a solution for large systems, where all of the examples in Babbar et al. (2013), Kumar et al. (2011a) and Ezzati et al. (2012) illustrated where $n = 2$ or 3. The details of the proposed method and verification of solution are provided in Appendix B.

Example 4.2.3. Consider the following 10×10 $FFLS$.

$$\begin{pmatrix} (-3,2,13) & (-2,2,11) & (-1,0,9) & (0,2,7) & (1,4,5) & (-2,4,3) & (3,4,1) & (4,4,1) & (5,4,3) & (6,6,5) \\ (-7,3,10) & (1,1,12) & (0,1,10) & (-1,1,8) & (0,1,6) & (-8,3,4) & (-2,5,2) & (3,7,0) & (-4,7,2) & (-5,7,4) \\ (-3,2,7) & (0,2,9) & (3,0,11) & (2,0,9) & (-1,0,7) & (0,2,5) & (1,2,3) & (2,4,1) & (3,6,1) & (4,8,3) \\ (-6,1,4) & (3,3,6) & (2,1,8) & (-5,1,10) & (4,1,8) & (3,1,6) & (-2,3,4) & (1,5,2) & (2,5,0) & (3,5,2) \\ (-8,1,2) & (-6,2,3) & (3,0,5) & (-4,0,7) & (7,2,9) & (6,2,7) & (5,2,5) & (4,4,3) & (-3,6,1) & (2,8,1) \\ (-9,2,1) & (8,2,1) & (6,1,2) & (3,3,4) & (6,1,6) & (9,3,8) & (8,3,6) & (7,3,4) & (6,5,2) & (5,7,0) \\ (-10,3,0) & (9,3,0) & (8,1,0) & (6,2,1) & (5,4,3) & (-8,2,5) & (-11,4,7) & (10,4,5) & (9,4,3) & (8,6,1) \\ (-11,4,1) & (10,2,1) & (-9,0,1) & (6,4,1) & (-6,5,2) & (7,5,2) & (10,3,4) & (13,5,6) & (4,5,4) & (11,5,2) \\ (-12,3,2) & (-6,1,2) & (8,1,2) & (-7,5,0) & (8,5,2) & (10,4,3) & (11,4,3) & (-12,4,3) & (5,4,5) & (-8,6,3) \\ (4,2,3) & (3,0,3) & (-1,2,3) & (-7,6,1) & (-3,6,1) & (2,4,3) & (14,3,4) & (-8,3,4) & (-9,5,4) & (-6,7,4) \end{pmatrix}$$

$$\otimes \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \tilde{x}_8 \\ \tilde{x}_9 \\ \tilde{x}_{10} \end{pmatrix} = \begin{pmatrix} (86,211,371) \\ (-99,373,305) \\ (50,240,377) \\ (31,269,329) \\ (77,354,324) \\ (186,288,418) \\ (11,302,429) \\ (226,403,410) \\ (52,471,371) \\ (-4,474,292) \end{pmatrix}.$$

The given $FFLS$ can be converted into crisp linear system using (3.20).

Then, the crisp solution can be obtained as,

$$X = J^{-1} B.$$

The fuzzy solution is:

$$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \tilde{x}_8 \\ \tilde{x}_9 \\ \tilde{x}_{10} \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \\ (m_4^x, \alpha_4^x, \beta_4^x) \\ (m_5^x, \alpha_5^x, \beta_5^x) \\ (m_6^x, \alpha_6^x, \beta_6^x) \\ (m_7^x, \alpha_7^x, \beta_7^x) \\ (m_8^x, \alpha_8^x, \beta_8^x) \\ (m_9^x, \alpha_9^x, \beta_9^x) \\ (m_{10}^x, \alpha_{10}^x, \beta_{10}^x) \end{pmatrix} = \begin{pmatrix} (1,2,3) \\ (0,2,5) \\ (0,1,4) \\ (0,2,8) \\ (1,4,5) \\ (5,1,1) \\ (7,4,1) \\ (4,4,1) \\ (5,4,3) \\ (6,6,5) \end{pmatrix}.$$

4.3 The Consistency of the Fully Fuzzy Linear System

In this section, the existence and fuzziness of the solution of the fully fuzzy linear system are checked, and the possibilities for $P - \tilde{X}$ for $NZ - FFLS$ are classified. Moreover, the proposed method is modified using row reduced method to find the solution whenever it is not unique.

Using *NZ-ALS* there are three cases for the solution for $NZ - FFLS$, according to the possibilities for solution of classical linear system (unique solution, infinite number of solution, no solution):

Case 1: Unique solution.

If $|J| \neq 0$. Then, J is invertible, thus $X = J^{-1}B$ provides $\tilde{X}$ as a unique solution for $NZ - FFLS$.

Case 2: Infinite number of solutions.

If $|J| = 0$ and $rank(J) \; = \; rank\;(J{:}B) < 3n$, then $JX = B$ has infinite number of solutions. In this case, row reduced method echelon for classical linear system is modified to provide infinite number of solutions for fuzzy system.

Case 3: No solution.

If $|J| = 0$ and $rank(J) \; < rank\;(J{:}B)$, then $JX = B$ has no solution. Thus the fuzzy system doesn't have a unique solution.

Therefore, using the linear system $\mathrm{J}X = B$, the solution of $\tilde{A} \otimes \tilde{X} = \tilde{B}$ can be obtained whenever it exists. The possibility for fuzzy unique solution is illustrated in

Examples 4.2.1. While the possibility for non fuzzy unique solution is illustrated in Examples 4.2.2.

The next section provides the fuzzy row reduced echelon method to obtain the infinity many solutions whenever they exist.

Fuzzy Row Reduced Echelon Method

In this section, the fuzzy row reduced echelon method is proposed to provide the infinity many solutions. The method is obtained by three steps, where $|J| = 0$.

Step 1 Computing the $rank\,(J)$ and $rank\,(J\!:\!B)$. Based on Case 2, if $rank(J) = rank\,(J\!:\!B)$ the system has infinitely many solutions.

Step 2 Transforming the *NZ-ALS* in Equation (4.20) to $3n$ linear equations. Then, reducing it.

Step 3 Solving linear equations with positive fuzzy inequalities in Equation (3.10).

The next example illustrates the fuzzy row reduced echelon method where the fuzzy system has infinite solutions. This example uses the same example as in Kumar et al. (2011a). However, the new method proposes infinite many solutions while Kumar et al. (2011a)'solution is a unique solution. Finally, Kumar et al. (2011a)'solution is revealed as particular solution of solution set using the proposed method.

Example 4.3.1. Kumar et al. (2011a) consider the following $FFLS$,

$$\begin{cases} (-1,1,0) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (-2,0,1) \otimes (m_2^x, \alpha_2^x, \beta_2^x) = (-6,4,2), \\ (-2,2,0) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (-4,0,2) \otimes (m_2^x, \alpha_2^x, \beta_2^x) = (-12,8,4). \end{cases}$$

The system is in matrix form,

$$\begin{pmatrix} (-1,1,0) & (-2,0,1) \\ (-2,2,0) & (-4,0,2) \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (-6,4,2) \\ (-12,8,4) \end{pmatrix}.$$

Since $|J| = 0$, then the fuzzy system does not have a unique solution.

Using step 1, $rank(J)$ and $rank(J : B)$ are computed as follows,

$$J = \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ -2 & -4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 0 & 4 & 4 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, rank(J) = 3,$$

$$(J : B) = \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 & -6 \\ -2 & -4 & 0 & 0 & 0 & 0 & -12 \\ 1 & 0 & 0 & 0 & 2 & 2 & 4 \\ 2 & 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 2 & 2 & 2 & 0 & 0 & 4 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 2 & 4 \\ 0 & 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, rank(J : B) = 3.$$

Since $rank(J) = rank(J : B) = 3$, and the number of rows is six, then the system has infinity many solutions.

Using Step 2 the six linear equations of *NZ-ALS* are,

$$\begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ -2 & -4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 0 & 4 & 4 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ \alpha_1^x \\ \alpha_2^x \\ \beta_1^x \\ \beta_2^x \end{pmatrix} = \begin{pmatrix} -6 \\ -12 \\ 4 \\ 8 \\ 2 \\ 4 \end{pmatrix} \Rightarrow \begin{cases} -m_1^x - 2m_2^x = -6, \\ -2m_1^x - 4m_2^x = -12, \\ m_1^x + 2\beta_1^x + 2\beta_2^x = 4, \\ 2m_1^x + 4\beta_1^x + 4\beta_2^x = 8, \\ m_2^x + \alpha_1^x + \alpha_2^x = 2, \\ 2m_2^x + 2\alpha_1^x + 2\alpha_2^x = 4. \end{cases}$$

which can be reduced to three linear equations,

$$\begin{cases} \alpha_1^x = 1 - \alpha_2^x - \beta_1^x - \beta_2^x, \\ m_2^x = 1 + \beta_1^x + \beta_2^x, \\ m_1^x = 4 - 2\beta_1^x - 2\beta_2^x. \end{cases}$$

Using Step 3, the previous linear equations should be solved with positive fuzzy inequalities in Equation (3.10) is as follows,

$$\begin{cases} \alpha_1^x = 1 - \alpha_2^x - \beta_1^x - \beta_2^x, \\ m_2^x = 1 + \beta_1^x + \beta_2^x, \\ m_1^x = 4 - 2\beta_1^x - 2\beta_2^x \\ \alpha_1^x \geq 0, \alpha_2^x \geq 0, \\ \beta_1^x \geq 0, \beta_2^x \geq 0, \\ m_1^x \geq \alpha_1^x, m_2^x \geq \alpha_2^x. \end{cases}$$

Hence, the general form solution $\tilde{X}_G$ for infinitely many solutions is:

$$\tilde{X}_G = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (4 - 2\beta_1^x - 2\beta_2^x, 1 - \alpha_2^x - \beta_1^x - \beta_2^x, \beta_1^x) \\ (1 + \beta_1^x + \beta_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix},$$

where $\alpha_2^x \in [0,1], \beta_1^x \in [0,1 - \alpha_2^x], \alpha_2^x \in [0,1 - \alpha_2^x - \beta_1^x]$.

Let $= \alpha_2^x, v = \beta_1^x, w = \alpha_2^x$, then $\tilde{X}_G$ can be written as follows,

$$\tilde{X}_G = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (4 - 2v - 2w, 1 - u - v - w, v) \\ (1 + v + w, u, w) \end{pmatrix},$$

where, $u \in [0,1], v \in [0,1 - u], w \in [0,1 - u - v]$.

Using this solution, we can obtain Kumar et al. (2011a)'solution where $u = 0$, $v = 1$ and $w = 0$ in $\tilde{X}_G$,

$$\begin{pmatrix} (4 - 2(1) - 0,1 - 0 - 1 - 0,1) \\ (1 + 1 + 0,0,0) \end{pmatrix} = \begin{pmatrix} (2,0,1) \\ (2,0,0) \end{pmatrix} = \tilde{X}_k.$$

The values of m_1^x is represented by Figure 4.1 where α_2^x is fixed $\frac{1}{2}$, while the value of m_1^x represented by a single values in Kumar et al. (2012)'method.

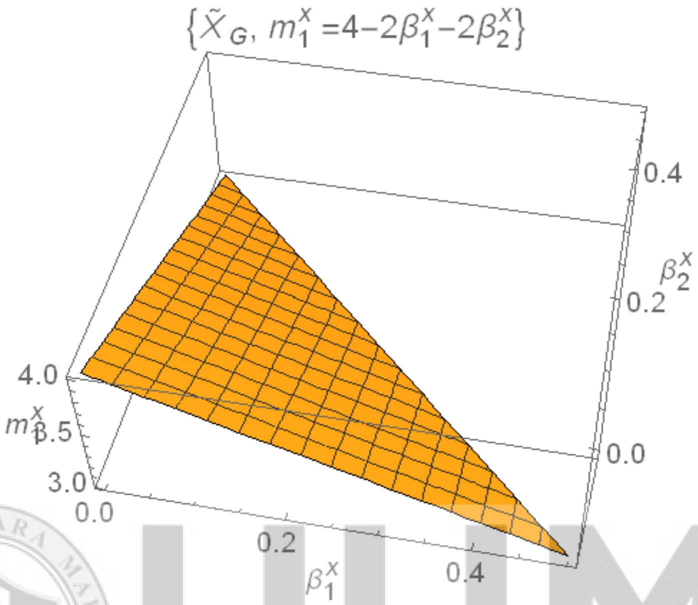

Figure 4.1. The values of m_1^x in general form solution $\tilde{X}_G$.

To verify the system has more than a unique solution. An particular solution $\tilde{X}_p$ which is different form $\tilde{X}_k$ using the proposed method is illustrated, let $u = \frac{1}{2},\ v = \frac{1}{4},\ w = \frac{1}{12}$. Then,

$$\tilde{X}_p = \begin{pmatrix} \left(\frac{10}{3}, \frac{1}{6}, \frac{1}{4}\right) \\ \left(\frac{4}{3}, \frac{1}{2}, \frac{1}{12}\right) \end{pmatrix}.$$

The verification of the solution $\tilde{X}_p$

$$\begin{cases} (-1,1,0) \otimes \left(\frac{10}{3}, \frac{1}{6}, \frac{1}{4}\right) \oplus (-2,0,1) \otimes \left(\frac{4}{3}, \frac{1}{2}, \frac{1}{12}\right) = \\ \quad \left(-\frac{10}{3}, \frac{23}{6}, \frac{1}{6}\right) \oplus \left(-\frac{8}{3}, \frac{1}{6}, \frac{11}{6}\right) = (-6,4,2), \\ \\ (-2,2,0) \otimes \left(\frac{10}{3}, \frac{1}{6}, \frac{1}{4}\right) \oplus (-4,0,2) \otimes \left(\frac{4}{3}, \frac{1}{2}, \frac{1}{12}\right) = \\ \quad \left(-\frac{20}{3}, \frac{23}{3}, \frac{1}{3}\right) \oplus \left(-\frac{16}{3}, \frac{1}{3}, \frac{11}{3}\right) = (-12,8,4). \end{cases}$$

Table 4.3 compares both methods in terms of fuzziness of solution, number of solution and independence between $\tilde{x}_1$ and $\tilde{x}_2$.

Table 4.3

Comparison of the solutions set $\tilde{X}_G$ and $\tilde{X}_k$.

	$\tilde{X}_G$	$\tilde{X}_k$
Fuzzy operation	No fuzzy operation.	Need to compute $\tilde{A} \otimes \tilde{X}$.
Number of solution	Infinitely many solutions.	Unique solution set.
fuzziness	All entries of $\tilde{x}_1$ and $\tilde{x}_2$ can be chosen nonzero.	The fuzziness cannot be provided by $\tilde{x}_2$ since the spreads α_2^x, β_2^x are zero, so it is represented only by its mean value as a crisp number m_2^x.
Independence between $\tilde{x}_1$ and $\tilde{x}_2$	α_1^x and β_1^x are independents, and the other entries are dependent for them.	Fixed solution.

As noted in Table 4.3, the general form solution shows that the new method provides infinitely many solutions, while the original work of Kumar et al. (2011a) provides a unique solution. In addition, the particular solution of $\tilde{X}_G$ can be considered symmetric or non symmetric, while the unique solution $\tilde{X}_k$ should be symmetric for $\tilde{x}_2$ because both spreads are equal. Also, all fuzzy number of particular solution in $\tilde{X}_G$ are nontrivial since the spreads in $\tilde{x}_1$, $\tilde{x}_2$ are nonzero, while $\tilde{x}_2$ in $\tilde{X}_k$ is represented by non fuzzy number (crisp number) based on Remark 2.2.1, because the both spreads are zero, $\alpha_2^x = \beta_2^x = 0$. Lastly, the entries of $\tilde{x}_1$ and $\tilde{x}_2$ have independent intervals to provide infinite many solutions in $\tilde{X}_G$, while all entries are fixed in $\tilde{X}_k$.

4.4 Conclusion and Contribution

In this chapter, we proposed a new method to obtain the $P-\tilde{X}$ of $NZ-FFLS$. The coefficients of fuzzy matrix $\tilde{A}$ and the entries of fuzzy vectors $\tilde{B}$ were represented in a using block matrix, to produce the associated linear system $NZ-ALS$ for $NZ-FFLS$. Thus, the solution can be obtained without fuzzy operation and without constructing or solving min-max system.

We can summarize the findings in this chapter by the following contribution:

1- Obtain the unique solution of $P-\tilde{X}$ to $NZ-FFLS$ without fuzzy operation.

2- Examine the existence of solution for $P-\tilde{X}$ before solving the system.

3- Find the possibilities of $P-\tilde{X}$ for $NZ-FFLS$.

4- Formulate fuzzy row reduced echelon method to provide infinite many solutions $P-\tilde{X}$ to $NZ-FFLS$.

CHAPTER FIVE

FINITE SOLUTION OF NEAR ZERO FULLY FUZZY LINEAR SYSTEM

In Chapter Three and Chapter Four $P - \tilde{X}$ for $P - FFLS$ and for $NZ - FFLS$ are solved, respectively. Chapter Five provides arbitrary solution for arbitrary $FFLS$ where the near zero is included ($NZ - \tilde{X}$ for $NZ - FFLS$). Arithmetic operations of fuzzy numbers are only used in order to avoid adding any restrictions to the system. As a result, all possible solutions for systems are detected. Hence we conclude that the nature of solution of linear system (unique, finite or infinitely many solutions) is no more sufficient to provide all the possible ways of solutions for $FFLS$, since $FFLS$ may have more than two distinct solutions but not infinitely many solutions.

The existing methods of solving $FFLS$ obtained the fuzzy solution is unique or infinitely many solutions due for determining the final solution of minimization problem. Thus this study provides an alternative solution for $FFLS$ namely finite solution of $FFLS$. The next section presents the fundamental definitions and main theorems to develop the new method for solving $NZ - \tilde{X}$ of $NZ - FFLS$.

5.1 Fundamental Concept for Min-Max System

In this section, a new concept for consistency which is called finite solution for $FFLS$ is defined, then we prove the possibility to find it in solving $NZ - FFLS$, where the $FFLS$ have more than two solutions, but not infinite solutions. This concept will be used to develop the proposed method.

In Chapter Four, a min-max system was defined while in this chapter the absolute system will be presented. Both definitions are used for solving $NZ-\tilde{X}$ of $NZ-FFLS$.

Definition 5.1.2. *An absolute equation is an equation has absolute term.*

Definition 5.1.2. *An absolute system (or a system of absolute equations) is a collection of equations such that at least one of them is an absolute equation.*

The next definition defines the finite solutions for systems, to add a new possible way of solutions for fuzzy systems.

Definition 5.1.3. *The solution set of a system is called a finite solution or alternative solutions, whereby the number of solution is more than one and not infinite solutions.*

The relation between min-max systems in Definition 4.1.3. and absolute systems is investigated in the next theorem.

Theorem 5.1.1. Consider the min-max system for $i=1,\dots,k$,

$$\begin{cases} \psi_1\left(h_{\psi_1}(x_1,..,x_k), g_{\psi_1}(x_1,..,x_k)\right)=\phi_1(x_1,x_2,..,x_k), \\ \psi_2\left(h_{\psi_2}(x_1,..,x_k), g_{\psi_2}(x_1,..,x_k)\right)=\phi_2(x_1,x_2,..,x_k), \\ \vdots \\ \vdots \\ \psi_n\left(h_{\psi_n}(x_1,..,x_k), g_{\psi_n}(x_1,..,x_k)\right)=\phi_n(x_1,x_2,..,x_k). \end{cases} \tag{5.1}$$

Then, the system can be reduced for an absolute linear system, where,

- ψ_j for $j = 1, .., n$ is minimum or maximum function.
- h_{ψ_j}, g_{ψ_j} and ϕ_j are linear functions for $j = 1, .. n$.
- x_i is variables for $i = 1, .. k$.

Proof. Since $\psi_j\left(h_{\psi_j}, g_{\psi_j}\right)$ is a minimum or a maximum function with just two variables h_{ψ_j} and g_{ψ_j}, the Remark 2.3.2. can be applied.

If ψ_j is a maximum function, then

$$\psi_j\left(h_{\psi_j}, g_{\psi_j}\right) = \left(\frac{h_{\psi_j} + g_{\psi_j}}{2}\right) + \left|\frac{h_{\psi_j} - g_{\psi_j}}{2}\right|, \tag{5.2}$$

or,

$$\left|h_{\psi_j} - g_{\psi_j}\right| = 2\,\psi_j\left(h_{\psi_j}, g_{\psi_j}\right) - \left(h_{\psi_j} + g_{\psi_j}\right), \tag{5.3}$$

but in Equation (5.1), $\psi_j\left(h_{\psi_j}, g_{\psi_j}\right) = \phi_j$, then,

$$\left|h_{\psi_j} - g_{\psi_j}\right| = 2\phi_j - \left(h_{\psi_j} + g_{\psi_j}\right). \tag{5.4}$$

Similarly, If ψ_j is a minimum function,

$$\psi_j\left(h_{\psi_j}, g_{\psi_j}\right) = \left(\frac{h_{\psi_j} + g_{\psi_j}}{2}\right) - \left|\frac{h_{\psi_j} - g_{\psi_j}}{2}\right|, \tag{5.5}$$

or,

$$\left|h_{\psi_j} - g_{\psi_j}\right| = \left(h_{\psi_j} + g_{\psi_j}\right) - 2\,\phi_j. \tag{5.6}$$

Then, any equation in Equation (5.1) for $j = 1, .. n$, may be written as Equations (5.4) or (5.6).

Suppose $h_{\psi_j} - g_{\psi_j}$ in Equations (5.4) and (5.6) is one function s_{ψ_j},

$$s_{\psi_j} = h_{\psi_j} - g_{\psi_j}. \tag{5.7}$$

Define, $2\phi_j - \left(h_{\psi_j} + g_{\psi_j}\right)$ and $\left(h_{\psi_j} + g_{\psi_j}\right) - 2\,\phi_j$ in Equation (5.4) is one piecewise function ξ_j,

$$\xi_j = \begin{cases} 2\phi_j - \left(h_{\psi_j} + g_{\psi_j}\right) & if\, \psi_j \; maximum\, function, \\ \left(h_{\psi_j} + g_{\psi_j}\right) - 2\,\phi_j & if\, \psi_j \; minimum\; function. \end{cases} \tag{5.8}$$

Hence, Equation (5.1) can be written as the following absolute linear system using Equations (5.7) and (5.8),

$$\begin{cases} \left|s_{\psi_1}(x_1, x_2, .., x_k)\right| = \xi_1(x_1, x_2, .., x_k), \\ \left|s_{\psi_2}(x_1, x_2, .., x_k)\right| = \xi_2(x_1, x_2, .., x_k), \\ \qquad \vdots \\ \qquad \vdots \\ |s_n(x_1, x_2, .., x_k)| = \xi_j(x_1, x_2, .., x_k). \end{cases} \tag{5.9}$$

□

The next theorem computes the finite solutions for the absolute system.

Theorem 5.1.2 Consider the absolute system,

$$\left|s_{\psi_j}(x_1, x_2, .., x_k)\right| = \xi_j(x_1, x_2, .., x_k), \quad j = 1, .. k. \tag{5.10}$$

Then the possible ways of solutions for the system are:

- No solution.
- Unique solution.
- Finite or alternative solutions not more than 2^n.
- Infinitely many solutions.

Proof. Since $s_{\psi_j} = h_{\psi_j}(x_1, x_2, .., x_k) - g_{\psi_j}(x_1, x_2, .., x_k)$ and h_{ψ_j}, g_{ψ_j} are linear functions, $\xi_j(x_1, x_2, .., x_n)$ is a piecewise function of linear functions, without loss of generalization, suppose s_{ψ_j} is a positive function, then Equation (5.10) may be written as the below $n \times n$ classical linear system.

$$s_{\psi_j}(x_1, x_2, .., x_k) = \xi_j(x_1, x_2, .., x_k), \qquad j = 1, \dots, n, \tag{5.11}$$

hence, the possible ways of solutions are:

- No solution.
- Unique solution.
- Infinite solutions.

The proof is completed when we obtain the finite solution less than 2^n. Since

$$\left|s_{\psi_j}(x_1, x_2, .., x_k)\right| = \begin{cases} s_{\psi_j}(x_1, x_2, .., x_k), & if\, s_{\psi_j} \geq 0, \\ -s_{\psi_j}(x_1, x_2, .., x_k), & if\, s_{\psi_j} < 0, \end{cases} \tag{5.12}$$

then, for every $j = 1, \dots, n$ we have,

$$s_{\psi_j}(x_1, x_2, .., x_k) = \xi_j(x_1, x_2, .., x_k) \text{ or } -s_{\psi_j}(x_1, x_2, .., x_k) = \xi_j(x_1, x_2, .., x_k).$$

Hence, according to counting principles, the likely number of $n \times n$ linear systems is 2^n (Weibel, 1967). each of them has three previous possible ways (no solution, unique solution, infinitely many solutions).

If at least two linear systems have unique solution and the other linear systems have no solution, then the finite solution is a collection for any linear system that has unique solution which will not be more than 2^n. Hence, the last possible way of solutions is:

- Finite solution. □

The next section provides the method for solving $NZ - \tilde{X}$ for $NZ - FFLS$. The discussion of finite solution is extended to possibility of $FFLS$ using Theorem 5.1.2.

5.2 Near Zero Solution for Near Zero Fully Fuzzy Linear System

The $NZ - \tilde{X}$ for $NZ - FFLS$ is obtained in this section. The method is proposed by reducing the min-max and absolute systems, to exclude any further restriction in the process. As a result, we will find out all possible solutions for system.

Consider the following $FFLS$ $\tilde{A} \otimes \tilde{X} = \tilde{B}$, where $\tilde{A} = (A, M, N), \tilde{B} = (m^b, \alpha^b, \beta^b)$ and $\tilde{X} = (m^x, \alpha^x, \beta^x)$ are arbitrary.

Then, the $n \times n$ $FFLS$ may be written as

$$\sum_{j=1}^{\oplus} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \ , \forall i = 1, 2, \ldots, n.$$

The solution is obtained by four steps:

Step 1 Separating the $FFLS$ for a linear system and a nonlinear system (min-max linear system).

Step 2 Solving the linear system to obtain the mean values m_i^x, substituting them in the min-max linear system and reducing it.

Step 3 Converting the reduced min-max system to absolute system using Remark 2.3.2.

Step 4 Reducing the absolute linear system, using the following fuzzy inequality,

$$\begin{cases} \alpha_i^x \geq 0, \\ \beta_i^x \geq 0. \qquad \forall i = 1, 2, \dots, n. \end{cases} \tag{5.13}$$

Then, Step 4 provides the values of spreads α_i^x and β_i^x, while the mean values m_i^x are obtained from Step 2, which completes the component of fuzzy solution $NZ - \tilde{X}$.

As noted, no restriction is adding to proposed method, we keep reducing the systems whether its min-max or absolute system to get the $NZ - \tilde{X}$ for $NZ - FFLS$.

The next corollary computes the finite solution of $NZ - \tilde{X}$ to $NZ - FFLS$.

Corollary 5.2.1. Consider the $NZ - \tilde{X}$ for $m \times m$ $NZ - FFLS$. Then, the system has possibility of finite solution less than 2^{2m}.

Proof. Applying Equation (2.12) for $m \times m$ $FFLS$ to obtain a $3m$ min-max system. Hence, $2m$ min-max equation is produced. Then, using Remark 2.3.2. an absolute system is constructed. The proof is concluded by recalculating Equation (5.12) for each absolute term. Thus according to counting principles, the likely number of the $m \times m$ linear systems is 2^{2m} solutions. □

The next example is used in Kumar et al. (2011b) and we obtain further different feasible solutions, while the proposed method in Kumar et al. (2011b) only one solution is obtained. We use geometrical analysis by graphs to verify the further second solution, and show that the system has finite solution.

Example 5.2.1. Kumar et al. (2011b) consider the following *FFLS*, $\big(written\ in\ \ the form\ (a,b,c)\big)$,

$$\begin{cases}(-2,3,4)\otimes(a_1{}^x,b_1{}^x,c_1{}^x)\ \oplus\ (-2,2,3)\otimes(a_2{}^x,b_2{}^x,c_2{}^x)=(-13,8,14),\\ \ (1,2,2)\otimes(a_1{}^x,b_1{}^x,c_1{}^x)\ \oplus\ (4,4,5)\otimes(a_2{}^x,b_2{}^x,c_2{}^x)=(-14,8,14),\end{cases}$$

where $\tilde{x}_i=(a_i{}^x,b_i{}^x,c_i{}^x), i=1,2$ are arbitrary *TFN*.

The system can be rewritten in matrix form as follows,

$$\begin{pmatrix}(-2,3,4) & (-2,2,3)\\ (1,2,2) & (4,4,5)\end{pmatrix}\otimes\begin{pmatrix}(a_1{}^x,b_1{}^x,c_1{}^x)\\ (a_2{}^x,b_2{}^x,c_2{}^x)\end{pmatrix}=\begin{pmatrix}(-13,8,14)\\ (-14,8,14)\end{pmatrix}.$$

Since we use (m,α,β) form for TFN in this study, the example is converted to form(m,α,β). We will denote $\tilde{A}',\tilde{X}'$ and $\tilde{B}'$ to the form of (a,b,c), while $\tilde{A},\tilde{X}$ and $\tilde{B}$ to the form of (m,α,β).

In matrix form $\tilde{A}\otimes\tilde{X}=\tilde{B}$ it becomes, $\big(in\ form\ (m,\alpha,\beta)\big)$,

$$\begin{pmatrix}(3,5,1) & (2,4,1)\\ (2,1,0) & (4,0,1)\end{pmatrix} \otimes \begin{pmatrix}(m_1{}^x, \alpha_1{}^x, \beta_1{}^x)\\ (m_2{}^x, \alpha_2{}^x, \beta_2{}^x)\end{pmatrix} = \begin{pmatrix}(8,21,6)\\ (8,22,6)\end{pmatrix}.$$

Using Step 1, the $FFLS$ is written as,

$$\begin{cases} 3m_1{}^x + 2m_2{}^x = 8, \\ -Min[4(m_1{}^x - \alpha_1{}^x), -2(m_1{}^x + \beta_1{}^x)] - Min[3(m_2{}^x - \alpha_2{}^x), -2(m_2{}^x + \beta_2{}^x)] \\ \quad +3m_1{}^x + 2m_2{}^x = 21, \\ Max[-2(m_1{}^x - \alpha_1{}^x), 4(m_1{}^x + \beta_1{}^x)] + Max[-2(m_2{}^x - \alpha_2{}^x), 3(m_2{}^x + \beta_2{}^x)] \\ \quad -3m_1{}^x - 2m_2{}^x = 6, \\ 2m_1{}^x + 4m_2{}^x = 8, \\ -Min[m_1{}^x - \alpha_1{}^x, 2(m_1{}^x - \alpha_1{}^x)] - Min[4(m_2{}^x - \alpha_2{}^x), 5(m_2{}^x - \alpha_2{}^x)] \\ \quad +2m_1{}^x + 4m_2{}^x = 22, \\ Max[m_1{}^x + \beta_1{}^x, 2(m_1{}^x + \beta_1{}^x)] + Max[4(m_2{}^x + \beta_2{}^x), 5(m_2{}^x + \beta_2{}^x)] \\ \quad -2m_1{}^x - 4m_2{}^x = 6. \end{cases} \tag{5.14}$$

The linear system is,

$$\begin{cases} 3m_1{}^x + 2m_2{}^x = 8, \\ 2m_1{}^x + 4m_2{}^x = 8. \end{cases} \tag{5.15a}$$

The nonlinear system is,

$$\begin{cases} -Min[4(m_1{}^x - \alpha_1{}^x), -2(m_1{}^x + \beta_1{}^x)] - Min[3(m_2{}^x - \alpha_2{}^x), -2(m_2{}^x + \beta_2{}^x)] \\ \quad +3m_1{}^x + 2m_2{}^x = 21, \\ Max[-2(m_1{}^x - \alpha_1{}^x), 4(m_1{}^x + \beta_1{}^x)] + Max[-2(m_2{}^x - \alpha_2{}^x), 3(m_2{}^x + \beta_2{}^x)] \\ \quad -3m_1{}^x - 2m_2{}^x = 6, \\ -Min[m_1{}^x - \alpha_1{}^x, 2(m_1{}^x - \alpha_1{}^x)] - Min[4(m_2{}^x - \alpha_2{}^x), 5(m_2{}^x - \alpha_2{}^x)] \\ \quad +2m_1{}^x + 4m_2{}^x = 22, \\ Max[m_1{}^x + \beta_1{}^x, 2(m_1{}^x + \beta_1{}^x)] + Max[4(m_2{}^x + \beta_2{}^x), 5(m_2{}^x + \beta_2{}^x)] \\ \quad -2m_1{}^x - 4m_2{}^x = 6. \end{cases} \tag{5.15b}$$

Using Step 2, solving the linear system (5.15a), we get ${m_1}^x = 2, {m_2}^x = 1$, then substituting the values of ${m_i}^x$ in the nonlinear system (5.15b),

$$\begin{cases} 8 - Min[4(2 - {\alpha_1}^x), -2(2 + {\beta_1}^x)] - Min[3(1 - {\alpha_2}^x), -2(1 + {\beta_2}^x)] = 21, \\ -8 + Max[-2(2 - {\alpha_1}^x), 4(2 + {\beta_1}^x)] + Max[-2(1 - {\alpha_2}^x), 3(1 + {\beta_2}^x)] = 6, \\ 8 - Min[2 - {\alpha_1}^x, 2(2 - {\alpha_1}^x)] - Min[4(1 - {\alpha_2}^x), 5(1 - {\alpha_2}^x)] = 22, \\ -8 + Max[2 + {\beta_1}^x, 2(2 + {\beta_1}^x)] + Max[4(1 + {\beta_2}^x), 5(1 + {\beta_2}^x)] = 6. \end{cases} \quad (5.16)$$

According to Step 3, the min and max functions in Remark 2.3.2. produces absolute system from min-max system in Equation (5.16),

$$\begin{cases} 4\alpha_1^x + 3\alpha_2^x + 2\beta_1^x + 2\beta_2^x + 2|-2\alpha_1^x + \beta_1^x + 6| + |-3\alpha_2^x + 2\beta_2^x + 5| = 31, \\ 2\alpha_1^x + 2\alpha_2^x + 4\beta_1^x + 3\beta_2^x + 2|\alpha_1^x - 2(\beta_1^x + 3)| + |-2\alpha_2^x + 3\beta_2^x + 5| = 23, \\ 3\alpha_1^x + 9\alpha_2^x + |\alpha_1^x - 2| + |\alpha_2^x - 1| = 43, \\ |2 + {\beta_1}^x| + |1 + {\beta_2}^x| + 3{\beta_1}^x + 9{\beta_2}^x = 13. \end{cases} \quad (5.17)$$

Now, based on Step 4, the absolute system in Equation (5.17) should be reduced to find fuzzy solution.

