BLOCK METHODS FOR DIRECT SOLUTION OF HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS USING INTERPOLATION AND COLLOCATION APPROACH

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Abstrak

Pelbagai permasalahan dalam situasi kehidupan nyata melibatkan kadar perubahan satu atau lebih pembolehubah tak bersandar. Kadar perubahan ini boleh diungkapkan dalam bentuk terbitan yang kemudiannya menjurus kepada pembentukan persamaan pembeza. Secara konvensional, masalah nilai awal bagi persamaan pembeza biasa peringkat tinggi diselesaikan dengan menurunkannya ke sistem persamaan pembeza peringkat pertama yang setara terlebih dahulu. Kemudian, kaedah berangka bersesuaian yang sedia ada bagi persamaan pembeza biasa peringkat pertama diguna untuk menyelesaikan persamaan yang terhasil. Walau bagaimanapun, pendekatan ini akan menambah bilangan persamaan lalu meningkat bebanan pengiraan yang boleh menjejaskan kejituan penyelesaian. Bagi mengatasi kelemahan ini, kaedah langsung dicadangkan. Malangnya, kebanyakan kaedah langsung sedia ada menganggar penyelesaian berangka pada satu titik sahaja pada satu masa. Kaedah blok kemudiannya diperkenalkan bertujuan untuk menganggar penyelesaian berangka pada beberapa titik serentak. Beberapa kaedah blok baharu yang menggunakan pendekatan interpolasi dan kolokasi untuk menyelesai masalahan nilai awal persamaan pembeza biasa peringkat tinggi secara langsung telah dibangunkan dalam kajian ini untuk meningkatkan kejituan penyelesaian. Dalam pembangunan kaedah ini, siri kuasa telah digunakan sebagai penyelesaian hampir kepada permasalahan persamaan pembeza biasa peringkat d. Siri kuasa diinterpolasi pada d titik sebelum dua titik terakhir dan terbitannya yang tertinggi dikolokasi pada semua titik grid bagi menerbitkan kaedah blok yang baharu. Di samping itu, sifat bagi kaedah baharu seperti peringkat, pemalar ralat, kestabilan-sifar, ketekalan, penumpuan dan rantau kestabilan mutlak turut dikaji. Kaedah baharu yang telah dibangunkan kemudiannya diaplikasi bagi menyelesai beberapa permasalahan nilai awal persamaan pembeza biasa peringkat tinggi. Keputusan berangka menunjukkan kaedah baharu menghasil kejituan yang lebih baik jika dibandingkan dengan kaedah sedia ada apabila menyelesai permasalahan yang sama. Oleh itu, kajian ini telah berjaya menghasilkan beberapa kaedah baharu bagi menyelesaikan masalah nilai awal persamaan pembeza peringkat tinggi.

Kata kunci: Interpolasi, kolokasi, kaedah blok, penyelesaian langsung, masalah nilai awal peringkat tinggi.

Abstract

Countless problems in real life situations involve rates of change of one or more independent variables. These rates of change can be expressed in terms of derivatives which lead to differential equations. Conventionally, initial value problems of higher order ordinary differential equations are solved by first reducing the equations to their equivalent systems of first order ordinary differential equations. Then, suitable existing numerical methods for first order ordinary differential equations will be employed to solve the resulting equations. However, this approach will enlarge the equations and thus increases computational burden which may jeopardise the accuracy of the solution. In overcoming the setbacks, direct methods were proposed. Disappointedly, most of the existing direct methods approximate the numerical solution at one point at a time. Block methods were then introduced with the aim of approximating numerical solutions at many points concurrently. Several new block methods using interpolation and collocation approach for solving initial value problems of higher order ordinary differential equations directly were developed in this study to increase the accuracy of the solution. In developing these methods, a power series was used as an approximate solution to the problems of ordinary differential equations of order d. The power series was interpolated at d points before the last two points while its highest derivative was collocated at all grid points in deriving the new block methods. In addition, the properties of the new methods such as order, error constant, zero-stability, consistency, convergence and region of absolute stability were also investigated. The developed methods were then applied to solve several initial value problems of higher order ordinary differential equations. The numerical results indicated that the new methods produced better accuracy than the existing methods when solving the same problems. Therefore, this study has successfully produced new methods for solving initial value problems of higher order ordinary differential equations.

Keywords: Interpolation, collocation, block method, direct solution, higher order initial value problems.

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CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Mathematical models in the field of science and engineering are usually developed to understand the physical phenomena. These models are always resulted to differential equations. Ross (1989) highlighted some of the problems that involved differential equations as follows:

1. The problem arising from determining the projectile motion, satellite, rocket or planet.
2. The problem of how to determine the charge or current in an electric circuit.
3. The study of chemical reactions.
4. The study of decomposition rate of radioactive substance or population growth rate.

The problems mentioned above obey certain scientific laws that involve rates of change of one or more quantities. Mathematically, these rates of change can be expressed by derivatives. When the problems are converted to mathematical equations they will form differential equations.

A differential equation can be defined as an equation containing an independent variable, a dependent variable and one or more derivative of the dependent variable with respect to the independent variable. In other words, a differential equation is an equation that contains the derivatives of one or more dependent variable with respect to one or more independent variable (Omar & Suleiman, 1999). It can be classified into two categories; *ordinary differential equations* (ODEs) and *partial differential equations* (PDEs).

An Ordinary differential equation is a differential equation in which the unknown function is a function of a single independent variable. On the other hand, a partial differential equation is a differential equation in which the unknown function is a function of two or more independent variables. A differential equation can further be classified according to the order and the degree of the differential equation. The *order* of a differential equation is the highest derivative included in a differential equation. The power to the highest derivative that occurs in the differential equation is known as *degree*. The below differential equations are some examples that will enhance the understanding of the terms defined above:

$$\frac{d^3y}{dx^3} - xy\left(\frac{dy}{dx}\right)^2 = 0 \tag{1.1}$$

$$\frac{d^4x}{dz^4} + 2\frac{d^3x}{dz^3} - \frac{dx}{dz} = \sin z \tag{1.2}$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial w} = v \tag{1.3}$$

$$\left(\frac{\partial^3 w}{\partial x^3}\right)^2 + \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial z} = 1. \tag{1.4}$$

Equations (1.1) and (1.2) are ODEs while equations (1.3) and (1.4) are PDEs. Equation (1.1) is of order 3 with degree 1 and equation (1.2) has an order 4 as well as degree 1. Meanwhile, equation (1.3) is having order and degree 1 while (1.4) is of order 3 with degree 2.

The d-th ODE can be generally denoted as

$$F(x, y(x), y^{'}(x), y^{''}(x), ..., y^{d-1}(x), y^{d}(x)) = 0 \tag{1.5}$$

where d is identified as the order of the differential equation, F is a continuous function, x is an independent variable and $y(x), y^{'}(x), ..., y^{d}(x)$ are the dependent variables. The canonical form of equation (1.5) is represented by

$$y^{d} = f(x, y(x), y^{'}(x), ..., y^{d-1}(x)) \tag{1.6}$$

In order for equation (1.6) to have a unique solution certain conditions need to be imposed at the initial point as follows

$$y^{s}(a) = \eta_{s}, s = 0(1)d-1 \tag{1.7}$$

Equation (1.6) together with (1.7) is called the d-th order Cauchy problems or the d-th order initial value problems (IVPs). When the conditions are given at more than the initial point, it is then called boundary value problems (BVPs).

Equation (1.5) is said to be *linear* if it can be expressed in the following form

$$a_{0}(x)y^{(d)} + a_{1}(x)y^{(d-1)} + ... + a_{d-1}(x)y' + a_{d}(x)y = b(x)$$

where a_{0} is not identically zero. An ordinary differential equation that is not linear is known as *nonlinear* ordinary differential equation (Omar, 1999). In the examples given previously, equation (1.1) is a nonlinear ODE and equation (1.2) is a linear ODE.

1.2 Existence and Uniqueness Theorem

The solutions to higher order ODEs as in (1.6) can be obtained by using two approaches:

1. Reduction to systems of first order ODEs.
2. Direct methods, i.e. without reduction.

Two theorems are going to be stated: Theorem (1.1) discusses about the existence and uniqueness of first order ODEs and Theorem (1.2) guarantees the uniqueness of higher order ordinary differential equations.

Theorem 1.1 (Henrici, 1962): Let $f(x,y)$ be a real function and continuous for all points (x,y) in the region D defined by $x \in [a,b]$, $y \in (-\infty,\infty)$, containing initial values (x_0, y_0) where *a, b* are finite. Let there exists a constant *L* called Lipschitz constant such that for any $x \in [a,b]$ and for any pairs y_1, y_2 for which (x, y_1), (x, y_2) are both in $D \left| f(x,y_1) - f(x,y_2) \right| \le L \left| y_1 - y_2 \right|$. Then for any given number $x \in [a,b]$, the first order initial value problem has a unique solution $y(x)$. The proof of Theorem 1.1 can be seen in Henrici (1962).

Theorem 1.2 (Wend, 1969): Let D be the region defined by the inequalities $x_0 \le x \le x_0 + a$, $\left| s_j - c_j \right| \le b$, $j = 0,1..,.d-1 \, (a > 0, b > 0)$. Suppose that

$f(x, c_0, ..., c_{d-1}) > 0$ and $f(x, s_0, ..., s_{d-1})$ is defined in $\Re$ and in addition

1. f is non-negative and non-decreasing in each of $x, s_0, ..., s_{d-1}$ in $\Re$.
2. $f(x, c_0, ..., c_{d-1}) > 0$ for $x_0 \le x \le x_0 + a$.
3. $c_k \ge 0$, $k = 1(1)d-1$.

Then the *d*-th order initial value problem has a unique solution in $\Re$. The proof of Theorem 1.2 can be found in Wend (1969).

Shortly after the introduction of calculus, it was discovered that not all ODEs can be solved analytically. In order to overcome this great challenge, numerical methods of solving ODEs were introduced with the aim of providing approximate solution to the ODEs. Numerical method is an approach by which difficult problems in mathematics are solved on a computer (Gerald & Wheatley, 1994). Some of the most famous numerical methods are the Euler method (Atkinson, 1989), linear multistep methods (Hairer, Norsett & Wanner, 1993) and Runge-Kutta methods (Lambert, 1991).

There are basically two types of numerical methods: *single step method* and *multistep method*. In a single step method, information computed from the previous x_n are used to approximate the numerical solution at x_{n+1}. Furthermore, the approximate value y_{n+1} to the solution $y(x_{n+1})$ at the grid point x_{n+1} is made if only the values of x_n, y_n and the step-size are known. Multistep method, on the other hand, requires the information used from the previous steps for the approximation of solution at the current step (Omar, 1999). Multistep method will be employed in this work and therefore it will be described in details.

1.3 Multistep Method

A multistep method is a computation method for determining the numerical solution of initial value problems of ODEs which form a linear relation between y_{n+j} and f_{n+j}, $j = 0(1)k$. This is a method which requires starting values from several previous steps for the approximation of solution at the current step. For instance, in the method k-step, the values of y computed at the previous k-step, that is $x_{n+j} = x_n + jh, \quad j = 0(1)k-1$ are used to calculate y_{n+k} (Omar, 1999).

The general form of linear multistep method is given as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^d \sum_{j=0}^{k} \beta_j(x) f_{n+j} \tag{1.8}$$

where

$$f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j}, ..., y_{n+j}^{d-1}),$$
$$y_{n+j} = y(x_{n+j}), j = 0(1)k,$$

d is the order of the differential equation, α_j and β_j are real constants where both α_0 and β_0 are not zero.

According to Lambert (1988), equation (1.8) can be represented by

$$\rho(r) = \sum_{j=0}^{k} \alpha_j y_{n+j}$$

and

$$\sigma(r) = h^d \sum_{j=0}^{k} \beta_j f_{n+j}.$$

$\rho(r)$ and $\sigma(r)$ are known as the *first* and *second characteristics polynomials* respectively.

Definition1.1 (Atkinson, 1989): Linear multistep method (1.8) is said to be implicit if $\beta_k \neq 0$, that is, the approximate solution at x_{n+k} which is y_{n+k} reflects on the both sides of (1.8). On the other hand, (1.8) is explicit if $\beta_k = 0$, that is, the approximate value of y_{n+k} can directly be determined in terms of y_{n+j}, f_{n+j} $j=0(1)k-1$. An implicit method requires the determination of initial values for $y_{n+j}, y'_{n+k}, ...y^{d-1}_{n+k}$ in terms of $f(x_{n+k}, y_{n+k}, ...y^{d-1}_{n+k})$.

Definition 1.2 (Adesanya, 2012): Numerical method for solving differential equation is based on principle of discretization in which the approximate solutions are evaluated at each grid point. We consider the sequence of point $\{x_n\}$ in the interval $I=[a,b]$ defined by $a=x_0<x_1<x_2,...,<x_n=b$; $h_i=x_{i+1}-x_i$, $i=0(1)n-1$. The parameter h_i is called the *step-size*. If the solution to linear multistep method is $y(x)$ approximated by y_{n+i}, $i=0(1)k$, any numerical method that computes y_{n+i} by using the information at $x_i, x_{i+1},..., x_{n+k}$ is called a $k-$ *step method*.

The development of approximate solution of ordinary differential equation led to comprehensive research into the analysis of the basic properties of numerical methods which involves the convergence and stability of the approximate solutions. These are defined below:

Definition 1.3 (Gear, 1971): The linear operator *L* associated with (1.8) is given by

$$L[y(x);h]=\sum_{j=0}^{k}[\alpha_j y(x+jh)-h^d\beta_j y^d(x+jh)] \tag{1.9}$$

where *d* is the order of the differential equation and *y(x)* is an arbitrary function that is continuous and differentiable on [a, b]. Using the Taylor series at point *x* to expand $y(x+jh)$ and $y^d(x+jh)$ gives

$$L[y(x);h] = C_0 y(x) + C_1 hy'(x) + ... + C_q h^q y^{(q)}(x) + C_{q+1} h^{q+1} y^{q+1}(x) + ...$$

where

$$C_0 = (\alpha_0 + \alpha_1 + \alpha_2 + ... + \alpha_k)$$
$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + ... + k\alpha_k$$
$$\vdots$$
$$C_q = \frac{1}{q!}(\alpha_1 + 2^q \alpha_2 + 3^q \alpha_3 + ... + k^q \alpha_k) - \frac{1}{(q-d)!}(\beta_1 + 2^{q-d} \beta_2 + ... + k^{q-d} \beta_k)$$

for $q = 2,3,...$

The linear multistep method (1.8) is said to have order p if

$C_0 = C_1 = C_2 = ... = C_p = C_{p+1} = ...C_{p+(d-1)} = 0,\ C_{p+d} \neq 0$. The first coefficient that does not vanish C_{p+d} is known to be the *error constant* and $C_{p+n} h^{p+d} y^{(p+d)}(x_n)$ is called the *principal local truncation error*.

Definition 1.4 (Lambert, 1973): A linear multistep method (1.8) is *consistent* if the following conditions stated below are satisfied:

1. $p \geq 1$

2. $\sum_{j=0}^{k} \alpha_j = 0$

Definition 1.5 (Adesanya, 2012): Linear multistep (1.8) is said to be *zero-stable* if no root of the first characteristic polynomial $\rho(r) = \det[rA^0 - A^1]$ is having a modulus greater than one and every root of modulus one is simple, where A^0 and A^1 are the coefficients of y-function in equation (1.10). The roots with modulus one is known as the principal roots and the other roots are called spurious roots.

Definition 1.6 (Lambert, 1973): A linear multistep method (1.8) is said be *absolutely stable* in $\Re$ region of the complex plane if, for all $\bar{h}(\lambda h) \in \Re$, all roots of the stability polynomial $\pi(r, \bar{h})$ associated with the method satisfy $|r_s| < 1, s = 1,2,...,k$ and $|r_s| < |r_1|,\ s = 2,3,...,k$.

Theorem 1.3 (Henrici, 1962): A linear multistep method (1.8) is said to be convergent if it is consistent and zero-stable.

Definition 1.7 (Lanczos, 1965): Collocation is the evaluation of the differential system of the basis or trial function at some selected grid points while interpolation is the evaluation of the approximate solution also at some selected grid points. This collocation method is widely considered as a means of providing numerical solution to ordinary differential equations.

1.4 Block Method

Block method was first proposed by Milne according to Olabode (2007). This method can be seen as a set of linear multistep method simultaneously applied to initial value problems and then combined to yield a better approximation. In other words, the set of new values derived by each application of the method is known as *block*. That is, at each iteration of the algorithm, the values of $y_{n+1}, y_{n+2}......., y_{n+k}$ are computed simultaneously.

Basically, there are two types of block methods, namely one-step and multistep block methods. In one-step block method, the value of the new block $y_{n+i} ...i = 1(1)k$

is derived from the information at y_n. On the other hand, when the values of the previous blocks are used to compute the next block, it is called a multistep block (Omar, 1999). In this work, block method of the form

$$A^0 Y_N = \sum_{i=0}^{d-1} h^i A^{i+1} Y_{N-1}^{(i)} + h^d \sum_{i=0}^{1} B^i F_{N-i} \tag{1.10}$$

is adopted to simultaneously generate the numerical solution at all the selected grid points. In equation (1.10), d is the order of differential equation, A^{i+1} and B^i are both squared matrices,

$Y_N = [y_{n+1}, y_{n+2}..., y_{n+k}]^T$, $Y_{N-1}^{(i)} = [y_{n-k+1}^{(i)}, y_{n-k+2}^{(i)}.., y_n^{(i)}]^T$, $F_N = [f_{n+1}, f_{n+2}..., f_{n+k}]^T$ and

$F_{N-1} = [f_{n-k+1}, f_{n-k+2}..., f_n]^T$.

1.5 Problem Statement

Higher order ordinary differential equations are conventionally solved by a reduction to a system of first order ordinary differential equations and then suitable numerical method for first order would be used to solve the system (Fatunla, 1988; Lambert, 1973; Awoyemi, 2001; Jator, 2007; Brugnano & Trigiante, 1998). This method computes the numerical solution at one point at a time. However, the major setbacks for this method are computational burden which affects the accuracy of the method in terms of the error. In addition, the computer program to examine the accuracy of the method is always found to be complicated (Awoyemi & Kayode, 2004).

In order to overcome these challenges and bring improvement on numerical method, scholars such as Olabode (2009a, 2009b), Mohammed (2010), Jator (2007), Omar and Suleiman (1999), Badmus and Yahaya (2009) and Omar (2004) developed block methods for direct solution of higher order ordinary differential equations whereby the accuracy of the methods is better than when it is reduced to system of first order ODEs.

In the work done by Olabode (2009a, 2009b) and Mohammed (2010), interpolation and collocation method was used in the derivation of blocks with step-length k=5 and k=6 for solving third and fourth order ordinary differential equations. The points of interpolation and collocation were taking at some selected points within the specified interval. It is noted that the accuracy of the methods were not efficient in terms of error.

Moreover, Adesanya et al (2012) used interpolation and collocation method for the development of block predictor-corrector of order six that solves second order ordinary differential equations. However, the implementation of predictor-corrector method causes computational burden which affects the accuracy of the method in terms of error (James et al., 2013).

Furthermore, in the method proposed by Badmus and Yahaya (2009), a uniform order six block method having a step-length k=5 was developed whereby the points of interpolation were made at the beginning of the interval. In the problems solved to examine the accuracy of the method, it is observed that the errors are too large. A family of implicit block integrators was also developed by Awari et al. (2014) using a step-length k=4(1)6 whereby collocation was only made at one point. The accuracy of the methods was very low. The accuracy of the methods can be improved if more points of collocation are considered.

Based on the drawbacks of these previous works, we attempt to increase the accuracy of the method by using different strategy which is interpolating at the d points before the last two points within the interval and collocating all grid points.

1.6 Objectives of the Research

The main objective of this research is to develop a set of block methods for direct solution of higher order initial value problems of ordinary differential equations. In order to achieve this, the following needs to be done.

1. To develop a continuous implicit schemes that solve second, third and fourth order ODEs by interpolating and collocating within the selected grids points.
2. To develop new block methods using the developed continuous implicit schemes for solving second, third and fourth order ODEs directly.
3. To establish the basic properties of the new block methods. That is, zero-stability, order, consistency, convergence, error constant and region of absolute stability.
4. To compare the results generated from the new block methods with the existing methods in terms of error.

1.7 Significance of the Study

New block methods for solving higher order ODEs directly without the rigor of developing separate predictors are developed. The methods are of step length $k = (d+1)(1)8$ for second, third and fourth order ODEs, where d is the order of differential equation. This increases the accuracy of the method in terms of error significantly. The methods are more accurate when compared with the existing methods for solving second, third and fourth order ODEs directly.

1.8 Limitation of the Study

This study considers comparison in terms of error as each runtime per program solution varies with respect to computer model and hence was not considered in this work. Furthermore, generalization of the method to d-th order ODEs is impossible because the point of interpolation depends on the order of differential equation. As the order of the differential equation increases the point of interpolation also increases. Therefore, because of this difficulty in generalization of the method and for the purpose of comparing with the existing methods in terms error, our scope of study in this research is limited to the steplength $k=(d+1)(1)8$ for single equation of linear and nonlinear second, third and fourth order ODEs.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Most of the modeled problems in ordinary differential equations do not have analytical solutions; therefore approximate solutions to the problems need to be considered. These approximations are computed through numerical methods which results in a discretization of solution. Discretization is when solution is represented by function value on the grid points which are mutually connected by interpolation of function. In this chapter, some literatures on the existing methods will be reviewed for solving ordinary differential equations

2.2 Review of the Existing Methods for solving Ordinary Differential Equations

In most application, higher order ordinary differential equations are solved by a reduction to a system of first order ordinary differential equation of the form

$$\begin{aligned}
&y_1 = y \\
&y_1^{'} = y^{'} = y_2 \\
&y_2^{'} = y^{''} = y_3 \\
&\vdots \\
&y_{d-1}^{'} = y^{(d-1)} = y_d \\
&y_d^{'} = y^{d} = f(x, y_1, y_2, \ldots y_{d-1}), \quad y(a) = y_0 \quad a \le x \le b
\end{aligned}$$

Then any suitable numerical methods would be used to solve the resulting equation. This approach is extensively discussed by scholars such as Spiegel (1971), Lambert (1973), Goult et al. (1973), Lambert and Watson (1976), Jain et al. (1984), Fatunla

(1988), Sarafyan (1990), Awoyemi (1999), Jator (2001) and Jaun (2002). It was noticed that this reduction process has a lots of setbacks such as difficulties in writing computer program for the method, computational burden which affects the accuracy of the method in terms of error and wastage of human effort.

Therefore, it will be appropriate and more efficient if direct method of solving (1.6) is employed as suggested by Dahlquist (1959), Henrici (1962), Chakravati and Worland (1971), Jeltsch (1976), Hall and Suleiman (1981), Twizeland and Khaliq (1984), Fatunla (1988), Taiwo and Onumanyi (1991), Awoyemi (1998, 1999), Onumanyi et al. (2001), Omar (2004), Kayode (2004, 2005, 2008), Adesanya et al. (2009), Sharp and Fine (1992) and Dormand et al. (2003). Much and considerable attention have been dedicated to solving higher order ordinary differential equations of the form (1.6) directly without being reduced to system of first order ordinary differential equation. For instance, Hairer and Wanner (1976) proposed Nystrom type method in which some additional conditions are being stated for determining the parameters of the methods. Explicit and implicit Runge-Kutta Nystrom type methods were independently developed by Chawla and Sharma (1981).

2.2.1 Predictor-Corrector Method

In predictor-corrector method, an explicit method is usually meant for predictor step while an implicit method for the corrector step. The development of Linear Multistep Method (LMM) through the predictor-corrector mode has been carefully considered by scholars such as Brown (1977), Lambert (1988), Awoyemi (1999, 2001, 2003, 2005), Kayode (2009), Udoh et al. (2007), Olabode (2007), Onumanyi et al. (2001) among others. These researchers independently suggested a multi-derivative Linear Multistep Method (LMM) which was implemented in a predictor-corrector mode where Taylor series algorithm is used to supply the starting values.

A P-stable linear multistep method for numerical solution of third order ordinary differential equations was developed by Awoyemi (2003), where the strategy of implementation was through predictor-corrector mode. The method is associated with computational burden due to the evaluation of many functions per iteration in both predictor and corrector method. Awoyemi (1998) stated the advantage of continuous linear multistep method over the discrete methods such that; it gives a simplified form of coefficients for additional analytical work at different points that guarantee easy approximation of solutions at all the interior points of the interval.

Two-step implicit method of order four for the solution of second order ODEs was developed by Omolehin et al. (2003) but the accuracy of the method is not efficient in terms of error. In order to improve the accuracy of the results generated by Omolehin et al. (2003), Kayode and Awoyemi (2005) presented a five-step method for solving second order directly with the use of power series method. Kayode and Adeyeye (2011) and Kayode (2011) also developed a three-step hybrid implicit method and a class of one-point zero-stable continuous hybrid methods respectively in which the implementation was through predictor-corrector whereby the accuracy of the methods were found better than Awoyemi and Kayode (2005). In addition, an efficient zero-stable numerical method for direct solution of fourth order ODEs was developed by Kayode (2008) which was also implemented in predictor-corrector scheme.

Although the methods implemented in a predictor-corrector mode produced good results, the process is more costly to implement. Furthermore, the subroutines are very difficult to write because of the special techniques required to supply the starting values which leads to a more human effort (Jator, 2007; Olabode, 2007). The major disadvantage of methods being implemented in predictor-corrector mode has been the use of predictors of lower order for the implementation of the scheme (Kayode & Adeyeye, 2013).

2.2.2 Block Method

In order to overcome the difficulties mentioned in predictor-corrector method, block method was developed (Fatunla, 1991). This method computes the discrete method at more than one grid point simultaneously. According to Olabode (2007) and Ehigie (2011) block method was firstly proposed by Milne (1953) who advocated the use of block as a means of getting a starting value for predictor-corrector algorithm and later adopted as a full method (Anake et al., 2012; Adesanya et al., 2013).

Authors who previously worked on block method for solving ODEs are Rosser (1967), Chu and Hamilton (1987), Fatunla (1991), Voss and Abbas (1997) and Omar and Suleiman (1999) to mention a few. They stated that numerical solutions on block method are produced with less computational efforts when compared with non-block method (Majid, 2004). This is because evaluation is made at more than one point simultaneously. Block Backward Difference Formula (BBDF) for the solution of stiff first order ODEs was proposed by Ibrahim et al. (2007) where two points were computed simultaneously using x_{n-1} and x_n as the back values in each block. Furthermore, Ibrahim et al. (2008) developed a fixed coefficients block backward differentiation formulas for the numerical solution of stiff ODEs. Likewise, convergence of the 2-point block backward differentiation formulas was also considered by Ibrahim et al. (2011). All these developed block methods could only cater for first order ODEs.

Fatunla (1991, 1994) proposed block method for the solution of second order ODEs. Omar and Suleiman (1999, 2003) later developed implicit and explicit parallel block methods for the solution of higher order ODEs through numerical integration where Newton backward difference is used as an interpolation polynomial. The accuracy of the methods in terms of error, however, is not encouraging. Furthermore, a parallel 3-point implicit block method for the solution of second order ODEs was proposed by Omar (2004) but the method is of lower accuracy. Ismail et al. (2009) adopted this method to develop a 3-point implicit and explicit block method for the solution of special second order ODEs. A 3-point implicit block method for the solution of first order ODEs based on Newton backward divided difference formulae was derived by Majid et al. (2006).

Jator (2007) and Jator and Li (2009) proposed a five-step and four-step self-starting methods respectively where continuous linear multistep method to obtain a block is adopted for the direct solution of second order ODEs. The accuracy of the methods can be improved if the step-length increases. A family of implicit uniformly accurate order block integrator for the solution of second order ODEs was examined by Awari et al. (2014). The accuracy of the methods developed are very low. A uniform order six block method of lower accuracy for solving general second order ODEs was developed by Badmus and Yahaya (2009). However, an improvement on the accuracy of the method can be achieved if another strategy of interpolating is considered.

Furthermore, in the work carried out by Yahaya and Sagir (2013), an order five implicit three-step block method with one off-grid point at f – function was examined. The results produced when the method was applied to second order ODEs were not efficient in terms of error. Mohammed, Jiya and Mohammed (2010) proposed the development of a class of six-step block method for the solution of second order ODEs directly. The method developed depends on the points of interpolation which leads to many functions to evaluate that resulted to the lower accuracy of the method (Kayode, 2008a). Furthermore, the research article entitled three-point block methods for direct integration of general second order ODEs was carried out by Ehigie, Okunuga and Sofoluwe (2011). The accuracy of the method is expected to be better when more points are considered. Adesanya et al. (2012) developed an order six block predictor-corrector method for direct solution of second order ODEs. The method is associated with much computational burden that reduced the accuracy of the method in terms of error. A two-step block method for solving second order ODEs was proposed by Adesanya et al. (2009). Higher accuracy can be accomplished if more step-length is considered.

Bolarinwa et al. (2012) developed a one-step implicit hybrid block method using collocation method for the solution of third order ODEs where three sub-step methods were considered. Bolaji et al. (2012) proceeded by developing a four-step implicit hybrid block for direct solution of the same third order ODEs. The method developed is of lower accuracy. The numerical solution of third order ODEs using an accurate block hybrid collocation method was examined by Yap et al. (2014). Moreover, Adesanya et al. (2012) developed a new block predictor-corrector algorithm for the solution of third order ODEs. The method resulted to more functions to be evaluated per iteration which in most cases affects the accuracy of the method in terms of error. Likewise, researchers like Olabode and Yusuph (2009) and Olabode (2009a, 2013) proposed block methods for solving third order ODEs using multistep collocation approach, the results generated are not efficient in terms of error. Additionally, Olabode (2009b) and Mohammed (2010) developed six-step block methods for solving fourth order ordinary differential equations whereby the accuracy of the methods are not encouraging. The accuracy could be better if more points were considered.

2.2.3 Collocation Method

The method of collocation was firstly introduced by Kantorovich in 1934 according to Faire and Meade (1989). The method was originally introduced for the solution of PDEs in two variables. The collocation method was considered by Frazer et al.

(1988) for the solution of ODEs and how it would be applied to the solution of ODEs. The use of collocation by polynomial as the roots of orthogonal polynomial rather than at the equidistant points was advocated by Lansczos in 1973. This type of collocation method is known as orthogonal collocation method. That is, it uses collocation at zeros of some orthogonal polynomial to convert the PDEs to a set of ODEs. The solution of ODEs, PDEs and integral equations through Chebyshev orthogonal collocation was described by Fox and Parker (1968).

Ever since then, research in the area of developing methods for the solution of ODEs has greatly improved. Awoyemi (2003) described collocation as now the most essential numerical process for obtaining continuous methods for the solution of ODEs. It has been a great interest to scholars in making use of power series as the basis function for the development of numerical methods. Power series method was adopted by Awoyemi (2003) to develop a numerical method for the solution of initial value problem of ODEs.

Researchers such as Yahaya (2007), Awoyemi et al. (2009), Kayode (2008, 2011), Olabode (2007, 2009b), Jator (2007), Mohammed (2010), Olabode (2009a, 2009b); Adesanya (2012), Adesanya (2013), James et al. (2013), Awari et al. (2014), Awoyemi et al. (2014), Sagir (2014) and Odekunle et al. (2014) adopted a power series polynomial for the generation of a continuous linear multistep method.

The highlights in the literature review is shown in the Table 2.1

Table 2.1

The Highlights in Literature Review

Authors	**Methods**	**Advantages**	**Disadvantages**
Awoyemi (1998, 1999, 2003)	Developed a class of continuous methods, a class of continuous stormer-cowell type methods for second order ODEs and a P-stable linear multistep method for third order ODEs which were implemented in predictor – corrector mode	The method solve second and third order ODEs directly	The methods are associated with much computational burden
Omolehin et al. (2003)	Developed a two-step Implicit method of order four using multistep collocation	The method solves second order ODEs without going through the process of reduction	The accuracy of the method is not efficient in terms of error. This was improved by Kayode and Awoyemi (2005)

Table 2.1 continued

Kayode and Awoyemi (2005)	Proposed a five-step multistep collocation method for solving second order ODEs which was implemented in predictor-corrector mode	The method solves second order ODEs directly	The method resulted to more functions to evaluate per step that leads to computational burden
Kayode and Adeyeye (2011)	Developed a class of hybrid implicit methods for the solution of second order ODEs which was also implemented in predictor-corrector mode.	The method solves second order ODEs without reduction method	The appearance of hybrid points at f–function increases the computational burden and this leads to lower accuracy of the method
Kayode (2008, 2011)	Developed an efficient zero-stable numerical method and a class of one-point zero stable continuous hybrid methods using predictor-corrector method	The methods directly applied to fourth and second order ODEs	The method requires much human effort compared to non-predictor-corrector method

Table 2.1 continued

Omar and Suleiman (1999, 2003, 2005)	Developed implicit and explicit parallel block methods using numerical integration	The methods solve higher order ODEs directly	The accuracy of the methods in terms of error is not encouraging
Ibrahim et al. (2007)	Developed a 2-point block method for solving stiff first order ODEs through numerical integration using backward difference formula as an interpolation polynomial	The method solves only first order ODEs	The method cannot solve higher order ODEs
Ismail et al. (2009)	Developed a 3-point implicit and explicit block method using linear difference operator	The method solves special second order ODEs directly	The accuracy of the method in terms of error is not encouraging

Table 2.1 continued

Majid et al. (2006)	Developed 3-point block method using numerical integration	The method solves first order ODEs	The method cannot solve higher order ODEs
Yahaya and Badmus (2009)	Developed a uniform order six block method for direct solution of second order ODEs	The method solves second order ODEs directly	The accuracy of the method is not efficient in terms of error.
Jator (2007)	Proposed a five-step self-starting block method for the solution of second order ODEs using multistep collocation method	The method does not associate with computational burden	The accuracy of the method can be better if the step-length increases
Jator and Li (2009)	Developed a four-step self-starting block method for second order ODEs using multistep collocation	The method solves second order ODEs directly	Lower accuracy of the method

Table 2.1 continued

Olabode (2009b)	Developed a six-step block for solving fourth order ODEs using multistep collocation	The method does not require more functions to evaluate per step	The method is of lower accuracy
Awari et al. (2014)	Developed a family of implicit uniformly accurate order block integrator	The methods directly solve second order ODEs	The accuracy of the methods developed are very low
Yahaya and Sagir (2013)	Proposed an order five implicit three-step block method with one off-grid point at f – function using multistep collocation method.	The method was applied to second order ODEs without reduction to system of first order	The method is of lower accuracy

Table 2.1 continued

Mohammed et al. (2010)	Developed a class of six-step block method with the use of interpolation of power series approximation approach	The method solves second order ODEs directly	The accuracy of the method is very low
Ehigie et al. (2011)	Developed a three-point block method for direct integration of general second order ODEs	The method is without reduction to system of first order	Accuracy of the method can be improved when more points are considered
Adesanya et al. (2009, 2012)	Proposed a two-step block method and an order six block predictor-corrector method using interpolation and collocation method	The methods solve second order ODEs directly	Higher accuracy can be accomplished if higher step-length is considered. The method associated with much computational burden that affected its accuracy

Table 2.1 continued

Mohammed (2010)	Proposed a six-step block method for the solution of fourth order ODEs	The method solves fourth order ODEs directly	The method is not efficient in terms of error.
Bolaji et al. (2012)	Developed one-step and four-step implicit hybrid block method using collocation method for solving third order ODEs.	The methods directly solve third order ODEs	The methods are of lower accuracy.
Yap et al. (2014)	Proposed a numerical solution of third order ODEs using an accurate block hybrid collocation method	The solution is without reduction to system of first order	Higher step-length will produce a better accuracy
Olabode and Yusuph (2009) and Olabode (2009a, 2013)	Proposed block methods for solving third order ODEs using multistep collocation approach	The rigor of developing separate predictor is not involved	The results generated are not efficient in terms of error

Table 2.1 continued

Awoyemi (2003), Yahaya (2007), Kayode (2008, 2009), Olabode (2007, 2009a, 2009b), Jator (2007), Mohammed (2010), Adesanya et al. (2013) and Odekunle et al. (2014)	Adopted power series for the development of multistep collocation method	It is used as a starting polynomial which helps in generating multistep collocation method.	The order of the power series approximate depends on the number of interpolation and collocation points

CHAPTER THREE

DEVELOPING BLOCK METHODS FOR SOLVING SECOND ORDER ODEs DIRECTLY

3.1 Introduction

In this chapter, the development of block methods with step-length $k = 3(1)8$ using interpolation and collocation approach for solving second order initial value problems of ODEs is discussed.

3.2 Three–Step Block Method for Second Order ODEs

In this section, the derivation of a three–step block method and the establishment of its properties are considered.

3.2.1 Derivation of Three–Step Block Method for Second Order ODEs

We consider power series of the form

$$y(x) = \sum_{j=0}^{k+2} a_j x^j \tag{3.2.1.1}$$

as an approximate solution to the general second order problems

$$y'' = f(x, y, y'); y(x_0) = y_0, y'(x_0) = y'_0 \tag{3.2.1.2}$$

where the step-length *k* in equation (3.2.1.1) is 3. The first and second derivatives of (3.2.1.1) are

$$y'(x) = \sum_{j=1}^{k+2} j a_j x^{j-1} \tag{3.2.1.3}$$

$$y''(x) = \sum_{j=2}^{k+2} j(j-1) a_j x^{j-2} = f(x, y, y') \tag{3.2.1.4}$$

Equation (3.2.1.1) is interpolated at the points $x = x_{n+i}, i = 0{,}1$ while equation (3.2.1.4) is collocated at the points $x = x_{n+i}, i = 0(1)3$ (refer to Figure 3.1), where C and I represent collocation and interpolation points respectively.

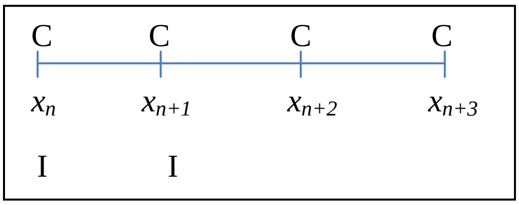

Figure 3.1. Three–step interpolation and collocation method for second order ODEs.

As a result, we will get

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \qquad (3.2.1.5)$$

In order to find the values of *a's* in (3.2.1.5), Gaussian elimination method is employed. The values of *a's* are given below:

$$\begin{aligned} a_0 = \quad & y_n + \frac{x_n^2 f_n}{2} + \frac{x_n^5}{120h^3}(f_n - 3f_{n+1} + 3f_{n+2} - f_{n+3}) + \frac{x_n}{36h}(36y_n - 36y_{n+1} + \\ & 11x_n^2 f_n - 18x_n^2 f_{n+1} + 9x_n^2 f_{n+1} - 2x_n^2 f_{n+3} + \frac{x_n^4}{24h^2}(2f_n - 5f_{n+1} + 4f_{n+2} - f_{n+3}) \\ & + \frac{97h}{360}x_n f_n + \frac{19h}{60}x_n f_{n+1} - \frac{13h}{120}x_n f_{n+2} + \frac{h}{45}x_n f_n \end{aligned}$$

$$\begin{aligned} a_1 = \quad & \frac{13h}{120}f_{n+2} - \frac{19h}{60}f_{n+1} - \frac{97h}{360}f_n - \frac{h}{45}f_n - x_n f_n - (y_n - y_{n+1}) + \frac{11}{12h}x_n^2 f_n - \\ & \frac{3}{2h}x_n^2 f_{n+1} + \frac{3}{4h}x_n^2 f_{n+2} - \frac{1}{6h}x_n^2 f_{n+3} - \frac{x_n^4}{24h^3}(f_n - 3f_{n+1} + 3f_{n+2} - f_{n+3}) - \\ & \frac{x_n^3}{6h^2}(2f_n - 5f_{n+1} + 4f_{n+2} - f_{n+3}) \end{aligned}$$

$$a_2 = \frac{1}{2} f_n + \frac{x_n}{12h}(11 f_n - 18 f_{n+1} + 9 f_{n+2} - 2 f_{n+3}) + \frac{x_n^3}{12h^3}(f_n - 3 f_{n+1} + 3 f_{n+2} - f_{n+3}) + \frac{x_n^2}{4h^2}(2 f_n - 5 f_{n+1} + 4 f_{n+2} - f_{n+3})$$

$$a_3 = -(\frac{11}{36h} f_n - \frac{1}{2h} f_{n+1} + \frac{1}{4h} f_{n+2} - \frac{1}{18h} f_{n+3} - \frac{x_n}{6h^2}(2 f_n - 5 f_{n+1} + 4 f_{n+2} - f_{n+3}) - \frac{x_n^2}{12h^3}(f_n - 3 f_{n+1} + 3 f_{n+2} - f_{n+3})$$

$$a_4 = \frac{1}{24h^2}(2 f_n - 5 f_{n+1} + 4 f_{n+2} - f_{n+3}) + \frac{x_n}{24h^3}(f_n - 3 f_{n+1} + 3 f_{n+2} - f_{n+3})$$

$$a_5 = -\frac{1}{120h^3}(f_n - 3 f_{n+1} + 3 f_{n+2} - f_{n+3})$$

Substituting the values of *a*'s into equation (3.2.1.1) and on simplifying, gives a continuous linear multistep method of the form:

$$y(x) = \sum_{j=0}^{k-2} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (3.2.1.6)$$

Using the transformation $z = \frac{x - x_{n+k-1}}{h}$, where k=3 implies

$$x = zh + x_n + 2h \qquad (3.2.1.7)$$

Substitute (3.2.1.7) into (3.2.1.6) and on simplifying gives

$$\begin{aligned}
\alpha_0(z) &= -1 - z \\
\alpha_1(z) &= 2 + z \\
\beta_0(z) &= \frac{1}{1080}(90 + 69z + 30z^3 - 9z^5) \\
\beta_1(z) &= \frac{1}{120}(100 + 122z - 10z^3 + 5z^4 + 3z^5) \\
\beta_2(z) &= \frac{1}{360}(30 + 159z + 180z^2 + 30z^3 - 30z^4 - 9z^5) \\
\beta_3(z) &= \frac{1}{360}(-8z + 20z^3 + 15z^4 + 3z^5)
\end{aligned} \qquad (3.2.1.8)$$

Evaluating (3.2.1.8) at the non-interpolating points. i.e at z =0 and 1 produces the following schemes

$$y_{n+2} = 2y_{n+1} - y_n + \frac{h^2}{12}(f_{n+2} + 10f_{n+1} + f_n) \quad (3.2.1.9)$$

$$y_{n+3} = 3y_{n+1} - 2y_n + \frac{h^2}{12}(f_{n+3} + 12f_{n+2} + 21f_{n+1} + 2f_n) \quad (3.2.1.10)$$

Differentiating (3.2.1.8) once gives

$$\begin{aligned}
\alpha_0'(z) &= -1 \\
\alpha_1'(z) &= 1 \\
\beta_0'(z) &= \frac{1}{360}(23 + 10z^2 - 15z^4) \\
\beta_1'(z) &= \frac{1}{120}(122 - 60z^2 + 20z^3 + 15z^4) \\
\beta_2'(z) &= \frac{1}{120}(53 + 120z + 30z^2 - 40z^3 - 15z^4) \\
\beta_3'(z) &= \frac{1}{360}(-8 + 60z^2 + 60z^3 + 15z^4)
\end{aligned} \quad (3.2.1.11)$$

Evaluating (3.2.1.11) at all the grid points, that is, at z = -2, -1, 0 and 1 yields

$$360hy_n' - 360y_{n+1} + 360y_n = h^2(-8f_{n+3} + 39f_{n+2} - 144f_{n+1} - 97f_n) \quad (3.2.1.12)$$

$$360hy_{n+1}' - 360y_{n+1} + 360y_n = h^2(7f_{n+3} - 36f_{n+2} + 171f_{n+1} + 38f_n) \quad (3.2.1.13)$$

$$360hy_{n+2}' - 360y_{n+1} + 360y_n = h^2(-8f_{n+3} + 159f_{n+2} + 366f_{n+1} + 23f_n) \quad (3.2.1.14)$$

$$360hy_{n+3}' - 360y_{n+1} + 360y_n = h^2(127f_{n+3} + 444f_{n+2} + 291f_{n+1} + 38f_n) \quad (3.2.1.15)$$

Joining equations (3.2.1.9), (3.2.1.10) and (3.2.1.12) produce a block of the form (1.10) as follows

$$\begin{pmatrix} -24 & 12 & 0 \\ -36 & 0 & 12 \\ -360 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -12 \\ 0 & 0 & -24 \\ 0 & 0 & -360 \end{pmatrix}\begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -360 \end{pmatrix}\begin{pmatrix} y_{n-2}' \\ y_{n-1}' \\ y_n' \end{pmatrix}$$
$$+ h^2\begin{pmatrix} 10 & 1 & 0 \\ 21 & 12 & 1 \\ -144 & 39 & -8 \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} + h^2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -97 \end{pmatrix}\begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$

Multiplying the above equation with the inverse of A^0 gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} y_{n-2}' \\ y_{n-1}' \\ y_n' \end{pmatrix} +$$

$$h^2\begin{pmatrix} \frac{57}{180} & \frac{-39}{360} & \frac{3}{135} \\ \frac{66}{45} & \frac{-6}{45} & \frac{6}{135} \\ \frac{81}{30} & \frac{81}{120} & \frac{9}{60} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} + h^2\begin{pmatrix} 0 & 0 & \frac{291}{1080} \\ 0 & 0 & \frac{168}{270} \\ 0 & 0 & \frac{117}{120} \end{pmatrix}\begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \tag{3.2.1.16}$$

whose solution is

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{1080}(24f_{n+3} - 117f_{n+2} + 342f_{n+1} + 291f_n) \tag{3.2.1.17}$$

$$y_{n+2} = y_n + 2hy'_n + \frac{h^2}{270}(12f_{n+3} - 36f_{n+2} + 396f_{n+1} + 168f_n) \tag{3.2.1.18}$$

$$y_{n+3} = y_n + 3hy'_n + \frac{h^2}{120}(18f_{n+3} + 81f_{n+2} + 324f_{n+1} + 117f_n) \tag{3.2.1.19}$$

Substituting (3.2.1.17) into (3.2.1.13) – (3.2.1.15), the derivative of the block is obtained

$$y'_{n+1} = y'_n + \frac{h}{72}(3f_{n+3} - 15f_{n+2} + 57f_{n+1} + 27f_n) \tag{3.2.1.20}$$

$$y'_{n+2} = y'_n + \frac{h}{9}(3f_{n+2} + 12f_{n+1} + 3f_n) \tag{3.2.1.21}$$

$$y'_{n+3} = y'_n + \frac{h}{24}(9f_{n+3} + 27f_{n+2} + 27f_{n+1} + 9f_n) \tag{3.2.1.22}$$

3.2.2 Properties of Three–Step Block Method for Second Order ODEs

The order, zero-stability, consistency and absolute region of three–step block method are described in this section.

3.2.2.1 Order of Three–Step Block Method for Second Order ODEs

In finding the order of the block (3.2.1.17 – 3.2.1.19), *y* and *f* functions are expanded about *x* using Taylor series as follows

$$\begin{pmatrix} \sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m - \sum_{m=0}^{1}\frac{h^m}{m!}y_n^m - \frac{291}{1080}h^2 y_n'' - \sum_{m=0}^{\infty}\frac{h^{2+m}}{(1080)(m!)}y_n^{(2+m)}\left(342(1)^m - 117(2)^m + 24(3)^m\right) \\ \sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m - \sum_{m=0}^{1}\frac{(2h)^m}{m!}y_n^m - \frac{168}{270}h^2 y_n'' - \sum_{m=0}^{\infty}\frac{h^{2+m}}{(270)(m!)}y_n^{(2+m)}\left(396(1)^m - 36(2)^m + 12(3)^m\right) \\ \sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m - \sum_{m=0}^{1}\frac{(3h)^m}{m!}y_n^m - \frac{117}{120}h^2 y_n'' - \sum_{m=0}^{\infty}\frac{h^{2+m}}{(120)(m!)}y_n^{(2+m)}\left(324(1)^m + 81(2)^m + 18(3)^m\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Comparing the coefficients of h^m and y_n^m gives

$$C_0 = \begin{pmatrix} 1-1 \\ 1-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{291}{1080} - \frac{1}{(1080)(0!)}\left(342(1)^0 - 117(2)^0 + 24(3)^0\right) \\ \frac{(2)^2}{2!} - \frac{168}{270} - \frac{1}{(270)(0!)}\left(396(1)^0 - 36(2)^0 + 12(3)^0\right) \\ \frac{(3)^2}{2!} - \frac{117}{120} - \frac{1}{(120)(0!)}\left(324(1)^0 + 81(2)^0 + 18(3)^0\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} \frac{1}{3!} - \frac{1}{(1080)(1!)}\left(342(1)^1 - 117(2)^1 + 24(3)^1\right) \\ \frac{(2)^3}{3!} - \frac{1}{(270)(1!)}\left(396(1)^1 - 36(2)^1 + 12(3)^1\right) \\ \frac{(3)^3}{3!} - \frac{1}{(120)(1!)}\left(324(1)^1 + 81(2)^1 + 18(3)^1\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_4 = \begin{pmatrix} \frac{1}{4!} - \frac{1}{(1080)(2!)}\left(342(1)^2 - 117(2)^2 + 24(3)^2\right) \\ \frac{(2)^4}{4!} - \frac{1}{(270)(2!)}\left(396(1)^2 - 36(2)^2 + 12(3)^2\right) \\ \frac{(3)^4}{4!} - \frac{1}{(120)(2!)}\left(324(1)^2 + 81(2)^2 + 18(3)^2\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_5 = \begin{pmatrix} \frac{1}{5!} - \frac{1}{(1080)(3!)}\left(342(1)^3 - 117(2)^3 + 24(3)^3\right) \\ \frac{(2)^5}{5!} - \frac{1}{(270)(3!)}\left(396(1)^3 - 36(2)^3 + 12(3)^3\right) \\ \frac{(3)^5}{5!} - \frac{1}{(120)(3!)}\left(324(1)^3 + 81(2)^3 + 18(3)^3\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6 = \begin{pmatrix} \frac{1}{6!} - \frac{1}{(1080)(4!)}\left(342(1)^4 - 117(2)^4 + 24(3)^4\right) \\ \frac{(2)^6}{6!} - \frac{1}{(270)(4!)}\left(396(1)^4 - 36(2)^4 + 12(3)^4\right) \\ \frac{(3)^6}{6!} - \frac{1}{(120)(4!)}\left(324(1)^4 + 81(2)^4 + 18(3)^4\right) \end{pmatrix} = \begin{pmatrix} \frac{-7}{480} \\ \frac{-1}{30} \\ \frac{-9}{160} \end{pmatrix}$$

By using definition 1.3, the block method has order $(4,4,4)^T$ with the error constants $\left(\frac{-7}{480}, \frac{-1}{30}, \frac{-9}{160}\right)^T$.

3.2.2.2 Zero Stability of Three–Step Block Method for Second Order ODEs.

In order to find the zero-stability of the block (1.10), we only put into consideration the coefficients of function *y* according to definition 1.5 that is

$$\rho(r) = \det[rA^{(0)} - A^{(1)}] = 0 \tag{3.2.2.2.1}$$

where $A^{(0)}$ and $A^{(1)}$ are the coefficients of $y_{n+i}, i = 1(1)k$ and y_n in (1.10). Therefore, for a three-step block method (3.2.1.17– 3.2.1.19), we have

$$\det[rA^{(0)} - A^{(1)}] = \left| r\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

which implies $r = 0,0,1$. Hence, the method is zero stable.

3.2.2.3 Consistency and Convergence of Three–Step Block Method for Second Order ODEs

The block method (3.2.1.17 – 3.2.1.19) is consistent because it satisfies the conditions listed in Definition 1.4. Hence, it is also convergent because it is zero-stable and consistent.

3.2.2.4 Region of Absolute Stability of Three–Step Block Method for Second Order ODEs.

The method known as boundary locus method proposed by Lambert (1973) and Henrici (1962) is adopted in finding the region of absolute stability of the block (1.10). This is given as $\bar{h}(r) = \dfrac{\ell(r)}{\sigma(r)}$ where $\ell(r)$ is the first characteristics polynomial and $\sigma(r)$ is the second characteristics polynomial. The test problem of the form $y^{(d)} = \lambda^d y$ is substituted into the block (1.10) to give

$$A^0 Y_N = \sum_{i=0}^{d-1} h^i A^{i+1} Y_{N-1}^{(i)} + h^d \sum_{i=0}^{1} B^i \lambda^d y_{N-i}$$

which produces

$$\bar{h}(r,h) = \frac{A^0 Y_N(r) - A^1 Y_{N-1}(r)}{B^0 y_N(r) + B^1 y_{N-1}(r)} \tag{3.2.2.4.1}$$

where $\bar{h} = \lambda^d h^d$. Equation (3.2.2.4.1) is written in Euler's form as

$$\bar{h}(\theta,h) = \frac{A^0 Y_N(\theta) - A^1 Y_{N-1}(\theta)}{B^0 y_N(\theta) + B^1 y_{N-1}(\theta)} \tag{3.2.2.4.2}$$

Equation (3.2.2.4.2) is called the characteristics matrix. Finding the determinant of (3.2.2.4.2) gives the stability polynomial of the method.

Applying (3.2.2.4.2) to three-step block (3.2.1.17– 3.2.1.19) gives

$$\bar{h}(\theta,h)=\frac{\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{2i\theta} & 0 \\ 0 & 0 & e^{3i\theta} \end{pmatrix}-\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}{\begin{pmatrix} \frac{342}{1080}e^{i\theta} & -\frac{117}{1080}e^{2i\theta} & \frac{24}{1080}e^{3i\theta} \\ \frac{396}{270}e^{i\theta} & -\frac{36}{270}e^{2i\theta} & \frac{12}{270}e^{3i\theta} \\ \frac{324}{120}e^{i\theta} & \frac{81}{120}e^{2i\theta} & \frac{18}{120}e^{3i\theta} \end{pmatrix}+\begin{pmatrix} 0 & 0 & \frac{291}{1080} \\ 0 & 0 & \frac{168}{270} \\ 0 & 0 & \frac{117}{120} \end{pmatrix}}$$

The above matrix is simplified and after finding the determinant we have

$$\bar{h}(\theta,h)=\frac{80(e^{3i\theta}-1)}{2e^{3i\theta}+13}$$

The above equation is expanded trigonometrically and the imaginary part are equated to zero to give

$$\bar{h}(\theta,h)=\frac{80\cos 3\theta-80}{2\cos 3\theta+13}$$

Evaluating $\bar{h}(\theta,h)$ at interval of θ of 30^0 produces results as tabulated in Table 3.1 below

Table 3.1

Interval of Absolute Stability of Three–Step Block Method for Second Order ODEs.

θ	0	30	60	90	120	150	180
$\bar{h}(\theta,h)$	0	-6.15	-14.55	-6.15	0	-6.15	-14.55

Therefore, the interval of absolute stability is (-14.55, 0). This is shown in the diagram below

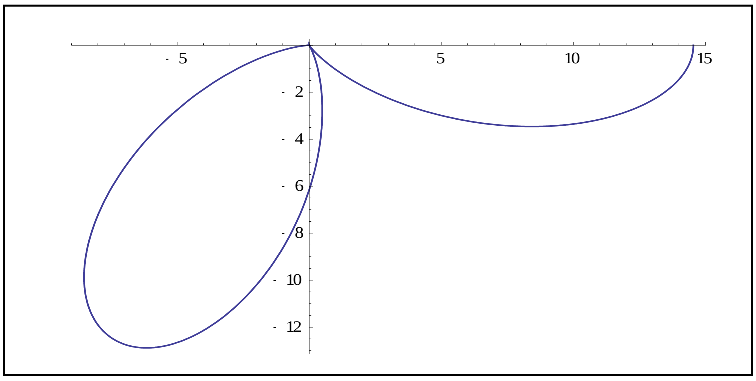

Figure 3.2. Region of absolute stability of three–step block method for second order ODEs.

3.3 Four–Step Block Method for Second Order ODEs

A four–step block method is derived in this segment. In addition, the properties of the method are also established.

3.3.1 Derivation of Four–Step Block Method for Second Order ODEs.

Power series of the form (3.2.1.1) is considered as an approximate solution to the general second order problem of the form (3.2.1.2) where the step-length k =4 is used. The first and second derivatives of (3.2.1.1) are given in (3.2.1.3) and (3.2.1.4). Interpolating equation (3.2.1.1) at the points $x = x_{n+i}, i = 1,2$ and collocating (3.2.1.4) at the points $x = x_{n+i}, i = 0(1)4$ produce Figure 3.3 below

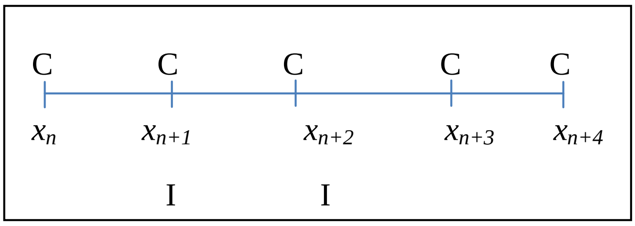

Figure 3.3. Four–step interpolation and collocation method for second order ODEs.

This strategy yields

$$\begin{pmatrix} 1 & x_{n+!} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+!}^5 & x_{n+!}^6 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 20x_{n+3}^4 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 20x_{n+4}^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \qquad (3.3.1.1)$$

In order to find the values of *a's*, Gaussian elimination method is applied which then gives the values of *a's* as below:

$$\begin{aligned} a_0 = \; & 2y_{n+1} - y_{n+2} + \frac{19}{240}h^2 f_n + \frac{17}{20}h^2 f_{n+1} + \frac{7}{120}h^2 f_{n+2} + \frac{1}{60}h^2 f_{n+3} - \frac{1}{240}h^2 f_{n+4} \\ & \frac{x_n^5}{240h^3}(5f_n - 18f_{n+1} + 24f_{n+2} - 14f_{n+3} + 3f_{n+4}) + \frac{x_n^4}{288h^2}(35f_n - 104f_{n+1} + 114f_{n+2} \\ & -56f_{n+3} + 11f_{n+4}) + \frac{x_n^6}{720h^4}(f_n - 4f_{n+1} + 6f_{n+2} - 4f_{n+3} + f_{n+4}) + \frac{481h}{1440}x_n f_n + \\ & \frac{49h}{40}x_n f_{n+1} - \frac{11h}{80}x_n f_{n+2} + \frac{7h}{72}x_n f_{n+3} - \frac{3h}{160}x_n f_{n+4} + \frac{x_n}{h}(y_{n+1} - y_{n+2}) + \frac{x_n^3}{72h}(25f_n \\ & -48f_{n+1} + 36f_{n+2} - 16f_{n+3} + 3f_{n+4}) \end{aligned}$$

$$\begin{aligned} a_1 = \; & \frac{11}{80}hf_{n+2} - \frac{49}{40}hf_{n+1} - \frac{481}{1440}hf_n - \frac{7}{72}hf_{n+3} + \frac{3}{160}hf_{n+4} - x_n f_n - \frac{1}{h}(y_{n+1} - y_{n+2}) \\ & -\frac{x_n^2}{24h}(25f_n - 48f_{n+1} + 36f_{n+2} - 16f_{n+3} + 3f_{n+4}) - \frac{x_n^4}{48h^3}(5f_n - 18f_{n+1} + 24f_{n+2} - \\ & 14f_{n+3} + 3f_{n+4}) - \frac{x_n^3}{72h^2}(35f_n - 104f_{n+1} + 114f_{n+2} - 56f_{n+3} + 11f_{n+4}) - \frac{x_n^5}{120h^4}(f_n \\ & -4f_{n+1} + 6f_{n+2} - 4f_{n+3} + f_{n+4}) \end{aligned}$$

$$\begin{aligned} a_2 = \; & \frac{1}{2}f_n + \frac{x_n^3}{24h^3}(5f_n - 18f_{n+1} + 24f_{n+2} - 14f_{n+3} + 3f_{n+4}) + \frac{x_n^2}{48h^2}(35f_n - \\ & 104f_{n+1} + 114f_{n+2} - 56f_{n+3} + 11f_{n+4}) + \frac{x_n^4}{48h^4}(f_n - 4f_{n+1} + 6f_{n+2} - 4f_{n+3} + \\ & f_{n+4}) + \frac{x_n}{24h}(25f_n - 48f_{n+1} + 36f_{n+2} - 16f_{n+3} + 3f_{n+4}) \end{aligned}$$

$$a_3 = \frac{-25}{72h} f_n + \frac{2}{3h} f_{n+1} - \frac{1}{2h} f_{n+2} + \frac{2}{9h} f_{n+3} - \frac{1}{24h} f_{n+4} - \frac{x_n^2}{24h^3}(5f_n - 18f_{n+1}$$

$$+24f_{n+2} - 14f_{n+3} + 3f_{n+4}) - \frac{x_n^3}{36h^4}(f_n - 4f_{n+1} + 6f_{n+2} - 4f_{n+3} + f_{n+4}) -$$

$$\frac{x_n}{72h^2}(35f_n - 104f_{n+1} + 114f_{n+2} - 56f_{n+3} + 11f_{n+4})$$

$$a_4 = \frac{35}{288h^2} f_n - \frac{13}{36h^2} f_{n+1} + \frac{19}{48h^2} f_{n+2} - \frac{7}{36h^2} f_{n+3} + \frac{11}{288h^2} f_{n+4} + \frac{x_n^2}{48h^4}(f_n$$

$$-4f_{n+1} + 6f_{n+2} - 4f_{n+3} + f_{n+4}) + \frac{x_n}{48h^3}(5f_n - 18f_{n+1} + 24f_{n+2} - 14f_{n+3} + 3f_{n+4})$$

$$a_5 = \frac{-1}{48h^3} f_n + \frac{3}{40h^3} f_{n+1} - \frac{1}{10h^3} f_{n+2} + \frac{7}{120h^3} f_{n+3} - \frac{1}{80h^3} f_{n+4} - \frac{x_n}{120h^4}(f_n -$$

$$4f_{n+1} + 6f_{n+2} - 4f_{n+3} + f_{n+4})$$

$$a_6 = \frac{x_n}{720h^4}(f_n - 4f_{n+1} + 6f_{n+2} - 4f_{n+3} + f_{n+4})$$

Substituting the values of *a's* into equation (3.2.1.1) and on simplifying gives a continuous linear multistep method of the form:

$$y(x) = \sum_{j=1}^{k-2} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (3.3.1.2)$$

Using the transformation $z = \dfrac{x - x_{n+k-1}}{h}$, where $k = 4$ this implies

$$x = zh + x_n + 3h \qquad (3.3.1.3)$$

Substitute (3.3.1.3) into (3.3.1.2), the following coefficients of y_{n+j} and f_{n+j} are obtained

$$\alpha_1(z) = -1 - z$$
$$\alpha_2(z) = 2 + z$$

$$\beta_0(z) = \frac{1}{4320}(-18 + 15z - 60z^3 - 15z^4 + 18z^5 + 6z^6)$$

$$\beta_1(z) = \frac{1}{360}(36 + 18z + 30z^3 + 5z^4 - 9z^5 - 2z^6)$$

$$\beta_2(z) = \frac{1}{720}(582 + 747z - 180z^3 + 15z^4 + 36z^5 + 6z^6)$$

$$\begin{aligned}\beta_3(z) &= \frac{1}{360}(36+154z+180z^2+50z^3-25z^4-15z^5-2z^6)\\ \beta_4(z) &= \frac{1}{1440}(-6-27z+60z^3+55z^4+18z^5+2z^6)\end{aligned} \qquad (3.3.1.4)$$

Evaluating (3.3.1.4) at the non-interpolating points, that is, at z = -3, 0 and 1 produces the following schemes

$$240y_{n+2}-480y_{n+1}+240y_n = h^2(-f_{n+4}+4f_{n+3}+14f_{n+2}+204f_{n+1}+19f_n) \qquad (3.3.1.5)$$

$$240y_{n+3}-480y_{n+2}+240y_{n+1} = h^2(-f_{n+4}+24f_{n+3}+194f_{n+2}+24f_{n+1}-f_n) \qquad (3.3.1.6)$$

$$240y_{n+4}-720y_{n+2}+480y_{n+1} = h^2(17f_{n+4}+252f_{n+3}+402f_{n+2}+52f_{n+1}-3f_n) \qquad (3.3.1.7)$$

Differentiating (3.3.1.4) once gives

$$\begin{aligned}\alpha_1'(z) &= -1\\ \alpha_2'(z) &= 1\\ \beta_0'(z) &= \frac{1}{1440}(5-60z^2-20z^3+30z^4+12z^5)\\ \beta_1'(z) &= \frac{1}{360}(18+90z^2+20z^3-45z^4-12z^5)\\ \beta_2'(z) &= \frac{1}{240}(249-180z^2+20z^3+60z^4+12z^5)\\ \beta_3'(z) &= \frac{1}{360}(154+360z+150z^2-100z^3-75z^4-12z^5)\\ \beta_4'(z) &= \frac{1}{1440}(-27z+180z^2+220z^3+90z^4+12z^5)\end{aligned} \qquad (3.3.1.8)$$

Equation (3.3.1.8) is evaluated at all the grid points. i.e, at z = -3, -2,-1, 0 and 1. As a result, the following schemes are obtained:

$$\begin{aligned}1440hy_n'-1440y_{n+2}+1440y_{n+1} &= h^2(27f_{n+4}-140f_{n+3}+198f_{n+2}\\ &\quad -1764f_{n+1}-481f_n)\end{aligned} \qquad (3.3.1.9)$$

$$\begin{aligned}1440hy_{n+1}'-1440y_{n+2}+1440y_{n+1} &= h^2(-11f_{n+4}+72f_{n+3}-330f_{n+2}\\ &\quad -472f_{n+1}+21f_n)\end{aligned} \qquad (3.3.1.10)$$

$$\begin{aligned}1440hy_{n+2}'-1440y_{n+2}+1440y_{n+1} &= h^2(11f_{n+4}-76f_{n+3}+582f_{n+2}\\ &\quad +220f_{n+1}-17f_n)\end{aligned} \qquad (3.3.1.11)$$

$$\begin{aligned}1440hy_{n+3}'-1440y_{n+2}+1440y_{n+1} &= h^2(-27f_{n+4}+616f_{n+3}+1494f_{n+2}\\ &\quad +72f_{n+1}+5f_n)\end{aligned} \qquad (3.3.1.12)$$

$$1440hy'_{n+4} - 1440y_{n+2} + 1440y_{n+1} = h^2(475f_{n+4} + 1908f_{n+3} + 966f_{n+2} + 284f_{n+1} - 33f_n) \qquad (3.3.1.13)$$

Joining equations (3.3.1.5) - (3.3.1.7) and (3.3.1.9) gives the following block method of the form (1.10)

$$\begin{pmatrix} -480 & 240 & 0 & 0 \\ 240 & -480 & 240 & 0 \\ 480 & -720 & 0 & 240 \\ 1440 & -1440 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -240 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} +$$

$$h\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1440 \end{pmatrix}\begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix} + h^2\begin{pmatrix} 204 & 14 & 4 & -1 \\ 24 & 194 & 24 & -1 \\ 52 & 402 & 252 & 17 \\ -1764 & 198 & -140 & 27 \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} +$$

$$h^2\begin{pmatrix} 0 & 0 & 0 & 19 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -481 \end{pmatrix}\begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}.$$

Multiplying the above equation with $(A^0)^{-1}$, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix} +$$

$$h^2\begin{pmatrix} \frac{162}{432} & \frac{-141}{720} & \frac{87}{1080} & \frac{-63}{4320} \\ \frac{24}{15} & \frac{-3}{9} & \frac{24}{135} & \frac{-3}{90} \\ \frac{351}{120} & \frac{81}{240} & \frac{9}{24} & \frac{-27}{480} \\ \frac{192}{45} & \frac{48}{45} & \frac{192}{135} & 0 \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} + h^2\begin{pmatrix} 0 & 0 & 0 & \frac{1101}{4320} \\ 0 & 0 & 0 & \frac{159}{270} \\ 0 & 0 & 0 & \frac{441}{480} \\ 0 & 0 & 0 & \frac{168}{135} \end{pmatrix}\begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \qquad (3.3.1.14)$$

which leads to

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{4320}(-63f_{n+4} + 348f_{n+3} - 846f_{n+2} + 1620f_{n+1} + 1101f_n) \qquad (3.3.1.15)$$

$$y_{n+2} = y_n + 2hy'_n + \frac{h^2}{270}(-9f_{n+4} + 48f_{n+3} - 90f_{n+2} + 432f_{n+1} + 159f_n) \qquad (3.3.1.16)$$

$$y_{n+3} = y_n + 3hy'_n + \frac{h^2}{480}(-27f_{n+4} + 180f_{n+3} + 162f_{n+2} + 1404f_{n+1} + 441f_n) \qquad (3.3.1.17)$$

$$y_{n+4} = y_n + 4hy'_n + \frac{h^2}{135}(192f_{n+3} + 144f_{n+2} + 576f_{n+1} + 168f_n) \qquad (3.3.1.18)$$

Substituting (3.3.1.15) and (3.3.1.16) into (3.3.1.10) – (3.3.1.13) gives the derivative of the block method

$$y'_{n+1} = y'_n + \frac{h}{720}(-19f_{n+4} + 106f_{n+3} - 264f_{n+2} + 646f_{n+1} + 251f_n) \qquad (3.3.1.19)$$

$$y'_{n+2} = y'_n + \frac{h}{90}(-f_{n+4} + 4f_{n+3} + 24f_{n+2} + 124f_{n+1} + 29f_n) \qquad (3.3.1.20)$$

$$y'_{n+3} = y'_n + \frac{h}{80}(-3f_{n+4} + 42f_{n+3} + 72f_{n+2} + 102f_{n+1} + 27f_n) \qquad (3.3.1.21)$$

$$y'_{n+4} = y'_n + \frac{h}{45}(14f_{n+4} + 64f_{n+3} + 24f_{n+2} + 64f_{n+1} + 14f_n) \qquad (3.3.1.22)$$

3.3.2 Properties of Four-Step Block Method for Second Order ODEs

In this section, the order, zero-stability and region of absolute stability of four –step block method are established.