Considering the fourth equation in Equation (5.17),

$$|2 + {\beta_1}^x| + |1 + {\beta_2}^x| + 3{\beta_1}^x + 9{\beta_2}^x = 13.$$

Using Equation (5.13) ${\beta_1}^x, {\beta_2}^x \geq 0$. Then $(2 + {\beta_1}^x)$, $(1 + {\beta_2}^x) \geq 0$,

it may be written as,

$$(2 + {\beta_1}^x) + (1 + {\beta_2}^x) + 3{\beta_1}^x + 9{\beta_2}^x = 13,$$

then,

$$-8+2(2+\beta_1{}^x)+5(1+\beta_2{}^x)=1+2\beta_1{}^x+5\,\beta_2{}^x=6,$$

or,

$$2\beta_1{}^x+5\,\beta_2{}^x=5.$$

Solving the system,

$$\begin{cases}2\beta_1{}^x+5\,\beta_2{}^x=5,\\ \quad \beta_1{}^x,\beta_2{}^x\geq 0.\end{cases} \tag{5.18}$$

Then, the fourth equation in Equation (5.17) can be written as one linear equation as shown in Figure 5.1,

$$\beta_2{}^x=1-\frac{2\beta_1{}^x}{5},\quad \beta_1{}^x\in\left[0,\frac{5}{2}\right]. \tag{5.19}$$

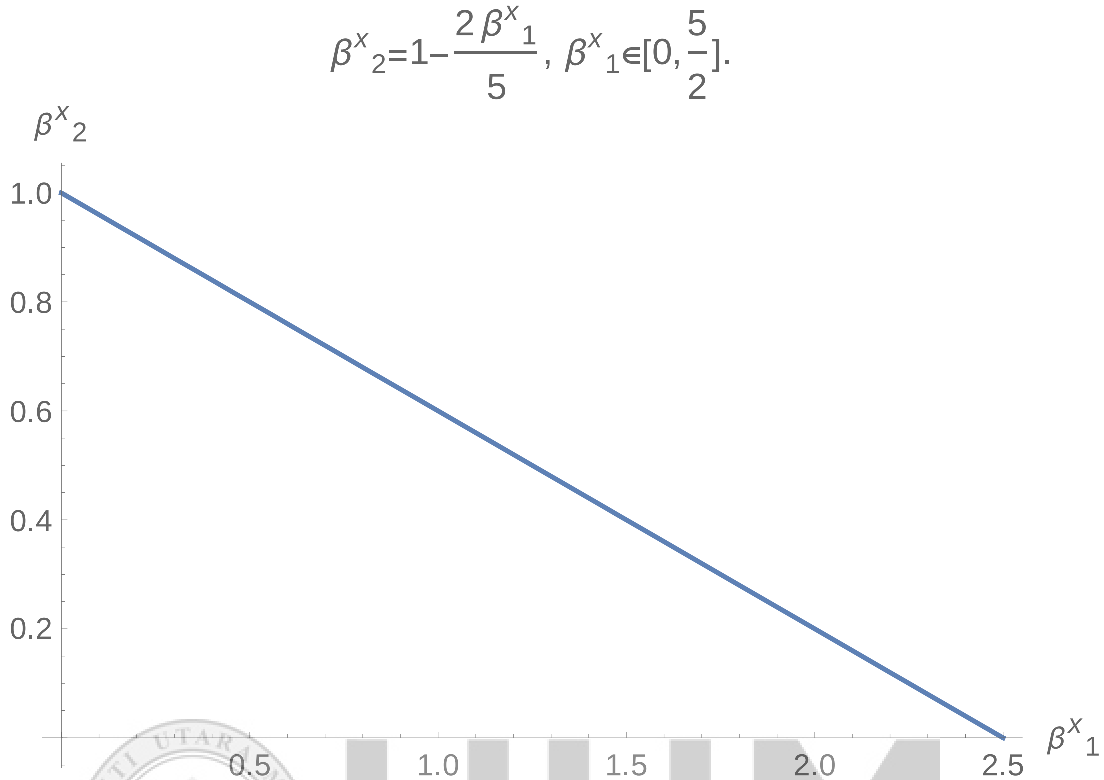

Figure 5.1. Representation of the fourth absolute value equation as a linear equation.

Similarly, by considering the third equation in Equation (5.17),

$$3\alpha_1^x + 9\alpha_2^x + |\alpha_1^x - 2| + |\alpha_2^x - 1| = 43,$$

because ${\alpha_1}^x, {\alpha_2}^x \geq 0$, we get three linear equations,

$$ {\alpha_2}^x = \begin{cases} \frac{1}{5}(21 - {\alpha_1}^x), & 0 \leq {\alpha_1}^x \leq 2, \\ \frac{1}{5}(23 - 2{\alpha_1}^x), & 2 < {\alpha_1}^x \leq 9, \\ \frac{1}{2}(11 - {\alpha_1}^x), & 9 < {\alpha_1}^x \leq 11. \end{cases} \tag{5.20}$$

The three linear equations for ${\alpha_2}^x$ is showed in Figure 5.2.

$$
{\alpha^x}_2 = \begin{cases} \frac{1}{5}\left(21 - {\alpha^x}_1\right), & 0 \leq {\alpha^x}_1 \leq 2, \\ \frac{1}{5}\left(23 - 2\,{\alpha^x}_1\right), & 2 < \alpha_1 \leq 9, \\ \frac{1}{2}\left(11 - {\alpha^x}_1\right), & 9 < {\alpha^x}_1 \leq 11. \end{cases}
$$

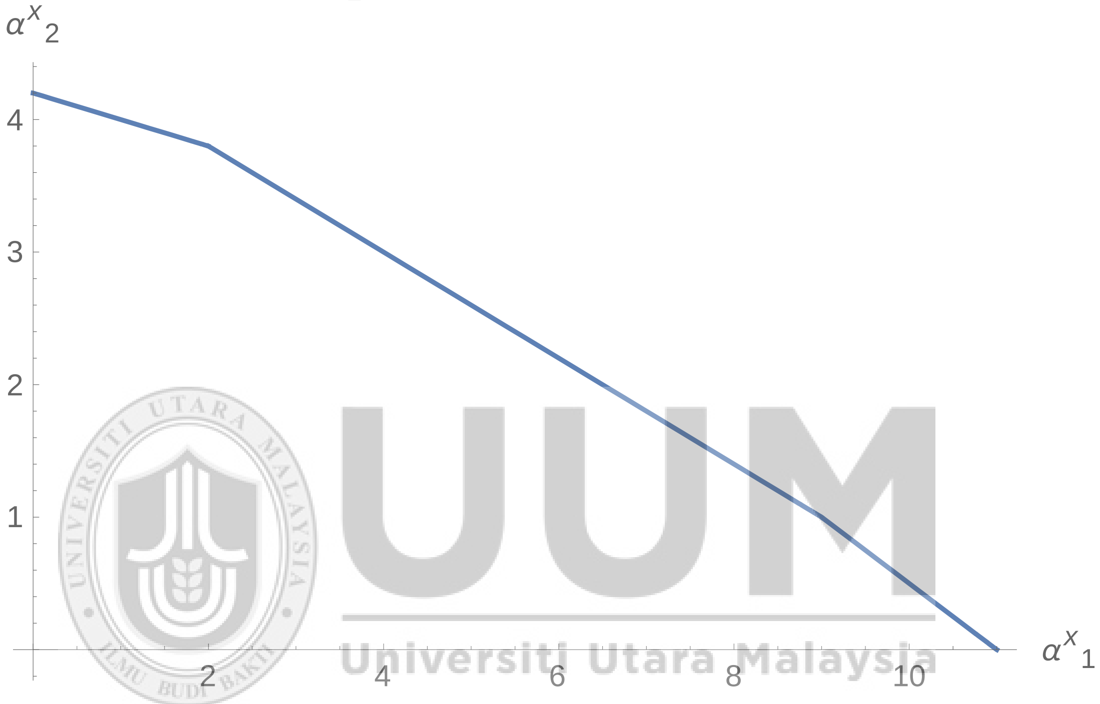

Figure 5.2. Representation of the third absolute value equation as a linear equation.

Now, the first and second equations are,

$$
\begin{cases} 4\alpha_1^x + 3\alpha_2^x + 2\beta_1^x + 2\beta_2^x + 2|-2\alpha_1^x + \beta_1^x + 6| \\ \quad + |-3\alpha_2^x + 2\beta_2^x + 5| = 31, \\ 2\alpha_1^x + 2\alpha_2^x + 4\beta_1^x + 3\beta_2^x + 2|\alpha_1^x - 2(\beta_1^x + 3)| \\ \quad + |-2\alpha_2^x + 3\beta_2^x + 5| = 23. \end{cases} \tag{5.21}
$$

Substituting the values ${\beta_2}^x$ in absolute system in Equation (5.21). Then, solving Equation (5.21) with the three equations of ${\alpha_1}^x$ in Equation (5.20) to provide three possible solutions:

- **For interval ${\alpha_1}^x \in [0, 2]$,**

$${\alpha_2}^x = \frac{1}{5}(21 - {\alpha_1}^x).$$

Hence, Equation (5.21) may be reduced to the following system,

$$\begin{cases} 10|-2\alpha_1^x + \beta_1^x + 6| + |3\alpha_1^x - 4(\beta_1^x + 7)| + 17\alpha_1^x + 6\beta_1^x = 82, \\ 5|\alpha_1^x - 2\beta_1^x - 6| + |\alpha_1^x - 3\beta_1^x - 1| + 4\alpha_1^x + 7\beta_1^x = 29, \\ 0 \le \alpha_1^x \le 2, \\ 0 \le \beta_1^x \le \frac{5}{2}. \end{cases} \quad (5.22a)$$

After solving the Equation (5.22a), we get,

$${\alpha_1}^x = 1, \qquad {\beta_1}^x = 0,$$

consequently,

$${\alpha_2}^x = \frac{1}{5}(21 - {\alpha_1}^x) = \frac{1}{5}(21 - 1) = 4,$$

$${\beta_2}^x = 1 - \frac{2{\beta_1}^x}{5} = 1 - \frac{2(0)}{5} = 1.$$

Then, we get the first fuzzy solution for *FFLS*,

$$\tilde{X}_1 = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} ({m_1}^x, {\alpha_1}^x, {\beta_1}^x) \\ ({m_2}^x, {\alpha_2}^x, {\beta_2}^x) \end{pmatrix} = \begin{pmatrix} (2, 1, 0) \\ (1, 4, 1) \end{pmatrix}.$$

Moreover, Figure 5.3, shows that there is an intersection between first and second equations in Equation (5.22a) in intervals ${\alpha_1}^x \in [0,2]$, ${\beta_1}^x \in \left[0, \frac{5}{2}\right]$, which means that the system has a fuzzy solution in intersection line:

$$\begin{cases} 10 \mid -2\,\alpha_1^x + \beta_1^x + 6 \mid + \mid 3\,\alpha_1^x - 4\left(\beta_1^x + 7\right) \mid + 17\,\alpha_1^x + 6\,\beta_1^x - 82, \\ 5 \mid \alpha_1^x - 2\,\beta_1^x - 6 \mid + \mid \alpha_1^x - 3\,\beta_1^x - 1 \mid + 4\,\alpha_1^x + 7\,\beta_1^x - 29, \\ \alpha_1^x \in [0,\, 2],\ \ {\beta^x}_1 \in \left[0,\, \frac{5}{2}\right]. \end{cases}$$

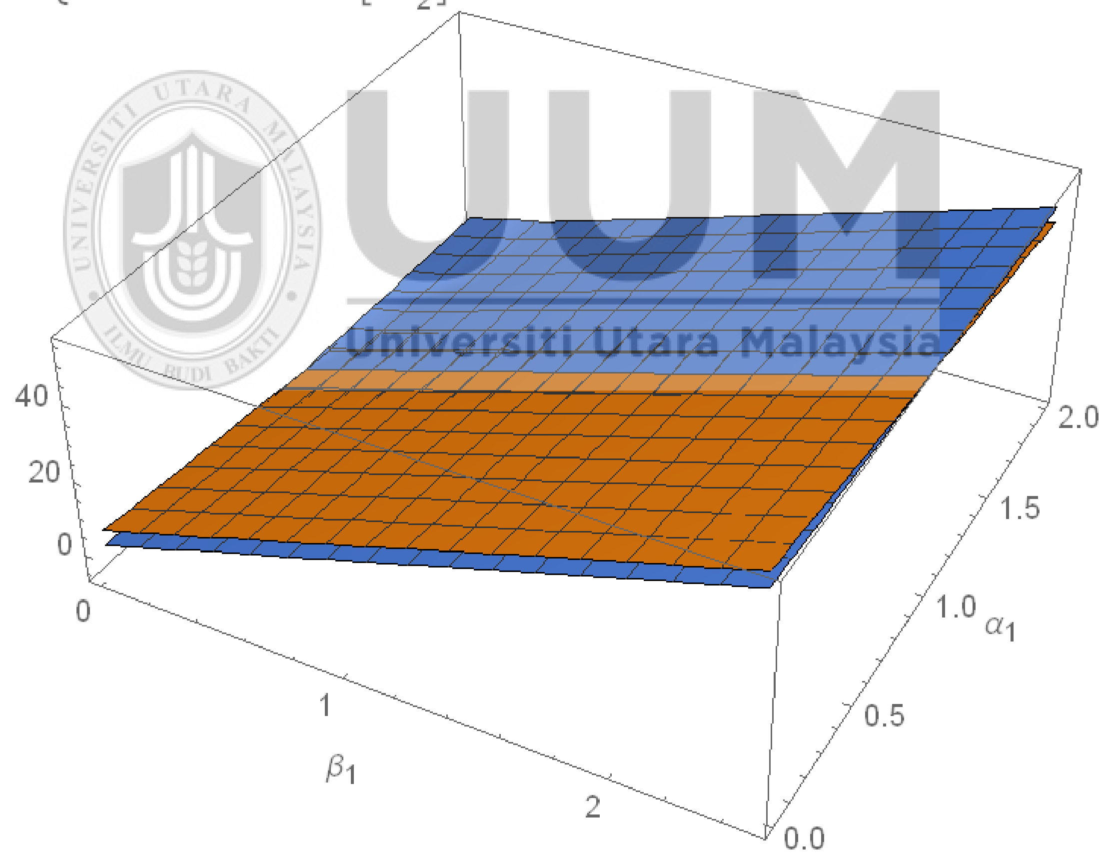

Figure 5.3. The first solution $\tilde{X}_1$ in ${\alpha_1}^x \in [0,2]$ and ${\beta_1}^x \in \left[0, \frac{5}{2}\right]$.

- **For interval ${\alpha_1}^x \in (2, 9]$,**

$$\alpha_2{}^x = \frac{1}{5}(23 - 2\alpha_1{}^x).$$

Hence, Equation (5.21) may be reduced to the following system,

$$\begin{cases} 5|-2\alpha_1^x + \beta_1^x + 6| + |-3\alpha_1^x + 2\beta_1^x + 17| + 7\alpha_1^x + 3\beta_1^x = 38, \\ |-2\alpha_1^x + 3\beta_1^x + 3| + 5|\alpha_1^x - 2(\beta_1^x + 3)| + 3\alpha_1^x + 7\beta_1^x = 27, \\ 2 < \alpha_1^x \leq 9, \\ 0 \leq \beta_1^x \leq \dfrac{5}{2}. \end{cases} \tag{5.22b}$$

After solving the previous system we get,

$$\alpha_1{}^x = \frac{51}{14}, \qquad \beta_1{}^x = 0,$$

consequently,

$$\alpha_2{}^x = \frac{1}{5}(23 - 2\alpha_1{}^x) = \frac{1}{5}\left(23 - 2 \times \frac{51}{14}\right) = \frac{22}{7},$$

$$\beta_2{}^x = 1 - \frac{2\beta_1{}^x}{5} = 1 - 0 = 1.$$

Then, the second fuzzy solution $\tilde{X}_2$ for fuzzy system,

$$\tilde{X}_2 = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1{}^x, \alpha_1{}^x, \beta_1{}^x) \\ (m_2{}^x, \alpha_2{}^x, \beta_2{}^x) \end{pmatrix} = \begin{pmatrix} \left(2, \frac{51}{14}, 0\right) \\ \left(1, \frac{22}{7}, 1\right) \end{pmatrix}.$$

The verification of solution

$$\begin{cases} (3,5,1)\otimes\left(2,\frac{51}{14},0\right)\oplus(2,4,1)\otimes\left(1,\frac{22}{7},1\right)= \\ \left(6,\frac{88}{7},2\right)\oplus\left(2,\frac{59}{7},4\right)=(8,21,6), \\ \\ (2,1,0)\otimes\left(2,\frac{51}{14},0\right)\oplus(4,0,1)\otimes\left(1,\frac{22}{7},1\right)= \\ \left(4,\frac{51}{7},0\right)\oplus\left(4,\frac{103}{7},6\right)=(8,22,6). \end{cases}$$

Moreover, Figure 5.4. shows that there is intersection between first and second equations in Equation (5.22b) in intervals ${\alpha_1}^x \in [2,9]$, ${\beta_1}^x \in \left[0,\frac{5}{2}\right]$, which means the system has a fuzzy solution in intersection line:

$$\begin{cases} 5\left|-2\alpha_1^x+\beta_1^x+6\right|+\left|-3\alpha_1^x+2\beta_1^x+17\right|+7\alpha_1^x+3\beta_1^x-38, \\ \left|-2\alpha_1^x+3\beta_1^x+3\right|+5\left|\alpha_1^x-2\left(\beta_1^x+3\right)\right|+3\alpha_1^x+7\beta_1^x-27, \\ \alpha_1^x\in(2,9],\ \beta^x{}_1\in\left[0,\frac{5}{2}\right]. \end{cases}$$

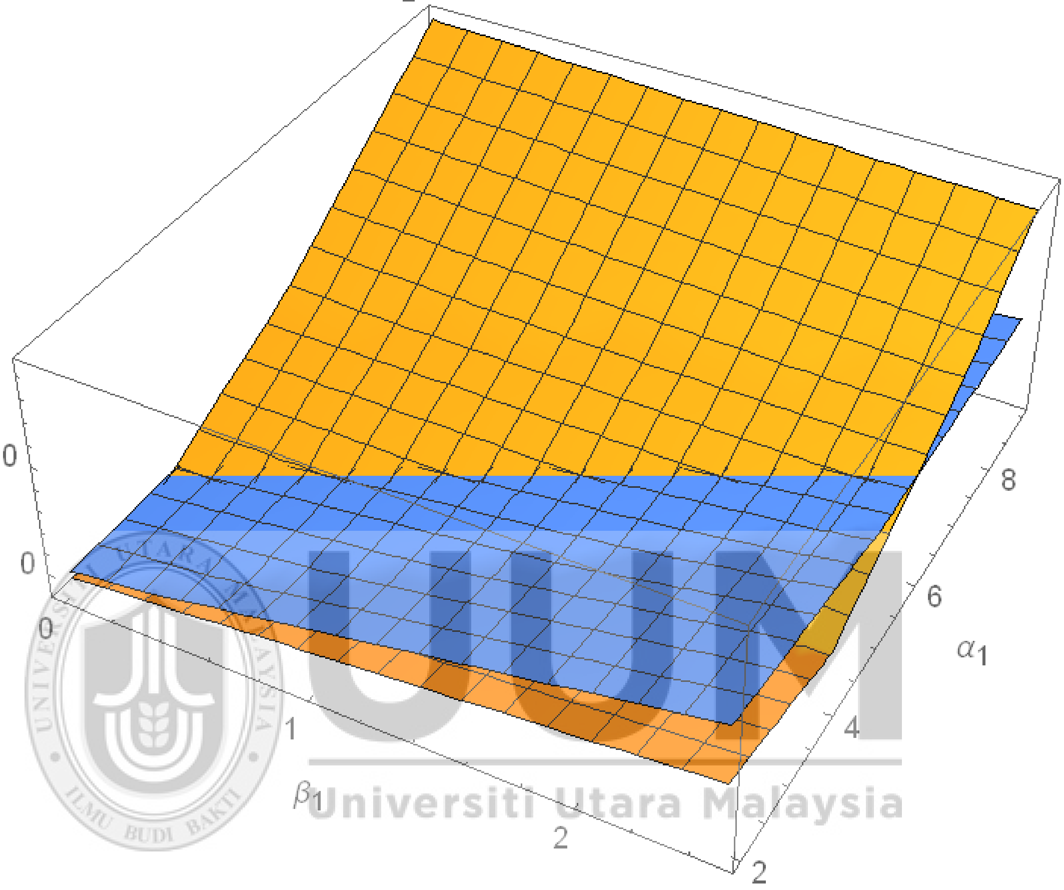

Figure 5.4. The second solution $\tilde{X}_2$ in $\alpha_1{}^x \in [2,9]$ and $\beta_1{}^x \in \left[0,\frac{5}{2}\right]$.

For interval $\boldsymbol{\alpha_1{}^x \in (9,11]}$,

$$\alpha_2{}^x = \frac{1}{2}(11 - \alpha_1{}^x).$$

Hence, Equation (5.21) may be reduced to the following system,

$$\begin{cases} 20|-2\alpha_1^x+\beta_1^x+6|+|-15\alpha_1^x+8\beta_1^x+95|+25\alpha_1^x+12\beta_1^x=125, \\ |\alpha_1^x-\frac{6\beta_1^x}{5}-3|+2|\alpha_1^x-2\beta_1^x-6|+\alpha_1^x+\frac{14\beta_1^x}{5}=9, \\ 9<\alpha_1{}^x\leq 11, \\ 0\leq \beta_1{}^x\leq \frac{5}{2}. \end{cases} \quad (5.22c)$$

We find that the system in Equation (5.22c) has no solution. Moreover, Figure 5.3. shows that there is no intersection between first and second equations in Equation (5.22c) in intervals $\alpha_1{}^x \in [9,11]$, $\beta_1{}^x \in \left[0,\frac{5}{2}\right]$, which means the system has no solution in intersection line.

$$\begin{cases} 20\left|-2\alpha_1^x+\beta_1^x+6\right|+\left|-15\alpha_1^x+8\beta_1^x+95\right|+25\alpha_1^x+12\beta_1^x-125, \\ \left|\alpha_1^x-\frac{6\beta_1^x}{5}-3\right|+2\left|\alpha_1^x-2\beta_1^x-6\right|+\alpha_1^x+\frac{14\beta_1^x}{5}-9, \\ \alpha_1^x\in(9,\ 11],\ \beta^x{}_1\in\left[0,\ \frac{5}{2}\right]. \end{cases}$$

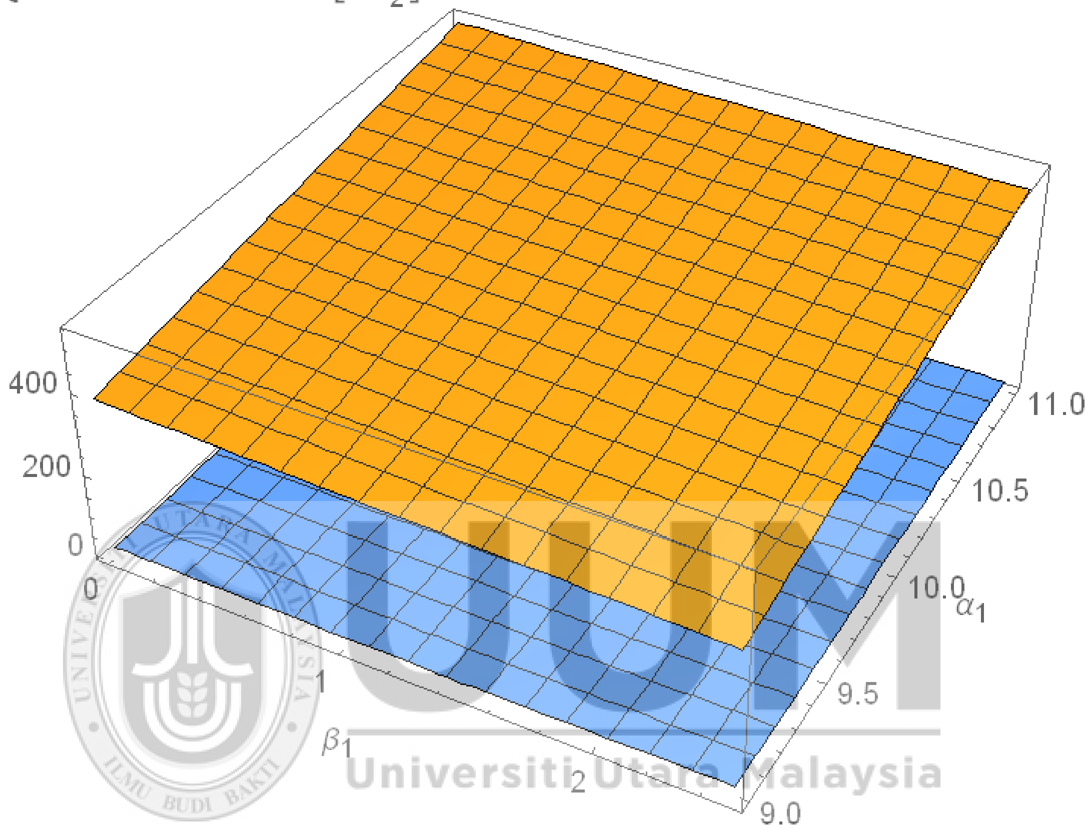

Figure 5.5. Shows the system has no solution in $\alpha_1{}^x \in [9,11]$ and $\beta_1{}^x \in \left[0, \frac{5}{2}\right]$.

The $\tilde{X}_1$ and $\tilde{X}_2$ are transformed to form (a, b, c) to compare with the solution in Kumar et al. (2011b).

We find the first solution $\tilde{X}_1$ as identical to his solution $\tilde{X}_1{}'$,

$$\tilde{X}_1{}' = \begin{pmatrix} \tilde{x}_1{}' \\ \tilde{x}_2{}' \end{pmatrix} = \begin{pmatrix} (a_1{}^x, b_1{}^x, c_1{}^x) \\ (a_2{}^x, b_2{}^x, c_2{}^x) \end{pmatrix} = \begin{pmatrix} (1, 2, 2) \\ (-3, 1, 2) \end{pmatrix}.$$

While, the solution $\tilde{X}_2$ represents the further solution $\tilde{X}_2{}'$ which cannot be determined through *LP* method in Kumar et al. (2011b)'method,

$$\tilde{X}_2{}' = \begin{pmatrix} \tilde{x}_1{}' \\ \\ \tilde{x}_2{}' \end{pmatrix} = \begin{pmatrix} (a_1{}^x, b_1{}^x, c_1{}^x) \\ \\ (a_2{}^x, b_2{}^x, c_2{}^x) \end{pmatrix} = \begin{pmatrix} \left(-\frac{23}{14}, 2, 2\right) \\ \\ \left(-\frac{15}{7}, 1, 2\right) \end{pmatrix}.$$

As it is noted, the proposed method can provide second solutions which satisfy the concept finite solution of *FFLS*, while Kumar et al. (2011b)'method cannot provide it.

Next example is used in Allahviranloo et al. (2014), they provided the solution as unique fuzzy solution. The same example is solved by proposed method, it is noted the system does not have a fuzzy solution. Moreover the verification of Allahviranloo et al. (2014)'solution shows that it doesn't satisfy the left hand side for the system.

Example 5.2.2. Allahviranloo et al. (2014) consider the following *FFLS*,

$$\begin{cases} (2,3,4) \otimes (a_1^x, b_1^x, c_1^x) \oplus (1,1,1) \otimes (a_2^x, b_2^x, c_2^x) \oplus (1,2,3) \otimes (a_3^x, b_3^x, c_3^x) \\ \qquad = (-9,2,18), \\ (5,6,7) \otimes (a_1^x, b_1^x, c_1^x) \oplus (-4,-3,-2) \otimes (a_2^x, b_2^x, c_2^x) \oplus (2,2,2) \otimes (a_3^x, b_3^x, c_3^x) \\ \qquad = (-13,8,34), \\ (-3,-2,-1) \otimes (a_1^x, b_1^x, c_1^x) \oplus (1,2,3) \otimes (a_2^x, b_2^x, c_2^x) \oplus (3,3,4) \otimes (a_3^x, b_3^x, c_3^x) \\ \qquad = (-3,-10,5). \end{cases}$$

The system can be rewritten in matrix form as follows,

$$\begin{pmatrix} (2,3,4) & (1,1,1) & (1,2,3) \\ (5,6,7) & (-4,-3,-2) & (2,2,2) \\ (-3,-2,-1) & (1,2,3) & (3,3,4) \end{pmatrix} \otimes \begin{pmatrix} (a_1^x, b_1^x, c_1^x) \\ (a_2^x, b_2^x, c_2^x) \\ (a_3^x, b_3^x, c_3^x) \end{pmatrix} = \begin{pmatrix} (-9,2,18) \\ (-13,8,34) \\ (-3,-10,5) \end{pmatrix}.$$

Familiar to Example 5.2.1, the form of (a, b, c) needs to be converted to the form of (m, α, β) for TFN. We will use $\tilde{A}', \tilde{X}'$ and $\tilde{B}'$ to the form of (a, b, c), while $\tilde{A}$, $\tilde{X}$ and $\tilde{B}$ to the form of (m, α, β).

In matrix form $\tilde{A} \otimes \tilde{X} = \tilde{B}$ it becomes, $\big(in\ form\ (m, \alpha, \beta)\big)$,

$$\begin{pmatrix} (3,1,1) & (1,0,0) & (2,1,1) \\ (6,1,1) & (-3,1,1) & (2,0,0) \\ (-2,1,1) & (2,1,1) & (3,0,1) \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (2,11,16) \\ (8,21,26) \\ (-10,20,15) \end{pmatrix}.$$

Using the proposed method,

$$
\begin{cases}
3m_1^x + m_2^x + 2m_3^x = 2, \\
-Min[2(m_1^x - \alpha_1^x), 4(m_1^x - \alpha_1^x)] - Min[m_3^x - \alpha_3^x, 3(m_3^x - \alpha_3^x)] + 3m_1^x + 2m_3^x \\
+\alpha_2^x = 11, \\
Max[2(m_1^x + \beta_1^x), 4(m_1^x + \beta_1^x)] + Max[m_3^x + \beta_3^x, 3(m_3^x + \beta_3^x)] - 3m_1^x - 2m_3^x \\
+\beta_2^x = 16, \\
6m_1^x - 3m_2^x + 2m_3^x = 8, \\
-Min[5(m_1^x - \alpha_1^x), 7(m_1^x - \alpha_1^x)] - Min[-4(m_2^x + \beta_2^x), -2(m_2^x + \beta_2^x)] + 6m_1^x \\
-3m_2^x + 2m_3^x - 2(m_3^x - \alpha_3^x) = 21, \\
Max[-4(m_2^x - \alpha_2^x), -2(m_2^x - \alpha_2^x)] + Max[5(m_1^x + \beta_1^x), 7(m_1^x + \beta_1^x)] - 6m_1^x \\
+3m_2^x - 2m_3^x + 2(m_3^x + \beta_3^x) = 26, \\
-2m_1^x + 2m_2^x + 3m_3^x = -10, \\
-Min[m_2^x - \alpha_2^x, 3(m_2^x - \alpha_2^x)] - Min[3(m_3^x - \alpha_3^x), 4(m_3^x - \alpha_3^x)] - \\
Min[-m_1^x - \beta_1^x, -3(m_1^x + \beta_1^x)] - 2m_1^x + 2m_2^x + 3m_3^x = -7, \\
Max[-3(m_1^x - \alpha_1^x), -m_1^x + \alpha_1^x] + Max[m_2^x + \beta_2^x, 3(m_2^x + \beta_2^x)] + \\
Max[3(m_3^x + \beta_3^x), 4(m_3^x + \beta_3^x)] + 2m_1^x - 2m_2^x - 3m_3^x = 15.
\end{cases}
$$

Substituting ${m^x}_1 = 2$, ${m^x}_2 = 0$ and ${m^x}_3 = -2$, in absolute system, we get,

$$
\begin{cases}
9 + Min[8 - 4{\alpha^x}_1, 4 - 2{\alpha^x}_1] + Min[-2 - {\alpha^x}_3, -3(2 + {\alpha^x}_3)] = {\alpha^x}_2, \\
Max[2(2 + {\beta^x}_1), 4(2 + {\beta^x}_1)] + Max[-2 + {\beta^x}_3, 3(-2 + {\beta^x}_3)] + {\beta^x}_2 = 18, \\
9 + Min[14 - 7{\alpha^x}_1, 10 - 5{\alpha^x}_1] + Min[-4{\beta^x}_2, -2{\beta^x}_2] = 2{\alpha^x}_3, \\
Max[2{\alpha^x}_2, 4{\alpha^x}_2] + Max[5(2 + {\beta^x}_1), 7(2 + {\beta^x}_1)] + 2{\beta^x}_3 = 38, \\
3 + Min[-3{\alpha^x}_2, -{\alpha^x}_2] + Min[-4(2 + {\alpha^x}_3), -3(2 + {\alpha^x}_3)] \\
+Min[-2 - {\beta^x}_1, -3(2 + {\beta^x}_1)] = 0, \\
Max[-2 + {\alpha^x}_1, -6 + 3{\alpha^x}_1] + Max[{\beta^x}_2, 3{\beta^x}_2] \\
+Max[3(-2 + {\beta^x}_3), 4(-2 + {\beta^x}_3)] = 5.
\end{cases}
$$

Transforming the min-max system to absolute system and reducing it, we get,

$$\begin{cases} |2-\alpha^x{}_1| + |2+\alpha^x{}_3| + 3\alpha^x{}_1 + \alpha^x{}_2 + 2\alpha^x{}_3 = 11, \\ |2+\beta^x{}_1| + |-2+\beta^x{}_3| + 3\beta^x{}_1 + \beta^x{}_2 + 2\beta^x{}_3 = 16, \\ |2-\alpha^x{}_1| + |\beta^x{}_2| + 6\alpha^x{}_1 + 2\alpha^x{}_3 + 3\beta^x{}_2 = 21, \\ |\alpha^x{}_2| + |2+\beta^x{}_1| + 3\alpha^x{}_2 + 6\beta^x{}_1 + 2\beta^x{}_3 = 26, \\ 16 + 2|\alpha^x{}_2| + |2+\alpha^x{}_3| + 2|2+\beta^x{}_1| + 4\alpha^x{}_2 + 7\alpha^x{}_3 + 4\beta^x{}_1 = 0, \\ 2|2-\alpha^x{}_1| + 2|\beta^x{}_2| + |-2+\beta^x{}_3| + 4\alpha^x{}_1 + 4\beta^x{}_2 + 7\beta^x{}_3 = 32. \end{cases}$$

Solving the system provides the non fuzzy solution, since the spreads are negative,

$$\alpha_1 = \frac{89}{17}, \beta^x{}_1 = \frac{50}{17}, \beta_3 = \frac{9}{17}, \beta^x{}_2 = -\frac{5}{17}, \alpha_3 = \frac{1}{2}\left(-\frac{232}{17} - |\beta^x{}_2| - 3\beta^x{}_2\right),$$

$$\alpha^x{}_2 = -\frac{135}{17} - |2+\alpha^x{}_3| - 2\alpha^x{}_3.$$

While, Allahviranloo et al. (2014) provided the following $\tilde{X}_{allv}{}'$ as a fuzzy solution,

$$\tilde{X}_{allv}{}' = \begin{pmatrix} (a_1^x, b_1^x, c_1^x) \\ (a_2^x, b_2^x, c_2^x) \\ (a_3^x, b_3^x, c_3^x) \end{pmatrix} = \begin{pmatrix} (1,2,4) \\ (-2,0,3) \\ (-3,-2,-1) \end{pmatrix}, \quad \tilde{X}_{allv} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (2,1,2) \\ (0,2,3) \\ (-2,1,1) \end{pmatrix}.$$

But, actually the verification of solution shows that $\tilde{A} \otimes \tilde{X}_{allv} \neq \tilde{B}$,

$$\begin{cases} (3,1,1)\otimes(2,1,2)\oplus(1,0,0)\otimes(0,2,3)\oplus(2,1,1)\otimes(-2,1,1)= \\ (6,4,10)\oplus(0,2,3)\oplus(-4,5,3)=(2,11,16), \\ (6,1,1)\otimes(2,1,2)\oplus(-3,1,1)\otimes(0,2,3)\oplus(2,0,0)\otimes(-2,1,1)= \\ (12,7,16)\oplus(0,12,8)\oplus(-4,2,2)=(8,21,26), \\ (-2,1,1)\otimes(2,1,2)\oplus(2,1,1)\otimes(0,2,3)\oplus(3,0,1)\otimes(-2,1,1)= \\ (-4,8,3)\oplus(0,6,9)\oplus(-6,6,3)=(-10,20,15). \end{cases}$$

As noted $\tilde{A}\otimes \tilde{X}_{allv} = \tilde{B}_{allv}$, where,

$$\tilde{B}_{allv}=\begin{pmatrix}(m_1^b,\alpha_1^b,\beta_1^b)\\(m_2^b,\alpha_2^b,\beta_2^b)\\(m_3^b,\alpha_3^b,\beta_3^b)\end{pmatrix}=\begin{pmatrix}(2,11,16)\\(8,21,26)\\(-10,20,15)\end{pmatrix},\tilde{B}_{allv}{}'=\begin{pmatrix}(a_1^b,b_1^b,c_1^b)\\(a_2^b,b_2^b,c_2^b)\\(a_3^b,b_3^b,c_3^b)\end{pmatrix}=\begin{pmatrix}(-9,2,18)\\(-13,8,34)\\(-30,-10,5)\end{pmatrix}.$$

But the left hand side in example is actually

$$\tilde{B}=\begin{pmatrix}(m_1^b,\alpha_1^b,\beta_1^b)\\(m_2^b,\alpha_2^b,\beta_2^b)\\(m_3^b,\alpha_3^b,\beta_3^b)\end{pmatrix}=\begin{pmatrix}(2,11,16)\\(8,21,26)\\(-10,-7,15)\end{pmatrix},\quad \tilde{B}'=\begin{pmatrix}(a_1^b,b_1^b,c_1^b)\\(a_2^b,b_2^b,c_2^b)\\(a_3^b,b_3^b,c_3^b)\end{pmatrix}=\begin{pmatrix}(-9,2,18)\\(-13,8,34)\\(-3,-10,5)\end{pmatrix}.$$

For that, this example has no fuzzy solution.

In next section, the proposed method is modified after Step 4 to produce all associated linear systems that may provide compatible solution satisfy the *FFLS*.

5.3 Associated Linear Systems of Near Zero Fully Fuzzy Linear System

In this section, we show that the possibility of finite solution can be obtained by Corollary 5.2.1. The method in Section 5.2 will be modified to produce all possible linear systems, so the modification will be after step 3, is as follows**,**

Step 4 Considering the Equation (5.13). Then, constructing and solving all possible linear systems from absolute system, using Equation (5.12).

Step 5 Considering only the solution which satisfies the fuzzy system $FFLS$, by deleting the solutions which do not satisfy the absolute system or have negative spreads (non fuzzy solution).

The next example is used in Babar et al. (2013). Using the proposed method in this section, we obtain further different feasible solutions, while the proposed method in Babar et al. (2013) obtained only one solution. The first and second solutions are also provided by previous method in Section 5.2 to confirm the result.

Example 5.3.1. Babar et al. (2013) consider the following *FFLS*,

$$\begin{cases} (3,2,3) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (-2,1,1) \otimes (m_2^x, \beta_2^x, \beta_2^x) = (5,16,17), \\ (-4,1,2) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (4,2,1) \otimes (m_2^x, \beta_2^x, \beta_2^x) = (-4,12,22), \end{cases}$$

where $\tilde{x}_i = ({m_i}^x, {\alpha_i}^x, {\beta_i}^x)$, $i = 1, 2$ are arbitrary triangular fuzzy numbers.

The system may be written in matrix form is as follows,

$$\begin{pmatrix} (3,2,3) & (-2,1,1) \\ (-4,1,2) & (4,2,1) \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (5,16,17) \\ (-4,12,22) \end{pmatrix}.$$

$$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} ({m_1}^x, {\alpha_1}^x, {\beta_1}^x) \\ ({m_2}^x, {\alpha_2}^x, {\beta_2}^x) \end{pmatrix}.$$

Then using Step 1,

$$\begin{cases} 3{m_1}^x - 2{m_2}^x = 5, \\ -Min[{m_1}^x - {\alpha_1}^x, 6({m_1}^x - {\alpha_1}^x)] - Min[-{m_2}^x - {\beta_2}^x, -3({m_2}^x + {\beta_2}^x)] \\ +3{m_1}^x - 2{m_2}^x = 16, \\ Max[-3({m_2}^x - {\alpha_2}^x), -{m_2}^x + {\alpha_2}^x] + Max[{m_1}^x + {\beta_1}^x, 6({m_1}^x + {\beta_1}^x)] \\ -3{m_1}^x + 2{m_2}^x = 17, \\ -4{m_1}^x + 4{m_2}^x = -4, \\ -Min[2({m_2}^x - {\alpha_2}^x), 5({m_2}^x - {\alpha_2}^x)] - Min[-5({m_1}^x + {\beta_1}^x), -2({m_1}^x + {\beta_1}^x)] \\ -4{m_1}^x + 4{m_2}^x = 12, \\ Max[-5({m_1}^x - {\alpha_1}^x), -2({m_1}^x - {\alpha_1}^x)] + Max[2({m_2}^x + {\beta_2}^x), 5({m_2}^x + {\beta_2}^x)] \\ +4{m_1}^x - 4{m_2}^x = 22. \end{cases}$$

Solving the following linear system,

$$\begin{cases} 3{m_1}^x - 2{m_2}^x = 5, \\ -4{m_1}^x + 4{m_2}^x = -4. \end{cases}$$

Then using Step 2, ${m_1}^x = 3, {m_2}^x = 2$.

Hence, the nonlinear system with values on ${m_i}^x$ is simplified as follows,

$$\begin{cases} 5 - Min[3 - {\alpha_1}^x, 6(3 - {\alpha_1}^x)] - Min[-2 - {\beta_2}^x, -3(2 + {\beta_2}^x)] = 16, \\ -5 + Max[-3(2 - {\alpha_2}^x), -2 + {\alpha_2}^x] + Max[3 + {\beta_1}^x, 6(3 + {\beta_1}^x)] = 17, \\ -4 - Min[2(2 - {\alpha_2}^x), 5(2 - {\alpha_2}^x)] - Min[-5(3 + {\beta_1}^x), -2(3 + {\beta_1}^x)] = 12, \\ 4 + Max[-5(3 - {\alpha_1}^x), -2(3 - {\alpha_1}^x)] + Max[2(2 + {\beta_2}^x), 5(2 + {\beta_2}^x)] = 22. \end{cases}$$

Using Step 3, the min-max system can be written as absolute system is as follows,

$$\begin{cases} \frac{1}{2}|-\alpha_1^x - 6(3-\alpha_1^x) + 3| + \frac{1}{2}|-\beta_2^x + 3(\beta_2^x + 2) - 2| \\ + \frac{1}{2}(\alpha_1^x - 6(3-\alpha_1^x) - 3) + \frac{1}{2}(\beta_2^x + 3(\beta_2^x + 2) + 2) + 5 = 16, \\ \frac{1}{2}|-\alpha_2^x - 3(2-\alpha_2^x) + 2| + \frac{1}{2}|\beta_1^x - 6(\beta_1^x + 3) + 3| \\ + \frac{1}{2}(\alpha_2^x - 3(2-\alpha_2^x) - 2) + \frac{1}{2}(\beta_1^x + 6(\beta_1^x + 3) + 3) - 5 = 17, \\ \frac{3|2-\alpha_2^x|}{2} + \frac{3|\beta_1^x + 3|}{2} - \frac{7}{2}(2-\alpha_2^x) + \frac{7}{2}(\beta_1^x + 3) - 4 = 12, \\ \frac{3|3-\alpha_1^x|}{2} + \frac{3|\beta_2^x + 2|}{2} - \frac{7}{2}(3-\alpha_1^x) + \frac{7}{2}(\beta_2^x + 2) + 4 = 22. \end{cases}$$

It can be simplified as follows

$$\begin{cases} 7\alpha_1^x + 4\beta_2^x + 5|\alpha_1^x - 3| + 2|\beta_2^x + 2| = 35, \\ 4\alpha_2^x + 7\beta_1^x + 2|\alpha_2^x - 2| + 5|\beta_1^x + 3| = 31, \\ 3|2-\alpha_2^x| + 3|\beta_1^x + 3| + 7(\alpha_2^x + \beta_1^x) = 25, \\ 3|3-\alpha_1^x| + 3|\beta_2^x + 2| + 7(\alpha_1^x + \beta_2^x) = 43. \end{cases}$$

Using Step 4, $(\beta_1^x + 3), (\beta_2^x + 2) \geq 0$, then,

$$\begin{cases} 7\alpha_1^x + 4\beta_2^x + 5|\alpha_1^x - 3| + 2(\beta_2^x + 2) = 35, \\ 4\alpha_2^x + 7\beta_1^x + 2|\alpha_2^x - 2| + 5(\beta_1^x + 3) = 31, \\ 3|2-\alpha_2^x| + 3(\beta_1^x + 3) + 7(\alpha_2^x + \beta_1^x) = 25, \\ 3|3-\alpha_1^x| + 3(\beta_2^x + 2) + 7(\alpha_1^x + \beta_2^x) = 43. \end{cases} \tag{5.23}$$

Since we have four absolute values, we can construct sixteen linear systems, the solution of sixteen possibly linear systems is as follows,

$$\begin{cases}
\beta^{x}{}_{1}=\frac{17}{15},\alpha^{x}{}_{2}=\frac{16}{15},\beta^{x}{}_{2}=\frac{23}{15},\alpha^{x}{}_{1}=\frac{46}{15},\\
\beta^{x}{}_{1}=\frac{17}{15},\alpha^{x}{}_{2}=\frac{16}{15},\beta^{x}{}_{2}=\frac{19}{12},\alpha^{x}{}_{1}=\frac{73}{24},\\
\beta^{x}{}_{1}=-\frac{5}{3},\alpha^{x}{}_{2}=\frac{20}{3},\beta^{x}{}_{2}=\frac{23}{15},\alpha^{x}{}_{1}=\frac{46}{15},\\
\beta^{x}{}_{1}=-\frac{5}{3},\alpha^{x}{}_{2}=\frac{20}{3},\beta^{x}{}_{2}=\frac{19}{12},\alpha^{x}{}_{1}=\frac{73}{24},\\
\beta^{x}{}_{1}=\frac{19}{25},\alpha^{x}{}_{2}=\frac{36}{25},\beta^{x}{}_{2}=\frac{23}{15},\alpha^{x}{}_{1}=\frac{46}{15},\\
\beta^{x}{}_{1}=\frac{19}{25},\alpha^{x}{}_{2}=\frac{36}{25},\beta^{x}{}_{2}=\frac{19}{12},\alpha^{x}{}_{1}=\frac{73}{24},\\
\beta^{x}{}_{1}=1,\alpha^{x}{}_{2}=0,\beta^{x}{}_{2}=\frac{23}{15},\alpha^{x}{}_{1}=\frac{46}{15},\\
\beta^{x}{}_{1}=1,\alpha^{x}{}_{2}=0,\beta^{x}{}_{2}=\frac{19}{12},\alpha^{x}{}_{1}=\frac{73}{24},\\
\beta^{x}{}_{1}=\frac{17}{15},\alpha^{x}{}_{2}=\frac{16}{15},\beta^{x}{}_{2}=\frac{17}{10},\alpha^{x}{}_{1}=\frac{29}{10},\\
\beta^{x}{}_{1}=\frac{17}{15},\alpha^{x}{}_{2}=\frac{16}{15},\beta^{x}{}_{2}=2,\alpha^{x}{}_{1}=2,\\
\beta^{x}{}_{1}=-\frac{5}{3},\alpha^{x}{}_{2}=\frac{20}{3},\beta^{x}{}_{2}=\frac{17}{10},\alpha^{x}{}_{1}=\frac{29}{10},\\
\beta^{x}{}_{1}=-\frac{5}{3},\alpha^{x}{}_{2}=\frac{20}{3},\beta^{x}{}_{2}=2,\alpha^{x}{}_{1}=2,\\
\beta^{x}{}_{1}=\frac{19}{25},\alpha^{x}{}_{2}=\frac{36}{25},\beta^{x}{}_{2}=\frac{17}{10},\alpha^{x}{}_{1}=\frac{29}{10},\\
\beta^{x}{}_{1}=\frac{19}{25},\alpha^{x}{}_{2}=\frac{36}{25},\beta^{x}{}_{2}=2,\alpha^{x}{}_{1}=2,\\
\beta^{x}{}_{1}=1,\alpha^{x}{}_{2}=0,\beta^{x}{}_{2}=\frac{17}{10},\alpha^{x}{}_{1}=\frac{29}{10},\\
\beta^{x}{}_{1}=1,\alpha^{x}{}_{2}=0,\beta^{x}{}_{2}=2,\alpha^{x}{}_{1}=2.
\end{cases}$$

Applying Step 5, we find the below linear systems only are provided fuzzy solutions for *FFLS*.

- The first fuzzy solution $\tilde{X}_1$ comes from the following linear system,

$$\begin{cases} 5(3-\alpha^x{}_1)+7\alpha^x{}_1+4\beta^x{}_2+2\left(2+\beta^x{}_2\right)=35, \\ 2(2-\alpha^x{}_2)+4\alpha^x{}_2+7\beta^x{}_1+5\left(3+\beta^x{}_1\right)=31, \\ 3(2-\alpha^x{}_2)+3\left(3+\beta^x{}_1\right)+7\left(\alpha^x{}_2+\beta^x{}_1\right)=25, \\ 3(3-\alpha^x{}_1)+3\left(2+\beta^x{}_2\right)+7\left(\alpha^x{}_1+\beta^x{}_2\right)=43, \end{cases} \longrightarrow \begin{cases} \beta^x{}_1=1, \\ \alpha^x{}_2=0, \\ \beta^x{}_2=2, \\ \alpha^x{}_1=2. \end{cases}$$

Thus the first fuzzy solution $\tilde{X}_1$ is,

$$\tilde{X}_1=\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}=\begin{pmatrix} (m_1{}^x,\alpha_1{}^x,\beta_1{}^x) \\ (m_2{}^x,\alpha_2{}^x,\beta_2{}^x) \end{pmatrix}=\begin{pmatrix} (3,2,1) \\ (2,0,2) \end{pmatrix}.$$

- The second fuzzy solution $\tilde{X}_2$ comes from the following linear system,

$$\begin{cases} -5(3-\alpha^x{}_1)+7\alpha^x{}_1+4\beta^x{}_2+2\left(2+\beta^x{}_2\right)=35, \\ 2(2-\alpha^x{}_2)+4\alpha^x{}_2+7\beta^x{}_1+5(3+\beta^x{}_1)=31, \\ 3(2-\alpha^x{}_2)+3\left(3+\beta^x{}_1\right)+7\left(\alpha^x{}_2+\beta^x{}_1\right)=25, \\ -3(3-\alpha^x{}_1)+3(2+\beta^x{}_2)+7(\alpha^x{}_1+\beta^x{}_2)=43, \end{cases} \longrightarrow \begin{cases} \beta^x{}_1=1, \\ \alpha^x{}_2=0, \\ \beta^x{}_2=\dfrac{23}{15}, \\ \alpha^x{}_1=\dfrac{46}{15}. \end{cases}$$

Thus the second fuzzy solution $\tilde{X}_2$ is,

$$\tilde{X}_2=\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}=\begin{pmatrix} (m_1{}^x,\alpha_1{}^x,\beta_1{}^x) \\ (m_2{}^x,\alpha_2{}^x,\beta_2{}^x) \end{pmatrix}=\begin{pmatrix} \left(3,\dfrac{46}{15},1\right) \\ \left(2,0,\dfrac{23}{15}\right) \end{pmatrix}.$$

The verification for the second fuzzy solution is

$$\begin{cases} (3,2,3) \otimes \left(3, \frac{46}{15}, 1\right) \oplus (-2,1,1) \otimes \left(2,0,\frac{23}{15}\right) = \\ \left(9, \frac{47}{5}, 15\right) \oplus (-4, \frac{33}{5}, 2) = (5, 16, 17), \\ (-4,1,2) \otimes \left(3, \frac{46}{15}, 1\right) \oplus (4,2,1) \otimes \left(2,0,\frac{23}{15}\right) = \\ \left(-12, 8, \frac{37}{3}\right) \oplus \left(8, 4, \frac{29}{3}\right) = (-4, 12, 22). \end{cases}$$

However, the method in Second 5.2 can be applied as follows:

From first equation in Equation (5.23),

$$\beta_2^x = \frac{1}{6}(-5|\alpha_1^x - 3| - 7\alpha_1^x + 31).$$

Also from second equation,

$$\beta_1^x = \frac{1}{12}(-|4 - 2\alpha_2^x| - 4\alpha_2^x + 16).$$

The third equation can be written is as follows,

$$22\alpha_2^x + 6|6 - 3\alpha_2^x| - 5|4 - 2\alpha_2^x| = 16,$$

we get ${\alpha_2}^x = 0$, then $\beta_1^x = 1$.

The fourth equation can be written as follows,

$$7\alpha_1^x + 8|\alpha_1^x - 3| = 22.$$

Then, we get two solutions ${\alpha_1}^x = 2$ or, ${\alpha_1}^x = \frac{46}{15}$.

Hence, we have fuzzy solutions as follows:

- When ${\alpha_1}^x = 2$ then ${\beta_2}^x = 2$,

  the first fuzzy solution $\tilde{X}_1$ is,

$$\tilde{X}_1 = \begin{pmatrix} \tilde{x}_1 \\ \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} ({m_1}^x, {\alpha_1}^x, {\beta_1}^x) \\ \\ ({m_2}^x, {\alpha_2}^x, {\beta_2}^x) \end{pmatrix} = \begin{pmatrix} (3, 2, 1) \\ \\ (2, 0, 2) \end{pmatrix}.$$

Which is obtained in Babar et al. (2013).

- When ${\alpha_1}^x = \frac{46}{15}$ then ${\beta_2}^x = \frac{23}{15}$,

  the second fuzzy solution $\tilde{X}_2$ is

$$\tilde{X}_2 = \begin{pmatrix} \tilde{x}_1 \\ \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} ({m_1}^x, {\alpha_1}^x, {\beta_1}^x) \\ \\ ({m_2}^x, {\alpha_2}^x, {\beta_2}^x) \end{pmatrix} = \begin{pmatrix} \left(3, \frac{46}{15}, 1\right) \\ \\ \left(2, 0, \frac{23}{15}\right) \end{pmatrix}.$$

In order to enhance the finite solution, an example of *FFLS* in size $n = 3$ is illustrated with two unique solutions.

Example 5.3.2. Consider the following 3×3 *FFLS*,

$$\begin{cases} (3,5,1) \otimes ({m_1}^x, {\alpha_1}^x, {\beta_1}^x) \oplus (2,4,1) \otimes ({m_2}^x, {\alpha_2}^x, {\beta_2}^x) \oplus (2,1,1) \\ \otimes ({m_3}^x, {\alpha_3}^x, {\beta_3}^x) = (12, 24, 11), \\ \\ (2,1,0) \otimes ({m_1}^x, {\alpha_1}^x, {\beta_1}^x) \oplus (4,0,1) \otimes ({m_2}^x, {\alpha_2}^x, {\beta_2}^x) \oplus (2,1,1) \\ \otimes ({m_3}^x, {\alpha_3}^x, {\beta_3}^x) = (12, 25, 11), \\ \\ (2,3,1) \otimes ({m_1}^x, {\alpha_1}^x, {\beta_1}^x) \oplus (2,6,1) \otimes ({m_2}^x, {\alpha_2}^x, {\beta_2}^x) \oplus (0,0,0) \\ \otimes ({m_3}^x, {\alpha_3}^x, {\beta_3}^x) = (6, 17, 12), \end{cases}$$

where $\tilde{x}_i = (m_i{}^x, \alpha_i{}^x, \beta_i{}^x)$, $i = 1, 2, 3$ are arbitrary *TFNs*.

The *FFLS* can be written in matrix form $\tilde{A} \otimes \tilde{X} = \tilde{B}$,

$$\begin{pmatrix} (3,5,1) & (2,4,1) & (2,1,1) \\ (2,1,0) & (4,0,1) & (2,1,1) \\ (2,3,1) & (2,6,1) & (0,0,0) \end{pmatrix} \otimes \begin{pmatrix} (m_1{}^x, \alpha_1{}^x, \beta_1{}^x) \\ (m_2{}^x, \alpha_2{}^x, \beta_2{}^x) \\ (m_3{}^x, \alpha_3{}^x, \beta_3{}^x) \end{pmatrix} = \begin{pmatrix} (12, 24, 11) \\ (12, 25, 11) \\ (6, 17, 12) \end{pmatrix}.$$

Using Step 1,

$$\begin{cases} 3m_1{}^x + 2m_2{}^x + 2m_3{}^x = 12, \\ -Min[4(m_1{}^x - \alpha_1{}^x), -2(m_1{}^x + \beta_1{}^x)] - Min[3(m_2{}^x - \alpha_2{}^x), -2(m_2{}^x + \beta_2{}^x)] \\ \quad -Min[m_3{}^x - \alpha_3{}^x, 3(m_3{}^x - \alpha_3{}^x)] + 3m_1{}^x + 2m_2{}^x + 2m_3{}^x = 24, \\ Max[-2(m_1{}^x - \alpha_1{}^x), 4(m_1{}^x + \beta_1{}^x)] + Max[-2(m_2{}^x - \alpha_2{}^x), 3(m_2{}^x + \beta_2{}^x)] \\ \quad +Max[m_3{}^x + \beta_3{}^x, 3(m_3{}^x + \beta_3{}^x)] - 3m_1{}^x - 2m_2{}^x - 2m_3{}^x = 11, \\ 2m_1{}^x + 4m_2{}^x + 2m_3{}^x = 12, \\ -Min[m_1{}^x - \alpha_1{}^x, 2(m_1{}^x - \alpha_1{}^x)] - Min[4(m_2{}^x - \alpha_2{}^x), 5(m_2{}^x - \alpha_2{}^x)] \\ \quad -Min[m_3{}^x - \alpha_3{}^x, 3(m_3{}^x - \alpha_3{}^x)] + 2m_1{}^x + 4m_2{}^x + 2m_3{}^x = 25, \\ Max[m_1{}^x + \beta_1{}^x, 2(m_1{}^x + \beta_1{}^x)] + Max[4(m_2{}^x + \beta_2{}^x), 5(m_2{}^x + \beta_2{}^x)] \\ \quad +Max[m_3{}^x + \beta_3{}^x, 3(m_3{}^x + \beta_3{}^x)] - 2m_1{}^x - 4m_2{}^x - 2m_3{}^x = 11, \\ 2m_1{}^x + 2m_2{}^x = 6, \\ -Min[3(m_1{}^x - \alpha_1{}^x), -m_1{}^x - \beta_1{}^x] - Min[3(m_2{}^x - \alpha_2{}^x), -4(m_2{}^x + \beta_2{}^x)] \\ \quad +2m_1{}^x + 2m_2{}^x = 17, \\ Max[-m_1{}^x + \alpha_1{}^x, 3(m_1{}^x + \beta_1{}^x)] + Max[-4(m_2{}^x - \alpha_2{}^x), 3(m_2{}^x + \beta_2{}^x)] \\ \quad -2m_1{}^x - 2m_2{}^x = 12. \end{cases}$$

The linear system is,

$$\begin{cases} 3m_1^x + 2m_2^x + 2m_3^x = 12, \\ 2m_1^x + 4m_2^x + 2m_3^x = 12, \\ \quad 2m_1^x + 2m_2^x = 6. \end{cases}$$

Using Step 2, ${m_3}^x = 2, {m_2}^x = 1, {m_1}^x = 2$.