3.3.2.1 Order of Four–Step Block Method for Second Order ODEs.

In finding the order of the block method (3.3.1.15 – 3.3.1.18), the same strategy mentioned in section 3.2.2.1 is used as displayed below

$$\begin{pmatrix} \sum_{m=0}^{\infty} \frac{h^m}{m!} y_n^m - \sum_{m=0}^{1} \frac{h^m}{m!} y_n^m - \frac{1101}{4320} h^2 y_n'' - \sum_{m=0}^{\infty} \frac{h^{2+m}}{(4320)(m!)} y_n^{(2+m)} \left(1620(1)^m - 846(2)^m + 348(3)^m - 63(4)^m\right) \\ \sum_{m=0}^{\infty} \frac{(2h)^m}{m!} y_n^m - \sum_{m=0}^{1} \frac{(2h)^m}{m!} y_n^m - \frac{159}{270} h^2 y_n'' - \sum_{m=0}^{\infty} \frac{h^{2+m}}{(270)(m!)} y_n^{(2+m)} \left(432(1)^m - 90(2)^m + 48(3)^m - 9(4)^m\right) \\ \sum_{m=0}^{\infty} \frac{(3h)^m}{m!} y_n^m - \sum_{m=0}^{1} \frac{(3h)^m}{m!} y_n^m - \frac{441}{480} h^2 y_n'' - \sum_{m=0}^{\infty} \frac{h^{2+m}}{(480)(m!)} y_n^{(2+m)} \left(1404(1)^m + 162(2)^m + 180(3)^m - 27(4)^m\right) \\ \sum_{m=0}^{\infty} \frac{(4h)^m}{m!} y_n^m - \sum_{m=0}^{1} \frac{(4h)^m}{m!} y_n^m - \frac{168}{135} h^2 y_n'' - \sum_{m=0}^{\infty} \frac{h^{2+m}}{(135)(m!)} y_n^{(2+m)} \left(576(1)^m + 144(2)^m + 192(3)^m\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Comparing the coefficients of h^m and y_n^m gives

$$C_0 = \begin{pmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{1101}{4320} - \frac{1}{(4320)(0!)} \left(1620(1)^0 - 846(2)^0 + 348(3)^0 - 63(4)^0\right) \\ \frac{(2)^2}{2!} - \frac{159}{270} - \frac{1}{(270)(0!)} \left(432(1)^0 - 90(2)^0 + 48(3)^0 - 9(4)^0\right) \\ \frac{(3)^2}{2!} - \frac{441}{480} - \frac{1}{(480)(0!)} \left(1404(1)^0 + 162(2)^0 + 180(3)^0 - 27(4)^0\right) \\ \frac{(4)^2}{2!} - \frac{168}{135} - \frac{1}{(135)(0!)} \left(576(1)^0 + 144(2)^0 + 192(3)^0\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} \frac{1}{3!} - \frac{1}{(4320)(1!)}\left(1620(1)^1 - 846(2)^1 + 348(3)^1 - 63(4)^1\right) \\ \frac{(2)^3}{3!} - \frac{1}{(270)(1!)}\left(432(1)^1 - 90(2)^1 + 48(3)^1 - 9(4)^1\right) \\ \frac{(3)^3}{3!} - \frac{1}{(480)(1!)}\left(1404(1)^1 + 162(2)^1 + 180(3)^1 - 27(4)^1\right) \\ \frac{(4)^3}{3!} - \frac{1}{(135)(1!)}\left(576(1)^1 + 144(2)^1 + 192(3)^1\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_4 = \begin{pmatrix} \frac{1}{4!} - \frac{1}{(4320)(2!)}\left(1620(1)^2 - 846(2)^2 + 348(3)^2 - 63(4)^2\right) \\ \frac{(2)^4}{4!} - \frac{1}{(270)(2!)}\left(432(1)^2 - 90(2)^2 + 48(3)^2 - 9(4)^2\right) \\ \frac{(3)^4}{4!} - \frac{1}{(480)(2!)}\left(1404(1)^2 + 162(2)^2 + 180(3)^2 - 27(4)^2\right) \\ \frac{(4)^4}{4!} - \frac{1}{(135)(2!)}\left(576(1)^2 + 144(2)^2 + 192(3)^2\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_5 = \begin{pmatrix} \frac{1}{5!} - \frac{1}{(4320)(3!)}\left(1620(1)^3 - 846(2)^3 + 348(3)^3 - 63(4)^3\right) \\ \frac{(2)^5}{5!} - \frac{1}{(270)(3!)}\left(432(1)^3 - 90(2)^3 + 48(3)^3 - 9(4)^3\right) \\ \frac{(3)^5}{5!} - \frac{1}{(480)(3!)}\left(1404(1)^3 + 162(2)^3 + 180(3)^3 - 27(4)^3\right) \\ \frac{(4)^5}{5!} - \frac{1}{(135)(3!)}\left(576(1)^3 + 144(2)^3 + 192(3)^3\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6 = \begin{pmatrix} \frac{1}{6!} - \frac{1}{(4320)(4!)}\left(1620(1)^4 - 846(2)^4 + 348(3)^4 - 63(4)^4\right) \\ \frac{(2)^6}{6!} - \frac{1}{(270)(4!)}\left(432(1)^4 - 90(2)^4 + 48(3)^4 - 9(4)^4\right) \\ \frac{(3)^6}{6!} - \frac{1}{(480)(4!)}\left(1404(1)^4 + 162(2)^4 + 180(3)^4 - 27(4)^4\right) \\ \frac{(4)^6}{6!} - \frac{1}{(135)(4!)}\left(576(1)^4 + 144(2)^4 + 192(3)^4\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_7 = \begin{pmatrix} \frac{1}{7!} - \frac{1}{(4320)(5!)}\left(1620(1)^5 - 846(2)^5 + 348(3)^5 - 63(4)^5\right) \\ \frac{(2)^7}{7!} - \frac{1}{(270)(5!)}\left(432(1)^5 - 90(2)^5 + 48(3)^5 - 9(4)^5\right) \\ \frac{(3)^7}{7!} - \frac{1}{(480)(5!)}\left(1404(1)^5 + 162(2)^5 + 180(3)^5 - 27(4)^5\right) \\ \frac{(4)^7}{7!} - \frac{1}{(135)(5!)}\left(576(1)^5 + 144(2)^5 + 192(3)^5\right) \end{pmatrix} = \begin{pmatrix} \frac{107}{10080} \\ \frac{16}{630} \\ \frac{9}{224} \\ \frac{32}{630} \end{pmatrix}$$

Hence, the block method has order $(5,5,5,5)^T$ with the following error constants

$$\left(\frac{107}{10080}, \frac{16}{630}, \frac{9}{224}, \frac{32}{630}\right)^T.$$

3.3.2.2 Zero Stability of Four–Step Block Method for Second Order ODEs.

Applying the equation (3.2.2.2.1) to the block (3.3.1.15 – 3.3.1.18), this gives

$$\det[rA^{(0)} - A^{(1)}] = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0$$

which implies $r = 0,0,0,1$. Hence, the method is zero stable.

3.3.2.3 Consistency and Convergence of Four–Step Block Method for Second Order ODEs

Based on the conditions listed in Definition 1.4, the block method (3.3.1.15–3.3.1.18) is consistent. It is also convergent because it is zero-stable and consistent.

3.3.2.4 Region of Absolute Stability of Four–Step Block Method for Second Order ODEs.

Equation (3.2.2.4.2) is applied to the block (3.3.1.15 – 3.3.1.18), this gives

$$\bar{h}(\theta,h)=\frac{\begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 \\ 0 & 0 & 0 & e^{4i\theta} \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\begin{pmatrix} \frac{1620}{4320}e^{i\theta} & -\frac{846}{4320}e^{2i\theta} & \frac{348}{4320}e^{3i\theta} & -\frac{63}{4320}e^{4i\theta} \\ \frac{432}{270}e^{i\theta} & -\frac{90}{270}e^{2i\theta} & \frac{48}{270}e^{3i\theta} & -\frac{9}{270}e^{4i\theta} \\ \frac{1404}{480}e^{i\theta} & \frac{162}{480}e^{2i\theta} & \frac{180}{480}e^{3i\theta} & -\frac{27}{480}e^{4i\theta} \\ \frac{576}{135}e^{i\theta} & \frac{144}{135}e^{2i\theta} & \frac{192}{135}e^{3i\theta} & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & \frac{1101}{4320} \\ 0 & 0 & 0 & \frac{159}{270} \\ 0 & 0 & 0 & \frac{441}{480} \\ 0 & 0 & 0 & \frac{168}{135} \end{pmatrix}}$$

The above matrix is simplified, after finding the determinant and equating the imaginary part to zero we have

$$\bar{h}(\theta,h)=\frac{225\cos 4\theta-225}{3\cos 4\theta-28}$$

Evaluating $\bar{h}(\theta,h)$ at intervals of θ of 30^0 gives results shown in Table 3.2

Table 3.2

Interval of Absolute Stability of Four–Step Block Method for Second Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta,h)$	0	11.44	11.44	0	11.44	11.44	0

Therefore, the interval of absolute stability is (0, 11.4). This is shown in the diagram below

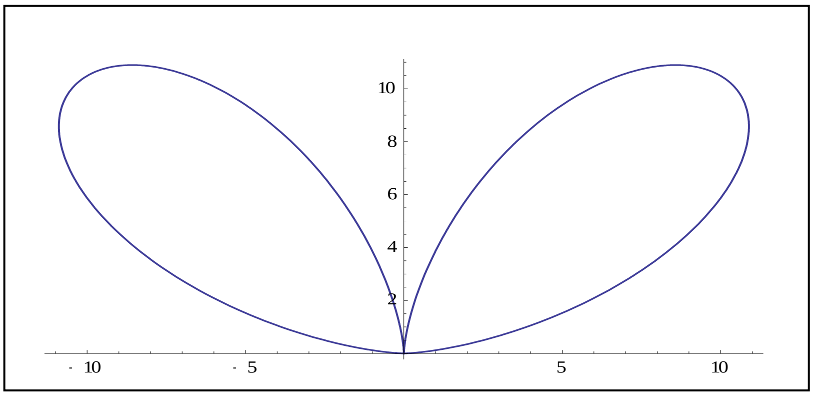

Figure 3.4. Region of absolute stability of four –step block method for second ODEs.

3.4 Five–Step Block Method for Second Order ODEs

This section considers the derivation of block method with *k*=5 for direct solution of second order ODEs. The properties of the method are also included.

3.4.1 Derivation of Five–Step Block Method for Second Order ODEs.

Power series of the form (3.2.1.1) is considered as an approximate solution to the general second order problem of the form (3.2.1.2) where k =5 is the step-length. The first and second derivatives of (3.2.1.1) are given in (3.2.1.3) and (3.2.1.4). Equation (3.2.1.1) is interpolated at $x = x_{n+i}, i = 2,3$ and (3.2.1.4) is collocated at $x = x_{n+i}, i = 0(1)5$. This diagram is demonstrated below:

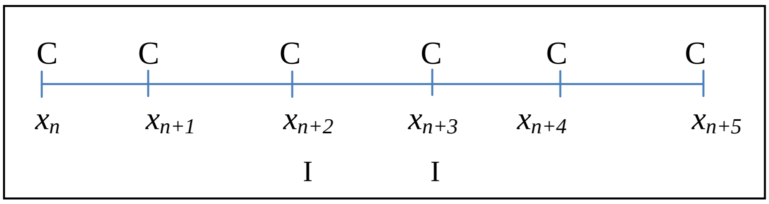

Figure 3.5. Five –step interpolation and collocation method for second order ODEs

As a result, the following is obtained

$$\begin{pmatrix} 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 \\ 0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_{n+2} \\ y_{n+3} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} \qquad (3.4.1.1)$$

In finding the values of *a's* in (3.4.1.1), Gaussian elimination method is employed.

The values of *a's* are below:

$$\begin{aligned} a_0 = \; & 3y_{n+2} - 2y_{n+3} + \frac{1}{15}h^2 f_n + \frac{257}{240}h^2 f_{n+1} + \frac{49}{30}h^2 f_{n+2} + \frac{31}{120}h^2 f_{n+3} - \frac{1}{30}h^2 f_{n+4} + \\ & \frac{1}{240}h^2 f_{n+5} + \frac{1}{2}x_n^2 f_n + \frac{x_n^6}{720h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) + \\ & \frac{x_n^5}{480h^3}(17f_n - 71f_{n+1} + 118f_{n+2} - 98f_{n+3} + 41f_{n+4} - 7f_{n+5}) + \frac{x_n^4}{288h^2}(45f_n - \\ & 154f_{n+1} + 214f_{n+2} - 156f_{n+3} + 61f_{n+4} - 10f_{n+5}) + \frac{x_n}{h}(y_{n+2} - y_{n+3}) + \frac{x_n^3}{360h}(137f_n \\ & -300f_{n+1} + 300f_{n+2} - 200f_{n+3} + 75f_{n+4} - 12f_{n+5}) + \frac{x_n^7}{5040h^5}(f_n - 5f_{n+1} + 10f_{n+2} \\ & -10f_{n+3} + 5f_{n+4} - f_{n+5}) + \frac{397h}{1260}x_n f_n + \frac{14101h}{10080}x_n f_{n+1} + \frac{659h}{1260}x_n f_{n+2} + \frac{1739h}{5040}x_n f_{n+3} \\ & + \frac{397h}{1260}x_n f_n + \frac{14101h}{10080}x_n f_{n+1} + \frac{659h}{1260}x_n f_{n+2} + \frac{1739h}{5040}x_n f_{n+3} - \frac{61h}{630}x_n f_{n+4} + \frac{149h}{10080}x_n f_{n+5} \end{aligned}$$

$$\begin{aligned} a_1 = \; & \frac{61}{630}hf_{n+4} - \frac{14101}{10080}hf_{n+1} - \frac{659}{1260}hf_{n+2} - \frac{1739}{5040}hf_{n+3} - \frac{397}{1260}hf_n - \frac{149}{10080}hf_{n+5} - x_n f_n - \\ & \frac{1}{h}(y_{n+2} - y_{n+3}) - \frac{x_n^2}{120h}(137f_n - 300f_{n+1} + 300f_{n+2} - 200f_{n+3} + 75f_{n+4} - 12f_{n+5}) - \frac{x_n^5}{120h^4}(3f_n \\ & -14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) - \frac{x_n^4}{96h^3}(17f_n - 71f_{n+1} + 118f_{n+2} - 98f_{n+3} + 41f_{n+4} \\ & -7f_{n+5}) - \frac{x_n^3}{72h^2}(45f_n - 154f_{n+1} + 214f_{n+2} - 156f_{n+3} + 61f_{n+4} - 10f_{n+5}) - \frac{x_n^6}{720h^5}(f_n - 5f_{n+1} + \\ & 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5}). \end{aligned}$$

$$a_2 = \frac{1}{2} f_n + \frac{x_n^4}{48h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) + \frac{x_n^3}{48h^3}(17f_n - 71f_{n+1} +$$

$$118f_{n+2} - 98f_{n+3} + 41f_{n+4} - 7f_{n+5}) + \frac{x_n^2}{48h^2}(45f_n - 154f_{n+1} + 214f_{n+2} - 156f_{n+3} + 61f_{n+4} -$$

$$0f_{n+5}) + \frac{x_n}{120h}(137f_n - 300f_{n+1} + 300f_{n+2} - 200f_{n+3} + 75f_{n+4} - 12f_{n+5}) + \frac{x_n^5}{240h^5}(f_n - 5f_{n+1}$$

$$+10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5}).$$

$$a_3 = \frac{-137}{360h} f_n + \frac{5}{6h} f_{n+1} - \frac{5}{6h} f_{n+2} + \frac{5}{9h} f_{n+3} - \frac{5}{24h} f_{n+4} + \frac{1}{30h} f_{n+5} - \frac{x_n^3}{36h^4}(3f_n - 14f_{n+1} +$$

$$26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) - \frac{x_n^2}{48h^3}(17f_n - 71f_{n+1} + 118f_{n+2} - 98f_{n+3} + 41f_{n+4} - 7f_{n+5})$$

$$-\frac{x_n}{72h^2}(45f_n - 154f_{n+1} + 214f_{n+2} - 156f_{n+3} + 61f_{n+4} - 10f_{n+5}) - \frac{x_n^4}{144h^5}(f_n - 5f_{n+1} + 10f_{n+2}$$

$$-10f_{n+3} + 5f_{n+4} - f_{n+5}).$$

$$a_4 = \frac{5}{32h^2} f_n - \frac{77}{144h^2} f_{n+1} + \frac{107}{144h^2} f_{n+2} - \frac{13}{24h^2} f_{n+3} + \frac{61}{288h^2} f_{n+4} - \frac{5}{144h^2} f_{n+5}$$

$$+\frac{x_n^2}{48h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) + \frac{x_n}{96h^3}(17f_n - 71f_{n+1}$$

$$+118f_{n+2} - 98f_{n+3} + 41f_{n+4} - 7f_{n+5}) + \frac{x_n^3}{144h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4}$$

$$-f_{n+5}).$$

$$a_5 = \frac{-17}{480h^3} f_n + \frac{71}{480h^3} f_{n+1} - \frac{59}{240h^3} f_{n+2} + \frac{49}{240h^3} f_{n+3} - \frac{41}{480h^3} f_{n+4} - \frac{7}{480h^3} f_{n+5}$$

$$-\frac{x_n}{120h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) - \frac{x_n^2}{240h^5}(f_n - 5f_{n+1} +$$

$$10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5}).$$

$$a_6 = \frac{1}{240h^4} f_n - \frac{7}{360h^4} f_{n+1} + \frac{13}{360h^4} f_{n+2} - \frac{1}{30h^4} f_{n+3} + \frac{11}{720h^4} f_{n+4} - \frac{1}{360h^4} f_{n+5} +$$

$$\frac{x_n}{720h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5}).$$

$$a_7 = -\frac{1}{5040h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5}).$$

Substituting the values of a's into equation (3.2.1.1) and simplifying, this gives a continuous linear multistep method of the form:

$$y(x)=\sum_{j=2}^{k-2}\alpha_j(x)y_{n+j}+h^2\sum_{j=0}^{k}\beta_j(x)f_{n+j} \qquad (3.4.1.2)$$

where $x=zh+x_n+4h$ (3.4.1.3)

Substituting (3.4.1.3) into (3.4.1.2) and on simplifying gives

$$\alpha_2(z)=-1-z$$
$$\alpha_3(z)=2+z$$
$$\beta_0(z)=\frac{1}{30240}(-120z+252z^3+105z^4-63z^5-42z^6-6z^7)$$
$$\beta_1(z)=\frac{1}{30240}(705z-1680z^3-630z^4+441z^5+252z^6+30z^7)$$
$$\beta_2(z)=\frac{1}{5040}(504+52z+840z^3+245z^4-231z^5-98z^6-10z^7) \qquad (3.4.1.4)$$
$$\beta_3(z)=\frac{1}{5040}(4074+5429z-1680z^3-70z^4+357z^5+112z^6+10z^7)$$
$$\beta_4(z)=\frac{1}{30240}(3024+12336z+15120z^2+5460z^3-1575z^4-1575z^5-378z^6 -10z^7)$$
$$\beta_5(z)=\frac{1}{10080}(-42-149z+336z^3+350z^4+147z^5+28z^6+2z^7)$$

Evaluating (3.4.1.4) at the non-interpolating points .that is, at z = -4, -3, 0 and 1. This produces the following schemes

$$480y_{n+3}-720y_{n+2}+240y_n=h^2(f_{n+5}-8f_{n+4}+62f_{n+3}+392f_{n+2}+257f_{n+1}+16f_n). \qquad (3.4.1.5)$$

$$240y_{n+3}-480y_{n+2}+240y_{n+1}=h^2(-f_{n+4}+24f_{n+3}+194f_{n+2}+24f_{n+1}-f_n). \qquad (3.4.1.6)$$

$$240y_{n+4}-480y_{n+3}+240y_{n+2}=h^2(-f_{n+5}+24f_{n+4}+194f_{n+3}+24f_{n+2}-f_{n+1}). \qquad (3.4.1.7)$$

$$240y_{n+5}-720y_{n+3}+480y_{n+2}=h^2(16f_{n+5}+257f_{n+4}+392f_{n+3}+62f_{n+2}-8f_{n+1}+f_n). \qquad (3.4.1.8)$$

The derivative of (3.4.1.4) are

$$\alpha_2'(z)=-1$$
$$\alpha_3'(z)=1$$
$$\beta_0'(z)=\frac{1}{10080}(-40+252z^2+140z^3-105z^4-84z^5-14z^6)$$

$$\beta_1'(z) = \frac{1}{10080}(235 - 1680z^2 - 840z^3 + 735z^4 + 504z^5 + 70z^6)$$

$$\beta_2'(z) = \frac{1}{5040}(52 + 2520z^2 + 980z^3 - 1155z^4 - 588z^5 - 70z^6) \qquad (3.4.1.9)$$

$$\beta_3'(z) = \frac{1}{5040}(5429 - 5040z^2 - 280z^3 + 1785z^4 + 672z^5 + 70z^6)$$

$$\beta_4'(z) = \frac{1}{10080}(4112 + 10080z + 5460z^2 - 2100z^3 - 2625z^4 - 756z^5 - 70z^6)$$

$$\beta_5'(z) = \frac{1}{10080}(-149 + 10080z^2 + 1400z^3 + 735z^4 + 168z^5 + 14z^6)$$

Evaluating (3.4.1.9) at all the grid points. i.e, at z = -4, -3,-2,-1, 0 and 1 produces

$$10080hy_n' - 10080y_{n+3} + 10080y_{n+2} = h^2(-149f_{n+5} + 976f_{n+4} - 3478f_{n+3} - 5272f_{n+2} - 14101f_{n+1} - 3176f_n). \qquad (3.4.1.10)$$

$$10080hy_{n+1}' - 10080y_{n+3} + 10080y_{n+2} = h^2(40f_{n+5} - 235f_{n+4} - 104f_{n+3} - 10858f_{n+2} - 4112f_{n+1} + 149f_n). \qquad (3.4.1.11)$$

$$10080hy_{n+2}' - 10080y_{n+3} + 10080y_{n+2} = h^2(-37f_{n+5} + 304f_{n+4} - 1910f_{n+3} - 3704f_{n+2} + 347f_{n+1} - 40f_n). \qquad (3.4.1.12)$$

$$10080hy_{n+3}' - 10080y_{n+3} + 10080y_{n+2} = h^2(40f_{n+5} - 347f_{n+4} + 3704f_{n+3} + 1910f_{n+2} - 304f_{n+1} + 37f_n). \qquad (3.4.1.13)$$

$$10080hy_{n+4}' - 10080y_{n+3} + 10080y_{n+2} = h^2(-149f_{n+5} + 4112f_{n+4} + 10858f_{n+3} + 104f_{n+2} + 235f_{n+1} - 40f_n). \qquad (3.4.1.14)$$

$$10080hy_{n+5}' - 10080y_{n+3} + 10080y_{n+2} = h^2(3176f_{n+5} + 14101f_{n+4} + 5272f_{n+3} + 3478f_{n+2} - 976f_{n+1} + 149f_n). \qquad (3.4.1.15)$$

Combining equations (3.4.1.5) - (3.4.1.8) and (3.4.1.10) gives a block of the form (1.10) as follows

$$\begin{pmatrix} 0 & -720 & 480 & 0 & 0 \\ 240 & -480 & 240 & 0 & 0 \\ 0 & 240 & -480 & 240 & 0 \\ 0 & 480 & -720 & 0 & 240 \\ 0 & 10080 & -10080 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & -240 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} +$$

$$
h\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -240 \end{pmatrix}\begin{pmatrix} y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_{n} \end{pmatrix} + h^2\begin{pmatrix} 257 & 392 & 62 & -8 & 1 \\ 24 & 194 & 24 & -1 & 0 \\ -1 & 24 & 194 & 24 & -1 \\ -8 & 62 & 392 & 257 & 16 \\ -14101 & -5272 & -3478 & 976 & -149 \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} +
$$

$$
h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -3176 \end{pmatrix}\begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}
$$

The above equation is multiplied by $(A^0)^{-1}$ to give

$$
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_{n} \end{pmatrix} +
$$

$$
h^2\begin{pmatrix} \frac{8630}{20160} & \frac{-411}{1361} & \frac{250}{1339} & \frac{-1364}{20160} & \frac{214}{20160} \\ \frac{1088}{630} & \frac{-37}{63} & \frac{272}{630} & \frac{-101}{630} & \frac{16}{630} \\ \frac{31509}{10080} & \frac{-9}{140} & \frac{87}{112} & \frac{-2592}{10080} & \frac{9}{224} \\ \frac{1424}{315} & \frac{176}{315} & \frac{608}{315} & \frac{-16}{63} & \frac{16}{315} \\ \frac{2203}{374} & \frac{625}{504} & \frac{1578}{509} & \frac{625}{1008} & \frac{275}{2016} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} + h^2\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{711}{2911} \\ 0 & 0 & 0 & 0 & \frac{355}{630} \\ 0 & 0 & 0 & 0 & \frac{8856}{10080} \\ 0 & 0 & 0 & 0 & \frac{376}{315} \\ 0 & 0 & 0 & 0 & \frac{1525}{1008} \end{pmatrix}\begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix} \qquad (3.4.1.16)
$$

which leads to

$$
y_{n+1} = y_n + hy'_n + \frac{h^2}{20160}(214 f_{n+5} - 1364 f_{n+4} + 3764 f_{n+3} - 6088 f_{n+2} + 8630 f_{n+1} + 4924 f_n). \qquad (3.4.1.17)
$$

$$
y_{n+2} = y_n + 2hy'_n + \frac{h^2}{630}(16 f_{n+5} - 101 f_{n+4} + 272 f_{n+3} - 370 f_{n+2} + 1088 f_{n+1} + 355 f_n). \qquad (3.4.1.18)
$$

$$y_{n+3} = y_n + 3hy'_n + \frac{h^2}{10080}(405 f_{n+5} - 2592 f_{n+4} + 7830 f_{n+3} - 648 f_{n+2} + 31509 f_{n+1} + 8856 f_n). \tag{3.4.1.19}$$

$$y_{n+4} = y_n + 4hy'_n + \frac{h^2}{630}(32 f_{n+5} - 160 f_{n+4} + 1216 f_{n+3} + 352 f_{n+2} + 2848 f_{n+1} + 752 f_n). \tag{3.4.1.20}$$

$$y_{n+5} = y_n + 5hy'_n + \frac{h^2}{10080}(1375 f_{n+5} + 6250 f_{n+4} + 31250 f_{n+3} + 12500 f_{n+2} + 59375 f_{n+1} + 1525 f_n). \tag{3.4.1.21}$$

Substituting (3.4.1.18) and (3.4.1.19) into (3.4.1.11) – (3.4.1.15) to give the derivative

$$y'_{n+1} = y'_n + \frac{h}{1440}(27 f_{n+5} - 173 f_{n+4} + 482 f_{n+3} - 798 f_{n+2} + 1427 f_{n+1} + 475 f_n). \tag{3.4.1.22}$$

$$y'_{n+2} = y'_n + \frac{h}{90}(f_{n+5} - 6 f_{n+4} + 14 f_{n+3} + 14 f_{n+2} + 129 f_{n+1} + 28 f_n). \tag{3.4.1.23}$$

$$y'_{n+3} = y'_n + \frac{h}{160}(3 f_{n+5} - 21 f_{n+4} + 114 f_{n+3} + 114 f_{n+2} + 219 f_{n+1} + 51 f_n). \tag{3.4.1.24}$$

$$y'_{n+4} = y'_n + \frac{h}{45}(14 f_{n+4} + 64 f_{n+3} + 24 f_{n+2} + 64 f_{n+1} + 14 f_n). \tag{3.4.1.25}$$

$$y'_{n+5} = y'_n + \frac{h}{288}(95 f_{n+5} + 375 f_{n+4} + 250 f_{n+3} + 250 f_{n+2} + 375 f_{n+1} + 95 f_n). \tag{3.4.1.26}$$

3.4.2 Properties of Five–Step Block Method for Second Order ODEs

This section establishes the order, zero-stability and region of absolute stability of five–step block method.

3.4.2.1 Order of Five–Step Block Method for Second Order ODEs.

The method used in section 3.2.2.1 is applied in finding the order of the block method (3.4.1.17 – 3.4.1.21) as shown below

$$\begin{pmatrix} \sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m - \sum_{m=0}^{1}\frac{h^m}{m!}y_n^m - \frac{4924}{20160}h^2y_n'' - \sum_{m=0}^{\infty}\frac{h^{2+m}}{(20160)(m!)}y_n^{(2+m)}\begin{pmatrix}8630(1)^m - 6088(2)^m + 3764(3)^m \\ -1364(4)^m + 214(5)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m - \sum_{m=0}^{1}\frac{(2h)^m}{m!}y_n^m - \frac{355}{630}h^2y_n'' - \sum_{m=0}^{\infty}\frac{h^{2+m}}{(630)(m!)}y_n^{(2+m)}\begin{pmatrix}1088(1)^m - 370(2)^m + 272(3)^m \\ -101(4)^m + 16(5)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m - \sum_{m=0}^{1}\frac{(3h)^m}{m!}y_n^m - \frac{8856}{10080}h^2y_n'' - \sum_{m=0}^{\infty}\frac{h^{2+m}}{(10080)(m!)}y_n^{(2+m)}\begin{pmatrix}31509(1)^m - 648(2)^m + 7830(3)^m \\ -2592(4)^m + 405(5)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m - \sum_{m=0}^{1}\frac{(4h)^m}{m!}y_n^m - \frac{752}{630}h^2y_n'' - \sum_{m=0}^{\infty}\frac{h^{2+m}}{(630)(m!)}y_n^{(2+m)}\begin{pmatrix}2848(1)^m + 352(2)^m + 1216(3)^m \\ -160(4)^m + 32(5)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^m - \sum_{m=0}^{1}\frac{(5h)^m}{m!}y_n^m - \frac{15250}{10080}h^2y_n'' - \sum_{m=0}^{\infty}\frac{h^{2+m}}{(10080)(m!)}y_n^{(2+m)}\begin{pmatrix}59375(1)^m + 12500(2)^m + 31250(3)^m \\ +6250(4)^m + 1375(5)^m\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

Comparing the coefficients of h^m and y_n^m gives

$$C_0 = \begin{pmatrix}1-1\\1-1\\1-1\\1-1\\1-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$C_1 = \begin{pmatrix}1-1\\2-2\\3-3\\4-4\\5-5\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{4924}{20160} - \frac{1}{(20160)(0!)}\left(8630(1)^0 - 6088(2)^0 + 3764(3)^0 - 1364(4)^0 + 214(5)^0\right) \\ \frac{(2h)^2}{2!} - \frac{355}{630} - \frac{1}{(630)(0!)}\left(1088(1)^0 - 370(2)^0 + 272(3)^0 - 101(4)^0 + 16(5)^0\right) \\ \frac{(3h)^2}{2!} - \frac{8856}{10080} - \frac{1}{(10080)(0!)}\left(31509(1)^0 - 648(2)^0 + 7830(3)^0 - 2592(4)^0 + 405(5)^0\right) \\ \frac{(4h)^2}{2!} - \frac{752}{630} - \frac{1}{(630)(0!)}\left(2848(1)^0 + 352(2)^0 + 1216(3)^0 - 160(4)^0 + 32(5)^0\right) \\ \frac{(5h)^2}{2!} - \frac{15250}{10080} - \frac{1}{(10080)(0!)}\left(59375(1)^0 + 12500(2)^0 + 31250(3)^0 + 6250(4)^0 + 1375(5)^0\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} \frac{1}{3!} - \frac{1}{(20160)(1!)}\left(8630(1)^1 - 6088(2)^1 + 3764(3)^1 - 1364(4)^1 + 214(5)^1\right) \\ \frac{(2h)^3}{3!} - \frac{1}{(630)(1!)}\left(1088(1)^1 - 370(2)^1 + 272(3)^1 - 101(4)^1 + 16(5)^1\right) \\ \frac{(3h)^3}{3!} - \frac{1}{(10080)(1!)}\left(31509(1)^1 - 648(2)^1 + 7830(3)^1 - 2592(4)^1 + 405(5)^1\right) \\ \frac{(4h)^3}{3!} - \frac{1}{(630)(1!)}\left(2848(1)^1 + 352(2)^1 + 1216(3)^1 - 160(4)^1 + 32(5)^1\right) \\ \frac{(5h)^3}{3!} - \frac{1}{(10080)(1!)}\left(59375(1)^1 + 12500(2)^1 + 31250(3)^1 + 6250(4)^1 + 1375(5)^1\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_4 = \begin{pmatrix} \frac{1}{4!} - \frac{1}{(20160)(2!)}\left(8630(1)^2 - 6088(2)^2 + 3764(3)^2 - 1364(4)^2 + 214(5)^2\right) \\ \frac{(2h)^4}{4!} - \frac{1}{(630)(2!)}\left(1088(1)^2 - 370(2)^2 + 272(3)^2 - 101(4)^2 + 16(5)^2\right) \\ \frac{(3h)^4}{4!} - \frac{1}{(10080)(2!)}\left(31509(1)^2 - 648(2)^2 + 7830(3)^2 - 2592(4)^2 + 405(5)^2\right) \\ \frac{(4h)^4}{4!} - \frac{1}{(630)(2!)}\left(2848(1)^2 + 352(2)^2 + 1216(3)^2 - 160(4)^2 + 32(5)^2\right) \\ \frac{(5h)^4}{4!} - \frac{1}{(10080)(2!)}\left(59375(1)^2 + 12500(2)^2 + 31250(3)^2 + 6250(4)^2 + 1375(5)^2\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_5 = \begin{pmatrix} \dfrac{1}{5!} - \dfrac{1}{(20160)(3!)}\left(8630(1)^3 - 6088(2)^3 + 3764(3)^3 - 1364(4)^3 + 214(5)^3\right) \\ \dfrac{(2h)^5}{5!} - \dfrac{1}{(630)(3!)}\left(1088(1)^3 - 370(2)^3 + 272(3)^3 - 101(4)^3 + 16(5)^3\right) \\ \dfrac{(3h)^5}{5!} - \dfrac{1}{(10080)(3!)}\left(31509(1)^3 - 648(2)^3 + 7830(3)^3 - 2592(4)^3 + 405(5)^3\right) \\ \dfrac{(4h)^5}{5!} - \dfrac{1}{(630)(3!)}\left(2848(1)^3 + 352(2)^3 + 1216(3)^3 - 160(4)^3 + 32(5)^3\right) \\ \dfrac{(5h)^5}{5!} - \dfrac{1}{(10080)(3!)}\left(59375(1)^3 + 12500(2)^3 + 31250(3)^3 + 6250(4)^3 + 1375(5)^3\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6 = \begin{pmatrix} \dfrac{1}{6!} - \dfrac{1}{(20160)(4!)}\left(8630(1)^4 - 6088(2)^4 + 3764(3)^4 - 1364(4)^4 + 214(5)^4\right) \\ \dfrac{(2h)^6}{6!} - \dfrac{1}{(630)(4!)}\left(1088(1)^4 - 370(2)^4 + 272(3)^4 - 101(4)^4 + 16(5)^4\right) \\ \dfrac{(3h)^6}{6!} - \dfrac{1}{(10080)(4!)}\left(31509(1)^4 - 648(2)^4 + 7830(3)^4 - 2592(4)^4 + 405(5)^4\right) \\ \dfrac{(4h)^6}{6!} - \dfrac{1}{(630)(4!)}\left(2848(1)^4 + 352(2)^4 + 1216(3)^4 - 160(4)^4 + 32(5)^4\right) \\ \dfrac{(5h)^6}{6!} - \dfrac{1}{(10080)(4!)}\left(59375(1)^4 + 12500(2)^4 + 31250(3)^4 + 6250(4)^4 + 1375(5)^4\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_7 = \begin{pmatrix} \dfrac{1}{7!} - \dfrac{1}{(20160)(5!)}\left(8630(1)^5 - 6088(2)^5 + 3764(3)^5 - 1364(4)^5 + 214(5)^5\right) \\ \dfrac{(2h)^7}{7!} - \dfrac{1}{(630)(5!)}\left(1088(1)^5 - 370(2)^5 + 272(3)^5 - 101(4)^5 + 16(5)^5\right) \\ \dfrac{(3h)^7}{7!} - \dfrac{1}{(10080)(5!)}\left(31509(1)^5 - 648(2)^5 + 7830(3)^5 - 2592(4)^5 + 405(5)^5\right) \\ \dfrac{(4h)^7}{7!} - \dfrac{1}{(630)(5!)}\left(2848(1)^5 + 352(2)^5 + 1216(3)^5 - 160(4)^5 + 32(5)^5\right) \\ \dfrac{(5h)^7}{7!} - \dfrac{1}{(10080)(5!)}\left(59375(1)^5 + 12500(2)^5 + 31250(3)^5 + 6250(4)^5 + 1375(5)^5\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_8 = \begin{pmatrix} \frac{1}{8!} - \frac{1}{(20160)(6!)}\left(8630(1)^6 - 6088(2)^6 + 3764(3)^6 - 1364(4)^6 + 214(5)^6\right) \\ \frac{(2h)^7}{7!} - \frac{1}{(630)(6!)}\left(1088(1)^6 - 370(2)^6 + 272(3)^6 - 101(4)^6 + 16(5)^6\right) \\ \frac{(3h)^8}{8!} - \frac{1}{(10080)(6!)}\left(31509(1)^6 - 648(2)^6 + 7830(3)^6 - 2592(4)^6 + 405(5)^6\right) \\ \frac{(4h)^8}{8!} - \frac{1}{(630)(6!)}\left(2848(1)^6 + 352(2)^6 + 1216(3)^6 - 160(4)^6 + 32(5)^6\right) \\ \frac{(5h)^8}{8!} - \frac{1}{(10080)(6!)}\left(59375(1)^6 + 12500(2)^6 + 31250(3)^6 + 6250(4)^6 + 1375(5)^6\right) \end{pmatrix} = \begin{pmatrix} \frac{-118}{14345} \\ \frac{-19}{945} \\ \frac{-141}{4480} \\ \frac{-8}{189} \\ \frac{-653}{11489} \end{pmatrix}$$

Hence, the order of the block gives $(6,6,6,6,6)^T$ with the error constants

$$\left(\frac{-118}{14345}, \frac{-19}{945}, \frac{-141}{4480}, \frac{-8}{189}, \frac{-653}{11489}\right)^T$$

3.4.2.2 Zero Stability of Five–Step Block Method for Second Order ODEs.

Applying equation (3.2.2.2.1), to the block (3.4.1.17– 3.4.1.21), we have

$$\det[rA^{(0)} - A^{(1)}] = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0$$

This implies $r = 0,0,0,0,1$. Hence, the method is zero stable.

3.4.2.3 Consistency and Convergence of Five–Step Block Method for Second Order ODEs

According to the conditions highlighted in Definition 1.4, the block method (3.4.1.17 – 3.4.1.21) is consistent. Hence, it is also convergent since it is zero-stable and consistent.

3.4.2.4 Region of Absolute Stability of Five–Step Block Method for Second Order ODEs.

Equation (3.2.2.4.2) is applied to the block (3.4.1.17– 3.4.1.21), we have

$$\bar{h}(\theta,h)=\frac{\begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 & 0 \\ 0 & 0 & 0 & e^{4i\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{5i\theta} \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}{\begin{pmatrix} \frac{8630}{20160}e^{i\theta} & -\frac{6088}{20160}e^{2i\theta} & \frac{3764}{20160}e^{3i\theta} & -\frac{1364}{20160}e^{4i\theta} & \frac{214}{20160}e^{5i\theta} \\ \frac{1088}{630}e^{i\theta} & -\frac{370}{630}e^{2i\theta} & \frac{272}{630}e^{3i\theta} & -\frac{101}{630}e^{4i\theta} & \frac{16}{630}e^{5i\theta} \\ \frac{31509}{10080}e^{i\theta} & -\frac{648}{10080}e^{2i\theta} & \frac{7830}{10080}e^{3i\theta} & -\frac{2592}{10080}e^{4i\theta} & \frac{405}{10080}e^{5i\theta} \\ \frac{2848}{630}e^{i\theta} & \frac{352}{630}e^{2i\theta} & \frac{1216}{630}e^{3i\theta} & -\frac{160}{630}e^{4i\theta} & \frac{32}{630}e^{5i\theta} \\ \frac{59375}{10080}e^{i\theta} & \frac{12500}{10080}e^{2i\theta} & \frac{31250}{10080}e^{3i\theta} & \frac{6250}{10080}e^{4i\theta} & \frac{1375}{10080}e^{5i\theta} \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{4924}{20160} \\ 0 & 0 & 0 & 0 & \frac{355}{630} \\ 0 & 0 & 0 & 0 & \frac{8856}{10080} \\ 0 & 0 & 0 & 0 & \frac{752}{630} \\ 0 & 0 & 0 & 0 & \frac{15250}{10080} \end{pmatrix}}$$

The above matrix is simplified and equating the imaginary part to zero, we have

$$\bar{h}(\theta,h)=\frac{1512\cos 5\theta-1512}{12\cos 5\theta+149}$$

Evaluating $\bar{h}(\theta,h)$ at interval of θ of 30^0 produces results as tabulated in Table 3.3

Table 3.3

Interval of Absolute Stability of Five–Step Block Method for Second Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta,h)$	0	-20.36	-4.88	-10.15	-15.86	-1.27	-22.07

Therefore, the interval of absolute stability is (-22.07, 0). This is shown in the diagram below

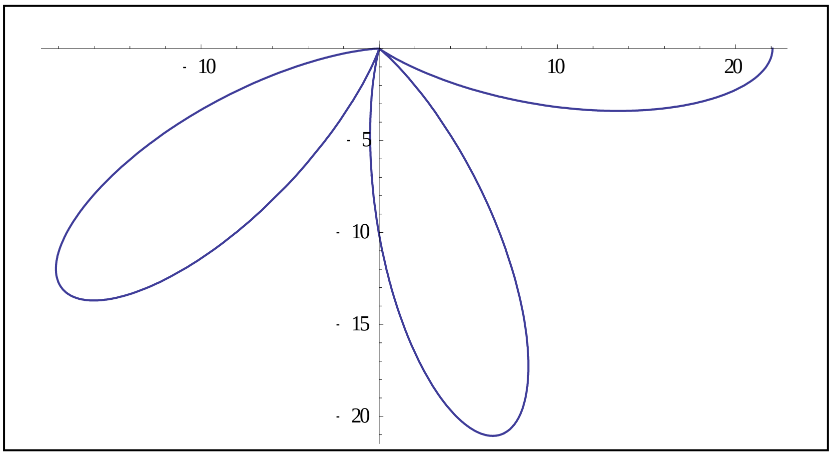

Figure 3.6. Region of absolute stability of five–step block method for second ODEs.

3.5 Six–Step Block Method for Second Order ODEs

In this section the derivation and properties of a six–step block method for solving second order ODEs are considered.

3.5.1 Derivation of Six–Step Block Method for Second Order ODEs.

Power series of the form (3.2.1.1) is considered as an approximate solution to the general second order problem of the form (3.2.1.2) where the step-length k =6. The first and second derivatives of (3.2.1.1) are given in (3.2.1.3) and (3.2.1.4).

Interpolating equation (3.2.1.1) at $x = x_{n+i}, i = 3,4$ and collocating (3.2.1.4) at $x = x_{n+i}, i = 0(1)6$ as illustrated below:

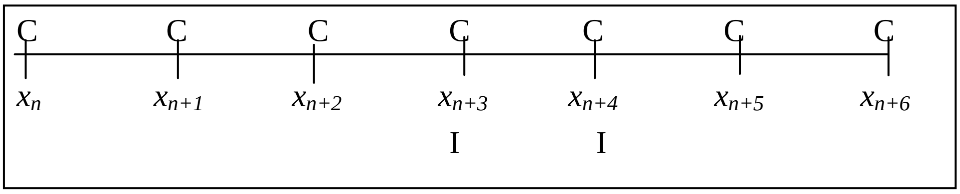

Figure 3.7. Six–step interpolation and collocation method for second order ODEs.

This approach gives the following results

$$
\begin{pmatrix}
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 \\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\
0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 \\
0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 \\
0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 \\
0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 \\
0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 & 56x_{n+5}^6 \\
0 & 0 & 2 & 6x_{n+6} & 12x_{n+6}^2 & 20x_{n+6}^3 & 30x_{n+6}^4 & 42x_{n+6}^5 & 56x_{n+6}^6
\end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix}
=
\begin{pmatrix} y_{n+3} \\ y_{n+4} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix}
\qquad (3.5.1.1)
$$

In finding the values of *a's* in (3.5.1.1) Gaussian elimination is employed. This produces the values of *a's* as follows

$$
\begin{aligned}
a_0 = \; & 4y_{n+3} - 3y_{n+4} + \frac{661}{10080}h^2 f_n + \frac{1789}{1680}h^2 f_{n+1} + \frac{2147}{1120}h^2 f_{n+2} + \frac{6817}{2520}h^2 f_{n+3} + \\
& \frac{841}{3360}h^2 f_{n+4} - \frac{1}{560}h^2 f_{n+5} - \frac{11}{10080}h^2 f_{n+6} + \frac{1}{2}x_n^2 f_n + \frac{x_n}{h}(y_{n+3} - y_{n+4}) + \\
& \frac{x_n^3}{360h}(147 f_n - 360 f_{n+1} + 450 f_{n+2} - 400 f_{n+3} + 225 f_{n+4} - 72 f_{n+5} + 10 f_{n+6}) + \\
& \frac{x_n^8}{40320h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) + \frac{5257h}{17280}x_n f_n \\
& + \frac{29431h}{20160}x_n f_{n+1} + \frac{1237h}{2688}x_n f_{n+2} + \frac{41443h}{30240}x_n f_{n+3} - \frac{6437h}{40320}x_n f_{n+4} + \frac{509h}{6720}x_n f_{n+5} \\
& - \frac{1313h}{120960}x_n f_{n+6} + \frac{x_n^7}{10080h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} \\
& + 5 f_{n+6}) + \frac{x_n^6}{4320h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} - 114 f_{n+5} \\
& + 17 f_{n+6}) + \frac{x_n^5}{960h^3}(49 f_n - 232 f_{n+1} + 461 f_{n+2} - 496 f_{n+3} + 307 f_{n+4} - 104 f_{n+5} \\
& + 15 f_{n+6}) + \frac{x_n^4}{4320h^2}(812 f_n - 3132 f_{n+1} + 5265 f_{n+2} - 5080 f_{n+3} + 2970 f_{n+4} - \\
& 972 f_{n+5} + 137 f_{n+6})
\end{aligned}
$$

$$
\begin{aligned}
a_1 = \; & \frac{6437}{40320}hf_{n+4} - \frac{5257}{17280}hf_n - \frac{29431}{20160}hf_{n+1} - \frac{1237}{2688}hf_{n+2} - \frac{41443}{30240}hf_{n+3} - \frac{509}{6720}hf_{n+5} \\
& + \frac{1313}{120960}hf_{n+6} - \frac{1}{h}(y_{n+3} - y_{n+4}) - \frac{x_n^2}{120h}(147 f_n - 360 f_{n+1} + 450 f_{n+2} - 400 f_{n+3} +
\end{aligned}
$$

$$225 f_{n+4} - 72 f_{n+5} + 10 f_{n+6}) - x_n f_n - \frac{x_n^7}{5040h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4}$$
$$-6 f_{n+5} + f_{n+6}) - \frac{x_n^6}{1440h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} +$$
$$5 f_{n+6}) - \frac{x_n^5}{720h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} - 114 f_{n+5} + 17 f_{n+6})$$
$$- \frac{x_n^4}{192h^3}(49 f_n - 232 f_{n+1} + 461 f_{n+2} - 496 f_{n+3} + 307 f_{n+4} - 104 f_{n+5} + 15 f_{n+6}) -$$
$$\frac{x_n^3}{10080h^2}(812 f_n - 3132 f_{n+1} + 5265 f_{n+2} - 5080 f_{n+3} + 2970 f_{n+4} - 972 f_{n+5} + 137 f_{n+6})$$

$$a_2 = \frac{1}{2} f_n + \frac{x_n^6}{1440h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) + \frac{x_n}{120h}(147 f_n$$
$$-360 f_{n+1} + 450 f_{n+2} - 400 f_{n+3} + 225 f_{n+4} - 72 f_{n+5} + 10 f_{n+6}) + \frac{x_n^5}{480h^5}(7 f_n - 40 f_{n+1} +$$
$$95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} + 5 f_{n+6}) + \frac{x_n^4}{288h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3}$$
$$+321 f_{n+4} - 114 f_{n+5} + 17 f_{n+6}) + \frac{x_n^3}{96h^3}(49 f_n - 232 f_{n+1} + 461 f_{n+2} - 496 f_{n+3} + 307 f_{n+4} -$$
$$104 f_{n+5} + 15 f_{n+6}) + \frac{x_n^2}{720h^2}(812 f_n - 3132 f_{n+1} + 5265 f_{n+2} - 5080 f_{n+3} + 2970 f_{n+4} - 972 f_{n+5}$$
$$+137 f_{n+6})$$

$$a_3 = \frac{-49}{120h} f_n + f_{n+1} - \frac{5}{4h} f_{n+2} + \frac{10}{9h} f_{n+3} - \frac{5}{8h} f_{n+4} + \frac{1}{5h} f_{n+5} - \frac{1}{36h} f_{n+6} - \frac{x_n^5}{720h^6}(f_n -$$
$$6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) - \frac{x_n}{1080h^2}(812 f_n - 3132 f_{n+1} + 5265 f_{n+2}$$
$$-5080 f_{n+3} + 2970 f_{n+4} - 972 f_{n+5} + 137 f_{n+6}) - \frac{x_n^4}{288h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3}$$
$$85 f_{n+4} - 32 f_{n+5} + 5 f_{n+6}) + \frac{x_n^3}{216h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} -$$
$$114 f_{n+5} + 17 f_{n+6}) - \frac{x_n^2}{96h^3}(49 f_n - 232 f_{n+1} + 461 f_{n+2} - 496 f_{n+3} + 307 f_{n+4} - 104 f_{n+5} +$$
$$15 f_{n+6}).$$

$$a_4 = \frac{203}{1080h^2} f_n - \frac{29}{40h^2} f_{n+1} + \frac{39}{32h^2} f_{n+2} - \frac{127}{108h^2} f_{n+3} + \frac{11}{16h^2} f_{n+4} - \frac{9}{40h^2} f_{n+5} + \frac{137}{4320h^2} f_{n+6}$$
$$+ \frac{x_n^4}{576h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) + \frac{x_n}{192h^3} 49 f_n - 232 f_{n+1} +$$

$$461 f_{n+2} - 496 f_{n+3} + 307 f_{n+4} - 104 f_{n+5} + 15 f_{n+6}) + \frac{x_n^3}{288h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3}$$
$$+85 f_{n+4} - 32 f_{n+5} + 5 f_{n+6}) + \frac{x_n^2}{288h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} - 114 f_{n+5}$$
$$+17 f_{n+6})$$

$$a_5 = \frac{-49}{960h^3} f_n + \frac{29}{120h^3} f_{n+1} - \frac{461}{960h^3} f_{n+2} + \frac{31}{60h^3} f_{n+3} - \frac{307}{960h^3} f_{n+4} + \frac{13}{120h^3} f_{n+5}$$
$$- \frac{1}{64h^3} f_{n+6} - \frac{x_n^3}{720h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) -$$
$$\frac{x_n}{720h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} - 114 f_{n+5} + 17 f_{n+6}) -$$
$$\frac{x_n^2}{480h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} + 5 f_{n+6})$$

$$a_6 = \frac{7}{864h^4} f_n - \frac{31}{720h^4} f_{n+1} + \frac{137}{1440h^4} f_{n+2} - \frac{121}{1080h^4} f_{n+3} + \frac{107}{1440h^4} f_{n+4} -$$
$$\frac{19}{720h^4} f_{n+5} + \frac{17}{4320h^4} f_{n+6} + + \frac{x_n^2}{1440h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4}$$
$$-6 f_{n+5} + f_{n+6}) + \frac{x_n}{1440h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5}$$
$$+5 f_{n+6})$$

$$a_7 = \frac{-1}{1440h^5} f_n + \frac{1}{252h^5} f_{n+1} - \frac{19}{2016h^5} f_{n+2} + \frac{1}{84h^5} f_{n+3} - \frac{17}{2016h^5} f_{n+4} + \frac{1}{315h^5} f_{n+5}$$
$$- \frac{1}{2016h^5} f_{n+6} - \frac{x_n}{5040h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6})$$

$$a_8 = \frac{1}{40320h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6})$$

The values of *a's* are substituted into equation (3.2.1.1) and simplified. This gives a continuous implicit scheme of the form:

$$y(x) = \sum_{j=3}^{k-2} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (3.5.1.2)$$

where $x = zh + x_n + 5h$ (3.5.1.3)

Substituting (3.5.1.3) into (3.51.2) and on simplifying produces

$$\alpha_3(z) = -1 - z$$
$$\alpha_4(z) = 2 + z$$
$$\beta_0(z) = \frac{1}{120960}(62 + 351z - 672z^3 - 364z^4 + 126z^5 + 140z^6 + 36z^7 + 3z^8)$$
$$\beta_1(z) = \frac{1}{120960}(372 - 2586z + 5040z^3 + 2604z^4 - 1008z^5 - 1008z^6 - 240z^7 - 18z^8)$$
$$\beta_2(z) = \frac{1}{120960}(426 + 8085z - 16800z^3 - 7980z^4 + 3654z^5 + 3108z^6 + 660z^7 + 45z^8)$$
$$\beta_3(z) = \frac{1}{120960}(10856 - 5772z + 33600z^3 + 13160z^4 - 8064z^5 - 5152z^6 - 960z^7 - 60z^8)$$
$$\beta_4(z) = \frac{1}{120960}(98706 + 135561z - 50400z^3 - 7140z^4 + 10458z^5 + 4788z^6 + 780z^7 + 45z^8)$$
$$\beta_5(z) = \frac{1}{120960}(11724 + 47238z + 60480z^2 + 25872z^3 - 4116z^4 - 7056z^5 - 2352z^6 - 336z^7 - 18z^8) \quad (3.5.1.4)$$
$$\beta_6(z) = \frac{1}{120960}(442 - 1437z + 3360z^3 + 3836z^4 + 1890z^5 + 476z^6 + 60z^7 + 3z^8)$$

Evaluating (3.5.1.4) at the non-interpolating points .that is, at z = -5, -4,-3, 0 and 1. This produces the following schemes

$$30240y_{n+4} - 40320y_{n+3} + 10080y_n = h^2(-11f_{n+6} - 18f_{n+5} + 2523f_{n+4} + 27268f_{n+3} + 19323f_{n+2} + 10734f_{n+1} + 661f_n). \quad (3.5.1.5)$$

$$40320y_{n+4} - 60480y_{n+3} + 20160y_{n+1} = h^2(31f_{n+6} - 354f_{n+5} + 4413f_{n+4} + 33988f_{n+3} + 20793f_{n+2} + 1662f_{n+1} - 53f_n). \quad (3.5.1.6)$$

$$60480y_{n+4} - 120960y_{n+3} + 60480y_{n+2} = h^2(31f_{n+6} - 438f_{n+5} + 6513f_{n+4} + 48268f_{n+3} + 6513f_{n+2} - 438f_{n+1} + 31f_n) \quad (3.5.1.7)$$

$$20160y_{n+6} - 60480y_{n+4} + 40320y_{n+3} = h^2(1291f_{n+6} + 21906f_{n+5} + 32133f_{n+4} + 6268f_{n+3} - 1467f_{n+2} + 402f_{n+1} - 53f_n). \quad (3.5.1.8)$$

$$60480y_{n+5} - 120960y_{n+4} + 60480y_{n+3} = h^2(-221f_{n+6} + 5862f_{n+5} + 49353f_{n+4} + 5428f_{n+3} + 213f_{n+2} - 186f_{n+1} + 31f_n). \quad (3.5.1.9)$$

The derivative of (3.5.1.4) gives

$$\alpha_2'(z) = -1$$
$$\alpha_3'(z) = 1$$
$$\beta_0'(z) = \frac{1}{120960}(351 - 2016z^2 - 1456z^3 + 630z^4 + 840z^5 + 252z^6 + 24z^7)$$
$$\beta_1'(z) = \frac{1}{120960}(-2586 + 15120z^2 + 10416z^3 - 5040z^4 - 6048z^5 - 1680z^6 - 144z^7)$$
$$\beta_2'(z) = \frac{1}{120960}(8085 - 50400z^2 - 31920z^3 + 18270z^4 + 18648z^5 + 4620z^6 + 360z^7)$$
$$\beta_3'(z) = \frac{1}{120960}(-5772 + 100800z^2 + 52640z^3 - 40320z^4 - 30912z^5 - 6720z^6 - 480z^7)$$
$$\beta_4'(z) = \frac{1}{120960}(135561 - 151200z^2 - 28560z^3 + 52290z^4 + 28728z^5 + 5460z^6 + 360z^7) \quad (3.5.1.10)$$
$$\beta_5'(z) = \frac{1}{120960}(47238 + 120960z + 77616z^2 - 16464z^3 - 35280z^4 - 14112z^5 - 2352z^6 + -144z^7)$$
$$\beta_6'(z) = \frac{1}{120960}(-1437 + 10080z^2 + 15344z^3 + 9450z^4 + 2856z^5 + 420z^6 + 24z^7)$$

Equation (3.5.1.10) is evaluated at all the grid points. i.e, at z = -5, -4, -3,-2,-1, 0 and 1. This produces the following schemes

$$120960hy_n' - 120960y_{n+4} + 120960y_{n+3} = h^2(-1313f_{n+6} + 9162f_{n+5} - 19311f_{n+4} + 165772f_{n+3} + 55665f_{n+2} + 176586f_{n+1} + 36799f_n). \quad (3.5.1.11)$$

$$120960hy_{n+1}' - 120960y_{n+4} + 120960y_{n+3} = h^2(-413f_{n+6} + 3462f_{n+5} - 21111f_{n+4} - 90764f_{n+3} - 148587f_{n+2} - 46362f_{n+1} + 1375f_n). \quad (3.5.1.12)$$

$$40320hy_{n+2}' - 40320y_{n+4} + 40320y_{n+3} = h^2(43f_{n+6} - 238f_{n+5} - 2171f_{n+4} - 41092f_{n+3} - 18203f_{n+2} + 1298f_{n+1} - 117f_n). \quad (3.5.1.13)$$

$$120960hy_{n+3}' - 120960y_{n+4} + 120960y_{n+3} = h^2(-253f_{n+6} + 2502f_{n+5} - 20055f_{n+4} - 48268f_{n+3} + 7029f_{n+2} - 1626f_{n+1} + 191f_n). \quad (3.5.1.14)$$

$$120960hy'_{n+4} - 120960y_{n+4} + 120960y_{n+3} = h^2(289f_{n+6} - 3018f_{n+5} + 41583f_{n+4} + 26740f_{n+3} - 6513f_{n+2} + 1590f_{n+1} - 191f_n). \tag{3.5.1.15}$$

$$40320hy'_{n+5} - 40320y_{n+4} + 40320y_{n+3} = h^2(-479f_{n+6} + 15746f_{n+5} + 45187f_{n+4} - 1924f_{n+3} + 2695f_{n+2} - 862f_{n+1} + 117f_n). \tag{3.5.1.16}$$

$$120960hy'_{n+6} - 120960y_{n+4} + 120960y_{n+3} = h^2(36737f_{n+6} + 177462f_{n+5} + 42639f_{n+4} + 69236f_{n+3} - 32337f_{n+2} + 10038f_{n+1} - 1375f_n). \tag{3.5.1.17}$$

Joining equations (3.5.1.5) - (3.5.1.9) and (3.5.1.11) gives a block of the form (1.10)

$$\begin{pmatrix} 0 & 0 & -40320 & 30240 & 0 & 0 \\ 10080 & 0 & -30240 & 20160 & 0 & 0 \\ 0 & 10080 & -20160 & 10080 & 0 & 0 \\ 0 & 0 & 10080 & -20160 & 10080 & 0 \\ 0 & 0 & 20160 & -30240 & 0 & 10080 \\ 0 & 0 & 10080 & -10080 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -10080 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -10080 \end{pmatrix} \begin{pmatrix} y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix} +$$

$$h^2 \begin{pmatrix} 10734 & 19323 & 27268 & 2523 & -18 & -11 \\ 1662 & 20793 & 33988 & 4413 & -354 & 31 \\ -438 & 6513 & 48268 & 6513 & -438 & 31 \\ -186 & 213 & 5428 & 49353 & 5862 & -221 \\ 402 & -1467 & 624 & 32133 & 21906 & 1291 \\ -176586 & -55665 & -165772 & 19311 & -9162 & 1313 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix} +$$

$$h^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 661 \\ 0 & 0 & 0 & 0 & 0 & -53 \\ 0 & 0 & 0 & 0 & 0 & 31 \\ 0 & 0 & 0 & 0 & 0 & 31 \\ 0 & 0 & 0 & 0 & 0 & -53 \\ 0 & 0 & 0 & 0 & 0 & -36799 \end{pmatrix} \begin{pmatrix} f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$

The above equation is multiplied by $(A^0)^{-1}$ to give

$$
\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}
\begin{pmatrix} y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4}\\ y_{n+5}\\ y_{n+6} \end{pmatrix}
=
\begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{pmatrix}
\begin{pmatrix} y_{n-5}\\ y_{n-4}\\ y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_{n} \end{pmatrix}
+h
\begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&2\\ 0&0&0&0&0&3\\ 0&0&0&0&0&4\\ 0&0&0&0&0&5\\ 0&0&0&0&0&6 \end{pmatrix}
\begin{pmatrix} y'_{n-5}\\ y'_{n-4}\\ y'_{n-3}\\ y'_{n-2}\\ y'_{n-1}\\ y'_{n} \end{pmatrix}
+
$$

$$
h^2
\begin{pmatrix}
\frac{257}{576} & \frac{-2830}{6653} & \frac{445}{1267} & \frac{-303}{1586} & \frac{403}{6720} & \frac{-118}{14345}\\
\frac{194}{105} & \frac{-8}{9} & \frac{788}{945} & \frac{-97}{210} & \frac{46}{315} & \frac{-19}{945}\\
\frac{1485}{448} & \frac{-715}{1333} & \frac{45}{32} & \frac{-3267}{4480} & \frac{513}{2240} & \frac{-141}{4480}\\
\frac{1504}{315} & \frac{-8}{105} & \frac{1455}{524} & \frac{-8}{9} & \frac{32}{105} & \frac{-8}{189}\\
\frac{754}{121} & \frac{584}{1507} & \frac{1216}{287} & \frac{-625}{2688} & \frac{275}{576} & \frac{-653}{11489}\\
\frac{54}{7} & \frac{27}{35} & \frac{204}{35} & \frac{27}{70} & \frac{54}{35} & 0
\end{pmatrix}
\begin{pmatrix} f_{n+1}\\ f_{n+2}\\ f_{n+3}\\ f_{n+4}\\ f_{n+5}\\ f_{n+6} \end{pmatrix}
+
$$

$$
h^2
\begin{pmatrix}
0&0&0&0&0&\frac{325}{1377}\\
0&0&0&0&0&\frac{1027}{1890}\\
0&0&0&0&0&\frac{759}{896}\\
0&0&0&0&0&\frac{1088}{945}\\
0&0&0&0&0&\frac{961}{660}\\
0&0&0&0&0&\frac{123}{70}
\end{pmatrix}
\begin{pmatrix} f_{n-5}\\ f_{n-4}\\ f_{n-3}\\ f_{n-2}\\ f_{n-1}\\ f_{n} \end{pmatrix}
\tag{3.5.1.18}
$$

whose solution is

$$
\begin{aligned}
y_{n+1} = y_n + hy'_n + \frac{h^2}{120960}(&-995 f_{n+6} + 7254 f_{n+5} - 23109 f_{n+4} + 42484 f_{n+3}\\
&- 51453 f_{n+2} + 57750 f_{n+1} + 28549 f_n).
\end{aligned}
\tag{3.5.1.19}
$$

$$
\begin{aligned}
y_{n+2} = y_n + 2hy'_n + \frac{h^2}{120960}(&-2432 f_{n+6} + 17664 f_{n+5} - 55872 f_{n+4} + 100864 f_{n+3}\\
&- 107520 f_{n+2} + 223488 f_{n+1} + 65728 f_n).
\end{aligned}
\tag{3.5.1.20}
$$

$$
\begin{aligned}
y_{n+3} = y_n + 3hy'_n + \frac{h^2}{4480}(&-141 f_{n+6} + 1026 f_{n+5} - 3267 f_{n+4} + 6300 f_{n+3}\\
&- 2403 f_{n+2} + 14850 f_{n+1} + 3795 f_n).
\end{aligned}
\tag{3.5.1.21}
$$

$$y_{n+4} = y_n + 4hy_n^{'} + \frac{h^2}{945}(-40f_{n+6} + 288f_{n+5} - 840f_{n+4} + 2624f_{n+3} - 72f_{n+2} + 4512f_{n+1} + 1088f_n). \quad (3.5.1.22)$$

$$y_{n+5} = y_n + 5hy_n^{'} + \frac{h^2}{24192}(-1375f_{n+6} + 11550f_{n+5} - 5625f_{n+4} + 102500f_{n+3} + 9375f_{n+2} + 150750f_{n+1} + 35225f_n). \quad (3.5.1.23)$$

$$y_{n+6} = y_n + 6hy_n^{'} + \frac{h^2}{140}(216f_{n+5} + 54f_{n+4} + 816f_{n+3} + 108f_{n+2} + 1080f_{n+1} + 246f_n). \quad (3.5.1.24)$$

Substituting (3.5.1.21) and (3.5.1.22) into (3.5.1.12) – (3.5.1.17) to give the derivative of the block

$$y_{n+1}^{'} = y_n^{'} + \frac{h}{60480}(-863f_{n+6} + 6312f_{n+5} - 20211f_{n+4} + 37504f_{n+3} - 46461f_{n+2} + 65112f_{n+1} + 19087f_n). \quad (3.5.1.25)$$

$$y_{n+2}^{'} = y_n^{'} + \frac{h}{3780}(-37f_{n+6} + 264f_{n+5} - 807f_{n+4} + 1328f_{n+3} + 33f_{n+2} + 5640f_{n+1} + 1139f_n). \quad (3.5.1.26)$$

$$y_{n+3}^{'} = y_n^{'} + \frac{h}{2240}(-29f_{n+6} + 216f_{n+5} - 729f_{n+4} + 2176f_{n+3} + 1161f_{n+2} + 3240f_{n+1} + 685f_n). \quad (3.5.1.27)$$

$$y_{n+4}^{'} = y_n^{'} + \frac{h}{1890}(-16f_{n+6} + 96f_{n+5} + 348f_{n+4} + 3008f_{n+3} + 768f_{n+2} + 572f_n). \quad (3.5.1.28)$$

$$y_{n+5}^{'} = y_n^{'} + \frac{h}{12096}(-275f_{n+6} + 5640_{n+5} + 11625f_{n+4} + 16000f_{n+3} + 6375f_{n+2} + 17400f_{n+1} + 3715f_n). \quad (3.5.1.29)$$

$$y_{n+6}^{'} = y_n^{'} + \frac{h}{140}(41f_{n+6} + 216f_{n+5} + 27f_{n+4} + 272f_{n+3} + 27f_{n+2} + 216f_{n+1} + 41f_n). \quad (3.5.1.30)$$

3.5.2 Properties of Six–Step Block Method for Second Order ODEs.

The order, zero-stability and absolute region of six–step block method are described in this section.