Hence, the nonlinear system can be written as,

$$\begin{cases} 12 - Min[4(2-\alpha_1^x), -2(2+\beta_1^x)] - Min[3(1-\alpha_2^x), -2(1+\beta_2^x)] \\ \quad -Min[2-\alpha_3^x, 3(2-\alpha_3^x)] = 24, \\ -12 + Max[-2(2-\alpha_1^x), 4(2+\beta_1^x)] + Max[-2(1-\alpha_2^x), 3(1+\beta_2^x)] \\ \quad +Max[2+\beta_3^x, 3(2+\beta_3^x)] = 11, \\ 12 - Min[2-\alpha_1^x, 2(2-\alpha_1^x)] - Min[4(1-\alpha_2^x), 5(1-\alpha_2^x)] \\ \quad -Min[2-\alpha_3^x, 3(2-\alpha_3^x)] = 25, \\ -12 + Max[2+\beta_1^x, 2(2+\beta_1^x)] + Max[4(1+\beta_2^x), 5(1+\beta_2^x)] \\ \quad +Max[2+\beta_3^x, 3(2+\beta_3^x)] = 11, \\ 6 - Min[3(2-\alpha_1^x), -2-\beta_1^x] - Min[3(1-\alpha_2^x), -4(1+\beta_2^x)] = 17, \\ -6 + Max[-2+\alpha_1^x, 3(2+\beta_1^x)] + Max[-4(1-\alpha_2^x), 3(1+\beta_2^x)] = 12. \end{cases}$$

Using Step 3, the absolute system is,

$$
\begin{cases}
\frac{1}{2}|-\alpha_3^x - 3(2-\alpha_3^x) + 2| + \frac{1}{2}|4(2-\alpha_1^x) + 2(\beta_1^x + 2)| \\
+\frac{1}{2}|3(1-\alpha_2^x) + 2(\beta_2^x + 1)| + \frac{1}{2}(\alpha_3^x - 3(2-\alpha_3^x) - 2) \\
+\frac{1}{2}\left(2(\beta_1^x + 2) - 4(2-\alpha_1^x)\right) + \frac{1}{2}\left(2(\beta_2^x + 1) - 3(1-\alpha_2^x)\right) + 12 = 24, \\
\\
\frac{1}{2}|-2(2-\alpha_1^x) - 4(\beta_1^x + 2)| + \frac{1}{2}|-2(1-\alpha_2^x) - 3(\beta_2^x + 1)| + \\
\frac{1}{2}|\beta_3^x - 3(\beta_3^x + 2) + 2| + \frac{1}{2}\left(4(\beta_1^x + 2) - 2(2-\alpha_1^x)\right) \\
+\frac{1}{2}\left(3(\beta_2^x + 1) - 2(1-\alpha_2^x)\right) + \frac{1}{2}(\beta_3^x + 3(\beta_3^x + 2) + 2) - 12 = 11, \\
\\
\frac{1}{2}|-\alpha_1^x - 2(2-\alpha_1^x) + 2| + \frac{|\alpha_2^x - 1|}{2} \\
+\frac{1}{2}|-\alpha_3^x - 3(2-\alpha_3^x) + 2| + \frac{1}{2}(\alpha_1^x - 2(2-\alpha_1^x) - 2) \\
-\frac{9}{2}(1-\alpha_2^x) + \frac{1}{2}(\alpha_3^x - 3(2-\alpha_3^x) - 2) + 12 = 25, \\
\\
\frac{1}{2}|\beta_1^x - 2(\beta_1^x + 2) + 2| + \frac{1}{2}|-\beta_2^x - 1| \\
+\frac{1}{2}|\beta_3^x - 3(\beta_3^x + 2) + 2| + \frac{1}{2}(\beta_1^x + 2(\beta_1^x + 2) + 2) \\
+\frac{9}{2}(\beta_2^x + 1) + \frac{1}{2}(\beta_3^x + 3(\beta_3^x + 2) + 2) - 12 = 11, \\
\\
\frac{1}{2}|\beta_1^x + 3(2-\alpha_1^x) + 2| + \frac{1}{2}|3(1-\alpha_2^x) + 4(\beta_2^x + 1)| \\
+\frac{1}{2}(\beta_1^x - 3(2-\alpha_1^x) + 2) + \frac{1}{2}\left(4(\beta_2^x + 1) - 3(1-\alpha_2^x)\right) + 6 = 17, \\
\\
\frac{1}{2}|\alpha_1^x - 3(\beta_1^x + 2) - 2| + \frac{1}{2}|-4(1-\alpha_2^x) - 3(\beta_2^x + 1)| \\
+\frac{1}{2}(\alpha_1^x + 3(\beta_1^x + 2) - 2) + \frac{1}{2}\left(3(\beta_2^x + 1) - 4(1-\alpha_2^x)\right) - 6 = 12.
\end{cases}
$$

After simplifying it,

$$\begin{cases} 2|-2\alpha_1^x + \beta_1^x + 6| + |-3\alpha_2^x + 2\beta_2^x + 5| + 2|\alpha_3^x - 2| \\ +4\alpha_1^x + 3\alpha_2^x + 4\alpha_3^x + 2\beta_1^x + 2\beta_2^x = 37, \\ 2|\alpha_1^x - 2(\beta_1^x + 3)| + |-2\alpha_2^x + 3\beta_2^x + 5| + 2|\beta_3^x + 2| \\ +2\alpha_1^x + 2\alpha_2^x + 4\beta_1^x + 3\beta_2^x + 4\beta_3^x = 33, \\ |\alpha_1^x - 2| + |\alpha_2^x - 1| + 2|\alpha_3^x - 2| + 3\alpha_1^x + 9\alpha_2^x + 4\alpha_3^x = 49, \\ |\beta_1^x + 2| + |\beta_2^x + 1| + 2|\beta_3^x + 2| + 3\beta_1^x + 9\beta_2^x + 4\beta_3^x = 23, \\ |-3\alpha_1^x + \beta_1^x + 8| + |-3\alpha_2^x + 4\beta_2^x + 7| + 3\alpha_1^x + 3\alpha_2^x + \beta_1^x + 4\beta_2^x = 25, \\ |\alpha_1^x - 3\beta_1^x - 8| + |-4\alpha_2^x + 3\beta_2^x + 7| + \alpha_1^x + 4\alpha_2^x + 3\beta_1^x + 3\beta_2^x = 33. \end{cases}$$

Solving the previous system using Step 4, Step 5, we have two solutions,

- The first fuzzy solution, where,

$$\alpha_1{}^x = 1, \beta_1{}^x = 0, \alpha_2{}^x = 4, \beta_2{}^x = 1, \alpha_3{}^x = 1, \beta_3{}^x = 1,$$

$$\tilde{X}_1 = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} (m_1{}^x, \alpha_1{}^x, \beta_1{}^x) \\ (m_2{}^x, \alpha_2{}^x, \beta_2{}^x) \\ (m_3{}^x, \alpha_3{}^x, \beta_3{}^x) \end{pmatrix} = \begin{pmatrix} (2,1,0) \\ (1,4,1) \\ (2,1,1) \end{pmatrix}.$$

The verifications of solution is,

$$\begin{cases} (3,5,1) \otimes (2,1,0) \oplus (2,4,1) \otimes (1,4,1) \oplus (2,1,1) \otimes (2,1,1) = \\ (6,10,2) \oplus (2,11,4) \oplus (4,3,5) = (12,24,11), \\ (2,1,0) \otimes (2,1,0) \oplus (4,0,1) \otimes (1,4,1) \oplus (2,1,1) \otimes (2,1,1) = \\ (4,3,0) \oplus (4,19,6) \oplus (4,3,5) = (12,25,11), \\ (2,3,1) \otimes (2,1,0) \oplus (2,6,1) \otimes (1,4,1) \oplus (0,0,0) \otimes (2,1,1) = \\ (4,6,2) \oplus (2,11,10) \oplus (0,0,0) = (6,17,12). \end{cases}$$

- The second fuzzy solution, where,

$$\alpha_1{}^x = \frac{17}{10}, \beta_1{}^x = \frac{1}{5}, \alpha_2{}^x = \frac{77}{20}, \beta_2{}^x = \frac{6}{5}, \alpha_3{}^x = \frac{21}{20}, \beta_3{}^x = \frac{8}{15}.$$

Then,

$$\tilde{X}_2 = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} (m_1{}^x, \alpha_1{}^x, \beta_1{}^x) \\ (m_2{}^x, \alpha_2{}^x, \beta_2{}^x) \\ (m_3{}^x, \alpha_3{}^x, \beta_3{}^x) \end{pmatrix} = \begin{pmatrix} \left(2, \frac{17}{10}, \frac{1}{5}\right) \\ \left(1, \frac{77}{20}, \frac{6}{5}\right) \\ \left(2, \frac{21}{20}, \frac{8}{15}\right) \end{pmatrix}.$$

The verification of solution is,

$$\begin{cases} (3,5,1) \otimes \left(2, \frac{17}{10}, \frac{1}{5}\right) \oplus (2,4,1) \otimes \left(1, \frac{77}{20}, \frac{6}{5}\right) \oplus (2,1,1) \otimes \left(2, \frac{21}{20}, \frac{8}{15}\right) = \\ \left(6, \frac{52}{5}, \frac{14}{5}\right) \oplus \left(2, \frac{211}{20}, \frac{23}{5}\right) \oplus \left(4, \frac{61}{20}, \frac{18}{5}\right) = (12, 24, 11), \\ (2,1,0) \otimes \left(2, \frac{17}{10}, \frac{1}{5}\right) \oplus (4,0,1) \otimes \left(1, \frac{77}{20}, \frac{6}{5}\right) \oplus (2,1,1) \otimes \left(2, \frac{21}{20}, \frac{8}{15}\right) = \\ \left(4, \frac{37}{10}, \frac{2}{5}\right) \oplus \left(4, \frac{73}{4}, 7\right) \oplus \left(4, \frac{61}{20}, \frac{18}{5}\right) = (12, 25, 11) \\ (2,3,1) \otimes \left(2, \frac{17}{10}, \frac{1}{5}\right) \oplus (2,6,1) \otimes \left(1, \frac{77}{20}, \frac{6}{5}\right) \oplus (0,0,0) \otimes \left(2, \frac{21}{20}, \frac{8}{15}\right) = \\ \left(4, \frac{31}{5}, \frac{13}{5}\right) \oplus \left(2, \frac{54}{5}, \frac{47}{5}\right) \oplus (0,0,0) = (6, 17, 12). \end{cases}$$

5.4 Conclusion and Contributions

In this chapter, we resolved the near zero *FFLS* without external restrictions in order to keep the nature of solution for consistency for *FFLS*. We found a new concept for consistency which was called finite solution of *FFLS*. The min-max system and absolute were applied to obtain the algorithm for solving an arbitrary *FFLS*.

We can summarize the findings in this chapter by the following contributions.

1- Provide the solution of near zero *FFLS.*

2- Adding new concept to the possibility of the $FFSE$.

3- Transfer the mix-max system to absolute system.

CHAPTER SIX

SOLUTIONS OF FULLY FUZZY MATRIX EQUATION AND FUZZY SYLVESTER EQUATION

This chapter develops new methods for solving fully fuzzy matrix equation (*FFME*) and fully fuzzy Sylvester equation (*FFSE*). In order to develop these methods, methods where used in the previous chapters are applied, in which the solution is obtained without any fuzzy operation.

6.1 Fundamental Concepts for *FFME* and *FFSE*

This section presents the definitions and theorems to develop methods for solving $FFME$ and *FFSE*.

Definition 6.1.1. *A vec operator generates a column vector from a fuzzy matrix* $\tilde{C}$ *by stacking the column vectors of* $= (\tilde{c}_1 \quad \tilde{c}_2 \quad \dots \quad \tilde{c}_n)$ *as*

$$vec(\tilde{C}) = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_n \end{pmatrix}. \tag{6.1}$$

Definition 6.1.2. *Let* $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ *and* $\tilde{B} = (\tilde{b}_{ij})_{m\times m}$ *be fuzzy matrices. The fuzzy Kronecker product, is defined as follows*

$$\tilde{A} \boxtimes \tilde{B} = \begin{pmatrix} \tilde{a}_{11}\otimes\tilde{B} & \tilde{a}_{12}\otimes\tilde{B} & \dots & \tilde{a}_{1n}\otimes\tilde{B} \\ \tilde{a}_{21}\otimes\tilde{B} & \tilde{a}_{22}\otimes\tilde{B} & \dots & \tilde{a}_{2n}\otimes\tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1}\otimes\tilde{B} & \tilde{a}_{n2}\otimes\tilde{B} & \cdots & \tilde{a}_{nn}\otimes\tilde{B} \end{pmatrix}. \tag{6.2}$$

Definition 6.1.3. *Let* $\tilde{A} = \left(\tilde{a}_{ij}\right)_{n\times m}$ *be a fuzzy matrix. The transpose fuzzy matrix* $\tilde{A}^T$ *is defined as follows*

$$\tilde{A}^T = \left(\tilde{a}_{ji}\right)_{m\times n}.$$

Definition 6.1.4. *The unitary fuzzy matrix is a square fuzzy matrix defined as* $\widetilde{U}_n = \left(\tilde{u}_{ij}\right)_{n\times n}$

$$\tilde{u}_{ij} = \begin{cases} (0,0,0) & i \neq j, \\ (1,0,0) & i = j. \end{cases}$$

In matrix form, $\widetilde{U}_n = \left(\tilde{u}_{ij}\right)_{n\times n}$ *is represented as follows,*

$$\widetilde{U} = \begin{pmatrix} (1,0,0) & (0,0,0) & \cdots & (0,0,0) \\ (0,0,0) & (1,0,0) & \cdots & (0,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0) & (0,0,0) & \cdots & (1,0,0) \end{pmatrix}.$$

The next theorem investigates the multiplication between fuzzy matrices and unitary matrix.

Theorem 6.1.1. Let $\tilde{A} = \left(\tilde{a}_{ij}\right)_{m\times m}$ be a fuzzy matrix. Then,

i- $\tilde{A}\widetilde{U} = \widetilde{U}\tilde{A} = \tilde{A}$.

ii- $\widetilde{U} = \widetilde{U}^T$.

Proof.

i- Assume $\tilde{A}\widetilde{U} = \tilde{C}$,

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1m} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{a}_{mm} \end{pmatrix} \begin{pmatrix} (1,0,0) & (0,0,0) & \cdots & (0,0,0) \\ (0,0,0) & (1,0,0) & \cdots & (0,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0) & (0,0,0) & \cdots & (1,0,0) \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1m} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{m1} & \tilde{c}_{m2} & \cdots & \tilde{c}_{mm} \end{pmatrix},$$

where $\tilde{c}_{ij}$ is obtained as follows:

$$\tilde{c}_{ij} = \sum_{k=1}^{\oplus} \tilde{a}_{ik} \otimes \tilde{u}_{kj} = \left(\sum_{k \neq j}^{\oplus} \tilde{a}_{ik} \otimes \tilde{u}_{kj} \right) \oplus \left(\tilde{a}_{ij} \otimes \tilde{u}_{jj} \right)$$

$$= \left(\sum_{i \neq j}^{\oplus} \tilde{a}_{ik} \otimes (0,0,0) \right) \oplus \left(\tilde{a}_{ij} \otimes (1,0,0) \right) = \tilde{a}_{ij} \otimes (1,0,0) = \tilde{a}_{ij},$$

Then, $\tilde{c}_{ij} = \tilde{a}_{ij}$. Hence,

$$\tilde{A}\widetilde{U} = \tilde{A}.$$

Similarly, let $\widetilde{U}\tilde{A} = \tilde{C}$ then $\widetilde{U}\tilde{A} = \tilde{C} = \tilde{A}$. Hence,

$$\widetilde{U}\tilde{A} = \tilde{A}.$$

ii- Since $\tilde{u}_{ij} = \tilde{u}_{ji} = (0,0,0)\ \forall i \neq j$, then,

$$\widetilde{U} = \widetilde{U}^T. \tag{6.3}$$

□

Theorem 6.1.2. Let $\tilde{A} = (\tilde{a}_{ij})_{s\times m}$ and $\tilde{X} = (\tilde{x}_{ij})_{m\times n}$ be fuzzy matrices where $\tilde{a}_{ij}, \tilde{x}_{ij}$ are *TFNs*. $\widetilde{U} = (\tilde{u}_{ij})_{n\times n}$ is a unitary fuzzy matrix. Then,

$$Vec(\tilde{A}\ \tilde{X}\ ) = (\widetilde{U} \boxtimes \tilde{A})Vec(\tilde{X}\ ). \quad (6.4)$$

Proof. Using Theorem 6.1.1.

$$\tilde{A}\ \tilde{X} = \tilde{A}\ \tilde{X}\ \widetilde{U},$$

then

$$Vec(\tilde{A}\ \tilde{X}\ ) = Vec(\tilde{A}\ \tilde{X}\ \widetilde{U}) = (\widetilde{U}^T \boxtimes \tilde{A}\ )Vec(\tilde{X}\ ),$$

but $\widetilde{U}^T = \widetilde{U}$, using Equation (6.3), then

$$Vec(\tilde{A}\ \tilde{X}\ ) = (\widetilde{U} \boxtimes \tilde{A})Vec(\tilde{X}\ ).$$ □

Theorem 6.1.3. Let $\tilde{A} = (\tilde{a}_{ij})_{s\times m}, \tilde{X} = (\tilde{x}_{ij})_{m\times n}$, where $\tilde{a}_{ij}$ and $\tilde{x}_{ij}$ are *TFNs* $\widetilde{U} = (\tilde{u}_{ij})_{n\times n}$ is a unitary fuzzy matrix, then

$$Vec(\tilde{X}\ \tilde{A}\ ) = (\tilde{A}^T \boxtimes \widetilde{U})\ Vec(\tilde{X}\ ).$$

Proof. Using Theorem 6.1.1,

$$\tilde{X}\ \tilde{A} = \widetilde{U}\ \tilde{X}\ \tilde{A},$$

then,

$$Vec(\tilde{X}\ \tilde{A}\ ) = Vec(\widetilde{U}\ \tilde{X}\ \tilde{A}\ ) = (\tilde{A}^T \boxtimes \widetilde{U})\ Vec(\tilde{X}\ ). \quad (6.5)$$

□

6.2 Solving Fully Fuzzy Matrix Equation

In this section, the *P-ALS*, $SX = B$, of *FFLS* in Section 3.2 is developed in order to obtain the solution for $FFME$. The vectors X and B are extended to matrices X_m and B_m, respectively.

Consider the *FFME* $\tilde{A} \otimes \tilde{X}_m = \tilde{B}_m$, where $\tilde{A} = \left(\tilde{a}_{ij}\right)_{i,j=1}^{n} = \left(m_{ij}^{a}, \alpha_{ij}^{a}, \beta_{ij}^{a}\right)_{i,j=1}^{n}$ and $\tilde{B}_m = \left(\tilde{b}_{ij}\right)_{i,j=1}^{n} = \left(m_{ij}^{b}, \alpha_{ij}^{b}, \beta_{ij}^{b}\right)_{i,j=1}^{n}$ are known positive fuzzy matrices,

$$\tilde{B}_m = \begin{pmatrix} \tilde{b}_{11} & \dots & \tilde{b}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{b}_{n1} & \cdots & \tilde{b}_{nn} \end{pmatrix} = \begin{pmatrix} \left(m_{1,1}^{b}, \alpha_{1,1}^{b}, \beta_{1,1}^{b}\right) & \dots & \left(m_{1,n}^{b}, \alpha_{1,n}^{b}, \beta_{1,n}^{b}\right) \\ \vdots & \ddots & \vdots \\ \left(m_{n,1}^{b}, \alpha_{n,1}^{b}, \beta_{n,1}^{b}\right) & \cdots & \left(m_{n,n}^{b}, \alpha_{n,n}^{b}, \beta_{n,n}^{b}\right) \end{pmatrix}.$$

While, $\tilde{X}_m = \left(\tilde{x}_{ij}\right)_{i,j=1}^{n} = \left(m_{ij}^{x}, \alpha_{ij}^{x}, \beta_{ij}^{x}\right)_{i,j=1}^{n}$ are unknown positive fuzzy matrix,

$$\tilde{X}_m = \begin{pmatrix} \tilde{x}_{11} & \dots & \tilde{x}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{x}_{n1} & \cdots & \tilde{x}_{nn} \end{pmatrix} = \begin{pmatrix} (m_{1,1}^{x}, \alpha_{1,1}^{x}, \beta_{1,1}^{x}) & \dots & (m_{1,n}^{x}, \alpha_{1,n}^{x}, \beta_{1,n}^{x}) \\ \vdots & \ddots & \vdots \\ (m_{n,1}^{x}, \alpha_{n,1}^{x}, \beta_{n,1}^{x}) & \cdots & (m_{n,n}^{x}, \alpha_{n,n}^{x}, \beta_{n,n}^{x}) \end{pmatrix}.$$

The solution is obtained in three steps:

Step 1 Separating the $FFME$ to $FFLSs$,

Assume that fuzzy vectors $(\tilde{x}_m)_j$ and $\left(\tilde{b}_m\right)_j$, for $j = 1, .., n$,

$$(\tilde{x}_m)_j = \begin{pmatrix} \tilde{x}_{1j} \\ \tilde{x}_{2j} \\ \vdots \\ \tilde{x}_{nj} \end{pmatrix} \text{ and } \left(\tilde{b}_m\right)_j = \begin{pmatrix} \tilde{b}_{1j} \\ \tilde{b}_{2j} \\ \vdots \\ \tilde{b}_{nj} \end{pmatrix}.$$

Thus, we get the following $FFLSs$

$$\tilde{A} \otimes (x_m)_j = (x_m)_j \quad \text{for } j = 1, \ldots n. \tag{6.6}$$

Step 2 Constructing the *P-ALSs* for all *FFLSs.*

Assume the crisp vectors $(x_m)_j$ and $(b_m)_j$ for $j = 1, \ldots, n$,

$$(x_m)_j = \begin{pmatrix} \begin{pmatrix} m^x_{1j} \\ \vdots \\ m^x_{n1} \end{pmatrix} \\ \begin{pmatrix} \alpha^x_{1j} \\ \vdots \\ \alpha^x_{nj} \end{pmatrix} \\ \begin{pmatrix} \beta^x_{11} \\ \vdots \\ \beta^x_{nj} \end{pmatrix} \end{pmatrix} \text{ and } (b_m)_j = \begin{pmatrix} \begin{pmatrix} m^b_{1j} \\ \vdots \\ m^b_{nj} \end{pmatrix} \\ \begin{pmatrix} \alpha^b_{1j} \\ \vdots \\ \alpha^b_{nj} \end{pmatrix} \\ \begin{pmatrix} \beta^b_{11} \\ \vdots \\ \beta^b_{nj} \end{pmatrix} \end{pmatrix}.$$

Thus, the solution for all *FFLSs* in (6.6) can be obtained by the following *P-ALSs* for $j = 1, \ldots, n$,

$$S(x_m)_j = (b_m)_j,$$

$$\begin{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} m_{11} & \dots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} & \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} n_{11} & \dots & n_{1n} \\ \vdots & \ddots & \vdots \\ n_{n1} & \cdots & n_{nn} \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m_{1j}^{x} \\ \vdots \\ m_{n1}^{x} \end{pmatrix} \\ \begin{pmatrix} \alpha_{1j}^{x} \\ \vdots \\ \alpha_{nj}^{x} \end{pmatrix} \\ \begin{pmatrix} \beta_{11}^{x} \\ \vdots \\ \beta_{nj}^{x} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} m_{1j}^{b} \\ \vdots \\ m_{nj}^{b} \end{pmatrix} \\ \begin{pmatrix} \alpha_{1j}^{b} \\ \vdots \\ \alpha_{nj}^{b} \end{pmatrix} \\ \begin{pmatrix} \beta_{11}^{b} \\ \vdots \\ \beta_{nj}^{b} \end{pmatrix} \end{pmatrix}. \tag{6.7}$$

Step 3 Collecting the *P-ALSs* in a matrix form.

The *P-ALSs* can be collected in one matrix form to obtain the solution of *FFME* as follows, assume the crisp matrices X_m and B_m are

$$X_m = ((x_m)_1 \;\; (x_m)_2 \;\; \dots (x_m)_n) = \begin{pmatrix} \begin{pmatrix} m_{11}^{x} \\ \vdots \\ m_{n1}^{x} \end{pmatrix} \begin{pmatrix} m_{12}^{x} \\ \vdots \\ m_{n2}^{x} \end{pmatrix} \dots \begin{pmatrix} m_{1n}^{x} \\ \vdots \\ m_{nn}^{x} \end{pmatrix} \\ \begin{pmatrix} \alpha_{11}^{x} \\ \vdots \\ \alpha_{n1}^{x} \end{pmatrix} \begin{pmatrix} \alpha_{12}^{x} \\ \vdots \\ \alpha_{n2}^{x} \end{pmatrix} \dots \begin{pmatrix} \alpha_{1n}^{x} \\ \vdots \\ \alpha_{nn}^{x} \end{pmatrix} \\ \begin{pmatrix} \beta_{11}^{x} \\ \vdots \\ \beta_{n1}^{x} \end{pmatrix} \begin{pmatrix} \beta_{12}^{x} \\ \vdots \\ \beta_{n2}^{x} \end{pmatrix} \dots \begin{pmatrix} \beta_{1n}^{x} \\ \vdots \\ \beta_{nn}^{x} \end{pmatrix} \end{pmatrix}$$

and

$$B_m = ((b_m)_1 \ \ (b_m)_2 \ \ \dots (b_m)_n) = \begin{pmatrix} \begin{pmatrix} m_{11}^b \\ \vdots \\ m_{n1}^b \end{pmatrix} \begin{pmatrix} m_{12}^b \\ \vdots \\ m_{n2}^b \end{pmatrix} \dots \begin{pmatrix} m_{1n}^b \\ \vdots \\ m_{nn}^b \end{pmatrix} \\ \begin{pmatrix} \alpha_{11}^b \\ \vdots \\ \alpha_{n1}^b \end{pmatrix} \begin{pmatrix} \alpha_{12}^b \\ \vdots \\ \alpha_{n2}^b \end{pmatrix} \dots \begin{pmatrix} \alpha_{1n}^b \\ \vdots \\ \alpha_{nn}^b \end{pmatrix} \\ \begin{pmatrix} \beta_{11}^b \\ \vdots \\ \beta_{n1}^b \end{pmatrix} \begin{pmatrix} \beta_{12}^b \\ \vdots \\ \beta_{n2}^b \end{pmatrix} \dots \begin{pmatrix} \beta_{1n}^b \\ \vdots \\ \beta_{nn}^b \end{pmatrix} \end{pmatrix}.$$

Hence, the solution of all *FFLSs* can be obtained by the following matrix form,

$$SX_m = B_m,$$

$$\begin{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} & \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} n_{11} & \cdots & n_{1n} \\ \vdots & \ddots & \vdots \\ n_{n1} & \cdots & n_{nn} \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m_{11}^b \\ \vdots \\ m_{n1}^b \end{pmatrix} \begin{pmatrix} m_{12}^b \\ \vdots \\ m_{n2}^b \end{pmatrix} \dots \begin{pmatrix} m_{1n}^b \\ \vdots \\ m_{nn}^b \end{pmatrix} \\ \begin{pmatrix} \alpha_{11}^b \\ \vdots \\ \alpha_{n1}^b \end{pmatrix} \begin{pmatrix} \alpha_{12}^b \\ \vdots \\ \alpha_{n2}^b \end{pmatrix} \dots \begin{pmatrix} \alpha_{1n}^b \\ \vdots \\ \alpha_{nn}^b \end{pmatrix} \\ \begin{pmatrix} \beta_{11}^b \\ \vdots \\ \beta_{n1}^b \end{pmatrix} \begin{pmatrix} \beta_{12}^b \\ \vdots \\ \beta_{n2}^b \end{pmatrix} \dots \begin{pmatrix} \beta_{1n}^b \\ \vdots \\ \beta_{nn}^b \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} m_{11}^b \\ \vdots \\ m_{n1}^b \end{pmatrix} \begin{pmatrix} m_{12}^b \\ \vdots \\ m_{n2}^b \end{pmatrix} \dots \begin{pmatrix} m_{1n}^b \\ \vdots \\ m_{nn}^b \end{pmatrix} \\ \begin{pmatrix} \alpha_{11}^b \\ \vdots \\ \alpha_{n1}^b \end{pmatrix} \begin{pmatrix} \alpha_{12}^b \\ \vdots \\ \alpha_{n2}^b \end{pmatrix} \dots \begin{pmatrix} \alpha_{1n}^b \\ \vdots \\ \alpha_{nn}^b \end{pmatrix} \\ \begin{pmatrix} \beta_{11}^b \\ \vdots \\ \beta_{n1}^b \end{pmatrix} \begin{pmatrix} \beta_{12}^b \\ \vdots \\ \beta_{n2}^b \end{pmatrix} \dots \begin{pmatrix} \beta_{1n}^b \\ \vdots \\ \beta_{nn}^b \end{pmatrix} \end{pmatrix}. \qquad (6.8)$$

By finding $X_m = S^{-1}B_m$ we can find the crisp matrix X_m which is equivalent to fuzzy matrix $\tilde{X}_m$, so the solution of $\tilde{A} \otimes \tilde{X}_m = \tilde{B}_m$ is obtained without any fuzzy operation.

Also, If the system has infinitely many solutions as in Section 3.4, fuzzy row reduced method can be applied for each *P-ALS* in Equation (6.7) instead of using Step 3.

The next example is solved by matrix form Equation (6.8) in Step 3 when *FFME* has unique solution.

Example 6.2.1. Consider the fully fuzzy matrix system $\tilde{A} \otimes \tilde{X}_m = \tilde{B}_m$,

$$\begin{pmatrix} (5,1,2) & (6,4,3) \\ (3,3,1) & (7,4,2) \end{pmatrix} \otimes \begin{pmatrix} (m_{1,1}^x, \alpha_{1,1}^x, \beta_{1,1}^x) & (m_{1,2}^x, \alpha_{1,2}^x, \beta_{1,2}^x) \\ (m_{2,1}^x, \alpha_{2,1}^x, \beta_{2,1}^x) & (m_{2,2}^x, \alpha_{2,2}^x, \beta_{2,2}^x) \end{pmatrix}$$

$$= \begin{pmatrix} (72,77,65) & (54,45,71) \\ (67,82,46) & (46,58,49) \end{pmatrix}.$$

The positive solution for $FFME$ can be solved using Equation (6.8),

$$A = \begin{pmatrix} 5 & 6 \\ 3 & 7 \end{pmatrix}, M = \begin{pmatrix} 1 & 4 \\ 3 & 4 \end{pmatrix}, N = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix},$$

$$S = \begin{pmatrix} A & 0 & 0 \\ N & A & 0 \\ M & 0 & A \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 5 & 6 \\ 3 & 7 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 4 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 5 & 6 \\ 3 & 7 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 5 & 6 \\ 3 & 7 \end{pmatrix} \end{pmatrix},$$

$|A| = 17$, using Theorem 3.1.1, $|A| \neq 0$, then $|S| \neq 0$, hence the system has unique solution,

$$B_m = \begin{pmatrix} \begin{pmatrix} m_{1,1}^b & m_{1,2}^b \\ m_{2,1}^b & m_{2,2}^b \end{pmatrix} \\ \begin{pmatrix} \alpha_{1,1}^b & \alpha_{1,2}^b \\ \alpha_{2,1}^b & \alpha_{2,2}^b \end{pmatrix} \\ \begin{pmatrix} \beta_{1,1}^b & \beta_{1,2}^b \\ \beta_{2,1}^b & \beta_{2,2}^b \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 72 & 54 \\ 67 & 46 \end{pmatrix} \\ \begin{pmatrix} 77 & 45 \\ 82 & 58 \end{pmatrix} \\ \begin{pmatrix} 65 & 71 \\ 46 & 49 \end{pmatrix} \end{pmatrix} \text{ and } X_m = \begin{pmatrix} \begin{pmatrix} m_{1,1}^x & m_{1,2}^x \\ m_{2,1}^x & m_{2,2}^x \end{pmatrix} \\ \begin{pmatrix} \alpha_{1,1}^x & \alpha_{1,2}^x \\ \alpha_{2,1}^x & \alpha_{2,2}^x \end{pmatrix} \\ \begin{pmatrix} \beta_{1,1}^x & \beta_{1,2}^x \\ \beta_{2,1}^x & \beta_{2,2}^x \end{pmatrix} \end{pmatrix}.$$

Then, $SX_m = B_m$,

$$\begin{pmatrix} \begin{pmatrix} 5 & 6 \\ 3 & 7 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 4 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 5 & 6 \\ 3 & 7 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 5 & 6 \\ 3 & 7 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m_{1,1}^x & m_{1,2}^x \\ m_{2,1}^x & m_{2,2}^x \end{pmatrix} \\ \begin{pmatrix} \alpha_{1,1}^x & \alpha_{1,2}^x \\ \alpha_{2,1}^x & \alpha_{2,2}^x \end{pmatrix} \\ \begin{pmatrix} \beta_{1,1}^x & \beta_{1,2}^x \\ \beta_{2,1}^x & \beta_{2,2}^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 72 & 54 \\ 67 & 46 \end{pmatrix} \\ \begin{pmatrix} 77 & 45 \\ 82 & 58 \end{pmatrix} \\ \begin{pmatrix} 65 & 71 \\ 46 & 49 \end{pmatrix} \end{pmatrix}.$$

By finding $X_m = S^{-1}B_m$,

$$X_m = \begin{pmatrix} \begin{pmatrix} m_{1,1}^x & m_{1,2}^x \\ m_{2,1}^x & m_{2,2}^x \end{pmatrix} \\ \begin{pmatrix} \alpha_{1,1}^x & \alpha_{1,2}^x \\ \alpha_{2,1}^x & \alpha_{2,2}^x \end{pmatrix} \\ \begin{pmatrix} \beta_{1,1}^x & \beta_{1,2}^x \\ \beta_{2,1}^x & \beta_{2,2}^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 6 & 6 \\ 7 & 4 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix} \\ \begin{pmatrix} 4 & 7 \\ 2 & 2 \end{pmatrix} \end{pmatrix},$$

then the fuzzy matrix solution $\tilde{X}_m$ is,

$$\tilde{X}_m = \begin{pmatrix} (m_{1,1}^x, \alpha_{1,1}^x, \beta_{1,1}^x) & (m_{1,2}^x, \alpha_{1,2}^x, \beta_{1,2}^x) \\ (m_{2,1}^x, \alpha_{2,1}^x, \beta_{2,1}^x) & (m_{2,2}^x, \alpha_{2,2}^x, \beta_{2,2}^x) \end{pmatrix} = \begin{pmatrix} (6,5,4) & (6,1,7) \\ (7,3,2) & (4,3,2) \end{pmatrix}.$$

Next remark shows that the scenarios of *FFME* which are similar to the scenarios of *FFLS* in Chapter Three, Four and Five are solved by same methods.

Remark 6.2.1. The methods of solving $FFLS$ can be extended to solve $FFME$ as follows:

a- If the matrix $\tilde{A}$ has near zero *TFNs*, then the positive solution $\tilde{X}_m$ can be obtained by replacing the matrix S in the left hand side in Equation (6.7) by the matrix J in Equation (4.20). Also, if the system has infinitely many solutions, fuzzy row reduced method can be applied for all *FFLSs* after Step 2

b- If the matrix $\tilde{A}$ has near zero *TFNs*, then the near zero solution $\tilde{X}_m$ can be obtained by applying Step 1 to produce the *FFLSs*, then solving them separately using the method in Section 5.2.

The next example was solved by Otadi and Mosleh (2012), and they obtained a unique solution. However, the proposed method provides infinitely many solutions.