3.5.2.1 Order of Six–Step Block Method for Second Order ODEs.

The approach used in section 3.2.2.1 is also applied to find the order of the block method (3.5.1.19 – 3.5.1.24) as shown below

$$
\begin{pmatrix}
\sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{h^m}{m!}y_n^m-\frac{28549}{120960}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(120960*m!)}y_n^{(2+m)}\begin{pmatrix}57750(1)^m-51453(2)^m+42484(3)^m\\-23109(4)^m+7254(5)^m-995(6)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(2h)^m}{m!}y_n^m-\frac{65728}{120960}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(120960*m!)}y_n^{(2+m)}\begin{pmatrix}223488(1)^m-107520(2)^m+100864(3)^m\\-55872(4)^m+17664(5)^m-2432(6)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(3h)^m}{m!}y_n^m-\frac{3795}{4480}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(4480*m!)}y_n^{(2+m)}\begin{pmatrix}14850(1)^m-2403(2)^m+6300(3)^m\\-3267(4)^m+1026(5)^m-141(6)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(4h)^m}{m!}y_n^m-\frac{1088}{945}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(945*m!)}y_n^{(2+m)}\begin{pmatrix}4512(1)^m-72(2)^m+2624(3)^m\\-840(4)^m+288(5)^m-40(6)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(5h)^m}{m!}y_n^m-\frac{35225}{24192}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(24192*m!)}y_n^{(2+m)}\begin{pmatrix}150750(1)^m+9375(2)^m+102500(3)^m\\-5625(4)^m+11550(5)^m-1375(6)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(6h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(6h)^m}{m!}y_n^m-\frac{246}{140}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(140*m!)}y_n^{(2+m)}\begin{pmatrix}1080(1)^m+108(2)^m+816(3)^m\\+54(4)^m+216(5)^m\end{pmatrix}
\end{pmatrix}
=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}
$$

Comparing the coefficients of h^m and y_n^m produces

$$
C_0=\begin{pmatrix}1-1\\1-1\\1-1\\1-1\\1-1\\1-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}
$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \\ 5-5 \\ 6-6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix}
\frac{1}{2!} - \frac{28549}{120960} - \frac{1}{(120960)(0!)} \begin{pmatrix} 57750(1)^0 - 51453(2)^0 + 42484(3)^0 - 23109(4)^0 + 7254(5)^0 \\ -995(6)^0 \end{pmatrix} \\
\frac{(2)^2}{2!} - \frac{65728}{120960} - \frac{1}{(120960)(0!)} \begin{pmatrix} 223488(1)^0 - 107520(2)^0 + 100864(3)^0 - 55872(4)^0 + \\ 17664(5)^0 - 2432(6)^0 \end{pmatrix} \\
\frac{(3)^2}{2!} - \frac{3795}{4480} - \frac{1}{(4480)(0!)} \begin{pmatrix} 14850(1)^0 - 2403(2)^0 + 6300(3)^0 - 3267(4)^0 + 1026(5)^0 \\ -141(6)^0 \end{pmatrix} \\
\frac{(4)^2}{2!} - \frac{1088}{945} - \frac{1}{(945)(0!)} \left(4512(1)^0 - 72(2)^0 + 2624(3)^0 - 840(4)^0 + 288(5)^0 - 40(6)^0\right) \\
\frac{(5)^2}{2!} - \frac{35225}{24192} - \frac{1}{(24192)(0!)} \begin{pmatrix} 150750(1)^0 + 9375(2)^0 + 102500(3)^0 - 5625(4)^0 + 11550(5)^0 \\ -1375(6)^0 \end{pmatrix} \\
\frac{(6)^2}{2!} - \frac{246}{140_n} - \frac{1}{(140)(0!)} \left(1080(1)^0 + 108(2)^0 + 816(3)^0 + 54(4)^0 + 216(5)^0\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \begin{pmatrix}
\frac{1}{3!} - \frac{1}{(120960)(1!)} \begin{pmatrix} 57750(1)^1 - 51453(2)^1 + 42484(3)^1 - 23109(4)^1 + 7254(5)^1 \\ -995(6)^1 \end{pmatrix} \\
\frac{(2)^3}{3!} - \frac{1}{(120960)(1!)} \begin{pmatrix} 223488(1)^1 - 107520(2)^1 + 100864(3)^1 - 55872(4)^1 + \\ 17664(5)^1 - 2432(6)^1 \end{pmatrix} \\
\frac{(3)^3}{3!} - \frac{1}{(4480)(1!)} \begin{pmatrix} 14850(1)^1 - 2403(2)^1 + 6300(3)^1 - 3267(4)^1 + 1026(5)^1 \\ -141(6)^1 \end{pmatrix} \\
\frac{(4)^3}{3!} - \frac{1}{(945)(1!)} \left(4512(1)^1 - 72(2)^1 + 2624(3)^1 - 840(4)^1 + 288(5)^1 - 40(6)^1\right) \\
\frac{(5)^3}{3!} - \frac{1}{(24192)(1!)} \begin{pmatrix} 150750(1)^1 + 9375(2)^1 + 102500(3)^1 - 5625(4)^1 + 11550(5)^1 \\ -1375(6)^1 \end{pmatrix} \\
\frac{(6)^3}{3!} - \frac{1}{(140)(1!)} \left(1080(1)^1 + 108(2)^1 + 816(3)^1 + 54(4)^1 + 216(5)^1\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
C_4 = \begin{pmatrix}
\frac{1}{4!} - \frac{1}{(120960)(2!)}\left(\begin{array}{l} 57750(1)^2 - 51453(2)^2 + 42484(3)^2 - 23109(4)^2 + 7254(5)^2 \\ -995(6)^2 \end{array}\right) \\
\frac{(2)^4}{4!} - \frac{1}{(120960)(2!)}\left(\begin{array}{l} 223488(1)^2 - 107520(2)^2 + 100864(3)^2 - 55872(4)^2 + \\ 17664(5)^2 - 2432(6)^2 \end{array}\right) \\
\frac{(3)^4}{4!} - \frac{1}{(4480)(2!)}\left(\begin{array}{l} 14850(1)^2 - 2403(2)^2 + 6300(3)^2 - 3267(4)^2 + 1026(5)^2 \\ -141(6)^2 \end{array}\right) \\
\frac{(4)^4}{4!} - \frac{1}{(945)(2!)}\left(4512(1)^2 - 72(2)^2 + 2624(3)^2 - 840(4)^2 + 288(5)^2 - 40(6)^2\right) \\
\frac{(5)^4}{4!} - \frac{1}{(24192)(2!)}\left(\begin{array}{l} 150750(1)^2 + 9375(2)^2 + 102500(3)^2 - 5625(4)^2 + 11550(5)^2 \\ -1375(6)^2 \end{array}\right) \\
\frac{(6)^4}{4!} - \frac{1}{(140)(2!)}\left(1080(1)^2 + 108(2)^2 + 816(3)^2 + 54(4)^2 + 216(5)^2\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_5 = \begin{pmatrix}
\frac{1}{5!} - \frac{1}{(120960)(3!)}\left(\begin{array}{l} 57750(1)^3 - 51453(2)^3 + 42484(3)^3 - 23109(4)^3 + 7254(5)^3 \\ -995(6)^3 \end{array}\right) \\
\frac{(2)^5}{5!} - \frac{1}{(120960)(3!)}\left(\begin{array}{l} 223488(1)^3 - 107520(2)^3 + 100864(3)^3 - 55872(4)^3 + \\ 17664(5)^3 - 2432(6)^3 \end{array}\right) \\
\frac{(3)^5}{5!} - \frac{1}{(4480)(3!)}\left(\begin{array}{l} 14850(1)^3 - 2403(2)^3 + 6300(3)^3 - 3267(4)^3 + 1026(5)^3 \\ -141(6)^3 \end{array}\right) \\
\frac{(4)^5}{5!} - \frac{1}{(945)(3!)}\left(4512(1)^3 - 72(2)^3 + 2624(3)^3 - 840(4)^3 + 288(5)^3 - 40(6)^3\right) \\
\frac{(5)^5}{5!} - \frac{1}{(24192)(3!)}\left(\begin{array}{l} 150750(1)^3 + 9375(2)^3 + 102500(3)^3 - 5625(4)^3 + 11550(5)^3 \\ -1375(6)^3 \end{array}\right) \\
\frac{(6)^5}{5!} - \frac{1}{(140)(3!)}\left(1080(1)^3 + 108(2)^3 + 816(3)^3 + 54(4)^3 + 216(5)^3\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_6 = \begin{pmatrix}
\frac{1}{6!} - \frac{1}{(120960)(4!)}\left(\begin{array}{l} 57750(1)^4 - 51453(2)^4 + 42484(3)^4 - 23109(4)^4 + 7254(5)^4 \\ -995(6)^4 \end{array}\right) \\
\frac{(2)^6}{6!} - \frac{1}{(120960)(4!)}\left(\begin{array}{l} 223488(1)^4 - 107520(2)^4 + 100864(3)^4 - 55872(4)^4 + \\ 17664(5)^4 - 2432(6)^4 \end{array}\right) \\
\frac{(3)^6}{6!} - \frac{1}{(4480)(4!)}\left(\begin{array}{l} 14850(1)^4 - 2403(2)^4 + 6300(3)^4 - 3267(4)^4 + 1026(5)^4 \\ -141(6)^4 \end{array}\right) \\
\frac{(4)^6}{6!} - \frac{1}{(945)(4!)}\left(4512(1)^4 - 72(2)^4 + 2624(3)^4 - 840(4)^4 + 288(5)^4 - 40(6)^4\right) \\
\frac{(5)^6}{6!} - \frac{1}{(24192)(4!)}\left(\begin{array}{l} 150750(1)^4 + 9375(2)^4 + 102500(3)^4 - 5625(4)^4 + 11550(5)^4 \\ -1375(6)^4 \end{array}\right) \\
\frac{(6)^6}{6!} - \frac{1}{(140)(4!)}\left(1080(1)^4 + 108(2)^4 + 816(3)^4 + 54(4)^4 + 216(5)^4\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$C_7=\begin{pmatrix}
\frac{1}{7!}-\frac{1}{(120960)(5!)}\begin{pmatrix}57750(1)^5-51453(2)^5+42484(3)^5-23109(4)^5+7254(5)^5\\-995(6)^5\end{pmatrix}\\
\frac{(2)^7}{7!}-\frac{1}{(120960)(5!)}\begin{pmatrix}223488(1)^5-107520(2)^5+100864(3)^5-55872(4)^5+\\17664(5)^5-2432(6)^5\end{pmatrix}\\
\frac{(3)^7}{7!}-\frac{1}{(4480)(5!)}\begin{pmatrix}14850(1)^5-2403(2)^5+6300(3)^5-3267(4)^5+1026(5)^5\\-141(6)^5\end{pmatrix}\\
\frac{(4)^7}{7!}-\frac{1}{(945)(5!)}\left(4512(1)^5-72(2)^5+2624(3)^5-840(4)^5+288(5)^5-40(6)^5\right)\\
\frac{(5)^7}{7!}-\frac{1}{(24192)(5!)}\begin{pmatrix}150750(1)^5+9375(2)^5+102500(3)^5-5625(4)^5+11550(5)^5\\-1375(6)^5\end{pmatrix}\\
\frac{(6)^7}{7!}-\frac{1}{(140)(5!)}\left(1080(1)^5+108(2)^5+816(3)^5+54(4)^5+216(5)^5\right)
\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_8=\begin{pmatrix}
\frac{1}{8!}-\frac{1}{(120960)(6!)}\begin{pmatrix}57750(1)^6-51453(2)^6+42484(3)^6-23109(4)^6+7254(5)^6\\-995(6)^6\end{pmatrix}\\
\frac{(2)^8}{8!}-\frac{1}{(120960)(6!)}\begin{pmatrix}223488(1)^6-107520(2)^6+100864(3)^6-55872(4)^6+\\17664(5)^6-2432(6)^6\end{pmatrix}\\
\frac{(3)^8}{8!}-\frac{1}{(4480)(6!)}\begin{pmatrix}14850(1)^6-2403(2)^6+6300(3)^6-3267(4)^6+1026(5)^6\\-141(6)^6\end{pmatrix}\\
\frac{(4)^8}{8!}-\frac{1}{(945)(6!)}\left(4512(1)^6-72(2)^6+2624(3)^6-840(4)^6+288(5)^6-40(6)^6\right)\\
\frac{(5)^8}{8!}-\frac{1}{(24192)(6!)}\begin{pmatrix}150750(1)^6+9375(2)^6+102500(3)^6-5625(4)^6+11550(5)^6\\-1375(6)^6\end{pmatrix}\\
\frac{(6)^8}{8!}-\frac{1}{(140)(6!)}\left(1080(1)^6+108(2)^6+816(3)^6+54(4)^6+216(5)^6\right)
\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_9=\begin{pmatrix}
\frac{1}{9!}-\frac{1}{(120960)(7!)}\begin{pmatrix}57750(1)^7-51453(2)^7+42484(3)^7-23109(4)^7+7254(5)^7\\-995(6)^7\end{pmatrix}\\
\frac{(2)^9}{9!}-\frac{1}{(120960)(7!)}\begin{pmatrix}223488(1)^7-107520(2)^7+100864(3)^7-55872(4)^7+\\17664(5)^7-2432(6)^7\end{pmatrix}\\
\frac{(3)^9}{9!}-\frac{1}{(4480)(7!)}\begin{pmatrix}14850(1)^7-2403(2)^7+6300(3)^7-3267(4)^7+1026(5)^7\\-141(6)^7\end{pmatrix}\\
\frac{(4)^9}{9!}-\frac{1}{(945)(7!)}\left(4512(1)^7-72(2)^7+2624(3)^7-840(4)^7+288(5)^7-40(6)^7\right)\\
\frac{(5)^9}{9!}-\frac{1}{(24192)(7!)}\begin{pmatrix}150750(1)^7+9375(2)^7+102500(3)^7-5625(4)^7+11550(5)^7\\-1375(6)^7\end{pmatrix}\\
\frac{(6)^9}{9!}-\frac{1}{(140)(7!)}\left(1080(1)^7+108(2)^7+816(3)^7+54(4)^7+216(5)^7\right)
\end{pmatrix}=\begin{pmatrix}\frac{149}{22413}\\\frac{92}{5597}\\\frac{9}{350}\\\frac{159}{4544}\\\frac{148}{3305}\\\frac{9}{175}\end{pmatrix}$$

Hence, the block has order $(7,7,7,7,7,7)^T$ with error constants

$$\left(\frac{149}{22413},\frac{92}{5597},\frac{9}{350},\frac{159}{4544},\frac{148}{3305},\frac{9}{175}\right)^T.$$

3.5.2.2 Zero Stability of Six–Step Block Method for Second Order ODEs.

Applying equation (3.2.2.2.1) to the block (3.5.1.19 – 3.5.1.24) we have

$$\det[rA^{(0)}-A^{(1)}]=\left|r\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{pmatrix}\right|=0$$

which implies $r=0,0,0,0,0,1$. Hence, the method is zero stable.

3.5.2.3 Consistency and Convergence of Six–Step Block Method for Second Order ODEs

The block method (3.5.1.19 – 3.5.1.24) is consistent since the conditions listed in Definition 1.4 are satisfied. This implies that the block method is convergent because it is zero-stable and consistent.

3.5.2.4 Region of Absolute Stability of Six–Step Block Method for Second Order ODEs.

Applying the equation (3.2.2.4.2) to the block (3.5.1.19 – 3.5.1.24) we have

$$\bar{h}(\theta,h)=\frac{\begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{4i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{5i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{6i\theta} \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}{\begin{pmatrix} \frac{57750}{120960}e^{i\theta} & -\frac{51453}{120960}e^{2i\theta} & \frac{42484}{120960}e^{3i\theta} & -\frac{23109}{120960}e^{4i\theta} & \frac{7254}{120960}e^{5i\theta} & -\frac{995}{120960}e^{6i\theta} \\ \frac{223488}{120960}e^{i\theta} & -\frac{107520}{120960}e^{2i\theta} & \frac{100864}{120960}e^{3i\theta} & -\frac{55872}{120960}e^{4i\theta} & \frac{17664}{120960}e^{5i\theta} & -\frac{2432}{120960}e^{6i\theta} \\ \frac{14850}{4480}e^{i\theta} & -\frac{2403}{4480}e^{2i\theta} & \frac{6300}{4480}e^{3i\theta} & -\frac{3267}{4480}e^{4i\theta} & \frac{1026}{4480}e^{5i\theta} & -\frac{141}{4480}e^{6i\theta} \\ \frac{4512}{945}e^{i\theta} & -\frac{72}{945}e^{2i\theta} & \frac{2624}{945}e^{3i\theta} & -\frac{840}{945}e^{4i\theta} & \frac{288}{945}e^{5i\theta} & -\frac{40}{945}e^{6i\theta} \\ \frac{150750}{24192}e^{i\theta} & \frac{9375}{24192}e^{2i\theta} & \frac{102500}{24192}e^{3i\theta} & -\frac{5625}{24192}e^{4i\theta} & \frac{11550}{24192}e^{5i\theta} & -\frac{1375}{24192}e^{6i\theta} \\ \frac{1080}{140}e^{i\theta} & \frac{108}{140}e^{2i\theta} & \frac{816}{140}e^{3i\theta} & \frac{54}{140}e^{4i\theta} & \frac{216}{140}e^{5i\theta} & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{28549}{120960} \\ 0 & 0 & 0 & 0 & 0 & \frac{65728}{120960} \\ 0 & 0 & 0 & 0 & 0 & \frac{3795}{4480} \\ 0 & 0 & 0 & 0 & 0 & \frac{1088}{945} \\ 0 & 0 & 0 & 0 & 0 & \frac{35225}{24192} \\ 0 & 0 & 0 & 0 & 0 & \frac{246}{140} \end{pmatrix}}$$

Finding the determinant of the above matrix and equating the imaginary part to zero we have

$$\bar{h}(\theta,h)=\frac{926840803312848076800\cos 6\theta-926840803312848076800}{472877960873902080\cos 6\theta-742418398572026399 21}$$

Evaluating $\bar{h}(\theta,h)$ at interval of θ of 30^0 gives results as tabulated in Table 3.4

Table 3.4

Interval of Absolute Stability of Six–Step Block Method for Second Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta,h)$	0	23.47	0	23.47	0	23.47	0

Therefore, the interval of absolute stability is (0, 23.47). This is shown in the diagram below

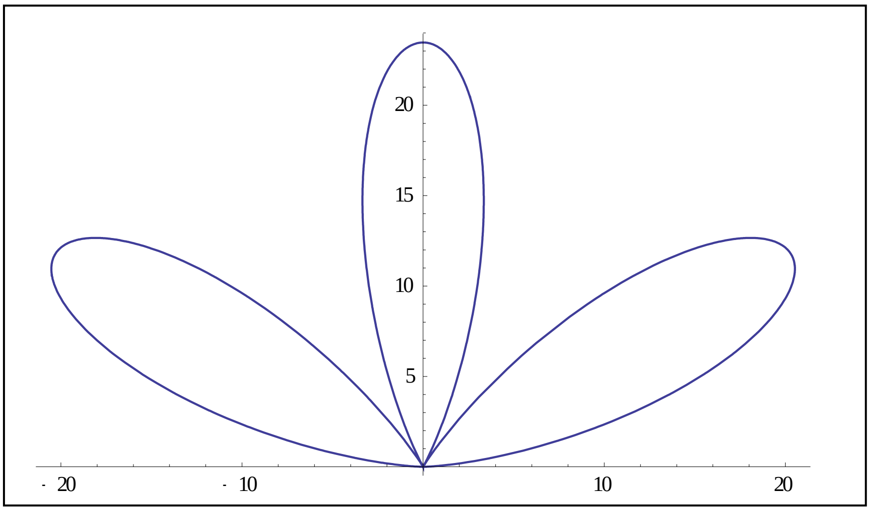

Figure 3.8. Region of absolute stability of six-step block method for second order ODEs.

3.6 Seven–Step Block Method for Second Order ODEs.

This section considers the derivation of block method with a step-length k=7 for direct solution of second order ODEs. The properties of the method are also established.

3.6.1 Derivation of Seven–Step Block Method for Second Order ODEs.

The use of power series of the form (3.2.1.1) is considered as an approximate solution to the general second order problem of the form (3.2.1.2) where k =7 is the step-length. The first and second derivatives of (3.2.1.1) are shown in (3.2.1.3) and (3.2.1.4).

Equation (3.2.1.1) is interpolated at $x = x_{n+i}, i = 4{,}5$ and (3.2.1.4) is collocated at $x = x_{n+i}, i = 0(1)7$ as shown in Fgure 3.9 below

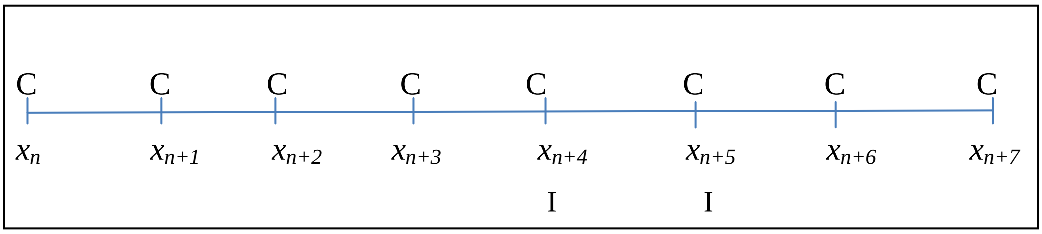

Figure 3.9. Seven–step interpolation and collocation method for second order ODEs.

This technique yields

$$\begin{pmatrix} 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 \\ 1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 & x_{n+5}^9 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 & 72x_{n+1}^7 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 & 72x_{n+2}^7 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 & 72x_{n+3}^7 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 & 72x_{n+4}^7 \\ 0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 & 56x_{n+5}^6 & 72x_{n+5}^7 \\ 0 & 0 & 2 & 6x_{n+6} & 12x_{n+6}^2 & 20x_{n+6}^3 & 30x_{n+6}^4 & 42x_{n+6}^5 & 56x_{n+6}^6 & 72x_{n+6}^7 \\ 0 & 0 & 2 & 6x_{n+7} & 12x_{n+7}^2 & 20x_{n+7}^3 & 30x_{n+7}^4 & 42x_{n+7}^5 & 56x_{n+7}^6 & 72x_{n+7}^7 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix} = \begin{pmatrix} y_{n+4} \\ y_{n+5} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \end{pmatrix} \quad (3.6.1.1)$$

In order to find the values of *a's*, the use of Gaussian elimination method is applied. The values of *a's* are shown below:

$$\begin{aligned} a_0 = \; & 5y_{n+4} - 4y_{n+5} + \frac{199}{30240}h^2 f_n + \frac{1363}{1260}h^2 f_{n+1} + \frac{18583}{10080}h^2 f_{n+2} + \frac{9707}{3024}h^2 f_{n+3} \\ & + \frac{6791}{2016}h^2 f_{n+4} + \frac{1193}{2520}h^2 f_{n+5} - \frac{1357}{30240}h^2 f_{n+6} + \frac{1}{240}h^2 f_{n+7} + \frac{1}{2}x_n^2 f_n + \frac{x_n^9}{362880h^7}(f_n \\ & -7f_{n+1} + 21f_{n+2} - 35f_{n+3} + 35f_{n+4} - 21f_{n+5} + 7f_{n+6} - f_{n+7}) + \frac{x_n^8}{40320h^6}(4f_n - 27f_{n+1} + \\ & 78f_{n+2} - 125f_{n+3} + 120f_{n+4} - 69f_{n+5} + 22f_{n+6} - 3f_{n+7}) + \frac{x_n^6}{4320h^4}(56f_n - 333f_{n+1} + 852f_{n+2} \\ & -1219f_{n+3} + 1056f_{n+4} - 555f_{n+5} + 164f_{n+6} - 21f_{n+7}) + \frac{x_n^7}{30240h^5}(46f_n - 295f_{n+1} + 810f_{n+2} \\ & -1235f_{n+3} + 1130f_{n+4} - 621f_{n+5} + 190f_{n+6} - 25f_{n+7}) + \frac{x_n^4}{4320h^2}(938f_n - 4014f_{n+1} + \\ & 7911f_{n+2} - 9490f_{n+3} + 7380f_{n+4} - 3618f_{n+5} + 1019f_{n+6} - 126f_{n+7}) + \frac{x_n^5}{14400h^3}(967f_n - \\ & 5104f_{n+1} + 11787f_{n+2} - 15560f_{n+3} + 12725f_{n+4} - 6432f_{n+5} + 1849f_{n+6} - 232f_{n+7}) + \end{aligned}$$

$$\frac{x_n^3}{2520h}(1089 f_n - 2940 f_{n+1} + 4410 f_{n+2} - 4900 f_{n+3} + 3675 f_{n+4} - 1764 f_{n+5} + 490 f_{n+6}$$
$$-60 f_{n+7}) + \frac{x_n}{h}(y_{n+4} - y_{n+5}) + \frac{535153h}{1814400} x_n f_n + \frac{345943h}{226800} x_n f_{n+1} + \frac{22303h}{86400} x_n f_{n+2} +$$
$$\frac{327109h}{181440} x_n f_{n+3} + \frac{113851h}{362880} x_n f_{n+4} + \frac{57191h}{151200} x_n f_{n+5} - \frac{150659h}{1814400} x_n f_{n+6} + \frac{8881h}{907200} x_n f_{n+7}$$

$$a_1 = \frac{150659}{1814400} hf_{n+6} - \frac{345943}{226800} hf_{n+1} - \frac{22303}{86400} hf_{n+2} - \frac{327109}{181440} hf_{n+3} - \frac{113851}{362880} hf_{n+4} - \frac{57191}{151200} hf_{n+5}$$
$$- \frac{535153}{1814400} hf_n - \frac{8881}{907200} hf_{n+7} - x_n f_n - \frac{1}{h}(y_{n+4} - y_{n+5}) - \frac{x_n^2}{840h}(1089 f_n - 2940 f_{n+1}$$
$$+4410 f_{n+2} - 4900 f_{n+3} + 3675 f_{n+4} - 1764 f_{n+5} + 490 f_{n+6} - 60 f_{n+7}) - \frac{x_n^8}{40320h^7}(f_n - 7 f_{n+1}$$
$$+21 f_{n+2} - 35 f_{n+3} + 35 f_{n+4} - 21 f_{n+5} + 7 f_{n+6} - f_{n+7}) - \frac{x_n^7}{5040h^6}(4 f_n - 27 f_{n+1} + 78 f_{n+2}$$
$$-125 f_{n+3} + 120 f_{n+4} - 69 f_{n+5} \quad +22 f_{n+6} - 3 f_{n+7}) - \frac{x_n^5}{720h^4}(56 f_n - 333 f_{n+1} + 852 f_{n+2}$$
$$-1219 f_{n+3} + 1056 f_{n+4} - 555 f_{n+5} + 164 f_{n+6} - 21 f_{n+7}) - \frac{x_n^6}{4320h^5}(46 f_n - 295 f_{n+1} + 810 f_{n+2}$$
$$-1235 f_{n+3} + 1130 f_{n+4} - 621 f_{n+5} + 190 f_{n+6} - 25 f_{n+7}).$$

$$a_2 = \frac{1}{2} f_n + \frac{x_n}{840h}(1089 f_n - 2940 f_{n+1} + 4410 f_{n+2} - 4900 f_{n+3} + 3675 f_{n+4} - 1764 f_{n+5}$$
$$+490 f_{n+6} - 60 f_{n+7}) + \frac{x_n^7}{10080h^7}(f_n - 7 f_{n+1} + 21 f_{n+2} - 35 f_{n+3} + 35 f_{n+4} - 21 f_{n+5} + 7 f_{n+6}$$
$$- f_{n+7}) + \frac{x_n^6}{1440h^6}(4 f_n - 27 f_{n+1} + 78 f_{n+2} - 125 f_{n+3} + 120 f_{n+4} - 69 f_{n+5} + 22 f_{n+6} - 3 f_{n+7})$$
$$+ \frac{x_n^4}{288h^4}(56 f_n - 333 f_{n+1} + 852 f_{n+2} - 1219 f_{n+3} + 1056 f_{n+4} - 555 f_{n+5} + 164 f_{n+6} - 21 f_{n+7})$$
$$+ \frac{x_n^5}{1440h^5}(46 f_n - 295 f_{n+1} + 810 f_{n+2} - 1235 f_{n+3} + 1130 f_{n+4} - 621 f_{n+5} + 190 f_{n+6} - 25 f_{n+7})$$
$$+ \frac{x_n^2}{720h^2}(938 f_n - 4014 f_{n+1} + 7911 f_{n+2} - 9490 f_{n+3} + 7380 f_{n+4} - 3618 f_{n+5} + 1019 f_{n+6}$$
$$-126 f_{n+7}) + \frac{x_n^3}{1440h^3}(967 f_n - 5104 f_{n+1} + 11787 f_{n+2} - 15560 f_{n+3} + 12725 f_{n+4} - 6432 f_{n+5} +$$
$$1849 f_{n+6} - 232 f_{n+7}).$$

$$a_3 = \frac{-121}{280h} f_n + \frac{7}{6h} f_{n+1} - \frac{7}{4h} f_{n+2} + \frac{35}{18h} f_{n+3} - \frac{35}{24h} f_{n+4} + \frac{7}{10h} f_{n+5} - \frac{7}{36h} f_{n+6}$$
$$+ \frac{1}{42h} f_{n+7} - \frac{x_n}{1080h^2}(938 f_n - 4014 f_{n+1} + 7911 f_{n+2} - 9490 f_{n+3} + 7380 f_{n+4} -$$

$$3618 f_{n+5} + 1019 f_{n+6} - 126 f_{n+7}) - 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) - \frac{x_n}{1080h^2}(812 f_n -$$

$$3132 f_{n+1} + 5265 f_{n+2} - 5080 f_{n+3} + 2970 f_{n+4} - 972 f_{n+5} + 137 f_{n+6}) - \frac{x_n^6}{4320h^7}(f_n$$

$$-7 f_{n+1} + 21 f_{n+2} - 35 f_{n+3} + 35 f_{n+4} - 21 f_{n+5} + 7 f_{n+6} - f_{n+7}) - \frac{x_n^5}{720h^6}(4 f_n$$

$$-27 f_{n+1} + 78 f_{n+2} - 3125 f_{n+3} + 120 f_{n+4} - 69 f_{n+5} + 22 f_{n+6} - 3 f_{n+7}) -$$

$$\frac{x_n^3}{216h^4}(56 f_n - 333 f_{n+1} + 852 f_{n+2} - 1219 f_{n+3} + 1056 f_{n+4} - 555 f_{n+5} + 164 f_{n+6}$$

$$-21 f_{n+7}) - \frac{x_n^4}{864h^5}(46 f_n - 295 f_{n+1} + 810 f_{n+2} - 1235 f_{n+3} + 1130 f_{n+4} - 621 f_{n+5}$$

$$+190 f_{n+6} - 25 f_{n+7}) - \frac{x_n^2}{1440h^3}(967 f_n - 5104 f_{n+1} + 11787 f_{n+2} - 15560 f_{n+3} +$$

$$12725 f_{n+4} - 6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7}) + 190 f_{n+6} - 25 f_{n+7}) - \frac{x_n^3}{1080h^2}(938 f_n$$

$$-4014 f_{n+1} + 7911 f_{n+2} - 9490 f_{n+3} + 7380 f_{n+4} - 3618 f_{n+5} + 1019 f_{n+6} - 126 f_{n+7})$$

$$-\frac{x_n^4}{2880h^3}(967 f_n - 5104 f_{n+1} + 11787 f_{n+2} - 15560 f_{n+3} + 12725 f_{n+4} - 6432 f_{n+5} +$$

$$1849 f_{n+6} - 232 f_{n+7}).$$

$$a_4 = \frac{469}{2160h^2} f_n - \frac{223}{240h^2} f_{n+1} + \frac{293}{160h^2} f_{n+2} - \frac{949}{432h^2} f_{n+3} + \frac{41}{24h^2} f_{n+4} - \frac{67}{80h^2} f_{n+5} +$$

$$\frac{1019}{4320h^2} f_{n+6} - \frac{7}{240h^2} f_{n+7} + \frac{x_n}{2880h^3}(967 f_n - 5104 f_{n+1} + 11787 f_{n+2} - 15560 f_{n+3} +$$

$$12725 f_{n+4} - 6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7}) - \frac{x_n^5}{2880h^7}(f_n - 7 f_{n+1} + 21 f_{n+2} - 35 f_{n+3}$$

$$+35 f_{n+4} - 21 f_{n+5} + 7 f_{n+6} - f_{n+7}) + \frac{x_n^4}{576h^6}(4 f_n - 27 f_{n+1} + 78 f_{n+2} - 125 f_{n+3} + 120 f_{n+4}$$

$$-69 f_{n+5} + 22 f_{n+6} - 3 f_{n+7}) + \frac{x_n^2}{288h^4}(56 f_n - 333 f_{n+1} + 852 f_{n+2} - 1219 f_{n+3} + 1056 f_{n+4}$$

$$555 f_{n+5} + 164 f_{n+6} - 21 f_{n+7}) + \frac{x_n^3}{864h^5}(46 f_n - 295 f_{n+1} + 810 f_{n+2} - 1235 f_{n+3} + 1130 f_{n+4}$$

$$-621 f_{n+5} + 190 f_{n+6} - 25 f_{n+7}).$$

$$a_5 = \frac{-967}{14400h^3} f_n + \frac{319}{900h^3} f_{n+1} - \frac{3929}{8400h^3} f_{n+2} + \frac{389}{360h^3} f_{n+3} - \frac{509}{576h^3} f_{n+4} + \frac{67}{150h^3} f_{n+5}$$

$$-\frac{1849}{14400h^3} f_{n+6} + \frac{29}{1800h^3} f_{n+7} - \frac{x_n}{720h^4}(56 f_n - 333 f_{n+1} + 852 f_{n+2} - 1219 f_{n+3} + 1056 f_{n+4}$$

$$-555 f_{n+5} + 164 f_{n+6} - 21 f_{n+7}) - \frac{x_n^4}{2880h^7}(f_n - 7 f_{n+1} + 21 f_{n+2} - 35 f_{n+3} + 35 f_{n+4} - 21 f_{n+5}$$

$$+7f_{n+6} - f_{n+7}) - \frac{x_n^3}{720h^6}(4f_n - 27f_{n+1} + 78f_{n+2} - 125f_{n+3} + 120f_{n+4} - 69f_{n+5} + 22f_{n+6} - 3f_{n+7}) - \frac{x_n^2}{1440h^5}(46f_n - 295f_{n+1} + 810f_{n+2} - 1235f_{n+3} + 1130f_{n+4} - 621f_{n+5} + 190f_{n+6} - 25f_{n+7}).$$

$$a_6 = \frac{7}{540h^4}f_n - \frac{37}{480h^4}f_{n+1} + \frac{71}{360h^4}f_{n+2} - \frac{1219}{4320h^4}f_{n+3} + \frac{11}{45h^4}f_{n+4} - \frac{37}{288h^4}f_{n+5} + \frac{41}{1080h^4}f_{n+6} - \frac{7}{1440h^4}f_{n+7} + \frac{x_n}{4320h^5}(46f_n - 295f_{n+1} + 810f_{n+2} - 1235f_{n+3} + 1130f_{n+4} - 621f_{n+5} + 190f_{n+6} - 25f_{n+7}) + \frac{x_n^3}{4320h^7}(f_n - 7f_{n+1} + 21f_{n+2} - 35f_{n+3} + 35f_{n+4} - 21f_{n+5} + 7f_{n+6} - f_{n+7}) + \frac{x_n^2}{1440h^6}(4f_n - 27f_{n+1} + 78f_{n+2} - 125f_{n+3} + 120f_{n+4} - 69f_{n+5} + 22f_{n+6} - 3f_{n+7}).$$

$$a_7 = \frac{-23}{15120h^5}f_n + \frac{59}{6048h^5}f_{n+1} - \frac{3}{112h^5}f_{n+2} + \frac{247}{6048h^5}f_{n+3} - \frac{113}{3024h^5}f_{n+4} + \frac{23}{1120h^5}f_{n+5} - \frac{19}{3024h^5}f_{n+6} - \frac{5}{6048h^5}f_{n+7} - \frac{x_n}{5040h^6}(4f_n - 27f_{n+1} + 78f_{n+2} - 125f_{n+3} + 120f_{n+4} - 69f_{n+5} + 22f_{n+6} - 3f_{n+7}) - \frac{x_n^2}{10080h^7}(f_n - 7f_{n+1} + 21f_{n+2} - 35f_{n+3} + 35f_{n+4} - 21f_{n+5} + 7f_{n+6} - f_{n+7}).$$

$$a_8 = \frac{1}{10080h^6}f_n - \frac{3}{4480h^6}f_{n+1} + \frac{13}{6720h^6}f_{n+2} - \frac{25}{8064h^6}f_{n+3} + \frac{1}{336h^6}f_{n+4} - \frac{23}{13440h^6}f_{n+5} + \frac{11}{20160h^6}f_{n+6} - \frac{7}{13440h^6}f_{n+7} + \frac{x_n}{40320h^7}(f_n - 7f_{n+1} + 21f_{n+2} - 35f_{n+3} + 35f_{n+4} - 21f_{n+5} + 7f_{n+6} - f_{n+7}).$$

$$a_9 = -\frac{1}{362880h^7}(f_n - 7f_{n+1} + 21f_{n+2} - 35f_{n+3} + 35f_{n+4} - 21f_{n+5} + 7f_{n+6} - f_{n+7}).$$

Substituting the values of a's into equation (3.2.1.1) and simplifying, this gives a continuous linear multistep method of the form:

$$y(x) = \sum_{j=4}^{k-2}\alpha_j(x)y_{n+j} + h^2\sum_{j=0}^{k}\beta_j(x)f_{n+j} \tag{3.6.1.2}$$

where $x = zh + x_n + 6h$ (3.6.1.3)

Substitute (3.6.1.3) into (3.6.1.2) and on simplifying we have

$$\alpha_4(z) = -1 - z$$
$$\alpha_5(z) = 2 + z$$
$$\beta_0(z) = \frac{1}{1814400}(-930 - 3793z + 7200z^3 + 4620z^4 - 882z^5 - 1680z^6$$
$$-600z^7 - 90z^8 - 5z^9)$$
$$\beta_1(z) = \frac{1}{1814400}(7440 + 31816z - 60480z^3 - 37800z^4 + 8064z^5 +$$
$$13860z^6 + 4740z^7 + 675z^8 + 35z^9)$$
$$\beta_2(z) = \frac{1}{604800}(8370 - 39481z + 75600z^3 + 45360z^4 - 11214z^5 -$$
$$16800z^6 - 5400z^7 - 720z^8 - 35z^9).$$
$$\beta_3(z) = \frac{1}{1814400}(38940 + 254030z - 504000z^3 - 281400z^4$$
$$+85680z^5 + 105420z^6 + 30900z^7 + 3825z^8 + 175z^9).$$
$$\beta_4(z) = \frac{1}{362880}(26058 - 43867z + 151200z^3 + 71820z^4$$
$$-30366z^5 - 27216z^6 - 7080z^7 - 810z^8 - 35z^9).$$
$$\beta_5(z) = \frac{1}{604800}(500040 + 704356z - 302400z^3 - 68040z^4$$
$$+58464z^5 + 35700z^6 + 8100z^7 + 855z^8 + 35z^9).$$
$$\beta_6(z) = \frac{1}{1814400}(169350 + 682019z + 907200z^2 + 438480z^3 -$$
$$29400z^4 - 112014z^5 - 47040z^6 - 9240z^7 - 840z^8 - 35z^9).$$
$$\beta_7(z) = \frac{1}{3628800}(-11400 - 35524z + 86400z^3 + 105840z^4 +$$
$$58464z^5 + 17640z^6 + 3000z^7 + 270z^8 + 10z^9). \quad (3.6.1.4)$$

Evaluating (3.6.1.4) at the non-interpolating points .i.e, at z = -6, -5, -4,-3, 0 and 1 gives

$$120960y_{n+5} - 151200y_{n+4} + 30240y_n = h^2(126f_{n+7} - 1357f_{n+6} + 14316f_{n+5}$$
$$+101865f_{n+4} + 97070f_{n+3} + 55749f_{n+2} + 32712f_{n+1} + 1919f_n). \quad (3.6.1.5)$$

$$90720y_{n+5} - 120960y_{n+4} + 30240y_{n+1} = h^2(31f_{n+7} - 502f_{n+6} + 8913f_{n+5}$$
$$+79564f_{n+4} + 60209f_{n+3} + 30858f_{n+2} + 2431f_{n+1} - 64f_n). \quad (3.6.1.6)$$

$$120960y_{n+5} - 181440y_{n+4} + 60480y_{n+2} = h^2(62f_{n+7} - 845f_{n+6} + 12588f_{n+5}$$
$$+103049f_{n+4} + 61294f_{n+3} + 5637f_{n+2} - 376f_{n+1} + 31f_n). \quad (3.6.1.7)$$

$$60480y_{n+5} - 120960y_{n+4} + 60480y_{n+3} = h^2(31f_{n+7} - 438f_{n+6} + 6513f_{n+5} + 48268f_{n+4} + 6513f_{n+3} - 438f_{n+2} + 31f_{n+1}). \tag{3.6.1.8}$$

$$60480y_{n+7} - 181440y_{n+5} + 120960y_{n+4} = h^2(3745f_{n+7} + 66614f_{n+6} + 93711f_{n+5} + 23284f_{n+4} - 8881f_{n+3} + 3894f_{n+2} - 1055f_{n+1} + 128f_n). \tag{3.6.1.9}$$

$$60480y_{n+6} - 120960y_{n+5} + 60480y_{n+4} = h^2(-190f_{n+7} + 5645f_{n+6} + 50004f_{n+5} + 4343f_{n+4} + 1298f_{n+3} - 837f_{n+2} + 248f_{n+1} - 31f_n). \tag{3.6.1.10}$$

The derivative of (3.6.1.4) gives

$$\alpha_4'(z) = -1$$
$$\alpha_5'(z) = 1$$

$$\begin{aligned}
\beta_0'(z) &= \frac{1}{1814400}(-3793 + 21600z^2 + 18480z^3 - 4410z^4 - 10080z^5 - 4200z^6 - 720z^7 - 45z^8).\\
\beta_1'(z) &= \frac{1}{1814400}(31816 - 181440z^2 - 151200z^3 + 40320z^4 + 83160z^5 + 33180z^6 + 5400z^7 + 315z^8).\\
\beta_2'(z) &= \frac{1}{1814400}(-118443 + 680400z^2 + 544320z^3 - 168210z^4 - 302400z^5 - 113400z^6 - 17280z^7 - 945z^8).\\
\beta_3'(z) &= \frac{1}{1814400}(254030 - 1512000z^2 - 1125600z^3 + 428400z^4 + 632520z^5 + 216300z^6 + 30600z^7 + 1575z^8).\\
\beta_4'(z) &= \frac{1}{1814400}(-219335 + 2268000z^2 + 1436400z^3 - 759150z^4 - 816480z^5 - 247800z^6 - 32400z^7 - 1575z^8).\\
\beta_5'(z) &= \frac{1}{1814400}(2113068 - 2721600z - 816480z^3 + 876960z^4 + 642600z^5 + 170100z^6 + 20520z^7 + 945z^8).\\
\beta_6'(z) &= \frac{1}{1814400}(682019 + 1814400z + 1315440z^2 - 117600z^3 - 560070z^4 - 282240z^5 - 64680z^6 - 7200z^7 - 315z^8).\\
\beta_7'(z) &= \frac{1}{1814400}(-17762 + 129600z^2 + 211680z^3 + 146160z^4 + 52920z^5 + 10500z^6 + 1080z^7 + 45z^8).
\end{aligned} \tag{3.6.1.11}$$

Equation (3.6.1.11) is evaluated at all the grid points. i.e, at z = -6, -5, -4, -3,-2,-1, 0 and 1 produces

$$1814400hy'_{n}-1814400y_{n+5}+1814400y_{n+4}=h^{2}(-17762f_{n+7}+150659f_{n+6}$$
$$-686292f_{n+5}-569255f_{n+4}-3271090f_{n+3}-468363f_{n+2}-2767544f_{n+1} \qquad (3.6.1.12)$$
$$-535153f_{n}).$$

$$1814400hy'_{n+1}-1814400y_{n+5}+1814400y_{n+4}=h^{2}(2863f_{n+7}-19606f_{n+6}$$
$$-63807f_{n+5}-1897460f_{n+4}-1424095f_{n+3}-2295318f_{n+2}-669809f_{n+1} \qquad (3.6.1.13)$$
$$+16832f_{n}).$$

$$1814400hy'_{n+2}-1814400y_{n+5}+1814400y_{n+4}=h^{2}(-2402f_{n+7}+25379f_{n+6}$$
$$-237012f_{n+5}-1494215f_{n+4}-2096050f_{n+3}-775083f_{n+2}+47176f_{n+1} \qquad (3.6.1.14)$$
$$-3793f_{n}).$$

$$1814400hy'_{n+3}-1814400y_{n+5}+1814400y_{n+4}=h^{2}(463f_{n+7}-406f_{n+6}$$
$$-128607f_{n+5}-1797620f_{n+4}-70655f_{n+3}+89322f_{n+2}-15569f_{n+1} \qquad (3.6.1.15)$$
$$+1472f_{n}).$$

$$1814400hy'_{n+4}-1814400y_{n+5}+1814400y_{n+4}=h^{2}(-2402f_{n+7}+27779f_{n+6}$$
$$-271572f_{n+5}-772775f_{n+4}+154190f_{n+3}-53643f_{n+2}+12616f_{n+1} \qquad (3.6.1.16)$$
$$-1393f_{n}).$$

$$1814400hy'_{n+5}-1814400y_{n+5}+1814400y_{n+4}=h^{2}(2863f_{n+7}-34966f_{n+6}$$
$$+592833f_{n+5}+452620f_{n+4}-149215f_{n+3}+54762f_{n+2}-13169f_{n+1} \qquad (3.6.1.17)$$
$$+1472f_{n}).$$

$$1814400hy'_{n+6}-1814400y_{n+5}+1814400y_{n+4}=h^{2}(-17762f_{n+7}+682019f_{n+6}$$
$$+2113068f_{n+5}-219335f_{n+4}+254030f_{n+3}-118443f_{n+2}+31816f_{n+1} \qquad (3.6.1.18)$$
$$-3793f_{n}).$$

$$1814400hy'_{n+7}-1814400y_{n+5}+1814400y_{n+4}=h^{2}(534223f_{n+7}+2779754f_{n+6}$$
$$+286113f_{n+5}+1627660f_{n+4}-1074175f_{n+3}+504042f_{n+2}-138449f_{n+1} \qquad (3.6.1.19)$$
$$+16832f_{n}).$$

Combining equations (3.6.1.5) - (3.6.1.10) and (3.6.1.12) to form a block of the form (1.10)

$$\begin{pmatrix} 0 & 0 & 0 & -151200 & 120960 & 0 & 0 \\ 30240 & 0 & 0 & -120960 & 90720 & 0 & 0 \\ 0 & 60480 & 0 & -181440 & 120960 & 0 & 0 \\ 0 & 0 & 60480 & -120960 & 60480 & 0 & 0 \\ 0 & 0 & 0 & 60480 & -120960 & 60480 & 0 \\ 0 & 0 & 0 & 120960 & -181440 & 0 & 60480 \\ 0 & 0 & 0 & 1814400 & -1814400 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \end{pmatrix} =$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -30240 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}
+ h
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1814400
\end{pmatrix}
\begin{pmatrix} y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}
+
$$

$$
h^2
\begin{pmatrix}
32712 & 55749 & 97070 & 101865 & 14316 & -1357 & 126 \\
2431 & 30858 & 60209 & 79564 & 8913 & -502 & 31 \\
-376 & 5637 & 61294 & 103049 & 12588 & -845 & 62 \\
31 & -438 & 6513 & 48268 & 6513 & -438 & 31 \\
248 & -837 & 1298 & 4343 & 50004 & 5645 & -190 \\
-1055 & 3894 & -8881 & 23284 & 93711 & 66614 & 3745 \\
-2767544 & -468363 & -3271090 & -569255 & -686292 & 150659 & -17762
\end{pmatrix}
\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \end{pmatrix}
$$

$$
+ h^2
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1919 \\
0 & 0 & 0 & 0 & 0 & 0 & -64 \\
0 & 0 & 0 & 0 & 0 & 0 & 31 \\
0 & 0 & 0 & 0 & 0 & 0 & 31 \\
0 & 0 & 0 & 0 & 0 & 0 & -31 \\
0 & 0 & 0 & 0 & 0 & 0 & 128 \\
0 & 0 & 0 & 0 & 0 & 0 & -535153
\end{pmatrix}
\begin{pmatrix} f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}
$$

The above equation is multiplied by the inverse of A^0 to give

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \end{pmatrix}
=
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}
+ h
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 7
\end{pmatrix}
\begin{pmatrix} y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}
$$

$$
+h^2\begin{pmatrix}
\frac{1115}{2128} & \frac{-513}{908} & \frac{943}{1615} & \frac{-1511}{3566} & \frac{283}{1418} & \frac{-226}{4127} & \frac{149}{22413}\\
\frac{2577}{1313} & \frac{-833}{675} & \frac{799}{567} & \frac{-1087}{1048} & \frac{783}{1594} & \frac{-113}{836} & \frac{141}{8578}\\
\frac{7961}{2278} & \frac{-1536}{1427} & \frac{369}{160} & \frac{-2641}{1621} & \frac{556}{723} & \frac{-623}{2946} & \frac{9}{350}\\
\frac{1541}{307} & \frac{-339}{418} & \frac{2837}{709} & \frac{-856}{405} & \frac{2653}{2552} & \frac{-661}{2301} & \frac{159}{4544}\\
\frac{5033}{769} & \frac{-1166}{2109} & \frac{1364}{235} & \frac{-2212}{1229} & \frac{1225}{864} & \frac{-995}{2687} & \frac{148}{3305}\\
\frac{1413}{175} & \frac{-54}{175} & \frac{267}{35} & \frac{-99}{70} & \frac{459}{175} & \frac{-9}{25} & \frac{9}{175}\\
\frac{2270}{237} & \frac{200}{7197} & \frac{12005}{1296} & \frac{-511}{649} & \frac{1065}{286} & \frac{734}{1299} & \frac{419}{3318}
\end{pmatrix}
\begin{pmatrix} f_{n+1}\\ f_{n+2}\\ f_{n+3}\\ f_{n+4}\\ f_{n+5}\\ f_{n+6}\\ f_{n+7}\end{pmatrix}+
$$

$$
h^2\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{581}{2533}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1183}{2245}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{653}{795}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{499}{447}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1026}{727}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{597}{350}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{609}{304}
\end{pmatrix}
\begin{pmatrix} f_{n-6}\\ f_{n-5}\\ f_{n-4}\\ f_{n-3}\\ f_{n-2}\\ f_{n-1}\\ f_{n}\end{pmatrix}. \qquad (3.6.1.20)
$$

which leads to

$$
\begin{aligned}
y_{n+1} = y_n + hy'_n + \frac{h^2}{1814400}(&12062 f_{n+7} - 99359 f_{n+6} + 36113 f_{n+5}\\
&- 768805 f_{n+4} + 1059430 f_{n+3} - 1025097 f_{n+2} + 950684 f_{n+1}\\
&+ 416173 f_n).
\end{aligned} \qquad (3.6.1.21)
$$

$$
\begin{aligned}
y_{n+2} = y_n + 2hy'_n + \frac{h^2}{28350}(&466 f_{n+7} - 3832 f_{n+6} + 13926 f_{n+5} - 29405 f_{n+4}\\
&+ 39950 f_{n+3} - 34986 f_{n+2} + 55642 f_{n+1} + 14939 f_n).
\end{aligned} \qquad (3.6.1.22)
$$

$$
\begin{aligned}
y_{n+3} = y_n + 3hy'_n + \frac{h^2}{604800}(&15552 f_{n+7} - 127899 f_{n+6} + 465102 f_{n+5}\\
&- 985365 f_{n+4} + 1394820 f_{n+3} - 650997 f_{n+2} + 2113614 f_{n+1}\\
&+ 496773 f_n).
\end{aligned} \qquad (3.6.1.23)
$$

$$y_{n+4} = y_n + 4hy_n' + \frac{h^2}{14175}(496 f_{n+7} - 4072 f_{n+6} + 14736 f_{n+5} - 29960 f_{n+4} + 56720 f_{n+3} - 11496 f_{n+2} + 71152 f_{n+1} + 15824 f_n). \tag{3.6.1.24}$$

$$y_{n+5} = y_n + 5hy_n' + \frac{h^2}{72576}(3250 f_{n+7} - 26875 f_{n+6} + 102900 f_{n+5} - 130625 f_{n+4} + 421250 f_{n+3} - 40125 f_{n+2} + 475000 f_{n+1} + 102425 f_n). \tag{3.6.1.25}$$

$$y_{n+6} = y_n + 6hy_n' + \frac{h^2}{350}(18 f_{n+7} - 126 f_{n+6} + 918 f_{n+5} - 495 f_{n+4} + 2670 f_{n+3} - 108 f_{n+2} + 2826 f_{n+1} + 597 f_n). \tag{3.6.1.26}$$

$$y_{n+7} = y_n + 7hy_n' + \frac{h^2}{259200}(32732 f_{n+7} - 146461 f_{n+6} + 965202 f_{n+5} - 204085 f_{n+4} + 2401000 f_{n+3} - 7203 f_{n+2} + 2482634 f_{n+1} + 519253 f_n). \tag{3.6.1.27}$$

Substituting (3.6.1.24) and (3.6.1.25) into (3.6.1.13) – (3.6.1.19) to give the derivative of the block

$$y'_{n+1} = y_n' + \frac{h}{120960}(1375 f_{n+7} - 11351 f_{n+6} + 41499 f_{n+5} - 88547 f_{n+4} + 123133 f_{n+3} - 121797 f_{n+2} + 139849 f_{n+1} + 36799 f_n). \tag{3.6.1.28}$$

$$y'_{n+2} = y_n' + \frac{h}{18900}(160 f_{n+7} - 1305 f_{n+6} + 4680 f_{n+5} - 9635 f_{n+4} + 12240 f_{n+3} - 3195 f_{n+2} + 29320 f_{n+1} + 5535 f_n). \tag{3.6.1.29}$$

$$y'_{n+3} = y_n' + \frac{h}{22400}(225 f_{n+7} - 1865 f_{n+6} + 6885 f_{n+5} - 15165 f_{n+4} + 29635 f_{n+3} + 6885 f_{n+2} + 33975 f_{n+1} + 6625 f_n). \tag{3.6.1.30}$$

$$y'_{n+4} = y_n' + \frac{h}{945}(8 f_{n+7} - 64 f_{n+6} + 216 f_{n+5} - 106 f_{n+4} + 1784 f_{n+3} + 216 f_{n+2} + 1448 f_{n+1} + 278 f_n). \tag{3.6.1.31}$$

$$y'_{n+5} = y_n' + \frac{h}{24192}(275 f_{n+7} - 2475 f_{n+6} + 17055 f_{n+5} + 13625 f_{n+4} + 41625 f_{n+3} + 6975 f_{n+2} + 36725 f_{n+1} + 7155 f_n). \tag{3.6.1.32}$$

$$y'_{n+6} = y_n' + \frac{h}{140}(41 f_{n+6} + 216 f_{n+5} + 27 f_{n+4} + 272 f_{n+3} + 27 f_{n+2} + 216 f_{n+1} + 41 f_n). \tag{3.6.1.33}$$

$$y'_{n+7} = y_n' + \frac{h}{17280}(5257 f_{n+7} + 25039 f_{n+6} + 9261 f_{n+5} + 20923 f_{n+4} + 20923 f_{n+3} + 9261 f_{n+2} + 25039 f_{n+1} + 5257 f_n). \tag{3.6.1.34}$$

3.6.2 Properties of Seven–Step Block Method for Second Order ODEs.

This section discusses the order, zero-stability and stability region of seven–step block method.

3.6.2.1 Order of Seven–Step Block Method for Second Order ODEs.

In finding the order of the block method (3.6.1.21 – 3.6.1.27), the strategy stated in section 3.2.2.1 is used and it is shown below

$$
\begin{pmatrix}
\sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{h^m}{m!}y_n^m-\frac{416173}{1814400}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(1814400)(m!)}y_n^{(2+m)}\begin{pmatrix}950684(1)^m-1025097(2)^m+1059430(3)^m\\-768805(4)^m+362112(5)^m-99359(6)^m\\+12062(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(2h)^m}{m!}y_n^m-\frac{14939}{28350}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(28350)(m!)}y_n^{(2+m)}\begin{pmatrix}55642(1)^m-34986(2)^m+39950(3)^m\\-29405(4)^m+13926(5)^m-3832(6)^m\\+466(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(3h)^m}{m!}y_n^m-\frac{496773}{604800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(604800)(m!)}y_n^{(2+m)}\begin{pmatrix}2113614(1)^m-650997(2)^m+1394820(3)^m\\-985365(4)^m+465102(5)^m-127899(6)^m\\+15552(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(4h)^m}{m!}y_n^m-\frac{15824}{14175}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(14175)(m!)}y_n^{(2+m)}\begin{pmatrix}71152(1)^m-11496(2)^m+56720(3)^m\\-29960(4)^m+14736(5)^m-4072(6)^m\\+496(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(5h)^m}{m!}y_n^m-\frac{102425}{72576}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(72576)(m!)}y_n^{(2+m)}\begin{pmatrix}475000(1)^m-40125(2)^m+421250(3)^m\\-130625(4)^m+102900(5)^m-26875(6)^m\\+3250(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(6h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(6h)^m}{m!}y_n^m-\frac{597}{350}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(350)(m!)}y_n^{(2+m)}\begin{pmatrix}2826(1)^m-108(2)^m+2670(3)^m\\-495(4)^m+918(5)^m-126(6)^m\\+18(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(7h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(7h)^m}{m!}y_n^m-\frac{519253}{259200}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(259200)(m!)}y_n^{(2+m)}\begin{pmatrix}2482634(1)^m-7203(2)^m+2401000(3)^m\\-204085(4)^m+965202(5)^m-146461(6)^m\\+32732(7)^m\end{pmatrix}
\end{pmatrix}
=
\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\end{pmatrix}
$$

Comparing the coefficients of h^m and y_n^m gives

$$C_0 = \begin{pmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \\ 5-5 \\ 6-6 \\ 7-7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{416173}{1814400} - \frac{1}{(1814400)(0!)} \begin{pmatrix} 950684(1)^0 - 1025097(2)^0 + 1059430(3)^0 - 768805(4)^0 + 362112(5)^0 - \\ 99359(6)^0 + 12062(7)^0 \end{pmatrix} \\ \frac{(2)^2}{2!} - \frac{14939}{28350} - \frac{1}{(28350)(0!)} \begin{pmatrix} 55642(1)^0 - 34986(2)^0 + 39950(3)^0 - 29405(4)^0 + 13926(5)^0 - 3832(6)^0 \\ + 466(7)^0 \end{pmatrix} \\ \frac{(3)^2}{2!} - \frac{496773}{604800} - \frac{1}{(604800)(0!)} \begin{pmatrix} 2113614(1)^0 - 650997(2)^0 + 1394820(3)^0 - 985365(4)^0 + 465102(5)^0 \\ -127899(6)^0 + 15552(7)^0 \end{pmatrix} \\ \frac{(4)^2}{2!} - \frac{15824}{14175} - \frac{1}{(14175)(0!)} \begin{pmatrix} 71152(1)^0 - 11496(2)^0 + 56720(3)^0 - 29960(4)^0 + 14736(5)^0 - 4072(6)^0 \\ + 496(7)^0 \end{pmatrix} \\ \frac{(5)^2}{2!} - \frac{102425}{72576} - \frac{1}{(72576)(0!)} \begin{pmatrix} 475000(1)^0 - 40125(2)^0 + 421250(3)^0 - 130625(4)^0 + 102900(5)^0 - \\ 26875(6)^0 + 3250(7)^0 \end{pmatrix} \\ \frac{(6)^2}{2!} - \frac{597}{350} - \frac{1}{(350)(0!)} \left(2826(1)^0 - 108(2)^0 + 2670(3)^0 - 495(4)^0 + 918(5)^0 - 126(6)^0 + 18(7)^0\right) \\ \frac{(7)^2}{2!} - \frac{102425}{259200} - \frac{1}{(259200)(0!)} \begin{pmatrix} 2482634(1)^0 - 7203(2)^0 + 2401000(3)^0 - 204085(4)^0 + 965202(5)^0 - \\ 146461(6)^0 + 32732(7)^0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
C_3 = \begin{pmatrix}
\frac{1}{3!} - \frac{1}{(1814400)(1!)}\begin{pmatrix} 950684(1)^1 - 1025097(2)^1 + 1059430(3)^1 - 768805(4)^1 + 362112(5)^1 - \\ 99359(6)^1 + 12062(7)^1 \end{pmatrix} \\
\frac{(2)^3}{3!} - \frac{1}{(28350)(1!)}\begin{pmatrix} 55642(1)^1 - 34986(2)^1 + 39950(3)^1 - 29405(4)^1 + 13926(5)^1 - 3832(6)^1 \\ + 466(7)^1 \end{pmatrix} \\
\frac{(3)^3}{3!} - \frac{1}{(604800)(1!)}\begin{pmatrix} 2113614(1)^1 - 650997(2)^1 + 1394820(3)^1 - 985365(4)^1 + 465102(5)^1 \\ -127899(6)^1 + 15552(7)^1 \end{pmatrix} \\
\frac{(4)^3}{3!} - \frac{1}{(14175)(1!)}\begin{pmatrix} 2482634(1)^1 - 7203(2)^1 + 2401000(3)^1 - 204085(4)^1 + 965202(5)^1 - \\ 146461(6)^1 + 32732(7)^1 \end{pmatrix} \\
\frac{(5)^2}{3!} - \frac{1}{(72576)(1!)}\begin{pmatrix} 475000(1)^1 - 40125(2)^1 + 421250(3)^1 - 130625(4)^1 + 102900(5)^1 - \\ 26875(6)^1 + 3250(7)^1 \end{pmatrix} \\
\frac{(6)^3}{3!} - \frac{1}{(350)(1!)}\left(2826(1)^1 - 108(2)^1 + 2670(3)^1 - 495(4)^1 + 918(5)^1 - 126(6)^1 + 18(7)^1\right) \\
\frac{(7)^3}{3!} - \frac{1}{(259200)(1!)}\begin{pmatrix} 2482634(1)^1 - 7203(2)^1 + 2401000(3)^1 - 204085(4)^1 + 965202(5)^1 - \\ 146461(6)^1 + 32732(7)^1 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_4 = \begin{pmatrix}
\frac{1}{4!} - \frac{1}{(1814400)(2!)}\begin{pmatrix} 950684(1)^2 - 1025097(2)^2 + 1059430(3)^2 - 768805(4)^2 + 362112(5)^2 - \\ 99359(6)^2 + 12062(7)^2 \end{pmatrix} \\
\frac{(2)^4}{4!} - \frac{1}{(28350)(2!)}\begin{pmatrix} 55642(1)^2 - 34986(2)^2 + 39950(3)^2 - 29405(4)^2 + 13926(5)^2 - 3832(6)^2 \\ + 466(7)^2 \end{pmatrix} \\
\frac{(3)^4}{4!} - \frac{1}{(604800)(2!)}\begin{pmatrix} 2113614(1)^2 - 650997(2)^2 + 1394820(3)^2 - 985365(4)^2 + 465102(5)^2 \\ -127899(6)^2 + 15552(7)^2 \end{pmatrix} \\
\frac{(4)^4}{4!} - \frac{1}{(14175)(2!)}\begin{pmatrix} 71152(1)^2 - 11496(2)^2 + 56720(3)^2 - 29960(4)^2 + 14736(5)^2 - 4072(6)^2 \\ + 496(7)^2 \end{pmatrix} \\
\frac{(5)^4}{4!} - \frac{1}{(72576)(2!)}\begin{pmatrix} 475000(1)^2 - 40125(2)^2 + 421250(3)^2 - 130625(4)^2 + 102900(5)^2 - \\ 26875(6)^2 + 3250(7)^2 \end{pmatrix} \\
\frac{(6)^4}{4!} - \frac{1}{(350)(2!)}\left(2826(1)^2 - 108(2)^2 + 2670(3)^2 - 495(4)^2 + 918(5)^2 - 126(6)^2 + 18(7)^2\right) \\
\frac{(7)^4}{4!} - \frac{1}{(259200)(2!)}\begin{pmatrix} 2482634(1)^2 - 7203(2)^2 + 2401000(3)^2 - 204085(4)^2 + 965202(5)^2 - \\ 146461(6)^2 + 32732(7)^2 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$C_5 = \begin{pmatrix} \frac{1}{5!} - \frac{1}{(1814400)(3!)}\begin{pmatrix} 950684(1)^3 - 1025097(2)^3 + 1059430(3)^3 - 768805(4)^3 + 362112(5)^3 - \\ 99359(6)^3 + 12062(7)^3 \end{pmatrix} \\ \frac{(2)^5}{5!} - \frac{1}{(28350)(3!)}\begin{pmatrix} 55642(1)^3 - 34986(2)^3 + 39950(3)^3 - 29405(4)^3 + 13926(5)^3 - 3832(6)^3 \\ + 466(7)^3 \end{pmatrix} \\ \frac{(3)^5}{5!} - \frac{1}{(604800)(3!)}\begin{pmatrix} 2113614(1)^3 - 650997(2)^3 + 1394820(3)^3 - 985365(4)^3 + 465102(5)^3 \\ -127899(6)^3 + 15552(7)^3 \end{pmatrix} \\ \frac{(4)^5}{5!} - \frac{1}{(14175)(3!)}\begin{pmatrix} 71152(1)^3 - 11496(2)^3 + 56720(3)^3 - 29960(4)^3 + 14736(5)^3 - 4072(6)^3 \\ + 496(7)^3 \end{pmatrix} \\ \frac{(5)^5}{5!} - \frac{1}{(72576)(3!)}\begin{pmatrix} 475000(1)^3 - 40125(2)^3 + 421250(3)^3 - 130625(4)^3 + 102900(5)^3 - \\ 26875(6)^3 + 3250(7)^3 \end{pmatrix} \\ \frac{(6)^5}{5!} - \frac{1}{(350)(3!)}\left(2826(1)^3 - 108(2)^3 + 2670(3)^3 - 495(4)^3 + 918(5)^3 - 126(6)^3 + 18(7)^3\right) \\ \frac{(7)^5}{5!} - \frac{1}{(259200)(3!)}\begin{pmatrix} 2482634(1)^3 - 7203(2)^3 + 2401000(3)^3 - 204085(4)^3 + 965202(5)^3 - \\ 146461(6)^3 + 32732(7)^3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6 = \begin{pmatrix} \frac{1}{6!} - \frac{1}{(1814400)(4!)}\begin{pmatrix} 950684(1)^4 - 1025097(2)^4 + 1059430(3)^4 - 768805(4)^4 + 362112(5)^4 - \\ 99359(6)^4 + 12062(7)^4 \end{pmatrix} \\ \frac{(2)^6}{6!} - \frac{1}{(28350)(4!)}\begin{pmatrix} 55642(1)^4 - 34986(2)^4 + 39950(3)^4 - 29405(4)^4 + 13926(5)^4 - 3832(6)^4 \\ + 466(7)^4 \end{pmatrix} \\ \frac{(3)^6}{6!} - \frac{1}{(604800)(4!)}\begin{pmatrix} 2113614(1)^4 - 650997(2)^4 + 1394820(3)^4 - 985365(4)^4 + 465102(5)^4 \\ -127899(6)^4 + 15552(7)^4 \end{pmatrix} \\ \frac{(4)^6}{6!} - \frac{1}{(14175)(4!)}\begin{pmatrix} 71152(1)^4 - 11496(2)^4 + 56720(3)^4 - 29960(4)^4 + 14736(5)^4 - 4072(6)^4 \\ + 496(7)^4 \end{pmatrix} \\ \frac{(5)^6}{5!} - \frac{1}{(72576)(4!)}\begin{pmatrix} 475000(1)^4 - 40125(2)^4 + 421250(3)^4 - 130625(4)^4 + 102900(5)^4 - \\ 26875(6)^4 + 3250(7)^4 \end{pmatrix} \\ \frac{(6)^6}{6!} - \frac{1}{(350)(4!)}\left(2826(1)^4 - 108(2)^4 + 2670(3)^4 - 495(4)^4 + 918(5)^4 - 126(6)^4 + 18(7)^4\right) \\ \frac{(7)^6}{6!} - \frac{1}{(259200)(4!)}\begin{pmatrix} 2482634(1)^4 - 7203(2)^4 + 2401000(3)^4 - 204085(4)^4 + 965202(5)^4 - \\ 146461(6)^4 + 32732(7)^4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
C_7 = \begin{pmatrix}
\dfrac{1}{7!} - \dfrac{1}{(1814400)(5!)}\begin{pmatrix} 950684(1)^5 - 1025097(2)^5 + 1059430(3)^5 - 768805(4)^5 + 362112(5)^5 - \\ 99359(6)^5 + 12062(7)^5 \end{pmatrix} \\
\dfrac{(2)^7}{7!} - \dfrac{1}{(28350)(5!)}\begin{pmatrix} 55642(1)^5 - 34986(2)^5 + 39950(3)^5 - 29405(4)^5 + 13926(5)^5 - 3832(6)^5 \\ + 466(7)^5 \end{pmatrix} \\
\dfrac{(3)^7}{7!} - \dfrac{1}{(604800)(5!)}\begin{pmatrix} 2113614(1)^5 - 650997(2)^5 + 1394820(3)^5 - 985365(4)^5 + 465102(5)^5 \\ -127899(6)^5 + 15552(7)^5 \end{pmatrix} \\
\dfrac{(4)^7}{7!} - \dfrac{1}{(14175)(5!)}\begin{pmatrix} 71152(1)^5 - 11496(2)^5 + 56720(3)^5 - 29960(4)^5 + 14736(5)^5 - 4072(6)^5 \\ + 496(7)^5 \end{pmatrix} \\
\dfrac{(5)^7}{5!} - \dfrac{1}{(72576)(5!)}\begin{pmatrix} 475000(1)^5 - 40125(2)^5 + 421250(3)^5 - 130625(4)^5 + 102900(5)^5 - \\ 26875(6)^5 + 3250(7)^5 \end{pmatrix} \\
\dfrac{(6)^7}{7!} - \dfrac{1}{(350)(5!)}\left(2826(1)^5 - 108(2)^5 + 2670(3)^5 - 495(4)^5 + 918(5)^5 - 126(6)^5 + 18(7)^5\right) \\
\dfrac{(7)^7}{7!} - \dfrac{1}{(259200)(5!)}\begin{pmatrix} 2482634(1)^5 - 7203(2)^5 + 2401000(3)^5 - 204085(4)^5 + 965202(5)^5 - \\ 146461(6)^5 + 32732(7)^5 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_8 = \begin{pmatrix}
\dfrac{1}{8!} - \dfrac{1}{(1814400)(6!)}\begin{pmatrix} 950684(1)^6 - 1025097(2)^6 + 1059430(3)^6 - 768805(4)^6 + 362112(5)^6 - \\ 99359(6)^6 + 12062(7)^6 \end{pmatrix} \\
\dfrac{(2)^8}{8!} - \dfrac{1}{(28350)(6!)}\begin{pmatrix} 55642(1)^6 - 34986(2)^6 + 39950(3)^6 - 29405(4)^6 + 13926(5)^6 - 3832(6)^6 \\ + 466(7)^6 \end{pmatrix} \\
\dfrac{(3)^8}{8!} - \dfrac{1}{(604800)(6!)}\begin{pmatrix} 2113614(1)^6 - 650997(2)^6 + 1394820(3)^6 - 985365(4)^6 + 465102(5)^6 \\ -127899(6)^6 + 15552(7)^6 \end{pmatrix} \\
\dfrac{(4)^8}{8!} - \dfrac{1}{(14175)(6!)}\begin{pmatrix} 71152(1)^6 - 11496(2)^6 + 56720(3)^6 - 29960(4)^6 + 14736(5)^6 - 4072(6)^6 \\ + 496(7)^6 \end{pmatrix} \\
\dfrac{(5)^8}{5!} - \dfrac{1}{(72576)(6!)}\begin{pmatrix} 475000(1)^6 - 40125(2)^6 + 421250(3)^6 - 130625(4)^6 + 102900(5)^6 - \\ 26875(6)^6 + 3250(7)^6 \end{pmatrix} \\
\dfrac{(6)^8}{8!} - \dfrac{1}{(350)(6!)}\left(2826(1)^6 - 108(2)^6 + 2670(3)^6 - 495(4)^6 + 918(5)^6 - 126(6)^6 + 18(7)^6\right) \\
\dfrac{(7)^8}{8!} - \dfrac{1}{(259200)(6!)}\begin{pmatrix} 2482634(1)^6 - 7203(2)^6 + 2401000(3)^6 - 204085(4)^6 + 965202(5)^6 - \\ 146461(6)^6 + 32732(7)^6 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$C_9 = \begin{pmatrix} \frac{1}{9!} - \frac{1}{(1814400)(7!)}\begin{pmatrix} 950684(1)^7 - 1025097(2)^7 + 1059430(3)^7 - 768805(4)^7 + 362112(5)^7 - \\ 99359(6)^7 + 12062(7)^7 \end{pmatrix} \\ \frac{(2)^9}{9!} - \frac{1}{(28350)(7!)}\begin{pmatrix} 55642(1)^7 - 34986(2)^7 + 39950(3)^7 - 29405(4)^7 + 13926(5)^7 - 3832(6)^7 \\ + 466(7)^7 \end{pmatrix} \\ \frac{(3)^9}{9!} - \frac{1}{(604800)(7!)}\begin{pmatrix} 2113614(1)^7 - 650997(2)^7 + 1394820(3)^7 - 985365(4)^7 + 465102(5)^7 \\ - 127899(6)^7 + 15552(7)^7 \end{pmatrix} \\ \frac{(4)^9}{9!} - \frac{1}{(14175)(7!)}\begin{pmatrix} 71152(1)^7 - 11496(2)^7 + 56720(3)^7 - 29960(4)^7 + 14736(5)^7 - 4072(6)^7 \\ + 496(7)^7 \end{pmatrix} \\ \frac{(5)^9}{5!} - \frac{1}{(72576)(7!)}\begin{pmatrix} 475000(1)^7 - 40125(2)^7 + 421250(3)^7 - 130625(4)^7 + 102900(5)^7 - \\ 26875(6)^7 + 3250(7)^7 \end{pmatrix} \\ \frac{(6)^9}{9!} - \frac{1}{(350)(7!)}\left(2826(1)^7 - 108(2)^7 + 2670(3)^7 - 495(4)^7 + 918(5)^7 - 126(6)^7 + 18(7)^7\right) \\ \frac{(7)^9}{9!} - \frac{1}{(259200)(7!)}\begin{pmatrix} 2482634(1)^7 - 7203(2)^7 + 2401000(3)^7 - 204085(4)^7 + 965202(5)^7 - \\ 146461(6)^7 + 32732(7)^7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_{10} = \begin{pmatrix} \frac{1}{10!} - \frac{1}{(1814400)(8!)}\begin{pmatrix} 950684(1)^8 - 1025097(2)^8 + 1059430(3)^8 - 768805(4)^8 + 362112(5)^8 - \\ 99359(6)^8 + 12062(7)^8 \end{pmatrix} \\ \frac{(2)^{10}}{10!} - \frac{1}{(28350)(8!)}\begin{pmatrix} 55642(1)^8 - 34986(2)^8 + 39950(3)^8 - 29405(4)^8 + 13926(5)^8 - 3832(6)^8 \\ + 466(7)^8 \end{pmatrix} \\ \frac{(3)^{10}}{10!} - \frac{1}{(604800)(8!)}\begin{pmatrix} 2113614(1)^8 - 650997(2)^8 + 1394820(3)^8 - 985365(4)^8 + 465102(5)^8 \\ - 127899(6)^8 + 15552(7)^8 \end{pmatrix} \\ \frac{(4)^{10}}{10!} - \frac{1}{(14175)(8!)}\begin{pmatrix} 71152(1)^8 - 11496(2)^8 + 56720(3)^8 - 29960(4)^8 + 14736(5)^8 - 4072(6)^8 \\ + 496(7)^8 \end{pmatrix} \\ \frac{(5)^{10}}{5!} - \frac{1}{(72576)(8!)}\begin{pmatrix} 475000(1)^8 - 40125(2)^8 + 421250(3)^8 - 130625(4)^8 + 102900(5)^8 - \\ 26875(6)^8 + 3250(7)^8 \end{pmatrix} \\ \frac{(6)^{10}}{10!} - \frac{1}{(350)(8!)}\left(2826(1)^8 - 108(2)^8 + 2670(3)^8 - 495(4)^8 + 918(5)^8 - 126(6)^8 + 18(7)^8\right) \\ \frac{(7)^{10}}{10!} - \frac{1}{(259200)(8!)}\begin{pmatrix} 2482634(1)^8 - 7203(2)^8 + 2401000(3)^8 - 204085(4)^8 + 965202(5)^8 - \\ 146461(6)^8 + 32732(7)^8 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{-47}{8488} \\ \frac{-161}{16181} \\ \frac{-82}{3797} \\ \frac{-113}{3832} \\ \frac{-167}{4458} \\ \frac{-9}{200} \\ \frac{-409}{7403} \end{pmatrix}$$

Hence, the block is having an order $(8,8,8,8,8,8,8)^T$ with error constants

$\left(\frac{-47}{8488}, \frac{-161}{16181}, \frac{-82}{3797}, \frac{-113}{3832}, \frac{-167}{4458}, \frac{-9}{200}, \frac{-367}{6639}\right)^T$.