Example 6.2.2. Otadi and Mosleh, (2012) consider the following *FFME*,

$$\begin{pmatrix} (2,3,4) & (1,2,4) \\ (-2,-1,1) & (-2,1,2) \end{pmatrix} \otimes \begin{pmatrix} \left(a_{1,1}^x, b_{1,1}^x, c_{1,1}^x\right) & \left(a_{1,2}^x, b_{1,2}^x, c_{1,2}^x\right) \\ \left(a_{2,1}^x, b_{2,1}^x, c_{2,1}^x\right) & \left(a_{2,2}^x, b_{2,2}^x, c_{2,2}^x\right) \end{pmatrix}$$

$$= \begin{pmatrix} (5,14,32) & (4,17,36) \\ (-16,2,13) & (-18,1,14) \end{pmatrix},$$

where $\tilde{x}_{i,j} = \left({a_{i,j}}^x, {b_{i,j}}^x, {c_{i,j}}^x\right) \geq 0, i,j = 1,2.$

Familiar to Example 5.2.1, because we follow (m, α, β) form for TFN in this study, the example is converted to form (m, α, β). We will use $\tilde{A}'$, $\tilde{B}_m'$ and $\tilde{X}_m'$ to form (a, b, c), while $\tilde{A}$, $\tilde{X}_m$ and $\tilde{B}_m$ to form (m, α, β).

$$\tilde{A} = \begin{pmatrix} (m_{1,1}^a, \alpha_{1,1}^a, \beta_{1,1}^a) & (m_{1,2}^a, \alpha_{1,2}^a, \beta_{1,2}^a) \\ (m_{2,1}^a, \alpha_{2,1}^a, \beta_{2,1}^a) & (m_{2,2}^a, \alpha_{2,2}^a, \beta_{2,2}^a) \end{pmatrix} = \begin{pmatrix} (3,1,1) & (2,1,2) \\ (-1,1,2) & (1,3,1) \end{pmatrix},$$

$$\tilde{B}_m = \begin{pmatrix} (m_{1,1}^b, \alpha_{1,1}^b, \beta_{1,1}^b) & (m_{1,2}^b, \alpha_{1,2}^b, \beta_{1,2}^b) \\ (m_{2,1}^b, \alpha_{2,1}^b, \beta_{2,1}^b) & (m_{2,2}^b, \alpha_{2,2}^b, \beta_{2,2}^b) \end{pmatrix} = \begin{pmatrix} (14,9,18) & (17,13,19) \\ (2,18,11) & (1,19,13) \end{pmatrix},$$

$$\tilde{X}_m = \begin{pmatrix} \tilde{x}_{1,1} & \tilde{x}_{1,2} \\ \tilde{x}_{2,1} & \tilde{x}_{2,2} \end{pmatrix} = \begin{pmatrix} (m_{1,1}^x, \alpha_{1,1}^x, \beta_{1,1}^x) & (m_{1,2}^x, \alpha_{1,2}^x, \beta_{1,2}^x) \\ (m_{2,1}^x, \alpha_{2,1}^x, \beta_{2,1}^x) & (m_{2,2}^x, \alpha_{2,2}^x, \beta_{2,2}^x) \end{pmatrix}.$$

Using Remark 6.2.1a, the block matrix J is constructed as follows,

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}, M = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, N = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

$$A - M = \begin{pmatrix} 2 & 1 \\ -2 & -2 \end{pmatrix}, P_c = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, N_c = \begin{pmatrix} 0 & 0 \\ -2 & -2 \end{pmatrix},$$

$$A + N = \begin{pmatrix} 4 & 4 \\ 1 & 2 \end{pmatrix}, P_D = \begin{pmatrix} 4 & 4 \\ 1 & 2 \end{pmatrix}, N_d = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then,

$$J = \begin{pmatrix} A & 0 & 0 \\ M & P_c & -N_c \\ N & -N_d & P_d \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 4 & 4 \\ 1 & 2 \end{pmatrix} \end{pmatrix}.$$

Because $|J| = 0$, the system is divided for two linear systems in order to obtain the solution using fuzzy row reduced method in Section 4.3.

$$(x_m)_1 = \begin{pmatrix} \begin{pmatrix} m_{1,1}^x \\ m_{2,1}^x \end{pmatrix} \\ \begin{pmatrix} \alpha_{1,1}^x \\ \alpha_{2,1}^x \end{pmatrix} \\ \begin{pmatrix} \beta_{11}^x \\ \beta_{1,2}^x \end{pmatrix} \end{pmatrix}, (x_m)_2 = \begin{pmatrix} \begin{pmatrix} m_{1,2}^x \\ m_{2,2}^x \end{pmatrix} \\ \begin{pmatrix} \alpha_{1,2}^x \\ \alpha_{2,2}^x \end{pmatrix} \\ \begin{pmatrix} \beta_{11}^x \\ \beta_{1,2}^x \end{pmatrix} \end{pmatrix},$$

and

$$(b_m)_1 = \begin{pmatrix} \begin{pmatrix} 14 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 9 \\ 18 \end{pmatrix} \\ \begin{pmatrix} 18 \\ 11 \end{pmatrix} \end{pmatrix}, (b_m)_2 = \begin{pmatrix} \begin{pmatrix} 17 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 13 \\ 19 \end{pmatrix} \\ \begin{pmatrix} 19 \\ 13 \end{pmatrix} \end{pmatrix}.$$

The first linear system is produced as follows

$$(J)\ (x_m)_1 = (b_m)_1,$$

$$\begin{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 4 & 4 \\ 1 & 2 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m_{1,1}^x \\ m_{2,1}^x \end{pmatrix} \\ \begin{pmatrix} \alpha_{1,1}^x \\ \alpha_{2,1}^x \end{pmatrix} \\ \begin{pmatrix} \beta_{11}^x \\ \beta_{1,2}^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 14 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 9 \\ 18 \end{pmatrix} \\ \begin{pmatrix} 18 \\ 11 \end{pmatrix} \end{pmatrix},$$

using fuzzy row reduced method in Section 4.3,

$$\begin{cases} 3m^{x}_{1,1} + 2m^{x}_{2,1} = 14, \\ -m^{x}_{1,1} + m^{x}_{2,1} = 2, \\ m^{x}_{1,1} + m^{x}_{2,1} + 2\alpha^{x}_{1,1} + \alpha^{x}_{2,1} = 9, \\ m^{x}_{1,1} + 3m^{x}_{2,1} + 2\beta^{x}_{1,1} + 2\beta^{x}_{2,1} = 18, \\ m^{x}_{1,1} + 2m^{x}_{2,1} + 4\beta^{x}_{1,1} + 4\beta^{x}_{2,1} = 18, \\ 2m^{x}_{1,1} + m^{x}_{2,1} + \beta^{x}_{1,1} + 2\beta^{x}_{2,1} = 11, \\ \alpha^{x}_{1,1} \geq 0, \beta^{x}_{1,1} \geq 0, m^{x}_{1,1} - \alpha^{x}_{1,1} \geq 0, \\ \alpha^{x}_{2,1} \geq 0, \beta^{x}_{2,1} \geq 0, m^{x}_{2,1} - \alpha^{x}_{2,1} \geq 0, \end{cases}$$

then $\tilde{x}_{1,1}$ and $\tilde{x}_{2,1}$ are,

$$\begin{pmatrix} \tilde{x}_{1,1} \\ \tilde{x}_{2,1} \end{pmatrix} = \begin{pmatrix} \left(m^{x}_{1,1}, \alpha^{x}_{1,1}, \beta^{x}_{1,1}\right) \\ \left(m^{x}_{2,1}, \alpha^{x}_{2,1}, \beta^{x}_{2,1}\right) \end{pmatrix} = \begin{pmatrix} (2,2-\theta,1) \\ (4,-1+2\theta,1) \end{pmatrix}, \theta \in \left[\frac{1}{2}, 2\right].$$

Similarly, the second linear system is produced as follows,

$$(J)\ (x_m)_2 = (b_m)_2,$$

$$\begin{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 4 & 4 \\ 1 & 2 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m^{x}_{1,2} \\ m^{x}_{2,2} \end{pmatrix} \\ \begin{pmatrix} \alpha^{x}_{1,2} \\ \alpha^{x}_{2,2} \end{pmatrix} \\ \begin{pmatrix} \beta^{x}_{11} \\ \beta^{x}_{1,2} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 17 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 13 \\ 19 \end{pmatrix} \\ \begin{pmatrix} 19 \\ 13 \end{pmatrix} \end{pmatrix},$$

Using fuzzy row reduced method,

$$\begin{cases} 3m_{1,2}^x + 2m_2^x = 17, \\ -m_{1,2}^x + m_2^x = 1, \\ m_{1,2}^x + m_2^x + 2\alpha_{1,2}^x + \alpha_{2,2}^x = 13, \\ m_{1,2}^x + 3m_2^x + 2\beta_{1,2}^x + 2\beta_{2,2}^x = 19, \\ m_{1,2}^x + 2m_2^x + 4\beta_{1,2}^x + 4\beta_{2,2}^x = 19, \\ 2m_{1,2}^x + m_2^x + \beta_{1,2}^x + 2\beta_{2,2}^x = 13, \\ \alpha_{1,2}^x \geq 0, \beta_{1,2}^x \geq 0, m_{1,2}^x - \alpha_{1,2}^x \geq 0, \\ \alpha_{2,2}^x \geq 0, \beta_{2,2}^x \geq 0, m_2^x - \alpha_{2,2}^x \geq 0, \end{cases}$$

then $\tilde{x}_{1,2}$ and $\tilde{x}_{2,2}$ are,

$$\begin{pmatrix} \tilde{x}_{1,2} \\ \tilde{x}_{2,2} \end{pmatrix} = \begin{pmatrix} (m_{1,2}^x, \alpha_{1,2}^x, \beta_{1,2}^x) \\ (m_{2,2}^x, \alpha_{2,2}^x, \beta_{2,2}^x) \end{pmatrix} = \begin{pmatrix} (3,3-\eta, 1) \\ (4,2\eta, 1) \end{pmatrix}, \eta \in [0, 2].$$

Hence, the fuzzy system has infinitely many solutions,

$$\widetilde{X}_G = \left(\begin{pmatrix} \tilde{x}_{1,1} \\ \tilde{x}_{2,1} \end{pmatrix} \begin{pmatrix} \tilde{x}_{1,2} \\ \tilde{x}_{2,2} \end{pmatrix} \right) \begin{pmatrix} \tilde{x}_{1,1} & \tilde{x}_{1,2} \\ \tilde{x}_{2,1} & \tilde{x}_{2,2} \end{pmatrix} = \begin{pmatrix} (m_{1,1}^x, \alpha_{1,1}^x, \beta_{1,1}^x) & (m_{1,2}^x, \alpha_{1,2}^x, \beta_{1,2}^x) \\ (m_{2,1}^x, \alpha_{2,1}^x, \beta_{2,1}^x) & (m_{2,2}^x, \alpha_{2,2}^x, \beta_{2,2}^x) \end{pmatrix}$$

$$\widetilde{X}_G = \begin{pmatrix} (2,2-\theta, 1) & (3,3-\eta, 1) \\ (4, -1+2\theta, 1) & (4,2\eta, 1) \end{pmatrix} \text{where, } \theta \in [\tfrac{1}{2}, 2], \eta \in [0, 2].$$

While, $\widetilde{X}_G$ in the form (a, b, c),

$$\widetilde{X}'_G = \begin{pmatrix} (a_{1,1}{}^x, b_{1,1}{}^x, c_{1,1}{}^x) & (a_{2,1}{}^x, b_{2,1}{}^x, c_{2,1}{}^x) \\ (a_{1,2}{}^x, b_{1,2}{}^x, c_{1,2}{}^x) & (a_{2,2}{}^x, b_{2,2}{}^x, c_{2,2}{}^x) \end{pmatrix} =$$

$$\begin{pmatrix} (\theta, 2,3) & (\eta, 3,4) \\ (5-2\theta, 4,5) & (4-2\eta, 4,5) \end{pmatrix}, \text{ where } \theta \in [\tfrac{1}{2}, 2] \text{ and } \eta \in [0, 2].$$

Otadi and Mosleh (2012) provided the below unique solution as mentioned before,

$$\tilde{X}_t{}' = \begin{pmatrix} \tilde{x}'_{1,1} & \tilde{x}'_{1,2} \\ \tilde{x}'_{2,1} & \tilde{x}'_{2,2} \end{pmatrix} = \begin{pmatrix} (a_{1,1}{}^x, b_{1,1}{}^x, c_{1,1}{}^x) & (a_{2,1}{}^x, b_{2,1}{}^x, c_{2,1}{}^x) \\ (a_{1,2}{}^x, b_{1,2}{}^x, c_{1,2}{}^x) & (a_{2,2}{}^x, b_{2,2}{}^x, c_{2,2}{}^x) \end{pmatrix}$$

$$= \begin{pmatrix} (1,2,3) & (1,3,4) \\ (3,4,5) & (2,4,5) \end{pmatrix}.$$

Which represents in the case of $\theta = 1, \eta = 1$ in $\widetilde{X}'_g$ as a particular solution,

$$\tilde{X}_t{}' = \begin{pmatrix} (a_{1,1}{}^x, b_{1,1}{}^x, c_{1,1}{}^x) & (a_{2,1}{}^x, b_{2,1}{}^x, c_{2,1}{}^x) \\ (a_{1,2}{}^x, b_{1,2}{}^x, c_{1,2}{}^x) & (a_{2,2}{}^x, b_{2,2}{}^x, c_{2,2}{}^x) \end{pmatrix}$$

$$\begin{pmatrix} (1,2,3) & (\eta, 3,4) \\ (5-2(1),4,5) & (4-2(1),4,5) \end{pmatrix} = \begin{pmatrix} (1,2,3) & (1,3,4) \\ (3,4,5) & (2,4,5) \end{pmatrix}.$$

The first fuzzy solution $\tilde{x}_1{}' \in \tilde{X}_t{}'$ and $x'_1 \in \widetilde{X}'_G$ are represented in Figure 6.1. and Figure 6.2. to compare the interval of solution for *FFME.*

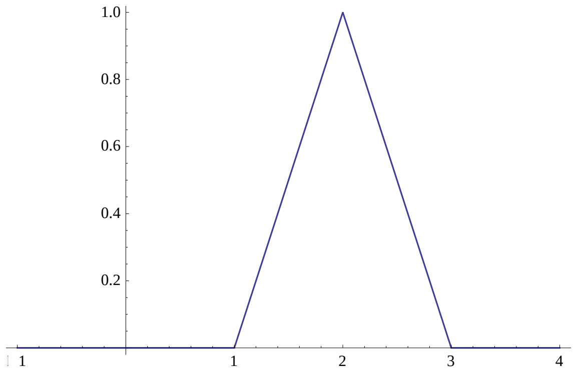

Figure 6.1. Representation of $\tilde{x}_1{}' = (1,2,3)$ at $\tilde{X}_t{}'$ using Otadi and Mosleh (2012)'method.

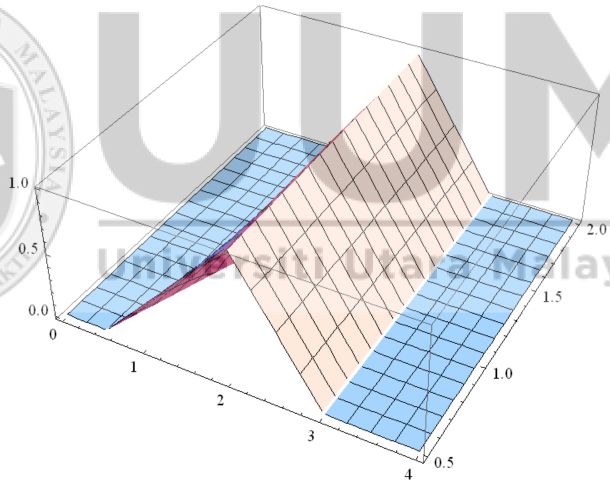

Figure 6.2. Representation of $x_1' = (\theta, 2,3)$, $\theta \in \left[\frac{1}{2}, 2\right]$ at $\widetilde{X}\,'_G$ using the proposed method.

Table 6.1 compares between the proposed method and Otadi and Mosleh (2012)'method.

Table 6.1

Comparison of the solutions set $\tilde{X}_G$ and $\tilde{X}_t$.

	$\tilde{\boldsymbol{X}}_{\boldsymbol{G}}$	$\tilde{\boldsymbol{X}}_{\boldsymbol{t}}$
Fuzzy operation	No fuzzy operation.	Need to compute $\tilde{A} \otimes \tilde{X}_m$.
Number of solution	Infinitely many solutions.	Unique solution set.
Method	Inversion matrix method.	*LP* method.

As noted in Table 6.2, no fuzzy operation is used in proposed method, while Otadi and Mosleh (2012), required to compute $\tilde{A} \otimes \tilde{X}_m$. In addition, the general form solution shows that the new method provides infinitely many solutions, while the original work of Otadi and Mosleh (2012) provided a unique solution. Lastly, the proposed method obtains the solution using the inversion matrix method, while Otadi and Mosleh (2012) found the solution using *LP* method.

The next section provides the solution for important matrix equation in control theory, Sylvester matrix equation, all coefficients are replaced by *TFN* to construct *FFSE.*

6.3 Kronecker Product for Solving Fully Fuzzy Sylvester Equation

This section develops the new method for solving $FFSE$. The rules of Kronecker product and vec operator in Section 2.1.2 are used to provide the solution.

Consider the following fuzzy matrices $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, $\tilde{X} = (\tilde{x}_{ij})_{n\times m}$, $\tilde{B} = (\tilde{b}_{ij})_{m\times m}$ and $\tilde{C} = (\tilde{c}_{ij})_{n\times m}$. $\widetilde{U}$ is fuzzy unitary matrix.

Step 1 Converting the *FFSE* to *FFLS.*

By taking vec operator for both sides $\tilde{A}\,\tilde{X} + \tilde{X}\,\tilde{B}$ and $\tilde{C}$ we have,

$$Vec(\tilde{A}\,\tilde{X} + \tilde{X}\,\tilde{B}) = Vec(\tilde{C}),$$

using Theorem 6.1.2,

$$Vec(\tilde{A}\,\tilde{X}) = (\widetilde{U} \boxtimes \tilde{A})Vec(\tilde{X}\,),$$

using Theorem 6.1.3,

$$Vec(\tilde{X}\,\tilde{B}) = (\tilde{B}^T \boxtimes \widetilde{U})\,Vec(\tilde{X}\,),$$

then,

$$Vec(\tilde{A}\,\tilde{X} + \tilde{X}\,\tilde{B}) = (\widetilde{U} \boxtimes \tilde{A})Vec(\tilde{X}\,) + (\tilde{B}^T \boxtimes \widetilde{U})\,Vec(\tilde{X}\,) =$$

$$\{(\widetilde{U} \boxtimes \tilde{A}) + (\tilde{B}^T \boxtimes \widetilde{U})\}\,Vec(\tilde{X}\,) = Vec(\tilde{C}).$$

Step 2: Solving the $FFLS$ by the proposed methods.

Since $\{(\tilde{U} \boxtimes \tilde{A}) + (\tilde{B}^T \boxtimes \tilde{U})\}$ is a fuzzy matrix, $Vec(\tilde{X})$ and $Vec(\tilde{C})$ are fuzzy vectors, the previous methods for solving $FFLS$ can be applied to solve fully fuzzy Sylvester equation.

The next example provides fully fuzzy Sylvester equation. This $FFSE$ is transferred to $FFLS$ and solved by associated linear which has been discussed in Section 3.2. The verification of solution is computed.

Example 6.3.1. Consider the fully fuzzy Sylvester equation,

$$\tilde{A}\,\tilde{X} + \tilde{X}\,\tilde{B} = \tilde{C},$$

where,

$$\tilde{A} = \begin{pmatrix} (8,4,7) & (5,4,5) \\ (9,6,1) & (7,2,7) \end{pmatrix}, \tilde{B} = \begin{pmatrix} (6,6,7) & (5,4,3) & (7,2,3) \\ (9,3,2) & (7,1,7) & (5,3,1) \\ (8,5,3) & (5,2,3) & (1,1,7) \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} (259,316,349) & (195,214,253) & (193,237,286) \\ (241,307,386) & (180,230,228) & (208,269,274) \end{pmatrix}$$

$$\tilde{X} = \begin{pmatrix} (m^x_{1,1}, \alpha^x_{1,1}, \beta^x_{1,1}) & (m^x_{1,2}, \alpha^x_{1,2}, \beta^x_{1,2}) & (m^x_{1,3}, \alpha^x_{1,3}, \beta^x_{1,3}) \\ (m^x_{2,1}, \alpha^x_{2,1}, \beta^x_{2,1}) & (m^x_{2,2}, \alpha^x_{2,2}, \beta^x_{2,2}) & (m^x_{2,3}, \alpha^x_{2,3}, \beta^x_{2,3}) \end{pmatrix}.$$

The positive solution will be obtained without fuzzy operation.

Using, Step 1 the *FFSE* is converted to *FFLS* as follows,

$$Vec(\tilde{A}\,\tilde{X} + \tilde{X}\,\tilde{B}) = (\tilde{U} \boxtimes \tilde{A})Vec(\tilde{X}\,) + (\tilde{B}^T \boxtimes \tilde{U})\,Vec(\tilde{X}\,)$$

$$= \{(\tilde{U} \boxtimes \tilde{A}) + (\tilde{B}^T \boxtimes \tilde{U})\}\,Vec(\tilde{X}\,) = Vec(\tilde{C}),$$

so, the pervious equation requires the following matrices $\widetilde{U}_3$, $(\tilde{U} \boxtimes \tilde{A}), Vec(\tilde{X}\,)$,$\tilde{B}^T$, $\widetilde{U}_2$, $(\tilde{B}^T \boxtimes \tilde{U})$ and $Vec(\tilde{C})$.

The unitary fuzzy matrix $\widetilde{U}_3$ is,

$$\widetilde{U}_3 = \begin{pmatrix} (1,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (1,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (1,0,0) \end{pmatrix},$$

Kronecker product $\widetilde{U}_3 \boxtimes \tilde{A}$ is as follows,

$$\widetilde{U}_3 \boxtimes \tilde{A} = \begin{pmatrix} (1,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (1,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (1,0,0) \end{pmatrix} \boxtimes \begin{pmatrix} (8,4,7) & (5,4,5) \\ (9,6,1) & (7,2,7) \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} (8,4,7) & (5,4,5) \\ (9,6,1) & (7,2,7) \end{pmatrix} & \begin{pmatrix} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{pmatrix} & \begin{pmatrix} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{pmatrix} \\ \begin{pmatrix} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{pmatrix} & \begin{pmatrix} (8,4,7) & (5,4,5) \\ (9,6,1) & (7,2,7) \end{pmatrix} & \begin{pmatrix} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{pmatrix} \\ \begin{pmatrix} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{pmatrix} & \begin{pmatrix} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{pmatrix} & \begin{pmatrix} (8,4,7) & (5,4,5) \\ (9,6,1) & (7,2,7) \end{pmatrix} \end{pmatrix}.$$

The vec-operator of fuzzy matrix $\tilde{X}$ is

$$Vec(\tilde{X}) = \begin{pmatrix} (m_{1,1}^x, \alpha_{1,1}^x, \beta_{1,1}^x) \\ (m_{2,1}^x, \alpha_{2,1}^x, \beta_{2,1}^x) \\ (m_{1,2}^x, \alpha_{1,2}^x, \beta_{1,2}^x) \\ (m_{2,2}^x, \alpha_{2,2}^x, \beta_{2,2}^x) \\ (m_{1,3}^x, \alpha_{1,3}^x, \beta_{1,3}^x) \\ (m_{2,3}^x, \alpha_{2,3}^x, \beta_{2,3}^x) \end{pmatrix}.$$

The transpose of matrix $\tilde{B}$ is

$$\tilde{B}^T = \begin{pmatrix} (6,6,7) & (9,3,2) & (8,5,3) \\ (5,4,3) & (7,1,7) & (5,2,3) \\ (7,2,3) & (5,3,1) & (1,1,7) \end{pmatrix}.$$

The unitary fuzzy matrix $\widetilde{U}_2$ is

$$\widetilde{U}_2 = \begin{pmatrix} (1,0,0) & (0,0,0) \\ (0,0,0) & (1,0,0) \end{pmatrix}.$$

Kronecker product $\tilde{B}^T \boxtimes \widetilde{U}_2$ is as follows,

$$\tilde{B}^T \boxtimes \widetilde{U}_2 = \begin{pmatrix} \begin{pmatrix} (6,6,7) & (0,0,0) \\ (0,0,0) & (6,6,7) \end{pmatrix} & \begin{pmatrix} (9,3,2) & (0,0,0) \\ (0,0,0) & (9,3,2) \end{pmatrix} & \begin{pmatrix} (8,5,3) & (0,0,0) \\ (0,0,0) & (8,5,3) \end{pmatrix} \\ \begin{pmatrix} (5,4,3) & (0,0,0) \\ (0,0,0) & (5,4,3) \end{pmatrix} & \begin{pmatrix} (7,1,7) & (0,0,0) \\ (0,0,0) & (7,1,7) \end{pmatrix} & \begin{pmatrix} (5,2,3) & (0,0,0) \\ (0,0,0) & (5,2,3) \end{pmatrix} \\ \begin{pmatrix} (7,2,3) & (0,0,0) \\ (0,0,0) & (7,2,3) \end{pmatrix} & \begin{pmatrix} (5,3,1) & (0,0,0) \\ (0,0,0) & (5,3,1) \end{pmatrix} & \begin{pmatrix} (1,1,7) & (0,0,0) \\ (0,0,0) & (1,1,7) \end{pmatrix} \end{pmatrix}.$$

The vec-operator of fuzzy matrix $\tilde{C}$ is,

$$Vec(\tilde{C}) = \begin{pmatrix} (259,316,349) \\ (241,307,386) \\ (195,214,253) \\ (180,230,228) \\ (193,237,286) \\ (208,269,274) \end{pmatrix}.$$

Hence, the Sylvester equation is transferred to the following $FFLS$.

Based on Step 2, the method in Section 3.2 is applied to solve the produced $FFLS$.

$$Vec(\tilde{A}\,\tilde{X} + \tilde{X}\,\tilde{B}) = (\tilde{U}_3 \boxtimes \tilde{A} + \tilde{B}^T \boxtimes \tilde{U}_2)\,Vec(\tilde{X}) = Vec(\tilde{C}) =$$

$$\begin{pmatrix} (14,10,14) & (5,4,5) & (9,3,2) & (0,0,0) & (8,5,3) & (0,0,0) \\ (9,6,1) & (13,8,14) & (0,0,0) & (9,3,2) & (0,0,0) & (8,5,3) \\ (5,4,3) & (0,0,0) & (15,5,14) & (5,4,5) & (5,2,3) & (0,0,0) \\ (0,0,0) & (5,4,3) & (9,6,1) & (14,3,14) & (0,0,0) & (5,2,3) \\ (7,2,3) & (0,0,0) & (5,3,1) & (0,0,0) & (9,5,14) & (5,4,5) \\ (0,0,0) & (7,2,3) & (0,0,0) & (5,3,1) & (9,6,1) & (8,3,14) \end{pmatrix}$$

$$\begin{pmatrix} (m_{1,1}^x, \alpha_{1,1}^x, \beta_{1,1}^x) \\ (m_{2,1}^x, \alpha_{2,1}^x, \beta_{2,1}^x) \\ (m_{1,2}^x, \alpha_{1,2}^x, \beta_{1,2}^x) \\ (m_{2,2}^x, \alpha_{2,2}^x, \beta_{2,2}^x) \\ (m_{1,3}^x, \alpha_{1,3}^x, \beta_{1,3}^x) \\ (m_{2,3}^x, \alpha_{2,3}^x, \beta_{2,3}^x) \end{pmatrix} = \begin{pmatrix} (259,316,349) \\ (241,307,386) \\ (195,214,253) \\ (180,230,228) \\ (193,237,286) \\ (208,269,274) \end{pmatrix}.$$

The $P - \tilde{X}$ for $P - FFLS$ can be obtained, using the *P-ALS* in Section (3.2),

$$A = \begin{pmatrix} 14 & 5 & 9 & 0 & 8 & 0 \\ 9 & 13 & 0 & 9 & 0 & 8 \\ 5 & 0 & 15 & 5 & 5 & 0 \\ 0 & 5 & 9 & 14 & 0 & 5 \\ 7 & 0 & 5 & 0 & 9 & 5 \\ 0 & 7 & 0 & 5 & 9 & 8 \end{pmatrix},$$

$$M = \begin{pmatrix} 10 & 4 & 3 & 0 & 5 & 0 \\ 6 & 8 & 0 & 3 & 0 & 5 \\ 4 & 0 & 5 & 4 & 2 & 0 \\ 0 & 4 & 6 & 3 & 0 & 2 \\ 2 & 0 & 3 & 0 & 5 & 4 \\ 0 & 2 & 0 & 3 & 6 & 3 \end{pmatrix}, N = \begin{pmatrix} 14 & 5 & 2 & 0 & 3 & 0 \\ 1 & 14 & 0 & 2 & 0 & 3 \\ 3 & 0 & 14 & 5 & 3 & 0 \\ 0 & 3 & 1 & 14 & 0 & 3 \\ 3 & 0 & 1 & 0 & 14 & 5 \\ 0 & 3 & 0 & 1 & 1 & 14 \end{pmatrix},$$

$$m^b=\begin{pmatrix}259\\241\\195\\180\\193\\208\end{pmatrix},\alpha^b=\begin{pmatrix}316\\307\\214\\230\\237\\269\end{pmatrix},\beta^b=\begin{pmatrix}349\\386\\253\\228\\286\\274\end{pmatrix}, \text{ then } B=\begin{pmatrix}\begin{pmatrix}259\\241\\195\\180\\193\\208\end{pmatrix}\\\begin{pmatrix}316\\307\\214\\230\\237\\269\end{pmatrix}\\\begin{pmatrix}349\\386\\253\\228\\286\\274\end{pmatrix}\end{pmatrix},$$

$|A| = -355455$, then using Theorem 3.1.1, $|A| \neq 0$, then $|S| \neq 0$.

Hence, the solution is obtained by $X = S^{-1}B$,

$$\begin{pmatrix}
14 & 5 & 9 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 13 & 0 & 9 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 15 & 5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 9 & 14 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 5 & 0 & 9 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 0 & 5 & 9 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 4 & 3 & 0 & 5 & 0 & 14 & 5 & 9 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 8 & 0 & 3 & 0 & 5 & 9 & 13 & 0 & 9 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 5 & 4 & 2 & 0 & 5 & 0 & 15 & 5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 6 & 3 & 0 & 2 & 0 & 5 & 9 & 14 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 3 & 0 & 5 & 4 & 7 & 0 & 5 & 0 & 9 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 3 & 6 & 3 & 0 & 7 & 0 & 5 & 9 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 5 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 & 5 & 9 & 0 & 8 & 0 \\
1 & 14 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 13 & 0 & 9 & 0 & 8 \\
3 & 0 & 14 & 5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 15 & 5 & 5 & 0 \\
0 & 3 & 1 & 14 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 9 & 14 & 0 & 5 \\
3 & 0 & 1 & 0 & 14 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 5 & 0 & 9 & 5 \\
0 & 3 & 0 & 1 & 1 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 5 & 9 & 8
\end{pmatrix}^{-1}
\begin{pmatrix}
259 \\ 241 \\ 195 \\ 180 \\ 193 \\ 208 \\ 316 \\ 307 \\ 214 \\ 230 \\ 237 \\ 269 \\ 349 \\ 386 \\ 253 \\ 228 \\ 286 \\ 274
\end{pmatrix}.$$

Hence, the crisp solution *X* is

$$X = \begin{pmatrix} m_{1,1}^{x}, \\ m_{2,1}^{x} \\ m_{1,2}^{x} \\ m_{2,2}^{x} \\ m_{1,3}^{x} \\ m_{2,3}^{x} \\ \alpha_{1,1}^{x}, \\ \alpha_{2,1}^{x} \\ \alpha_{1,2}^{x} \\ \alpha_{2,2}^{x} \\ \alpha_{1,3}^{x} \\ \alpha_{2,3}^{x} \\ \beta_{1,1}^{x}, \\ \beta_{2,1}^{x} \\ \beta_{1,2}^{x} \\ \beta_{2,2}^{x} \\ \beta_{1,3}^{x} \\ \beta_{2,3}^{x} \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 7 \\ 3 \\ 9 \\ 7 \\ 2 \\ 6 \\ 4 \\ 3 \\ 8 \\ 5 \\ 7 \\ 9 \\ 1 \\ 5 \\ 4 \\ 2 \end{pmatrix} \text{ or } X = \begin{pmatrix} \begin{pmatrix} m_{1,1}^{x} \\ m_{2,1}^{x} \\ m_{1,2}^{x} \\ m_{2,2}^{x} \\ m_{1,3}^{x} \\ m_{2,3}^{x} \end{pmatrix} \\ \begin{pmatrix} \alpha_{1,1}^{x} \\ \alpha_{2,1}^{x} \\ \alpha_{1,2}^{x} \\ \alpha_{2,2}^{x} \\ \alpha_{1,3}^{x} \\ \alpha_{2,3}^{x} \end{pmatrix} \\ \begin{pmatrix} \beta_{1,1}^{x} \\ \beta_{2,1}^{x} \\ \beta_{1,2}^{x} \\ \beta_{2,2}^{x} \\ \beta_{1,3}^{x} \\ \beta_{2,3}^{x} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 6 \\ 8 \\ 7 \\ 3 \\ 9 \\ 7 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 6 \\ 4 \\ 3 \\ 8 \\ 5 \end{pmatrix} \\ \begin{pmatrix} 7 \\ 9 \\ 1 \\ 5 \\ 4 \\ 2 \end{pmatrix} \end{pmatrix}.$$

Thus, the $Vec(\tilde{X})$ is,

$$Vec(\tilde{X}) = \begin{pmatrix} (m_{1,1}^x, \alpha_{1,1}^x, \beta_{1,1}^x) \\ (m_{2,1}^x, \alpha_{2,1}^x, \beta_{2,1}^x) \\ (m_{1,2}^x, \alpha_{1,2}^x, \beta_{1,2}^x) \\ (m_{2,2}^x, \alpha_{2,2}^x, \beta_{2,2}^x) \\ (m_{1,3}^x, \alpha_{1,3}^x, \beta_{1,3}^x) \\ (m_{2,3}^x, \alpha_{2,3}^x, \beta_{2,3}^x) \end{pmatrix} = \begin{pmatrix} (6,2,7) \\ (8,6,9) \\ (7,4,1) \\ (3,3,5) \\ (9,8,4) \\ (7,5,2) \end{pmatrix}.$$

Then, the fuzzy matrix solution $\tilde{X}$ is,

$$\tilde{X} = \begin{pmatrix} (m_{1,1}^x, \alpha_{1,1}^x, \beta_{1,1}^x) & (m_{1,2}^x, \alpha_{1,2}^x, \beta_{1,2}^x) & (m_{1,3}^x, \alpha_{1,3}^x, \beta_{1,3}^x) \\ (m_{2,1}^x, \alpha_{2,1}^x, \beta_{2,1}^x) & (m_{2,2}^x, \alpha_{2,2}^x, \beta_{2,2}^x) & (m_{2,3}^x, \alpha_{2,3}^x, \beta_{2,3}^x) \end{pmatrix}$$
$$= \begin{pmatrix} (6,2,7) & (7,4,1) & (9,8,4) \\ (8,6,9) & (3,3,5) & (7,5,2) \end{pmatrix}.$$

The verification of solution

$$\tilde{A}\tilde{X} = \begin{pmatrix} (88,102,183) & (71,87,97) & (107,153,140) \\ (110,112,188) & (84,105,72) & (130,175,108) \end{pmatrix},$$

$$\tilde{X}\tilde{B} = \begin{pmatrix} (171,214,166) & (124,127,156) & (86,84,146) \\ (131,195,198) & (96,125,156) & (78,94,166) \end{pmatrix},$$

then,

$$\tilde{A}\tilde{X}+\tilde{X}\tilde{B}=\begin{pmatrix}(259,316,349) & (195,214,253) & (193,237,286)\\(241,307,386) & (180,230,228) & (208,269,274)\end{pmatrix}=\tilde{C}.$$

6.4 Conclusion and Contribution

In this chapter, we proposed new methods to solve $FFME$ and $FFSE$. We showed the pervious methods developed in Chapters Three, Four and Five for solving $FFLS$, can be developed for solving $FFME$. The vec operator and Kronecker product were used to transfer $FFSE$ to $FFLS$.

We can summarize the findings in this chapter by the following contributions.

4- Transfer the $FFME$ and $FFSE$ to equivalent $FFLS$.

5- Obtain the unique and infinitely many solutions of $FFME$ and $FFSE$.

6- Examine the existence of solutions for $FFME$ and $FFSE$ before solving the system, using the pervious discussion for $FFLS$ as Section 4.3.

CHAPTER SEVEN

SOLUTIONS OF FUZZY LINEAR SYSTEMS BASED ON TRAPEZOIDAL FUZZY

In all previous chapters the solutions of fuzzy systems are presented with triangular fuzzy numbers. In this chapter, the solutions of $LR - TFLS$ with trapezoidal fuzzy numbers are constructed. It shows that the previous developed methods can be employed on other fuzzy numbers. For this purpose an associated linear system is established for $LR{-}TFLS$. Also, the sufficient and necessary conditions of LR-$TFLS$ to have a fuzzy solution are investigated based on the fuzziness of the solution. In addition, the possibilities of the solution are classified (i.e., unique solution, infinite number of solutions, no solution). Lastly, in the case of non fuzzy solutions, a minimization problem is proposed to provide the nearest approximation solution.

7.1 Fundamental Concepts

This section presents the fundamental concepts for developing new methods for solving $LR - TFLS$ with trapezoidal fuzzy numbers.

Definition 7.1.1. Consider the matrix $A = \left(a_{ij}\right)_{n\times n}$. The matrices P and N are defined according to the sign of a_{ij} where,

$$P = \left(a_{i,j}^{+}\right)_{n\times n} = \begin{cases} a_{i,j}^{+}, & a_{ij} \geq 0, \\ 0, & a_{ij} < 0, \end{cases}$$

$$N = \left(a_{i,j}^{-}\right)_{n\times n} = \begin{cases} 0, & a_{ij} \geq 0, \\ a_{i,j}^{-}, & a_{ij} < 0. \end{cases}.$$

The next corollary is needed in order to find the structure of the inverse of the block matrices. It is based on Theorem 2.1.2.

Corollary 7.1.1. Assuming P and N are matrices with a common size. If P and $(P - NP^{-1}N)$ are invertible in the block matrices F and R:

$$F = \begin{pmatrix} P & N \\ N & P \end{pmatrix}, R = \begin{pmatrix} P & -N \\ -N & P \end{pmatrix},$$

then the inverse of matrices F and R are

$$F^{-1} = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}, \qquad (7.1a)$$

$$R^{-1} = \begin{pmatrix} f_1 & -f_2 \\ -f_2 & f_1 \end{pmatrix}, \qquad (7.1b)$$

where $f_1 = (P - NP^{-1}N)^{-1}$ and $f_2 = -f_1\, NP^{-1}$.

Proof. Suppose $P = A = D$ and $N = B = C$ in matrix H in Equation (2.3), then

$$F^{-1} = \begin{pmatrix} (P - NP^{-1}N)^{-1} & -(P - NP^{-1}N)^{-1}NP^{-1} \\ -(P - NP^{-1}N)^{-1}NP^{-1} & (P - NP^{-1}N)^{-1} \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}.$$

Replace matrix N in Equation (7.1a), by matrix $-N$ in f_1 and f_2, then

$$(P - (-N)P^{-1}(-N))^{-1} = (P - NP^{-1}N)^{-1} = f_1,$$

$$-f_1(-N)P^{-1} = f_1\, NP^{-1} = -f_2,$$

$$R^{-1} = \begin{pmatrix} (P - NP^{-1}N)^{-1} & (P - NP^{-1}N)^{-1}NP^{-1} \\ (P - NP^{-1}N)^{-1}NP^{-1} & (P - NP^{-1}N)^{-1} \end{pmatrix} = \begin{pmatrix} f_1 & -f_2 \\ -f_2 & f_1 \end{pmatrix}. \quad \square$$

The next section provides method to solve *LR-TFLS* using an associated linear system.

7.2 Solution of Left Right-Trapezoidal Fuzzy Linear System

This section discusses the solution of LR fuzzy linear systems with trapezoidal fuzzy numbers. We introduce new computational method to solve the *LR-TFLS* using the equivalent linear system.

Consider the *LR-TFLS* $A \otimes \tilde{X} = \tilde{B}$, where $A = (a_{ij})_{n\times n}, \tilde{X} = (\tilde{x}_j)_{n\times 1}$ and $\tilde{B}_i = (\tilde{b}_i)_{n\times 1}$,

$$\tilde{x}_j = \left(m_j^x, n_j^x, \alpha_j^x, \beta_j^x\right) \text{ and } \tilde{b}_i = \left(m_i^b, n_i^b, \alpha_i^b, \beta_i^b\right).$$

The $n \times n$ *LR-TFLS* can be written as

$$\sum_{j=1}^{\oplus} a_{ij} \otimes \tilde{x}_j = \tilde{b}_j, \forall i = 1, 2, \dots, n$$

$$\sum_{j=1}^{\oplus} a_{ij} \otimes \left(m_j^x, n_{i,j}^x, \alpha_j^x, \beta_j^x\right) = \left(m_i^b, n_i^b, \alpha_i^b, \beta_i^b\right), \forall i = 1, 2, \dots, n. \quad (7.2)$$

The solution is obtained in five steps.

Step 1 Given $a_{i,j}^+, a_{i,j}^-$.

Let,

$$a_{ij} = \begin{cases} a_{i,j}^+, & a_{ij} \geq 0, \\ a_{i,j}^-, & a_{ij} < 0. \end{cases} \quad (7.3)$$

Clearly, either $a_{i,j}^+ = 0$ or $a_{i,j}^- = 0$ (or both).

Then, $a_{i,j}^+ + a_{i,j}^- = a_{ij}$ and $(a_{i,j}^+)\,(a_{i,j}^-) = 0$.

Step 2 Assuming m_{ij}^{a} and n_{ij}^{a} according to the sign of a_{ij}.

m_j^x and n_j^x are written in piecewise function using Equation (7.3).

$$m_{ij}^{a} = \begin{cases} a_{i,j}^{+} m_j^x, & a_{ij} \geq 0, \\ a_{i,j}^{-} n_j^x, & a_{ij} < 0. \end{cases} \tag{7.4}$$

Thus,

$$m_{ij}^{a} = a_{i,j}^{+} m_j^x + a_{i,j}^{-} n_j^x. \tag{7.5}$$

Since $\left(a_{i,j}^{+} m_j^x\right)\left(a_{i,j}^{-} n_j^x\right) = 0$.

Similarly,

$$n_{ij}^{a} = \begin{cases} a_{i,j}^{+} n_j^x, & a_{ij} \geq 0, \\ a_{i,j}^{-} m_j^x, & a_{ij} < 0, \end{cases} \tag{7.6}$$

since $\left(a_{i,j}^{+} m_j^x\right)\left(a_{i,j}^{-} n_j^x\right) = 0$, then

$$n_{ij}^{a} = a_{i,j}^{+} n_j^x + a_{i,j}^{-} m_j^x. \tag{7.7}$$

Step 3 Assuming $\alpha_{i,j}^{a}$ and $\beta_{i,j}^{a}$ according to the sign of a_{ij}.

The mean values α_j^x and β_j^x are functionalized by piecewise function using Equation (7.3).

$$\alpha_{i,j}^{a} = \begin{cases} a_{i,j}^{+} \alpha_j^x, & a_{ij} \geq 0, \\ -a_{i,j}^{-} \beta_j^x, & a_{ij} < 0. \end{cases} \tag{7.8}$$

Since $\left(-a_{i,j}^{+} \alpha_j^x\right)\left(a_{i,j}^{-}\, \beta_{i,j}^{a}\right) = 0$, then

$$\alpha_{i,j}^{a} = a_{i,j}^{+} \alpha_j^x - a_{i,j}^{-}\, \beta_{i,j}^{a}. \tag{7.9}$$

Similarly,

$$\beta_{i,j}^{a} = \begin{cases} a_{i,j}^{+}\beta_j^{x}, & a_{i,j}^{+} \geq 0, \\ -a_{i,j}^{-}\alpha_j^{x}, & a_{i,j}^{-} < 0, \end{cases} \tag{7.10}$$

therefore,

$$\beta_{i,j}^{a} = a_{i,j}^{+}\beta_j^{x} - a_{i,j}^{-}\alpha_j^{x}. \tag{7.11}$$

Because $\left(a_{i,j}^{+}\beta_j^{x}\right)\left(-a_{i,j}^{-}\alpha_j^{x}\right) = 0$.

Step 4 Writing $a_{ij} \otimes \tilde{x}_j$ in Equation (7.1) as piecewise function using Equations (7.4), (7.6), (7.8) and (7.10):

$$a_{ij} \otimes \tilde{x}_j = \left(m_{ij}^{a}, n_{ij}^{a}, \alpha_{i,j}^{a}, \beta_{i,j}^{a}\right)$$

$$= \begin{cases} \left(a_{i,j}^{+}m_j^{x}, a_{i,j}^{+}n_j^{x}, a_{i,j}^{+}\alpha_j^{x}, a_{i,j}^{+}\beta_j^{x}\right), & a_{ij} \geq 0, \\ \left(a_{i,j}^{-}n_j^{x}, a_{i,j}^{-}m_j^{x}, -a_{i,j}^{-}\beta_j^{x}, -a_{i,j}^{-}\alpha_j^{x}\right), & a_{ij} < 0 \end{cases} \tag{7.12}$$

Using Equations (7.5), (7.7), (7.9) and (7.11), $a_{ij} \otimes \tilde{x}_j$ in Equation (7.12) may be written by two *TZFNs*, where at least one of them is zero,

$$a_{ij} \otimes \tilde{x}_j = \left(a_{i,j}^{+}m_j^{x}, a_{i,j}^{+}n_j^{x}, a_{i,j}^{+}\alpha_j^{x}, a_{i,j}^{+}\beta_j^{x}\right) \oplus \left(a_{i,j}^{-}n_j^{x}, a_{i,j}^{-}m_j^{x}, -a_{i,j}^{-}\beta_j^{x}, -a_{i,j}^{-}\alpha_j^{x}\right).$$

Then using Equation ($2.9a$),

$$a_{ij} \otimes \tilde{x}_j =$$
$$= \left(a_{i,j}^{+}m_j^{x} + a_{i,j}^{-}n_j^{x}, a_{i,j}^{+}n_j^{x} + a_{i,j}^{-}m_j^{x}, a_{i,j}^{+}\alpha_j^{x} + -a_{i,j}^{-}\beta_j^{x}, a_{i,j}^{+}\beta_j^{x} + -a_{i,j}^{-}\alpha_j^{x}\right). \tag{7.13}$$

Step 5 $A \otimes \tilde{X} = \tilde{B}$ can be represented using Equations (7.1) and (7.13):

$$\sum_{j=1}^{n}\left(a_{i,j}^{+}m_j^{x} + a_{i,j}^{-}n_j^{x}, a_{i,j}^{+}n_j^{x} + a_{i,j}^{-}m_j^{x}, a_{i,j}^{+}\alpha_j^{x} \pm a_{i,j}^{-}\beta_j^{x}, a_{i,j}^{+}\beta_j^{x} + -a_{i,j}^{-}\alpha_j^{x}\right)$$
$$= \left(m_i^{b}, n_i^{b}, \alpha_i^{b}, \beta_i^{b}\right) \quad \forall i = 1, 2, \dots, n. \tag{7.14}$$

We obtain the following linear equations:

$$\sum_{j=1}^{n}\left(a_{i,j}^{+}m_j^x + a_{i,j}^{-}n_j^x\right) = m_i^b, \quad (7.15a)$$

$$\sum_{j=1}^{n}\left(a_{i,j}^{+}n_j^x + a_{i,j}^{-}m_j^x\right) = n_i^b, \quad (7.15b)$$

$$\sum_{j=1}^{n}\left(a_{i,j}^{+}\alpha_j^x - a_{i,j}^{-}\beta_j^x\right) = \alpha_i^b, \quad (7.15c)$$

$$\sum_{j=1}^{n}\left(a_{i,j}^{+}\beta_j^x - a_{i,j}^{-}\alpha_j^x\right) = \beta_i^b. \quad (7.15d)$$

Solve the linear system in Equations $(7.15a, b, c \text{ and } d)$ to find $m_j^x, n_{i,j}^x, \alpha_j^x$ and $\beta_j^x, \forall j = 1, 2, \dots, n$.

The next theorems transform the *LR-TFLS* for two independence crisp linear systems.

Theorem 7.2.1.

i. The mean values m^x and n^x can be obtained using the following independence $2n \times 2n$ linear system:

$$\begin{cases} Pm^x + Nn^x = m^b, \\ Pn^x + Nm^x = n^b. \end{cases} \quad (7.16)$$

ii. The spread values α^x and β^x can be obtained using the following independence $2n \times 2n$ linear system:

$$\begin{cases} P\alpha^x - N\beta^x = \alpha^b, \\ P\beta^x - N\alpha^x = \beta^b, \end{cases} \quad (7.17)$$

where,

$$P = \left(a_{i,j}^{+}\right)_{n\times n}, N = \left(a_{i,j}^{-}\right)_{n\times n}.$$

Proof.

i. To find the mean values, using Equation $(7.15a)$,

$$\sum_{j=1}^{n}\left(a_{i,j}^{+}m_j^x + a_{i,j}^{-}n_j^x\right) = \sum_{j=1}^{n} a_{i,j}^{+}m_j^x + \sum_{j=1}^{n} a_{i,j}^{-}n_j^x = m_i^b, \forall i = 1, 2, \dots, n,$$

then,

$$P\, m^x + N\, n^x = m^b.$$

Similarly, using Equation $(7.15b)$,

$$\sum_{j=1}^{n}\left(a_{i,j}^{+}n_j^x + a_{i,j}^{-}m_j^x\right) = \sum_{j=1}^{n} a_{i,j}^{+}n_j^x + \sum_{j=1}^{n} a_{i,j}^{-}m_j^x = n_i^b, \forall i = 1, 2, \dots, n,$$

then,

$$P\, n^x + N m^x = n_i^b,$$

which provides the following linear system:

$$\begin{cases} P\, m^x + N\, n^x = m^b, \\ P\, n^x + N\, m^x = n^b. \end{cases}$$

Hence, the mean values m^x and n^x are obtained using the above system.

ii. To find the spread values, using Equation $(7.15c)$,

$$\sum_{j=1}^{n}\left(a_{i,j}^{+}\alpha_j^x - a_{i,j}^{-}\beta_j^x\right) = \sum_{j=1}^{n} a_{i,j}^{+}\alpha_j^x - \sum_{j=1}^{n} a_{i,j}^{-}\beta_j^x = \alpha_i^b, \qquad \forall i = 1, 2, \dots, n,$$

then,

$$P\, \alpha^x - N\, \beta^x = \alpha^b.$$

Similarly, using Equation $(7.15d)$,

$$\sum_{j=1}^{n}\left(a_{i,j}^{+}\beta_j^x - a_{i,j}^{-}\alpha_j^x\right) = \sum_{j=1}^{n} a_{i,j}^{+}\beta_j^x - \sum_{j=1}^{n} -a_{i,j}^{-}\alpha_j^x = \beta_i^b, \forall i = 1, 2, \dots, n,$$

$$P\, \beta^x - N\, \alpha^x = \beta^b,$$

then,

$$\begin{cases} P\,\alpha^x - N\,\beta^x = \alpha^b, \\ P\,\beta^x - N\,\alpha^x = \beta^b. \end{cases}$$

Hence, the spreads values α^x and β^x are obtained using the above system.

The following remark finds the solution of mean values and spread values in matrix form, independently.

Remark 7.2.1.

i. The solution for the mean values m^x and n^x of *LR-TFLS* is obtained using block matrix F as follows:

$$\begin{pmatrix} P & N \\ N & P \end{pmatrix}\begin{pmatrix} m^x \\ n^x \end{pmatrix} = \begin{pmatrix} m^b \\ n^b \end{pmatrix}. \tag{7.18}$$

ii. The solution for the spreads values α^x and β^x of *LR-TFLS* is obtained using block matrix R as follows:

$$\begin{pmatrix} P & -N \\ -N & P \end{pmatrix}\begin{pmatrix} \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} \alpha^b \\ \beta^b \end{pmatrix}. \tag{7.19}$$

Proof. The proof is straight forward from the Theorem 7.2.1.

Using the above discussion, the associated crisp linear system for *LR-TFLS* is introduced in the below definition, using the matrices P, N and vectors m^x, n^x, α^x, β^x, m^b, n^b, α^b and β^b.

Definition 7.2.1. Consider the following linear system,

$$TX = B,$$

$$\begin{pmatrix} P & N & 0 & 0 \\ N & P & 0 & 0 \\ 0 & 0 & P & -N \\ 0 & 0 & -N & P \end{pmatrix}\begin{pmatrix} m^x \\ n^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} m^b \\ n^b \\ \alpha^b \\ \beta^b \end{pmatrix}. \tag{7.20}$$

This linear system is called the associated linear system (*LR-ALS*) of *LR-TFLS*.

The following theorem shows the relation between the solution of *LR-ALS* and *LR-TFLS*.

Theorem 7.2.2. The unique crisp vectors solution m^x, n^x, α^x and β^x of X in *LR-ALS* and fuzzy solution $\tilde{X}$ of *LR-TFLS* is equivalent.

Proof. Equation (7.20) can be written as

$$\begin{cases} P\,m^x + N\,n^x + 0\alpha^x + 0\beta^x = m^b, \\ P\,n^x + N\,m^x + 0\alpha^x + 0\beta^x = n^b, \\ 0\,m^x + 0\,n^x + P\alpha^x - N\beta^x = \alpha^b, \\ 0\,m^x + 0\,n^x - N\alpha^x + P\beta^x = \alpha^b, \end{cases} \longrightarrow \begin{cases} P\,m^x + N\,n^x = m^b, \\ P\,n^x + N\,m^x = n^b, \\ P\alpha^x - N\beta^x = \alpha^b, \\ P\beta^x - N\alpha^x = \alpha^b, \end{cases}$$

which is equivalent to Equations (7.16) and (7.17). Moreover, it is clear that the system has a unique solution if $|T| \neq 0$. □

The next example is obtained in Nasseri and Gholami (2011). The same example is solved to show the efficiency of the proposed method in the case of unique solution for *LR-TFLS*. The similar solution is obtained.

Example 7.2.1. Nasseri and Gholami, (2011) consider the following $LR - TFLS$,

$$\begin{cases} 1 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus 1 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \oplus -1 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \\ \qquad = (6, 11, 4, 4), \\ 1 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus -2 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \oplus 1 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \\ \qquad = (-11, -5, 4, 7), \\ 2 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus 1 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \oplus 3 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \\ \qquad = (-7, 4, 7, 8), \end{cases}$$

where $\tilde{x}_i = (m_i{}^x, n_i{}^x, \alpha_i{}^x, \beta_i{}^x),\ i = 1, \dots, 3$ are arbitrary trapezoidal fuzzy numbers.

The *LR-TFLS* may be written in the following matrix form:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (6, 11, 4, 4) \\ (-11, -5, 4, 7) \\ (-7, 4, 7, 8) \end{pmatrix}.$$

The crisp matrices P and N are defined using crisp matrix A, as follows:

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}, N = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The block crisp matrix T is then constructed as follows:

$$T = \begin{pmatrix} P & N & 0 & 0 \\ N & P & 0 & 0 \\ 0 & 0 & P & -N \\ 0 & 0 & -N & P \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix} \end{pmatrix},$$

and

$$|T| = 169,$$

then the system has a unique solution since $|T| \neq 0$.

The crisp vectors m^x, n^x, α^x and β^x are made using fuzzy vector $\tilde{X}$ to construct the crisp block vector X.

$$m^x = \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix}, n^x = \begin{pmatrix} n_1^x \\ n_2^x \\ n_3^x \end{pmatrix}, \alpha^x = \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix}, \beta^x = \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix}, \text{then } X = \begin{pmatrix} m^x \\ n^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix} \\ \begin{pmatrix} n_1^x \\ n_2^x \\ n_3^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} \end{pmatrix}.$$

The crisp vectors m^b, n^b, α^b and β^b are made using fuzzy vector $\tilde{B}$.

$$m^b = \begin{pmatrix} m_1^b \\ m_2^b \\ m_3^b \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \\ -7 \end{pmatrix}, \quad n^b = \begin{pmatrix} n_1^b \\ n_2^b \\ n_3^b \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \\ 4 \end{pmatrix},$$

$$\alpha^b = \begin{pmatrix} \alpha_1^b \\ \alpha_2^b \\ \alpha_3^b \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 7 \end{pmatrix}, \quad \beta^b = \begin{pmatrix} \beta_1^b \\ \beta_2^b \\ \beta_3^b \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix},$$

thus, the crisp vector B is constructed as follows:

$$B=\begin{pmatrix} m^b \\ n^b \\ \alpha^b \\ \beta^b \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} m_1^b \\ m_2^b \\ m_3^b \end{pmatrix} \\ \begin{pmatrix} n_1^b \\ n_2^b \\ n_3^b \end{pmatrix} \\ \begin{pmatrix} \alpha_1^b \\ \alpha_2^b \\ \alpha_3^b \end{pmatrix} \\ \begin{pmatrix} \beta_1^b \\ \beta_2^b \\ \beta_3^b \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 6 \\ -11 \\ -7 \end{pmatrix} \\ \begin{pmatrix} 11 \\ -5 \\ 4 \end{pmatrix} \\ \begin{pmatrix} 4 \\ 4 \\ 7 \end{pmatrix} \\ \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \\ -7 \\ 11 \\ -5 \\ 4 \\ 4 \\ 4 \\ 7 \\ 4 \\ 7 \\ 8 \end{pmatrix}.$$

Thus, the original *LR-TFLS* is equivalent to the *LR-ALS*, $TX = B$,

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \\ n_1^x \\ n_2^x \\ n_3^x \\ \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \\ \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \\ -7 \\ 11 \\ -5 \\ 4 \\ 4 \\ 4 \\ 7 \\ 4 \\ 7 \\ 8 \end{pmatrix}.$$

By computing $X = T^{-1}B$, the crisp solution for the linear system is

$$X = T^{-1}B = \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \\ n_1^x \\ n_2^x \\ n_3^x \\ \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \\ \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -4 \\ 3 \\ 4 \\ -2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \text{ or } X = \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix} \\ \begin{pmatrix} n_1^x \\ n_2^x \\ n_3^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \end{pmatrix}.$$

By collecting the fuzzy solution $\tilde{X}$ according to the corresponding entries of each vectors m^x, n^x, α^x and β^x in X, $\tilde{X}$ is then given as follows,

$$\tilde{X} = \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (1, 3, 1, 2) \\ (3, 4, 2, 1) \\ (-4, -2, 1, 1) \end{pmatrix}.$$

The proposed method is compared to Nasseri and Gholami (2011)'method, as shown in Table 7.1, in terms of fuzzy operation, the used system and size of systems.

Table 7.1

Comparison between the proposed method and Nasseri and Gholami (2011)'method.

	The proposed method	**Nasseri and Gholami (2011)'method**
Fuzzy operation	No fuzzy operation.	Computing associated triangular fuzzy number and particular form.
The used system	Matrix form.	The *LR-TFLS* transformed to *FLS*.
The size of system	Large system as $n = 10$.	All examples do not exceeded $n = 3$.

As shown in Table 7.1, in the proposed method, the *LR-TFLS* is transformed to associated linear system, where the solution can be obtained by matrix inversion method, in which no fuzzy operation is used. While Nasseri and Gholami (2011)'method computed the associated triangular fuzzy numbers and particular form of fuzzy numbers to provide *FLS*, then used the Friedman et al.(1998)'method to provide the solution in particular form of fuzzy number, then develop the solution for trapezoidal fuzzy number. The proposed method can solve any size of *LR-TFLS* regardless of the size of matrix A as $n = 10$ in Example 7.2.2, while in Nasseri and Gholami, (2011)'method, the size of matrix A is not more than $n = 3$.

In Examples 7.2.2, we show that the efficiency of the proposed method in obtaining a solution for large systems, where all of the examples in Nasseri and Gholami, (2011), Allahviranloo et al. (2012a,b, 2013) are illustrated where $n = 2$ or 3. The details of the proposed method and verification of solution are provided in Appendix C.

Examples 7.2.2. Consider the following 10×10 *LR-TFLS*,

$$\begin{pmatrix} -3 & 6 & 7 & 8 & 4 & 2 & -3 & 4 & 1 & 5 \\ 2 & 3 & 5 & 7 & 2 & 7 & 9 & 0 & -4 & -1 \\ -1 & -2 & 8 & 5 & -6 & -3 & 8 & 1 & 5 & 6 \\ 7 & 8 & 4 & 8 & -3 & 6 & 7 & 8 & 4 & 4 \\ 6 & 7 & -8 & 4 & 7 & 9 & 0 & 3 & 2 & 7 \\ 7 & -3 & 6 & 7 & -7 & 1 & 4 & 7 & 9 & 0 \\ 7 & 9 & 0 & 5 & 7 & 5 & 0 & 3 & 3 & -7 \\ 5 & 7 & 8 & -4 & 2 & 3 & 7 & 4 & 4 & 2 \\ 1 & 7 & 9 & 0 & -3 & 0 & 7 & 2 & 2 & 7 \\ -3 & -6 & 5 & 7 & 0 & 4 & -3 & 7 & 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \\ (m_4^x, n_4^x, \alpha_4^x, \beta_4^x) \\ (m_5^x, n_5^x, \alpha_5^x, \beta_5^x) \\ (m_6^x, n_6^x, \alpha_6^x, \beta_6^x) \\ (m_7^x, n_7^x, \alpha_7^x, \beta_7^x) \\ (m_8^x, n_8^x, \alpha_8^x, \beta_8^x) \\ (m_9^x, n_9^x, \alpha_9^x, \beta_9^x) \\ (m_{10}^x, n_{10}^x, \alpha_{10}^x, \beta_{10}^x) \end{pmatrix}$$

$$= \begin{pmatrix} (-167, 23, 122, 226) \\ (-136, 80, 174, 120) \\ (-177, 22, 128, 199) \\ (-168, 98, 224, 253) \\ (51, 207, 216, 151) \\ (-222, 13, 197, 273) \\ (-83, 90, 205, 148) \\ (-145, 51, 216, 184) \\ (-109, 13, 140, 167) \\ (-205, -49, 96, 231) \end{pmatrix},$$

then the fuzzy solution, using the proposed method is given as follows:

$$\tilde{X}=\begin{pmatrix}(m_1^x,n_1^x,\alpha_1^x,\beta_1^x)\\(m_2^x,n_2^x,\alpha_2^x,\beta_2^x)\\(m_3^x,n_3^x,\alpha_3^x,\beta_3^x)\\(m_4^x,n_4^x,\alpha_4^x,\beta_4^x)\\(m_5^x,n_5^x,\alpha_5^x,\beta_5^x)\\(m_6^x,n_6^x,\alpha_6^x,\beta_6^x)\\(m_7^x,n_7^x,\alpha_7^x,\beta_7^x)\\(m_8^x,n_8^x,\alpha_8^x,\beta_8^x)\\(m_9^x,n_9^x,\alpha_9^x,\beta_9^x)\\(m_{10}^x,n_{10}^x,\alpha_{10}^x,\beta_{10}^x)\end{pmatrix}=\begin{pmatrix}(-1,8,9,0)\\(2,3,7,7)\\(-10,-9,3,6)\\(-7,5,3,6)\\(0,3,6,1)\\(0,0,0,0)\\(-4,6,6,1)\\(-6,-6,1,8)\\(-2,0,4,8)\\(3,5,0,1)\end{pmatrix}.$$

In the next section, the fuzziness of the solution is investigated, since the exact solution may be a non fuzzy solution.

7.3 The Sufficient and Necessary Conditions for Obtaining A Fuzzy Solution to *LR-TFLS*

Sufficient and necessary conditions is constructed to check the fuzziness of solution. In doing these conditions, the sub-vector solutions m^x, n^x, α^x and β^x for *LR-TFLS* are required.

The inverse of matrix F in Corollary 7.1.1. is used to provide the sub-vectors solutions m^x, n^x, α^x and β^x for *LR–TFLS* in the Theorem 7.3.1.