3.6.2.2 Zero Stability of Seven–Step Block Method for Second Order ODEs.

Equation (3.2.2.2.1) is applied to the block (3.6.1.21 – 3.6.1.27), this gives

$$\det[rA^{(0)} - A^{(1)}] = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0$$

This implies $r = 0,0,0,0,0,0,1$. Hence, the method is zero stable.

3.6.2.3 Consistency and Convergence of Seven–Step Block Method for Second Order ODEs

The block method (3.6.1.21 – 3.6.1.27) is found to be consistent because it satisfies the conditions listed in Definition 1.4. Hence, it is also convergent since it is zero-stable and consistent.

3.6.2.4 Region of Absolute Stability of Seven–Step Block Method for Second Order ODEs.

Applying the equation (3.2.2.4.2) to the block (3.6.1.21 – 3.6.1.27), we have

$$\bar{h}(\theta, h) = \frac{A - B}{C + D}$$

where

$$
A=\begin{pmatrix}
e^{i\theta} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & e^{3i\theta} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & e^{4i\theta} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & e^{5i\theta} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & e^{6i\theta} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & e^{7i\theta}
\end{pmatrix},\quad
B=\begin{pmatrix}
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1
\end{pmatrix}
$$

$$
C=\begin{pmatrix}
\frac{950684}{1814400}e^{i\theta} & -\frac{1025097}{1814400}e^{2i\theta} & \frac{1059430}{1814400}e^{3i\theta} & -\frac{768805}{1814400}e^{4i\theta} & \frac{362112}{1814400}e^{5i\theta} & -\frac{99359}{1814400}e^{6i\theta} & \frac{12062}{1814400}e^{7i\theta}\\
\frac{55642}{28350}e^{i\theta} & -\frac{34986}{28350}e^{2i\theta} & \frac{39950}{28350}e^{3i\theta} & -\frac{29405}{28350}e^{4i\theta} & \frac{13926}{28350}e^{5i\theta} & -\frac{3832}{28350}e^{6i\theta} & \frac{466}{28350}e^{7i\theta}\\
\frac{2113614}{604800}e^{i\theta} & -\frac{650997}{604800}e^{2i\theta} & \frac{1394820}{604800}e^{3i\theta} & -\frac{985365}{604800}e^{4i\theta} & \frac{465102}{604800}e^{5i\theta} & -\frac{127899}{604800}e^{6i\theta} & \frac{15552}{604800}e^{7i\theta}\\
\frac{71152}{14175}e^{i\theta} & -\frac{11496}{14175}e^{2i\theta} & \frac{56720}{14175}e^{3i\theta} & -\frac{29960}{14175}e^{4i\theta} & \frac{14736}{14175}e^{5i\theta} & -\frac{4072}{14175}e^{6i\theta} & \frac{496}{14175}e^{7i\theta}\\
\frac{475000}{72576}e^{i\theta} & -\frac{40125}{72576}e^{2i\theta} & \frac{421250}{72576}e^{3i\theta} & -\frac{130625}{72576}e^{4i\theta} & \frac{102900}{72576}e^{5i\theta} & -\frac{26875}{72576}e^{6i\theta} & \frac{3250}{72576}e^{7i\theta}\\
\frac{2826}{350}e^{i\theta} & -\frac{108}{350}e^{2i\theta} & \frac{2670}{350}e^{3i\theta} & -\frac{495}{350}e^{4i\theta} & \frac{918}{350}e^{5i\theta} & -\frac{126}{350}e^{6i\theta} & \frac{18}{350}e^{7i\theta}\\
\frac{2482634}{259200}e^{i\theta} & \frac{7203}{259200}e^{2i\theta} & \frac{2401000}{259200}e^{4i\theta} & -\frac{204085}{259200}e^{5i\theta} & \frac{965202}{259200}e^{5i\theta} & \frac{146461}{259200}e^{6i\theta} & \frac{32732}{259200}e^{7i\theta}
\end{pmatrix}
$$

$$
D=\begin{pmatrix}
0&0&0&0&0&0&\frac{416173}{1814400}\\
0&0&0&0&0&0&\frac{14939}{28350}\\
0&0&0&0&0&0&\frac{496773}{604800}\\
0&0&0&0&0&0&\frac{15824}{14175}\\
0&0&0&0&0&0&\frac{102425}{72576}\\
0&0&0&0&0&0&\frac{597}{350}\\
0&0&0&0&0&0&\frac{519253}{259200}
\end{pmatrix}
$$

The above matrix is simplified and equating the imaginary part to zero we have

$$
\bar{h}(\theta,h)=\frac{=1.464987989910687\text{E}+121\cos 7\theta-1.464987989910687\text{E}+121}{5.086736208603886\text{E}+118\cos 7\theta+9.741168310279197\text{E}+119}
$$

Evaluating $\bar{h}(\theta,h)$ at intervals of θ of 30^{0} gives results as tabulated in Table 3.5.

Table 3.5

Interval of Absolute Stability of Seven–Step Block Method for Second Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta, h)$	0	-29.39	-7.33	-15.04	-23.16	-1.93	-31.74

Therefore, the interval of absolute stability is (-31.74, 0). This is shown in the diagram below

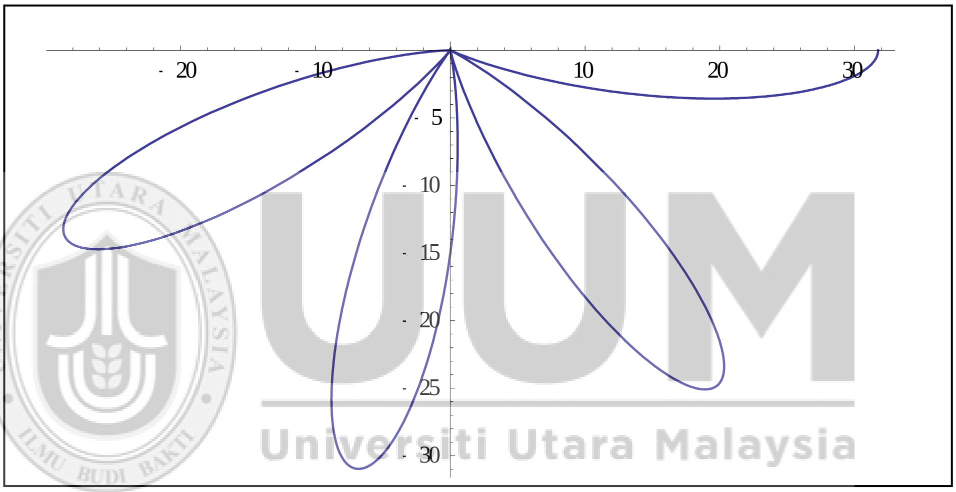

Figure 3.10. Region of absolute stability of seven–step block method for second ODEs.

3.7 Eight–Step Block Method for Second Order ODEs.

The derivation and properties of eight–step block method for solving second order ODEs are established in this section.

3.7.1 Derivation of Eight–Step Block Method for Second Order ODEs.

Power series of the form (3.2.1.1) is also considered as an approximate solution to the general second order problem of the form (3.2.1.2) where the step-length k =8. The first and second derivatives of (3.2.1.1) are shown in (3.2.1.3) and (3.2.1.4). Interpolating (3.2.1) at $x = x_{n+i}, i = 5,6$ and collocating (3.2.4) at $x = x_{n+i}, i = 0(1)8$.

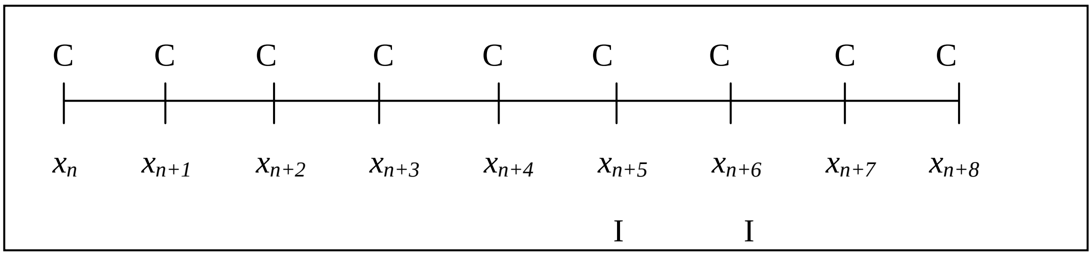

Figure 3.11. Eight–step interpolation and collocation method for second order ODEs

This approach produces the result displayed in the matix below

$$\begin{pmatrix}
1 & x_{n+5} & x^2_{n+5} & x^3_{n+5} & x^4_{n+5} & x^5_{n+4} & x^6_{n+5} & x^7_{n+5} & x^8_{n+5} & x^9_{n+5} & x^{10}_{n+5} \\
1 & x_{n+6} & x^2_{n+6} & x^3_{n+6} & x^4_{n+6} & x^5_{n+6} & x^6_{n+6} & x^7_{n+6} & x^8_{n+6} & x^9_{n+6} & x^{10}_{n+6} \\
0 & 0 & 2 & 6x_n & 12x^2_n & 20x^3_n & 30x^4_n & 42x^5_n & 56x^6_n & 72x^7_n & 90x^8_n \\
0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} & 42x^5_{n+1} & 56x^6_{n+1} & 72x^7_{n+1} & 90x^8_{n+1} \\
0 & 0 & 2 & 6x_{n+2} & 12x^2_{n+2} & 20x^3_{n+2} & 30x^4_{n+2} & 42x^5_{n+2} & 56x^6_{n+2} & 72x^7_{n+2} & 90x^8_{n+2} \\
0 & 0 & 2 & 6x_{n+3} & 12x^2_{n+3} & 20x^3_{n+3} & 30x^4_{n+3} & 42x^5_{n+3} & 56x^6_{n+3} & 72x^7_{n+3} & 90x^8_{n+3} \\
0 & 0 & 2 & 6x_{n+4} & 12x^2_{n+4} & 20x^3_{n+4} & 30x^4_{n+4} & 42x^5_{n+4} & 56x^6_{n+4} & 72x^7_{n+4} & 90x^8_{n+4} \\
0 & 0 & 2 & 6x_{n+5} & 12x^2_{n+5} & 20x^3_{n+5} & 30x^4_{n+5} & 42x^5_{n+5} & 56x^6_{n+5} & 72x^7_{n+5} & 90x^8_{n+5} \\
0 & 0 & 2 & 6x_{n+6} & 12x^2_{n+6} & 20x^3_{n+6} & 30x^4_{n+6} & 42x^5_{n+6} & 56x^6_{n+6} & 72x^7_{n+6} & 90x^8_{n+6} \\
0 & 0 & 2 & 6x_{n+7} & 12x^2_{n+7} & 20x^3_{n+7} & 30x^4_{n+7} & 42x^5_{n+7} & 56x^6_{n+7} & 72x^7_{n+7} & 90x^8_{n+7} \\
0 & 0 & 2 & 6x_{n+8} & 12x^2_{n+8} & 20x^3_{n+8} & 30x^4_{n+8} & 42x^5_{n+8} & 56x^6_{n+8} & 72x^7_{n+8} & 90x^8_{n+8}
\end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{pmatrix}
=
\begin{pmatrix} y_{n+5} \\ y_{n+6} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \\ f_{n+8} \end{pmatrix} \quad (3.7.1.1)$$

In finding the values of *a's* in (3.7.1.1) Gaussian elimination is employed. The values of *a's* are given below:

$$\begin{aligned}
a_0 = \quad & 6y_{n+5} - 5y_{n+6} + \frac{2935}{48384}h^2 f_n + \frac{33389}{30240}h^2 f_{n+1} + \frac{17819}{10080}h^2 f_{n+2} + \frac{50357}{15120}h^2 f_{n+3} \\
& + \frac{89779}{24192}h^2 f_{n+4} + \frac{1035}{224}h^2 f_{n+5} + \frac{3139}{7560}h^2 f_{n+6} - \frac{73}{7560}h^2 f_{n+7} - \frac{19}{80640}h^2 f_{n+8}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}x_n^2 f_n+\frac{x_n}{h}(y_{n+4}-y_{n+5})+\frac{x_n^3}{5040h}(2283f_n-6720f_{n+1}+11760f_{n+2}-15680f_{n+3}\\
&+14700f_{n+4}-9408f_{n+5}+3920f_{n+6}-960f_{n+7}+105f_{n+8})+\frac{x_n^9}{725760h^7}(9f_n-70f_{n+1}\\
&+238f_{n+2}-462f_{n+3}+560f_{n+4}-434f_{n+5}+210f_{n+6}-58f_{n+7}+7f_{n+8})+\frac{x_n^8}{161280h^6}(39f_n\\
&-292f_{n+1}+956f_{n+2}-1788f_{n+3}+2090f_{n+4}-1564f_{n+5}+732f_{n+6}-196f_{n+7}+23f_{n+8})\\
&+\frac{x_n^7}{30240h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}-2581f_{n+5}+1170f_{n+6}\\
&-305f_{n+7}+35f_{n+8})+\frac{x_n^5}{28800h^3}(2403f_n-13960f_{n+1}+36706f_{n+2}-57384f_{n+3}+\\
&58280f_{n+4}-39128f_{n+5}+16830f_{n+6}-4216f_{n+7}+469f_{n+8})+\frac{x_n^6}{172800h^4}(3207f_n-\\
&21056f_{n+1}+61156f_{n+2}-102912f_{n+3}+109930f_{n+4}-76352f_{n+5}+33636f_{n+6}-\\
&8576f_{n+7}+967f_{n+8})+\frac{x_n^4}{120960h^2}(29531f_n-138528f_{n+1}+312984f_{n+2}-448672f_{n+3}\\
&+435330f_{n+4}-284256f_{n+5}+120008f_{n+6}-29664f_{n+7}+3267f_{n+8}).\\
&+\frac{x_n^{10}}{3628800h^6}(f_n-8f_{n+1}+28f_{n+2}-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})\\
&+\frac{83287h}{290304}x_n f_n+\frac{1442209h}{907200}x_n f_{n+1}+\frac{10039h}{302400}x_n f_{n+2}+\frac{1019017h}{453600}x_n f_{n+3}-\frac{14743h}{103680}x_n f_{n+4}\\
&+\frac{729h}{448}x_n f_{n+5}-\frac{45541h}{226800}x_n f_{n+6}+\frac{15187h}{226800}x_n f_{n+7}-\frac{18239h}{2419200}x_n f_{n+8}.
\end{aligned}
$$

$$
\begin{aligned}
a_1 =\ &\frac{14743}{103680}hf_{n+4}-\frac{1442209}{907200}hf_{n+1}-\frac{10039}{302400}hf_{n+2}-\frac{1019017}{453600}hf_{n+3}\\
&-\frac{83287}{290304}hf_n-\frac{729}{448}hf_{n+5}+\frac{45541}{226800}hf_{n+6}-\frac{15187}{226800}hf_{n+7}\\
&+\frac{18239}{2419200}hf_{n+8}-x_n f_n-\frac{1}{h}(y_{n+5}-y_{n+6})-\frac{x_n^2}{1680h}(2283f_n\\
&-6720f_{n+1}+11760f_{n+2}-15680f_{n+3}+14700f_{n+4}-9408f_{n+5}\\
&+3920f_{n+6}-960f_{n+7}+105f_{n+8})-\frac{x_n^8}{80640h^7}(9f_n-70f_{n+1}\\
&+238f_{n+2}-462f_{n+3}+560f_{n+4}-434f_{n+5}+210f_{n+6}-58f_{n+7}\\
&+7f_{n+8})-\frac{x_n^7}{20160h^6}(39f_n-292f_{n+1}+956f_{n+2}-1788f_{n+3}+2090f_{n+4}\\
&-1564f_{n+5}+732f_{n+6}-196f_{n+7}+23f_{n+8})-\frac{x_n^6}{4320h^5}(81f_n-575f_{n+1}\\
&+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}-2581f_{n+5}+1170f_{n+6}-305f_{n+7}\\
&+35f_{n+8})-\frac{x_n^4}{5760h^3}(2403f_n-13960f_{n+1}+36706f_{n+2}-57384f_{n+3}\\
&+58280f_{n+4}-39128_{n+5}+16830f_{n+6}-4216f_{n+7}+469f_{n+8})
\end{aligned}
$$

$$
\begin{aligned}
a_2 = \;& \frac{1}{2} f_n + \frac{x_n^7}{20160h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} \\
& + 210 f_{n+6} - 58 f_{n+7} + 7 f_{n+8}) + \frac{x_n^6}{5760h^6}(39 f_n - 292 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3} \\
& + 2090 f_{n+4} - 1564 f_{n+5} + 732 f_{n+6} - 196 f_{n+7} + 23 f_{n+8}) + \frac{x_n^5}{1440h^5}(81 f_n - 575 f_{n+1} \\
& + 1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4} - 2581 f_{n+5} + 1170 f_{n+6} - 305 f_{n+7} \\
& + 35 f_{n+8}) - \frac{x_n^5}{28800h^4}(3207 f_n - 21056 f_{n+1} + 61156 f_{n+2} - 102912 f_{n+3} + 109930 f_{n+4} \\
& - 76352 f_{n+5} + 33636 f_{n+6} - 8576 f_{n+7} + 967 f_{n+8}) - \frac{x_n^3}{30240h^2}(29531 f_n \\
& - 138528 f_{n+1} + 312984 f_{n+2} - 448672 f_{n+3} + 435330 f_{n+4} - 284256 f_{n+5} + \\
& 120008 f_{n+6} - 29664 f_{n+7} + 3267 f_{n+8}) - \frac{x_n^9}{362880h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} \\
' \;& - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8}).
\end{aligned}
$$

$$
\begin{aligned}
& + 1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4} - 2581 f_{n+5} + 1170 f_{n+6} - 305 f_{n+7} + \\
& 35 f_{n+8}) + \frac{x_n^3}{2880h^3}(2403 f_n - 13960 f_{n+1} + 36706 f_{n+2} - 57384 f_{n+3} + 58280 f_{n+4} \\
& - 39128 f_{n+5} + 16830 f_{n+6} - 4216 f_{n+7} + 469 f_{n+8}) + \frac{x_n^4}{11520h^4}(3207 f_n - 21056 f_{n+1} \\
& + 61156 f_{n+2} - 102912 f_{n+3} + 109930 f_{n+4} - 76352 f_{n+5} + 33636 f_{n+6} - 8576 f_{n+7} \\
& + 967 f_{n+8}) + \frac{x_n^2}{20160h^2}(29531 f_n - 138528 f_{n+1} + 312984 f_{n+2} - 448672 f_{n+3} \\
& + 435330 f_{n+4} - 284256 f_{n+5} + 120008 f_{n+6} - 29664 f_{n+7} + 3267 f_{n+8}) + \frac{x_n^8}{80640h^8}(f_n \\
& - 8 f_{n+1} + 28 f_{n+2}' - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8}) + \frac{x_n}{1680h}(2283 f_n \\
& - 6720 f_{n+1} + 11760 f_{n+2} - 15680 f_{n+3} + 14700 f_{n+4} - 9408 f_{n+5} + 3920 f_{n+6} - 960 f_{n+7} + \\
& 105 f_{n+8})
\end{aligned}
$$

$$
\begin{aligned}
a_3 = \;& \frac{-761}{1680h} f_n + \frac{5}{3h} f_{n+1} - \frac{7}{3h} f_{n+2} + \frac{28}{9h} f_{n+3} - \frac{35}{12h} f_{n+4} + \frac{28}{15h} f_{n+5} - \frac{7}{9h} f_{n+6} + \frac{4}{21h} f_{n+7} \\
& - \frac{1}{48h} f_{n+8} - \frac{x_n^6}{8640h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} \\
& + 210 f_{n+6} - 58 f_{n+7} + 7 f_{n+8}) - \frac{x_n^5}{2880h^6}(39 f_n - 292 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3}
\end{aligned}
$$

$$+2090 f_{n+4} - 1564 f_{n+5} + 732 f_{n+6} - 196 f_{n+7} + 23 f_{n+8}) - \frac{x_n^4}{864h^5}(81 f_n - 575 f_{n+1}$$
$$+1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4} - 2581 f_{n+5} + 1170 f_{n+6} - 305 f_{n+7} +$$
$$35 f_{n+8}) - \frac{x_n^2}{2880h^3}(2403 f_n - 13960 f_{n+1} + 36706 f_{n+2} - 57384 f_{n+3} + 58280 f_{n+4}$$
$$-39128 f_{n+5} + 16830 f_{n+6} - 4216 f_{n+7} + 469 f_{n+8}) - \frac{x_n^3}{8640h^4}(3207 f_n - 21056 f_{n+1}$$
$$+61156 f_{n+2} - 102912 f_{n+3} + 109930 f_{n+4} - 76352 f_{n+5} + 133636 f_{n+6} - 8576 f_{n+7}$$
$$+967 f_{n+8}) - \frac{x_n^7}{30240h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6}$$
$$-8 f_{n+7} + f_{n+8}) - \frac{x_n^2}{1440h^5}(81 f_n - 575 f_{n+1} + 1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4}$$
$$-2581 f_{n+5} + 1170 f_{n+6} - 305 f_{n+7} + 35 f_{n+8}).$$

$$a_4 = \frac{29531}{120960h^2} f_n - \frac{481}{420h^2} f_{n+1} + \frac{207}{80h^2} f_{n+2} - \frac{2003}{540h^2} f_{n+3} + \frac{691}{192h^2} f_{n+4} - \frac{47}{20h^2} f_{n+5}$$
$$+ \frac{2143}{2160h^2} f_{n+6} - \frac{103}{240h^2} f_{n+7} + \frac{121}{4480h^2} f_{n+8} - \frac{x_n^5}{5760h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2}$$
$$-462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + 210 f_{n+6} - 58 f_{n+7} + 7 f_{n+8}) + \frac{x_n^4}{2304h^6}(39 f_n$$
$$-292 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3} + 2090 f_{n+4} - 1564 f_{n+5} + 732 f_{n+6} - 196 f_{n+7} +$$
$$23 f_{n+8}) + \frac{x_n^3}{864h^5}(81 f_n - 575 f_{n+1} + 1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4} - 2581 f_{n+5}$$
$$+1170 f_{n+6} - 305 f_{n+7} + 35 f_{n+8}) + \frac{x_n^2}{11520h^4}(3207 f_n - 21056 f_{n+1} + 61156 f_{n+2}$$
$$-102912 f_{n+3} + 109930 f_{n+4} - 76352 f_{n+5} + 133636 f_{n+6} - 8576 f_{n+7} + 967 f_{n+8})$$
$$- \frac{x_n^6}{17280h^8}(f_n - 8 f_{n+1} + 28 f_{n+2}' - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8})$$
$$+ \frac{x_n}{5760h^3}(2403 f_n - 13960 f_{n+1} + 36706 f_{n+2} - 57384 f_{n+3} + 58280 f_{n+4} - 39128 f_{n+5}$$
$$+16830 f_{n+6} - 4216 f_{n+7} + 469 f_{n+8}).$$

$$a_5 = \frac{-267}{3200h^3} f_n + \frac{349}{720h^3} f_{n+1} - \frac{18353}{14400h^3} f_{n+2} + \frac{797}{400h^3} f_{n+3} - \frac{1457}{720h^3} f_{n+4}$$
$$+ \frac{4891}{3600h^3} f_{n+5} - \frac{187}{320h^3} f_{n+6} + \frac{527}{3600h^3} f_{n+7} - - \frac{469}{28800h^3} f_{n+8} -$$
$$\frac{x_n^4}{5760h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5}$$
$$+210 f_{n+6} - 58 f_{n+7} + 7 f_{n+8}) + \frac{x_n^3}{2880h^6}(39 f_n - 292 f_{n+1} + 956 f_{n+2}$$

$$-1788f_{n+3}+2090f_{n+4}-1564f_{n+5}+732f_{n+6}-196f_{n+7}+23f_{n+8})$$
$$-\frac{x_n^2}{1440h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}$$
$$-2581f_{n+5}+1170f_{n+6}-305f_{n+7}+35f_{n+8})-\frac{x_n^5}{14400h^8}(f_n-8f_{n+1}+28f_{n+2}$$
$$'-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})-\frac{x_n}{28800h^4}(3207f_n$$
$$-21056f_{n+1}+61156f_{n+2}-102912f_{n+3}+109930f_{n+4}-76352f_{n+5}$$
$$+133636f_{n+6}-8576f_{n+7}+967f_{n+8}).$$

$$a_6=\quad \frac{1069}{57600h^4}f_n-\frac{329}{2700h^4}f_{n+1}+\frac{15289}{43200h^4}f_{n+2}-\frac{134}{225h^4}f_{n+3}+\frac{10993}{17280h^4}f_{n+4}$$
$$-\frac{1193}{2700h^4}f_{n+5}+\frac{2803}{14400h^4}f_{n+6}-\frac{67}{1350h^4}f_{n+7}++\frac{967}{172800h^4}f_{n+8}+$$
$$\frac{x_n^3}{8640h^7}(9f_n-70f_{n+1}+238f_{n+2}-462f_{n+3}+560f_{n+4}-434f_{n+5}+210f_{n+6}$$
$$-58f_{n+7}+7f_{n+8})+\frac{x_n^2}{5760h^6}(39f_n-292f_{n+1}+956f_{n+2}-1788f_{n+3}$$
$$+2090f_{n+4}-1564f_{n+5}+732f_{n+6}-196f_{n+7}+23f_{n+8})-\frac{x_n^4}{17280h^8}(f_n$$
$$-8f_{n+1}+28f_{n+2}-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})+$$
$$\frac{x_n}{4320h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}-2581f_{n+5}$$
$$+1170f_{n+6}-305f_{n+7}+35f_{n+8}).$$

$$a_7=\quad \frac{-3}{1120h^5}f_n+\frac{115}{6048h^5}f_{n+1}-\frac{179}{3024h^5}f_{n+2}+\frac{71}{672h^5}f_{n+3}-\frac{179}{1512h^5}f_{n+4}+$$
$$\frac{2581}{30240h^5}f_{n+5}-\frac{13}{336h^5}f_{n+6}+\frac{61}{6048h^5}f_{n+7}-\frac{1}{864h^5}f_{n+8}-\frac{x_n^2}{20160h^7}(9f_n$$
$$-70f_{n+1}+238f_{n+2}-462f_{n+3}+560f_{n+4}-434f_{n+5}+210f_{n+6}-58f_{n+7}+$$
$$7f_{n+8})-\frac{x_n^3}{30240h^8}(f_n-8f_{n+1}+28f_{n+2}-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}$$
$$-8f_{n+7}+f_{n+8})-\frac{x_n}{20160h^6}(39f_n-292f_{n+1}+956f_{n+2}-1788f_{n+3}+2090f_{n+4}$$
$$-1564f_{n+5}+732f_{n+6}-196f_{n+7}+23f_{n+8}).$$

$$a_8 = \frac{13}{53760h^6} f_n - \frac{73}{40320h^6} f_{n+1} + \frac{239}{40320h^6} f_{n+2} - \frac{149}{13440h^6} f_{n+3} + \frac{209}{16128h^6} f_{n+4}$$
$$- \frac{391}{40320h^6} f_{n+5} + \frac{61}{134400h^6} f_{n+6} - \frac{7}{5760h^6} f_{n+7} + \frac{23}{161280h^6} f_{n+8} - \frac{x_n^2}{80640h^8}(f_n$$
$$-8f_{n+1} + 28f_{n+2} - 56f_{n+3} + 70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} + f_{n+8}) + \frac{x_n}{80640h^7}(9f_n$$
$$-70f_{n+1} + 238f_{n+2} - 462f_{n+3} + 560f_{n+4} - 434f_{n+5} + 210f_{n+6} - 58f_{n+7} + 7f_{n+8}).$$

$$a_9 = \frac{-1}{80640h^7} f_n + \frac{1}{10368h^7} f_{n+1} - \frac{17}{51840h^7} f_{n+2} + \frac{11}{17280h^7} f_{n+3} - \frac{1}{1296h^7} f_{n+4}$$
$$+ \frac{31}{51840h^7} f_{n+5} - \frac{1}{3456h^7} f_{n+6} + \frac{29}{362880h^7} f_{n+7} - \frac{1}{103680h^7} f_{n+8} -$$
$$\frac{x_n}{362880h^8}(f_n - 8f_{n+1} + 28f_{n+2}{}' - 56f_{n+3} + 70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7}$$
$$+ f_{n+8}).$$

$$a_{10} = \frac{1}{3628800h^8}(f_n - 8f_{n+1} + 28f_{n+2}{}' - 56f_{n+3} + 70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} + f_{n+8}).$$

The values of *a's* are substituted into equation (3.2.1.1) and simplified, this gives a continuous linear multistep method of the form:

$$y(x) = \sum_{j=5}^{k-2} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \tag{3.7.1.2}$$

where $x = zh + x_n + 7h$ (3.7.1.3)

Equation (3.7.1.3) is substituted into (3.7.1.2) and simplified as follows

$$\alpha_5(z) = -1 - z$$
$$\alpha_6(z) = 2 + z$$
$$\beta_0(z) = \frac{1}{7257600}(3142 + 11203z - 21600z^3 - 15660z^4 + 1260z^5$$
$$+5334z^6 + 2400z^7 + 495z^8 + 50z^9 + 2z^{10}).$$
$$\beta_1(z) = \frac{1}{1814400}(-7214 - 26199z + 50400z^3 + 35940z^4 - 3402z^5$$
$$-12348z^6 - 5400z^7 - 1080z^8 + 105z^9 - 4z^{10}).$$
$$\beta_2(z) = \frac{1}{1814400}(29434 + 110237z - 211680z^3 - 147420z^4$$
$$+16884z^5 + 51198z^6 + 2154z^7 + 4140z^8 + 385z^9 + 14z^{10}).$$

$$\beta_3(z)=\frac{1}{1814400}(-69098-275285z+529200z^3+355320z^4-51282z^5$$
$$-125076z^6-49800z^7-9090z^8-805z^9-28z^{10}).$$
$$\beta_4(z)=\frac{1}{7257600}(375700+1800330z-3528000z^3-2221800z^4+430920z^5$$
$$+795060z^6+291600z^7+49950z^8+4200z^9+140z^{10}).$$
$$\beta_5(z)=\frac{1}{1814400}(86302-376177z+1058400z^3+578340z^4-169470z^5$$
$$-210756z^6-69000z^7-10980z^8-875z^9-28z^{10}).$$
$$\beta_6(z)=\frac{1}{1814400}(1522114+2191489z-1058400z^3-313740z^4+184212z^5$$
$$+144438z^6+41100z^7+6030z^8+455z^9+14z^{10}).$$
$$\beta_7(z)=\frac{1}{1814400}(163066+659613z+907200z^2+481680z^3+1920z^4$$
$$-114534z^5-57708z^6-14040z^7-1890z^8-135z^9-4z^{10}).$$
$$\beta_8(z)=\frac{1}{7257600}(-19658-59845z+151200z^3+196020z^4+118188z^5$$
$$+40614z^6+8400z^7+1035z^8+70z^9+2z^{10}). \tag{3.7.1.4}$$

Evaluating (3.7.4) at the non-interpolating points .i.e, at z = -7, -6, -5, -4, -3, 0 and 1 produces

$$1209600y_{n+6}-1451520y_{n+5}+241920y_n=h^2(-57f_{n+8}-2336f_{n+7}$$
$$-100448f_{n+6}+1117800f_{n+5}+897790f_{n+4}+805712f_{n+3}+427656f_{n+2}$$
$$+267112f_{n+1}+14675f_n). \tag{3.7.1.5}$$

$$1451520y_{n+6}-1814400y_{n+5}+362880y_{n+1}=h^2(641f_{n+8}-9316f_{n+7}$$
$$+147404f_{n+6}+1271156f_{n+5}+1103870f_{n+4}+717764f_{n+3}+368156f_{n+2}$$
$$+29996f_{n+1}-871f_n). \tag{3.7.1.6}$$

$$1814400\,y_{n+6}-2419200y_{n+5}+362880y_{n+2}=h^2(641f_{n+8}-10208f_{n+7}$$
$$+178848f_{n+6}+1590104f_{n+5}+1205650f_{n+4}+615984f_{n+3}+49208f_{n+2}$$
$$-1448f_{n+1}+21f_n). \tag{3.7.1.7}$$

$$2419200y_{n+6}-3628800y_{n+5}+120960y_{n+3}=h^2(951f_{n+8}-14588f_{n+7}$$
$$+243668f_{n+6}+2077164f_{n+5}+1205650f_{n+4}+128924f_{n+3}-15612f_{n+2}$$
$$+2932f_{n+1}-289f_n). \tag{3.7.1.8}$$

$$3628800y_{n+6}-7257600y_{n+5}+3628800y_{n+4}=h^2(1571f_{n+8}-23968f_{n+7}$$
$$+382688f_{n+6}+2912264f_{n+5}+370550f_{n+4}-10096f_{n+3}-6232f_{n+2}+2312f_{n+1}$$
$$-289f_n). \tag{3.7.1.9}$$

$$3628800y_{n+7} - 7257600y_{n+6} + 3628800y_{n+5} = h^2(-9829f_{n+8} + 326132f_{n+7} + 3044228f_{n+6} + 172604f_{n+5} + 187850f_{n+4} - 138196f_{n+3} + 58868f_{n+2} - 14428f_{n+1} + 1571f_n). \tag{3.7.1.10}$$

$$1209600y_{n+8} - 3628800y_{n+6} + 2419200y_{n+5} = h^2(72671f_{n+8} + 1350112f_{n+7} + 1811808f_{n+6} + 590504f_{n+5} - 333650f_{n+4} + 202704f_{n+3} - 83512f_{n+2} + 20392f_{n+1} - 2229f_n). \tag{3.7.1.11}$$

The derivative of (3.7.1.4) gives

$$\alpha_5'(z) = -1$$
$$\alpha_6'(z) = 1$$
$$\beta_0'(z) = \frac{1}{7257600}(11203 - 64800z^2 - 62640z^3 + 6300z^4 + 32004z^5 + 16800z^6 + 3960z^7 + 450z^8 + 20z^9).$$

$$\beta_1'(z) = \frac{1}{1814400}(-26199 + 151200z^2 + 143760z^3 - 17010z^4 - 74088z^5 - 37800z^6 - 8640z^7 + 945z^8 - 40z^9).$$

$$\beta_2'(z) = \frac{1}{1814400}(110237 - 635040z^2 - 589680z^3 + 84420z^4 + 307188z^5 + 15078z^6 + 33120z^7 + 3465z^8 + 140z^9).$$

$$\beta_3'(z) = \frac{1}{1814400}(-275285 + 1587600z^2 + 1421280z^3 - 256410z^4 - 750456z^5 - 348600z^6 + -72720z^7 - 7245z^8 - 280z^9).$$

$$\beta_4'(z) = \frac{1}{7257600}(1800330 - 10584000z^2 - 8887200z^3 + 2154600z^4 + 4770360z^5 + 2041200z^6 + 399600z^7 + 37800z^8 + 1400z^9).$$

$$\beta_5'(z) = \frac{1}{1814400}(-376177 + 3175200z^2 + 2313360z^3 - 847350z^4 - 1264536z^5 - 483000z^6 + -87840z^7 - 7875z^8 - 280z^9).$$

$$\beta_6'(z) = \frac{1}{1814400}(2191489 - 3175200z^2 - 1254960z^3 + 921060z^4 + 866628z^5 + 287700z^6 + 48240z^7 + 4095z^8 + 140z^9).$$

$$\beta_7'(z) = \frac{1}{1814400}(659613 + 1814400z + 1445040z^2 + 7680z^3 - 572670z^4 - 346248z^5 - 98280z^6 + -15120z^7 - 1215z^8 - 40z^9).$$

$$\beta_8'(z) = \frac{1}{7257600}(-59845 + 453600z^2 + 784080z^3 + 590940z^4 + 243684z^5 + 58800z^6 + 8280z^7 + 630z^8 + 20z^9). \tag{3.7.1.12}$$

Evaluating (3.7.1.12) at all the grid points. i.e, at z =-7, -6, -5, -4, -3,

-2,-1, 0 and 1 gives

$$7257600hy'_n - 7257600y_{n+6} + 7257600y_{n+5} = h^2(54717f_{n+8} - 485984f_{n+7} + 1457312f_{n+6} - 11809800f_{n+5} + 1032010f_{n+4} - 16304272f_{n+3} - 240936f_{n+2} - 11537672f_{n+1} - 2082175f_n). \quad (3.7.1.13)$$

$$7257600hy'_{n+1} - 7257600y_{n+6} + 7257600y_{n+5} = h^2(-13189f_{n+8} + 139764f_{n+7} - 1125116f_{n+6} - 5517124f_{n+5} + 9034230f_{n+4} - 5113556f_{n+3} - 9450124f_{n+2} - 2603484f_{n+1} + 57859f_n). \quad (3.7.1.14)$$

$$7257600hy'_{n+2} - 7257600y_{n+6} + 7257600y_{n+5} = h^2(1405f_{n+8} + 1952f_{n+7} - 536544f_{n+6} - 7027208f_{n+5} - 6399670f_{n+4} - 8618640f_{n+3} - 2960552f_{n+2} + 147704f_{n+1} - 10047f_n). \quad (3.7.1.15)$$

$$7257600hy'_{n+3} - 7257600y_{n+6} + 7257600y_{n+5} = h^2(-5061f_{n+8} + 65140f_{n+7} - 820732f_{n+6} - 6231492f_{n+5} - 8065910f_{n+4} - 3354964f_{n+3} + 316020f_{n+2} - 51548f_{n+1} + 4547f_n). \quad (3.7.1.16)$$

$$7257600hy'_{n+4} - 7257600y_{n+6} + 7257600y_{n+5} = h^2(-67f_{n+8} + 13728f_{n+7} - 568160f_{n+6} - 7083016f_{n+5} - 3616950f_{n+4} + 464752f_{n+3} - 116008f_{n+2} + 21240f_{n+1} - 1919f_n). \quad (3.7.1.17)$$

$$7257600hy'_{n+5} - 7257600y_{n+6} + 7257600y_{n+5} = h^2(-6533f_{n+8} + 86516f_{n+7} - 1000188f_{n+6} - 3263300f_{n+5} + 832010f_{n+4} - 386772f_{n+3} + 136564f_{n+2} - 30172f_{n+1} + 3075f_n). \quad (3.7.1.18)$$

$$7257600hy'_{n+6} - 7257600y_{n+6} + 7257600y_{n+5} = h^2(8061f_{n+8} - 112736f_{n+7} + 2276384f_{n+6} + 2000376f_{n+5} - 834230f_{n+4} + 408944f_{n+3} - 147624f_{n+2} + 33016f_{n+1} - 3391f_n). \quad (3.7.1.19)$$

$$7257600hy'_{n+7} - 7257600y_{n+6} + 7257600y_{n+5} = h^2(-59845f_{n+8} + 2638452f_{n+7} + 8765956f_{n+6} - 1504708f_{n+5} + 1800330f_{n+4} - 1101140f_{n+3} + 440948f_{n+2} - 104796f_{n+1} + 11203f_n). \quad (3.7.1.20)$$

$$7257600hy'_{n+8} - 7257600y_{n+6} + 7257600y_{n+5} = h^2(2080189f_{n+8} + 11572640f_{n+7} - 443232f_{n+6} + 9686008f_{n+5} - 8265910f_{n+4} + 5191536f_{n+3} - 2141480f_{n+2} + 520952f_{n+1} - 56703f_n). \quad (3.7.1.21)$$

Joining equations (3.7.1.5) - (3.7.1.11) and (3.7.1.13) to form a block (1.10)

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & -1451520 & 1209600 & 0 & 0 \\
362880 & 0 & 0 & 0 & -1814400 & 1451520 & 0 & 0 \\
0 & 362880 & 0 & 0 & -2419200 & 1814400 & 0 & 0 \\
0 & 0 & 120960 & 0 & -3628800 & 2419200 & 0 & 0 \\
0 & 0 & 0 & 3628800 & -7257600 & 3628800 & 0 & 0 \\
0 & 0 & 0 & 0 & 3628800 & -7257600 & 3628800 & 0 \\
0 & 0 & 0 & 0 & 2419200 & -3628800 & 0 & 1209600 \\
0 & 0 & 0 & 0 & 7257600 & -7257600 & 0 & 0
\end{pmatrix}
\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ y_{n+8} \end{pmatrix} =
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -241920 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} y_{n-7} \\ y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix}
+ h
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -7257600
\end{pmatrix}
\begin{pmatrix} y'_{n-7} \\ y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_{n} \end{pmatrix} +
$$

$$
h^2
\begin{pmatrix}
267112 & 427656 & 805712 & 897790 & 1117800 & -100448 & -2336 & -57 \\
29996 & 368156 & 717764 & 1103870 & 1271156 & 147404 & -9316 & 641 \\
-1448 & 49208 & 615984 & 1205650 & 1590104 & 178848 & -10208 & 641 \\
2932 & -15612 & 128924 & 1205650 & 2077164 & 243668 & -14588 & 951 \\
2312 & -6232 & -10096 & 370550 & 2912264 & 382688 & -23968 & 1571 \\
-14428 & 58868 & -138196 & 187850 & 172604 & 3044228 & 326132 & -9829 \\
20392 & -83512 & 202704 & -333650 & 590504 & 1811808 & 1350112 & 72671 \\
-11537672 & -240936 & -16304272 & 1032010 & -11809800 & 1457312 & -485984 & 54717
\end{pmatrix}
\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \\ f_{n+8} \end{pmatrix}
$$

$$
+ h^2
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 14675 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -871 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -289 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -289 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1571 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2229 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2082175
\end{pmatrix}
\begin{pmatrix} f_{n-7} \\ f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}
$$

The above equation is multiplied by $(A^0)^{-1}$ to give

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ y_{n+8} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-7} \\ y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix} +$$

$$h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} y'_{n-7} \\ y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_{n} \end{pmatrix}$$

$$+ \; h^2 \begin{pmatrix} \frac{1103}{1941} & \frac{-1381}{1941} & \frac{877}{981} & \frac{-1604}{1977} & \frac{211}{414} & \frac{-107}{510} & \frac{202}{3965} & \frac{-47}{8488} \\ \frac{2245}{1083} & \frac{-81}{50} & \frac{1747}{801} & \frac{-987}{493} & \frac{749}{593} & \frac{-877}{1683} & \frac{335}{2644} & \frac{-161}{11681} \\ \frac{1467}{400} & \frac{-817}{486} & \frac{225}{64} & \frac{-1181}{376} & \frac{2930}{1481} & \frac{-1909}{2339} & \frac{340}{1713} & \frac{-82}{3797} \\ \frac{1687}{321} & \frac{-928}{567} & \frac{9084}{1607} & \frac{-188}{45} & \frac{1959}{728} & \frac{-1281}{1151} & \frac{256}{945} & \frac{-113}{3832} \\ \frac{2950}{431} & \frac{-181}{113} & \frac{1533}{194} & \frac{-2923}{661} & \frac{225}{64} & \frac{-2099}{1479} & \frac{691}{2006} & \frac{-167}{4458} \\ \frac{1476}{175} & \frac{-549}{350} & \frac{1776}{175} & \frac{-639}{140} & \frac{36}{7} & \frac{-81}{50} & \frac{72}{175} & \frac{-9}{200} \\ \frac{4499}{449} & \frac{-2935}{1932} & \frac{6092}{493} & \frac{-2467}{530} & \frac{5870}{861} & \frac{-488}{497} & \frac{1103}{1941} & \frac{-409}{7403} \\ \frac{8281}{712} & \frac{-1059}{674} & \frac{1940}{131} & \frac{-1317}{257} & \frac{1164}{131} & \frac{-353}{674} & \frac{1183}{712} & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \\ f_{n+8} \end{pmatrix} +$$

$$h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{293}{1309} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{838}{1633} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{755}{944} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{413}{380} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1018}{741} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{93}{56} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2887}{1482} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{998}{447} \end{pmatrix}\begin{pmatrix} f_{n-7} \\ f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}. \tag{3.7.1.22}$$

which produces

$$\begin{aligned} y_{n+1} = y_n + hy_n' + \frac{h^2}{7257600}(&-40187 f_{n+8} + 369744 f_{n+7} - 1522673 f_{n+6} \\ &+ 3698922 f_{n+5} - 64888311 f_{n+4} + 6488191 f_{n+3} - 5225623 f_{n+2} \\ &+ 4124231 f_{n+1}. + 1624505 f_n). \end{aligned} \tag{3.7.1.23}$$

$$\begin{aligned} y_{n+2} = y_n + 2hy_n' + \frac{h^2}{113400}(&-1563 f_{n+8} + 14368 f_{n+7} - 59092 f_{n+6} \\ &+ 143232 f_{n+5} - 227030 f_{n+4} + 247328 f_{n+3} - 183708 f_{n+2} \\ &+ 235072 f_{n+1}. + 58193 f_n). \end{aligned} \tag{3.7.1.24}$$

$$\begin{aligned} y_{n+3} = y_n + 3hy_n' + \frac{h^2}{89600}(&-1935 f_{n+8} + 17784 f_{n+7} - 73128 f_{n+6} \\ &+ 177264 f_{n+5} - 281430 f_{n+4} + 315000 f_{n+3} - 150624 f_{n+2} \\ &+ 328608 f_{n+1} + 71661 f_n). \end{aligned} \tag{3.7.1.25}$$

$$\begin{aligned} y_{n+4} = y_n + 4hy_n' + \frac{h^2}{28350}(&-836 f_{n+8} + 7680 f_{n+7} - 31552 f_{n+6} \\ &+ 76288 f_{n+5} - 118440 f_{n+4} + 160256 f_{n+3} - 46400 f_{n+2} \\ &+ 148992 f_{n+1}. + 30812 f_n). \end{aligned} \tag{3.7.1.26}$$

$$\begin{aligned} y_{n+5} = y_n + 5hy_n' + \frac{h^2}{290304}(&-10875 f_{n+8} + 100000 f_{n+7} - 412000 f_{n+6} \\ &+ 1020600 f_{n+5} - 1283750 f_{n+4} + 2294000 f_{n+3} - 465000 f_{n+2} \\ &+ 1987000 f_{n+1}. + 398825 f_n). \end{aligned} \tag{3.7.1.27}$$

$$\begin{aligned} y_{n+6} = y_n + 6hy_n' + \frac{h^2}{1400}(&-63 f_{n+8} + 576 f_{n+7} - 2268 f_{n+6} + 7200 f_{n+5} \\ &- 6390 f_{n+4} + 14208 f_{n+3} - 2196 f_{n+2} + 11808 f_{n+1} + 2325 f_n). \end{aligned} \tag{3.7.1.28}$$

$$y_{n+7} = y_n + 7hy'_n + \frac{7h^2}{1036800}(-8183 f_{n+8} + 84168 f_{n+7} - 145432 f_{n+6} + 1009792 f_{n+5} - 689430 f_{n+4} + 1830248 f_{n+3} - 225008 f_{n+2} + 1484112 f_{n+1} + 288533 f_n). \quad (3.7.1.29)$$

$$y_{n+8} = y_n + 8hy'_n + \frac{h^2}{28350}(47104 f_{n+7} - 14848 f_{n+6} + 251904 f_{n+5} - 145280 f_{n+4} + 419840 f_{n+3} - 44544 f_{n+2} + 329728 f_{n+1} + 63296 f_n). \quad (3.7.1.30)$$

Substituting (3.7.1.27) and (3.7.1.28) into (3.7.1.14) – (3.7.1.21) to give the derivative of the block

$$y'_{n+1} = y'_n + \frac{h}{1069200}(-10004 f_{n+8} + 92186 f_{n+7} - 380447 f_{n+6} + 927046 f_{n+5} - 1482974 f_{n+4} + 1648632 f_{n+3} - 1356711 f_{n+2} + 1316197 f_{n+1} + 315273 f_n). \quad (3.7.1.31)$$

$$y'_{n+2} = y'_n + \frac{h}{113400}(-833 f_{n+8} + 7624 f_{n+7} - 31154 f_{n+6} + 74728 f_{n+5} - 116120 f_{n+4} + 120088 f_{n+3} - 42494 f_{n+2} + 182584 f_{n+1} + 32377 f_n). \quad (3.7.1.32)$$

$$y'_{n+3} = y'_n + \frac{h}{44800}(-369 f_{n+8} + 3402 f_{n+7} - 14062 f_{n+6} + 34434 f_{n+5} - 56160 f_{n+4} + 79934 f_{n+3} + 3438 f_{n+2} + 70902 f_{n+1} + 12881 f_n). \quad (3.7.1.33)$$

$$y'_{n+4} = y'_n + \frac{h}{28350}(-214 f_{n+8} + 1952 f_{n+7} - 7912 f_{n+6} + 18464 f_{n+5} - 18160 f_{n+4} + 65504 f_{n+3} + 488 f_{n+2} + 45152 f_{n+1} + 8126 f_n). \quad (3.7.1.34)$$

$$y'_{n+5} = y'_n + \frac{h}{145152}(-1225 f_{n+8} + 11450 f_{n+7} - 49150 f_{n+6} + 170930 f_{n+5} - 4000 f_{n+4} + 318350 f_{n+3} + 7550 f_{n+2} + 230150 f_{n+1} + 41705 f_n). \quad (3.7.1.35)$$

$$y'_{n+6} = y'_n + \frac{h}{1400}(-9 f_{n+8} + 72 f_{n+7} + 158 f_{n+6} + 2664 f_{n+5} - 360 f_{n+4} + 3224 f_{n+3} + 18 f_{n+2} + 2232 f_{n+1} + 401 f_n). \quad (3.7.1.36)$$

$$y'_{n+7} = y'_n + \frac{h}{518400}(-8183 f_{n+8} + 223174 f_{n+7} + 522046 f_{n+6} + 736078 f_{n+5} + 54880 f_{n+4} + 1085937 f_{n+3} + 48706 f_{n+2} + 816634 f_{n+1} + 149527 f_n). \quad (3.7.1.37)$$

$$y'_{n+8} = y'_n + \frac{h}{28350}(7912 f_{n+8} + 47104 f_{n+7} - 7424 f_{n+6} + 83968 f_{n+5} - 36320 f_{n+4} + 83968 f_{n+3} - 7424 f_{n+2} + 47104 f_{n+1} + 7912 f_n). \quad (3.7.1.38)$$

3.7.2 Properties of an Eight–Step Block Method for Second Order ODEs.

This segment considers the order, zero-stability and region of absolute stability of eight–step block method.

3.7.2.1 Order of Eight–Step Block Method for Second Order ODEs.

The approach discussed in section 3.2.2.1 is applied in finding the order of the block method (3.7.1.23 – 3.7.1.30) as shown below

$$
\begin{pmatrix}
\sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{h^m}{m!}y_n^m-\frac{1624505}{7257600}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(7257600)(m!)}y_n^{(2+m)}\begin{pmatrix}4124231(1)^m-5225623(2)^m+6488191(3)^m\\-5888311(4)^m+3698922(5)^m-1522673(6)^m\\+369744(7)^m-40187(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(2h)^m}{m!}y_n^m-\frac{58193}{113400}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(113400)(m!)}y_n^{(2+m)}\begin{pmatrix}235072(1)^m-183708(2)^m+247328(3)^m\\-227030(4)^m+143232(5)^m-59092(6)^m\\+14368(7)^m-1563(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(3h)^m}{m!}y_n^m-\frac{71661}{89600}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(89600)(m!)}y_n^{(2+m)}\begin{pmatrix}328608(1)^m-150624(2)^m+315000(3)^m\\-281430(4)^m+177264(5)^m-73128(6)^m\\+17784(7)^m-1935(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(4h)^m}{m!}y_n^m-\frac{30812}{28350}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(28350)(m!)}y_n^{(2+m)}\begin{pmatrix}148992(1)^m-46400(2)^m+160256(3)^m\\-118440(4)^m+76288(5)^m-31552(6)^m\\+7680(7)^m-836(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(5h)^m}{m!}y_n^m-\frac{398825}{290304}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(290304)(m!)}y_n^{(2+m)}\begin{pmatrix}1987000(1)^m-465000(2)^m+2294000(3)^m\\-1283750(4)^m+1020600(5)^m-412000(6)^m\\+100000(7)^m-10875(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(6h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(6h)^m}{m!}y_n^m-\frac{2325}{1400}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(1400)(m!)}y_n^{(2+m)}\begin{pmatrix}11808(1)^m-2196(2)^m+14208(3)^m\\-6390(4)^m+7200(5)^m-2268(6)^m\\+576(7)^m-63(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(7h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(7h)^m}{m!}y_n^m-\frac{7(288533)}{1036800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(1036800)(m!)}y_n^{(2+m)}\begin{pmatrix}1484112(1)^m-225008(2)^m+1830248(3)^m\\-689430(4)^m+1009792(5)^m-145432(6)^m\\+84168(7)^m-8183(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(8h)^m}{m!}y_n^m-\sum_{m=0}^{1}\frac{(8h)^m}{m!}y_n^m-\frac{63296}{28350}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{2+m}}{(28350)(m!)}y_n^{(2+m)}\begin{pmatrix}329728(1)^m-44544(2)^m+419840(3)^m\\-145280(4)^m+251904(5)^m-14848(6)^m\\+47104(7)^m\end{pmatrix}
\end{pmatrix}
=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}
$$

Comparing the coefficients of h^m and y_n^m gives

$$C_0 = \begin{pmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \\ 5-5 \\ 6-6 \\ 7-7 \\ 8-8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{1624505}{7257600} - \frac{1}{(7257600)(0!)}\begin{pmatrix} 4124231(1)^0 - 5225623(2)^0 + 6488191(3)^0 - 5888311(4)^0 + \\ 3698922(5)^0 - 1522673(6)^0 + 369744(7)^0 - 40187(8)^0 \end{pmatrix} \\ \frac{(2)^2}{2!} - \frac{58193}{113400} - \frac{1}{(113400)(0!)}\begin{pmatrix} 235072(1)^0 - 183708(2)^0 + 247328(3)^0 - 227030(4)^0 + 143232(5)^0 \\ -59092(6)^0 + 14368(7)^0 - 1563(8)^0 \end{pmatrix} \\ \frac{(3)^2}{2!} - \frac{71661}{89600} - \frac{1}{(89600)(0!)}\begin{pmatrix} 328608(1)^0 - 150624(2)^0 + 315000(3)^0 - 281430(4)^0 + 177264(5)^0 \\ -73128(6)^0 + 17784(7)^0 - 1935(8)^0 \end{pmatrix} \\ \frac{(4)^2}{2!} - \frac{30812}{28350} - \frac{1}{(28350)(0!)}\begin{pmatrix} 148992(1)^0 - 46400(2)^0 + 160256(3)^0 - 118440(4)^0 + 76288(5)^0 - \\ 31552(6)^0 + 7680(7)^0 - 836(8)^0 \end{pmatrix} \\ \frac{(5)^2}{2!} - \frac{398825}{290304} - \frac{1}{(290304)(0!)}\begin{pmatrix} 1987000(1)^0 - 465000(2)^0 + 2294000(3)^0 - 1283750(4)^0 + \\ 1020600(5)^0 - 412000(6)^0 + 100000(7)^0 - 10875(8)^0 \end{pmatrix} \\ \frac{(6)^2}{2!} - \frac{2325}{1400} - \frac{1}{(1400)(0!)}\begin{pmatrix} 11808(1)^0 - 2196(2)^0 + 14208(3)^0 - 6390(4)^0 + 7200(5)^0 - 2268(6)^0 \\ +576(7)^0 - 63(8)^0 \end{pmatrix} \\ \frac{(7)^2}{2!} - \frac{7(288533)}{1036800} - \frac{1}{(1036800)(0!)}\begin{pmatrix} 1484112(1)^0 - 225008(2)^0 + 1830248(3)^0 - 689430(4)^0 + \\ 1009792(5)^0 - 145432(6)^0 + 84168(7)^0 - 8183(8)^0 \end{pmatrix} \\ \frac{(8)^2}{2!} - \frac{63296}{28350} - \frac{1}{(28350)(0!)}\begin{pmatrix} 329728(1)^0 - 44544(2)^0 + 419840(3)^0 - 145280(4)^0 + 251904(5)^0 \\ -14848(6)^0 + 47104(7)^0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \begin{pmatrix}
\frac{1}{3!} - \frac{1}{(7257600)(1!)}\left(\begin{array}{l} 4124231(1)^1 - 5225623(2)^1 + 6488191(3)^1 - 5888311(4)^1 + \\ 3698922(5)^1 - 1522673(6)^1 + 369744(7)^1 - 40187(8)^1 \end{array}\right) \\
\frac{(2)^3}{3!} - \frac{1}{(113400)(1!)}\left(\begin{array}{l} 235072(1)^1 - 183708(2)^1 + 247328(3)^1 - 227030(4)^1 + 143232(5)^1 \\ - 59092(6)^1 + 14368(7)^1 - 1563(8)^1 \end{array}\right) \\
\frac{(3)^3}{3!} - \frac{1}{(89600)(1!)}\left(\begin{array}{l} 328608(1)^1 - 150624(2)^1 + 315000(3)^1 - 281430(4)^1 + 177264(5)^1 \\ - 73128(6)^1 + 17784(7)^1 - 1935(8)^1 \end{array}\right) \\
\frac{(4)^3}{3!} - \frac{1}{(28350)(1!)}\left(\begin{array}{l} 148992(1)^1 - 46400(2)^1 + 160256(3)^1 - 118440(4)^1 + 76288(5)^1 - \\ 31552(6)^1 + 7680(7)^1 - 836(8)^1 \end{array}\right) \\
\frac{(5)^3}{3!} - \frac{1}{(290304)(1!)}\left(\begin{array}{l} 1987000(1)^1 - 465000(2)^1 + 2294000(3)^1 - 1283750(4)^1 + \\ 1020600(5)^1 - 412000(6)^1 + 100000(7)^1 - 10875(8)^1 \end{array}\right) \\
\frac{(6)^3}{3!} - \frac{1}{(1400)(1!)}\left(\begin{array}{l} 11808(1)^1 - 2196(2)^1 + 14208(3)^1 - 6390(4)^1 + 7200(5)^1 - 2268(6)^1 \\ + 576(7)^1 - 63(8)^1 \end{array}\right) \\
\frac{(7)^3}{3!} - \frac{1}{(1036800)(1!)}\left(\begin{array}{l} 1484112(1)^1 - 225008(2)^1 + 1830248(3)^1 - 689430(4)^1 + \\ 1009792(5)^1 - 145432(6)^1 + 84168(7)^1 - 8183(8)^1 \end{array}\right) \\
\frac{(8)^3}{3!} - \frac{1}{(28350)(1!)}\left(\begin{array}{l} 329728(1)^1 - 44544(2)^1 + 419840(3)^1 - 145280(4)^1 + 251904(5)^1 \\ - 14848(6)^1 + 47104(7)^1 \end{array}\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_4 = \begin{pmatrix}
\frac{1}{4!} - \frac{1}{(7257600)(2!)}\left(\begin{array}{l} 4124231(1)^2 - 5225623(2)^2 + 6488191(3)^2 - 5888311(4)^2 + \\ 3698922(5)^2 - 1522673(6)^2 + 369744(7)^2 - 40187(8)^2 \end{array}\right) \\
\frac{(2)^4}{4!} - \frac{1}{(113400)(2!)}\left(\begin{array}{l} 235072(1)^2 - 183708(2)^2 + 247328(3)^2 - 227030(4)^2 + 143232(5)^2 \\ - 59092(6)^2 + 14368(7)^2 - 1563(8)^2 \end{array}\right) \\
\frac{(3)^4}{4!} - \frac{1}{(89600)(2!)}\left(\begin{array}{l} 328608(1)^2 - 150624(2)^2 + 315000(3)^2 - 281430(4)^2 + 177264(5)^2 \\ - 73128(6)^2 + 17784(7)^2 - 1935(8)^2 \end{array}\right) \\
\frac{(4)^4}{4!} - \frac{1}{(28350)(2!)}\left(\begin{array}{l} 148992(1)^2 - 46400(2)^2 + 160256(3)^2 - 118440(4)^2 + 76288(5)^2 - \\ 31552(6)^2 + 7680(7)^2 - 836(8)^2 \end{array}\right) \\
\frac{(5)^4}{4!} - \frac{1}{(290304)(2!)}\left(\begin{array}{l} 1987000(1)^2 - 465000(2)^2 + 2294000(3)^2 - 1283750(4)^2 + \\ 1020600(5)^2 - 412000(6)^2 + 100000(7)^2 - 10875(8)^2 \end{array}\right) \\
\frac{(6)^4}{4!} - \frac{1}{(1400)(2!)}\left(\begin{array}{l} 11808(1)^2 - 2196(2)^2 + 14208(3)^2 - 6390(4)^2 + 7200(5)^2 - 2268(6)^2 \\ + 576(7)^2 - 63(8)^2 \end{array}\right) \\
\frac{(7)^4}{4!} - \frac{1}{(1036800)(2!)}\left(\begin{array}{l} 1484112(1)^2 - 225008(2)^2 + 1830248(3)^2 - 689430(4)^2 + \\ 1009792(5)^2 - 145432(6)^2 + 84168(7)^2 - 8183(8)^2 \end{array}\right) \\
\frac{(8)^4}{4!} - \frac{1}{(28350)(2!)}\left(\begin{array}{l} 329728(1)^2 - 44544(2)^2 + 419840(3)^2 - 145280(4)^2 + 251904(5)^2 \\ - 14848(6)^2 + 47104(7)^2 \end{array}\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
C_5 = \begin{pmatrix}
\frac{1}{5!} - \frac{1}{(7257600)(3!)}\left(\begin{array}{l} 4124231(1)^3 - 5225623(2)^3 + 6488191(3)^3 - 5888311(4)^3 + \\ 3698922(5)^3 - 1522673(6)^3 + 369744(7)^3 - 40187(8)^3 \end{array}\right) \\
\frac{(2)^5}{5!} - \frac{1}{(113400)(3!)}\left(\begin{array}{l} 235072(1)^3 - 183708(2)^3 + 247328(3)^3 - 227030(4)^3 + 143232(5)^3 \\ - 59092(6)^3 + 14368(7)^3 - 1563(8)^3 \end{array}\right) \\
\frac{(3)^5}{5!} - \frac{1}{(89600)(3!)}\left(\begin{array}{l} 328608(1)^3 - 150624(2)^3 + 315000(3)^3 - 281430(4)^3 + 177264(5)^3 \\ - 73128(6)^3 + 17784(7)^3 - 1935(8)^3 \end{array}\right) \\
\frac{(4)^5}{5!} - \frac{1}{(28350)(3!)}\left(\begin{array}{l} 148992(1)^3 - 46400(2)^3 + 160256(3)^3 - 118440(4)^3 + 76288(5)^3 - \\ 31552(6)^3 + 7680(7)^3 - 836(8)^3 \end{array}\right) \\
\frac{(5)^5}{5!} - \frac{1}{(290304)(3!)}\left(\begin{array}{l} 1987000(1)^3 - 465000(2)^3 + 2294000(3)^3 - 1283750(4)^3 + \\ 1020600(5)^3 - 412000(6)^3 + 100000(7)^3 - 10875(8)^3 \end{array}\right) \\
\frac{(6)^5}{5!} - \frac{1}{(1400)(3!)}\left(\begin{array}{l} 11808(1)^3 - 2196(2)^3 + 14208(3)^3 - 6390(4)^3 + 7200(5)^3 - 2268(6)^3 \\ + 576(7)^3 - 63(8)^3 \end{array}\right) \\
\frac{(7)^5}{5!} - \frac{1}{(1036800)(3!)}\left(\begin{array}{l} 1484112(1)^3 - 225008(2)^3 + 1830248(3)^3 - 689430(4)^3 + \\ 1009792(5)^3 - 145432(6)^3 + 84168(7)^3 - 8183(8)^3 \end{array}\right) \\
\frac{(8)^5}{5!} - \frac{1}{(28350)(3!)}\left(\begin{array}{l} 329728(1)^3 - 44544(2)^3 + 419840(3)^3 - 145280(4)^3 + 251904(5)^3 \\ - 14848(6)^3 + 47104(7)^3 \end{array}\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_6 = \begin{pmatrix}
\frac{1}{6!} - \frac{1}{(7257600)(4!)}\left(\begin{array}{l} 4124231(1)^4 - 5225623(2)^4 + 6488191(3)^4 - 5888311(4)^4 + \\ 3698922(5)^4 - 1522673(6)^4 + 369744(7)^4 - 40187(8)^4 \end{array}\right) \\
\frac{(2)^6}{6!} - \frac{1}{(113400)(4!)}\left(\begin{array}{l} 235072(1)^4 - 183708(2)^4 + 247328(3)^4 - 227030(4)^4 + 143232(5)^4 \\ - 59092(6)^4 + 14368(7)^4 - 1563(8)^4 \end{array}\right) \\
\frac{(3)^6}{6!} - \frac{1}{(89600)(4!)}\left(\begin{array}{l} 328608(1)^4 - 150624(2)^4 + 315000(3)^4 - 281430(4)^4 + 177264(5)^4 \\ - 73128(6)^4 + 17784(7)^4 - 1935(8)^4 \end{array}\right) \\
\frac{(4)^6}{6!} - \frac{1}{(28350)(4!)}\left(\begin{array}{l} 148992(1)^4 - 46400(2)^4 + 160256(3)^4 - 118440(4)^4 + 76288(5)^4 - \\ 31552(6)^4 + 7680(7)^4 - 836(8)^4 \end{array}\right) \\
\frac{(5)^6}{6!} - \frac{1}{(290304)(4!)}\left(\begin{array}{l} 1987000(1)^4 - 465000(2)^4 + 2294000(3)^4 - 1283750(4)^4 + \\ 1020600(5)^4 - 412000(6)^4 + 100000(7)^4 - 10875(8)^4 \end{array}\right) \\
\frac{(6)^6}{6!} - \frac{1}{(1400)(4!)}\left(\begin{array}{l} 11808(1)^4 - 2196(2)^4 + 14208(3)^4 - 6390(4)^4 + 7200(5)^4 - 2268(6)^4 \\ + 576(7)^4 - 63(8)^4 \end{array}\right) \\
\frac{(7)^6}{6!} - \frac{1}{(1036800)(4!)}\left(\begin{array}{l} 1484112(1)^4 - 225008(2)^4 + 1830248(3)^4 - 689430(4)^4 + \\ 1009792(5)^4 - 145432(6)^4 + 84168(7)^4 - 8183(8)^4 \end{array}\right) \\
\frac{(8)^6}{6!} - \frac{1}{(28350)(4!)}\left(\begin{array}{l} 329728(1)^4 - 44544(2)^4 + 419840(3)^4 - 145280(4)^4 + 251904(5)^4 \\ - 14848(6)^4 + 47104(7)^4 \end{array}\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$C_7 = \begin{pmatrix} \dfrac{1}{7!} - \dfrac{1}{(7257600)(5!)}\begin{pmatrix} 4124231(1)^5 - 5225623(2)^5 + 6488191(3)^5 - 5888311(4)^5 + \\ 3698922(5)^5 - 1522673(6)^5 + 369744(7)^5 - 40187(8)^5 \end{pmatrix} \\ \dfrac{(2)^7}{7!} - \dfrac{1}{(113400)(5!)}\begin{pmatrix} 235072(1)^5 - 183708(2)^5 + 247328(3)^5 - 227030(4)^5 + 143232(5)^5 \\ -59092(6)^5 + 14368(7)^5 - 1563(8)^5 \end{pmatrix} \\ \dfrac{(3)^7}{7!} - \dfrac{1}{(89600)(5!)}\begin{pmatrix} 328608(1)^5 - 150624(2)^5 + 315000(3)^5 - 281430(4)^5 + 177264(5)^5 \\ -73128(6)^5 + 17784(7)^5 - 1935(8)^5 \end{pmatrix} \\ \dfrac{(4)^7}{7!} - \dfrac{1}{(28350)(5!)}\begin{pmatrix} 148992(1)^5 - 46400(2)^5 + 160256(3)^5 - 118440(4)^5 + 76288(5)^5 - \\ 31552(6)^5 + 7680(7)^5 - 836(8)^5 \end{pmatrix} \\ \dfrac{(5)^7}{7!} - \dfrac{1}{(290304)(5!)}\begin{pmatrix} 1987000(1)^5 - 465000(2)^5 + 2294000(3)^5 - 1283750(4)^5 + \\ 1020600(5)^5 - 412000(6)^5 + 100000(7)^5 - 10875(8)^5 \end{pmatrix} \\ \dfrac{(6)^7}{7!} - \dfrac{1}{(1400)(5!)}\begin{pmatrix} 11808(1)^5 - 2196(2)^5 + 14208(3)^5 - 6390(4)^5 + 7200(5)^5 - 2268(6)^5 \\ +576(7)^5 - 63(8)^5 \end{pmatrix} \\ \dfrac{(7)^7}{7!} - \dfrac{1}{(1036800)(5!)}\begin{pmatrix} 1484112(1)^5 - 225008(2)^5 + 1830248(3)^5 - 689430(4)^5 + \\ 1009792(5)^5 - 145432(6)^5 + 84168(7)^5 - 8183(8)^5 \end{pmatrix} \\ \dfrac{(8)^7}{7!} - \dfrac{1}{(28350)(5!)}\begin{pmatrix} 329728(1)^5 - 44544(2)^5 + 419840(3)^5 - 145280(4)^5 + 251904(5)^5 \\ -14848(6)^5 + 47104(7)^5 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_8 = \begin{pmatrix} \dfrac{1}{8!} - \dfrac{1}{(7257600)(6!)}\begin{pmatrix} 4124231(1)^6 - 5225623(2)^6 + 6488191(3)^6 - 5888311(4)^6 + \\ 3698922(5)^6 - 1522673(6)^6 + 369744(7)^6 - 40187(8)^6 \end{pmatrix} \\ \dfrac{(2)^8}{8!} - \dfrac{1}{(113400)(6!)}\begin{pmatrix} 235072(1)^6 - 183708(2)^6 + 247328(3)^6 - 227030(4)^6 + 143232(5)^6 \\ -59092(6)^6 + 14368(7)^6 - 1563(8)^6 \end{pmatrix} \\ \dfrac{(3)^8}{8!} - \dfrac{1}{(89600)(6!)}\begin{pmatrix} 328608(1)^6 - 150624(2)^6 + 315000(3)^6 - 281430(4)^6 + 177264(5)^6 \\ -73128(6)^6 + 17784(7)^6 - 1935(8)^6 \end{pmatrix} \\ \dfrac{(4)^8}{8!} - \dfrac{1}{(28350)(6!)}\begin{pmatrix} 148992(1)^6 - 46400(2)^6 + 160256(3)^6 - 118440(4)^6 + 76288(5)^6 - \\ 31552(6)^6 + 7680(7)^6 - 836(8)^6 \end{pmatrix} \\ \dfrac{(5)^8}{8!} - \dfrac{1}{(290304)(6!)}\begin{pmatrix} 1987000(1)^6 - 465000(2)^6 + 2294000(3)^6 - 1283750(4)^6 + \\ 1020600(5)^6 - 412000(6)^6 + 100000(7)^6 - 10875(8)^6 \end{pmatrix} \\ \dfrac{(6)^8}{8!} - \dfrac{1}{(1400)(6!)}\begin{pmatrix} 11808(1)^6 - 2196(2)^6 + 14208(3)^6 - 6390(4)^6 + 7200(5)^6 - 2268(6)^6 \\ +576(7)^6 - 63(8)^6 \end{pmatrix} \\ \dfrac{(7)^8}{8!} - \dfrac{1}{(1036800)(6!)}\begin{pmatrix} 1484112(1)^6 - 225008(2)^6 + 1830248(3)^6 - 689430(4)^6 + \\ 1009792(5)^6 - 145432(6)^6 + 84168(7)^6 - 8183(8)^6 \end{pmatrix} \\ \dfrac{(8)^8}{8!} - \dfrac{1}{(28350)(6!)}\begin{pmatrix} 329728(1)^6 - 44544(2)^6 + 419840(3)^6 - 145280(4)^6 + 251904(5)^6 \\ -14848(6)^6 + 47104(7)^6 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_9 = \begin{pmatrix} \dfrac{1}{9!} - \dfrac{1}{(7257600)(7!)} \begin{pmatrix} 4124231(1)^7 - 5225623(2)^7 + 6488191(3)^7 - 5888311(4)^7 + \\ 3698922(5)^7 - 1522673(6)^7 + 369744(7)^7 - 40187(8)^7 \end{pmatrix} \\ \dfrac{(2)^8}{9!} - \dfrac{1}{(113400)(7!)} \begin{pmatrix} 235072(1)^7 - 183708(2)^7 + 247328(3)^7 - 227030(4)^7 + 143232(5)^7 \\ - 59092(6)^7 + 14368(7)^7 - 1563(8)^7 \end{pmatrix} \\ \dfrac{(3)^8}{9!} - \dfrac{1}{(89600)(7!)} \begin{pmatrix} 328608(1)^7 - 150624(2)^7 + 315000(3)^7 - 281430(4)^7 + 177264(5)^7 \\ - 73128(6)^7 + 17784(7)^7 - 1935(8)^7 \end{pmatrix} \\ \dfrac{(4)^8}{9!} - \dfrac{1}{(28350)(7!)} \begin{pmatrix} 148992(1)^7 - 46400(2)^7 + 160256(3)^7 - 118440(4)^7 + 76288(5)^7 - \\ 31552(6)^7 + 7680(7)^7 - 836(8)^7 \end{pmatrix} \\ \dfrac{(5)^8}{9!} - \dfrac{1}{(290304)(7!)} \begin{pmatrix} 1987000(1)^7 - 465000(2)^7 + 2294000(3)^7 - 1283750(4)^7 + \\ 1020600(5)^7 - 412000(6)^7 + 100000(7)^7 - 10875(8)^7 \end{pmatrix} \\ \dfrac{(6)^8}{9!} - \dfrac{1}{(1400)(7!)} \begin{pmatrix} 11808(1)^7 - 2196(2)^7 + 14208(3)^7 - 6390(4)^7 + 7200(5)^7 - 2268(6)^7 \\ + 576(7)^7 - 63(8)^7 \end{pmatrix} \\ \dfrac{(7)^8}{9!} - \dfrac{1}{(1036800)(7!)} \begin{pmatrix} 1484112(1)^7 - 225008(2)^7 + 1830248(3)^7 - 689430(4)^7 + \\ 1009792(5)^7 - 145432(6)^7 + 84168(7)^7 - 8183(8)^7 \end{pmatrix} \\ \dfrac{(8)^8}{9!} - \dfrac{1}{(28350)(7!)} \begin{pmatrix} 329728(1)^7 - 44544(2)^7 + 419840(3)^7 - 145280(4)^7 + 251904(5)^7 \\ - 14848(6)^7 + 47104(7)^7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_{10} = \begin{pmatrix} \dfrac{1}{10!} - \dfrac{1}{(7257600)(8!)} \begin{pmatrix} 4124231(1)^8 - 5225623(2)^8 + 6488191(3)^8 - 5888311(4)^8 + \\ 3698922(5)^8 - 1522673(6)^8 + 369744(7)^8 - 40187(8)^8 \end{pmatrix} \\ \dfrac{(2)^{10}}{10!} - \dfrac{1}{(113400)(8!)} \begin{pmatrix} 235072(1)^8 - 183708(2)^8 + 247328(3)^8 - 227030(4)^8 + 143232(5)^8 \\ - 59092(6)^8 + 14368(7)^8 - 1563(8)^8 \end{pmatrix} \\ \dfrac{(3)^{10}}{10!} - \dfrac{1}{(89600)(8!)} \begin{pmatrix} 328608(1)^8 - 150624(2)^8 + 315000(3)^8 - 281430(4)^8 + 177264(5)^8 \\ - 73128(6)^8 + 17784(7)^8 - 1935(8)^8 \end{pmatrix} \\ \dfrac{(4)^{10}}{10!} - \dfrac{1}{(28350)(8!)} \begin{pmatrix} 148992(1)^8 - 46400(2)^8 + 160256(3)^8 - 118440(4)^8 + 76288(5)^8 - \\ 31552(6)^8 + 7680(7)^8 - 836(8)^8 \end{pmatrix} \\ \dfrac{(5)^{10}}{10!} - \dfrac{1}{(290304)(8!)} \begin{pmatrix} 1987000(1)^8 - 465000(2)^8 + 2294000(3)^8 - 1283750(4)^8 + \\ 1020600(5)^8 - 412000(6)^8 + 100000(7)^8 - 10875(8)^8 \end{pmatrix} \\ \dfrac{(6)^{10}}{10!} - \dfrac{1}{(1400)(8!)} \begin{pmatrix} 11808(1)^8 - 2196(2)^8 + 14208(3)^8 - 6390(4)^8 + 7200(5)^8 - 2268(6)^8 \\ + 576(7)^8 - 63(8)^8 \end{pmatrix} \\ \dfrac{(7)^{10}}{10!} - \dfrac{1}{(1036800)(8!)} \begin{pmatrix} 1484112(1)^8 - 225008(2)^8 + 1830248(3)^8 - 689430(4)^8 + \\ 1009792(5)^8 - 145432(6)^8 + 84168(7)^8 - 8183(8)^8 \end{pmatrix} \\ \dfrac{(8)^{10}}{10!} - \dfrac{1}{(28350)(8!)} \begin{pmatrix} 329728(1)^8 - 44544(2)^8 + 419840(3)^8 - 145280(4)^8 + 251904(5)^8 \\ - 14848(6)^8 + 47104(7)^8 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_{11}=\begin{pmatrix}\frac{1}{11!}-\frac{1}{(7257600)(9!)}\begin{pmatrix}4124231(1)^9-5225623(2)^9+6488191(3)^9-5888311(4)^9+\\3698922(5)^9-1522673(6)^9+369744(7)^9-40187(8)^9\end{pmatrix}\\ \frac{(2)^{11}}{11!}-\frac{1}{(113400)(9!)}\begin{pmatrix}235072(1)^9-183708(2)^9+247328(3)^9-227030(4)^9+143232(5)^9\\-59092(6)^9+14368(7)^9-1563(8)^9\end{pmatrix}\\ \frac{(3)^{11}}{11!}-\frac{1}{(89600)(9!)}\begin{pmatrix}328608(1)^9-150624(2)^9+315000(3)^9-281430(4)^9+177264(5)^9\\-73128(6)^9+17784(7)^9-1935(8)^9\end{pmatrix}\\ \frac{(4)^{11}}{11!}-\frac{1}{(28350)(9!)}\begin{pmatrix}148992(1)^9-46400(2)^9+160256(3)^9-118440(4)^9+76288(5)^9-\\31552(6)^9+7680(7)^9-836(8)^9\end{pmatrix}\\ \frac{(5)^{11}}{11!}-\frac{1}{(290304)(9!)}\begin{pmatrix}1987000(1)^9-465000(2)^9+2294000(3)^9-1283750(4)^9+\\1020600(5)^9-412000(6)^9+100000(7)^9-10875(8)^9\end{pmatrix}\\ \frac{(6)^{11}}{11!}-\frac{1}{(1400)(9!)}\begin{pmatrix}11808(1)^9-2196(2)^9+14208(3)^9-6390(4)^9+7200(5)^9-2268(6)^9\\+576(7)^9-63(8)^9\end{pmatrix}\\ \frac{(7)^{11}}{11!}-\frac{1}{(1036800)(9!)}\begin{pmatrix}1484112(1)^9-225008(2)^9+1830248(3)^9-689430(4)^9+\\1009792(5)^9-145432(6)^9+84168(7)^9-8183(8)^9\end{pmatrix}\\ \frac{(8)^{11}}{11!}-\frac{1}{(28350)(9!)}\begin{pmatrix}329728(1)^9-44544(2)^9+419840(3)^9-145280(4)^9+251904(5)^9\\-14848(6)^9+47104(7)^9\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\frac{63}{13346}\\ \frac{140}{11873}\\ \frac{148}{7995}\\ \frac{187}{7388}\\ \frac{155}{4827}\\ \frac{299}{7700}\\ \frac{51}{1111}\\ \frac{187}{3694}\end{pmatrix}$$

Hence, the block has an order $(9,9,9,9,9,9,9,9)^T$ with error constants

$$\left(\frac{63}{13346},\frac{140}{11873},\frac{148}{7995},\frac{187}{7388},\frac{155}{4827},\frac{299}{7700},\frac{51}{1111},\frac{187}{3694}\right)^T.$$

3.7.2.2 Zero Stability of Eight–Step Block Method for Second Order ODEs

Equation (3.2.2.2.1) is applied to the block (3.7.1.23– 3.7.1.30), this gives

$$\det[rA^{(0)}-A^{(1)}]=\left|r\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{pmatrix}\right|=0$$

which implies $r=0,0,0,0,0,0,0,1$. Hence, the method is zero stable.

3.7.2.3 Consistency and Convergence of Eight–Step Block Method for Second Order ODEs

Referring to the conditions listed in Definition 1.4, the block method (3.7.1.23 – 3.7.1.30) is consistent. Therefore, it is convergent beacuse it is zero-stable and consistent.

3.7.2.4 Region of Absolute Stability of Eight–Step Block Method for Second Order ODEs.

Applying the equation (3.2.2.4.2) to the block (3.7.1.23 – 3.7.1.30), we have

$$\bar{h}(\theta,h)=\frac{A-B}{C+D}$$

$$A=\begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{4i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{5i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{6i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{7i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{8i\theta} \end{pmatrix},\ B=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C=\begin{pmatrix} \frac{4124231}{7257600}e^{i\theta} & -\frac{5225623}{7257600}e^{2i\theta} & \frac{6488191}{7257600}e^{3i\theta} & -\frac{5888311}{7257600}e^{4i\theta} & \frac{3698922}{7257600}e^{5i\theta} & -\frac{1522673}{7257600}e^{6i\theta} & \frac{369744}{7257600}e^{7i\theta} & -\frac{40187}{7257600}e^{8i\theta} \\ \frac{235072}{113400}e^{i\theta} & -\frac{183708}{113400}e^{2i\theta} & \frac{247328}{113400}e^{3i\theta} & -\frac{227030}{113400}e^{4i\theta} & \frac{143232}{113400}e^{5i\theta} & -\frac{59092}{113400}e^{6i\theta} & \frac{14368}{113400}e^{7i\theta} & -\frac{1563}{113400}e^{8i\theta} \\ \frac{328608}{89600}e^{i\theta} & -\frac{150624}{89600}e^{2i\theta} & \frac{315000}{89600}e^{3i\theta} & -\frac{281430}{89600}e^{4i\theta} & \frac{177264}{89600}e^{5i\theta} & -\frac{73128}{89600}e^{6i\theta} & \frac{17784}{89600}e^{7i\theta} & -\frac{1935}{89600}e^{8i\theta} \\ \frac{148992}{28350}e^{i\theta} & -\frac{46400}{28350}e^{2i\theta} & \frac{160256}{28350}e^{3i\theta} & -\frac{118440}{28350}e^{4i\theta} & \frac{76288}{28350}e^{5i\theta} & -\frac{31552}{28350}e^{6i\theta} & \frac{7680}{28350}e^{7i\theta} & -\frac{836}{28350}e^{8i\theta} \\ \frac{1987000}{290304}e^{i\theta} & -\frac{465000}{290304}e^{2i\theta} & \frac{2294000}{290304}e^{3i\theta} & -\frac{1283750}{290304}e^{4i\theta} & \frac{1020600}{290304}e^{5i\theta} & -\frac{412000}{290304}e^{6i\theta} & \frac{100000}{290304}e^{7i\theta} & -\frac{10875}{290304}e^{8i\theta} \\ \frac{11808}{1400}e^{i\theta} & -\frac{2196}{1400}e^{2i\theta} & \frac{14208}{1400}e^{3i\theta} & -\frac{6390}{1400}e^{4i\theta} & \frac{7200}{1400}e^{5i\theta} & -\frac{2268}{1400}e^{6i\theta} & \frac{576}{1400}e^{7i\theta} & -\frac{63}{1400}e^{8i\theta} \\ \frac{10388784}{1036800}e^{i\theta} & -\frac{1575056}{1036800}e^{2i\theta} & \frac{12811736}{1036800}e^{3i\theta} & -\frac{4826010}{1036800}e^{4i\theta} & \frac{7068544}{1036800}e^{5i\theta} & -\frac{1018024}{1036800}e^{6i\theta} & \frac{589176}{1036800}e^{7i\theta} & -\frac{57281}{1036800}e^{8i\theta} \\ \frac{329728}{28350}e^{i\theta} & -\frac{44544}{28350}e^{2i\theta} & \frac{419840}{28350}e^{3i\theta} & -\frac{145280}{28350}e^{4i\theta} & \frac{251904}{28350}e^{5i\theta} & -\frac{14848}{28350}e^{6i\theta} & \frac{47104}{28350}e^{7i\theta} & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1624505}{7257600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{58193}{113400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{71661}{89600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{30812}{28350} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{398825}{290304} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2325}{1400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2019731}{1036800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{63296}{28350} \end{pmatrix}$$

Simplifying the above matrix and equating the imaginary part to zero, we have

$$\bar{h}(\theta,h) = \frac{4.250951589891620\text{E}+138\cos 8\theta - 4.250951589891620\text{E}+138}{1.049651485059028\text{E}+136\cos 8\theta - 2.387197812730720\text{E}+137}$$

Evaluating $\bar{h}(\theta,h)$ at intervals of θ of 30^0, the following tabulation are obtained

Table 3.6

Interval of Absolute Stability of Eight–Step Block Method for Second Order ODEs.

θ	0	30	60	90	120	150	180
$\bar{h}(\theta,h)$	0	26.14	26.14	0	26.14	26.14	0

Therefore, the interval of absolute stability is (0, 26.14). This is shown in the diagram below

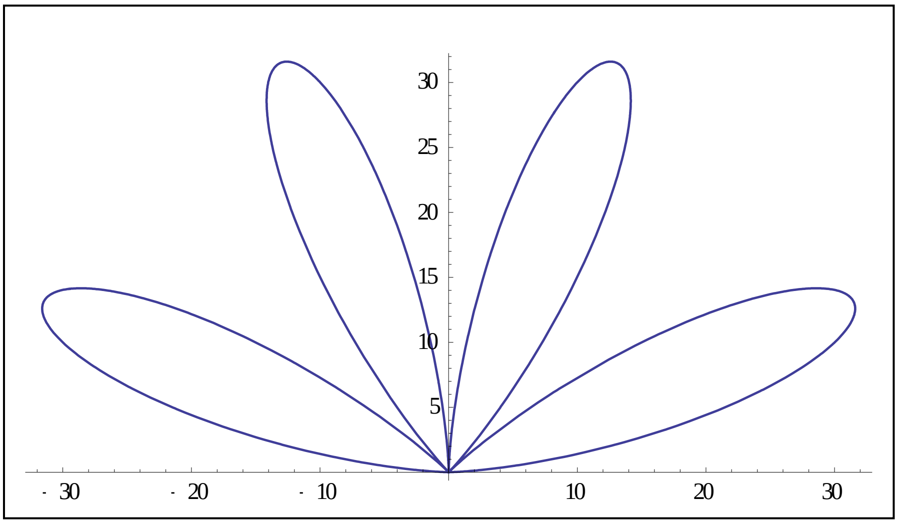

Figure 3.12. Region of absolute stability of eight–step block method for second ODEs.

3.8 Comments on the Properties of the Block Methods for Second Order ODEs.

It can be seen from the above that as the step-length k increases, the order of the new developed block methods increases to be k+1 which implies that the methods are consistent because the order is greater than one. The new block methods converged since they are zero-stable and consistent. Furthermore, it is observed that for odd step-length k, the interval of absolute stability for second order ODEs becomes larger as k increases as well as when k is even. These are shown in Tables 3.1, 3.3, 3.5 (odd step-length k) and Tables 3.2, 3.4, 3.6 (even step-length k). It is also noticed that the region of absolute stability of the new methods for odd step-length k is below the line because of their negative stability function over $[0, \ \pi]$. On the other hand, for even step-length k, the region of absolute stability is above the line because of its positive stability over $[0, \ \pi]$.

3.9 Test Problems for Second Order ODEs

In order to test the accuracy of the block methods above, the following second order ODEs are examined. The same problems the existing methods solved are also considered in order to compare our results in terms of error.

Problem 1: $y'' - x(y')^2 = 0,\ y(0) = 1, y'(0) = \frac{1}{2}, h = 0.003125$

Exact Solution: $y(x) = 1 + \frac{1}{2} In\left(\frac{2+x}{2-x}\right)$

Problem 2: $y'' + \lambda^2 y = 0,\ \ \lambda = 2,\ y(0) = 1, y'(0) = \frac{1}{2}, h = 0.003125$

Exact Solution: $y(x) = \cos 2x + \sin 2x$

Problem 3: $y'' = y',\ \ y(0) = 0, y'(0) = -1,\ h = 0.1$

Exact Solution: $y(x) = 1 - e^{x}$

Problem 4: $y'' - 100y = 0,\ \ y(0) = 1, y'(0) = -10,\ h = 0.01$

Exact Solution: $y(x) = e^{-10x}$

Problem 5: $2yy'' - (y')^2 + 4y^2 = 0,\ \ y\left(\frac{\pi}{6}\right) = \frac{1}{4}, y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},\ h = \frac{1}{320}$

Exact Solution: $y(x) = \sin^2 x$

Problem 6: $y'' = -y + 2\cos x,\ \ y(0) = 1, y'(0) = 0,\ 0 \le x \le 1$

Exact Solution: $y(x) = \cos x + x\sin x$

Problem 7: $y'' + \left(\frac{6}{x}\right)y' + \left(\frac{4}{x^2}\right)y = 0, y(1) = 1, y'(1) = 1,\ h = \frac{0.1}{32}$

Exact Solution: $y(x) = \frac{5x^3 - 2}{3x^4}$

Problem 8: $y'' = y' + 2e^x (x+1), y(0) = 1, y'(0) = 1,\ 0 \le x \le 1$

Exact Solution: $y(x) = (x^2 + 1)e^x$

Problem 9: $y'' = 6x,\ y(0) = 0, y'(0) = 0,\ 0 \le x \le 1$

Exact Solution: $y(x) = x^3$

Problem 10: $y'' = y,\ y(0) = 1, y'(0) = 1,\ 0 \le x \le 1$

Exact Solution: $y(x) = e^x$

Problem 11: $y'' + y = 0,\ y(0) = 1, y'(0) = 1,\ 0 \le x \le 1.2$

Exact Solution: $y(x) = \cos x + \sin x$

3.10 Numerical Results for Second Order ODEs

The tables displayed below show the numerical results when the new block methods with step-length $k = 3(1)8$ were applied to differential equations above. The generated numerical results are compared with the existing methods of the same step-length or higher step-length. The following notations are used in some of the tables

S2PEB	Sequential implementation of the 2-point explicit block method
P2PEB	Parallel implementation of the 2-point explicit block method
S3PEB	Sequential implementation of the 3-point explicit block method
P3PEB	Parallel implementation of the 3-point explicit block method

Table 3.7

Comparison of the New Block Method k=3 with Predictor-Corrector Method (Kayode & Adeyeye, 2011) and Block Method (Badmus and Yahaya, 2009) for Solving Problem 1

x-values	Exact Solution	Computed Solution	Error in new Method, k=3, h=1/30	Error in Kayode and Adeyeye (2011), k= 3, h=1/30	Error in Badmus and Yahaya (2009) k=5, h=1/30
0.1	1.0500417292784914	1.0500417292783137	1.776357E-13	4.684675E-11	5.891000E-06
0.2	1.1003353477310756	1.1003353477278701	3.205436E-12	3.307625E-09	8.239900E-05
0.3	1.1511404359364668	1.1511404359191078	1.735900E-11	1.555013E-08	3.464210E-04
0.4	1.2027325540540823	1.2027325539952658	5.881651E-11	4.276702E-08	7.521010E-04
0.5	1.2554128118829957	1.2554128117277092	1.552865E-10	2.281139E-08	1.380283E-03

Table 3.8

Comparison of the New Block Method k=3 with Block Predictor-Corrector Method (Adesanya et al., 2012) and Modified Block Method (Awoyemi et al., 2011) for Solving Problem 2

x-values	Exact Solution	Computed Solution	Error in New Method k=3, h=0.01	Error in Adesanya et al. (2012), k= 3, h=0.01	Awoyemi et al. (2011), k=3, h=0.01
0.01	1.039189440847612100	1.01979867335895410	9.567902E-13	9.5379E-13	-
0.02	1.039189440847612100	1.03918944084542720	2.184919E-12	2.1846E-12	2.65E-06
0.03	1.058164546414648700	1.05816454641095950	3.689271E-12	3.6890E-12	3.98E-06
0.04	1.094837581924853900	1.07671640026597900	5.813128E-12	7.1798E-12	5.30E-06
0.05	1.094837581924853900	1.09483758191663210	8.221868E-12	1.0965E-11	6.62E-06
0.06	1.112520843142785500	1.11252084313186650	1.091904E-11	1.5016E-11	7.94E-06
0.07	1.146545489989872800	1.12975911084245410	1.441958E-11	2.1162E-11	9.25E-06
0.08	1.146545489989872800	1.14654548997165740	1.821543E-11	2.7600E-11	1.06E-06
0.09	1.162873266213945600	1.16287326619163540	2.231015E-11	3.4333E-11	1.19E06
0.10	1.194127072411474800	1.17873590860891580	2.738698E-11	4.3238E-11	1.32E-06

Table 3.9

Comparison of the New Block Method k=3 with Block Method (Awoyemi et al., 2011) and Predictor-Corrector Method (Awoyemi & Kayode, 2005) for Solving Problem 1

x-values	Exact Solution	Computed Solution	Error in new Method, *k*=3, *h*=0.003125	Error in Awoyemi et al. (2011) *k*=3, *h*=0.003125	Error in Awoyemi and Kayode (2005), *k*=4, *h*=0.003125
0.1	1.050041729278491400	1.050041729278484000	7.327472E-15	6.5501E-11	6.6391E-13
0.2	1.100335347731075600	1.100335347731016300	5.928591E-14	5.4803E-10	2.0012E-09
0.3	1.151140435936466800	1.151140435936253800	2.129408E-13	1.9256E-09	1.7201E-09
0.4	1.202732554054082300	1.202732554053524300	5.579981E-13	4.8029E-09	5.8946E-09
0.5	1.255412811882995700	1.255412811881757600	1.238121E-12	1.0006E-08	1.4435E-08
0.6	1.309519604203112100	1.309519604200610300	2.501777E-12	1.8727E-08	4.1864E-08
0.7	1.365443754271396900	1.365443754266602000	4.794831E-12	3.2746E-08	5.3110E-08
0.8	1.423648930193602600	1.423648930184683300	8.919310E-12	5.3969E-08	9.1317E-08
0. 9	1.484700278594052600	1.484700278577664800	1.638778E-11	8.8004E-08	1.4924E-07
1.0	1.549306144334055900	1.549306144303872300	3.018363E-11	1.4353E-07	2.3719E-07

Table 3.10

Comparison of the New Block Method k= 4 with Block Methods (Awari et al., 2014) for Solving Problem 4

x-values	Exact Solution	Computed Solution	Error in new Method, k=4, h=0.01	Error in Awari et. al (2014), k=4, h=0.01	Error in Awari et. al (2014), k=5, h=0.01
0.01	0.904837418035959520	0.904837418933206020	8.972465E-10	1.1067E-05	1.2413E-06
0.02	0.818730753077981820	0.818730755238518570	2.160537E-09	3.1403E-05	3.4226E-06
0.03	0.740818220681717880	0.740818224125312510	3.443595E-09	5.2700E-05	5.7008E-06
0.04	0.670320046035639330	0.670320050447407390	4.411768E-09	7.4521E-05	8.0308E-06
0.05	0.606530659712633420	0.606530633744317260	2.596832E-08	8.2312E-05	1.0439E-05
0.06	0.548811636094026500	0.548811579725073550	5.636895E-08	9.7067E-05	1.1244E-05
0.07	0.496585303791409470	0.496585216456416680	8.733499E-08	1.1323E-04	1.2725E-05
0.08	0.449328964117221560	0.449328844707934880	1.194093E-07	1.3052E-04	1.4369E-05
0. 09	0.406569659740599170	0.406569486018967530	1.737216E-07	1.3614E-04	1.6156E-05
0.10	0.367879441171442330	0.367879211559240750	2.296122E-07	1.4725E-04	1.8102E-05
0.11	0.332871083698079500	0.332870795896438030	2.878016E-07	1.6012E-04	1.8649E-05
0.12	0.301194211912202080	0.301193862883732230	3.490285E-07	1.7459E-04	1.9725E-05

Table 3.11

Comparison of the New Block Method k=4 with Block Method (Adesanya et al., 2013) and Predictor-Corrector Method (Awoyemi & Kayode, 2005) for Solving Problem 1

x-values	Exact Solution	Computed Solution	Error in new Method, k=4, h=0.003125	Error in Adesanya et al. (2013), k=4, h=0.003125	Error in Awoyemi and Kayode (2005), k=4, h=0.003125
0.1	1.050041729278491400	1.050041729278491600	2.220446E-16	9.992E-15	6.550E-11
0.2	1.100335347731075600	1.100335347731075800	2.220446E-16	5.456E-14	5.480E-10
0.3	1.151140435936467000	1.151140435936465600	1.332268E-15	4.700E-13	1.925E-09
0.4	1.202732554054082300	1.202732554054076800	5.551115E-15	1.637E-12	4.802E-09
0.5	1.255412811882995500	1.255412811882979900	1.554312E-14	4.664E-12	1.000E-08
0.6	1.309519604203111600	1.309519604203074800	3.685940E-14	1.116E-11	1.872E-08
0. 7	1.365443754271396000	1.365443754271314000	8.193446E-14	2.501E-11	3.274E-08
0.8	1.423648930193601300	1.423648930193427600	1.736389E-13	5.215E-11	5.396E-08
0.9	1.484700278594051100	1.484700278593693100	3.579359E-13	1.076E-11	8.800E-08
1.0	1.549306144334053700	1.549306144333322500	7.311929E-13	2.170E-10	1.435E-07

Table 3.12

Comparison of the New Block Method k=4 with Predictor-Corrector Method (Awoyemi & Kayode, 2005) and Predictor-Corrector Method (Awoyemi, 2001) for Solving Problem 5

x-values	Exact Solution	Computed Solution	Error in new Method, k=4, h=1/320	Error in Awoyemi and Kayode (2005), k=4, h=1/320	Error in Awoyemi (2001), k=4, h=1/320
1.1	0.795642473567278130	0.795642473550267400	1.701073E-11	0.416327E-06	0.469215E-06
1.2	0.869859011321088560	0.869859011298550480	2.253808E-11	0.458667E-06	0.408029E-06
1.3	0.929330437451130750	0.929330437423069090	2.806166E-11	0.409282E-06	0.228974E-06
1.4	0.971685813870537320	0.971685813837236290	3.330103E-11	0.262955E-06	0.812872E-07
1.5	0.995236565381586220	0.995236565343565420	3.802081E-11	0.455387E-07	0.524472E-06
1.6	0.999043797840225900	0.999043797798192860	4.203304E-11	0.480549E-06	0.108974E-05
1.7	0.982955728902747180	0.982955728857558770	4.518841E-11	0.103225E-05	0.175373E-05
1.8	0.947613739108849830	0.947613739061481160	4.736866E-11	0.167850E-05	0.248148E-05
1.9	0.894426802063518540	0.894426802015042990	4.847556E-11	0.238575E-05	0.322842E-05
2.0	0.825515313105665170	0.825515313057240800	4.842438E-11	0.311084E-05	0.394301E-05

Table 3.13

Comparison of the New Block Method k=5 with Block Hybrid Backward Difference Formula (Mohammed & Adeniyi, 2014) and Block Method Mohammed, 2011) for Solving Problem 3

x-values	Exact Solution	Computed Solution	Error in new Method, k=5 h=0.1	Error in Mohammed and Adeniyi (2014), k=5, h=0.1	Error in Mohammed (2011), k=5, h=0.1
0.1	-0.105170918075647710	-0.105170918075396830	2.508826E-13	2.004000000E-07	2.198000000E-05
0.2	-0.221402758160169850	-0.221402758095238100	6.493175E-11	5.386000000E-07	6.070400000E-06
0.3	-0.349858807576003180	-0.349858805892857230	1.683146E-09	8.840000000E-07	1.005100000E-05
0.4	-0.491824697641270350	-0.491824680634920690	1.700635E-08	1.229700000E-06	1.402530000E-05
0.5	-0.648721270700128190	-0.648721168154762000	1.025454E-07	1.575200000E-06	1.799340000E-05
0.6	-0.822118800390508890	-0.822116241679589410	2.558711E-06	1.920400000E-06	2.161620000E-05
0.7	-1.013752707470476600	-1.013747434170635200	5.273300E-06	2.506000000E-06	2.799300000E-05
0.8	-1.225540928492467900	-1.225532652557803400	8.275935E-06	3.106000000E-06	3.456100000E-05
0.9	-1.459603111156949900	-1.459591494482804300	1.161667E-05	3.705000000E-06	4.111400000E-05
0.1	-1.718281828459045500	-1.718266406589084500	1.542187E-05	4.304000000E-06	4.765600000E-05

Table 3.14

Comparison of the New Block Method k=5 with Block Method (Omar, 2004) in which Maximum errors were considered for Solving Problem 6

h-values	New Method	Omar (2004)	Number of Steps	Error in new Method, k=5	Error in Omar (2004) k=5
10^{-2}	5-Step Method	S2PEB	53	4.886702E-12	1.43153E-03
		P2PEB	53	4.886702E-12	1.43153E-03
		S3PEB	36	1.187872E-11	1.43153E-03
		P3PEB	36	1.187872E-11	1.43153E-03
10^{-3}	5-Step Method	S2PEB	503	4.518608E-14	1.43166E-04
		P2PEB	503	4.518608E-14	1.43166E-04
		S3PEB	336	2.220446E-16	1.43166E-04
		P3PEB	336	2.220446E-16	1.43166E-04
10^{-4}	5-Step Method	S2PEB	5003	2.101652E-13	1.43167E-05
		P2PEB	5003	2.101652E-13	1.43167E-05
		S3PEB	3336	3.197442E-14	1.43167E-05
		P3PEB	3336	3.197442E-14	1.43167E-05
10^{-5}	5-Step Method	S2PEB	50003	6.672440E-14	1.43167E-06
		P2PEB	50003	6.672440E-14	1.43167E-06
		S3PEB	33336	9.459100E-14	1.43167E-06
		P3PEB	33336	9.459100E-14	1.43167E-06

Table 3.15

Comparison of the New Block Method k=5 with Block Method (Badmus & Yahaya, 2009) for Solving Problem 7

x-values	Exact Solution	Computed Solution	Error in new Method, k=5, h=0.1/32	Error in Badmus and Yahaya (2009) k=5, h=0.1/32
0.003125	1.003076525857696400	1.00307652585769610	2.220446E-16	3.8354E-05
0.00625	1.006057503083516400	1.00605750308351590	4.440892E-16	7.5004E-05
0.009375	1.008944995088837600	1.00894499508883870	1.110223E-15	1.0592E-04
0.0125	1.011741018167988400	1.01174101816798650	1.998401E-15	1.35476E-04
0.015625	1.014447542686413900	1.01444754268640770	6.217249E-15	1.55567E-04
0.01875	1.017066494235672400	1.01706649423568110	8.659740E-15	1.86372E-04
0.025	1.011741018167988400	1.01174101816798160	6.883383E-15	1.96055E-04
0.028125	1.024416518738402700	1.02441651873847910	7.638334E-14	2.21045E-04
0.03125	1.026703577500806200	1.02670357750087080	6.461498E-14	2.05628E-04

Table 3.16

Comparison of the New Block Method k=6 with Numerical Methods (Adeniyi & Alabi, 2011) where two Continuous Collocation Methods for k=6 were considered for Solving Problem 1

x-values	Exact Solution	Computed Solution	Error in new Method, k=6, h=0.1	Error in Adeniyi and Alabi (2011) k=6, h=0.1	Error in Adeniyi and Alabi (2011) k=6, h=0.1
0.1	1.050041729278491400	1.050041728320724600	9.577668E-10	0.1329867326E-09	0.1708719055E-09
0.2	1.100335347731075600	1.100335345362366100	2.368709E-09	0.5872691257E-08	0.6836010114E-08
0.3	1.151140435936466800	1.151140432204224200	3.732243E-09	0.1327845616E-07	0.1555757709E-07
0.4	1.202732554054082100	1.202732548578963300	5.475119E-09	0.2317829012E-07	0.2880198295E-07
0.5	1.255412811882995200	1.255412800461100500	1.142189E-08	0.3218793564E-07	0.4802328029E-07
0.6	1.309519604203111900	1.309519558523672100	4.567944E-08	0.6871246012E-07	0.7628531256E-07
0.7	1.365443754271396400	1.365441698433357300	2.055838E-06	0.1012728156E-06	0.1157914170E-06
0.8	1.423648930193601700	1.423644681894230500	4.248299E-06	0.1231093271E-06	0.1727046080E-06
0.9	1.484700278594052000	1.484693618135657400	6.660458E-06	0.2019286712E-06	0.2561456831E-06
1.0	1.549306144334054800	1.549296699167710000	9.445166E-06	0.2990871645E-06	0.3815695118E-06

Table 3.17

Comparison of the New Block Method k=6 with Uniform Accurate Block Integrators (Awari et al., 2014) and Zero Stable Continuous Block Method (Awari & Abada, 2014) for Solving Problem 4

x-values	Exact Solution	Computed Solution	Error in new Method k=6, h=0.01	Error in Awari et al. (2014) k=6, h=0.01	Error in Awari & Abada (2014) k=7, h=0.01
0.1	0.367879441171442330	0.367879441216022340	4.458001E-11	1.353E-07	1.440E-08
0.2	0.135335283236612730	0.135335283360837970	1.242252E-10	3.658E-07	3.850E-08
0.3	0.049787068367863924	0.049787068629654027	2.617901E-10	6.051E-07	6.330E-08
0.4	0.018315638888734147	0.018315639588088788	6.993546E-10	8.502E-07	8.800E-08
0.5	0.006737946999085455	0.006737948897670706	1.898585E-09	1.104E-06	1.151E-07
0.6	0.002478752176666350	0.002478757332916922	5.156251E-09	1.369E-06	1.427E-07
0.7	0.000911881965554513	0.000911895979841017	1.401429E-08	1.450E-06	1.716E-07
0.8	0.000335462627902510	0.000335500721945338	3.809404E-08	1.597E-06	1.796E-07
0.9	0.000123409804086679	0.000123513354121929	1.035500E-07	1.763E-06	1.941E-07
0.1	0.000045399929762485	0.000045681407874946	2.814781E-07	1.946E-06	2.109E-07

Table 3.18

Comparison of the New Block Method k=6 with Block Method (Mohammed et al., 2010) for Solving Problem 3

x-values	Exact Solution	Computed Solution	Error in new Method k=6, h=0.1	Error in Mohammed et al. (2010) k=6, h=0.1
0.1	-0.105170918075647710	-0.105170918075644850	2.858824E-15	5.7269E-06
0.2	-0.221402758160169850	-0.221402758158730170	1.439682E-12	6.6391E-06
0.3	-0.349858807576003180	-0.349858807520089350	5.591383E-11	7.0283E-06
0.4	-0.491824697641270350	-0.491824696888888910	7.523814E-10	7.4539E-06
0.5	-0.648721270700128190	-0.648721265035962390	5.664166E-09	7.8935E-06
0.6	-0.822118800390509110	-0.822118770857143020	2.953337E-08	8.1942E-06
0.7	-1.013752707470476600	-1.013752046870527000	6.605999E-07	8.1810E-06
0.8	-1.225540928492467900	-1.225539570453469700	1.358039E-06	8.1810E-06
0.09	-1.459603111156949900	-1.459600982232234200	2.128925E-06	8.1730E-06
0.1	-1.718281828459045500	-1.718278846414507600	2.982045E-06	8.1650E-06

Table 3.19

Comparison of the New Block Method k=7 with Zero Stable Continuous Block Method (Awari & Abada, 2014) for Solving Problem 4

x-values	Exact Solution	Computed Solution	Error in new Method k=7, h=0.01	Error in Awari & Abada (2014), k=7, h=0.01
0.01	0.904837418035959520	0.904837418035385090	5.744294E-13	1.440E-08
0.02	0.818730753077981820	0.818730753200521470	1.225396E-10	3.850E-08
0.03	0.740818220681717880	0.740818220899703510	2.179856E-10	6.330E-08
0.04	0.670320046035639330	0.670320046349561880	3.139226E-10	8.800E-08
0.05	0.606530659712633420	0.606530660132277630	4.196442E-10	1.151E-07
0.06	0.548811636094026500	0.548811636683720680	5.896942E-10	1.427E-07
0.07	0.496585303791409470	0.496585303587777190	2.036323E-10	1.716E-07
0.08	0.449328964117221560	0.449328963932432490	1.847891E-10	1.796E-07
0.09	0.406569659740599050	0.406569659572844410	1.677546E-10	1.941E-07
0.1	0.367879441171442330	0.367879441019080040	1.523623E-10	2.109E-07

Table 3.20

Comparison of the New Block Method k=7 with Zero Stable Continuous Block Method (Awari & Abada, 2014) for Solving Problem 11

x-values	Exact Solution	Computed Solution	Error in new Method, k=7, h=0.1	Error in Awari and Abada (2014) k=7, h=0.1
0.1	1.094837581924853900	1.094837581924300400	5.535572E-13	2.770E-08
0.2	1.178735908636302700	1.178735908634924900	1.377787E-12	7.330E-08
0.3	1.250856695786945600	1.250856695784829300	2.116307E-12	1.192E-07
0.4	1.310479336311535700	1.310479336309601200	1.934453E-12	1.637E-07
0.5	1.357008100494575800	1.357008100502533600	7.957857E-12	2.067E-07
0.6	1.389978088304713700	1.389978088386357300	8.164358E-11	2.476E-07
0.7	1.409059874522179600	1.409059874981646100	4.594665E-10	2.859E-07
0.8	1.414062800246688200	1.414062669791839700	1.304548E-07	3.133E-07
0.9	1.404936877898147900	1.404936617832056400	2.600661E-07	3.561E-07
1.0	1.381773290676036300	1.381772903597241700	3.870788E-07	3.963E-07
1.1	1.344803481487012700	1.344802971263106400	5.102239E-07	4.325E-07
1.2	1.294396840443899700	1.294396212173423800	6.282705E-07	4.644E-07

Table 3.21

Comparison of the New Block Method k=7 with Block Method (Omar, 1999) whereby Maximum Errors were selected for Solving Problem 6

h-values	New Method	Omar (2004)	Number of Steps	Error in new Method, k=7	Error in Omar (1999) k=8
10^{-2}	7-Step Method	S2PEB	54	4.932632E-11	3.34334E-03
		P2PEB	54	4.932632E-11	3.34334E-03
		S3PEB	39	2.491618E-11	6.01615E-04
		P3PEB	39	2.491618E-11	6.01615E-04
10^{-3}	7-Step Method	S2PEB	504	5.595524E-14	4.19854E-04
		P2PEB	504	5.595524E-14	4.19854E-04
		S3PEB	339	4.551914E-15	4.16853E-04
		P3PEB	339	4.551914E-15	4.16853E-04
10^{-4}	7-Step Method	S2PEB	5004	8.104628E-13	4.20740E-05
		P2PEB	5004	8.104628E-13	4.20740E-05
		S3PEB	3339	1.054712E-14	4.20700E-05
		P3PEB	3339	1.054712E-14	4.20700E-05
10^{-5}	7-Step Method	S2PEB	50004	1.287859E-12	4.20736E-06
		P2PEB	50004	1.287859E-12	4.20736E-06
		S3PEB	33339	2.704503E-13	4.20736E-06
		P3PEB	33339	2.704503E-13	4.20736E-06

Table 3.22

Comparison of the New Block Method k=8 with Block Method (Omar, 1999) whereby Maximum Errors were selected for Solving Problem 8

h-values	New Method	Omar (2004)	Number of Steps	Error in new Method, k=8	Error in Omar (1999) k=8
10^{-2}	8-Step Method	S2PEB	54	7.306049E-06	7.29714E+03
		P2PEB	54	7.306049E-06	7.29714E+03
		S3PEB	39	1.504936E-06	1.10599E+09
		P3PEB	39	1.504936E-06	1.10599E+09
10^{-3}	8-Step Method	S2PEB	504	2.225931E-09	1.25832E-03
		P2PEB	504	2.225931E-09	1.25832E-03
		S3PEB	339	3.856115E-10	3.93051E-02
		P3PEB	339	3.856115E-10	3.93051E-02
10^{-4}	8-Step Method	S2PEB	5004	1.100489E-10	1.19981E-04
		P2PEB	5004	1.100489E-10	1.19981E-04
		S3PEB	3339	9.805490E-12	3.51680E-04
		P3PEB	3339	9.805490E-12	3.51680E-04
10^{-5}	8-Step Method	S2PEB	50004	1.200419E-09	2.44078E-05
		P2PEB	50004	1.200419E-09	2.44078E-05
		S3PEB	33339	1.218723E-10	1.93968E-05
		P3PEB	33339	1.218723E-10	1.93968E-05

Table 3.23

Comparison of the New Block Method k=8 with Block Method (Omar, 1999) where selection of Maximum Errors were considered for Solving Problem 9

h-values	New Method	Omar (2004)	Number of Steps	Error in new Method, k=8	Error in Omar (1999) k=8
10^{-2}	8-Step Method	S2PEB	54	8.526513E-14	5.00000E-05
		P2PEB	54	8.526513E-14	5.00000E-05
		S3PEB	39	1.421085E-14	5.00000E-05
		P3PEB	39	1.421085E-14	5.00000E-05
10^{-3}	8-Step Method	S2PEB	504	4.263256E-14	5.00000E-07
		P2PEB	504	4.263256E-14	5.00000E-07
		S3PEB	339	1.776357E-14	5.00000E-07
		P3PEB	339	1.776357E-14	5.00000E-07
10^{-4}	8-Step Method	S2PEB	5004	8.235190E-12	4.99983E-09
		P2PEB	5004	8.235190E-12	4.99983E-09
		S3PEB	3339	1.991296E-12	4.99992E-09
		P3PEB	3339	1.991296E-12	4.99992E-09
10^{-5}	8-Step Method	S2PEB	50004	7.423751E-11	5.18058E-11
		P2PEB	50004	7.423751E-11	5.18058E-11
		S3PEB	33339	1.807621E-11	4.84884E-11
		P3PEB	33339	1.807621E-11	4.84884E-11

Table 3.24

Comparison of the New Block Method k=8 with Block Method (Omar, 1999) whereby Maximum Errors were selected for Solving Problem 10

h-values	New Method	Omar (2004)	Number of Steps	Error in new Method, k=8	Error in Omar (1999) k=8
10^{-2}	8-Step Method	S2PEB	54	3.446132E-11	7.00654E-03
		P2PEB	54	3.446132E-11	7.00654E-03
		S3PEB	39	8.148149E-12	1.06947E-02
		P3PEB	39	8.148149E-12	1.06947E-02
10^{-3}	8-Step Method	S2PEB	504	1.421085E-14	5.88915E-04
		P2PEB	504	1.421085E-14	5.88915E-04
		S3PEB	339	3.552714E-15	5.93077E-04
		P3PEB	339	3.552714E-15	5.93077E-04
10^{-4}	8-Step Method	S2PEB	5004	8.377299E-12	5.87617E-05
		P2PEB	5004	8.377299E-12	5.87617E-05
		S3PEB	3339	1.268319E-12	5.87659E-05
		P3PEB	3339	1.268319E-12	5.87659E-05
10^{-5}	8-Step Method	S2PEB	50004	7.828049E-11	5.87601E-06
		P2PEB	50004	7.828049E-11	5.87601E-06
		S3PEB	33339	1.246470E-11	5.87601E-06
		P3PEB	33339	1.246470E-11	5.87601E-06

3.11 Comments on the Results

Two non-linear ODEs are simply considered because of a small number of literatures on it while nine linear ODEs are examined by the new developed block methods. It is apparent in Tables 3.7 and 3.9 that the results of the new block method k=3 outperform Kayode and Adeyeye (2011) k=3, Badmus and Yahaya (2009) k=5, Awoyemi et al. (2011) k=3 and Awoyemi and Kayode (2005) k=4 for solving Problem 1 despite the higher step-length in Badmus and Yahaya (2009) and Awoyemi and Kayode (2005). Furthermore, in Table 3.8, the results of the new block method k=3 are better when compared with Adesanya et al. (2012) k=3 and Awoyemi et al. (2011) k=3 for solving Problem 2.

In term of accuracy, the generated numerical results of the new block method k=4 shown in Table 3.10 claim superiority over Awari et al. (2014) k =4 and 5 for solving Problem 4. In Table 3.11, the results produced when the new method k=4 was applied to Problem 1 are found better than Adesanya et al. (2013) k=4 and Awoyemi and Kayode (2005) k=4. Additionally, the numerical results of the new block method k=4 in Table 3,12 compared fovourably than Awoyemi and Kayode (2005) k=4 and Awoyemi et al. (2001) k=4 when Problem 5 was solved.

The results displayed in Table 3.13 for solving Problem 3 implies that the new block method k=5 is high in accuracy than Mohammed and Adeniyi (2014) and Mohammed (2012) even with the same step-length k=5. Problem 6 was also solved by the new block method k=5 and the created numerical results are better than Omar

(2004) k=5 which are presented in Table 3.14. Furthermore, the results of the new method in Table 3.15 have better accuracy in terms of error when compared with Badmus and Yahaya(2009) k=5 for solving Problem 7. In Table 3.16, the numerical results derived from the new block method k=6 when the method was applied to Problem 1 show the efficiency of the method in terms of error over Adeniyi and Alabi (2011) k=6. The accuracy of the new method in Table 3.17 is better when comparison was made with Awari et al. (2014) k=6 and Awari and Abada (2014) k=7 for solving Problem 4 in spite of the higher step-length in the former method. In addition, Problem 3 was also considered by the new block method k=6 and the results generated outperformed Mohammed et al. (2010). This can be seen in Table 3.18.

In Tables 3.19 and 3.20, it can be observed that the accuracy of the new block method k=7 for solving Problems 4 and 11 is higher than Awari and Abada (2014) k=7. The application of the new method to Problem 6 in Table 3.21 is also better in terms of error than Omar (1999) k=8. Furthermore, by comparing errors in Tables 3.22, 3.23 and 3.24 for solving Problems 8, 9 and 10, it can be seen that the accuracy of the new block method k=8 is more advanced than Omar (1999) k=8.

3.12 Summary

The derivation of block methods with step-length k=3(1)8 for solving second order initial value problems of ODEs has been examined in this chapter. This also includes the analysis of the properties of the methods. The results obtained when the new block methods were applied to second order initial value problems are compared with some existing methods displayed above. The new methods performed better in terms of accuracy than the existing methods even when comparison of some of the results generated are made with the existing methods of higher step-length shown in Tables 3.7, 3.9, 3.10, 3.17 and 3.21.

CHAPTER FOUR

DEVELOPING BLOCK METHODS FOR SOLVING THIRD ORDER ODEs DIRECTLY

4.1 Introduction

This chapter considers the derivation of block methods with step-length $k = 4(1)8$ using interpolation and collocation approach for the direct solution of third order initial value problems of ODEs. These are shown below

4.2 Four–Step Block Method for Third Order ODEs.

This section includes the derivation of four–step block method and establishment of its properties.

4.2.1 Derivation of Four–Step Block Method for Third Order ODEs.

Power series of the form

$$y(x) = \sum_{j=0}^{k+3} a_j x^j \tag{4.2.1.1}$$

is considered as an approximate solution to the general third order problem of the form

$$y''' = f(x, y, y', y''); y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0 \tag{4.2.1.2}$$

where in (4.2.1.1) k =4 is the step-length. The first, second and third derivative of (4.2.1.1) give

$$y'(x) = \sum_{j=1}^{k+3} j a_j x^{j-1} \tag{4.2.1.3}$$

$$y''(x) = \sum_{j=2}^{k+3} j(j-1) a_j x^{j-2} \tag{4.2.1.4}$$

$$y'''(x)=\sum_{j=3}^{k+3} j(j-1)(j-2)a_j x^{j-3} = f(x,y,y',y'') \qquad (4.2.1.5)$$

Equation (4.2.1.1) is interpolated at points $x = x_{n+i}, i = 0(1)2$ and (4.2.1.5) is collocated at $x = x_{n+i}, i = 0(1)4$ as demonstrated in Figure 4.1 below

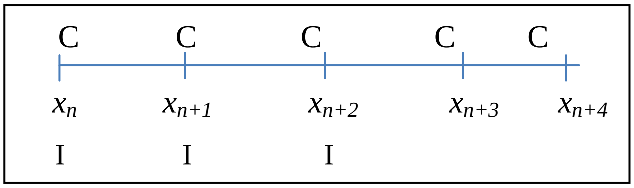

Figure 4.1. Four–step interpolation and collocation method for third order ODEs

As a result, we get

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \qquad (4.2.1.6)$$

Employing the Gaussian eliminated method in finding the values of a's in (4.2.1.6), we have

$$\begin{aligned} a_0 = {} & y_n - \frac{x_n^3}{6} f_n - \frac{233h}{1440} x_n^2 f_n - \frac{307h^2}{5040} x_n f_n - \frac{101h}{240} x_n^2 f_{n+1} - \frac{793h^2}{2520} x_n f_{n+1} + \frac{31h}{240} x_n^2 f_{n+2} \\ & + \frac{19h^2}{280} x_n f_{n+2} - \frac{41h}{720} x_n^2 f_{n+3} - \frac{79h^2}{2520} x_n f_{n+3} + \frac{h}{96} x_n^2 f_{n+4} + \frac{29h^2}{5040} x_n f_{n+4} - \frac{x_n^6}{1440h^3}(5f_n - \\ & 18f_{n+1} + 24f_{n+2} - 14f_{n+3} + 3f_{n+4}) - \frac{x_n^7}{5040h^4}(f_n - 4f_{n+1} + 6f_{n+2} - 4f_{n+3} + f_{n+4}) + \\ & \frac{x_n^4}{288h}(-25f_n + 48f_{n+1} - 36f_{n+2} + 16f_{n+3} - 3f_{n+4}) + \frac{x_n}{2h}(3y_n - 4y_{n+1} + y_{n+2}) + \frac{x_n^2}{2h^2}(y_n \\ & -2y_{n+1} + y_{n+2}) + \frac{x_n^5}{1440h^2}(-35f_n + 104f_{n+1} - 114f_{n+2} + 56f_{n+3} - 11f_{n+4}). \end{aligned}$$

$$\begin{aligned} a_1 = {} & -\frac{x_n}{2h}(3y_n - 4y_{n+1} + y_{n+2}) + \frac{x_n^3}{72h}(-25f_n + 48f_{n+1} - 36f_{n+2} + 16f_{n+3} - 3f_{n+4}) \\ & + \frac{h^2}{5040}(307f_n + 1586f_{n+1} - 342f_{n+2} + 158f_{n+3} - 29f_{n+4}) - \frac{x_n^2}{2} f_n + \frac{x_n^5}{240h^3}(5f_n - \end{aligned}$$

$18f_{n+1}+24f_{n+2}-14f_{n+3}+3f_{n+4})+\frac{x_n^6}{720h^4}(f_n-4f_{n+1}+6f_{n+2}-4f_{n+3}+f_{n+4})-\frac{x_n}{h^2}(y_n$

$-2y_{n+1}+y_{n+2})-\frac{x_n^4}{288h^2}(-35f_n+104f_{n+1}-114f_{n+2}+56f_{n+3}-11f_{n+4})+\frac{x_n h}{720}(233f_n$

$+606f_{n+1}-186f_{n+2}+82f_{n+3}-15f_{n+4})$

$a_2 = \frac{1}{2}(y_n-2y_{n+1}+y_{n+2})+\frac{x_n^3}{144h^2}(-35f_n+104f_{n+1}-114f_{n+2}+56f_{n+3}-11f_{n+4})-$

$\frac{h}{1440}(233f_n+606f_{n+1}-186f_{n+2}+82f_{n+3}-15f_{n+4})-\frac{x_n}{2}f_n-\frac{x_n^4}{96h^3}(5f_n-18f_{n+1}$

$+24f_{n+2}-14f_{n+3}+3f_{n+4})+\frac{x_n^2}{48h}(-25f_n+48f_{n+1}-36f_{n+2}+16f_{n+3}-3f_{n+4})-\frac{x_n^5}{240h^4}(f_n$

$-4f_{n+1}+6f_{n+2}-4f_{n+3}+f_{n+4})$

$a_3 = \frac{1}{6}f_n+\frac{x_n^3}{72h^3}(5f_n-18f_{n+1}+24f_{n+2}-14f_{n+3}+3f_{n+4})-\frac{x_n^2}{144h^2}(-35f_n+104f_{n+1}$

$-114f_{n+2}+56f_{n+3}-11f_{n+4})+\frac{x_n^4}{144h^4}(f_n-4f_{n+1}+6f_{n+2}-4f_{n+3}+f_{n+4})-\frac{x_n}{72h}(-25f_n$

$+48f_{n+1}-36f_{n+2}+16f_{n+3}-3f_{n+4})$

$a_4 = \frac{1}{288h}(-25f_n+48f_{n+1}-36f_{n+2}+16f_{n+3}-3f_{n+4})-\frac{x_n^2}{96h^3}(5f_n-18f_{n+1}+24f_{n+2}$

$-14f_{n+3}+3f_{n+4})-\frac{x_n^3}{144h^4}(f_n-4f_{n+1}+6f_{n+2}-4f_{n+3}+f_{n+4})+\frac{x_n}{288h^2}(-35f_n+104f_{n+1}$

$-114f_{n+2}+56f_{n+3}-11f_{n+4})$

$a_5 = -\frac{1}{1440h^2}(-35f_n+104f_{n+1}-114f_{n+2}+56f_{n+3}-11f_{n+4})+\frac{x_n^2}{240h^4}(f_n-4f_{n+1}+$

$6f_{n+2}-4f_{n+3}+f_{n+4})+\frac{x_n}{240h^3}(5f_n-18f_{n+1}+24f_{n+2}-14f_{n+3}+3f_{n+4})$

$a_6 = \frac{1}{1440h^3}(5f_n-18f_{n+1}+24f_{n+2}-14f_{n+3}+3f_{n+4})-\frac{x_n}{720h^4}(f_n-4f_{n+1}+6f_{n+2}-$

$4f_{n+3}+f_{n+4})$

$a_7 = \frac{1}{5040h^4}(f_n-4f_{n+1}+6f_{n+2}-4f_{n+3}+f_{n+4})$

Substituting the values of *a's* into equation (4.2.1.1) and simplifying, this gives a continuous linear multistep method of the form:

$$y(x)=\sum_{j=0}^{k-2}\alpha_j(x)y_{n+j}+h^3\sum_{j=0}^{k}\beta_j(x)f_{n+j} \tag{4.2.1.7}$$

where $x=zh+x_n+3h$ (4.2.1.8)

Equation (4.2.1.8) is substituted into (4.2.1.7) and simplified as follows

$$\begin{aligned}
\alpha_0(z)&=1+\frac{z^2}{2}+\frac{3z}{2}\\
\alpha_1(z)&=-3-4z-z^2\\
\alpha_2(z)&=3+\frac{5z}{2}+\frac{z^2}{2}\\
\beta_0(z)&=\frac{1}{10080}(42+89z+70z^2-35z^4-7z^5+7z^6+2z^7)\\
\beta_1(z)&=\frac{1}{10080}(4872+7204z+2184z^2+210z^4+28z^5-42z^6-8z^7)\\
\beta_2(z)&=\frac{1}{5040}(2646+5265z+2919z^2-315z^4+21z^5+42z^6+6z^7)\\
\beta_3(z)&=\frac{1}{5040}(-84+326z+1036z^2+840z^3+175z^4-70z^5-35z^6-4z^7)\\
\beta_4(z)&=\frac{1}{10080}(42+5z-84z^2+105z^4+77z^5+21z^6+2z^7)
\end{aligned} \tag{4.2.1.9}$$

Equation (4.2.1.9) is evaluated at the non-interpolating points .i.e, at $z=0$ and 1 to give

$$\begin{aligned}240y_{n+3}-720y_{n+2}+720y_{n+1}-240y_n&=h^3(f_{n+4}-4f_{n+3}+126f_{n+2}\\&+116f_{n+1}+f_n).\end{aligned} \tag{4.2.1.10}$$

$$\begin{aligned}60y_{n+4}-360y_{n+2}+480y_{n+1}-180y_n&=h^3(f_{n+4}+26f_{n+3}+126f_{n+2}\\&+86f_{n+1}+f_n).\end{aligned} \tag{4.2.1.11}$$

The first derivative of (4.2.1.9) gives

$$\begin{aligned}
\alpha_0'(z)&=z+\frac{3}{2}\\
\alpha_1'(z)&=-4-2z\\
\alpha_2'(z)&=z+\frac{5}{2}\\
\beta_0'(z)&=\frac{1}{10080}(89+140z-140z^3-35z^4+42z^5+14z^6)
\end{aligned}$$

$$\beta_1'(z)=\frac{1}{10080}(7204+4368z+840z^3+140z^4-252z^5-56z^6)$$

$$\beta_2'(z)=\frac{1}{5040}(5265+5838z-1260z^3+105z^4+252z^5+42z^6)$$

$$\beta_3'(z)=\frac{1}{5040}(326+2072z+2520z^2+700z^3-350z^4-210z^5-28z^6)$$

$$\beta_4'(z)=\frac{1}{10080}(5-168z+420z^3+385z^4+126z^5+14z^6) \qquad (4.2.1.12)$$

Evaluating (4.2.1.12) at all the grid points. i.e, at z=-3, -2,-1, 0 and 1 gives

$$5040hy_n'-2520y_{n+2}+10080y_{n+1}-7560y_n=h^3(-29f_{n+4}+158f_{n+3} \qquad (4.2.1.13)$$
$$-342f_{n+2}+1586f_{n+1}+307f_n).$$

$$10080hy_{n+1}'-5040y_{n+2}+5040y_n=h^3(5f_{n+4}-20f_{n+3}-54f_{n+2} \qquad (4.2.1.14)$$
$$-1532f_{n+1}-79f_n).$$

$$5040hy_{n+2}'-7560y_{n+2}+10080y_{n+1}-2520y_n=h^3(13f_{n+4}-94f_{n+3} \qquad (4.2.1.15)$$
$$+582f_{n+2}+13f_n).$$

$$10080hy_{n+3}'-25200y_{n+2}+40320y_{n+1}-15120y_n=h^3(5f_{n+4}+652f_{n+3} \qquad (4.2.1.16)$$
$$+10530f_{n+2}+7204f_{n+1}+89f_n).$$

$$5040hy_{n+4}'-17640y_{n+2}+30240y_{n+1}-12600y_n=h^3(391f_{n+4}+5030f_{n+3} \qquad (4.2.1.17)$$
$$+10242f_{n+2}+6122f_{n+1}+55f_n).$$

The second derivative of (4.2.1.9) gives

$$\alpha_0''(z)=1$$
$$\alpha_1''(z)=-2$$
$$\alpha_2''(z)=1$$
$$\beta_0''(z)=\frac{1}{10080}(140-420z^2-140z^3+210z^4+84z^5)$$
$$\beta_1''(z)=\frac{1}{10080}(4368+2520z^2+560z^3-1260z^4-336z^5) \qquad (4.2.1.18)$$
$$\beta_2''(z)=\frac{1}{5040}(5838-3780z^2+420z^3+1260z^4+252z^5)$$
$$\beta_3''(z)=\frac{1}{5040}(2072+5040z+2100z^2-1400z^3-1050z^4-168z^5)$$
$$\beta_4''(z)=\frac{1}{10080}(-168+1260z^2+1540z^3+630z^4+84z^5)$$

Evaluating (4.2.1.18) at all the grid points. i.e, at z=-3, -2,-1, 0 and 1 produces

$$720h^2y_n''-720y_{n+2}+1440y_{n+1}-720y_n=h^3(15f_{n+4}-82f_{n+3}+186f_{n+2} \qquad (4.2.1.19)$$
$$-602f_{n+1}-233f_n).$$

$$360h^2 y''_{n+1} - 360y_{n+2} + 720y_{n+1} - 360y_n = h^3(-2f_{n+4} + 12f_{n+3} - 39f_{n+2} + 20f_{n+1} + 9f_n). \tag{4.2.1.20}$$

$$720h^2 y''_{n+2} - 720y_{n+2} + 1440y_{n+1} - 720y_n = h^3(7f_{n+4} - 50f_{n+3} + 378f_{n+2} + 386f_{n+1} - f_n). \tag{4.2.1.21}$$

$$360h^2 y''_{n+3} - 360y_{n+2} + 720y_{n+1} - 360y_n = h^3(-6f_{n+4} + 148f_{n+3} + 417f_{n+2} + 156f_{n+1} + 5f_n). \tag{4.2.1.22}$$

$$720h^2 y''_{n+4} - 720y_{n+2} + 1440y_{n+1} - 720y_n = h^3(239f_{n+4} + 942f_{n+3} + 570f_{n+2} + 418f_{n+1} - 9f_n). \tag{4.2.1.23}$$

Joining equations (4.2.1.10), (4.2.1.11), (4.2.1.13) and (4.2.1.19) to form a block of the form (1.10)

$$\begin{pmatrix} 720 & -720 & 240 & 0 \\ 480 & -360 & 0 & 60 \\ -10080 & 2520 & 0 & 0 \\ 1440 & -720 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 240 \\ 0 & 0 & 0 & 180 \\ 0 & 0 & 0 & -7560 \\ 0 & 0 & 0 & 720 \end{pmatrix}\begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5040 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix} +$$

$$h^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -720 \end{pmatrix}\begin{pmatrix} y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix} + h^3\begin{pmatrix} 116 & 126 & -4 & 1 \\ 86 & 126 & 26 & 1 \\ 1586 & -342 & 158 & -29 \\ -606 & 186 & -82 & 15 \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} + h^3\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 307 \\ 0 & 0 & 0 & -233 \end{pmatrix}\begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$

The equation above is multiplied by $(A^0)^{-1}$ to give

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix} +$$

$$h^2\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 8 \end{pmatrix}\begin{pmatrix} y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix} h^3\begin{pmatrix} \frac{3210}{30420} & \frac{-1854}{30240} & \frac{258}{10080} & \frac{-141}{10080} \\ \frac{1992}{1890} & \frac{-720}{1890} & \frac{104}{630} & \frac{-57}{1890} \\ \frac{11178}{3360} & \frac{-1458}{3360} & \frac{45}{112} & \frac{-243}{3360} \\ \frac{2176}{315} & \frac{32}{105} & \frac{128}{105} & \frac{-8}{63} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} +$$

$$h^3\begin{pmatrix} 0 & 0 & 0 & \frac{3051}{30240} \\ 0 & 0 & 0 & \frac{993}{1890} \\ 0 & 0 & 0 & \frac{4293}{3360} \\ 0 & 0 & 0 & \frac{248}{105} \end{pmatrix}\begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \qquad (4.2.1.24)$$

This gives

$$y_{n+1} = y_n + hy_n' + \frac{1}{2}h^2 y_n'' + \frac{h^3}{30240}(-141 f_{n+4} + 774 f_{n+3} - 1854 f_{n+2} + 3210 f_{n+1} + 3051 f_n). \qquad (4.2.1.25)$$

$$y_{n+2} = y_n + 2hy_n' + 2h^2 y_n'' + \frac{h^3}{1890}(-57 f_{n+4} + 312 f_{n+3} - 720 f_{n+2} + 1992 f_{n+1} + 993 f_n). \qquad (4.2.1.26)$$

$$y_{n+3} = y_n + 3hy_n' + \frac{9}{2}h^2 y_n'' + \frac{h^3}{3360}(-243 f_{n+4} + 1350 f_{n+3} - 1458 f_{n+2} + 11178 f_{n+1} + 4293 f_n). \qquad (4.2.1.27)$$

$$y_{n+4} = y_n + 4hy_n' + 8h^2 y_n'' + \frac{h^3}{2205}(-280 f_{n+4} + 2688 f_{n+3} + 672 f_{n+2} + 15232 f_{n+1} + 5208 f_n). \qquad (4.2.1.28)$$

Substituting (4.2.1.25) and (4.2.1.26) into (4.2.1.14) – (4.2.1.17) to give the first derivative of the block

$$y'_{n+1} = y_n' + hy_n'' + \frac{h^2}{1440}(-21 f_{n+4} + 116 f_{n+3} - 282 f_{n+2} + 540 f_{n+1} + 367 f_n) \qquad (4.2.1.29)$$

$$y'_{n+2} = y_n' + 2hy_n'' + \frac{h^2}{270}(-9 f_{n+4} + 48 f_{n+3} - 90 f_{n+2} + 432 f_{n+1} + 159 f_n) \qquad (4.2.1.30)$$

$$y'_{n+3} = y_n' + 3hy_n'' + \frac{h^2}{160}(-9 f_{n+4} + 60 f_{n+3} + 54 f_{n+2} + 468 f_{n+1} + 147 f_n) \qquad (4.2.1.31)$$

$$y'_{n+4} = y_n' + 4hy_n'' + \frac{h^2}{45}(64 f_{n+3} + 48 f_{n+2} + 192 f_{n+1} + 56 f_n) \qquad (4.2.1.32)$$

Substituting (4.2.1.25) and (4.2.1.26) into (4.2.1.20) – (4.2.1.23) to give the second derivative of the block

$$y''_{n+1} = y_n'' + \frac{h}{720}(-19 f_{n+4} + 106 f_{n+3} - 264 f_{n+2} + 646 f_{n+1} + 251 f_n) \qquad (4.2.1.33)$$

$$y''_{n+2} = y''_n + \frac{h}{90}(-f_{n+4} + 4f_{n+3} + 24f_{n+2} + 124f_{n+1} + 29f_n) \qquad (4.2.1.34)$$

$$y''_{n+3} = y''_n + \frac{h}{80}(-3f_{n+4} + 42f_{n+3} + 72f_{n+2} + 102f_{n+1} + 27f_n) \qquad (4.2.1.35)$$

$$y''_{n+4} = y''_n + \frac{h}{45}(14f_{n+4} + 64f_{n+3} + 24f_{n+2} + 64f_{n+1} + 14f_n) \qquad (4.2.1.36)$$

4.2.2 Properties of Four–Step Block Method for Third Order ODEs.

The section contains the establishment of the order, zero-stability and region of absolute stability of four–step block method for third order ODEs.