Corollary 7.3.1. Let P and N be crisp matrices in common size. If P^{-1} and $(P - NP^{-1}N)^{-1}$ exist, then:

i- The sub-vectors m^x and n^x are computed independently as follows:

$$f_1 m^b + f_2 n^b = m^x, \tag{7.21a}$$

$$f_2 m^b + f_1 n^b = n^x. \tag{7.21b}$$

ii- The sub-vectors α^x and β^x are computed independently as follows:

$$f_1 \alpha^b - f_2 \beta^b = \alpha^x, \tag{7.21c}$$

$$f_1 \beta^b - f_2 \alpha^b = \beta^x. \tag{7.21d}$$

Proof.

i- Apply the inverse of matrix F in Equation (7.1a) on the matrix form in Equation (7.18).

$$\begin{pmatrix} m^x \\ n^x \end{pmatrix} = \begin{pmatrix} P & N \\ N & P \end{pmatrix}^{-1} \begin{pmatrix} m^b \\ n^b \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix} \begin{pmatrix} m^b \\ n^b \end{pmatrix},$$

Hence, m^x and n^x are obtained as follows,

$$f_1 m^b + f_2 n^b = m^x,$$

$$f_2 m^b + f_1 n^b = n^x.$$

ii- Similarly, apply the inverse of matrix R in Equation (7.1b),

$$\begin{pmatrix} \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} P & -N \\ -N & P \end{pmatrix}^{-1} \begin{pmatrix} m^b \\ n^b \end{pmatrix} = \begin{pmatrix} f_1 & -f_2 \\ -f_2 & f_1 \end{pmatrix} \begin{pmatrix} \alpha^b \\ \beta^b \end{pmatrix},$$

hence, α^x and β^x are obtained as follows,

$$f_1 \alpha^b - f_2 \beta^b = \alpha^x,$$

$$f_1 \beta^b - f_2 \alpha^b = \beta^x. \qquad \square$$

Using the sub-vector solutions m^x, n^x, α^x and β^x, the necessary and sufficient conditions to check the fuzziness of solution for *LR-TFLS* are provided in the next theorem.

Theorem 7.3.1. The solution of *LR-TFLS* is a fuzzy solution if and only if the following conditions are satisfied:

i- $(f_1 - f_2)m^b \leq (f_1 - f_2)n^b$,

ii- $f_1\alpha^b \geq f_2\beta^b$,

iii- $f_1\beta^b \geq f_2\alpha^b$.

Proof. Using Equations (7.21a and b), then

$$m^x \leq n^x \iff f_1m^b + f_2n^b \leq f_2m^b + f_1n^b \iff (f_1 - f_2)m^b \leq (f_1 - f_2)n^b.$$

i- Using Equation (7.21c), then

$$\alpha^x \geq 0 \iff \text{if } f_1\alpha^b - f_2\beta^b \geq 0 \iff f_1\alpha^b \geq f_2\beta^b.$$

ii- Using Equation (7.21d), then

$$\beta^x \geq 0 \iff f_1\beta^b - f_2\alpha^b \geq 0 \iff f_1\beta^b \geq f_2\alpha^b.$$ □

The next example is taken from Allahviranloo et al. (2012b). The existence of the fuzzy solution is investigated using Theorem 7.3.1. Also, our approach are verified by computing the sub-vectors m^x, n^x, α^x and β^x using Corollary 7.3.1.

Example 7.3.1. Allahviranloo et al. (2012b) consider the following *LR-TFLS*,

$$\begin{cases} -1 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus 2 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \oplus 1 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \\ \qquad = (-6, 3, 8, 9), \\ 3 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus 1 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \oplus -2 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \\ \qquad = (-14, 0, 14, 7), \\ 1 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus -1 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \oplus 4 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \\ \qquad = (4, 20, 7, 20), \end{cases}$$

where $\tilde{x}_i = (m_i{}^x, n_i{}^x, \alpha_i{}^x, \beta_i{}^x), i = 1, \dots, 3$ are arbitrary trapezoidal fuzzy numbers.

The *LR-TFLS* may be written in the following matrix form:

$$\begin{pmatrix} -1 & 2 & 1 \\ 3 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (-6, 3, 8, 9) \\ (-14, 0, 14, 7) \\ (4, 20, 7, 20) \end{pmatrix}.$$

The crisp vectors m^b, n^b, α^b and β^b are

$$m^b = \begin{pmatrix} m_1^b \\ m_2^b \\ m_3^b \end{pmatrix} = \begin{pmatrix} -6 \\ -14 \\ 4 \end{pmatrix}, \qquad n^b = \begin{pmatrix} n_1^b \\ n_2^b \\ n_3^b \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 20 \end{pmatrix},$$

$$\alpha^b = \begin{pmatrix} \alpha_1^b \\ \alpha_2^b \\ \alpha_3^b \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \\ 7 \end{pmatrix}, \qquad \beta^b = \begin{pmatrix} \beta_1^b \\ \beta_2^b \\ \beta_3^b \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \\ 20 \end{pmatrix}.$$

The crisp matrices N and P are

$$N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & 2 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix},$$

then the inverse of matrix P is

$$P^{-1} = \begin{pmatrix} -\frac{4}{25} & \frac{8}{25} & \frac{1}{25} \\ \frac{12}{25} & \frac{1}{25} & -\frac{3}{25} \\ \frac{1}{25} & -\frac{2}{25} & \frac{6}{25} \end{pmatrix}.$$

According to Corollary 7.1.1, since P^{-1} and $(P - NP^{-1}N)^{-1}$ are exist, the matrices f_1 and f_2 are computed as follows:

$$f_1 = (P - NP^{-1}N)^{-1} = \begin{pmatrix} -\frac{25}{272} & \frac{191}{544} & -\frac{11}{544} \\ \frac{141}{272} & -\frac{11}{544} & -\frac{25}{544} \\ -\frac{1}{272} & -\frac{25}{544} & \frac{141}{544} \end{pmatrix},$$

$$f_2 = -f_1\, NP^{-1} = \begin{pmatrix} \frac{9}{272} & -\frac{47}{544} & \frac{91}{544} \\ -\frac{29}{272} & \frac{91}{544} & \frac{9}{544} \\ \frac{33}{272} & \frac{9}{544} & -\frac{29}{544} \end{pmatrix}.$$

Now to apply Theorem 7.3.1, $f_1 - f_2$, $f_1\, m^b$, $f_2\, n^b$, $f_1\, \alpha^b$, $f_2\, \beta^b$, $f_1\, \beta^b$ and $f_2\, \alpha^b$ are computed:

$$f_1 - f_2 = \begin{pmatrix} -\frac{1}{8} & \frac{7}{16} & -\frac{3}{16} \\ \frac{5}{8} & -\frac{3}{16} & -\frac{1}{16} \\ -\frac{1}{8} & -\frac{1}{16} & \frac{5}{16} \end{pmatrix},$$

$$f_1\, m^b = \begin{pmatrix} -\frac{1209}{272} \\ -\frac{819}{272} \\ \frac{463}{272} \end{pmatrix}, f_2\, n^b = \begin{pmatrix} \frac{937}{272} \\ \frac{3}{272} \\ -\frac{191}{272} \end{pmatrix},$$

$$f_1\, \alpha^b = \begin{pmatrix} \frac{2197}{544} \\ \frac{1927}{544} \\ \frac{621}{544} \end{pmatrix}, f_2\, \beta^b = \begin{pmatrix} \frac{1653}{544} \\ \frac{295}{544} \\ \frac{77}{544} \end{pmatrix} \text{ and } f_1\, \beta^b = \begin{pmatrix} \frac{667}{544} \\ \frac{1961}{544} \\ \frac{2627}{544} \end{pmatrix}, f_2\, \alpha^b = \begin{pmatrix} \frac{123}{544} \\ \frac{873}{544} \\ \frac{451}{544} \end{pmatrix}.$$

The three conditions in Theorem 7.3.1. are checked as follows

i- $(f_1 - f_2)m^b \leq (f_1 - f_2)n^b$,

$$(f_1 - f_2)m^b = \begin{pmatrix} -\frac{49}{8} \\ -\frac{11}{8} \\ \frac{23}{8} \end{pmatrix} \text{ and } (f_1 - f_2)n^b =, \begin{pmatrix} -\frac{33}{8} \\ \frac{5}{8} \\ \frac{47}{8} \end{pmatrix}.$$

Then,

$$\begin{pmatrix} -\frac{49}{8} \\ -\frac{11}{8} \\ \frac{23}{8} \end{pmatrix} \leq \begin{pmatrix} -\frac{33}{8} \\ \frac{5}{8} \\ \frac{47}{8} \end{pmatrix}.$$

ii- $f_1\, \alpha^b \geq f_2\, \beta^b$,

$$\begin{pmatrix} \frac{2197}{544} \\ \frac{1927}{544} \\ \frac{621}{544} \end{pmatrix} \geq \begin{pmatrix} \frac{1653}{544} \\ \frac{295}{544} \\ \frac{77}{544} \end{pmatrix}.$$

iii- $f_1\, \beta^b \geq f_2\, \alpha^b$,

$$\begin{pmatrix} \frac{667}{544} \\ \frac{1961}{544} \\ \frac{2627}{544} \end{pmatrix} \geq \begin{pmatrix} \frac{123}{544} \\ \frac{873}{544} \\ \frac{451}{544} \end{pmatrix}.$$

Since the three conditions are satisfied, the solution of $LR - TFLS$ is a fuzzy solution.

Next, our approach is verified, where the sub vector solution m^x, n^x, α^x and β^x are independently computed and providing similar solution in Allahviranloo et al. (2012b).

The sub vector solution n^x is independently obtained using Equation (7.21a),

$$m^x = f_1\, m^b + f_2\, n^b = \begin{pmatrix} -\frac{1209}{272} \\ -\frac{819}{272} \\ \frac{463}{272} \end{pmatrix} + \begin{pmatrix} \frac{937}{272} \\ \frac{3}{272} \\ -\frac{191}{272} \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}.$$

The sub vector solution n^x is independently obtained using Equation (7.21b),

$$n^x = f_2\, m^b + f_1\, n^b = \begin{pmatrix} -\frac{185}{272} \\ \frac{173}{272} \\ \frac{1407}{272} \end{pmatrix} + \begin{pmatrix} \frac{457}{272} \\ -\frac{445}{272} \\ -\frac{319}{272} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}.$$

The sub vector solution α^x is independently obtained using Equation (7.21c),

$$\alpha^x = f_1\, \alpha^b - f_2\, \beta^b = \begin{pmatrix} \frac{2197}{544} \\ \frac{1927}{544} \\ \frac{621}{544} \end{pmatrix} - \begin{pmatrix} \frac{1653}{544} \\ \frac{295}{544} \\ \frac{77}{544} \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

The sub vector solution β^x is independently obtained using Equation (7.21d),

$$\beta^x = f_1\, \beta^b - f_2\, \alpha^b = \begin{pmatrix} \frac{667}{544} \\ \frac{1961}{544} \\ \frac{2627}{544} \end{pmatrix} - \begin{pmatrix} \frac{123}{544} \\ \frac{873}{544} \\ \frac{451}{544} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

Hence, the fuzzy solution $\tilde{X}$ is as follows,

$$\tilde{X} = \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (-1, 1, 1, 1) \\ (-3, -1, 3, 2) \\ (1, 4, 1, 4) \end{pmatrix}.$$

Table 7.2 compares the proposed method with Allahviranloo et al. (2012b)'method in terms of existence of fuzzy solution, possibility of solution, method of solution and the size of the system.

Table 7.2

Comparison between the proposed method and Allahviranloo et al. (2012b)'method.

	The proposed method	**Allahviranloo et al. (2012b)'method**
Existence of fuzzy solution	Investigated.	Not investigated.
possibility of solution	Investigated.	Not investigated.
Method of solution	Linear system.	LP method.
The size of system	Large system as $n = 10$.	All examples don't exceed $n = 3$.

The solution of *LR-TFLS* is non fuzzy solution if one of the conditions in Theorem 7.3.1. is not satisfied. This motivated us to provide the approximation fuzzy solution. In Section 7.5, the approximated fuzzy solution is provided using a minimization problem.

The conditions for obtaining a fuzzy solution to *LR-TFLS* are listed in Theorem 7.3.1. In order to complete the elaboration of the nature of the solution for *LR-TFLS*, in the next section, the consistency of the solution (fuzzy or non fuzzy) is investigated to list the possibilities for the solution (i.e., unique solution, infinite number of solutions, no solution).

7.4 The Consistency of the Left Right-Trapezoidal Fuzzy Linear System

In this section, the consistency of the *LR-TFLS* solution are checked, and the possibilities of the solution are classified.

Since the solution of *LR-TFLS* is equivalent to the solution of *LR-ALS*, three cases for the solution of *LR-TFLS* can be studied according to the possibilities of the solution of the *LR-ALS*. They are unique solution, infinite number of solutions, no solution:

Case 1: Unique solution.

If $|T| \neq 0$, then T is invertible. Thus, $TX = B$ provides a unique solution $X = T^{-1}B$, then *LR–TFLS* has a unique solution.

Case 2: Infinite number of solutions.

If $|T| = 0$ and $m = rank(T) = rank\ (T{:}B)$, $m < 4n$, then $TX = B$ provides infinite number of solutions. In this case, fuzzy row reduced echelon method is applied to provide infinite number of solutions.

Case 3: No solution.

If $|T| = 0$ and $rank(T) < rank\ (T{:}B)$, then $TX = B$ has no solution.

Previous examples in this chapter illustrated the unique solution based on Case 1. In the next method, we show the efficiency of the *LR-ALS* in producing a general form solution based on Case 2, where infinite number of solutions exist.

Fuzzy Row Reduced Echelon Method

In this section, the fuzzy row reduced echelon method is proposed to solve *LR-TFLS* where $|T| = 0$ or T is rectangle. So, infinitely many solutions can be found whenever it exist. The method consist of three steps:

Step 1 Computing the $rank\,(T)$ and $rank\,(T\colon B)$. Based on Case 2, if $rank(S) = rank\,(S\colon B) = m$, the system has infinitely many solutions.

Step 2 Transforming the *LR-ALS* in Equation (7.20) to $4n$ linear equations. Then, simplifying it to m linear equation.

Step 3 Solving m linear equations with positive fuzzy inequalities,

$$\begin{cases} m_i^x \leq n_i^x, \\ \alpha_i^x \geq 0, \\ \beta_i^x \geq 0, \qquad \forall i = 1, 2, \dots, n. \end{cases} \tag{7.22}$$

The next example illustrates the fuzzy row reduced echelon method where the fuzzy system has an infinitely many solutions. This example uses the same example in Allahviranloo et al. (2013). The proposed method provides a different solution sets for infinitely many solutions for the original work in Allahviranloo et al. (2013). We observe that Allahviranloo et al. (2013)'solution set is shown as a subset of a solution set using the proposed method.

Example 7.4.1. Allahviranloo et al. (2013) consider the following *LR-TFLS*,

$$\begin{cases} 1 \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus -1 \otimes (m_2^x, \alpha_2^x, \beta_2^x) = (1, 2, 3), \\ 1 \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus 1 \otimes (m_2^x, \alpha_2^x, \beta_2^x) = (3, 3, 2). \end{cases}$$

The system may be written in the following matrix form:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (1, 2, 3) \\ (3, 3, 2) \end{pmatrix}.$$

The system may be written in the matrix form $A \otimes \tilde{X} = \tilde{B}$ in trapezoidal fuzzy numbers:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (1,1,2,3) \\ (3,3,3,2) \end{pmatrix}.$$

Using Equation (7.20),

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ n_1^x \\ n_2^x \\ \alpha_1^x \\ \alpha_2^x \\ \beta_1^x \\ \beta_2^x \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \\ 2 \\ 3 \\ 3 \\ 2 \end{pmatrix}.$$

However, the linear system is reduced to obtain infinite number of solutions.

Using Step 1,

$$T = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, rank(T) = 6,$$

$$(T{:}\,B) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, rank(T{:}\,B) = 6,$$

because $rank(T) = rank\,(T{:}\,B) = 6$, where $4n = 8$, $6 \leq 8$. The linear system provides infinite number of solutions.

Using Step 2, *LR-ALS* is transformed to eight linear equations,

$$\begin{cases} m_1^x - n_2^x = 1, \\ m_1^x + m_2^x = 3, \\ -m_2^x + n_1^x = 1, \\ n_1^x + n_2^x = 3, \\ \alpha_1^x + \beta_2^x = 2, \\ \alpha_1^x + \alpha_2^x = 3, \\ \alpha_2^x + \beta_1^x = 3, \\ \beta_1^x + \beta_2^x = 2. \end{cases}$$

The eight linear equations can be reduced to six linear equations,

$$\begin{cases} \beta_1^x = 2 - \beta_2^x, \\ \alpha_2^x = 1 + \beta_2^x, \\ \alpha_1^x = 2 - \beta_2^x, \\ n_1^x = 3 - n_2^x, \\ m_2^x = 2 - n_2^x, \\ m_1^x = 1 + n_2^x \end{cases}$$

Using Step 3, we reduce the six linear equations using fuzzy inequality for trapezoidal fuzzy number in Equation (7.22). Hence, the general form solution is obtained as follows:

$$\tilde{X}_{\boldsymbol{g}} = \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (2,2,\theta,\theta) \\ (1,1,3-\theta,2-\theta) \end{pmatrix}, \theta \in [0,2].$$

The original system is *LR-FLS*, so $m_1^x = n_1^x$ and $m_2^x = n_2^x$, then we get,

$$\tilde{X}_{\boldsymbol{g}} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (2,\theta,\theta) \\ (1,3-\theta,2-\theta) \end{pmatrix}, \theta \in [0,2].$$

Meanwhile, the interval solution in Allahviranloo et al. (2013) cannot exceed $\lambda \in [0,1]$ because the solution was constructed through a convex set.

$$\tilde{X}_{allv} = \begin{pmatrix} \lambda(2,1,1) + (1-\lambda)(2,2,2) \\ \lambda(1,2,1) + (1-\lambda)(1,1,0) \end{pmatrix} = \begin{pmatrix} (2,2-\lambda,2-\lambda) \\ (1,1+\lambda,\lambda) \end{pmatrix}, \lambda \in [0,1].$$

It can be showed the solution set for infinitely many solutions in Allahviranloo et al. (2012b) is subset of solution set for infinitely many solutions in the proposed method, is as follows:

Suppose $2-\lambda=\theta$, then

$$\begin{pmatrix} (2,\theta,\theta) \\ (1,1+2-\theta,2-\theta) \end{pmatrix} = \begin{pmatrix} (2,\theta,\theta) \\ (1,3-\theta,2-\theta) \end{pmatrix} = \tilde{X}_{allv}, \qquad \theta \in [1,2].$$

Thus, the solution of Allahviranloo et al. (2013) cannot provide all the particular solutions such as $\tilde{X}_1$ where $\theta = 0$,

$$\tilde{X}_1 = \begin{pmatrix} (2,0,0) \\ (1,3,2) \end{pmatrix}.$$

The verifications of solution $\tilde{X}_1$

$$\begin{cases} (1) \otimes (2,0,0) \oplus (-1) \otimes (1,3,2) = (2,0,0) \oplus (-1,2,3) = (1,2,3), \\ (1) \otimes (2,0,0) \oplus (1) \otimes (1,3,2) = (2,0,0) \oplus (1,3,2) = (3,3,2). \end{cases}$$

Moreover, the method of Allahviranloo et al. (2013) cannot determine if the solution is infinite or unique prior to obtaining the final solution.

Table 7.3 shows the comparison between the proposed method and Allahviranloo et al. (2012b)'method, in terms of existence of fuzzy solution, possibility of solution, solution set of infinity many solution and method of solution.

Table 7.3

Comparison between the proposed method and Allahviranloo et al. (2012b)'method.

	The proposed method	**Allahviranloo et al. (2012b)'method**
Existence of fuzzy solution	Investigated.	Not investigated.
Possibility of solution	Investigated.	Not investigated.
Solution set of infinitely many solution	$\theta \in [0, 2]$.	$\theta \in [1, 2]$.
Method of solution	Fuzzy row reduced echelon method.	*LP* method.
The size of system	Large system as $n = 10$.	$n \leq 3$.

As noted in Table 7.3, the solution set for infinitely many solutions in Allahviranloo et al. (2012b) is subset of solution set for infinitely many solutions in the proposed method. In addition, the possibility and existence of fuzzy solution are investigated in the proposed method before solving the system, which are not being investigated in Allahviranloo et al. (2012b). Lastly, the proposed methods is able to solve *LR-FLS* where the size of matrix A is $n = 10$, while all examples in Allahviranloo et al. (2012b) do not exceed $n = 3$.

The next example illustrates Case 3, when the *LR-TFLS* has no solution.

Example 7.4.2. Consider the following *LR-TFLS*:

$$\begin{cases} -1 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus\ -1 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) =\ (2,7,6,6), \\ -3(m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus\ 3 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) =\ (-7,1,11,11). \end{cases}$$

The system may be written in the matrix form $A \otimes \tilde{X} = \tilde{B}$ in trapezoidal fuzzy numbers:

$$\begin{pmatrix} -1 & -1 \\ -3 & 3 \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (2,7,6,6) \\ (-7,1,11,11) \end{pmatrix}$$

Using Equation (7.20),

$$\begin{pmatrix} 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ n_1^x \\ n_2^x \\ \alpha_1^x \\ \alpha_2^x \\ \beta_1^x \\ \beta_2^x \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \\ 7 \\ 1 \\ 6 \\ 11 \\ 6 \\ 11 \end{pmatrix},$$

then,

$$T = \begin{pmatrix} 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} rank(T) = 6,$$

$$(T{:}B) = \begin{pmatrix} 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & -7 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ -3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 11 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 & 11 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, rank(T{:}B) = 7.$$

Given that $rank(T) < rank(T{:}B)$, according to Case 3, the *LR-TFLS* contains no unique solution.

The next section provides the approximation fuzzy solution of *LR-TFLS*. If the exact solution is non fuzzy based on Theorem 7.3.1, a minimization problem is proposed using *LP* approach.

7.5 Approximate Solution in the Case of Non fuzzy Solution

In this section, an minimization problem is presented to obtain the nearest fuzzy solution where the exact solution is non fuzzy.

The next lemma is used to provide a minimization function in optimization problem.

Lemma 7.5.1. If x_1, x_2, x_3 and x_4 are arbitrary real numbers, then

$$\sqrt{\frac{{x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2}{4}} \leq \frac{|x_1| + |x_2| + |x_3| + |x_4|}{2}.$$

Proof. Given

$$(|x_1| + |x_2| + |x_3| + |x_4|)^2 = |x_1|^2 + 2|x_2||x_1| + 2|x_3||x_1| + 2|x_4||x_1| + |x_2|^2 + |x_3|^2 + |x_4|^2 + 2|x_2||x_3| + 2|x_2||x_4| + 2|x_3||x_4|,$$

then

$$(|x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2) \leq (|x_1| + |x_2| + |x_3| + |x_4|)^2,$$

$$\frac{{x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2}{4} = \frac{(|x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2)}{4}$$

$$\leq \frac{(|x_1| + |x_2| + |x_3| + |x_4|)^2}{4}.$$

Thus,

$$\sqrt{\frac{{x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2}{4}} \leq \sqrt{\frac{(|x_1| + |x_2| + |x_3| + |x_4|)^2}{4}}$$

$$= \frac{|x_1| + |x_2| + |x_3| + |x_4|}{2}. \qquad \square$$

Using the distance metric function in Definition 2.2.14. we provide a function that will be used in constructing the optimization problem in next theorem.

Theorem 7.5.1. If $\tilde{A} = (m_1, n_1, \alpha_1, \beta_1), \tilde{B} = (m_2, n_2, \alpha_2, \beta_2)$ are two trapezoidal fuzzy numbers, then

$$d(\tilde{A}, \tilde{B}) \leq Td(\tilde{A}, \tilde{B}),$$

where

$$Td(\tilde{A},\tilde{B}) = \frac{|m_1 - m_2 - \alpha_1 + \alpha_2| + |n_1 - n_2 + \beta_1 - \beta_2| + |m_1 - m_2| + |n_1 - n_2|}{2}.$$

Proof.

$$d^2(\tilde{A},\tilde{B}) = \frac{[(m_1 - m_2) - (\alpha_1 - \alpha_2)]^2 + [(n_1 - n_2) + (\beta_1 - \beta_2)]^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2}{4}$$

$$= \frac{[m_1 - m_2 - \alpha_1 + \alpha_2]^2 + [n_1 - n_2 + \beta_1 - \beta_2]^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2}{4}.$$

Hence,

$$d(\tilde{A},\tilde{B}) = \sqrt{\frac{[m_1 - m_2 - \alpha_1 + \alpha_2]^2 + [n_1 - n_2 + \beta_1 - \beta_2]^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2}{4}}.$$

Suppose that

$x_1 = (m_1 - m_2 - \alpha_1 + \alpha_2)$, $x_2 = (n_1 - n_2 + \beta_1 - \beta_2)$ $x_3 = (m_1 - m_2)$ and $x_4 = (n_1 - n_2)$, then $d(\tilde{A},\tilde{B})$ is equal

$$d(\tilde{A},\tilde{B}) = \sqrt{\frac{{x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2}{4}}.$$

Using Lemma 7.5.1,

$$d(\tilde{A},\tilde{B}) = \sqrt{\frac{{x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2}{4}} \leq \frac{|x_1| + |x_2| + |x_3| + |x_4|}{2},$$

Thus,

$$d(\tilde{A},\tilde{B}) \leq \frac{|m_1 - m_2 - \alpha_1 + \alpha_2| + |n_1 - n_2 + \beta_1 - \beta_2| + |m_1 - m_2| + |n_1 - n_2|}{2},$$

hence,

$$d(\tilde{A},\tilde{B}) \leq Td(\tilde{A},\tilde{B}).$$ □

Next section provides the optimization problem which can provide the approximation fuzzy solutions, in case the exact solution is non fuzzy.

Optimization Problem for Non fuzzy Solution

Based on Theorem 7.3.1, if one condition is not satisfied, the solution of *LR-TFLS* is a non fuzzy solution. In this case, in this section, we construct an Minimization problem in order to obtain the nearest approximation fuzzy solution.

If the *LR-TFLS* has a non fuzzy solution, the Minimization problem is constructed as follows. Suppose $\tilde{y}_i = \left(m_{y_i}, n_{y_i}, \alpha_{y_i}, \beta_{y_i}\right)$:

the minimization function Z is

$$Minimize \quad Z = \sum_{i=1}^{n} Td\left(\tilde{y}_i, \tilde{b}_i\right). \tag{7.23}$$

Using Equations $(7.15a), (7.15b), (7.15c)$ and $(7.15d)$, the constrains for subject are provided as follows:

$$\sum_{j=1}^{n}\left(a_{i,j}^{+} m_j^x + a_{i,j}^{-} n_j^x\right) = m_{y_i}, \tag{7.24a}$$

$$\sum_{j=1}^{n}\left(a_{i,j}^{+} n_j^x + a_{i,j}^{-} m_j^x\right) = n_{y_i}, \tag{7.24b}$$

$$\sum_{j=1}^{n}\left(a_{i,j}^{+} \alpha_j^x - a_{i,j}^{-} \beta_j^x\right) = \alpha_{y_i}, \tag{7.24c}$$

$$\sum_{j=1}^{n}\left(a_{i,j}^{+} \beta_j^x - a_{i,j}^{-} \alpha_j^x\right) = \beta_{y_i}, \tag{7.24d}$$

$$\begin{cases} 0 \le \alpha_i^x, \alpha_{y_i}, \\ 0 \le \beta_i^x, \beta_{y_i}, \\ m_i^x \le n_i^x, m_{y_i} \le n_{y_i} \quad \forall i = 1, 2, \dots, n. \end{cases} \tag{7.25}$$

Solving the minimization problem provides the approximation fuzzy solution $\tilde{x}_i = (m_i{}^x, n_i{}^x, \alpha_i{}^x, \beta_i{}^x), i = 1, \dots, n$ for *LR-TFLS*.

Allahviranloo et al. (2012a) illustrated an example that contains a non fuzzy solution. They used a minimization problem for symmetric solution only. The existence of a fuzzy solution is investigated using Theorem 7.2.1 to show that *LR-TFLS* does not have an exact fuzzy solution. However, the exact non fuzzy solution is provided using the equivalent linear system in Equation (7.20). Then, we use the proposed minimization problem to provide the nearest approximation symmetric and non symmetric fuzzy solution. The distance metric in Definition 2.2.14. is used to show that our solution is nearer than that of Allahviranloo et al. (2012a). The verification solution is provided.

Example 7.5.1. Consider the following *LR-TFLS* (Allahviranloo et al. 2012a),

$$\begin{cases} 1 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus -2 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \oplus 1 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \\ \qquad = (-4,1,5,5), \\ -1 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus -1 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \oplus 1 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \\ \qquad = (-5,-2,4,4), \\ -2 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus 1 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \oplus 1 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \\ \qquad = (-2,-1,6,6). \end{cases}$$

where $\tilde{x}_i = (m_i{}^x, n_i{}^x, \alpha_i{}^x, \beta_i{}^x), i = 1, \dots, 3$ are arbitrary trapezoidal fuzzy numbers.

In matrix form $A \otimes \tilde{X} = \tilde{B}$:

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (-4,1,5,5) \\ (-5,-2,4,4) \\ (-2,-1,6,6) \end{pmatrix}.$$

Using Theorem 7.3.1, we check whether or not the solution is fuzzy prior to solving:

$$N = \begin{pmatrix} 0 & -2 & 0 \\ -1 & -1 & 0 \\ -2 & 0 & 0 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \text{ then } P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$f_1 = \begin{pmatrix} \frac{1}{3} & -1 & \frac{2}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \\ 0 & 1 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 & \frac{1}{3} \\ 1 & -2 & 1 \end{pmatrix},$$

$$f_1 - f_2 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{pmatrix},$$

$$(f_1 - f_2)n^b = \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} \text{ and } (f_1 - f_2)m^b = \begin{pmatrix} 3 \\ 1 \\ -9 \end{pmatrix},$$

$$(f_1 - f_2)n^b - (f_1 - f_2)m^b = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}.$$

Given that $(f_1 - f_2)n^b - (f_1 - f_2)m^b$ has a negative entry -2, then

$$(f_1 - f_2)m^b \nleq (f_1 - f_2)n^b,$$

hence, the exact solution is non fuzzy. We can enhance that using the equivalent linear system in Equation (7.20), the exact solution is not a fuzzy solution because $m_1^x \nleq n_1^x$,

$$\tilde{X} = \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (3,1,2,2) \\ (1,3,1,1) \\ (-1,2,1,1) \end{pmatrix}.$$

Now, we can find the nearest approximation that is a symmetric and non symmetric fuzzy as follows,

Minimize

$$Z = \frac{1}{2}\left(\left|-\alpha_{y_1} - n_{y_1} + 6\right| + \left|-m_{y_1} + \beta_{y_1} - 9\right| + \left|-m_{y_1} - 4\right| + \left|1 - n_{y_1}\right|\right)$$

$$+\frac{1}{2}\left(\left|-\alpha_{y_2} - n_{y_2} + 2\right| + \left|-m_{y_2} + \beta_{y_2} - 9\right| + \left|-m_{y_2} - 5\right| + \left|-n_{y_2} - 2\right|\right) +$$

$$+\frac{1}{2}\left(\left|-\alpha_{y_3} - n_{y_3} + 5\right| + \left|-m_{y_3} + \beta_{y_3} - 8\right| + \left|-m_{y_3} - 2\right| + \left|-n_{y_3} - 1\right|\right).$$

The subject is

$$\begin{cases} m_1^x + m_3^x - 2\alpha_1^x &= m_{y_1}, \\ m_3^x - m_4^x - \alpha_1^x &= m_{y_2}, \\ m_2^x + m_3^x - 2m_4^x &= m_{y_3}, \\ -2m_2^x + m_4^x + \alpha_2^x &= n_{y_1}, \\ -m_1^x - m_2^x + \alpha_2^x &= n_{y_2}, \\ -2m_1^x + \alpha_1^x + \alpha_2^x &= n_{y_3}, \\ \alpha_3^x + \beta_1^x + 2\beta_3^x &= \alpha_{y_1}, \\ \beta_1^x + \beta_2^x + \beta_3^x &= \alpha_{y_2}, \\ \alpha_4^x + \beta_1^x + 2\beta_2^x &= \alpha_{y_3}, \\ 2\alpha_4^x + \beta_2^x + \beta_4^x &= \beta_{y_1}, \\ \alpha_3^x + \alpha_4^x + \beta_4^x &= \beta_{y_1}, \\ 2\alpha_3^x + \beta_3^x + \beta_4^x &= \beta_{y_1}. \end{cases}$$

where, $\alpha_i^x, \beta_i^x, \alpha_{y_i}, \beta_{y_i} \geq 0$ and $m_i^x \geq n_i^x, n_{y_i} \geq m_{y_i}$ for $i = 1, \ldots, 3$.

The minimization problem is solved in two cases (symmetric and non symmetric solution).