4.2.2.1 Order of Four–Step Block Method for Third Order ODEs.

The method used in section 3.2.2.1 is applied in finding the order of the block method (4.2.1.25 – 4.2.1.28) as shown below

$$\begin{pmatrix} \sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m - \sum_{m=0}^{2}\frac{h^m}{m!}y_n^{(m)} - \frac{3051}{30240}h^3 y'''_n - \sum_{m=0}^{\infty}\frac{h^{3+m}}{(30240)(m!)}y_n^{(3+m)}\begin{pmatrix}3210(1)^m - 1854(2)^m + 774(3)^m \\ -141(4)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m - \sum_{m=0}^{2}\frac{(2h)^m}{m!}y_n^{(m)} - \frac{993}{1890}h^3 y'''_n - \sum_{m=0}^{\infty}\frac{h^{3+m}}{(1890)(m!)}y_n^{(3+m)}\begin{pmatrix}1992(1)^m - 720(2)^m + 312(3)^m \\ -57(4)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m - \sum_{m=0}^{2}\frac{(3h)^m}{m!}y_n^{(m)} - \frac{4293}{3360}h^3 y'''_n - \sum_{m=0}^{\infty}\frac{h^{3+m}}{(3360)(m!)}y_n^{(3+m)}\begin{pmatrix}11178(1)^m - 1458(2)^m + 1350(3)^m \\ -342(4)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m - \sum_{m=0}^{2}\frac{(4h)^m}{m!}y_n^{(m)} - \frac{5208}{2205}h^3 y'''_n - \sum_{m=0}^{\infty}\frac{h^{3+m}}{(2205)(m!)}y_n^{(3+m)}\begin{pmatrix}15232(1)^m + 672(2)^m + 2688(3)^m \\ -280(4)^m\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Comparing the coefficients of h^m and y_n^m. This gives

$$C_0 = \begin{pmatrix}1-1\\1-1\\1-1\\1-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{1}{2!} \\ \frac{(2)^2}{2!} - \frac{(2)^2}{2!} \\ \frac{(3)^2}{2!} - \frac{(3)^2}{2!} \\ \frac{(4)^2}{2!} - \frac{(4)^2}{2!} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} \frac{1}{3!} - \frac{3051}{30240} - \frac{1}{(30240)(0!)}\left(3210(1)^0 - 1854(2)^0 + 774(3)^0 - 141(4)^0\right) \\ \frac{(2)^3}{3!} - \frac{993}{1890} - \frac{1}{(1890)(0!)}\left(1992(1)^0 - 720(2)^0 + 312(3)^0 - 57(4)^0\right) \\ \frac{(3)^3}{3!} - \frac{4293}{3360} - \frac{1}{(3360)(0!)}\left(11178(1)^0 - 1458(2)^0 + 1350(3)^0 - 243(4)^0\right) \\ \frac{(4)^3}{3!} - \frac{5208}{2205} - \frac{1}{(2205)(0!)}\left(15232(1)^0 + 672(2)^0 + 2688(3)^0 - 280(4)^0\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_4 = \begin{pmatrix} \frac{1}{4!} - \frac{1}{(30240)(1!)}\left(3210(1)^1 - 1854(2)^1 + 774(3)^1 - 141(4)^1\right) \\ \frac{(2)^4}{4!} - \frac{1}{(1890)(1!)}\left(1992(1)^1 - 720(2)^1 + 312(3)^1 - 57(4)^1\right) \\ \frac{(3)^4}{4!} - \frac{1}{(3360)(1!)}\left(11178(1)^1 - 1458(2)^1 + 1350(3)^1 - 243(4)^1\right) \\ \frac{(4)^4}{4!} - \frac{1}{(2205)(1!)}\left(15232(1)^1 + 672(2)^1 + 2688(3)^1 - 280(4)^1\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_5 = \begin{pmatrix} \frac{1}{5!} - \frac{1}{(30240)(2!)}\left(3210(1)^2 - 1854(2)^2 + 774(3)^2 - 141(4)^2\right) \\ \frac{(2)^5}{5!} - \frac{1}{(1890)(2!)}\left(1992(1)^2 - 720(2)^2 + 312(3)^2 - 57(4)^2\right) \\ \frac{(3)^5}{5!} - \frac{1}{(3360)(2!)}\left(11178(1)^2 - 1458(2)^2 + 1350(3)^2 - 243(4)^2\right) \\ \frac{(4)^5}{5!} - \frac{1}{(2205)(2!)}\left(15232(1)^2 + 672(2)^2 + 2688(3)^2 - 280(4)^2\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6 = \begin{pmatrix} \frac{1}{6!} - \frac{1}{(30240)(3!)}\left(3210(1)^3 - 1854(2)^3 + 774(3)^3 - 141(4)^3\right) \\ \frac{(2)^6}{6!} - \frac{1}{(1890)(3!)}\left(1992(1)^3 - 720(2)^3 + 312(3)^3 - 57(4)^3\right) \\ \frac{(3)^6}{6!} - \frac{1}{(3360)(3!)}\left(11178(1)^3 - 1458(2)^3 + 1350(3)^3 - 243(4)^3\right) \\ \frac{(4)^6}{6!} - \frac{1}{(2205)(3!)}\left(15232(1)^3 + 672(2)^3 + 2688(3)^3 - 280(4)^3\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_7 = \begin{pmatrix} \frac{1}{7!} - \frac{1}{(30240)(4!)}\left(3210(1)^4 - 1854(2)^4 + 774(3)^4 - 141(4)^4\right) \\ \frac{(2)^7}{7!} - \frac{1}{(1890)(4!)}\left(1992(1)^4 - 720(2)^4 + 312(3)^4 - 57(4)^4\right) \\ \frac{(3)^7}{7!} - \frac{1}{(3360)(4!)}\left(11178(1)^4 - 1458(2)^4 + 1350(3)^4 - 243(4)^4\right) \\ \frac{(4)^7}{7!} - \frac{1}{(2205)(4!)}\left(15232(1)^4 + 672(2)^4 + 2688(3)^4 - 280(4)^4\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_8 = \begin{pmatrix} \frac{1}{8!} - \frac{1}{(30240)(5!)}\left(3210(1)^5 - 1854(2)^5 + 774(3)^5 - 141(4)^5\right) \\ \frac{(2)^8}{8!} - \frac{1}{(1890)(5!)}\left(1992(1)^5 - 720(2)^5 + 312(3)^5 - 57(4)^5\right) \\ \frac{(3)^8}{8!} - \frac{1}{(3360)(5!)}\left(11178(1)^5 - 1458(2)^5 + 1350(3)^5 - 243(4)^5\right) \\ \frac{(4)^8}{8!} - \frac{1}{(2205)(5!)}\left(15232(1)^5 + 672(2)^5 + 2688(3)^5 - 280(4)^5\right) \end{pmatrix} = \begin{pmatrix} \frac{139}{40320} \\ \frac{1}{45} \\ \frac{243}{4480} \\ \frac{32}{315} \end{pmatrix}$$

Therefore, the block is having an order $(5,5,5,5)^T$ with error constants

$$\left(\frac{139}{40320}, \frac{1}{45}, \frac{243}{4480}, \frac{32}{315}\right)^T.$$

4.2.2.2 Zero Stability of Four–Step Block Method for Third Order ODEs

Applying the equation (3.2.2.2.1) to four-step block method (4.2.1.25 – 4.2.1.28) this gives

$$\det[rA^{(0)} - A^{(1)}] = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0$$

This implies $r = 0,0,0,1$. Hence, the method is zero stable.

4.2.2.3 Consistency and Convergence of Four–Step Block Method for Third Order ODEs

The block method (4.2.1.25 – 4.2.1.28) is found to be consistent because it satisfies the conditions listed in Definition 1.4. Hence, it is also convergent since it is zero-stable and consistent.

4.2.2.4 Region of Absolute Stability of Four–Step Block Method for Third Order ODEs.

Applying the equation (3.2.2.4.2) to four-step block (4.2.1.25– 4.2.1.28) we have

$$\bar{h}(\theta, h) = \frac{\begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 \\ 0 & 0 & 0 & e^{4i\theta} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\begin{pmatrix} \frac{3210}{30240}e^{i\theta} & -\frac{1854}{30240}e^{2i\theta} & \frac{774}{30240}e^{3i\theta} & -\frac{141}{30240}e^{4i\theta} \\ \frac{1992}{1890}e^{i\theta} & -\frac{720}{1890}e^{2i\theta} & \frac{312}{1890}e^{3i\theta} & -\frac{57}{1890}e^{4i\theta} \\ \frac{11178}{3360}e^{i\theta} & -\frac{1458}{3360}e^{2i\theta} & \frac{1350}{3360}e^{3i\theta} & -\frac{243}{3360}e^{4i\theta} \\ \frac{15232}{2205}e^{i\theta} & \frac{672}{2205}e^{2i\theta} & \frac{2688}{2205}e^{3i\theta} & \frac{-280}{2205}e^{4i\theta} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \frac{3051}{30240} \\ 0 & 0 & 0 & \frac{993}{1890} \\ 0 & 0 & 0 & \frac{4293}{3360} \\ 0 & 0 & 0 & \frac{5208}{2205} \end{pmatrix}}$$

Simplifying the above matrix and equating the imaginary part to zero we have

$$\bar{h}(\theta, h) = \frac{2625\cos 4\theta - 2625}{\cos 4\theta - 41}$$

Evaluating $\bar{h}(\theta, h)$ at intervals of θ of 30^0 gives results as tabulated in Table 4.1

Table 4.1

Interval of Absolute Stability of Four–Step Block Method for Third Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta, h)$	0	95.85	95.85	0	95.85	95.85	0

Therefore, the interval of absolute stability is (0, 95.85). This is shown in the diagram below

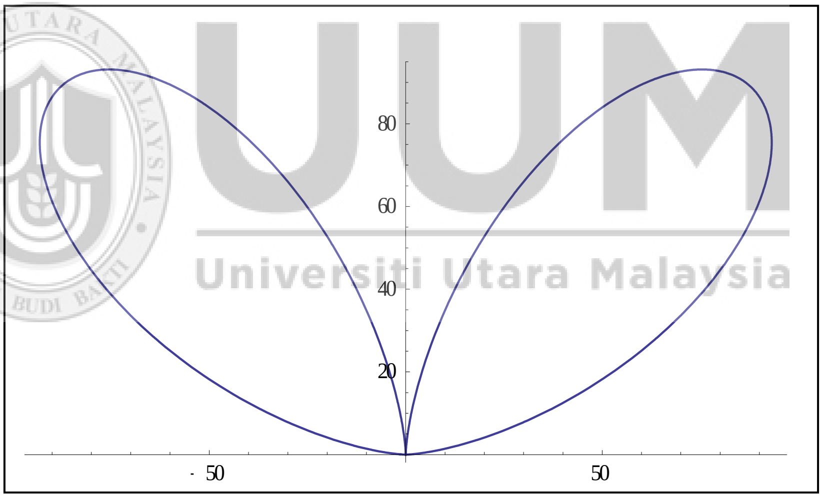

Figure 4.2. Region of absolute stability of four-step block method for third order ODEs

4.3 Five–Step Block Method for Third Order ODEs.

This section includes the derivation of five–step block method for third order ODEs and establishment of its properties.

4.3.1 Derivation of Five–Step Block Method for Third Order ODEs.

We consider power series of the form (4.2.1.1) as an approximate solution to the general third order ODEs of the form (4.2.1.2) where k =5 is the step-length. The first, second and third derivatives of (4.2.1.1) are given in (4.2.1.3), (4.2.1.4) and (4.2.1.5).

Interpolating equation (4.2.1.1) at $x = x_{n+i}, i = 1(1)3$ and collocating (4.2.1.5) at $x = x_{n+i}, i = 0(1)5$ as illustrated in Figure 4.3 below

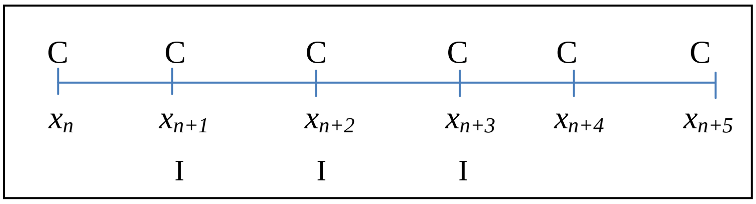

Figure 4.3. Five–step interpolation and collocation method for third ODEs.

This approach gives

$$\begin{pmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & 336x_{n+4}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+5} & 60x_{n+5}^2 & 120x_{n+5}^3 & 210x_{n+5}^4 & 336x_{n+5}^5 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} \quad (4.3.1.1)$$

In order to find the values of *a's* in (4.3.1.1), Gaussian elimination method is applied and this gives the following

$$
\begin{aligned}
a_0 = \; & 3y_{n+1} - 3y_{n+2} + y_{n+3} + \frac{h^3}{480}(-3f_n - 227f_{n+1} - 262f_{n+2} + 18f_{n+3} - 7f_{n+4} + f_{n+5}) \\
& + \frac{x_n^3}{6} f_n - \frac{x_n^7}{5040h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) - \frac{x_n^6}{2880h^3}(17f_n \\
& - 71f_{n+1} + 118f_{n+2} - 98f_{n+3} + 41f_{n+4} - 7f_{n+5}) + \frac{x_n^2 h}{403200}(-63410f_n - 280470f_{n+1} \\
& - 27060f_{n+2} - 46300f_{n+3} + 16710f_{n+4} - 2670f_{n+5}) + \frac{x_n h^2}{10080}(-666f_n - 10535f_{n+1} \\
& - 7144f_{n+2} - 174f_{n+3} + 50f_{n+4} - 11f_{n+5}) + \frac{x_n}{2h}(5y_{n+1} - 8y_{n+2} + 3y_{n+3}) + \frac{x_n^4}{1440h}(-137f_n \\
& + 300f_{n+1} - 300f_{n+2} + 200f_{n+3} - 75f_{n+4} + 12f_{n+5}) + \frac{x_n^2}{2h^2}(y_{n+1} - 2y_{n+2} + y_{n+3}) + \\
& \frac{x_n^5}{1440h^2}(-45f_n + 154f_{n+1} - 214f_{n+2} + 156f_{n+3} - 61f_{n+4} + 10f_{n+5}) - \frac{x_n^8}{40320h^5}(f_n - 5f_{n+1} \\
& + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5})
\end{aligned}
$$

$$
\begin{aligned}
a_1 = \; & -\frac{h^2}{10080}(-666f_n - 10535f_{n+1} - 7144f_{n+2} - 174f_{n+3} + 50f_{n+4} - 11f_{n+5}) + \frac{x_n^2}{2} f_n - \\
& \frac{1}{2h}(5y_{n+1} - 8y_{n+2} + 3y_{n+3}) + \frac{x_n^3}{360h}(-137f_n + 300f_{n+1} - 300f_{n+2} + 200f_{n+3} - 75f_{n+4} + \\
& 12f_{n+5}) + \frac{x_n^6}{720h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) + \frac{x_n^5}{480h^3}(17f_n - \\
& 71f_{n+1} + 118f_{n+2} - 98f_{n+3} + 41f_{n+4} - 7f_{n+5}) - \frac{x_n}{2h^2}(y_{n+1} - 2y_{n+2} + y_{n+3}) - \frac{x_n^4}{288h^2}(-45f_n \\
& + 154f_{n+1} - 214f_{n+2} + 156f_{n+3} - 61f_{n+4} + 10f_{n+5}) + \frac{x_n^7}{5040h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} \\
& + 5f_{n+4} - f_{n+5}) + \frac{x_n h}{20160}(6341f_n + 28047f_{n+1} + 2706f_{n+2} + 4630f_{n+3} - 1671f_{n+4} - \\
& 267f_{n+5})
\end{aligned}
$$

$$
\begin{aligned}
a_2 = \; & \frac{h}{40320}(-6341f_n - 28047f_{n+1} - 2706f_{n+2} - 4630f_{n+3} + 1671f_{n+4} - 267f_{n+5}) - \frac{x_n}{2} f_n \\
& - \frac{1}{2h}(y_{n+1} - 2y_{n+2} + y_{n+3}) + \frac{x_n^3}{144h^2}(-45f_n + 154f_{n+1} - 214f_{n+2} + 156f_{n+3} - 61f_{n+4} + \\
& 10f_{n+5}) - \frac{x_n^5}{240h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) - \frac{x_n^4}{192h^3}(17f_n - \\
& 71f_{n+1} + 118f_{n+2} - 98f_{n+3} + 41f_{n+4} - 7f_{n+5}) - \frac{x_n^2}{240h}(-137f_n + 300f_{n+1} - 300f_{n+2} \\
& + 200f_{n+3} - 75f_{n+4} + 12f_{n+5}) - \frac{x_n^6}{1440h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5})
\end{aligned}
$$

$$a_3 = \frac{1}{6}f_n + \frac{x_n^4}{144h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) + \frac{x_n^3}{144h^3}(17f_n$$
$$-71f_{n+1} + 118f_{n+2} - 98f_{n+3} + 41f_{n+4} - 7f_{n+5}) + \frac{x_n^2}{144h^2}(-45f_n + 154f_{n+1} - 214f_{n+2}$$
$$+156f_{n+3} - 61f_{n+4} + 10f_{n+5}) - \frac{x_n}{360h}(-137f_n + 300f_{n+1} - 300f_{n+2} + 200f_{n+3} - 75f_{n+4}$$
$$+12f_{n+5}) + \frac{x_n^5}{720h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5})$$

$$a_4 = \frac{1}{1440h}(-137f_n + 300f_{n+1} - 300f_{n+2} + 200f_{n+3} - 75f_{n+4} + 12f_{n+5}) - \frac{x_n^3}{144h^4}(3f_n$$
$$-14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) - \frac{x_n^2}{192h^3}(17f_n - 71f_{n+1} + 118f_{n+2} - 98f_{n+3}$$
$$+41f_{n+4} - 7f_{n+5}) - \frac{x_n}{288h^2}(-45f_n + 154f_{n+1} - 214f_{n+2} + 156f_{n+3} - 61f_{n+4} + 10f_{n+5})$$
$$-\frac{x_n^4}{576h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5}).$$

$$a_5 = -\frac{1}{1440h^2}(-45f_n + 154f_{n+1} - 214f_{n+2} + 156f_{n+3} - 61f_{n+4} + 10f_{n+5}) + \frac{x_n^2}{240h^4}(3f_n$$
$$-14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) + \frac{x_n}{480h^3}(17f_n - 71f_{n+1} + 118f_{n+2} - 98f_{n+3}$$
$$+41f_{n+4} - 7f_{n+5}) + \frac{x_n^3}{720h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5})$$

$$a_6 = -\frac{1}{2880h^3}(17f_n - 71f_{n+1} + 118f_{n+2} - 98f_{n+3} + 41f_{n+4} - 7f_{n+5}) - \frac{x_n}{720h^4}(3f_n -$$
$$14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) - \frac{x_n^2}{1440h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3}$$
$$+5f_{n+4} - f_{n+5})$$

$$a_7 = \frac{1}{5040h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) + \frac{x_n}{5040h^5}(f_n$$
$$-5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5})$$

$$a_8 = -\frac{1}{40320h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5})$$

Substituting the values of a's into equation (4.2.1.1) and simplifying, this gives a continuous linear multistep method of the form:

$$y(x)=\sum_{j=1}^{k-2}\alpha_j(x)y_{n+j}+h^3\sum_{j=0}^{k}\beta_j(x)f_{n+j} \qquad (4.3.1.2)$$

where $x = zh + x_n + 4h$ (4.3.1.3)

Substituting (4.3.1.3) into (4.3.1.2) and on simplifying gives

$$\begin{aligned}
\alpha_1(z) &= 1+\frac{z^2}{2}+\frac{3z}{2}\\
\alpha_2(z) &= -3-4z-z^2\\
\alpha_3(z) &= 3+\frac{5z}{2}+\frac{z^2}{2}\\
\beta_0(z) &= \frac{1}{40320}(84+64z-69z^2+84z^4+28z^5-14z^6-8z^7-z^8).\\
\beta_1(z) &= \frac{1}{40320}(-252+36z+625z^2-560z^4-168z^5+98z^6+48z^7\\
&\quad +5z^8).\\
\beta_2(z) &= \frac{1}{20160}(10164+14728z+4023z^2+840z^4+196z^5-154z^6\\
&\quad -56z^7-5z^8).\\
\beta_3(z) &= \frac{1}{20160}(10164+20740z+12021z^2-1680z^4-56z^5+238z^6\\
&\quad +64z^7+5z^8).\\
\beta_4(z) &= \frac{1}{40320}(-252+2928z+7943z^2+6720z^3+1820z^4-420z^5\\
&\quad -350z^6-72z^7-5z^8).
\end{aligned} \qquad (4.3.1.4)$$

Evaluating (4.3.1.4) at the non-interpolating points .that is, at z= -4, 0 and 1 yields

$$\begin{aligned}&-480y_{n+3}+1440y_{n+2}-1440y_{n+1}+480y_n=h^3(f_{n+5}-7f_{n+4}+18f_{n+3}\\&-262f_{n+2}-227f_{n+1}-3f_n).\end{aligned} \qquad (4.3.3.5)$$

$$\begin{aligned}&480y_{n+4}-1440y_{n+3}+1440y_{n+2}-480y_{n+1}=h^3(f_{n+5}-3f_{n+4}+242f_{n+3}\\&+242f_{n+2}-3f_{n+1}+f_n).\end{aligned} \qquad (4.3.1.6)$$

$$\begin{aligned}&240y_{n+5}-1440y_{n+3}+1920y_{n+2}-720y_n=h^3(3f_{n+5}+109f_{n+4}+494f_{n+3}\\&+354f_{n+2}-f_{n+1}+f_n).\end{aligned} \qquad (4.3.1.7)$$

The first derivative of (4.3.1.4) gives

$$\alpha_1'(z) = z + \frac{3}{2}$$
$$\alpha_2'(z) = -4 - 2z$$
$$\alpha_3'(z) = z + \frac{5}{2}$$
$$\beta_0'(z) = \frac{1}{40320}(64 - 138z + 336z^3 + 140z^4 - 84z^5 - 56z^6 - 8z^7).$$
$$\beta_1'(z) = \frac{1}{40320}(36 + 1250z - 2240z^3 - 840z^4 + 588z^5 + 336z^6 + 40z^7).$$
$$\beta_2'(z) = \frac{1}{20160}(14728 + 8046z + 3360z^3 + 980z^4 - 924z^5 - 392z^6 - 40z^7).$$
$$\beta_3'(z) = \frac{1}{20160}(20740 + 24042z - 6720z^3 - 280z^4 + 1428z^5 + 448z^6 + 40z^7).$$
$$\beta_4'(z) = \frac{1}{40320}(2928 + 15886z + 20160z^2 + 7280z^3 - 2100z^4 - 2100z^5 - 504z^6 - 40z^7). \tag{4.3.1.8}$$
$$\beta_5'(z) = \frac{1}{40320}(-44 - 534z + 1344z^3 + 1400z^4 + 588z^5 + 112z^6 + 8z^7).$$

Evaluating (4.3.1.8) at all the grid points. That is, at z=-4, -3, -2,-1, 0 and 1 gives

$$10080hy_n' + 15120y_{n+3} - 40320y_{n+2} + 25200y_{n+1} = h^3(11f_{n+5} - 50f_{n+4} + 174f_{n+3} + 7144f_{n+2} + 10535f_{n+1} + 666f_n). \tag{4.3.1.9}$$

$$20160hy_{n+1}' + 10080y_{n+3} - 40320y_{n+2} + 30240y_{n+1} = h^3(-31f_{n+5} + 207f_{n+4} - 518f_{n+3} + 5494f_{n+2} + 1653f_{n+1} - 85f_n). \tag{4.3.1.10}$$

$$10080hy_{n+2}' - 5040y_{n+3} + 5040y_{n+1} = h^3(5f_{n+4} - 104f_{n+3} - 1482f_{n+2} - 104f_{n+1} + 5f_n). \tag{4.3.1.11}$$

$$20160hy_{n+3}' - 30240y_{n+3} + 40320y_{n+2} - 10080y_{n+1} = h^3(31f_{n+5} - 271f_{n+4} + 2118f_{n+3} + 4874f_{n+2} - 53f_{n+1} + 85f_n). \tag{4.3.1.12}$$

$$10080hy_{n+4}' - 25200y_{n+3} + 40320y_{n+2} - 15120y_{n+1} = h^3(-11f_{n+5} + 732f_{n+4} + 10370f_{n+3} + 7364f_{n+2} + 9f_{n+1} + 16f_n). \tag{4.3.1.13}$$

$$20160hy_{n+5}' - 70560y_{n+3} + 120960y_{n+2} - 50400y_{n+1} = h^3(1437f_{n+5} + 20755f_{n+4} + 39698f_{n+3} + 25758f_{n+2} - 415f_{n+1} + 127f_n). \tag{4.3.1.14}$$

The second derivative of (4.3.1.4) gives

$$\alpha_1''(z) = 1$$
$$\alpha_2''(z) = -2$$
$$\alpha_3''(z) = 1$$
$$\beta_0''(z) = \frac{1}{40320}(-138 + 10080z^2 + 560z^3 - 420z^4 - 336z^5 - 56z^6)$$
$$\beta_1''(z) = \frac{1}{40320}(1250 - 6720z^2 - 3360z^3 + 2940z^4 + 2016z^5 + 280z^6)$$
$$\beta_2''(z) = \frac{1}{20160}(8046 + 10080z^2 + 3920z^3 - 4620z^4 - 2352z^5 - 280z^6)$$
$$\beta_3''(z) = \frac{1}{20160}(24042 - 20160z^2 - 1120z^3 + 7140z^4 + 2688z^5 + 280z^6).$$
$$\beta_4''(z) = \frac{1}{40320}(15886 + 40320z + 21840z^2 - 8400z^3 - 10500z^4 - 3024z^5 - 280z^6). \tag{4.3.1.15}$$
$$\beta_5''(z) = \frac{1}{40320}(-534 + 4032z^2 + 5600z^3 + 2940z^4 + 672z^5 + 56z^6).$$

Evaluating (4.3.1.15) at all the grid points. i.e is, at z=-4, -3, -2,-1, 0 and 1 yields

$$20160h^2 y_n'' - 20160y_{n+3} + 40320y_{n+2} - 20160y_{n+1} = h^3(-267f_{n+5} + 1671f_{n+4} - 4630f_{n+3} - 2706f_{n+2} - 28047f_{n+1} - 6341f_n). \tag{4.3.1.16}$$

$$20160h^2 y_{n+1}'' - 20160y_{n+3} + 40320y_{n+2} - 20160y_{n+1} = h^3(111f_{n+5} - 751f_{n+4} + 2118f_{n+3} - 13878f_{n+2} - 8069f_{n+1} + 309f_n). \tag{4.3.1.17}$$

$$20160h^2 y_{n+2}'' - 20160y_{n+3} + 40320y_{n+2} - 20160y_{n+1} = h^3(-43f_{n+5} + 327f_{n+4} - 1494f_{n+3} + 430f_{n+2} + 849f_{n+1} - 69f_n). \tag{4.3.1.18}$$

$$20160h^2 y_{n+3}'' - 20160y_{n+3} + 40320y_{n+2} - 20160y_{n+1} = h^3(111f_{n+5} - 975f_{n+4} + 9734f_{n+3} + 11658f_{n+2} - 453f_{n+1} + 85f_n). \tag{4.3.1.19}$$

$$20160h^2 y_{n+4}'' - 20160y_{n+3} + 40320y_{n+2} - 20160y_{n+1} = h^3(-267f_{n+5} + 7943f_{n+4} + 24042f_{n+3} + 8046f_{n+2} + 625f_{n+1} - 69f_n). \tag{4.3.1.20}$$

$$20160h^2 y_{n+5}'' - 20160y_{n+3} + 40320y_{n+2} - 20160y_{n+1} = h^3(6383f_{n+5} + 27921f_{n+4} + 12870f_{n+3} + 14794f_{n+2} - 1797f_{n+1} + 309f_n). \tag{4.3.1.21}$$

Joining equations (4.3.1.5) - (4.3.1.7), (4.3.1.9) and (4.3.1.16) to produce a block of the form (1.10)

$$\begin{pmatrix} -1440 & 1440 & -480 & 0 & 0 \\ -480 & 1440 & -1440 & 480 & 0 \\ -720 & 1920 & -1440 & 0 & 240 \\ 25200 & -40320 & 15120 & 0 & 0 \\ -20160 & 40320 & -20160 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & -480 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix} +$$

$$h\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10080 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_{n} \end{pmatrix} + h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20160 \end{pmatrix}\begin{pmatrix} y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_{n} \end{pmatrix} +$$

$$h^3\begin{pmatrix} -227 & -262 & 18 & -7 & 1 \\ -3 & 242 & 242 & -3 & 1 \\ -1 & 354 & 494 & 109 & 3 \\ 10535 & 7144 & 174 & -50 & 11 \\ -28047 & -2706 & -4630 & 1671 & -267 \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} + h^3\begin{pmatrix} 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 666 \\ 0 & 0 & 0 & 0 & -6341 \end{pmatrix}\begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}$$

The above equation is multiplied by the inverse of A^0 to give

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_{n} \end{pmatrix} +$$

$$h^2\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & \frac{25}{2} \end{pmatrix}\begin{pmatrix} y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_{n} \end{pmatrix} + h^3\begin{pmatrix} \frac{29850}{241920} & \frac{-23172}{241920} & \frac{173}{2880} & \frac{-80}{3653} & \frac{834}{241920} \\ \frac{2202}{1890} & \frac{-1140}{1890} & \frac{122}{315} & \frac{-267}{1890} & \frac{42}{1890} \\ \frac{48357}{13440} & \frac{-13122}{13440} & \frac{423}{448} & \frac{-4617}{13440} & \frac{243}{4480} \\ \frac{7008}{945} & \frac{-672}{945} & \frac{704}{315} & \frac{-600}{945} & \frac{96}{945} \\ \frac{16916}{1339} & \frac{11250}{72576} & \frac{543}{113} & \frac{-39375}{72576} & \frac{791}{4639} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} +$$

$$h^3 \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{23574}{241920} \\ 0 & 0 & 0 & 0 & \frac{317}{630} \\ 0 & 0 & 0 & 0 & \frac{16443}{13440} \\ 0 & 0 & 0 & 0 & \frac{2136}{945} \\ 0 & 0 & 0 & 0 & \frac{262125}{72576} \end{pmatrix} \begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \tag{4.3.1.22}$$

which leads to

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2 y''_n + \frac{h^3}{241920}(834 f_{n+5} - 5298 f_{n+4} + 14532 f_{n+3} - 23172 f_{n+2} + 29850 f_{n+1} + 23574 f_n). \tag{4.3.1.23}$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2 y''_n + \frac{h^3}{1890}(42 f_{n+5} - 267 f_{n+4} + 732 f_{n+3} - 1140 f_{n+2} + 2202 f_{n+1} + 951 f_n). \tag{4.3.1.24}$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2}h^2 y''_n + \frac{h^3}{13440}(729 f_{n+5} - 4617 f_{n+4} + 12690 f_{n+3} - 13122 f_{n+2} + 48357 f_{n+1} + 16443 f_n). \tag{4.3.1.25}$$

$$y_{n+4} = y_n + 4hy'_n + 8h^2 y''_n + \frac{h^3}{945}(96 f_{n+5} - 601 f_{n+4} + 2112 f_{n+3} - 672 f_{n+2} + 7008 f_{n+1} + 2136 f_n). \tag{4.3.1.26}$$

$$y_{n+5} = y_n + 5hy'_n + \frac{25}{2}h^2 y''_n + \frac{h^3}{72576}(12375 f_{n+5} - 39375 f_{n+4} + 348750 f_{n+3} + 11250 f_{n+2} + 916875 f_{n+1} + 262125 f_n). \tag{4.3.1.27}$$

Substituting (4.3.1.23) - (4.3.1.25) into (4.3.1.10) – (4.3.1.14) to give the first derivative of the block

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{20160}(214 f_{n+5} - 1364 f_{n+4} + 3764 f_{n+3} - 6088 f_{n+2} + 8630 f_{n+1} + 4924 f_n). \tag{4.3.1.28}$$

$$y'_{n+2} = y'_n + 2hy''_n + \frac{h^2}{630}(16 f_{n+5} - 101 f_{n+4} + 272 f_{n+3} - 370 f_{n+2} + 1088 f_{n+1} + 355 f_n). \tag{4.3.1.29}$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{h^2}{10080}(405 f_{n+5} - 2592 f_{n+4} + 7830 f_{n+3} - 648 f_{n+2} + 31509 f_{n+1} + 8856 f_n). \tag{4.3.1.30}$$

$$y'_{n+4} = y'_n + 4hy''_n + \frac{h^2}{630}(32 f_{n+5} - 160 f_{n+4} + 1216 f_{n+3} + 352 f_{n+2} + 2848 f_{n+1} + 752 f_n). \tag{4.3.1.31}$$

$$y'_{n+5} = y'_n + 5hy''_n + \frac{h^2}{10080}(1375 f_{n+5} + 6250 f_{n+4} + 31250 f_{n+3} + 12500 f_{n+2} + 59375 f_{n+1} + 1525 f_n). \tag{4.3.1.32}$$

Substituting (4.3.1.23) - (4.3.1.25) into (4.3.1.17) – (4.3.1.21) to give the second derivative of the block

$$y''_{n+1} = y''_n + \frac{h}{1440}(27 f_{n+5} - 173 f_{n+4} + 482 f_{n+3} - 798 f_{n+2} + 1427 f_{n+1} + 475 f_n). \tag{4.3.1.33}$$

$$y''_{n+2} = y''_n + \frac{h}{90}(f_{n+5} - 6 f_{n+4} + 14 f_{n+3} + 14 f_{n+2} + 129 f_{n+1} + 28 f_n) \tag{4.3.1.34}$$

$$y''_{n+3} = y''_n = y'_n + \frac{h}{160}(3 f_{n+5} - 21 f_{n+4} + 114 f_{n+3} + 114 f_{n+2} + 219 f_{n+1} + 51 f_n) \tag{4.3.1.35}$$

$$y''_{n+4} = y''_n + \frac{h}{45}(14 f_{n+4} + 64 f_{n+3} + 24 f_{n+2} + 64 f_{n+1} + 14 f_n) \tag{4.3.1.36}$$

$$y''_{n+5} = y''_n + \frac{h}{288}(95 f_{n+5} + 375 f_{n+4} + 250 f_{n+3} + 250 f_{n+2} + 375 f_{n+1} + 95 f_n). \tag{4.3.1.37}$$

4.3.2 Properties of Five–Step Block Method for Third Order ODEs

The order, zero-stability and region of absolute stability of five–step block method for third order ODEs are considered in this section.

4.3.2.1 Order of Five–Step Block Method for Third Order ODEs

Using the approach stated in section 3.2.2.1 in finding the order of the block method (4.3.1.23 – 4.3.1.27) as shown below

$$\begin{pmatrix} \sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m - \sum_{m=0}^{2}\frac{h^m}{m!}y_n^{(m)} - \frac{23574}{241920}h^2y_n'' - \sum_{m=0}^{\infty}\frac{h^{3+m}}{(241920)(m!)}y_n^{(3+m)}\begin{pmatrix}29850(1)^m - 23172(2)^m + 14532(3)^m \\ -5298(4)^m + 834(5)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m - \sum_{m=0}^{2}\frac{(2h)^m}{m!}y_n^{(m)} - \frac{951}{1890}h^2y_n'' - \sum_{m=0}^{\infty}\frac{h^{3+m}}{(1890)(m!)}y_n^{(3+m)}\begin{pmatrix}2202(1)^m - 1140(2)^m + 732(3)^m \\ -267(4)^m + 42(5)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m - \sum_{m=0}^{2}\frac{(3h)^m}{m!}y_n^{(m)} - \frac{16443}{13440}h^2y_n'' - \sum_{m=0}^{\infty}\frac{h^{3+m}}{(13440)(m!)}y_n^{(3+m)}\begin{pmatrix}48357(1)^m - 13122(2)^m + 12690(3)^m \\ -4617(4)^m + 729(5)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m - \sum_{m=0}^{2}\frac{(4h)^m}{m!}y_n^{(m)} - \frac{2136}{945}h^2y_n'' - \sum_{m=0}^{\infty}\frac{h^{3+m}}{(945)(m!)}y_n^{(3+m)}\begin{pmatrix}7008(1)^m - 672(2)^m + 2112(3)^m \\ -600(4)^m + 96(5)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^m - \sum_{m=0}^{2}\frac{(5h)^m}{m!}y_n^{(m)} - \frac{262125}{72576}h^2y_n'' - \sum_{m=0}^{\infty}\frac{h^{3+m}}{(72576)(m!)}y_n^{(3+m)}\begin{pmatrix}916875(1)^m + 11250(2)^m + 348750(3)^m \\ -39375(4)^m + 12375(5)^m\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

Comparing the coefficients of h^m and y_n^m. This gives

$$C_0 = \begin{pmatrix}1-1\\1-1\\1-1\\1-1\\1-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$C_1 = \begin{pmatrix}1-1\\2-2\\3-3\\4-4\\5-5\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$C_2 = \begin{pmatrix}\frac{1}{2!}-\frac{1}{2!}\\ \frac{(2)^2}{2!}-\frac{(2)^2}{2!}\\ \frac{(3)^2}{2!}-\frac{(3)^2}{2!}\\ \frac{(4)^2}{2!}-\frac{(4)^2}{2!}\\ \frac{(5)^2}{2!}-\frac{(5)^2}{2!}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$C_3 = \begin{pmatrix} \frac{1}{3!} - \frac{23574}{241920} - \frac{1}{(241920)(0!)}\left(29850(1)^0 - 23172(2)^0 + 14532(3)^0 - 5298(4)^0 + 834(5)^0\right) \\ \frac{(2)^3}{3!} - \frac{951}{1890} - \frac{1}{(1890)(0!)}\left(2202(1)^0 - 1140(2)^0 + 732(3)^0 - 267(4)^0 + 42(5)^0\right) \\ \frac{(3)^3}{3!} - \frac{16443}{13440} - \frac{1}{(13440)(0!)}\left(48357(1)^0 - 13122(2)^0 + 12690(3)^0 - 4617(4)^0 + 729(5)^0\right) \\ \frac{(4)^3}{3!} - \frac{2136}{945} - \frac{1}{(945)(0!)}\left(7008(1)^0 - 672(2)^0 + 2112(3)^0 - 600(4)^0 + 96(5)^0\right) \\ \frac{(5)^3}{3!} - \frac{262125}{72576} - \frac{1}{(72576)(0!)}\left(916875(1)^0 + 11250(2)^0 + 348750(3)^0 - 39375(4)^0 + 12375(5)^0\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_4 = \begin{pmatrix} \frac{1}{4!} - \frac{1}{(241920)(1!)}\left(29850(1)^1 - 23172(2)^1 + 14532(3)^1 - 5298(4)^1 + 834(5)^1\right) \\ \frac{(2)^4}{4!} - \frac{1}{(1890)(1!)}\left(2202(1)^1 - 1140(2)^1 + 732(3)^1 - 267(4)^1 + 42(5)^1\right) \\ \frac{(3)^4}{4!} - \frac{1}{(13440)(1!)}\left(48357(1)^1 - 13122(2)^1 + 12690(3)^1 - 4617(4)^1 + 729(5)^1\right) \\ \frac{(4)^4}{4!} - \frac{1}{(945)(1!)}\left(7008(1)^1 - 672(2)^1 + 2112(3)^1 - 600(4)^1 + 96(5)^1\right) \\ \frac{(5)^4}{4!} - \frac{1}{(72576)(1!)}\left(916875(1)^1 + 11250(2)^1 + 348750(3)^1 - 39375(4)^1 + 12375(5)^1\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_5 = \begin{pmatrix} \frac{1}{5!} - \frac{1}{(241920)(2!)}\left(29850(1)^2 - 23172(2)^2 + 14532(3)^2 - 5298(4)^2 + 834(5)^2\right) \\ \frac{(2)^5}{5!} - \frac{1}{(1890)(2!)}\left(2202(1)^2 - 1140(2)^2 + 732(3)^2 - 267(4)^2 + 42(5)^2\right) \\ \frac{(3)^5}{5!} - \frac{1}{(13440)(2!)}\left(48357(1)^2 - 13122(2)^2 + 12690(3)^2 - 4617(4)^2 + 729(5)^2\right) \\ \frac{(4)^5}{5!} - \frac{1}{(945)(2!)}\left(7008(1)^2 - 672(2)^2 + 2112(3)^2 - 600(4)^2 + 96(5)^2\right) \\ \frac{(5)^5}{5!} - \frac{1}{(72576)(2!)}\left(916875(1)^2 + 11250(2)^2 + 348750(3)^2 - 39375(4)^2 + 12375(5)^2\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6 = \begin{pmatrix} \frac{1}{6!} - \frac{1}{(241920)(3!)}\left(29850(1)^3 - 23172(2)^3 + 14532(3)^3 - 5298(4)^3 + 834(5)^3\right) \\ \frac{(2)^6}{6!} - \frac{1}{(1890)(3!)}\left(2202(1)^3 - 1140(2)^3 + 732(3)^3 - 267(4)^3 + 42(5)^3\right) \\ \frac{(3)^6}{6!} - \frac{1}{(13440)(3!)}\left(48357(1)^3 - 13122(2)^3 + 12690(3)^3 - 4617(4)^3 + 729(5)^3\right) \\ \frac{(4)^6}{6!} - \frac{1}{(945)(3!)}\left(7008(1)^3 - 672(2)^3 + 2112(3)^3 - 600(4)^3 + 96(5)^3\right) \\ \frac{(5)^6}{6!} - \frac{1}{(72576)(3!)}\left(916875(1)^3 + 11250(2)^3 + 348750(3)^3 - 39375(4)^3 + 12375(5)^3\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_7 = \begin{pmatrix} \frac{1}{7!} - \frac{1}{(241920)(4!)}\left(29850(1)^4 - 23172(2)^4 + 14532(3)^4 - 5298(4)^4 + 834(5)^4\right) \\ \frac{(2)^7}{7!} - \frac{1}{(1890)(4!)}\left(2202(1)^4 - 1140(2)^4 + 732(3)^4 - 267(4)^4 + 42(5)^4\right) \\ \frac{(3)^7}{7!} - \frac{1}{(13440)(4!)}\left(48357(1)^4 - 13122(2)^4 + 12690(3)^4 - 4617(4)^4 + 729(5)^4\right) \\ \frac{(4)^7}{7!} - \frac{1}{(945)(4!)}\left(7008(1)^4 - 672(2)^4 + 2112(3)^4 - 600(4)^4 + 96(5)^4\right) \\ \frac{(5)^7}{7!} - \frac{1}{(72576)(4!)}\left(916875(1)^4 + 11250(2)^4 + 348750(3)^4 - 39375(4)^4 + 12375(5)^4\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_8 = \begin{pmatrix} \frac{1}{8!} - \frac{1}{(241920)(5!)}\left(29850(1)^5 - 23172(2)^5 + 14532(3)^5 - 5298(4)^5 + 834(5)^5\right) \\ \frac{(2)^8}{8!} - \frac{1}{(1890)(5!)}\left(2202(1)^5 - 1140(2)^5 + 732(3)^5 - 267(4)^5 + 42(5)^5\right) \\ \frac{(3)^8}{8!} - \frac{1}{(13440)(5!)}\left(48357(1)^5 - 13122(2)^5 + 12690(3)^5 - 4617(4)^5 + 729(5)^5\right) \\ \frac{(4)^8}{8!} - \frac{1}{(945)(5!)}\left(7008(1)^5 - 672(2)^5 + 2112(3)^5 - 600(4)^5 + 96(5)^5\right) \\ \frac{(5)^8}{8!} - \frac{1}{(72576)(5!)}\left(916875(1)^5 + 11250(2)^5 + 348750(3)^5 - 39375(4)^5 + 12375(5)^5\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_9 = \begin{pmatrix} \frac{1}{9!} - \frac{1}{(241920)(6!)}\left(29850(1)^6 - 23172(2)^6 + 14532(3)^6 - 5298(4)^6 + 834(5)^6\right) \\ \frac{(2)^9}{9!} - \frac{1}{(1890)(6!)}\left(2202(1)^6 - 1140(2)^6 + 732(3)^6 - 267(4)^6 + 42(5)^6\right) \\ \frac{(3)^9}{9!} - \frac{1}{(13440)(6!)}\left(48357(1)^6 - 13122(2)^6 + 12690(3)^6 - 4617(4)^6 + 729(5)^6\right) \\ \frac{(4)^9}{9!} - \frac{1}{(945)(6!)}\left(7008(1)^6 - 672(2)^6 + 2112(3)^6 - 600(4)^6 + 96(5)^6\right) \\ \frac{(5)^9}{9!} - \frac{1}{(72576)(6!)}\left(916875(1)^6 + 11250(2)^6 + 348750(3)^6 - 39375(4)^6 + 12375(5)^6\right) \end{pmatrix} = \begin{pmatrix} \frac{-37}{13688} \\ \frac{-491}{28350} \\ \frac{-1917}{44800} \\ \frac{-1136}{14175} \\ \frac{-121}{943} \end{pmatrix}$$

Hence, the block has order $(6,6,6,6,6)^T$ together with the following error constants $\left(\frac{-37}{13688}, \frac{-491}{28350}, \frac{-1917}{44800}, \frac{-1136}{14175}, \frac{-121}{943}\right)^T$.

4.3.2.2 Zero Stability of Five–Step Block Method for Third Order ODEs.

Equation (3.2.2.2.1) is applied to five-step block method (4.3.1.23 – 4.3.1.27), we have

$$\det[rA^{(0)} - A^{(1)}] = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0$$

This implies $r = 0,0,0,0,1$. Hence, the method is zero stable.

4.3.2.3 Consistency and Convergence of Five–Step Block Method for Third Order ODEs.

The block method (4.3.1.23 – 4.3.1.27) fulfills the conditions itemized in Definition 1.4 and this makes the method to be consistent. Since it is consistent and zero-stable, it is therefore convergent.

4.3.2.4 Region of Absolute Stability of Five–Step Block Method for Third Order ODEs.

Applying the equation (3.2.2.4.2) to five-step block method (4.3.1.23 –4.3.1.27), we have

$$\bar{h}(\theta,h)=\frac{\begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 & 0 \\ 0 & 0 & 0 & e^{4i\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{5i\theta} \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}{\begin{pmatrix} \frac{28950}{241920}e^{i\theta} & -\frac{23172}{241920}e^{2i\theta} & \frac{14532}{241920}e^{3i\theta} & -\frac{5298}{241920}e^{4i\theta} & \frac{834}{241920}e^{5i\theta} \\ \frac{2202}{1890}e^{i\theta} & -\frac{1140}{1890}e^{2i\theta} & \frac{732}{1890}e^{3i\theta} & -\frac{267}{1890}e^{4i\theta} & \frac{42}{1890}e^{5i\theta} \\ \frac{48357}{13440}e^{i\theta} & -\frac{13122}{13440}e^{2i\theta} & \frac{12690}{13440}e^{3i\theta} & -\frac{4617}{13440}e^{4i\theta} & \frac{729}{13440}e^{5i\theta} \\ \frac{7008}{945}e^{i\theta} & -\frac{672}{945}e^{2i\theta} & \frac{2112}{945}e^{3i\theta} & -\frac{600}{945}e^{4i\theta} & \frac{96}{945}e^{5i\theta} \\ \frac{916875}{72576}e^{i\theta} & \frac{11250}{72576}e^{2i\theta} & \frac{348750}{72576}e^{3i\theta} & -\frac{39375}{72576}e^{4i\theta} & \frac{12375}{72576}e^{5i\theta} \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{23574}{241920} \\ 0 & 0 & 0 & 0 & \frac{951}{1890} \\ 0 & 0 & 0 & 0 & \frac{16443}{13440} \\ 0 & 0 & 0 & 0 & \frac{2136}{945} \\ 0 & 0 & 0 & 0 & \frac{262125}{72576} \end{pmatrix}}$$

The above matrix is simplified, after finding the determinant and equating the imaginary part to zero we have

$$\bar{h}(\theta,h)=\frac{2.3461E+23\cos 5\theta-2.3461E+23}{3.3249E+19\cos 5\theta+2.3510E+21}$$

Evaluating $\bar{h}(\theta,h)$ at intervals of θ of 30^{0}, the following tabulation are obtained

Table 4.2

Interval of Absolute Stability of Five–Step Block Method for Third Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta,h)$	0	-189.52	-51.55	-101.80	-152.77	-14.21	-204.44

Therefore, the region of absolute stability is (-204.44, 0). This is shown in the diagram below

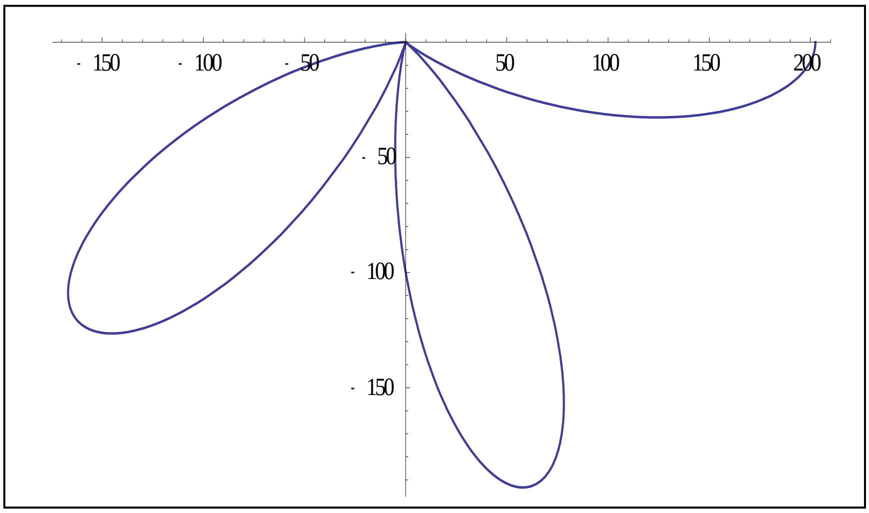

Figure 4.4. Region of absolute stability of five–step block method for third order ODEs.

4.4 Six–Step Block Method for Third Order ODEs.

This section considers the derivation of block method with a step-length k=6 for direct solution of third order ODEs. The properties of the block method are also established.

4.4.1 Derivation of Six–Step Block Method for Third Order ODEs.

The approximate power series of the form (4.2.1.1) is considered as an approximate solution to the general third order ODEs of the form (4.2.1.2) where k =6 is the step-length. The first, second and third derivatives of (4.2.1.1) are given in (4.2.1.3), (4.2.1.4) and (4.2.1.5).

Equation (4.2.1.1) is interpolated at $x = x_{n+i}, i = 2(1)4$ and (4.2.1.5) is collocated at $x = x_{n+i}, i = 0(1)6$ as shown in Figure 4.5 below

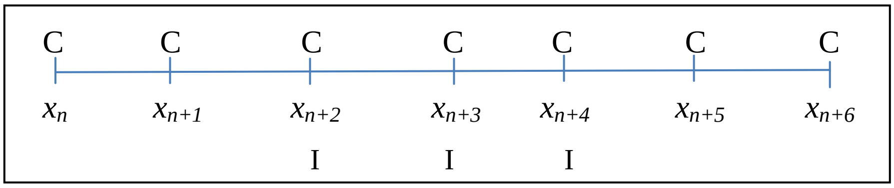

Figure 4.5. Six–step interpolation and collocation method for third order ODEs.

This strategy produces the result below

$$\begin{pmatrix}
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 \\
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 \\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 \\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 & 504x_n^6 \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 & 504x_{n+1}^6 \\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 & 504x_{n+2}^6 \\
0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 & 504x_{n+3}^6 \\
0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & 336x_{n+4}^5 & 504x_{n+4}^6 \\
0 & 0 & 0 & 6 & 24x_{n+5} & 60x_{n+5}^2 & 120x_{n+5}^3 & 210x_{n+5}^4 & 336x_{n+5}^5 & 504x_{n+5}^6 \\
0 & 0 & 0 & 6 & 24x_{n+6} & 60x_{n+6}^2 & 120x_{n+6}^3 & 210x_{n+6}^4 & 336x_{n+6}^5 & 504x_{n+6}^6
\end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix}
=
\begin{pmatrix} y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix} \quad (4.4.1.1)$$

Employing the Gaussian eliminated method in finding the values of *a's* in (4.4.1.1), we have the following

$$\begin{aligned}
a_0 = \quad & 6y_{n+2} - 8y_{n+3} + 3y_{n+4} + \frac{h^3}{30240}(-317f_n - 14100f_{n+1} - 61329f_{n+2} - 45824f_{n+3} \\
& +1041f_{n+4} - 492f_{n+5} + 61f_{n+6}) - \frac{x_n^3}{6}f_n + \frac{x_n^2 h}{80640}(12287f_n - 58716f_{n+1} - 18555f_{n+2} \\
& -29376f_{n+3} + 10779f_{n+4} - 3492f_{n+5} + 479f_{n+6}) + \frac{x_n h^2}{1814400}(-121657f_n - 1913988f_{n+1} \\
& -3457725f_{n+2} - 2471680f_{n+3} + 152445f_{n+4} - 57468f_{n+5} + 7673f_{n+6}) - \frac{x_n^9}{362880h^6}(f_n \\
& -6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6}) + \frac{x_n^2}{2h^2}(y_{n+2} - 2y_{n+3} + y_{n+4}) + \\
& \frac{x_n^5}{21600h^2}(-812f_n + 3132f_{n+1} - 5265f_{n+2} + 5080f_{n+3} - 2970f_{n+4} + 792f_{n+5} - 137f_{n+6}) - \\
& \frac{x_n^8}{80640h^5}(7f_n - 40f_{n+1} + 95f_{n+2} - 120f_{n+3} + 85f_{n+4} - 32f_{n+5} + 5f_{n+6}) - \frac{x_n^7}{30240h^4}(35f_n \\
& -186f_{n+1} + 411f_{n+2} - 484f_{n+3} + 321f_{n+4} - 114f_{n+5} + 17f_{n+6}) - \frac{x_n^6}{5760h^3}(49f_n - 232f_{n+1}
\end{aligned}$$

$$+461 f_{n+2} - 496 f_{n+3} + 307 f_{n+4} - 104 f_{n+5} + 15 f_{n+6}) + \frac{x_n}{2h}(7y_{n+1} - 12y_{n+2} + 5y_{n+3}) +$$
$$\frac{x_n^4}{1440h}(-147 f_n + 360 f_{n+1} - 450 f_{n+2} + 400 f_{n+3} - 225 f_{n+4} + 72 f_{n+5} - 10 f_{n+6})$$

$$a_1 = \quad -\frac{h^2}{1814400}(-121657 f_n - 1913988 f_{n+1} - 3457725 f_{n+2} - 2471680 f_{n+3} + 152445 f_{n+4}$$
$$-57468 f_{n+5} + 7673 f_{n+6}) - \frac{1}{2h}(7y_{n+2} - 12y_{n+3} + 5y_{n+4}) - \frac{x_n^3}{360h}(-147 f_n + 360 f_{n+1}$$
$$-450 f_{n+2} + 400 f_{n+3} - 225 f_{n+4} + 72 f_{n+5} - 10 f_{n+6}) + \frac{x_n^2}{2} f_n + \frac{x_n^8}{40320h^6}(f_n - 6 f_{n+1}$$
$$+15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) + \frac{x_n h}{40320}(12287 f_n - 58716 f_{n+1} - 18555 f_{n+2}$$
$$-29376 f_{n+3} + 10779 f_{n+4} - 3492 f_{n+5} + 479 f_{n+6}) + \frac{x_n^7}{10080h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} -$$
$$120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} + 5 f_{n+6}) + \frac{x_n^6}{4320h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3} +$$
$$321 f_{n+4} - 114 f_{n+5} + 17 f_{n+6}) + \frac{x_n^5}{960h^3}(49 f_n - 232 f_{n+1} + 461 f_{n+2} - 496 f_{n+3} + 307 f_{n+4} -$$
$$104 f_{n+5} + 15 f_{n+6}) - \frac{x_n^2}{2h^2}(y_{n+2} - 2y_{n+3} + y_{n+4}) - \frac{x_n^4}{4320h^2}(-812 f_n + 3132 f_{n+1} - 5265 f_{n+2}$$
$$+5080 f_{n+3} - 2970 f_{n+4} + 972 f_{n+5} - 137 f_{n+6})$$

$$a_2 = \quad \frac{h}{80640}(-12287 f_n - 58716 f_{n+1} - 18555 f_{n+2} - 29376 f_{n+3} + 10779 f_{n+4} - 3492 f_{n+5}$$
$$+479 f_{n+6}) - \frac{x_n}{2} f_n + \frac{1}{2}(y_{n+2} - 2y_{n+3} + y_{n+4}) + \frac{x_n^3}{2160h^2}(-812 f_n + 3132 f_{n+1} - 5265 f_{n+2}$$
$$+5080 f_{n+3} - 2970 f_{n+4} + 972 f_{n+5} - 137 f_{n+6}) - \frac{x_n^7}{10080h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} +$$
$$15 f_{n+4} - 6 f_{n+5} + f_{n+6}) - \frac{x_n^6}{2880h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} +$$
$$5 f_{n+6}) - \frac{x_n^5}{1440h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} - 114 f_{n+5} + 17 f_{n+6}) +$$
$$\frac{x_n^2}{240h}(-147 f_n + 360 f_{n+1} - 450 f_{n+2} + 400 f_{n+3} - 225 f_{n+4} + 72 f_{n+5} - 10 f_{n+6}) + \frac{x_n^4}{384h^3}(49 f_n$$
$$-232 f_{n+1} + 461 f_{n+2} - 496 f_{n+3} + 307 f_{n+4} - 104 f_{n+5} + 15 f_{n+6}).$$

$$a_3 = \frac{1}{6}f_n + \frac{x_n^6}{4320h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6}) - \frac{x_n}{360h}(-147f_n$$
$$+360f_{n+1} - 450f_{n+2} + 400f_{n+3} - 225f_{n+4} + 72f_{n+5} - 10f_{n+6}) + \frac{x_n^5}{1440h^5}(7f_n - 40f_{n+1} +$$
$$95f_{n+2} - 120f_{n+3} + 85f_{n+4} - 32f_{n+5} + 5f_{n+6}) + \frac{x_n^4}{864h^4}(35f_n - 186f_{n+1} + 411f_{n+2} -$$
$$484f_{n+3} + 321f_{n+4} - 114f_{n+5} + 17f_{n+6}) + \frac{x_n^3}{288h^3}(49f_n - 232f_{n+1} + 461f_{n+2} - 496f_{n+3}$$
$$+307f_{n+4} - 104f_{n+5} + 15f_{n+6}) - \frac{x_n^2}{2160h^2}(-812f_n + 3132f_{n+1} - 5265f_{n+2} + 5080f_{n+3}$$
$$-2970f_{n+4} + 972f_{n+5} - 137f_{n+6})$$

$$a_4 = \frac{1}{1440h}(-147f_n + 360f_{n+1} - 450f_{n+2} + 400f_{n+3} - 225f_{n+4} + 72f_{n+5} - 10f_{n+6}) -$$
$$-\frac{x_n^5}{2880h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6}) - \frac{x_n}{4320h^2}(-812f_n$$
$$+3132f_{n+1} - 5265f_{n+2} + 5080f_{n+3} - 2970f_{n+4} + 972f_{n+5} - 137f_{n+6}) - \frac{x_n^4}{1152h^5}(7f_n$$
$$-40f_{n+1} + 95f_{n+2} - 120f_{n+3} + 85f_{n+4} - 32f_{n+5} + 5f_{n+6}) - \frac{x_n^3}{864h^4}(35f_n - 186f_{n+1}$$
$$+411f_{n+2} - 484f_{n+3} + 321f_{n+4} - 114f_{n+5} + 17f_{n+6}) - \frac{x_n^2}{384h^3}(49f_n - 232f_{n+1} +$$
$$461f_{n+2} - 496f_{n+3} + 307f_{n+4} - 104f_{n+5} + 15f_{n+6})$$

$$a_5 = -\frac{1}{12600h^2}(-812f_n + 3132f_{n+1} - 5265f_{n+2} + 5080f_{n+3} - 2970f_{n+4} + 972f_{n+5}$$
$$-137f_{n+6}) + \frac{x_n^4}{2880h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6}) +$$
$$\frac{x_n}{960h^3}(49f_n - 232f_{n+1} + 461f_{n+2} - 496f_{n+3} + 307f_{n+4} - 104f_{n+5} + 15f_{n+6}) +$$
$$\frac{x_n^3}{1440h^5}(7f_n - 40f_{n+1} + 95f_{n+2} - 120f_{n+3} + 85f_{n+4} - 32f_{n+5} + 5f_{n+6}) +$$
$$\frac{x_n^2}{1440h^4}(35f_n - 186f_{n+1} + 411f_{n+2} - 484f_{n+3} + 321f_{n+4} - 114f_{n+5} + 17f_{n+6})$$

$$a_6 = -\frac{1}{5760h^3}(49f_n - 232f_{n+1} + 461f_{n+2} - 496f_{n+3} + 307f_{n+4} - 104f_{n+5} + 15f_{n+6})$$
$$-\frac{x_n^3}{4320h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6}) - \frac{x_n}{4320h^4}(35f_n -$$
$$186f_{n+1} + 411f_{n+2} - 484f_{n+3} + 321f_{n+4} - 114f_{n+5} + 17f_{n+6}) - \frac{x_n^2}{2880h^5}(7f_n - 40f_{n+1}$$
$$+95f_{n+2} - 120f_{n+3} + 85f_{n+4} - 32f_{n+5} + 5f_{n+6})$$

$$a_7 = \frac{1}{30240h^4}(35f_n - 186f_{n+1} + 411f_{n+2} - 484f_{n+3} + 321f_{n+4} - 114f_{n+5} + 17f_{n+6})$$
$$+\frac{x_n^2}{10080h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6}) + \frac{x_n}{10080h^5}(7f_n$$
$$-40f_{n+1} + 95f_{n+2} - 120f_{n+3} + 85f_{n+4} - 32f_{n+5} + 5f_{n+6})$$

$$a_8 = -\frac{1}{80640h^5}(7f_n - 40f_{n+1} + 95f_{n+2} - 120f_{n+3} + 85f_{n+4} - 32f_{n+5} + 5f_{n+6}) -$$
$$\frac{x_n}{40320h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6})$$

$$a_9 = \frac{1}{362880h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6})$$

Substituting the values of a's into equation (4.2.1.1) and simplifying, this gives a continuous linear multistep method of the form:

$$y(x) = \sum_{j=2}^{k-2} \alpha_j(x) y_{n+j} + h^3 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \tag{4.4.1.2}$$

where $x = zh + x_n + 4h$ (4.4.1.3)

Equation (4.4.1.3) is substituted into (4.4.1.2) and simplified to produce

$$\alpha_2(z) = 1 + \frac{z^2}{2} + \frac{3z}{2}$$
$$\alpha_3(z) = -3 - 4z - z^2$$
$$\alpha_4(z) = 3 + \frac{5z}{2} + \frac{z^2}{2}$$

$$\beta_0(z) = \frac{1}{3628800}(3720 + 2086z + 4335z^2 - 5040z^4 - 2184z^5 + 630z^6$$
$$+600z^7 + 135z^8 + 10z^9).$$

$$\beta_1(z) = \frac{1}{302400}(2490 + 1523z + 2685z^2 + 3150z^4 + 1302z^5 - 402z^6$$
$$-360z^7 - 75z^8 - 5z^9).$$

$$\beta_2(z) = \frac{1}{241920}(-5232 - 1870z + 8085z^2 - 8400z^4 - 3192z^5 + 1218z^6$$
$$+888z^7 + 165z^8 + 10z^9).$$

$$\beta_3(z) = \frac{1}{907200}(475980 + 673190z + 159360z^2 + 63000z^4 + 19740z^5$$
$$-10080z^6 - 5520z^7 - 90z^8 - 50z^9).$$

$$\beta_4(z) = \frac{1}{241920}(118248 + 246794z + 148587z^2 - 25200z^4 - 2856z^5$$
$$+3486z^6 + 1368z^7 195z^8 + 10z^9).$$

$$\beta_5(z) = \frac{1}{302400}(-30 + 23003z + 574050z^2 + 50400z^3 + 16170z^4$$
$$-2058z^5 - 2940z^6 - 840z^7 - 105z^8 - 5z^9).$$

$$\beta_6(z) = \frac{1}{3628800}(3840 - 6046z - 19695z^2 + 25200z^4 + 23016z^5 \qquad (4.4.1.4)$$
$$+9450z^6 + 2040z^7 + 225z^8 + 10z^9).$$

Evaluating (4.4.1.4) at the non-interpolating points. i.e, at z= -5, -4, 0 and 1. gives

$$-90720y_{n+4} + 241920y_{n+3} - 181440y_{n+2} + 30240y_n = h^3(61f_{n+6} - 492f_{n+5} \qquad (4.4.1.5)$$
$$+1041f_{n+4} - 45824f_{n+3} - 61329f_{n+2} - 14100f_{n+1} - 317f_n).$$

$$-30240y_{n+4} + 90720y_{n+3} - 90720y_{n+2} + 30240y_{n+1} = h^3(31f_{n+6} - 249f_{n+5} \qquad (4.4.1.6)$$
$$+654f_{n+4} - 15866f_{n+3} - 14781f_{n+2} + 3f_{n+1} - 32f_n).$$

$$30240y_{n+5} - 90720y_{n+4} + 90720y_{n+3} - 30240y_{n+2} = h^3(32f_{n+6} - 3f_{n+5} \qquad (4.4.1.7)$$
$$+14781f_{n+4} + 15866f_{n+3} - 654f_{n+2} + 249f_{n+1} - 31f_n).$$

$$30240y_{n+6} - 181440y_{n+4} + 241920y_{n+3} - 90720y_{n+2} = h^3(317f_{n+6} + 14100f_{n+5} \qquad (4.4.1.8)$$
$$+61329f_{n+4} + 45824f_{n+3} - 1041f_{n+2} + 492f_{n+1} - 61f_n).$$

The first derivative of (4.4.1.4) gives

$$\alpha_2'(z) = z + \frac{3}{2}$$
$$\alpha_3'(z) = -4 - 2z$$
$$\alpha_4'(z) = z + \frac{5}{2}$$
$$\beta_0'(z) = \frac{1}{1814400}(-1043 + 4335z - 10080z^3 - 5460z^4 + 1890z^5$$
$$+2100z^6 + 540z^7 + 45z^8).$$

$$\beta_1'(z) = \frac{1}{1814400}(9138 - 32220z + 75600z^3 + 39060z^4 - 15120z^5$$
$$-15120z^6 - 3600z^7 - 270z^8).$$
$$\beta_2'(z) = \frac{1}{1814400}(-10425 + 121275z - 252000z^3 - 119700z^4 +$$
$$54810z^5 + 46620z^6 + 9900z^7 + 675z^8).$$
$$\beta_3'(z) = \frac{1}{1814400}(1346380 + 637440z + 504000z^3 + 197400z^4$$
$$-120960z^5 - 77280z^6 - 14400z^7 - 900z^8).$$
$$\beta_4'(z) = \frac{1}{1814400}(1850955 + 2228805z - 756000z^3 - 107100z^4$$
$$+156870z^5 + 71820z^6 + 11700z^7 + 675z^8).$$
$$\beta_5'(z) = \frac{1}{1814400}(138018 + 688860z + 907200z^2 + 388080z^3$$
$$-61740z^4 - 105840z^5 - 35280z^6 - 5040z^7 - 270z^8).$$
$$\beta_6'(z) = \frac{1}{1814400}(-3023 - 19695z + 50400z^3 + 57540z^4 + \tag{4.4.1.9}$$
$$28350z^5 + 7140z^6 + 900z^7 + 45z^8).$$

Equation (4.4.1.9) is evaluated at all the grid points. i.e, at z=-5, -4, -3, -2,-1, 0 and 1 produces

$$1814400hy_n' + 4536000y_{n+4} - 10886400y_{n+3} + 6350400y_{n+2} = h^3(-7673f_{n+6}$$
$$+57468f_{n+5} - 152445f_{n+4} + 2471680f_{n+3} + 3457725f_{n+2} + 1913988f_{n+1} \tag{4.4.1.10}$$
$$+121657f_n).$$

$$1814400hy_{n+1}' + 2721600y_{n+4} - 7257600y_{n+3} + 4536000y_{n+2} = h^3(-1043f_{n+6} \tag{4.4.1.11}$$
$$+9138f_{n+5} - 14025f_{n+4} + 1346380f_{n+3} + 1850955f_{n+2} + 138018f_{n+1} - 3023f_n).$$

$$1814400hy_{n+2}' + 907200y_{n+4} - 3628800y_{n+3} + 2721600y_{n+2} = h^3(-1043f_{n+6} \tag{4.4.1.12}$$
$$+8148f_{n+5} - 20415f_{n+4} + 459520f_{n+3} + 174975f_{n+2} - 18132f_{n+1} + 1747f_n).$$

$$1814400hy_{n+3}' - 907200y_{n+4} + 907200y_{n+2} = h^3(-113f_{n+6} + 1578f_{n+5} \tag{4.4.1.13}$$
$$-20415f_{n+4} - 264500f_{n+3} - 20415f_{n+2} + 1578f_{n+1} - 113f_n).$$

$$1814400hy_{n+4}' - 2721600y_{n+4} + 3628800y_{n+3} - 907200y_{n+2} = h^3(1747f_{n+6} \tag{4.4.1.14}$$
$$-18132f_{n+5} + 174975f_{n+4} + 459520f_{n+3} - 20415f_{n+2} + 8184f_{n+1} - 1043f_n).$$

$$1814400hy_{n+5}' - 4536000y_{n+4} + 7257600y_{n+3} - 2721600y_{n+2} = h^3(-3023f_{n+6} \tag{4.4.1.15}$$
$$+138018f_{n+5} + 1850955f_{n+4} + 1346380f_{n+3} - 14025f_{n+2} + 9138f_{n+1} - 1043f_n).$$

$$1814400hy'_{n+6} - 6350400y_{n+4} + 10886400y_{n+3} - 4536000y_{n+2} = h^3(121657f_{n+6} + 1913988f_{n+5} + 3457725f_{n+4} + 2471680f_{n+3} - 152445f_{n+2} + 57468f_{n+1} - 7673f_n). \quad (4.4.1.16)$$

The second derivative of (4.4.1.4) gives

$$\begin{aligned}
\alpha_2''(z) &= 1 \\
\alpha_3''(z) &= -2 \\
\alpha_4''(z) &= 1 \\
\beta_0''(z) &= \frac{1}{120960}(289 - 2016z^2 - 1456z^3 + 630z^4 + 840z^5 \\
&\quad +252z^6 + 24z^7). \\
\beta_1''(z) &= \frac{1}{120960}(-2148 + 15120z^2 + 10416z^3 - 5040z^4 \\
&\quad -6048z^5 - 1680z^6 - 144z^7). \\
\beta_2''(z) &= \frac{1}{120960}(8085 - 50400z^2 - 31920z^3 + 18270z^4 \\
&\quad +18648z^5 + 4620z^6 + 360z^7). \\
\beta_3''(z) &= \frac{1}{120960}(42496 + 100800z^2 + 52640z^3 - 40320z^4 \\
&\quad -30912z^5 - 6720z^6 - 480z^7). \\
\beta_4''(z) &= \frac{1}{120960}(148587 - 151200z^2 - 28560z^3 + 52290z^4 \\
&\quad +28728z^5 + 5460z^6 + 360z^7). \\
\beta_5''(z) &= \frac{1}{120960}(45924 + 120960z + 77616z^2 - 16464z^3 \\
&\quad -35280z^4 - 14112z^5 - 2352z^6 - 144z^7). \\
\beta_6''(z) &= \frac{1}{120960}(-1313 + 10080z^2 + 15344z^3 + 9450z^4 \\
&\quad +2856z^5 + 420z^6 + 27z^7)
\end{aligned} \quad (4.4.1.17)$$

Evaluating (4.4.1.17) at all the grid points. i.e, at z=-5, -4, -3, -2,-1, 0 and 1 produces

$$40320h^2y''_n - 40320y_{n+4} + 80640y_{n+3} - 40320y_{n+2} = h^3(479f_{n+6} - 3492f_{n+5} + 10779f_{n+4} - 39168f_{n+3} - 18555f_{n+2} - 58716f_{n+1} - 12287f_n). \quad (4.4.1.18)$$

$$120960h^2y''_{n+1} - 120960y_{n+4} - 241920y_{n+3} - 120960y_{n+2} = h^3(-289f_{n+6} + 2148f_{n+5} - 8085f_{n+4} - 42496f_{n+3} - 148587f_{n+2} - 45924f_{n+1} + 1313f_n). \quad (4.4.1.19)$$

$$120960h^2y''_{n+2} - 120960y_{n+4} - 241920y_{n+3} - 120960y_{n+2} = h^3(253f_{n+6} - 2028f_{n+5} + 6513f_{n+4} - 75008f_{n+3} - 54609f_{n+2} + 4332f_{n+1} - 413f_n). \quad (4.4.1.20)$$

$$40320h^2 y''_{n+3} - 40320y_{n+4} + 80640y_{n+3} - 40320y_{n+2} = h^3(-43f_{n+6} + 396f_{n+5} - 2343f_{n+4} + 2343f_{n+2} - 396f_{n+1} + 43f_n). \quad (4.4.1.21)$$

$$120960h^2 y''_{n+4} - 120960y_{n+4} - 241920yy_{n+2} = h^3(413f_{n+6} - 4332f_{n+5} + 54609f_{n+4} + 75008f_{n+3} - 6513f_{n+2} + 2028f_{n+1} - 253f_n). \quad (4.4.1.22)$$

$$120960h^2 y''_{n+5} - 120960y_{n+4} - 241920y_{n+3} - 120960y_{n+2} = h^3(-1313f_{n+6} + 45924f_{n+5} + 148587f_{n+4} + 42496f_{n+3} + 8085f_{n+2} - 2148f_{n+1} + 289f_n). \quad (4.4.1.23)$$

$$40320h^2 y''_{n+6} - 40320y_{n+4} + 80640y_{n+3} - 40320y_{n+2} = h^3(12287f_{n+6} + 58716f_{n+5} + 18555f_{n+4} + 39168f_{n+3} - 10779f_{n+2} + 3492f_{n+1} - 479f_n). \quad (4.4.1.24)$$

Combining equations (4.4.1.5) - (4.4.1.8), (4.4.1.10) and (4.4.1.18) to produce a block of the form as follows

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -30240 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1814400 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}$$

$$+ h^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -40320 \end{pmatrix} \begin{pmatrix} y''_{n-5} \\ y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix} +$$

$$\begin{pmatrix} 0 & -181440 & 241920 & -90720 & 0 & 0 \\ 30240 & -90720 & 90720 & -30240 & 0 & 0 \\ 0 & 30240 & 90720 & -90720 & 30240 & 0 \\ 0 & -90720 & 241920 & -181440 & 0 & 30240 \\ 0 & 6350400 & -10886400 & 4536000 & 0 & 0 \\ 0 & -40320 & 80640 & -40320 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \end{pmatrix} =$$

$$h^3 \begin{pmatrix} -14100 & -61329 & -45824 & 1041 & -492 & 61 \\ 3 & -14781 & -15866 & 654 & -249 & 31 \\ 249 & -654 & 15866 & 14781 & -3 & 32 \\ 492 & -1041 & 45824 & 61329 & 14100 & 317 \\ 1913988 & 3457725 & 2471680 & -152445 & 57468 & -7673 \\ -58716 & -18555 & -39168 & 10779 & -3492 & 479 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix} +$$

$$
h^3\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -317 \\ 0 & 0 & 0 & 0 & 0 & -32 \\ 0 & 0 & 0 & 0 & 0 & -31 \\ 0 & 0 & 0 & 0 & 0 & -61 \\ 0 & 0 & 0 & 0 & 0 & 121657 \\ 0 & 0 & 0 & 0 & 0 & -12287 \end{pmatrix}\begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}
$$

The equation above is multiplied by the inverse of A^0 and this gives

$$
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix} +
$$

$$
h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & \frac{25}{2} \\ 0 & 0 & 0 & 0 & 0 & 18 \end{pmatrix}\begin{pmatrix} y''_{n-5} \\ y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix} + h^3\begin{pmatrix} \frac{227}{1626} & \frac{-185}{1357} & \frac{655}{5739} & \frac{-218}{3491} & \frac{179}{9102} & \frac{-37}{13688} \\ \frac{618}{487} & \frac{-233}{270} & \frac{416}{567} & \frac{-379}{945} & \frac{596}{4725} & \frac{-257}{14839} \\ \frac{1141}{296} & \frac{-2882}{1781} & \frac{9}{5} & \frac{-337}{342} & \frac{589}{1894} & \frac{-73}{1706} \\ \frac{1682}{213} & \frac{-1808}{945} & \frac{2176}{567} & \frac{-248}{135} & \frac{1565}{2687} & \frac{-159}{1984} \\ \frac{5884}{439} & \frac{-292}{165} & \frac{1349}{183} & \frac{-1431}{580} & \frac{1625}{1728} & \frac{-121}{943} \\ \frac{3564}{175} & \frac{-81}{70} & \frac{432}{35} & \frac{-81}{35} & \frac{324}{175} & \frac{-9}{50} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix}
$$

$$
+h^3\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{182}{1921} \\ 0 & 0 & 0 & 0 & 0 & \frac{1082}{2227} \\ 0 & 0 & 0 & 0 & 0 & \frac{3575}{3028} \\ 0 & 0 & 0 & 0 & 0 & \frac{5191}{2381} \\ 0 & 0 & 0 & 0 & 0 & \frac{3466}{995} \\ 0 & 0 & 0 & 0 & 0 & \frac{891}{175} \end{pmatrix}\begin{pmatrix} f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \qquad (4.4.1.25)
$$

whose solution is

$$
y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2 y''_n + \frac{h^3}{3628800}(-9809 f_{n+6} + 71364 f_{n+5} - 226605 f_{n+4} + 414160 f_{n+3} - 494715 f_{n+2} + 506604 f_{n+1} + 343801 f_n) \qquad (4.4.1.26)
$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2 y''_n + \frac{h^3}{28350}(-491 f_{n+6} + 3576 f_{n+5} - 11370 f_{n+4} + 20800 f_{n+3} - 24465 f_{n+2} + 35976 f_{n+1} + 13774 f_n). \quad (4.4.1.27)$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2}h^2 y''_n + \frac{h^3}{44800}(-1917 f_{n+6} + 13932 f_{n+5} - 44145 f_{n+4} + 80640 f_{n+3} - 72495 f_{n+2} + 172692 f_{n+1} + 52893 f_n). \quad (4.4.1.28)$$

$$y_{n+4} = y_n + 4hy'_n + 8h^2 y''_n + \frac{h^3}{28350}(-2272 f_{n+6} + 16512 f_{n+5} - 52080 f_{n+4} + 108800 f_{n+3} - 54240 f_{n+2} + 223872 f_{n+1} + 61808 f_n). \quad (4.4.1.29)$$

$$y_{n+5} = y_n + 5hy'_n + \frac{25}{2}h^2 y''_n + \frac{h^3}{145152}(-18625 f_{n+6} + 136500 f_{n+5} - 358125 f_{n+4} + 1070000 f_{n+3} - 256875 f_{n+2} + 1945500 f_{n+1} + 505625 f_n). \quad (4.4.1.30)$$

$$y_{n+6} = y_n + 6hy'_n + 18h^2 y''_n + \frac{h^3}{700}(-126 f_{n+6} + 1296 f_{n+5} - 1620 f_{n+4} + 8640 f_{n+3} - 810 f_{n+2} + 14256 f_{n+1} + 3564 f_n). \quad (4.4.1.31)$$

Substituting (4.4.1.27) - (4.4.1.29) into (4.4.1.11) – (4.4.1.16) to give the first derivative of the block as follows

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{120960}(-995 f_{n+6} + 7254 f_{n+5} - 23109 f_{n+4} + 42484 f_{n+3} - 51453 f_{n+2} + 57750 f_{n+1} + 28549 f_n). \quad (4.4.1.32)$$

$$y'_{n+2} = y'_n + 2hy''_n + \frac{h^2}{120960}(-2432 f_{n+6} + 17664 f_{n+5} - 55872 f_{n+4} + 100864 f_{n+3} - 107520 f_{n+2} + 223488 f_{n+1} + 65728 f_n). \quad (4.4.1.33)$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{h^2}{4480}(-141 f_{n+6} + 1026 f_{n+5} - 3267 f_{n+4} + 6300 f_{n+3} - 2403 f_{n+2} + 14850 f_{n+1} + 3795 f_n). \quad (4.4.1.34)$$

$$y'_{n+4} = y'_n + 4hy''_n + \frac{h^2}{945}(-40 f_{n+6} + 288 f_{n+5} - 840 f_{n+4} + 2624 f_{n+3} - 72 f_{n+2} + 4512 f_{n+1} + 1088 f_n). \quad (4.4.1.35)$$

$$y'_{n+5} = y'_n + 5hy''_n + \frac{h^2}{24192}(-1375 f_{n+6} + 11550 f_{n+5} - 5625 f_{n+4} + 102500 f_{n+3} + 9375 f_{n+2} + 150750 f_{n+1} + 35225 f_n). \quad (4.4.1.36)$$

$$y'_{n+6} = y'_n + 6hy''_n + \frac{h^2}{140}(216 f_{n+5} + 54 f_{n+4} + 816 f_{n+3} + 108 f_{n+2} + 1080 f_{n+1} + 246 f_n). \quad (4.4.1.37)$$

Equations (4.4.1.27) - (4.4.1.29) are substituted into (4.4.1.19) – (4.4.1.24) to give the second derivative of the block below

$$y''_{n+1} = y''_n + \frac{h}{60480}(-863 f_{n+6} + 6312 f_{n+5} - 20211 f_{n+4} + 37504 f_{n+3} - 46461 f_{n+2} + 65112 f_{n+1} + 19087 f_n). \quad (4.4.1.38)$$

$$y''_{n+2} = y''_n + \frac{h}{3780}(-37 f_{n+6} + 264 f_{n+5} - 807 f_{n+4} + 1328 f_{n+3} + 33 f_{n+2} + 5640 f_{n+1} + 1139 f_n). \quad (4.4.1.39)$$

$$y''_{n+3} = y''_n + \frac{h}{2240}(-29 f_{n+6} + 216 f_{n+5} - 729 f_{n+4} + 2176 f_{n+3} + 1161 f_{n+2} + 3240 f_{n+1} + 685 f_n). \quad (4.4.1.40)$$

$$y''_{n+4} = y''_n + \frac{h}{1890}(-16 f_{n+6} + 96 f_{n+5} + 348 f_{n+4} + 3008 f_{n+3} + 768 f_{n+2} + 572 f_n). \quad (4.4.1.41)$$

$$y''_{n+5} = y''_n + \frac{h}{12096}(-275 f_{n+6} + 5640_{n+5} + 11625 f_{n+4} + 16000 f_{n+3} + 6375 f_{n+2} + 17400 f_{n+1} + 3715 f_n). \quad (4.4.1.42)$$

$$y''_{n+6} = y''_n + \frac{h}{140}(41 f_{n+6} + 216 f_{n+5} + 27 f_{n+4} + 272 f_{n+3} + 27 f_{n+2} + 216 f_{n+1} + 41 f_n). \quad (4.4.1.43)$$

4.4.2 Properties of Six–Step Block Method for Third Order ODEs.

This section establishes the order, zero-stability and region of absolute stability of six–step block method for third order ODEs

4.4.2.1 Order of Block Method Six–Step Block Method for Third Order ODEs.

The technique used in section 3.2.2.1 is applied in finding the order of the block method (4.4.1.26 – 4.4.1.31) as shown below

$$\begin{pmatrix} \sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{h^m}{m!}y_n^{(m)}-\frac{343801}{3628800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(3628800)(m!)}y_n^{(3+m)}\begin{pmatrix}506604(1)^m-494715(2)^m+414160(3)^m\\-226605(4)^m+71364(5)^m-9809(6)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(2h)^m}{m!}y_n^{(m)}-\frac{13774}{28350}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(28350)(m!)}y_n^{(3+m)}\begin{pmatrix}35976(1)^m-24465(2)^m+20800(3)^m\\-11370(4)^m+3576(5)^m-491(6)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(3h)^m}{m!}y_n^{(m)}{}_n-\frac{52893}{44800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(44800)(m!)}y_n^{(3+m)}\begin{pmatrix}172692(1)^m-72495(2)^m+80640(3)^m\\-44145(4)^m+13932(5)^m-1917(6)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(4h)^m}{m!}y_n^{(m)}-\frac{61808}{28350}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(28350)(m!)}y_n^{(3+m)}\begin{pmatrix}223872(1)^m-54240(2)^m+108800(3)^m\\-52080(4)^m+16512(5)^m-2272(6)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(5h)^m}{m!}y_n^{(m)}-\frac{505625}{145152}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(145152)(m!)}y_n^{(3+m)}\begin{pmatrix}1945500(1)^m-256875(2)^m+1070000(3)^m\\-358125(4)^m+136500(5)^m-18625(6)^m\end{pmatrix} \\ \sum_{m=0}^{\infty}\frac{(6h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(6h)^m}{m!}y_n^{(m)}-\frac{3564}{700}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(700)(m!)}y_n^{(3+m)}\begin{pmatrix}14256(1)^m-810(2)^m+8640(3)^m\\-1620(4)^m+1296(5)^m-126(6)^m\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

Comparing the coefficients of h^m and y_n^m gives

$$C_0 = \begin{pmatrix}1-1\\1-1\\1-1\\1-1\\1-1\\1-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \\ 5-5 \\ 6-6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{1}{2!} \\ \frac{(2)^2}{2!} - \frac{(2)^2}{2!} \\ \frac{(3)^2}{2!} - \frac{(3)^2}{2!} \\ \frac{(4)^2}{2!} - \frac{(4)^2}{2!} \\ \frac{(5)^2}{2!} - \frac{(5)^2}{2!} \\ \frac{(6)^2}{2!} - \frac{(6)^2}{2!} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} \frac{1}{3!} - \frac{343801}{3628800} - \frac{1}{(3628800)(0!)} \begin{pmatrix} 506604(1)^0 - 494715(2)^0 + 414160(3)^0 - 226605(4)^0 + 71364(5)^0 \\ -9809(6)^0 \end{pmatrix} \\ \frac{(2)^3}{3!} - \frac{13774}{28350} - \frac{1}{(28350)(0!)} \left(35976(1)^0 - 24465(2)^0 + 20800(3)^0 - 11370(4)^0 + 3576(5)^0 - 491(6)^0\right) \\ \frac{(3)^3}{3!} - \frac{52893}{44800} - \frac{1}{(44800)(0!)} \begin{pmatrix} 172692(1)^0 - 72495(2)^0 + 80640(3)^0 - 44145(4)^0 + 13932(5)^0 - \\ 1917(6)^0 \end{pmatrix} \\ \frac{(4)^3}{3!} - \frac{61808}{28350} - \frac{1}{(28350)(0!)} \begin{pmatrix} 223872(1)^0 - 54240(2)^0 + 108800(3)^0 - 52080(4)^0 + 16512(5)^0 - \\ 2272(6)^0 \end{pmatrix} \\ \frac{(5)^3}{3!} - \frac{505625}{145152} - \frac{1}{(145152)(0!)} \begin{pmatrix} 1945500(1)^0 - 256875(2)^0 + 1070000(3)^0 - 358125(4)^0 + 136500(5)^0 \\ -18625(6)^0 \end{pmatrix} \\ \frac{(6)^3}{3!} - \frac{3564}{700} - \frac{1}{(700)(0!)} \left(14256(1)^0 - 810(2)^0 + 8640(3)^0 - 1620(4)^0 + 1296(5)^0 - 126(6)^0\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_4 = \begin{pmatrix} \dfrac{1}{4!} - \dfrac{1}{(3628800)(1!)} \begin{pmatrix} 506604(1)^1 - 494715(2)^1 + 414160(3)^1 - 226605(4)^1 + 71364(5)^1 \\ -9809(6)^0 \end{pmatrix} \\ \dfrac{(2)^4}{4!} - \dfrac{1}{(28350)(1!)} \left(35976(1)^1 - 24465(2)^1 + 20800(3)^1 - 11370(4)^1 + 3576(5)^1 - 491(6)^1\right) \\ \dfrac{(3)^4}{4!} - \dfrac{1}{(44800)(1!)} \begin{pmatrix} 172692(1)^1 - 72495(2)^1 + 80640(3)^1 - 44145(4)^1 + 13932(5)^1 - \\ 1917(6)^1 \end{pmatrix} \\ \dfrac{(4)^4}{4!} - \dfrac{1}{(28350)(1!)} \begin{pmatrix} 223872(1)^1 - 54240(2)^1 + 108800(3)^1 - 52080(4)^1 + 16512(5)^1 - \\ 2272(6)^1 \end{pmatrix} \\ \dfrac{(5)^4}{4!} - \dfrac{1}{(145152)(1!)} \begin{pmatrix} 1945500(1)^1 - 256875(2)^1 + 1070000(3)^1 - 358125(4)^1 + 136500(5)^1 \\ -18625(6)^1 \end{pmatrix} \\ \dfrac{(6)^4}{4!} - \dfrac{1}{(700)(1!)} \left(14256(1)^1 - 810(2)^1 + 8640(3)^1 - 1620(4)^1 + 1296(5)^1 - 126(6)^1\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_5 = \begin{pmatrix} \dfrac{1}{5!} - \dfrac{1}{(3628800)(2!)} \begin{pmatrix} 506604(1)^2 - 494715(2)^2 + 414160(3)^2 - 226605(4)^2 + 71364(5)^2 \\ -9809(6)^2 \end{pmatrix} \\ \dfrac{(2)^5}{5!} - \dfrac{1}{(28350)(2!)} \left(35976(1)^2 - 24465(2)^2 + 20800(3)^2 - 11370(4)^2 + 3576(5)^2 - 491(6)^2\right) \\ \dfrac{(3)^5}{4!} - \dfrac{1}{(44800)(2!)} \begin{pmatrix} 172692(1)^2 - 72495(2)^2 + 80640(3)^2 - 44145(4)^2 + 13932(5)^2 - \\ 1917(6)^2 \end{pmatrix} \\ \dfrac{(4)^5}{5!} - \dfrac{1}{(28350)(2!)} \begin{pmatrix} 223872(1)^2 - 54240(2)^2 + 108800(3)^2 - 52080(4)^2 + 16512(5)^2 - \\ 2272(6)^2 \end{pmatrix} \\ \dfrac{(5)^5}{5!} - \dfrac{1}{(145152)(2!)} \begin{pmatrix} 1945500(1)^2 - 256875(2)^2 + 1070000(3)^2 - 358125(4)^2 + 136500(5)^2 \\ -18625(6)^2 \end{pmatrix} \\ \dfrac{(6)^5}{5!} - \dfrac{1}{(700)(2!)} \left(14256(1)^2 - 810(2)^2 + 8640(3)^2 - 1620(4)^2 + 1296(5)^2 - 126(6)^2\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6=\begin{pmatrix} \frac{1}{6!}-\frac{1}{(3628800)(3!)}\begin{pmatrix}506604(1)^3-494715(2)^3+414160(3)^3-226605(4)^3+71364(5)^3\\ -9809(6)^3\end{pmatrix}\\ \frac{(2)^6}{6!}-\frac{1}{(28350)(3!)}\left(35976(1)^3-24465(2)^3+20800(3)^3-11370(4)^3+3576(5)^3-491(6)^3\right)\\ \frac{(3)^6}{6!}-\frac{1}{(44800)(3!)}\begin{pmatrix}172692(1)^3-72495(2)^3+80640(3)^3-44145(4)^3+13932(5)^3-\\ 1917(6)^3\end{pmatrix}\\ \frac{(4)^6}{6!}-\frac{1}{(28350)(3!)}\begin{pmatrix}223872(1)^3-54240(2)^3+108800(3)^3-52080(4)^3+16512(5)^3-\\ 2272(6)^3\end{pmatrix}\\ \frac{(5)^6}{6!}-\frac{1}{(145152)(3!)}\begin{pmatrix}1945500(1)^3-256875(2)^3+1070000(3)^3-358125(4)^3+136500(5)^3\\ -18625(6)^3\end{pmatrix}\\ \frac{(6)^6}{6!}-\frac{1}{(700)(3!)}\left(14256(1)^3-810(2)^3+8640(3)^3-1620(4)^3+1296(5)^3-126(6)^3\right)\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_7=\begin{pmatrix} \frac{1}{7!}-\frac{1}{(3628800)(4!)}\begin{pmatrix}506604(1)^4-494715(2)^4+414160(3)^4-226605(4)^4+71364(5)^4\\ -9809(6)^4\end{pmatrix}\\ \frac{(2)^7}{7!}-\frac{1}{(28350)(4!)}\left(35976(1)^4-24465(2)^4+20800(3)^4-11370(4)^4+3576(5)^4-491(6)^4\right)\\ \frac{(3)^7}{7!}-\frac{1}{(44800)(4!)}\begin{pmatrix}172692(1)^4-72495(2)^4+80640(3)^4-44145(4)^4+13932(5)^4-\\ 1917(6)^4\end{pmatrix}\\ \frac{(4)^7}{7!}-\frac{1}{(28350)(4!)}\begin{pmatrix}223872(1)^4-54240(2)^4+108800(3)^4-52080(4)^4+16512(5)^4-\\ 2272(6)^4\end{pmatrix}\\ \frac{(5)^7}{7!}-\frac{1}{(145152)(4!)}\begin{pmatrix}1945500(1)^4-256875(2)^4+1070000(3)^4-358125(4)^4+136500(5)^4\\ -18625(6)^4\end{pmatrix}\\ \frac{(6)^7}{7!}-\frac{1}{(700)(4!)}\left(14256(1)^4-810(2)^4+8640(3)^4-1620(4)^4+1296(5)^4-126(6)^4\right)\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_8 = \begin{pmatrix} \dfrac{1}{8!} - \dfrac{1}{(3628800)(5!)}\begin{pmatrix} 506604(1)^5 - 494715(2)^5 + 414160(3)^5 - 226605(4)^5 + 71364(5)^5 \\ -9809(6)^5 \end{pmatrix} \\ \dfrac{(2)^8}{8!} - \dfrac{1}{(28350)(5!)}\left(35976(1)^5 - 24465(2)^5 + 20800(3)^5 - 11370(4)^5 + 3576(5)^5 - 491(6)^5\right) \\ \dfrac{(3)^8}{8!} - \dfrac{1}{(44800)(5!)}\begin{pmatrix} 172692(1)^5 - 72495(2)^5 + 80640(3)^5 - 44145(4)^5 + 13932(5)^5 - \\ 1917(6)^4 \end{pmatrix} \\ \dfrac{(4)^8}{8!} - \dfrac{1}{(28350)(5!)}\begin{pmatrix} 223872(1)^5 - 54240(2)^5 + 108800(3)^5 - 52080(4)^5 + 16512(5)^5 - \\ 2272(6)^5 \end{pmatrix} \\ \dfrac{(5)^8}{8!} - \dfrac{1}{(145152)(5!)}\begin{pmatrix} 1945500(1)^5 - 256875(2)^5 + 1070000(3)^5 - 358125(4)^5 + 136500(5)^5 \\ -18625(6)^5 \end{pmatrix} \\ \dfrac{(6)^8}{8!} - \dfrac{1}{(700)(5!)}\left(14256(1)^5 - 810(2)^5 + 8640(3)^5 - 1620(4)^5 + 1296(5)^5 - 126(6)^5\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_9 = \begin{pmatrix} \dfrac{1}{9!} - \dfrac{1}{(3628800)(6!)}\begin{pmatrix} 506604(1)^6 - 494715(2)^6 + 414160(3)^6 - 226605(4)^6 + 71364(5)^6 \\ -9809(6)^6 \end{pmatrix} \\ \dfrac{(2)^9}{9!} - \dfrac{1}{(28350)(6!)}\left(35976(1)^6 - 24465(2)^6 + 20800(3)^6 - 11370(4)^6 + 3576(5)^6 - 491(6)^6\right) \\ \dfrac{(3)^9}{9!} - \dfrac{1}{(44800)(6!)}\begin{pmatrix} 172692(1)^6 - 72495(2)^6 + 80640(3)^6 - 44145(4)^6 + 13932(5)^6 - \\ 1917(6)^6 \end{pmatrix} \\ \dfrac{(4)^9}{9!} - \dfrac{1}{(28350)(6!)}\begin{pmatrix} 223872(1)^6 - 54240(2)^6 + 108800(3)^6 - 52080(4)^6 + 16512(5)^6 - \\ 2272(6)^6 \end{pmatrix} \\ \dfrac{(5)^9}{9!} - \dfrac{1}{(145152)(6!)}\begin{pmatrix} 1945500(1)^6 - 256875(2)^6 + 1070000(3)^6 - 358125(4)^6 + 136500(5)^6 \\ -18625(6)^6 \end{pmatrix} \\ \dfrac{(6)^9}{9!} - \dfrac{1}{(700)(6!)}\left(14256(1)^6 - 810(2)^6 + 8640(3)^6 - 1620(4)^6 + 1296(5)^6 - 126(6)^6\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
C_{10} = \begin{pmatrix}
\frac{1}{10!} - \frac{1}{(3628800)(7!)}\begin{pmatrix} 506604(1)^7 - 494715(2)^7 + 414160(3)^7 - 226605(4)^7 + 71364(5)^7 \\ -9809(6)^7 \end{pmatrix} \\
\frac{(2)^{10}}{10!} - \frac{1}{(28350)(7!)}\left(35976(1)^7 - 24465(2)^7 + 20800(3)^7 - 11370(4)^7 + 3576(5)^7 - 491(6)^7\right) \\
\frac{(3)^{10}}{10!} - \frac{1}{(44800)(7!)}\begin{pmatrix} 172692(1)^7 - 72495(2)^7 + 80640(3)^7 - 44145(4)^7 + 13932(5)^7 - \\ 1917(6)^7 \end{pmatrix} \\
\frac{(4)^{10}}{10!} - \frac{1}{(28350)(7!)}\begin{pmatrix} 223872(1)^7 - 54240(2)^7 + 108800(3)^7 - 52080(4)^7 + 16512(5)^7 - \\ 2272(6)^7 \end{pmatrix} \\
\frac{(5)^{10}}{10!} - \frac{1}{(145152)(7!)}\begin{pmatrix} 1945500(1)^7 - 256875(2)^7 + 1070000(3)^7 - 358125(4)^7 + 136500(5)^7 \\ -18625(6)^7 \end{pmatrix} \\
\frac{(6)^{10}}{10!} - \frac{1}{(700)(7!)}\left(14256(1)^7 - 810(2)^7 + 8640(3)^7 - 1620(4)^7 + 1296(5)^7 - 126(6)^7\right)
\end{pmatrix}
= \begin{pmatrix} \frac{37}{16779} \\ \frac{199}{14175} \\ \frac{278}{7953} \\ \frac{353}{5392} \\ \frac{523}{4978} \\ \frac{27}{175} \end{pmatrix}
$$

Hence, the block has order $(7,7,7,7,7,7)^T$ with error constants

$$\left(\frac{37}{16779}, \frac{199}{14175}, \frac{278}{7953}, \frac{353}{5392}, \frac{523}{4978}, \frac{27}{175}\right)^T$$

4.4.2.2 Zero Stability of Six–Step Block Method for Third Order ODEs

Equation (3.2.2.2.1) is applied to six-step block method (4.4.1.26 – 4.4.1.31) we have

$$
\det[rA^{(0)} - A^{(1)}] = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0
$$

which implies $r = 0,0,0,0,0,1$. Hence, the method is zero stable.

4.4.2.3 Consistency and Convergence of Six–Step Block Method for Third Order ODEs.

The conditions enumerated in Definition 1.4 are satisfied on the block method (4.4.1.26 –4.4.1.31) and this implies that the method is consistent. Subsequently, it is convergent because it is zero-stable and consistent.

4.4.2.4 Region of Absolute Stability of Six–Step Block Method for Third Order ODEs.

Equation (3.2.2.4.2) is applied to six-step block (4.4.1.26 – 4.4.1.31), we have

$$\bar{h}(\theta, h) = \frac{A - B}{C + D}$$

where

$$A = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{4i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{5i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{6i\theta} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{506604}{3628800}e^{i\theta} & -\frac{494715}{3628800}e^{2i\theta} & \frac{414160}{3628800}e^{3i\theta} & -\frac{226605}{3628800}e^{4i\theta} & \frac{71364}{3628800}e^{5i\theta} & -\frac{9809}{3628800}e^{6i\theta} \\ \frac{35976}{28350}e^{i\theta} & -\frac{24465}{28350}e^{2i\theta} & \frac{20800}{28350}e^{3i\theta} & -\frac{11370}{28350}e^{4i\theta} & \frac{3576}{28350}e^{5i\theta} & -\frac{491}{28350}e^{6i\theta} \\ \frac{172692}{44800}e^{i\theta} & -\frac{72495}{44800}e^{2i\theta} & \frac{80640}{44800}e^{3i\theta} & -\frac{44145}{44800}e^{4i\theta} & \frac{13932}{44800}e^{5i\theta} & -\frac{1917}{44800}e^{6i\theta} \\ \frac{223872}{28350}e^{i\theta} & -\frac{54240}{28350}e^{2i\theta} & \frac{108800}{28350}e^{3i\theta} & -\frac{52080}{28350}e^{4i\theta} & \frac{16512}{28350}e^{5i\theta} & -\frac{2272}{28350}e^{6i\theta} \\ \frac{1945500}{145152}e^{i\theta} & -\frac{256875}{145152}e^{2i\theta} & \frac{1070000}{145152}e^{3i\theta} & -\frac{358125}{145152}e^{4i\theta} & \frac{136500}{145152}e^{5i\theta} & -\frac{18625}{145152}e^{6i\theta} \\ \frac{14256}{700}e^{i\theta} & -\frac{810}{700}e^{2i\theta} & \frac{8640}{700}e^{3i\theta} & -\frac{1620}{700}e^{4i\theta} & \frac{1296}{700}e^{5i\theta} & -\frac{126}{700}e^{6i\theta} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{343801}{3628800} \\ 0 & 0 & 0 & 0 & 0 & \frac{13774}{28350} \\ 0 & 0 & 0 & 0 & 0 & \frac{52893}{44800} \\ 0 & 0 & 0 & 0 & 0 & \frac{61808}{28350} \\ 0 & 0 & 0 & 0 & 0 & \frac{505625}{145152} \\ 0 & 0 & 0 & 0 & 0 & \frac{14256}{700} \end{pmatrix}$$

Simplifying the above matrix and equating the imaginary part to zero, we have

$$\bar{h}(\theta, h) = \frac{3.1063\text{E} + 44\cos 6\theta - 3.1063\text{E} + 44}{1.8867E + 40\cos 6\theta - 2.1056\text{E} + 42}$$

The value of $\bar{h}(\theta, h)$ is evaluated at intervals of θ of 30^0 and this produced results tabulated in Table 4.3 as follows

Table 4.3

Interval of Absolute Stability of Six–Step Block Method for Third Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta, h)$	0	290.41	0	290.41	0	290.41	0

Therefore, the interval of absolute stability is (0, 290.41). This is demonstrated in the diagram below

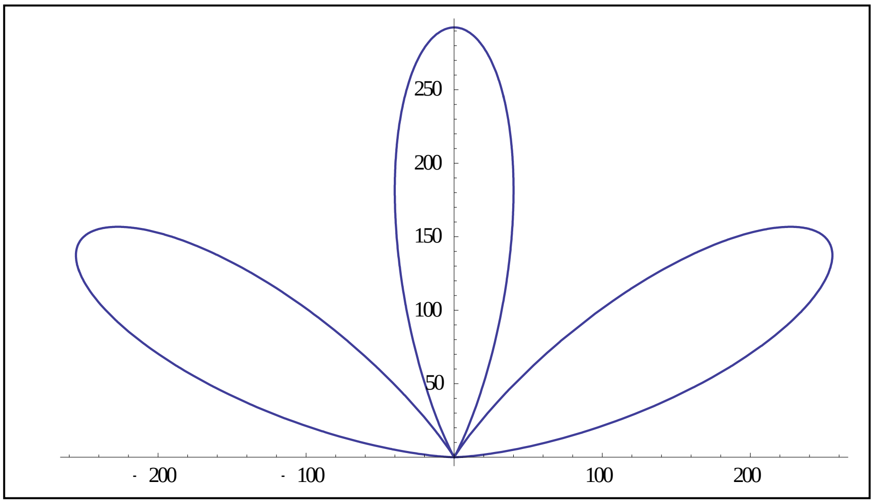

Figure 4.6. Region of absolute stability of six–step block method for third order ODEs

4.5 Seven –Step Block Method for Third Order ODEs.

This section contains the derivation of seven–step block method for solving third order ODEs. It also includes the properties of the block method.

4.5.1 Derivation of Seven–Step Block Method for Third Order ODEs.

The approximate power series of the form (4.2.1.1) is considered as an approximate solution to the general third order ODEs of the form (4.2.1.2) where the step-length k =7 is used. The first, second and third derivatives of (4.2.1.1) are shown in (4.2.1.3), (4.2.1.4) and (4.2.1.5).

Interpolating (4.2.1.1) at the points $x = x_{n+i}, i = 3(1)5$ and collocating (4.2.1.5) at $x = x_{n+i}, i = 0(1)7$ as displayed in Figure 4.7 below

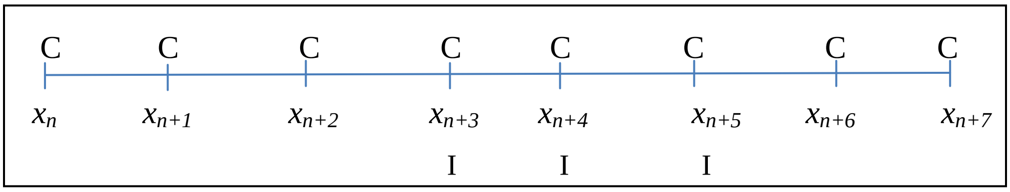

Figure 4.7. Seven–step interpolation and collocation method for third order ODEs.

This technique leads to

$$AX = B \qquad (4.5.1.1)$$

where

$$A = \begin{pmatrix}
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 & x_{n+3}^{10} \\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 & x_{n+4}^{10} \\
1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 & x_{n+5}^9 & x_{n+5}^{10} \\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 & 504x_n^6 & 720x_n^7 \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 & 504x_{n+1}^6 & 720x_{n+1}^7 \\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 & 504x_{n+2}^6 & 720x_{n+2}^7 \\
0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 & 504x_{n+3}^6 & 720x_{n+3}^7 \\
0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & 336x_{n+4}^5 & 504x_{n+4}^6 & 720x_{n+4}^7 \\
0 & 0 & 0 & 6 & 24x_{n+5} & 60x_{n+5}^2 & 120x_{n+5}^3 & 210x_{n+5}^4 & 336x_{n+5}^5 & 504x_{n+5}^6 & 720x_{n+5}^7 \\
0 & 0 & 0 & 6 & 24x_{n+6} & 60x_{n+6}^2 & 120x_{n+6}^3 & 210x_{n+6}^4 & 336x_{n+6}^5 & 504x_{n+6}^6 & 720x_{n+6}^7 \\
0 & 0 & 0 & 6 & 24x_{n+7} & 60x_{n+7}^2 & 120x_{n+7}^3 & 210x_{n+7}^4 & 336x_{n+7}^5 & 504x_{n+7}^6 & 720x_{n+7}^7
\end{pmatrix}$$

$$X = \left(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\right)^T$$

$$B = \left(y_{n+3}, y_{n+4}, y_{n+5}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}, f_{n+7}\right)^T$$

In order to find the values of *a's* in (4.5.1.1), Gaussian elimination method is applied and this gives the following

$$\begin{aligned}
a_0 = \; & 10y_{n+3} - 15y_{n+4} + 6y_{n+5} + \frac{h^3}{30240}(-194f_n - 15153f_{n+1} - 58728f_{n+2} - 138815f_{n+3} - \\
& 89850f_{n+4} + 849f_{n+5} - 572f_{n+6} + 63f_{n+7}) - \frac{x_n^3}{6}f_n + \frac{x_n^2 h}{3628800}(-534575f_n - 2772520f_{n+1} \\
& -449655f_{n+2} - 3291320f_{n+3} + 174995f_{n+4} - 503040f_{n+5} + 134995f_{n+6} - 16480f_{n+7}) + \\
& \frac{x_n h^2}{3628800}(-225656f_n - 3965022f_{n+1} - 6543372f_{n+2} - 11769410f_{n+3} - 5740800f_{n+4} - \\
& 211074f_{n+5} + 34372f_{n+6} - 4638f_{n+7}) - \frac{x_n^{10}}{3628800h^7}(f_n - 7f_{n+1} + 21f_{n+2} - 35f_{n+3} +
\end{aligned}$$

$$35f_{n+4}-21f_{n+5}+7f_{n+6}-f_{n+7})+\frac{x_n}{2h}(9y_{n+3}-16y_{n+4}+7y_{n+5})+\frac{x_n^4}{10080h}(-1089f_n+$$
$$2940f_{n+1}-4410f_{n+2}+4900f_{n+3}-3675f_{n+4}+1764f_{n+5}-490f_{n+6}+60f_{n+7})-$$
$$\frac{x_n^9}{362880h^6}(4f_n-27f_{n+1}+78f_{n+2}-125f_{n+3}+120f_{n+4}-69f_{n+5}+22f_{n+6}-3f_{n+7})$$
$$-\frac{x_n^7}{30240h^4}(56f_n-333f_{n+1}+852f_{n+2}-1219f_{n+3}+1056f_{n+4}-555f_{n+5}+164f_{n+6}$$
$$-21f_{n+7})-\frac{x_n^8}{241920h^5}(46f_n-295f_{n+1}+810f_{n+2}-1235f_{n+3}+1130f_{n+4}-621f_{n+5}$$
$$+190f_{n+6}-25f_{n+7})-\frac{x_n^6}{86400h^3}(967f_n-5104f_{n+1}+11787f_{n+2}-15560f_{n+3}+$$
$$12725f_{n+4}-6432f_{n+5}+1849f_{n+6}-232f_{n+7})+\frac{x_n^2}{2h^2}(y_{n+3}-2y_{n+4}+y_{n+5})+$$
$$\frac{x_n^5}{21600h^2}(-938f_n+4014f_{n+1}-7911f_{n+2}+9490f_{n+3}-7380f_{n+4}+3618f_{n+5}$$
$$-1019f_{n+6}+126f_{n+7}).$$

$$a_1 = -\frac{h^2}{3628800}(-225656f_n-3965022f_{n+1}-6543372f_{n+2}-11769410f_{n+3}-$$
$$5740800f_{n+4}-211074f_{n+5}+34372f_{n+6}-4638f_{n+7})+\frac{x_n^2}{2}f_n-\frac{x_n}{2h}(9y_{n+3}-16y_{n+4}$$
$$+7y_{n+5})+\frac{x_n^3}{2520h}(-1089f_n+2940f_{n+1}-4410f_{n+2}+4900f_{n+3}-3675f_{n+4}+$$
$$1764f_{n+5}-490f_{n+6}+60f_{n+7})+\frac{x_n^9}{362880h^7}(f_n-7f_{n+1}+21f_{n+2}-35f_{n+3}+35f_{n+4}$$
$$-21f_{n+5}+7f_{n+6}-f_{n+7})-\frac{x_n}{2h^2}(y_{n+3}-2y_{n+4}+y_{n+5})-\frac{x_n^4}{4320h^2}(-938f_n+4014f_{n+1}$$
$$-7911f_{n+2}+9490f_{n+3}-7380f_{n+4}+3618f_{n+5}-1019f_{n+6}+126f_{n+7})+\frac{x_n^8}{40320h^6}(4f_n$$
$$-27f_{n+1}+78f_{n+2}-125f_{n+3}+120f_{n+4}-69f_{n+5}+22f_{n+6}-3f_{n+7})+\frac{x_n^6}{30240h^4}(56f_n$$
$$-333f_{n+1}+852f_{n+2}-1219f_{n+3}+1056f_{n+4}-555f_{n+5}+164f_{n+6}-21f_{n+7})+$$
$$\frac{x_n^7}{30240h^5}(46f_n-295f_{n+1}+810f_{n+2}-1235f_{n+3}+1130f_{n+4}-621f_{n+5}+190f_{n+6}$$
$$-25f_{n+7})+\frac{x_n^5}{14400h^3}(967f_n-5104f_{n+1}+11787f_{n+2}-15560f_{n+3}+12725f_{n+4}-$$
$$6432f_{n+5}+1849f_{n+6}-232f_{n+7})-\frac{x_nh}{1814400}(-534575f_n-2772520f_{n+1}-$$
$$449655f_{n+2}-3291320f_{n+3}+174995f_{n+4}-503040f_{n+5}+134995f_{n+6}-16480f_{n+7}).$$

$$
\begin{aligned}
a_2 = \;& \frac{h}{3628800}(-534575 f_n - 2772520 f_{n+1} - 449655 f_{n+2} - 3291320 f_{n+3} + \\
& 174995 f_{n+4} - 503040 f_{n+5} + 134995 f_{n+6} - 16480 f_{n+7}) - \frac{x_n^2}{2} f_n + \frac{1}{2h^2}(y_{n+3} - 2y_{n+4} \\
& + y_{n+5}) + \frac{x_n^3}{2160h^2}(-938 f_n + 4014 f_{n+1} - 7911 f_{n+2} + 9490 f_{n+3} - 7380 f_{n+4} + 3618 f_{n+5} \\
& -1019 f_{n+6} + 126 f_{n+7}) - \frac{x_n^8}{80640h^7}(f_n - 7 f_{n+1} + 21 f_{n+2} - 35 f_{n+3} + 35 f_{n+4} - 21 f_{n+5} + \\
& 7 f_{n+6} - f_{n+7}) - \frac{x_n^7}{10080h^6}(4 f_n - 27 f_{n+1} + 78 f_{n+2} - 125 f_{n+3} + 120 f_{n+4} - 69 f_{n+5} + 22 f_{n+6} \\
& -3 f_{n+7}) - \frac{x_n^5}{1440h^4}(56 f_n - 333 f_{n+1} + 852 f_{n+2} - 1219 f_{n+3} + 1056 f_{n+4} - 555 f_{n+5} + \\
& 164 f_{n+6} - 21 f_{n+7}) - \frac{x_n^6}{8640h^5}(46 f_n - 295 f_{n+1} + 810 f_{n+2} - 1235 f_{n+3} + 1130 f_{n+4} - 621 f_{n+5} \\
& +190 f_{n+6} - 25 f_{n+7}) + \frac{x_n^2}{1680h}(-1089 f_n + 2940 f_{n+1} - 4410 f_{n+2} + 4900 f_{n+3} - 3675 f_{n+4} \\
& +1764 f_{n+5} - 490 f_{n+6} + 60 f_{n+7}) - \frac{x_n^4}{5760h^3}(967 f_n - 5104 f_{n+1} + 11787 f_{n+2} - 15560 f_{n+3} \\
& +12725 f_{n+4} - 6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7}).
\end{aligned}
$$

$$
\begin{aligned}
a_3 = \;& \frac{1}{6} f_n - \frac{x_n}{2520h}(-1089 f_n + 2940 f_{n+1} - 4410 f_{n+2} + 4900 f_{n+3} - 3675 f_{n+4} + 1764 f_{n+5} \\
& -490 f_{n+6} + 60 f_{n+7}) + \frac{x_n^7}{30240h^7}(f_n - 7 f_{n+1} + 21 f_{n+2} - 35 f_{n+3} + 35 f_{n+4} - 21 f_{n+5} + 7 f_{n+6} \\
& f_{n+7}) + \frac{x_n^6}{4320h^6}(4 f_n - 27 f_{n+1} + 78 f_{n+2} - 125 f_{n+3} + 120 f_{n+4} - 69 f_{n+5} + 22 f_{n+6} - 3 f_{n+7}) \\
& + \frac{x_n^4}{864h^4}(56 f_n - 333 f_{n+1} + 852 f_{n+2} - 1219 f_{n+3} + 1056 f_{n+4} - 555 f_{n+5} + 164 f_{n+6} - 21 f_{n+7}) \\
& + \frac{x_n^5}{4320h^5}(46 f_n - 295 f_{n+1} + 810 f_{n+2} - 1235 f_{n+3} + 1130 f_{n+4} - 621 f_{n+5} + 190 f_{n+6} - 25 f_{n+7}) \\
& - \frac{x_n^2}{2160h^2}(-938 f_n + 4014 f_{n+1} - 7911 f_{n+2} + 9490 f_{n+3} - 7380 f_{n+4} + 3618 f_{n+5} - 1019 f_{n+6} \\
& +126 f_{n+7}) + \frac{x_n^3}{4320h^3}(967 f_n - 5104 f_{n+1} + 11787 f_{n+2} - 15560 f_{n+3} + 12725 f_{n+4} - 6432 f_{n+5} \\
& +1849 f_{n+6} - 232 f_{n+7}).
\end{aligned}
$$

$$
\begin{aligned}
a_4 = \;& \frac{1}{10080h}(-1089 f_n + 2940 f_{n+1} - 4410 f_{n+2} + 4900 f_{n+3} - 3675 f_{n+4} + 1764 f_{n+5} \\
& -490 f_{n+6} + 60 f_{n+7}) + \frac{x_n}{4320h^2}(-938 f_n + 4014 f_{n+1} - 7911 f_{n+2} + 9490 f_{n+3} -
\end{aligned}
$$

$$7380f_{n+4}+3618f_{n+5}-1019f_{n+6}+126f_{n+7})-\frac{x_n^{6}}{17280h^7}(f_n-7f_{n+1}+21f_{n+2}-$$

$$35f_{n+3}+35f_{n+4}-21f_{n+5}+7f_{n+6}-f_{n+7})-\frac{x_n^5}{2880h^6}(4f_n-27f_{n+1}+78f_{n+2}-125f_{n+3}$$

$$+120f_{n+4}-69f_{n+5}+22f_{n+6}-3f_{n+7})-\frac{x_n^3}{864h^4}(56f_n-333f_{n+1}+852f_{n+2}-1219f_{n+3}$$

$$+1056f_{n+4}-555f_{n+5}+164f_{n+6}-21f_{n+7})-\frac{x_n^4}{3456h^5}(46f_n-295f_{n+1}+810f_{n+2}-$$

$$1235f_{n+3}+1130f_{n+4}-621f_{n+5}+190f_{n+6}-25f_{n+7})-\frac{x_n^2}{5760h^3}(967f_n-5104f_{n+1}+$$

$$11787f_{n+2}-15560f_{n+3}+12725f_{n+4}-6432f_{n+5}+1849f_{n+6}-232f_{n+7}).$$

$$a_5=\quad -\frac{1}{21600h^2}(-938f_n+4014f_{n+1}-7911f_{n+2}+9490f_{n+3}-7380f_{n+4}+3618f_{n+5}$$

$$-1019f_{n+6}+126f_{n+7})+\frac{x_n}{14400h^3}(967f_n-5104f_{n+1}+11787f_{n+2}-15560f_{n+3}+$$

$$12725f_{n+4}-6432f_{n+5}+1849f_{n+6}-232f_{n+7})+\frac{x_n^{5}}{14400h^7}(f_n-7f_{n+1}+21f_{n+2}-35f_{n+3}$$

$$+35f_{n+4}-21f_{n+5}+7f_{n+6}-f_{n+7})+\frac{x_n^4}{2880h^6}(4f_n-27f_{n+1}+78f_{n+2}-125f_{n+3}+120f_{n+4}$$

$$-69f_{n+5}+22f_{n+6}-3f_{n+7})+\frac{x_n^2}{1440h^4}(56f_n-333f_{n+1}+852f_{n+2}-1219f_{n+3}+1056f_{n+4}$$

$$-555f_{n+5}+164f_{n+6}-21f_{n+7})+\frac{x_n^3}{4320h^5}(46f_n-295f_{n+1}+810f_{n+2}-1235f_{n+3}+$$

$$1130f_{n+4}-621f_{n+5}+190f_{n+6}-25f_{n+7}).$$

$$a_6=\quad -\frac{1}{86400h^3}(967f_n-5104f_{n+1}+11787f_{n+2}-15560f_{n+3}+12725f_{n+4}-6432f_{n+5}$$

$$+1849f_{n+6}-232f_{n+7})-\frac{x_n}{4320h^4}(56f_n-333f_{n+1}+852f_{n+2}-1219f_{n+3}+1056f_{n+4}-$$

$$555f_{n+5}+164f_{n+6}-21f_{n+7})-\frac{x_n^{4}}{17280h^7}(f_n-7f_{n+1}+21f_{n+2}-35f_{n+3}+35f_{n+4}-21f_{n+5}$$

$$+7f_{n+6}-f_{n+7})-\frac{x_n^3}{4320h^6}(4f_n-27f_{n+1}+78f_{n+2}-125f_{n+3}+120f_{n+4}-69f_{n+5}+22f_{n+6}$$

$$-3f_{n+7})-\frac{x_n^2}{8640h^5}(46f_n-295f_{n+1}+810f_{n+2}-1235f_{n+3}+1130f_{n+4}-621f_{n+5}+190f_{n+6}$$

$$-25f_{n+7}).$$

$$a_7=\quad \frac{1}{30240h^4}(56f_n-333f_{n+1}+852f_{n+2}-1219f_{n+3}+1056f_{n+4}-555f_{n+5}+$$

$$164f_{n+6}-21f_{n+7})+\frac{x_n}{30240h^5}(46f_n-295f_{n+1}+810f_{n+2}-1235f_{n+3}+1130f_{n+4}$$

$$-621f_{n+5}+190f_{n+6}-25f_{n+7})+\frac{x_n^3}{30240h^7}(f_n-7f_{n+1}+21f_{n+2}-35f_{n+3}+35f_{n+4}$$

$$-21f_{n+5}+7f_{n+6}-f_{n+7})+\frac{x_n^2}{10080h^6}(4f_n-27f_{n+1}+78f_{n+2}-125f_{n+3}+120f_{n+4}$$
$$-69f_{n+5}+22f_{n+6}-3f_{n+7}).$$

$$a_8 = -\frac{1}{241920h^5}(46f_n-295f_{n+1}+810f_{n+2}-1235f_{n+3}+1130f_{n+4}-621f_{n+5}$$
$$+190f_{n+6}-25f_{n+7})-\frac{x_n}{40320h^6}(4f_n-27f_{n+1}+78f_{n+2}-125f_{n+3}+120f_{n+4}-$$
$$69f_{n+5}+22f_{n+6}-3f_{n+7})-\frac{x_n^2}{80640h^7}(f_n-7f_{n+1}+21f_{n+2}-35f_{n+3}+35f_{n+4}-$$
$$21f_{n+5}+7f_{n+6}-f_{n+7}).$$

$$a_9 = \frac{1}{362880h^6}(4f_n-27f_{n+1}+78f_{n+2}-125f_{n+3}+120f_{n+4}-69f_{n+5}+22f_{n+6}-3$$
$$f_{n+7})+\frac{x_n}{362880h^7}(f_n-7f_{n+1}+21f_{n+2}-35f_{n+3}+35f_{n+4}-21f_{n+5}+7f_{n+6}-f_{n+7}).$$

$$a_{10} = \frac{1}{3628800h^7}(f_n-7f_{n+1}+21f_{n+2}-35f_{n+3}+35f_{n+4}-21f_{n+5}+7f_{n+6}-f_{n+7})$$

Substituting the values of *a's* into equation (4.2.1.1) and simplifying, this gives a continuous linear multistep method of the form:

$$y(x)=\sum_{j=3}^{k-2}\alpha_j(x)y_{n+j}+h^3\sum_{j=0}^{k}\beta_j(x)f_{n+j} \tag{4.5.1.2}$$

where $x = zh + x_n + 6h$ (4.5.1.3)

Equation (4.5.1.3) is substituted into (4.5.1.2) to produce the following coefficients of y_{n+j} and f_{n+j}

$$\alpha_3(z)=1+\frac{z^2}{2}+\frac{3z}{2}$$
$$\alpha_4(z)=-3-4z-z^2$$
$$\alpha_5(z)=3+\frac{5z}{2}+\frac{z^2}{2}$$
$$\beta_0(z)=\frac{1}{7257600}(3720+904z-6430z^2+7200z^4+3696z^5$$
$$-588z^6-960z^7-300z^8-40z^9-2z^{10}).$$

$$\beta_1(z)=\frac{1}{3628800}(-16740-5250z+26840z^2-30240z^4-15120z^5$$
$$+2688z^6+3960z^7+1185z^8+150z^9+7z^{10}).$$
$$\beta_2(z)=\frac{1}{3628800}(68940+27768z-99737z^2+113400z^4+54432z^5$$
$$-11214z^6-14400z^7-4050z^8-480z^9-21z^{10}).$$
$$\beta_3(z)=\frac{1}{3628800}(-143580-43870z+233800z^2-252000z^4-$$
$$112560z^5+28560z^6+30120z^7+7725z^8+850z^9+35z^{10}).$$
$$\beta_4(z)=\frac{1}{3628800}(1969020+2708580z+524915z^2+378000z^4$$
$$+143640z^5-50610z^6-38880z^7-8850z^8-900z^9-35z^{10}).$$
$$\beta_5(z)=\frac{1}{3628800}(1734660+3692418z+2296320z^2-453600z^4$$
$$-81648z^5+58464z^6+30600z^7+6075z^8+570z^9+21z^{10}).$$
$$\beta_6(z)=\frac{1}{3628800}(12660+279200z+666355z^2+604800z^3$$
$$+219240z^4-11760z^5-37338z^6-13440z^7-2310z^8$$
$$-200z^9-7z^{10}). \tag{4.5.1.4}$$
$$\beta_7(z)=\frac{1}{3628800}(1980-6498z-16480z^2+21600z^4+21168z^5$$
$$+9744z^6+2520z^7+375z^8+30z^9+z^{10}).$$

Evaluating (4.5.1.4) at the non-interpolating points .i.e, at z=-6, -5, -4, 0 and 1 gives

$$-181440y_{n+5}+453600y_{n+4}-302400y_{n+3}+30240y_n=h^3(63f_{n+7}-572f_{n+6}$$
$$+849f_{n+5}-89850f_{n+4}-138815f_{n+3}-58728f_{n+2}-15153f_{n+1}-194f_n). \tag{4.5.1.5}$$

$$-90720y_{n+5}+241920y_{n+4}-181440y_{n+3}+30240y_{n+1}=h^3(31f_{n+7}-282f_{n+6}$$
$$+411f_{n+5}-44774f_{n+4}-62379f_{n+3}-13470f_{n+2}-527f_{n+1}+30f_n). \tag{4.5.1.6}$$

$$-60480y_{n+5}+181440y_{n+4}-181440y_{n+3}+60480y_{n+2}=h^3(31f_{n+7}-281f_{n+6}$$
$$+657f_{n+5}-30647f_{n+4}-30647f_{n+3}+657f_{n+2}-281f_{n+1}+31f_n). \tag{4.5.1.7}$$

$$60480y_{n+6}-181440y_{n+5}+181440y_{n+4}-60480y_{n+3}=h^3(33f_{n+7}+211f_{n+6}$$
$$+28911f_{n+5}+32817f_{n+4}-2393f_{n+3}+1149f_{n+2}-279f_{n+1}+31f_n). \tag{4.5.1.8}$$

$$30240y_{n+7}-181440y_{n+5}+241920y_{n+4}-90720y_{n+3}=h^3(287f_{n+7}+14310f_{n+6}$$
$$+60699f_{n+5}+46874f_{n+4}-2091f_{n+3}+1122f_{n+2}-271f_{n+1}+30f_n). \tag{4.5.1.9}$$

The first derivative of (4.5.1.4) gives

$$\alpha_3'(z) = z + \frac{3}{2}$$
$$\alpha_4'(z) = -4 - 2z$$
$$\alpha_5'(z) = z + \frac{5}{2}$$

$$\beta_3'(z) = \frac{1}{3628800}(-43870 + 467600z - 1008000z^3 - 562800z^4 \\ +171360z^5 + 210840z^6 + 61800z^7 + 7650z^8 + 350z^9).$$

$$\beta_0'(z) = \frac{1}{7257600}(904 - 12860z + 28800z^3 + 18480z^4 \\ -3528z^5 - 6720z^6 - 2400z^7 - 360z^8 - 20z^9).$$

$$\beta_1'(z) = \frac{1}{3628800}(-5250 + 53680z - 120960z^3 - 75600z^4 \\ +16128z^5 + 27720z^6 + 9480z^7 + 1350z^8 + 70z^9).$$

$$\beta_2'(z) = \frac{1}{3628800}(27768 - 199474z + 453600z^3 + 272160z^4 \\ -67284z^5 - 100800z^6 - 32400z^7 - 4320z^8 - 210z^9).$$

$$\beta_4'(z) = \frac{1}{3628800}(2708580 + 1049830z + 1512000z^3 + 718200z^4 \\ -303660z^5 - 272160z^6 - 70800z^7 - 8100z^8 - 350z^9).$$

$$\beta_5'(z) = \frac{1}{3628800}(3692418 + 4592640z - 1814400z^3 - 408240z^4 \\ +350784z^5 + 214200z^6 + 48600z^7 + 5130z^8 + 210z^9).$$

$$\beta_6'(z) = \frac{1}{3628800}(279200 + 1332710z + 1814400z^2 + 876960z^3 \\ -58800z^4 - 224028z^5 - 94080z^6 - 18480z^7 - 1800z^8 - 70z^9). \qquad (4.5.1.10)$$

$$\beta_7'(z) = \frac{1}{3628800}(-6498 - 32960z + 86400z^3 + 105840z^4 \\ +58464z^5 + 17640z^6 + 3000z^7 + 270z^8 + 10z^9).$$

Equation (4.5.1.10) is evaluated at all the grid points.i.e, at z=-6, -5, -4, -3, -2,-1, 0 and 1 produces

$$1814400hy_n' + 6350400y_{n+5} - 14515200y_{n+4} + 8164800y_{n+3} = h^3(2319f_{n+7} \\ -17186f_{n+6} + 105537f_{n+5} + 2870400f_{n+4} + 5884705f_{n+3} + 3271686f_{n+2} \\ +1982511f_{n+1} + 112828f_n). \qquad (4.5.1.11)$$

$$1814400hy'_{n+1} + 4536000y_{n+5} - 10886400y_{n+4} + 6350400y_{n+3} = h^3(-2099f_{n+7} + 18450f_{n+6} - 35391f_{n+5} + 2276590f_{n+4} + 3652815f_{n+3} + 1796934f_{n+2} + 160675f_{n+1} - 5574f_n). \quad (4.5.1.12)$$

$$1814400hy'_{n+2} + 2721600y_{n+5} - 7257600y_{n+4} + 4536000y_{n+3} = h^3(-817f_{n+7} + 7556f_{n+6} - 9279f_{n+5} + 1338470f_{n+4} + 1858865f_{n+3} + 133272f_{n+2} - 1441f_{n+1} - 226f_n). \quad (4.5.1.13)$$

$$1814400hy'_{n+3} + 907200y_{n+5} - 3628800y_{n+4} + 2721600y_{n+3} = h^3(-465f_{n+7} + 4102f_{n+6} - 8277f_{n+5} + 439290f_{n+4} + 195205f_{n+3} - 30270f_{n+2} + 5793f_{n+1} - 578f_n). \quad (4.5.1.14)$$

$$1814400hy'_{n+4} - 907200y_{n+5} + 907200y_{n+3} = h^3(-113f_{n+7} + 1578f_{n+6} - 20415f_{n+5} - 264500f_{n+4} - 20415f_{n+3} + 1578f_{n+2} - 113f_{n+1}). \quad (4.5.1.15)$$

$$1814400hy'_{n+5} - 2721600y_{n+5} + 3628800y_{n+4} - 907200y_{n+3} = h^3(1169f_{n+7} - 14086f_{n+6} + 162837f_{n+5} + 479750f_{n+4} - 40645f_{n+3} + 20286f_{n+2} - 5089f_{n+1} + 578f_n). \quad (4.5.1.16)$$

$$1814400hy'_{n+6} - 4536000y_{n+5} + 7257600y_{n+4} - 2721600y_{n+3} = h^3(-3249f_{n+7} + 139600f_{n+6}1846209f_{n+5} + 1354290f_{n+4} - 21935f_{n+3} + 13884f_{n+2} - 2625f_{n+1} + 226f_n). \quad (4.5.1.17)$$

$$1814400hy'_{n+7} - 6350400y_{n+5} + 10886400y_{n+4} - 4536000y_{n+3} = h^3(116083f_{n+7} + 1953006f_{n+6} + 3340671f_{n+5} + 2666770f_{n+4} - 347535f_{n+3} + 174522f_{n+2} - 46691f_{n+1} + 5574f_n). \quad (4.5.1.18)$$