Case 1: Symmetric approximate solution $\tilde{X}_s$

For the symmetric solution $\tilde{X}_s$, we add $\alpha_{y_i} = \beta_{y_i}, \alpha_i^x = \beta_i^x$ for $i = 1, \dots, 3$, to the subject. Then,

$$Z = 1,$$

$$m_1^x = 2, n_1^x = 2, \alpha_1^x = 1, \beta_1^x = 1, m_2^x = \frac{3}{4}, n_2^x = \frac{13}{4}, \alpha_2^x = \frac{3}{4}, \beta_2^x = \frac{3}{4}, m_3^x = \frac{1}{2}, n_3^x = \frac{1}{2}, \alpha_3^x = \frac{5}{2}, \beta_3^x = \frac{5}{2}.$$

$$m_{y_1} = -4, n_{y_1} = 1, \alpha_{y_1} = 5, \beta_{y_1} = 5, m_{y_2} = -\frac{19}{4}, n_{y_2} = -\frac{9}{4}, \alpha_{y_2} = \frac{17}{4}, \beta_{y_2} = \frac{17}{4}, m_{y_3} = -\frac{11}{4}, n_{y_3} = -\frac{1}{4}, \alpha_{y_3} = \frac{21}{4}, \beta_{y_3} = \frac{21}{4}.$$

$$\tilde{X}_s = \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (2,2,1,1) \\ \left(\frac{3}{4}, \frac{13}{4}, \frac{3}{4}, \frac{3}{4}\right) \\ \left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}\right) \end{pmatrix}$$

and $A \otimes \tilde{X}_s = \tilde{B}_s$, where $\tilde{B}_s$ is

$$\tilde{B}_s = \begin{pmatrix} \left(m_{y_1}, n_{y_1}, \alpha_{y_1}, \beta_{y_1}\right) \\ \left(m_{y_2}, n_{y_2}, \alpha_{y_2}, \beta_{y_2}\right) \\ \left(m_{y_3}, n_{y_3}, \alpha_{y_3}, \beta_{y_3}\right) \end{pmatrix} = \begin{pmatrix} (-4,1,5,5) \\ \left(-\frac{19}{4}, -\frac{9}{4}, \frac{17}{4}, \frac{17}{4}\right) \\ \left(-\frac{11}{4}, -\frac{1}{4}, \frac{21}{4}, \frac{21}{4}\right) \end{pmatrix}.$$

By comparing the distance metric function in Definition 2.2.14. with the right side fuzzy vector:

$$\tilde{B} = \begin{pmatrix} (-4,1,5,5) \\ (-5,-2,4,4) \\ (-2,-1,6,6) \end{pmatrix}.$$

$$D_2(\tilde{B}_s, \tilde{B}) = \sqrt{\left(\frac{1}{4\sqrt{2}}\right)^2 + \left(\frac{3}{4\sqrt{2}}\right)^2} = 0.559017.$$

Verification of the solution of $\tilde{X}_s$

$$\begin{cases} (1) \otimes (2,2,1,1) \oplus (-2) \otimes \left(\frac{3}{4}, \frac{13}{4}, \frac{3}{4}, \frac{3}{4}\right) \oplus (1) \otimes \left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}\right) \\ = (2,2,1,1) \oplus \left(-\frac{13}{2}, -\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right) \oplus \left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}\right) = (-4,1,5,5), \\ \\ (-1) \otimes (2,2,1,1) \oplus (-1) \otimes \left(\frac{3}{4}, \frac{13}{4}, \frac{3}{4}, \frac{3}{4}\right) \oplus (1) \otimes \left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}\right) \\ = (-2,-2,1,1) \oplus \left(-\frac{13}{4}, -\frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right) \oplus \left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}\right) = \left(-\frac{19}{4}, -\frac{9}{4}, \frac{17}{4}, \frac{17}{4}\right), \\ \\ (-2) \otimes (2,2,1,1) \oplus (1) \otimes \left(\frac{3}{4}, \frac{13}{4}, \frac{3}{4}, \frac{3}{4}\right) \oplus (1) \otimes \left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}\right) \\ = (-4,-4,2,2) \oplus \left(\frac{3}{4}, \frac{13}{4}, \frac{3}{4}, \frac{3}{4}\right) \oplus \left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}\right) = \left(-\frac{11}{4}, -\frac{1}{4}, \frac{21}{4}, \frac{21}{4}\right). \end{cases}$$

Case 2: Non symmetric approximate solution $\tilde{X}_{ns}$

To obtain the non symmetrical solution $\tilde{X}_{ns}$, we omit $\alpha_{y_i} = \beta_{y_i}$, $\alpha_i^x = \beta_i^x$ for $i = 1, \ldots, 3$, then,

$$\tilde{X}_{ns}=\begin{pmatrix}(m_1^x,n_1^x,\alpha_1^x,\beta_1^x)\\ \\ (m_2^x,n_2^x,\alpha_2^x,\beta_2^x)\\ \\ (m_3^x,n_3^x,\alpha_3^x,\beta_3^x)\end{pmatrix}=\begin{pmatrix}\left(\frac{25}{12},\frac{25}{12},\frac{13}{12},\frac{11}{12}\right)\\ \\ \left(\frac{11}{12}\quad\frac{41}{12}\quad\frac{11}{12}\quad\frac{7}{12}\right)\\ \\ \left(\frac{3}{4}\quad\frac{3}{4}\quad\frac{11}{4}\quad\frac{9}{4}\right)\end{pmatrix},$$

and $A\otimes\ \tilde{X}_{ns}=\tilde{B}_{ns}$, where $\tilde{B}_{ns}$ is

$$\tilde{B}_{ns}=\begin{pmatrix}\left(m_{y_1},n_{y_1},\alpha_{y_1},\beta_{y_1}\right)\\ \\ \left(m_{y_2},n_{y_2},\alpha_{y_2},\beta_{y_2}\right)\\ \\ \left(m_{y_3},n_{y_3},\alpha_{y_3},\beta_{y_3}\right)\end{pmatrix}=\begin{pmatrix}(-4,1,5,5)\\ \\ \left(-\frac{19}{4},-\frac{9}{4},\frac{17}{4},\frac{17}{4}\right)\\ \\ \left(-\frac{5}{2},0,\frac{11}{2},5\right)\end{pmatrix}.$$

Using the metric function in Definition 2.2.14, we get

$$D_2\left(\tilde{B}_{ns},\tilde{B}\right)=\sqrt{0+(\tfrac{1}{4\sqrt{2}})^2+(\tfrac{\sqrt{5}}{4})^2}=0.586302.$$

Verification of the solution of $\tilde{X}_{ns}$

$$\begin{cases}(1)\otimes\left(\frac{25}{12},\frac{25}{12},\frac{13}{12},\frac{11}{12}\right)\oplus(-2)\otimes\left(\frac{11}{12},\frac{41}{12},\frac{11}{12},\frac{7}{12}\right)\oplus(1)\\ \otimes\left(\frac{3}{4},\frac{3}{4},\frac{11}{4},\frac{9}{4}\right)=\left(\frac{25}{12},\frac{25}{12},\frac{13}{12},\frac{11}{12}\right)\oplus\left(-\frac{41}{6},-\frac{11}{6},\frac{7}{6},\frac{11}{6}\right)\oplus\left(\frac{3}{4},\frac{3}{4},\frac{11}{4},\frac{9}{4}\right)\\ =(-4,1,5,5),\\ \\ (-1)\otimes\left(\frac{25}{12},\frac{25}{12},\frac{13}{12},\frac{11}{12}\right)\oplus(-1)\otimes\left(\frac{11}{12},\frac{41}{12},\frac{11}{12},\frac{7}{12}\right)\oplus(1)\\ \otimes\left(\frac{3}{4},\frac{3}{4},\frac{11}{4},\frac{9}{4}\right)=\left(-\frac{25}{12},-\frac{25}{12},\frac{11}{12},\frac{13}{12}\right)\oplus\left(-\frac{41}{12},-\frac{11}{12},\frac{7}{12},\frac{11}{12}\right)\oplus\left(\frac{3}{4},\frac{3}{4},\frac{11}{4},\frac{9}{4}\right)\\ =\left(-\frac{19}{4},-\frac{9}{4},\frac{17}{4},\frac{17}{4}\right),\\ \\ (-2)\otimes\left(\frac{25}{12},\frac{25}{12},\frac{13}{12},\frac{11}{12}\right)\oplus(1)\otimes\left(\frac{11}{12},\frac{41}{12},\frac{11}{12},\frac{7}{12}\right)\oplus(1)\\ \otimes\left(\frac{3}{4},\frac{3}{4},\frac{11}{4},\frac{9}{4}\right)=\left(-\frac{25}{6},-\frac{25}{6},\frac{11}{6},\frac{13}{6}\right)\oplus\left(\frac{11}{12},\frac{41}{12},\frac{11}{12},\frac{7}{12}\right)\oplus\left(\frac{3}{4},\frac{3}{4},\frac{11}{4},\frac{9}{4}\right)\\ =\left(-\frac{5}{2},0,\frac{11}{2},5\right).\end{cases}$$

According to Allahviranloo et al. (2012a), the nearest approximation symmetric fuzzy solution is

$$\tilde{X}_v = \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (2,2,2,2) \\ (0.8333,3.1667,1,1) \\ (0.5,0.5,1,1) \end{pmatrix},$$

and $A \otimes \tilde{X}_v = \tilde{B}_v$, where $\tilde{B}_v$ is

$$\tilde{B}_v = \begin{pmatrix} (-3.8334,0.8334;5,5) \\ (-4.6667,-2.3333,4,4) \\ (-2.6667,-0.3333,6,6) \end{pmatrix}.$$

Using the metric function in Definition 2.2.14,

$$D_2(\tilde{B}_v, \tilde{B}) = \sqrt{(0.16659)^2 + (0.33329)^2 + (0.6667)^2} = 0.763756.$$

Figures 7.1, 7.2 and 7.3 show the three right-hand sides $\tilde{B}_v$, $\tilde{B}_s$ and $\tilde{B}_{ns}$ which are obtained from the three solutions $\tilde{X}v$, $\tilde{X}s$, and $\tilde{X}ns$, respectively, with the left hand side $\tilde{B}$ of *LR-TFLS*.

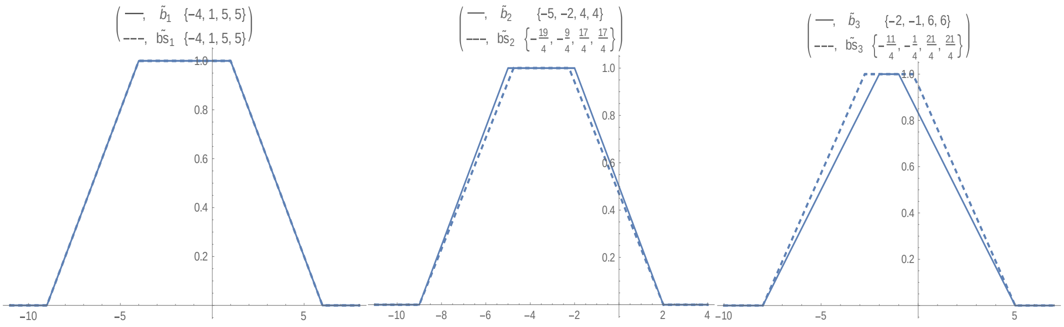

Figure 7.1. Comparison between $\tilde{B}$ and symmetric $\tilde{B}_s$ using the proposed method.

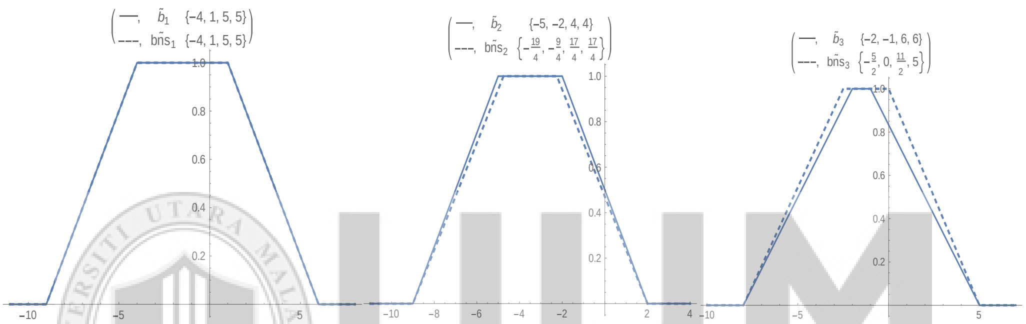

Figure 7.2. Comparison between $\tilde{B}$ and non symmetric $\tilde{B}_{ns}$ using the proposed method.

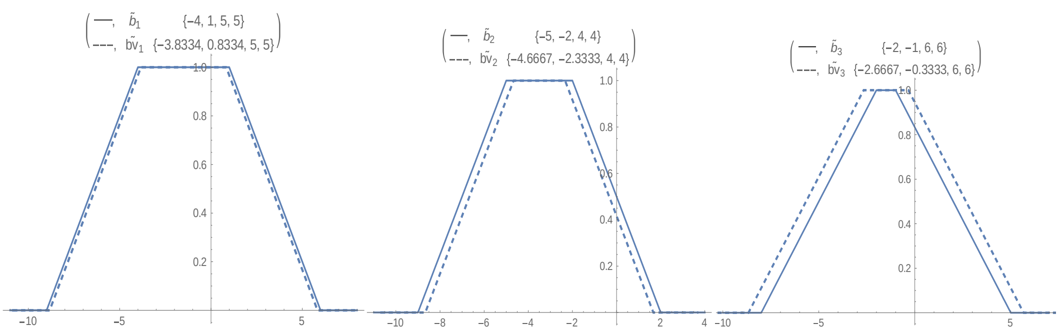

Figure 7.3. Comparison of $\tilde{B}$ and the $\tilde{B}v$ in Allahviranloo et al. (2012a)'approximation.

Table 7.4 shows the comparison between the proposed method and Allahviranloo et al. (2012a)'method, in terms of distance metric function for symmetric solution, distance metric function for non symmetric solution, non fuzzy solution and existence of fuzzy solution and possibility of solution.

Table 7.4

Comparison between the proposed method and Allahviranloo et al. (2012a)'method.

	The proposed method	**Allahviranloo et al. (2012b)'method**
Distance metric function for Symmetric solution	0.559017	0.763756
Distance metric function for non symmetric solution	0.586302	Not proposed.
Non fuzzy solution	Proposed.	Not proposed.
Existence of fuzzy solution	Investigated.	Not investigated.
possibility of solution	Investigated.	Not investigated.

As shown in Table 7.4, the proposed method provided more than nearer approximation symmetric fuzzy solution that of Allahviranloo et al. (2012a). In addition, the proposed method provides an approximation symmetric fuzzy solution and the exact non fuzzy solution, which are not provided in Allahviranloo et al. (2012a). In addition, the possibility and existence of fuzzy solution are investigated in the proposed method, before solving the system, while, in Allahviranloo et al. (2012a), these are not investigated.

7.6 Conclusion and Contributions

This chapter studied the *LR-TFLF* in trapezoidal fuzzy numbers that is an extension of *FLS* in particular form and *LR-FLS* in triangular fuzzy numbers. An associated linear system which is equivalent to *LR-TFLS* was derived.

We can summarize the finding in this chapter by the following contributions:

1- Obtain the fuzzy solution of $LR - TFLS$ without fuzzy operation.

2- The nature of the solution of *LR-TFLS* is distinguished for fuzziness of solution and possibilities of the solution.

3- Provide the sufficient and necessary conditions of *LR-TFLS* needed to have a fuzzy solution based on the fuzziness of the solution.

4- Classify the possibilities of the solution (i.e. unique solution, infinite number of solutions, no solution).

5- Obtain the nearest approximation fuzzy solution using a minimization problem when the exact solution is non fuzzy.

CHAPTER EIGHT

CONCLUSION

This section presents the summary of this study. A conclusion of the whole study is introduced. Also, the main contributions are provided. Lastly, the suggestions of future works are presented.

8.1 Conclusion of the Study

Many researchers have conducted studies on methods for solving fuzzy system but few studies have attempted to construct methods by associated linear systems, whereby most methods employed *LP* technique for solving linear system. Moreover, the methods avoid dealing with near zero. In addition, because of the complexity of fuzzy operation in mentioned previous studies, they restricted the systems only for $n = 2$ or 3.

Thus, this study attempts to develop methods that did not rely on *LP*, *NLP* or without fuzzy operation, no restrictions employed to the developed methods. Most of the constructed methods in this study can use the classical linear system. Since no complexity on operation occur, we manage to solve large fuzzy systems such as $n = 10$. Besides, in the case of solving non fuzzy exact solution, *LP* is employed to obtain approximate fuzzy solution. Fuzzy operations are needed only when providing the unrestricted solution of unrestricted fuzzy systems in order to consider the sign near zero fuzzy number.

In next section, the main contributions of the proposed methods are presented.

8.2 Main Contributions

This thesis mainly focused on the new methods for solving fuzzy systems. The following contributions are achieved in this study.

- **Obtain the $P - \widetilde{X}$ without fuzzy operations for $P - FFLS$ and $NZ - FFLS$**

This study proposed the methods of solving $P - \tilde{X}$ for $P - FFLS$ and $NZ - FFLS$, using associated linear systems in Chapter Three and Four. Moreover, all coefficients in $P - FFLS$ and $NZ - FFLS$ are transformed to matrix form by only rearranging the coefficients in block matrix and crisp vector, which resulted in the following finding.

1- Solving fuzzy systems without any use of fuzzy operations.

2- Solving large system such as $n = 10$.

3- The developed methods can be used for developing methods to obtain either exact fuzzy solution or exact non fuzzy solution.

4- Distinguishing the nature of the solution of these system for fuzziness of solution and possibilities of the solution:

i- Providing the sufficient and necessary conditions of systems in order to check fuzziness of the solution. These conditions are used to examine prior solving the systems.

ii- Classifying the possibilities of the solution (i.e., unique solution, infinite number of solutions, no solution). These possibilities are used to classify the solutions prior solving the systems.

• **Adding the Finite Fuzzy Concept in $FFLS$**

This study added a finite fuzzy concept in the fuzzy systems, which explained in Chapter Five. This concept is formulated without external restrictions in order to keep the nature of solution. Then the theoretical works for finding this concept are used to construct method for solving $NZ - \tilde{X}$ for $NZ - FFLS$ by using min-max system, absolute system, and fuzzy operations. It means this method able to obtain unrestricted solution for unrestricted fuzzy system.

• **Transforming the $FFMS$ and $FFSE$ to Equivalent $FFLS$**

In Chapter Six we transfer the $FFME$ and $FFSE$ to $FFLS$, and solve them using the proposed methods for solving $FFLS$.

• **Replication of Pervious Methods with Other Type of Fuzzy Numbers.**

In Chapter Seven, the solution of $LR - TFLS$ with trapezoidal fuzzy numbers was proposed. We replicated the pervious methods of solving fuzzy systems in triangular fuzzy number. All above findings can be extended to $LR - TFLS$.

• **Certifying the priority of using LP method in approximation solution only.**

The LP method is used in Chapter Seven, to provide the nearest approximation solution when the exact solution is non fuzzy. We verify the priority of this technique is able to offer the approximation for $LR - TFLS$.

8.3 Suggestions for Future Work

We now turn our attention to comply with future problems that can be done, our suggestion of the future work are classified according to three opinions; types of coefficients and operations, solved systems and methods of solution.

- In this thesis we represent the coefficients of fuzzy systems as triangular fuzzy numbers and trapezoidal fuzzy numbers, it is suggested to use the following coefficients and operations as future works:

  1- Append fuzzy complex numbers for mean values and spreads values of triangular fuzzy numbers by using operations of complex fuzzy numbers in Fu and Shen (2011).

  2- Replace the triangular fuzzy numbers by type-2 triangular fuzzy with operation in Dinagar and Latha (2013). Also, extend the triangular fuzzy number which represented by ordered three crisp real numbers to fuzzy number represents by ordered six real numbers such as hexagonal fuzzy numbers with operations in Thamaraiselvi and Santhi (2015).

  3- Solve the $P-\tilde{X}$ for $P-FFLS$ by multiple types of fuzzy numbers and compare the fuzziness and positivity of solution, since they are most used in application

  4- We follow (m, α, β) form for triangular fuzzy numbers, it is suggested to formulate methods that can deal directly with the other form (a, b, c).

- This thesis solved $FFLS, FFME$ and $FFSE,$ it is suggested to solve the fully fuzzy Lyapunov Equation, fully fuzzy Riccati Equation and fully fuzzy Stein Equation.

- In this study, the exact solution is proposed by embedding methods, it is suggested to solve the previous systems by iterative techniques like Gauss–Seidel and Jacobi Adomian decomposition methods, Extrapolated Richardson, and followed by convergence theorems. Also, In this thesis we obtain exact solution whenever exist, it is suggested to provide Moore–Penrose pseudo inverse for case of no solution.

REFERENCES

Abadir, K. M., & Magnus, J. R. (2005). *Matrix algebra* (Vol. 1). Cambridge University Press.

Abbasbandy, S., & Hashemi, M. S. (2012). Solving fully fuzzy linear systems by using implicit Gauss–Cholesky algorithm. *Computational mathematics and modeling*, *23*(1), 107-124.

Abdolmaleki, E., & Edalatpanah, S. A. (2014). Chebyshev Semi-iterative Method to Solve Fully Fuzzy linear Systems. *Journal of Information and Computing Science*, *9*(1), 067-074.

Allahviranloo, T. (2004). Numerical methods for fuzzy system of linear equations. *Applied Mathematics and Computation*, *155*(2), 493-502.

Allahviranloo, T. (2005). Successive over relaxation iterative method for fuzzy system of linear equations. *Applied Mathematics and Computation*, *162*(1), 189-196.

Allahviranloo, T., & Kermani, M. A. (2006). Solution of a fuzzy system of linear equation. *Applied Mathematics and Computation*, *175*(1), 519-531.

Allahviranloo, T., & Mikaeilvand, N. (2006). Positive solutions of fully fuzzy linear systems. *Lahijan Journal of Applied Mathematics*, *3*(11), 1-12.

Allahviranloo, T., Ghanbari, M., Hosseinzadeh, A. A., Haghi, E., & Nuraei, R. (2011a). A note on “Fuzzy linear systems”. *Fuzzy sets and systems*, *177*(1), 87-92.

Allahviranloo, T., Haghi, E., & Ghanbari, M. (2012a). The nearest symmetric fuzzy solution for a symmetric fuzzy linear system. *Analele Universitatii" Ovidius" Constanta-Seria Matematica*, *20*(1), 151-172.

Allahviranloo, T., Hosseinzadeh, A. A., Ghanbari, M., Haghi, E., & Nuraei, R. (2014). On the new solutions for a fully fuzzy linear system. *Soft Computing*,*18*(1), 95-107.

Allahviranloo, T., Lotfi, F. H., Kiasari, M. K., & Khezerloo, M. (2013). On the fuzzy solution of LR fuzzy linear systems. *Applied Mathematical Modelling*,*37*(3), 1170-1176.

Allahviranloo, T., Mikaeilvand, N., Kiani, N. A., & Shabestari, R. M. (2008). Signed decomposition of fully fuzzy linear systems. *An International journal of Applications and Applied Mathematics*, *3*(1), 77-88.

Allahviranloo, T., Nuraei, R., Ghanbari, M., Haghi, E., & Hosseinzadeh, A. A. (2012b). A new metric for L–R fuzzy numbers and its application in fuzzy linear systems. *Soft Computing*, *16*(10), 1743-1754.

Allahviranloo, T., Salahshour, S., & Khezerloo, M. (2011b). Maximal-and minimal symmetric solutions of fully fuzzy linear systems. *Journal of Computational and Applied Mathematics*, *235*(16), 4652-4662.

Amirfakhrian, M. (2012). Analyzing the solution of a system of fuzzy linear equations by a fuzzy distance. *Soft Computing*, *16*(6), 1035-1041.

Asady, B., and P. Mansouri. (2009). "Numerical solution of fuzzy linear system."International *Journal of Computer Mathematics 86*(1),151-162.

Babbar, N., Kumar, A., & Bansal, A. (2013). Solving fully fuzzy linear system with arbitrary triangular fuzzy numbers ({m,\ alpha,\ beta}). *Soft Computing*,*17*(4), 691-702.

Benner, P. (2004). Factorized solution of Sylvester equations with applications in control. *sign (H)*, *1* (2), 1-10.

Dafchahi, F. N. (2008). A New Refinement of Jacobi Method for Solution of Linear System Equations AX= b. *Int. J. Contemp. Math. Sciences*, *3*(17), 819-827.

Darouach, M. (2006). Solution to Sylvester equation associated to linear descriptor systems. *Systems & control letters*, *55*(10), 835-838.

Dehghan, M., & Hashemi, B. (2006a). Solution of the fully fuzzy linear systems using the decomposition procedure. *Applied Mathematics and Computation*,*182*(2), 1568-1580.

Dehghan, M., & Hashemi, B. (2006b). Iterative solution of fuzzy linear systems.*Applied Mathematics and Computation*, *175*(1), 645-674.

Dehghan, M., Hashemi, B., & Ghatee, M. (2006). Computational methods for solving fully fuzzy linear systems. *Applied Mathematics and Computation*,*179*(1), 328-343.

Dehghan, M., Hashemi, B., & Ghatee, M. (2007). Solution of the fully fuzzy linear systems using iterative techniques. *Chaos, Solitons & Fractals*, *34*(2), 316-336.

Dinagar, D. S., & Latha, K. (2013). Some types of type-2 triangular fuzzy matrices. *International Journal of Pure and Applied Mathematics*, 82(1), 21-32.

Dubois, D., & Prade, H. (1980). *Fuzzy sets and systems: theory and applications* (Vol. 144). Academic press.

Ezzati, R. (2011). Solving fuzzy linear systems. *Soft computing*, *15*(1), 193-197.

Ezzati, R., Khezerloo, S., & Yousefzadeh, A. (2012). Solving fully fuzzy linear system of equations in general form. *Journal of Fuzzy Set Valued Analysis*, *2012*.1-11.

Ezzati, R., Khezerloo, S., Mahdavi-Amiri, N., & Valizadeh, Z. (2014). Approximate Nonnegative Symmetric Solution of Fully Fuzzy Systems Using Median Interval Defuzzification. *Fuzzy Information and Engineering*, *6*(3), 331-358.

Fortin, J., Dubois, D., & Fargier, H. (2008). Gradual numbers and their application to fuzzy interval analysis. *Fuzzy Systems, IEEE Transactions on*,*16*(2), 388-402.

Friedman, M., Ming, M., & Kandel, A. (1998). Fuzzy linear systems. *Fuzzy sets and systems*, *96*(2), 201-209.

Fu, X., & Shen, Q. (2011). Fuzzy complex numbers and their application for classifiers performance evaluation. *Pattern Recognition*, *44*(7), 1403-1417.

Gao, J., & Zhang, Q. (2009). A unified iterative scheme for solving fully fuzzy linear system. *Global Congress on Intelligent Systems* . 431-435.

Ghanbari, R., & Mahdavi-Amiri, N. (2015). Fuzzy LR linear systems: quadratic and least squares models to characterize exact solutions and an algorithm to compute approximate solutions. *Soft Computing*, 19(1), 205-216.

Ghanbari, R., Mahdavi-Amiri, N., & Yousefpour, R. (2010). Exact and approximate solutions of fuzzy LR linear systems: new algorithms using a least squares model and the abs approach. *Iranian Journal of Fuzzy Systems*, *7*(2), 1-18.

Guo, X. B., & Jin, T. X. (2014). Fuzzy minimal solution of semi fuzzy linear matrix equations. *American Journal Of Mathematics And Mathematical Sciences*,*3*(2). 197-204.

Guo, X., & Shang, D. (2013a). Fuzzy Approximate Solution of Positive Fully Fuzzy Linear Matrix Equations. *Journal of Applied Mathematics*, *2013*.1-7.

Guo, X., & Shang, D. (2013b). Approximate Solution of LR Fuzzy Sylvester Matrix Equations. *Journal of Applied Mathematics*, *2013*.1-10.

Hogben, L. (2006). *Handbook of linear algebra*. CRC Press.

Jafari, H., Saeidy, M., & Vahidi, J. (2009). The Homotopy analysis method for solving fuzzy system of linear equations. *International Journal of Fuzzy Systems*, *11*(4), 308-313.

Kargar, R., Allahviranloo, T., Rostami-Malkhalifeh, M., & Jahanshaloo, G. R. (2014). A Proposed Method for Solving Fuzzy System of Linear Equations.*The Scientific World Journal*, *2014*.

Kaufmann, A., & Gupta, M. M. (1991). Introduction to fuzzy arithmetic: theory and applications. Arden Shakespeare.

Kumar, A., Babbar, N., & Bansal, A. (2012). A new computational method to solve fully fuzzy linear systems for negative coefficient matrix. *International Journal of Manufacturing Technology and Management*, *25*(1-3), 19-32.

Kumar, A., Bansal, A. and Neetu (2012). A new computational method for solving fully fuzzy linear systems of triangular fuzzy numbers. *Fuzzy Information and Engineering*, *4*(1), 63-73.

Kumar, A., Bansal, A., & Babbar, N. (2011a). Solution of fully fuzzy linear system with arbitrary coefficients. *International Journal of Applied Mathematics and Computation*, *3*(3), 232-237.

Kumar, A., Bansal, A., & Neetu, N. (2010). A method for solving fully fuzzy linear system with trapezoidal fuzzy numbers. *Iranian Journal of Optimization, 2*. 359-374.

Kumar, A., Neetu, & Bansal, A. (2011b). A new approach for solving fully fuzzy linear systems. *Advances in Fuzzy Systems*, *2011*, 1-8.

Lazim, and Hakimah (2010). A numerical method for solving fuzzy linear system. *Intelligent and Advanced Systems (ICIAS), 2010 International Conference*. 1-5.

Liu, H. K. (2010). On the solution of fully fuzzy linear systems. *International Journal of Computational and Mathematical Sciences*, *4*(1), 29-33.

Mansouri, P. and Asady, B. (2011). Iterative Methods for Solving Fuzzy Linear Systems. *Australian Journal of Basic and Applied Sciences, 5*(7), 1036-1049.

Matinfar, M., Nasseri, S. H., & Sohrabi, M. (2008). Solving fuzzy linear system of equations by using Householder decomposition method. *Applied Mathematical Sciences*, *51*, 2569-2575.

Minc, H. (1988). Non-negative matrices. *Wiley, New York.*

Ming, M., Friedman, M., & Kandel, A. (1997). General fuzzy least squares. *Fuzzy sets and systems*, *88*(1), 107-118.

Moore, R. E. (1979). *Methods and applications of interval analysis* (Vol. 2). Philadelphia: Siam.

Muzzioli, S., & Reynaerts, H. (2006). Fuzzy linear systems of the form A1x+ b1= A2x+ b2. *Fuzzy Sets and Systems, 157*(7), 939-951.

Najafi, H. S., & Edalatpanah, S. A. (2013). An improved model for iterative algorithms in fuzzy linear systems. *Computational Mathematics and Modeling*,*24*(3), 443-451.

Nasseri, S. H., & Zahmatkesh, F. (2010). Huang method for solving fully fuzzy linear system of equations. *The Journal of Mathematics and Computer Science*, *1*(1), 1-5.

Nasseri, S. H., Behmanesh, E., & Sohrabi, M. (2012). A new method, for system of fully fuzzy linear equations based on a certain decomposition of its coefficient matrix. *Annals of fuzzy Mathematics and Infromatics*.

Nasseri, S. H., Gholami, M., (2011). Linear system of equations with trapezoidal fuzzy numbers. *The Journal of Mathematics and Computer Science*, *3*(1), 71-79.

Nasseri, S. H., Sohrabi, M., & Ardil, E. (2008). Solving fully fuzzy linear systems by use of a certain decomposition of the coefficient matrix.*International Journal of Computational and Mathematical Sciences*, *2*(1), 140-142.

Nikuie, M., & Ahmad, M. Z. (2014). Minimal Solution of Singular LR Fuzzy Linear Systems. *The Scientific World Journal*, *2014*,1-7.

Otadi, M., & Mosleh, M. (2012). Solving fully fuzzy matrix equations. *Applied Mathematical Modelling*, *36*(12), 6114-6121.

Otadi, M., Mosleh, M., & Abbasbandy, S. (2011). Numerical solution of fully fuzzy linear systems by fuzzy neural network. *Soft computing*, *15*(8), 1513-1522

Radhakrishnan, S., Gajivaradhan, P., & Govindarajan, R. (2014) A New and Simple Method of Solving Fully Fuzzy Linear System,8(2), 193-199

Thamaraiselvi, A., & Santhi, R. (2015). R. On Intuitionistic Fuzzy Transportation Problem Using Hexagonal Intuitionistic Fuzzy Numbers. *International Journal of Fuzzy Logic Systems (IJFLS)*, *5*(1), 15-28.

Weibel, E. R. (1967). Introduction to counting principles. In *Quantitative Methods in Morphology/Quantitative Methoden in der Morphologie* (pp. 55-57). Springer Berlin Heidelberg.

Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.

Zhang, F. (2011). *Matrix theory: basic results and techniques*. Springer Science & Business Media.