The second derivative of (4.1.5.4) gives

$$\alpha''_3(z) = 1$$
$$\alpha''_4(z) = -2$$
$$\alpha''_5(z) = 1$$

$$\beta''_0(z) = \frac{1}{362880}(-643 + 4320z^2 + 3696z^3 - 882z^4 - 2016z^5 - 840z^6 - 144z^7 - 9z^8).$$

$$\beta''_1(z) = \frac{1}{362880}(5368 - 36288z^2 - 30240z^3 + 8064z^4 + 16632z^5 + 6636z^6 + 1080z^7 + 63z^8).$$

$$\beta''_2(z) = \frac{1}{362880}(-19947 + 136080z^2 + 108864z^3 - 33642z^4 - 60480z^5 - 22680z^6 - 3456z^7 - 189z^8).$$

$$\beta''_3(z) = \frac{1}{362880}(46760 - 302400z^2 - 225120z^3 + 85680z^4 + 126504z^5 + 43260z^6 + 6120z^7 + 315z^8).$$

$$\beta_4''(z) = \frac{1}{362880}(104983 + 453600z^2 + 287280z^3 - 151830z^4$$
$$-163296z^5 - 49560z^6 - 6480z^7 - 315z^8).$$
$$\beta_5''(z) = \frac{1}{362880}(459264 - 544320z^2 - 163296z^3 + 175392z^4$$
$$+128520z^5 + 34020z^6 + 4104z^7 + 189z^8). \qquad (4.5.1.19)$$
$$\beta_6''(z) = \frac{1}{362880}(133271 + 362880z + 263088z^2 - 23520z^3$$
$$-112014z^4 - 56448z^5 - 12936z^6 - 1440z^7 - 63z^8).$$
$$\beta_7''(z) = \frac{1}{362880}(-3296 + 25920z^2 + 42336z^3 + 29232z^4$$
$$+10584z^5 + 2100z^6 + 216z^7 + 9z^8).$$

Evaluating (4.5.1.19) at all the grid points. i.e, at z=-6, -5, -4, -3, -2, -1, 0 and 1 gives

$$362880h^2 y''_n - 362880y_{n+5} + 725760y_{n+4} - 362880y_{n+3} = h^3(-3296 f_{n+7}$$
$$+ 26999 f_{n+6} - 100608 f_{n+5} + 34999 f_{n+4} - 658264 f_{n+3} - 89931 f_{n+2} \qquad (4.5.1.20)$$
$$- 554504 f_{n+1} - 106915 f_n).$$

$$362880h^2 y''_{n+1} - 362880y_{n+5} + 725760y_{n+4} - 362880y_{n+3} = h^3(829 f_{n+7}$$
$$- 7054 f_{n+6} + 23889 f_{n+5} - 230642 f_{n+4} - 288865 f_{n+3} - 455322 f_{n+2} \qquad (4.5.1.21)$$
$$-134957 f_{n+1} + 3482 f_n).$$

$$362880h^2 y''_{n+2} - 362880y_{n+5} + 725760y_{n+4} - 362880y_{n+3} = h^3(-224 f_{n+7}$$
$$+1943 f_{n+6} - 10752 f_{n+5} - 149993 f_{n+4} - 423256 f_{n+3} - 151275 f_{n+2} \qquad (4.5.1.22)$$
$$+ 8440 f_{n+1} - 643 f_n).$$

$$362880h^2 y''_{n+3} - 362880y_{n+5} + 725760y_{n+4} - 362880y_{n+3} = h^3(349 f_{n+7}$$
$$- 3214 f_{n+6} + 10929 f_{n+5} - 210674 f_{n+4} - 178177 f_{n+3} + 21606 f_{n+2} \qquad (4.5.1.23)$$
$$- 4109 f_{n+1} + 410 f_n).$$

$$362880h^2 y''_{n+4} - 362880y_{n+5} + 725760y_{n+4} - 362880y_{n+3} = h^3(-224 f_{n+7}$$
$$+ 2423 f_{n+6} - 17664 f_{n+5} - 5705 f_{n+4} + 26792 f_{n+3} - 6987 f_{n+2} \qquad (4.5.1.24)$$
$$+1528 f_{n+1} - 163 f_n).$$

$$362880h^2 y''_{n+5} - 362880y_{n+5} + 725760y_{n+4} - 362880y_{n+3} = h^3(829 f_{n+7}$$
$$-10126 f_{n+6} + 155217 f_{n+5} + 239374 f_{n+4} - 33889 f_{n+3} + 14694 f_{n+2} \qquad (4.5.1.25)$$
$$-3629 f_{n+1} + 410 f_n).$$

$$362880h^2 y''_{n+6} - 362880y_{n+5} + 725760y_{n+4} - 362880y_{n+3} = h^3(-3296 f_{n+7}$$
$$+133271 f_{n+6} + 459264 f_{n+5} + 104983 f_{n+4} + 46760 f_{n+3} - 19947 f_{n+2} \qquad (4.5.1.26)$$
$$+ 5368 f_{n+1} - 643 f_n).$$

$$362880h^2 y''_{n+7} - 362880y_{n+5} + 725760y_{n+4} - 362880y_{n+3} = h^3(107101f_{n+7} + 552818f_{n+6} + 93873f_{n+5} + 474382f_{n+4} - 218881f_{n+3} + 104550f_{n+2} - 28685f_{n+1} + 3482f_n). \tag{4.5.1.27}$$

Combining equations (4.5.1.5) - (4.5.1.9), (4.5.1.11) and (4.5.1.20) to produce a block of the form (1.10) as shown below

$$\begin{pmatrix} 0 & 0 & -302400 & 453600 & -181440 & 0 & 0 \\ 30240 & 0 & -181440 & 241920 & -90720 & 0 & 0 \\ 0 & 60480 & -181440 & 181440 & -60480 & 0 & 0 \\ 0 & 0 & -60480 & 181440 & -181440 & 60480 & 0 \\ 0 & 0 & -90720 & 241920 & -181440 & 0 & 30240 \\ 0 & 0 & 8164800 & -14515200 & 6350400 & 0 & 0 \\ 0 & 0 & -362880 & 725760 & -362880 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -30240 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1814400 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}$$

$$+ h^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -362880 \end{pmatrix} \begin{pmatrix} y''_{n-6} \\ y''_{n-5} \\ y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix} +$$

$$h^3 \begin{pmatrix} -15153 & -58728 & -138815 & -89850 & 849 & -572 & 63 \\ -527 & -13470 & -62379 & -44774 & 411 & -282 & 31 \\ -281 & 657 & -30647 & -30647 & 657 & -281 & 31 \\ -279 & 1149 & -2393 & 32817 & 28911 & 211 & 33 \\ -271 & 1122 & -2091 & 46874 & 60699 & 14310 & 287 \\ 1982511 & 3271686 & 5884705 & 2870400 & 105537 & -17186 & 2319 \\ 554504 & 89931 & 658264 & -34999 & 100608 & -26999 & 3296 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \end{pmatrix} +$$

$$
h^3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -194 \\ 0 & 0 & 0 & 0 & 0 & 0 & 30 \\ 0 & 0 & 0 & 0 & 0 & 0 & 31 \\ 0 & 0 & 0 & 0 & 0 & 0 & 31 \\ 0 & 0 & 0 & 0 & 0 & 0 & 30 \\ 0 & 0 & 0 & 0 & 0 & 0 & 112828 \\ 0 & 0 & 0 & 0 & 0 & 0 & -106915 \end{pmatrix} \begin{pmatrix} f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}
$$

The equation above is multiplied by the inverse of A^0 and this gives

$$
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} +
$$

$$
h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix} + h^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{49}{2} \end{pmatrix} \begin{pmatrix} y''_{n-6} \\ y''_{n-5} \\ y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix} +
$$

$$
+h^3 \begin{pmatrix} \frac{329}{2122} & \frac{-162}{887} & \frac{731}{3821} & \frac{-463}{3316} & \frac{419}{6351} & \frac{-54}{2977} & \frac{37}{16779} \\ \frac{685}{501} & \frac{-521}{450} & \frac{2074}{1693} & \frac{-506}{567} & \frac{221}{525} & \frac{-539}{4663} & \frac{199}{14175} \\ \frac{5649}{1378} & \frac{-1409}{599} & \frac{387}{128} & \frac{-952}{431} & \frac{116}{111} & \frac{-163}{567} & \frac{278}{7953} \\ \frac{5272}{631} & \frac{-605}{184} & \frac{2801}{457} & \frac{-1672}{405} & \frac{1053}{538} & \frac{-848}{1575} & \frac{353}{5392} \\ \frac{1428}{101} & \frac{-7125}{1792} & \frac{15181}{1374} & \frac{-2808}{457} & \frac{1287}{409} & \frac{-691}{800} & \frac{523}{4978} \\ \frac{3753}{175} & \frac{-1539}{350} & \frac{621}{35} & \frac{-54}{7} & \frac{891}{175} & \frac{-63}{50} & \frac{27}{175} \\ \frac{7540}{249} & \frac{-8137}{1780} & \frac{16807}{640} & \frac{-2434}{273} & \frac{7135}{853} & \frac{-899}{711} & \frac{419}{1896} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \end{pmatrix}
$$

$$
+h^3\begin{pmatrix}
0 & 0 & 0 & 0 & & 0 & \frac{2475}{26746}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{226}{479}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2194}{1915}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2286}{1081}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{3902}{1155}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{864}{175}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{3728}{549}
\end{pmatrix}
\begin{pmatrix}
f_{n-6}\\ f_{n-5}\\ f_{n-4}\\ f_{n-3}\\ f_{n-2}\\ f_{n-1}\\ f_n
\end{pmatrix} \tag{4.5.1.28}
$$

which leads to

$$
\begin{aligned}
y_{n+1} = y_n + hy_n' + \frac{1}{2}h^2 y_n'' + \frac{h^3}{3628800}(8002 f_{n+7} - 65823 f_{n+6} + 239406 f_{n+5}\\
- 506675 f_{n+4} + 694230 f_{n+3} - 662757 f_{n+2} + 562618 f_{n+1} + 335799 f_n).
\end{aligned} \tag{4.5.1.29}
$$

$$
\begin{aligned}
y_{n+2} = y_n + 2hy_n' + 2h^2 y_n'' + \frac{h^3}{28350}(398 f_{n+7} - 3277 f_{n+6} + 11934 f_{n+5}\\
- 25300 f_{n+4} + 34730 f_{n+3} - 32823 f_{n+2} + 38762 f_{n+1} + 13376 f_n).
\end{aligned} \tag{4.5.1.30}
$$

$$
\begin{aligned}
y_{n+3} = y_n + 3hy_n' + \frac{9}{2}h^2 y_n'' + \frac{h^3}{44800}(1566 f_{n+7} - 12879 f_{n+6} + 46818 f_{n+5}\\
- 98955 f_{n+4} + 135450 f_{n+3} - 105381 f_{n+2} + 183654 f_{n+1} + 51327 f_n).
\end{aligned} \tag{4.5.1.31}
$$

$$
\begin{aligned}
y_{n+4} = y_n + 4hy_n' + 8h^2 y_n'' + \frac{h^3}{14175}(928 f_{n+7} - 7632 f_{n+6} + 27744 f_{n+5}\\
- 58520 f_{n+4} + 86880 f_{n+3} - 46608 f_{n+2} + 118432 f_{n+1} + 29976 f_n).
\end{aligned} \tag{4.5.1.32}
$$

$$
\begin{aligned}
y_{n+5} = y_n + 5hy_n' + \frac{25}{2}h^2 y_n'' + \frac{5h^3}{145152}(3050 f_{n+7} - 25075 f_{n+6} + 91350 f_{n+5}\\
- 178375 f_{n+4} + 320750 f_{n+3} - 115425 f_{n+2} + 410450 f_{n+1} + 98075 f_n).
\end{aligned} \tag{4.5.1.33}
$$

$$
\begin{aligned}
y_{n+6} = y_n + 6hy_n' + 18h^2 y_n'' + \frac{h^3}{1400}(216 f_{n+7} - 1764 f_{n+6} + 7125 f_{n+5}\\
- 10800 f_{n+4} + 24840 f_{n+3} - 6156 f_{n+2} + 30024 f_{n+1} + 6912 f_n).
\end{aligned} \tag{4.5.1.34}
$$

$$
\begin{aligned}
y_{n+7} = y_n + 7hy_n' + \frac{49}{2}h^2 y_n'' + \frac{7h^3}{518400}(16366 f_{n+7} - 93639 f_{n+6} + 619458 f_{n+5}\\
- 660275 f_{n+4} + 1944810 f_{n+3} - 338541 f_{n+2} + 2242534 f_{n+1} + 502887 f_n).
\end{aligned} \tag{4.5.1.35}
$$

Substituting (4.5.1.31) - (4.5.1.33) into (4.5.1.12) – (4.5.1.18) to give the first derivative of the block

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{1814400}(12062 f_{n+7} - 99359 f_{n+6} + 36113 f_{n+5} - 768805 f_{n+4} + 1059430 f_{n+3} - 1025096 f_{n+2} + 950684 f_{n+1} + 416173 f_n). \tag{4.5.1.36}$$

$$y'_{n+2} = y'_n + 2hy''_n + \frac{h^2}{28350}(466 f_{n+7} - 3832 f_{n+6} + 13926 f_{n+5} - 29405 f_{n+4} + 39950 f_{n+3} - 34986 f_{n+2} + 55642 f_{n+1} + 14939 f_n). \tag{4.5.1.37}$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{h^2}{604800}(15552 f_{n+7} - 127899 f_{n+6} + 465102 f_{n+5} - 985365 f_{n+4} + 1394820 f_{n+3} - 650997 f_{n+2} + 2113614 f_{n+1} + 496773 f_n). \tag{4.5.1.38}$$

$$y'_{n+4} = y'_n + 4hy''_n + \frac{h^2}{14175}(496 f_{n+7} - 4072 f_{n+6} + 14736 f_{n+5} - 29960 f_{n+4} + 56720 f_{n+3} - 11496 f_{n+2} + 71152 f_{n+1} + 15824 f_n). \tag{4.5.1.39}$$

$$y'_{n+5} = y'_n + 5hy''_n + \frac{h^2}{72576}(3250 f_{n+7} - 26875 f_{n+6} + 102900 f_{n+5} - 130625 f_{n+4} + 421250 f_{n+3} - 40125 f_{n+2} + 475000 f_{n+1} + 102425 f_n). \tag{4.5.1.40}$$

$$y'_{n+6} = y'_n + 6hy''_n + \frac{h^2}{350}(18 f_{n+7} - 126 f_{n+6} + 918 f_{n+5} - 495 f_{n+4} + 2670 f_{n+3} - 108 f_{n+2} + 2826 f_{n+1} + 597 f_n). \tag{4.5.1.41}$$

$$y'_{n+7} = y'_n + 7hy''_n + \frac{h^2}{259200}(32732 f_{n+7} - 146461 f_{n+6} + 965202 f_{n+5} - 204085 f_{n+4} + 2401000 f_{n+3} + 7203 f_{n+2} + 2482634 f_{n+1} + 519253 f_n). \tag{4.5.1.42}$$

Substituting (4.5.1.31) - (4.5.1.33) into (4.5.1.21) – (4.5.1.27) to give the second derivative of the block

$$y''_{n+1} = y''_n + \frac{h}{120960}(1375 f_{n+7} - 11351 f_{n+6} + 41499 f_{n+5} - 88547 f_{n+4} + 123133 f_{n+3} - 121797 f_{n+2} + 139849 f_{n+1} + 36799 f_n). \tag{4.5.1.43}$$

$$y''_{n+2} = y''_n + \frac{h}{18900}(160 f_{n+7} - 1305 f_{n+6} + 4680 f_{n+5} - 9635 f_{n+4} + 12240 f_{n+3} - 3195 f_{n+2} + 29320 f_{n+1} + 5535 f_n). \tag{4.5.1.44}$$

$$y''_{n+3} = y''_n + \frac{h}{22400}(225 f_{n+7} - 1865 f_{n+6} + 6885 f_{n+5} - 15165 f_{n+4} + 29635 f_{n+3} + 6885 f_{n+2} + 33975 f_{n+1} + 6625 f_n). \quad (4.5.1.45)$$

$$y''_{n+4} = y''_n + \frac{h}{945}(8 f_{n+7} - 64 f_{n+6} + 216 f_{n+5} - 106 f_{n+4} + 1784 f_{n+3} + 216 f_{n+2} + 1448 f_{n+1} + 278 f_n). \quad (4.5.1.46)$$

$$y''_{n+5} = y''_n + \frac{h}{24192}(275 f_{n+7} - 2475 f_{n+6} + 17055 f_{n+5} + 13625 f_{n+4} + 41625 f_{n+3} + 6975 f_{n+2} + 36725 f_{n+1} + 7155 f_n). \quad (4.5.1.47)$$

$$y''_{n+6} = y'_n + \frac{h}{140}(41 f_{n+6} + 216 f_{n+5} + 27 f_{n+4} + 272 f_{n+3} + 27 f_{n+2} + 216 f_{n+1} + 41 f_n). \quad (4.5.1.48)$$

$$y''_{n+7} = y''_n + \frac{h}{17280}(5257 f_{n+7} + 25039 f_{n+6} + 9261 f_{n+5} + 20923 f_{n+4} + 20923 f_{n+3} + 9261 f_{n+2} + 25039 f_{n+1} + 5257 f_n). \quad (4.5.1.49)$$

4.5.2 Properties of Seven–Step Block Method for Third Order ODEs.

This segment deals with the establishment of the order, zero-stability and region of absolute stability of seven–step block for third order ODEs

4.5.2.1 Order of Seven–Step Block Method for Third Order ODEs

The method used in section 3.2.2.1 is applied in finding the order of the block method (4.5.1.29 – 4.5.1.35) as shown below

$$
\begin{pmatrix}
\sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{h^m}{m!}y_n^{(m)}-\frac{335799}{3628800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(3628800)(m!)}y_n^{(3+m)}\begin{pmatrix}562618(1)^m-662757(2)^m+694230(3)^m\\-506675(4)^m+239406(5)^m-65823(6)^m\\+8002(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(2h)^m}{m!}y_n^{(m)}-\frac{13376}{28350}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(28350)(m!)}y_n^{(3+m)}\begin{pmatrix}38762(1)^m-32823(2)^m+34730(3)^m\\-25300(4)^m+11934(5)^m-3277(6)^m\\+398(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(3h)^m}{m!}y_n^{(m)}-\frac{51327}{44800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(44800)(m!)}y_n^{(3+m)}\begin{pmatrix}183654(1)^m-105381(2)^m+135450(3)^m\\-98955(4)^m+46818(5)^m-12879(6)^m\\+1566(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(4h)^m}{m!}y_n^{(m)}-\frac{29976}{14175}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(14175)(m!)}y_n^{(3+m)}\begin{pmatrix}118432(1)^m-46608(2)^m+86880(3)^m\\-58520(4)^m+27744(5)^m-7632(6)^m\\+928(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(5h)^m}{m!}y_n^{(m)}-\frac{98075}{145152}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(145152)(m!)}y_n^{(3+m)}\begin{pmatrix}410450(1)^m-115425(2)^m+320750(3)^m\\-178375(4)^m+91350(5)^m-25075(6)^m\\+3050(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(6h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(6h)^m}{m!}y_n^{(m)}-\frac{6912}{1400}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(1400)(m!)}y_n^{(3+m)}\begin{pmatrix}30024(1)^m-6156(2)^m+24840(3)^m\\-10800(4)^m+7128(5)^m-1764(6)^m\\+216(7)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(7h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(7h)^m}{m!}y_n^{(m)}-\frac{7(502887)}{518400}h^2y_n''-\sum_{m=0}^{\infty}\frac{7h^{3+m}}{(518400)(m!)}y_n^{(3+m)}\begin{pmatrix}2242534(1)^m-338541(2)^m+1944810(3)^m\\-660275(4)^m+619458(5)^m-93639(6)^m\\+16366(7)^m\end{pmatrix}
\end{pmatrix}
=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\end{pmatrix}
$$

Comparing the coefficients of h^m and y_n^m. This gives

$$C_0 = \begin{pmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \\ 5-5 \\ 6-6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{1}{2!} \\ \frac{(2)^2}{2!} - \frac{(2)^2}{2!} \\ \frac{(3)^2}{2!} - \frac{(3)^2}{2!} \\ \frac{(4)^2}{2!} - \frac{(4)^2}{2!} \\ \frac{(5)^2}{2!} - \frac{(5)^2}{2!} \\ \frac{(6)^2}{2!} - \frac{(6)^2}{2!} \\ \frac{(7)^2}{2!} - \frac{(7)^2}{2!} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
C_3 = \begin{pmatrix}
\frac{1}{3!} - \frac{335799}{3628800} - \frac{1}{(3628800)(0!)}\begin{pmatrix} 562618(1)^0 - 662757(2)^0 + 694230(3)^0 - 506675(4)^0 + \\ 239406(5)^0 - 65823(6)^0 + 8002(7)^0 \end{pmatrix} \\
\frac{(2)^3}{3!} - \frac{13376}{28350} - \frac{1}{(28350)(0!)}\begin{pmatrix} 38762(1)^0 - 32823(2)^0 + 34730(3)^0 - 25300(4)^0 + 11934(5)^0 \\ -3277(6)^0 + 398(7)^0 \end{pmatrix} \\
\frac{(3)^3}{3!} - \frac{51327}{44800} - \frac{1}{(44800)(0!)}\begin{pmatrix} 183654(1)^0 - 105381(2)^0 + 135450(3)^0 - 98955(4)^0 + 46818(5)^0 \\ -12879(6)^0 + 1566(7)^0 \end{pmatrix} \\
\frac{(4)^3}{3!} - \frac{29976}{14175} - \frac{1}{(14175)(0!)}\begin{pmatrix} 118432(1)^0 - 46608(2)^0 + 86880(3)^0 - 58520(4)^0 + 27744(5)^0 - \\ 7632(6)^0 + 928(7)^0 \end{pmatrix} \\
\frac{(5)^3}{3!} - \frac{98075}{145152} - \frac{1}{(145152)(0!)}\begin{pmatrix} 410450(1)^0 - 115425(2)^0 + 320750(3)^0 - 178375(4)^0 + 91350(5)^0 \\ -25075(6)^0 + 3050(7)^0 \end{pmatrix} \\
\frac{(6)^3}{3!} - \frac{6912}{1400} - \frac{1}{(1400)(0!)}\begin{pmatrix} 30024(1)^0 - 6156(2)^0 + 24840(3)^0 - 10800(4)^0 + 7128(5)^0 - 1764(6)^0 \\ +216(7)^0 \end{pmatrix} \\
\frac{(7)^3}{3!} - \frac{7(502887)}{518400} - \frac{7}{(518400)(0!)}\begin{pmatrix} 2242534(1)^0 - 338541(2)^0 + 1944810(3)^0 - 660275(4)^0 + \\ 619458(5)^0 - 93639(6)^0 + 16366(7)^0 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_4 = \begin{pmatrix}
\frac{1}{4!} - \frac{1}{(3628800)(1!)}\begin{pmatrix} 562618(1)^1 - 662757(2)^1 + 694230(3)^1 - 506675(4)^1 + \\ 239406(5)^1 - 65823(6)^1 + 8002(7)^1 \end{pmatrix} \\
\frac{(2)^4}{4!} - \frac{1}{(28350)(1!)}\begin{pmatrix} 38762(1)^1 - 32823(2)^1 + 34730(3)^1 - 25300(4)^1 + 11934(5)^1 \\ -3277(6)^1 + 398(7)^1 \end{pmatrix} \\
\frac{(3)^4}{4!} - \frac{1}{(44800)(1!)}\begin{pmatrix} 183654(1)^1 - 105381(2)^1 + 135450(3)^1 - 98955(4)^1 + 46818(5)^1 \\ -12879(6)^1 + 1566(7)^1 \end{pmatrix} \\
\frac{(4)^4}{4!} - \frac{1}{(14175)(1!)}\begin{pmatrix} 118432(1)^1 - 46608(2)^1 + 86880(3)^1 - 58520(4)^1 + 27744(5)^1 - \\ 7632(6)^1 + 928(7)^1 \end{pmatrix} \\
\frac{(5)^4}{4!} - \frac{1}{(145152)(1!)}\begin{pmatrix} 410450(1)^1 - 115425(2)^1 + 320750(3)^1 - 178375(4)^1 + 91350(5)^1 \\ -25075(6)^1 + 3050(7)^1 \end{pmatrix} \\
\frac{(6)^4}{4!} - \frac{1}{(1400)(1!)}\begin{pmatrix} 30024(1)^1 - 6156(2)^1 + 24840(3)^1 - 10800(4)^1 + 7128(5)^1 - 1764(6)^1 \\ +216(7)^1 \end{pmatrix} \\
\frac{(7)^4}{4!} - \frac{7}{(518400)(1!)}\begin{pmatrix} 2242534(1)^1 - 338541(2)^1 + 1944810(3)^1 - 660275(4)^1 + \\ 619458(5)^1 - 93639(6)^1 + 16366(7)^1 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$C_5 = \begin{pmatrix} \frac{1}{5!} - \frac{1}{(3628800)(2!)}\begin{pmatrix} 562618(1)^2 - 662757(2)^2 + 694230(3)^2 - 506675(4)^2 + \\ 239406(5)^2 - 65823(6)^2 + 8002(7)^2 \end{pmatrix} \\ \frac{(2)^5}{5!} - \frac{1}{(28350)(2!)}\begin{pmatrix} 38762(1)^2 - 32823(2)^2 + 34730(3)^2 - 25300(4)^2 + 11934(5)^2 \\ -3277(6)^2 + 398(7)^2 \end{pmatrix} \\ \frac{(3)^5}{5!} - \frac{1}{(44800)(2!)}\begin{pmatrix} 183654(1)^2 - 105381(2)^2 + 135450(3)^2 - 98955(4)^2 + 46818(5)^2 \\ -12879(6)^2 + 1566(7)^2 \end{pmatrix} \\ \frac{(4)^5}{5!} - \frac{1}{(14175)(2!)}\begin{pmatrix} 118432(1)^2 - 46608(2)^2 + 86880(3)^2 - 58520(4)^2 + 27744(5)^2 - \\ 7632(6)^2 + 928(7)^2 \end{pmatrix} \\ \frac{(5)^5}{5!} - \frac{1}{(145152)(2!)}\begin{pmatrix} 410450(1)^2 - 115425(2)^2 + 320750(3)^2 - 178375(4)^2 + 91350(5)^2 \\ -25075(6)^2 + 3050(7)^2 \end{pmatrix} \\ \frac{(6)^5}{5!} - \frac{1}{(1400)(2!)}\begin{pmatrix} 30024(1)^2 - 6156(2)^2 + 24840(3)^2 - 10800(4)^2 + 7128(5)^2 - 1764(6)^2 \\ +216(7)^2 \end{pmatrix} \\ \frac{(7)^5}{5!} - \frac{7}{(518400)(2!)}\begin{pmatrix} 2242534(1)^2 - 338541(2)^2 + 1944810(3)^2 - 660275(4)^2 + \\ 619458(5)^2 - 93639(6)^2 + 16366(7)^2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6 = \begin{pmatrix} \frac{1}{6!} - \frac{1}{(3628800)(3!)}\begin{pmatrix} 562618(1)^3 - 662757(2)^3 + 694230(3)^3 - 506675(4)^3 + \\ 239406(5)^3 - 65823(6)^3 + 8002(7)^3 \end{pmatrix} \\ \frac{(2)^6}{6!} - \frac{1}{(28350)(3!)}\begin{pmatrix} 38762(1)^3 - 32823(2)^3 + 34730(3)^3 - 25300(4)^3 + 11934(5)^3 \\ -3277(6)^3 + 398(7)^3 \end{pmatrix} \\ \frac{(3)^6}{6!} - \frac{1}{(44800)(3!)}\begin{pmatrix} 183654(1)^3 - 105381(2)^3 + 135450(3)^3 - 98955(4)^3 + 46818(5)^3 \\ -12879(6)^3 + 1566(7)^3 \end{pmatrix} \\ \frac{(4)^6}{6!} - \frac{1}{(14175)(3!)}\begin{pmatrix} 118432(1)^3 - 46608(2)^3 + 86880(3)^3 - 58520(4)^3 + 27744(5)^3 - \\ 7632(6)^3 + 928(7)^3 \end{pmatrix} \\ \frac{(5)^6}{6!} - \frac{1}{(145152)(3!)}\begin{pmatrix} 410450(1)^3 - 115425(2)^3 + 320750(3)^3 - 178375(4)^3 + 91350(5)^3 \\ -25075(6)^3 + 3050(7)^3 \end{pmatrix} \\ \frac{(6)^6}{6!} - \frac{1}{(1400)(3!)}\begin{pmatrix} 30024(1)^3 - 6156(2)^3 + 24840(3)^3 - 10800(4)^3 + 7128(5)^3 - 1764(6)^3 \\ +216(7)^3 \end{pmatrix} \\ \frac{(7)^6}{6!} - \frac{7}{(518400)(3!)}\begin{pmatrix} 2242534(1)^3 - 338541(2)^3 + 1944810(3)^3 - 660275(4)^3 + \\ 619458(5)^3 - 93639(6)^3 + 16366(7)^3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_7 = \begin{pmatrix} \frac{1}{7!} - \frac{1}{(3628800)(4!)}\begin{pmatrix} 562618(1)^4 - 662757(2)^4 + 694230(3)^4 - 506675(4)^4 + \\ 239406(5)^4 - 65823(6)^4 + 8002(7)^4 \end{pmatrix} \\ \frac{(2)^7}{7!} - \frac{1}{(28350)(4!)}\begin{pmatrix} 38762(1)^4 - 32823(2)^4 + 34730(3)^4 - 25300(4)^4 + 11934(5)^4 \\ -3277(6)^4 + 398(7)^4 \end{pmatrix} \\ \frac{(3)^7}{7!} - \frac{1}{(44800)(4!)}\begin{pmatrix} 183654(1)^4 - 105381(2)^4 + 135450(3)^4 - 98955(4)^4 + 46818(5)^4 \\ -12879(6)^4 + 1566(7)^4 \end{pmatrix} \\ \frac{(4)^7}{7!} - \frac{1}{(14175)(4!)}\begin{pmatrix} 118432(1)^4 - 46608(2)^4 + 86880(3)^4 - 58520(4)^4 + 27744(5)^4 - \\ 7632(6)^4 + 928(7)^4 \end{pmatrix} \\ \frac{(5)^7}{7!} - \frac{1}{(145152)(4!)}\begin{pmatrix} 410450(1)^4 - 115425(2)^4 + 320750(3)^4 - 178375(4)^4 + 91350(5)^4 \\ -25075(6)^4 + 3050(7)^4 \end{pmatrix} \\ \frac{(6)^7}{7!} - \frac{1}{(1400)(4!)}\begin{pmatrix} 30024(1)^4 - 6156(2)^4 + 24840(3)^4 - 10800(4)^4 + 7128(5)^4 - 1764(6)^4 \\ + 216(7)^4 \end{pmatrix} \\ \frac{(7)^7}{7!} - \frac{7}{(518400)(4!)}\begin{pmatrix} 2242534(1)^4 - 338541(2)^4 + 1944810(3)^4 - 660275(4)^4 + \\ 619458(5)^4 - 93639(6)^4 + 16366(7)^4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_8 = \begin{pmatrix} \frac{1}{8!} - \frac{1}{(3628800)(5!)}\begin{pmatrix} 562618(1)^5 - 662757(2)^5 + 694230(3)^5 - 506675(4)^5 + \\ 239406(5)^5 - 65823(6)^5 + 8002(7)^5 \end{pmatrix} \\ \frac{(2)^8}{8!} - \frac{1}{(28350)(5!)}\begin{pmatrix} 38762(1)^5 - 32823(2)^5 + 34730(3)^5 - 25300(4)^5 + 11934(5)^5 \\ -3277(6)^5 + 398(7)^5 \end{pmatrix} \\ \frac{(3)^8}{8!} - \frac{1}{(44800)(5!)}\begin{pmatrix} 183654(1)^5 - 105381(2)^5 + 135450(3)^5 - 98955(4)^5 + 46818(5)^5 \\ -12879(6)^5 + 1566(7)^5 \end{pmatrix} \\ \frac{(4)^8}{8!} - \frac{1}{(14175)(5!)}\begin{pmatrix} 118432(1)^5 - 46608(2)^5 + 86880(3)^5 - 58520(4)^5 + 27744(5)^5 - \\ 7632(6)^5 + 928(7)^5 \end{pmatrix} \\ \frac{(5)^8}{8!} - \frac{1}{(145152)(5!)}\begin{pmatrix} 410450(1)^5 - 115425(2)^5 + 320750(3)^5 - 178375(4)^5 + 91350(5)^5 \\ -25075(6)^5 + 3050(7)^5 \end{pmatrix} \\ \frac{(6)^8}{8!} - \frac{1}{(1400)(5!)}\begin{pmatrix} 30024(1)^5 - 6156(2)^5 + 24840(3)^5 - 10800(4)^5 + 7128(5)^5 - 1764(6)^5 \\ + 216(7)^5 \end{pmatrix} \\ \frac{(7)^8}{8!} - \frac{7}{(518400)(5!)}\begin{pmatrix} 2242534(1)^5 - 338541(2)^5 + 1944810(3)^5 - 660275(4)^5 + \\ 619458(5)^5 - 93639(6)^5 + 16366(7)^5 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_9 = \begin{pmatrix} \frac{1}{9!} - \frac{1}{(3628800)(6!)}\begin{pmatrix} 562618(1)^6 - 662757(2)^6 + 694230(3)^6 - 506675(4)^6 + \\ 239406(5)^6 - 65823(6)^6 + 8002(7)^6 \end{pmatrix} \\ \frac{(2)^9}{9!} - \frac{1}{(28350)(6!)}\begin{pmatrix} 38762(1)^6 - 32823(2)^6 + 34730(3)^6 - 25300(4)^6 + 11934(5)^6 \\ -3277(6)^6 + 398(7)^6 \end{pmatrix} \\ \frac{(3)^9}{9!} - \frac{1}{(44800)(6!)}\begin{pmatrix} 183654(1)^6 - 105381(2)^6 + 135450(3)^6 - 98955(4)^6 + 46818(5)^6 \\ -12879(6)^6 + 1566(7)^6 \end{pmatrix} \\ \frac{(4)^9}{9!} - \frac{1}{(14175)(6!)}\begin{pmatrix} 118432(1)^6 - 46608(2)^6 + 86880(3)^6 - 58520(4)^6 + 27744(5)^6 - \\ 7632(6)^6 + 928(7)^6 \end{pmatrix} \\ \frac{(5)^9}{9!} - \frac{1}{(145152)(6!)}\begin{pmatrix} 410450(1)^6 - 115425(2)^6 + 320750(3)^6 - 178375(4)^6 + 91350(5)^6 \\ -25075(6)^6 + 3050(7)^6 \end{pmatrix} \\ \frac{(6)^9}{9!} - \frac{1}{(1400)(6!)}\begin{pmatrix} 30024(1)^6 - 6156(2)^6 + 24840(3)^6 - 10800(4)^6 + 7128(5)^6 - 1764(6)^6 \\ +216(7)^6 \end{pmatrix} \\ \frac{(7)^9}{9!} - \frac{7}{(518400)(6!)}\begin{pmatrix} 2242534(1)^6 - 338541(2)^6 + 1944810(3)^6 - 660275(4)^6 + \\ 619458(5)^6 - 93639(6)^6 + 16366(7)^6 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_{10} = \begin{pmatrix} \frac{1}{10!} - \frac{1}{(3628800)(7!)}\begin{pmatrix} 562618(1)^7 - 662757(2)^7 + 694230(3)^7 - 506675(4)^7 + \\ 239406(5)^7 - 65823(6)^7 + 8002(7)^7 \end{pmatrix} \\ \frac{(2)^{10}}{10!} - \frac{1}{(28350)(7!)}\begin{pmatrix} 38762(1)^7 - 32823(2)^7 + 34730(3)^7 - 25300(4)^7 + 11934(5)^7 \\ -3277(6)^7 + 398(7)^7 \end{pmatrix} \\ \frac{(3)^{10}}{10!} - \frac{1}{(44800)(7!)}\begin{pmatrix} 183654(1)^7 - 105381(2)^7 + 135450(3)^7 - 98955(4)^7 + 46818(5)^7 \\ -12879(6)^7 + 1566(7)^7 \end{pmatrix} \\ \frac{(4)^{10}}{10!} - \frac{1}{(14175)(7!)}\begin{pmatrix} 118432(1)^7 - 46608(2)^7 + 86880(3)^7 - 58520(4)^7 + 27744(5)^7 - \\ 7632(6)^7 + 928(7)^7 \end{pmatrix} \\ \frac{(5)^{10}}{10!} - \frac{1}{(145152)(7!)}\begin{pmatrix} 410450(1)^7 - 115425(2)^7 + 320750(3)^7 - 178375(4)^7 + 91350(5)^7 \\ -25075(6)^7 + 3050(7)^7 \end{pmatrix} \\ \frac{(6)^{10}}{10!} - \frac{1}{(1400)(7!)}\begin{pmatrix} 30024(1)^7 - 6156(2)^7 + 24840(3)^7 - 10800(4)^7 + 7128(5)^7 - 1764(6)^7 \\ +216(7)^7 \end{pmatrix} \\ \frac{(7)^{10}}{10!} - \frac{7}{(518400)(7!)}\begin{pmatrix} 2242534(1)^7 - 338541(2)^7 + 1944810(3)^7 - 660275(4)^7 + \\ 619458(5)^7 - 93639(6)^7 + 16366(7)^7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_{11}=\begin{pmatrix}\frac{1}{11!}-\frac{1}{(3628800)(8!)}\begin{pmatrix}562618(1)^8-662757(2)^8+694230(3)^8-506675(4)^8+\\239406(5)^8-65823(6)^8+8002(7)^8\end{pmatrix}\\\frac{(2)^{11}}{11!}-\frac{1}{(28350)(8!)}\begin{pmatrix}38762(1)^8-32823(2)^8+34730(3)^8-25300(4)^8+11934(5)^8\\-3277(6)^8+398(7)^8\end{pmatrix}\\\frac{(3)^{11}}{11!}-\frac{1}{(44800)(8!)}\begin{pmatrix}183654(1)^8-105381(2)^8+135450(3)^8-98955(4)^8+46818(5)^8\\-12879(6)^8+1566(7)^8\end{pmatrix}\\\frac{(4)^{11}}{11!}-\frac{1}{(14175)(8!)}\begin{pmatrix}118432(1)^8-46608(2)^8+86880(3)^8-58520(4)^8+27744(5)^8-\\7632(6)^8+928(7)^8\end{pmatrix}\\\frac{(5)^{11}}{11!}-\frac{1}{(145152)(8!)}\begin{pmatrix}410450(1)^8-115425(2)^8+320750(3)^8-178375(4)^8+91350(5)^8\\-25075(6)^8+3050(7)^8\end{pmatrix}\\\frac{(6)^{11}}{11!}-\frac{1}{(1400)(8!)}\begin{pmatrix}30024(1)^8-6156(2)^8+24840(3)^8-10800(4)^8+7128(5)^8-1764(6)^8\\+216(7)^8\end{pmatrix}\\\frac{(7)^{11}}{11!}-\frac{7}{(518400)(8!)}\begin{pmatrix}2242534(1)^8-338541(2)^8+1944810(3)^8-660275(4)^8+\\619458(5)^8-93639(6)^8+16366(7)^8\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\frac{-165}{89141}\\\frac{-179}{15283}\\\frac{-2889}{98560}\\\frac{-256}{4661}\\\frac{-460}{5209}\\\frac{-999}{7700}\\\frac{-2402}{13423}\end{pmatrix}$$

Hence, the block is of order $(8,8,8,8,8,8,8)^T$ with error constants

$$\left(\frac{-165}{89141},\frac{-179}{15283},\frac{-2889}{98560},\frac{-256}{4661},\frac{-460}{5209},\frac{-999}{7700},\frac{2402}{13423}\right)^T$$

4.5.2.2 Zero Stability of Seven–Step Block Method for Third Order ODEs.

Applying the equation (3.2.2.2.1) to seven-step block method (4.5.1.29–4.5.1.35) this gives

$$\det[rA^{(0)}-A^{(1)}]=\left|r\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\end{pmatrix}\right|=0$$

This implies $r=0,0,0,0,0,0,1$. Hence, the method is zero stable.

4.5.2.3 Consistency and Convergence of Seven–Step Block Method for Third Order ODEs.

The block method (4.5.1.29– 4.5.1.35) is consistent because it accomplishes the conditions highlighted in Definition 1.4. Hence, it is also convergent since it is zero-stable and consistent.

4.5.2.4 Region of Absolute Stability of Seven–Step Block Method for Third Order ODEs

Applying the equation (3.2.2.4.2) for seven-step block (4.5.1.29– 4.5.1.35), we have

$$\bar{h}(\theta,h)=\frac{A-B}{C+D}$$

where

$$A=\begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{4i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{5i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{6i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{7i\theta} \end{pmatrix},\quad B=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C=\begin{pmatrix} \frac{562618}{3628800}e^{i\theta} & -\frac{662757}{3628800}e^{2i\theta} & \frac{694230}{3628800}e^{3i\theta} & -\frac{506675}{3628800}e^{4i\theta} & \frac{239406}{3628800}e^{5i\theta} & -\frac{65823}{3628800}e^{6i\theta} & \frac{8002}{3628800}e^{7i\theta} \\ \frac{38762}{28350}e^{i\theta} & -\frac{32823}{28350}e^{2i\theta} & \frac{34730}{28350}e^{3i\theta} & -\frac{25300}{28350}e^{4i\theta} & \frac{11934}{28350}e^{5i\theta} & -\frac{3277}{28350}e^{6i\theta} & \frac{398}{28350}e^{7i\theta} \\ \frac{183654}{44800}e^{i\theta} & -\frac{105381}{44800}e^{2i\theta} & \frac{135450}{44800}e^{3i\theta} & -\frac{98955}{44800}e^{4i\theta} & \frac{46818}{44800}e^{5i\theta} & -\frac{12879}{44800}e^{6i\theta} & \frac{1566}{44800}e^{7i\theta} \\ \frac{118432}{14175}e^{i\theta} & -\frac{46608}{14175}e^{2i\theta} & \frac{86880}{14175}e^{3i\theta} & -\frac{58520}{14175}e^{4i\theta} & \frac{27744}{14175}e^{5i\theta} & -\frac{7632}{14175}e^{6i\theta} & \frac{928}{14175}e^{7i\theta} \\ \frac{2052250}{145152}e^{i\theta} & -\frac{577125}{145152}e^{2i\theta} & \frac{1603750}{145152}e^{3i\theta} & -\frac{891875}{145152}e^{4i\theta} & \frac{456750}{145152}e^{5i\theta} & -\frac{125375}{145152}e^{6i\theta} & \frac{15250}{145152}e^{7i\theta} \\ \frac{30024}{700}e^{i\theta} & -\frac{6156}{700}e^{2i\theta} & \frac{24840}{700}e^{3i\theta} & -\frac{10800}{700}e^{4i\theta} & \frac{7128}{700}e^{5i\theta} & -\frac{1764}{700}e^{6i\theta} & \frac{216}{700}e^{7i\theta} \\ \frac{15697738}{518400}e^{i\theta} & \frac{2369787}{518400}e^{2i\theta} & \frac{13613670}{518400}e^{3i\theta} & -\frac{4621925}{518400}e^{4i\theta} & \frac{4336206}{518400}e^{5i\theta} & \frac{655473}{518400}e^{6i\theta} & \frac{114562}{518400}e^{7i\theta} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{335799}{3628800} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{13376}{28350} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{51327}{44800} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{29976}{14175} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{490375}{145152} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6912}{700} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3520209}{518400} \end{pmatrix}$$

The above matrix is simplified and the imaginary part is equated to zero. This gives

$$\bar{h}(\theta, h) = \frac{= 1.4775\text{E} + 113\cos 7\theta - 1.4775\text{E} + 113}{4.2753\text{E} + 108\cos 7\theta + 7.0527\text{E} + 110}$$

Evaluating $\bar{h}(\theta, h)$ at intervals of θ of 30^0 gives results as tabulated in Table 4.4.

Table 4.4

Interval of Absolute Stability of Seven–Step Block Method for Third Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta, h)$	0	-29.39	-7.33	-15.04	-23.16	-1.93	-31.74

Therefore, the interval of absolute stability is (-31.74, 0). This is shown in the diagram below

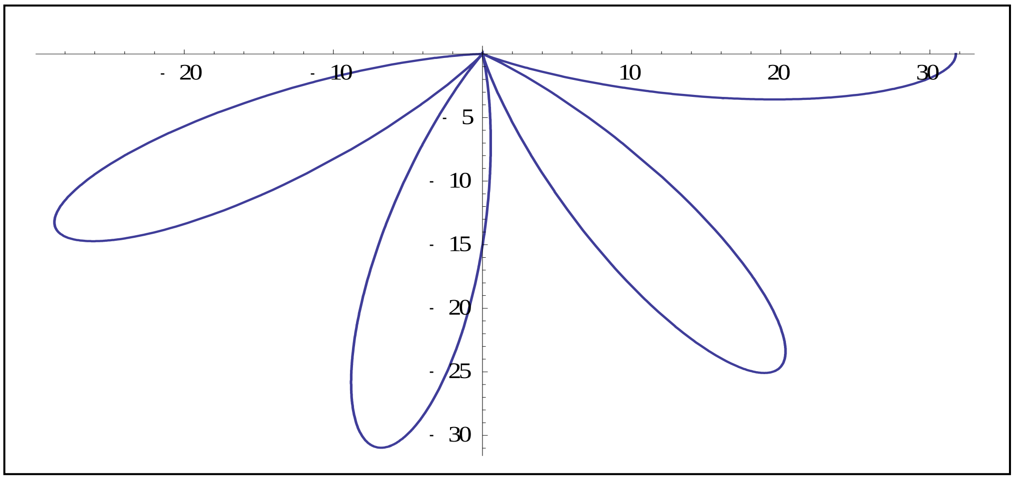

Figure 4.8. Region of absolute stability of seven–step block method for third order ODEs.

4.6 Eight–Step Block Method for Third Order ODEs.

This section includes the derivation of eight–step block method for third order ODEs and establishment of its properties.

4.6.1 Derivation of Eight–Step Block Method for Third Order ODEs.

Power series of the form (4.2.1.1) is considered as an approximate solution to the general third order ODEs of the form (4.2.1.2) where k =8 is the step-length. The first, second and third derivatives of (4.2.1.1) are given in (4.2.1.3), (4.2.1.4) and (4.2.1.5).

Equation (4.2.1.1) is interpolated at $x = x_{n+i}, i = 4(1)6$ and (4.2.1.5) is collocated at $x = x_{n+i}, i = 0(1)8$ which is shown in Figure 4.9 below

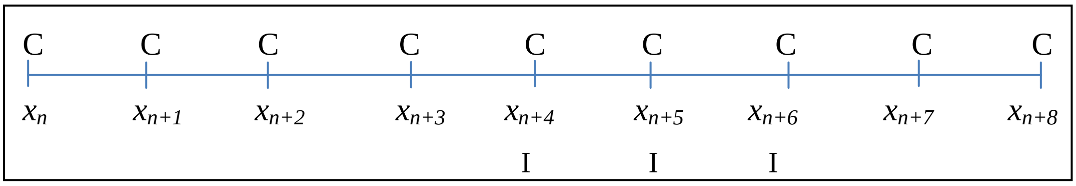

Figure 4.9. Eight–step interpolation and collocation method for third order ODEs.

This procedure gives

$$AX = B \tag{4.6.1.1}$$

where

$$A = \begin{pmatrix}
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 & x_{n+4}^{10} & x_{n+4}^{11} \\
1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 & x_{n+5}^9 & x_{n+5}^{10} & x_{n+5}^{11} \\
1 & x_{n+6} & x_{n+6}^2 & x_{n+6}^3 & x_{n+6}^4 & x_{n+6}^5 & x_{n+6}^6 & x_{n+6}^7 & x_{n+6}^8 & x_{n+6}^9 & x_{n+6}^{10} & x_{n+6}^{11} \\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 & 504x_n^6 & 720x_n^7 & 990x_n^8 \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 & 504x_{n+1}^6 & 720x_{n+1}^7 & 990x_{n+1}^8 \\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 & 504x_{n+2}^6 & 720x_{n+2}^7 & 990x_{n+2}^8 \\
0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 & 504x_{n+3}^6 & 720x_{n+3}^7 & 990x_{n+3}^8 \\
0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & 336x_{n+4}^5 & 504x_{n+4}^6 & 720x_{n+4}^7 & 990x_{n+4}^8 \\
0 & 0 & 0 & 6 & 24x_{n+5} & 60x_{n+5}^2 & 120x_{n+5}^3 & 210x_{n+5}^4 & 336x_{n+5}^5 & 504x_{n+5}^6 & 720x_{n+5}^7 & 990x_{n+5}^8 \\
0 & 0 & 0 & 6 & 24x_{n+6} & 60x_{n+6}^2 & 120x_{n+6}^3 & 210x_{n+6}^4 & 336x_{n+6}^5 & 504x_{n+6}^6 & 720x_{n+6}^7 & 990x_{n+6}^8 \\
0 & 0 & 0 & 6 & 24x_{n+7} & 60x_{n+7}^2 & 120x_{n+7}^3 & 210x_{n+7}^4 & 336x_{n+7}^5 & 504x_{n+7}^6 & 720x_{n+7}^7 & 990x_{n+7}^8 \\
0 & 0 & 0 & 6 & 24x_{n+8} & 60x_{n+8}^2 & 120x_{n+8}^3 & 210x_{n+8}^4 & 336x_{n+8}^5 & 504x_{n+8}^6 & 720x_{n+8}^7 & 990x_{n+8}^8
\end{pmatrix}$$

$$X = \left(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}\right)^T$$

$$B = \left(y_{n+4}, y_{n+5}, y_{n+6}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}, f_{n+7}, f_{n+8}\right)^T$$

In order to find the values *a's* in (4.6.1.1), Gaussian elimination method is employed.

The values of *a's* are below:

$$\begin{aligned}
a_0 = \quad & 15y_{n+4} - 24y_{n+5} + 10y_{n+6} + \frac{h^3}{30240}(-293f_n - 14206f_{n+1} - 62905f_{n+2} - 129986f_{n+3} \\
& -250015f_{n+4} - 25162f_{n+5} - 59f_{n+6} - 550f_{n+7} + 56f_{n+8}) - \frac{x_n^3}{6}f_n + \frac{x_n^2 h}{725760}(-104181f_n \\
& -576190f_{n+1} - 15065f_{n+2} - 807426f_{n+3} + 42497f_{n+4} - 437594f_{n+5} + 107493f_{n+6} - \\
& 26854f_{n+7} + 2920f_{n+8}) + \frac{x_n h^2}{19958400}(-1230751f_n - 36115835f_{n+1} - 21844442f_{n+2} - \\
& 64336102f_{n+3} - 76360205f_{n+4} - 47251694f_{n+5} + 1454687f_{n+6} - 525650f_{n+7} + 56392f_{n+8})
\end{aligned}$$

$$+\frac{x_n}{2h}(11y_{n+4}-20y_{n+5}+9y_{n+6})+\frac{x_n^4}{20160h}(-2283f_n+6720f_{n+1}-11760f_{n+2}+15680f_{n+3}$$

$$-14700f_{n+4}+9408f_{n+5}-3920f_{n+6}+960f_{n+7}+105f_{n+8})-\frac{x_n^{10}}{7257600h^7}(9f_n-70f_{n+1}+$$

$$238f_{n+2}-462f_{n+3}+560f_{n+4}-434f_{n+5}+210f_{n+6}-58f_{n+7}+7f_{n+8})-\frac{x_n^9}{1451520h^6}(39f_n$$

$$-292f_{n+1}+956f_{n+2}-1788f_{n+3}+2090f_{n+4}-1564f_{n+5}+732f_{n+6}-196f_{n+7}+23f_{n+8})-$$

$$\frac{x_n^8}{241920h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}-2581f_{n+5}+1170f_{n+6}$$

$$-305f_{n+7}+35f_{n+8})-\frac{x_n^6}{172800h^3}(2403f_n-13960f_{n+1}+36709f_{n+2}-57384f_{n+3}$$

$$+58280f_{n+4}-39128f_{n+5}+16830f_{n+6}-4216f_{n+7}+469f_{n+8})-\frac{x_n^7}{1209600h^4}(3207f_n$$

$$-21056f_{n+1}+61156f_{n+2}-102912f_{n+3}+109930f_{n+4}-76352f_{n+5}+33636f_{n+6}-$$

$$8576f_{n+7}+967f_{n+8})+\frac{x_n^2}{2h^2}(y_{n+4}-2y_{n+5}+y_{n+6})+\frac{x_n^5}{604800h^2}(-29531f_n+138528f_{n+1}$$

$$-312984f_{n+2}+448672f_{n+3}-435330f_{n+4}+284256f_{n+5}-120008f_{n+6}+29664f_{n+7}$$

$$-3267f_{n+8})-\frac{x_n^{11}}{39916800h^8}(f_n-8f_{n+1}+28f_{n+2}-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}$$

$$-8f_{n+7}+f_{n+8}).$$

$$a_1=-\frac{h^2}{19958400}(-1230751f_n-36115835f_{n+1}-21844442f_{n+2}-64336102f_{n+3}-$$

$$76360205f_{n+4}-47251694f_{n+5}+1454687f_{n+6}-525650f_{n+7}+56392f_{n+8})-$$

$$\frac{1}{2h}(11y_{n+4}-20y_{n+5}+9y_{n+6})+\frac{x_n^3}{5040h}(-2283f_n+6720f_{n+1}-11760f_{n+2}+15680f_{n+3}$$

$$-14700f_{n+4}+9408f_{n+5}-3920f_{n+6}+960f_{n+7}+105f_{n+8})+\frac{x_n^2}{2}f_n-\frac{x_n}{2h^2}(y_{n+4}-2y_{n+5}$$

$$+y_{n+6})+\frac{x_n^4}{120960h^2}(-29531f_n+138528f_{n+1}-312984f_{n+2}+448672f_{n+3}-$$

$$435330f_{n+4}+284256f_{n+5}-120008f_{n+6}+29664f_{n+7}-3267f_{n+8})+\frac{x_n^9}{725760h^7}(9f_n$$

$$-70f_{n+1}+238f_{n+2}-462f_{n+3}+560f_{n+4}-434f_{n+5}+210f_{n+6}-58f_{n+7}+7f_{n+8})+$$

$$\frac{x_n^8}{161280h^6}(39f_n-292f_{n+1}+956f_{n+2}-1788f_{n+3}+2090f_{n+4}-1564f_{n+5}+732f_{n+6}-$$

$$196f_{n+7}+23f_{n+8})+\frac{x_n^7}{30240h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}-$$

$$2581f_{n+5}+1170f_{n+6}-305f_{n+7}+35f_{n+8})+\frac{x_n^5}{28800h^3}(2403f_n-13960f_{n+1}+36709f_{n+2}$$
$$-57384f_{n+3}+58280f_{n+4}-39128f_{n+5}+16830f_{n+6}-4216f_{n+7}+469f_{n+8})+$$
$$\frac{x_n^6}{172800h^4}(3207f_n-21056f_{n+1}+61156f_{n+2}-102912f_{n+3}+109930f_{n+4}-76352f_{n+5}$$
$$+33636f_{n+6}-8576f_{n+7}+967f_{n+8})+\frac{x_n^{10}}{3628800h^8}(f_n-8f_{n+1}+28f_{n+2}-56f_{n+3}+$$
$$70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})-\frac{x_n h}{3628800}(-104181f_n-576190f_{n+1}$$
$$-15065f_{n+2}-807426f_{n+3}+42497f_{n+4}-437594f_{n+5}+107493f_{n+6}-26854f_{n+7}$$
$$+2920f_{n+8}).$$

$$a_2=\frac{h}{725760}(-104181f_n-576190f_{n+1}-15065f_{n+2}-807426f_{n+3}+42497f_{n+4}$$
$$-437594f_{n+5}+107493f_{n+6}-26854f_{n+7}+2920f_{n+8})-\frac{x_n}{2}f_n+\frac{1}{2h^2}(y_{n+4}-$$
$$2y_{n+5}+y_{n+6})+\frac{x_n^3}{60480h^2}(-29531f_n+138528f_{n+1}-312984f_{n+2}+448672f_{n+3}$$
$$-435330f_{n+4}+284256f_{n+5}-120008f_{n+6}+29664f_{n+7}-3267f_{n+8})-\frac{x_n^8}{161280h^7}(9f_n$$
$$-70f_{n+1}+238f_{n+2}-462f_{n+3}+560f_{n+4}-434f_{n+5}+210f_{n+6}-58f_{n+7}+7f_{n+8})-$$
$$\frac{x_n^7}{40320h^6}(39f_n-292f_{n+1}+956f_{n+2}-1788f_{n+3}+2090f_{n+4}-1564f_{n+5}+732f_{n+6}-$$
$$196f_{n+7}+23f_{n+8})-\frac{x_n^6}{8640h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}-$$
$$2581f_{n+5}+1170f_{n+6}-305f_{n+7}+35f_{n+8})+\frac{x_n^2}{3360h}(-2283f_n+6720f_{n+1}-11760f_{n+2}$$
$$+15680f_{n+3}-14700f_{n+4}+9408f_{n+5}-3920f_{n+6}+960f_{n+7}+105f_{n+8})-$$
$$\frac{x_n^4}{11520h^3}(2403f_n-13960f_{n+1}+36709f_{n+2}-57384f_{n+3}+58280f_{n+4}-39128f_{n+5}$$
$$+16830f_{n+6}-4216f_{n+7}+469f_{n+8})-\frac{x_n^5}{57600h^4}(3207f_n-21056f_{n+1}+61156f_{n+2}$$
$$-102912f_{n+3}+109930f_{n+4}-76352f_{n+5}+33636f_{n+6}-8576f_{n+7}+967f_{n+8})-$$
$$\frac{x_n^9}{725760h^8}(f_n-8f_{n+1}+28f_{n+2}-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8}).$$

$$
\begin{aligned}
a_3 = \; & \frac{1}{6} f_n + \frac{x_n^7}{60480h^7}(9f_n - 70f_{n+1} + 238f_{n+2} - 462f_{n+3} + 560f_{n+4} - 434f_{n+5} + \\
& 210f_{n+6} - 58f_{n+7} + 7f_{n+8}) + \frac{x_n^{6}}{17280h^6}(39f_n - 292f_{n+1} + 956f_{n+2} - 1788f_{n+3} + \\
& 2090f_{n+4} - 1564f_{n+5} + 732f_{n+6} - 196f_{n+7} + 23f_{n+8}) + \frac{x_n^5}{4320h^5}(81f_n - 575f_{n+1} + \\
& 1790f_{n+2} - 3195f_{n+3} + 3580f_{n+4} - 2581f_{n+5} + 1170f_{n+6} - 305f_{n+7} + 35f_{n+8}) + \\
& \frac{x_n^3}{8640h^3}(2403f_n - 13960f_{n+1} + 36709f_{n+2} - 57384f_{n+3} + 58280f_{n+4} \\
& -39128f_{n+5} + 16830f_{n+6} - 4216f_{n+7} + 469f_{n+8}) + \frac{x_n^4}{34560h^4}(3207f_n - 21056f_{n+1} \\
& +61156f_{n+2} - 102912f_{n+3} + 109930f_{n+4} - 76352f_{n+5} + 33636f_{n+6} - 8576f_{n+7} + \\
& 967f_{n+8}) - \frac{x_n^2}{60480h^2}(-29531f_n + 138528f_{n+1} - 312984f_{n+2} + 448672f_{n+3} - \\
& 435330f_{n+4} + 284256f_{n+5} - 120008f_{n+6} + 29664f_{n+7} - 3267f_{n+8}) + \frac{x_n^{8}}{241920h^8}(f_n \\
& -8f_{n+1} + 28f_{n+2} - 56f_{n+3} + 70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} + f_{n+8}) - \\
& \frac{x_n}{5040h}(-2283f_n + 6720f_{n+1} - 11760f_{n+2} + 15680f_{n+3} - 14700f_{n+4} + 9408f_{n+5} - \\
& 3920f_{n+6} + 960f_{n+7} + 105f_{n+8}).
\end{aligned}
$$

$$
\begin{aligned}
a_4 = \; & \frac{1}{20160h}(-2283f_n + 6720f_{n+1} - 11760f_{n+2} + 15680f_{n+3} - 14700f_{n+4} + 9408f_{n+5} \\
& -3920f_{n+6} + 960f_{n+7} + 105f_{n+8}) - \frac{x_n^6}{34560h^7}(9f_n - 70f_{n+1} + 238f_{n+2} - 462f_{n+3} + \\
& 560f_{n+4} - 434f_{n+5} + 210f_{n+6} - 58f_{n+7} + 7f_{n+8}) - \frac{x_n^{5}}{11520h^6}(39f_n - 292f_{n+1} + 956f_{n+2} \\
& -1788f_{n+3} + 2090f_{n+4} - 1564f_{n+5} + 732f_{n+6} - 196f_{n+7} + 23f_{n+8}) - \frac{x_n^4}{3456h^5}(81f_n \\
& -575f_{n+1} + 1790f_{n+2} - 3195f_{n+3} + 3580f_{n+4} - 2581f_{n+5} + 1170f_{n+6} - 305f_{n+7} + \\
& 35f_{n+8}) - \frac{x_n^2}{11520h^3}(2403f_n - 13960f_{n+1} + 36709f_{n+2} - 57384f_{n+3} + 58280f_{n+4} \\
& -39128f_{n+5} + 16830f_{n+6} - 4216f_{n+7} + 469f_{n+8}) - \frac{x_n^3}{34560h^4}(3207f_n - 21056f_{n+1} \\
& +61156f_{n+2} - 102912f_{n+3} + 109930f_{n+4} - 76352f_{n+5} + 33636f_{n+6} - 8576f_{n+7} +
\end{aligned}
$$

$$967 f_{n+8}) - \frac{x_n^{7}}{120960h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} -$$

$$8 f_{n+7} + f_{n+8}) + \frac{x_n}{120960h^2}(-29531 f_n + 138528 f_{n+1} - 312984 f_{n+2} + 448672 f_{n+3} -$$

$$435330 f_{n+4} + 284256 f_{n+5} - 120008 f_{n+6} + 29664 f_{n+7} - 3267 f_{n+8})$$

$$a_5 = -\frac{1}{604800h^2}(-29531 f_n + 138528 f_{n+1} - 312984 f_{n+2} + 448672 f_{n+3} - 435330 f_{n+4} +$$

$$284256 f_{n+5} - 120008 f_{n+6} + 29664 f_{n+7} - 3267 f_{n+8}) + \frac{x_n^5}{28800h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2}$$

$$-462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + 210 f_{n+6} - 58 f_{n+7} + 7 f_{n+8}) + \frac{x_n^{4}}{11520h^6}(39 f_n - 292 f_{n+1}$$

$$+956 f_{n+2} - 1788 f_{n+3} + 2090 f_{n+4} - 1564 f_{n+5} + 732 f_{n+6} - 196 f_{n+7} + 23 f_{n+8}) +$$

$$\frac{x_n^3}{4320h^5}(81 f_n - 575 f_{n+1} + 1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4} - 2581 f_{n+5} + 1170 f_{n+6}$$

$$-305 f_{n+7} + 35 f_{n+8}) + \frac{x_n^2}{57600h^4}(3207 f_n - 21056 f_{n+1} + 61156 f_{n+2} - 102912 f_{n+3}$$

$$+109930 f_{n+4} - 76352 f_{n+5} + 33636 f_{n+6} - 8576 f_{n+7} + 967 f_{n+8}) + \frac{x_n^{6}}{86400h^8}(f_n -$$

$$8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8}) + \frac{x_n}{28800h^3}(2403 f_n$$

$$-13960 f_{n+1} + 36709 f_{n+2} - 57384 f_{n+3} + 58280 f_{n+4} 39128 f_{n+5} + 16830 f_{n+6} - 4216 f_{n+7}$$

$$+469 f_{n+8}).$$

$$a_6 = -\frac{1}{172800h^3}(2403 f_n - 13960 f_{n+1} + 36709 f_{n+2} - 57384 f_{n+3} + 58280 f_{n+4} -$$

$$39128 f_{n+5} + 16830 f_{n+6} - 4216 f_{n+7} + 469 f_{n+8}) - \frac{x_n^4}{34560h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2}$$

$$-462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + 210 f_{n+6} - 58 f_{n+7} + 7 f_{n+8}) - \frac{x_n^{3}}{17280h^6}(39 f_n -$$

$$92 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3} + 2090 f_{n+4} - 1564 f_{n+5} + 732 f_{n+6} - 196 f_{n+7} + 23 f_{n+8})$$

$$-\frac{x_n^2}{8640h^5}(81 f_n - 575 f_{n+1} + 1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4} - 2581 f_{n+5} + 1170 f_{n+6}$$

$$-305 f_{n+7} + 35 f_{n+8}) - \frac{x_n^{5}}{86400h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} +$$

$$28 f_{n+6} - 8 f_{n+7} + f_{n+8}) - \frac{x_n}{172800h^4}(3207 f_n - 21056 f_{n+1} + 61156 f_{n+2} - 102912 f_{n+3}$$

$$+109930 f_{n+4} - 76352 f_{n+5} + 33636 f_{n+6} - 8576 f_{n+7} + 967 f_{n+8}).$$

$$
\begin{aligned}
a_7 = \quad & \frac{1}{1209600h^4}(3207 f_n - 21056 f_{n+1} + 61156 f_{n+2} - 102912 f_{n+3} + 109930 f_{n+4} \\
& 76352 f_{n+5} + 33636 f_{n+6} - 8576 f_{n+7} + 967 f_{n+8}) + \frac{x_n^3}{60480h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2} \\
& -462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + 210 f_{n+6} - 58 f_{n+7} + 7 f_{n+8}) + \frac{x_n^2}{40320h^6}(39 f_n - \\
& 292 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3} + 2090 f_{n+4} - 1564 f_{n+5} + 732 f_{n+6} - 196 f_{n+7} + 23 f_{n+8}) \\
& + \frac{x_n^4}{120960h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + \\
& f_{n+8}) + \frac{x_n}{30240h^5}(81 f_n - 575 f_{n+1} + 1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4} - 2581 f_{n+5} \\
& +1170 f_{n+6} - 305 f_{n+7} + 35 f_{n+8}).
\end{aligned}
$$

$$
\begin{aligned}
a_8 = \quad & -\frac{1}{241920h^5}(81 f_n - 575 f_{n+1} + 1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4} - 2581 f_{n+5} \\
& +1170 f_{n+6} - 305 f_{n+7} + 35 f_{n+8}) - \frac{x_n^2}{161280h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + \\
& 560 f_{n+4} - 434 f_{n+5} + 210 f_{n+6} - 58 f_{n+7} + 7 f_{n+8}) - \frac{x_n^3}{241920h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} \\
& -56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8}) - \frac{x_n}{161280h^6}(39 f_n - 292 f_{n+1} \\
& +956 f_{n+2} - 1788 f_{n+3} + 2090 f_{n+4} - 1564 f_{n+5} + 732 f_{n+6} - 196 f_{n+7} + 23 f_{n+8})
\end{aligned}
$$

$$
\begin{aligned}
a_9 = \quad & \frac{1}{1451520h^6}(39 f_n - 292 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3} + 2090 f_{n+4} - 1564 f_{n+5} + \\
& 732 f_{n+6} - 196 f_{n+7} + 23 f_{n+8}) + \frac{x_n^2}{725760h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} \\
& -56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8}) + \frac{x_n}{725760h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} \\
& +560 f_{n+4} - 434 f_{n+5} + 210 f_{n+6} - 58 f_{n+7} + 7 f_{n+8})
\end{aligned}
$$

$$
\begin{aligned}
a_{10} = \quad & \frac{1}{7257600h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + \\
& 210 f_{n+6} - 58 f_{n+7} + 7 f_{n+8}) - \frac{x_n}{3628800h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} \\
& -56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8})
\end{aligned}
$$

$$
\begin{aligned}
a_{11} = \quad & \frac{1}{39916800h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} \\
& + f_{n+8})
\end{aligned}
$$

Substituting the values of a's into equation (4.2.1.1) and simplifying, this produces a continuous linear multistep method of the form:

$$y(x)=\sum_{j=4}^{k-2}\alpha_j(x)y_{n+j}+h^3\sum_{j=0}^{k}\beta_j(x)f_{n+j} \qquad (4.6.1.2)$$

where $x=zh+x_n+7h$ (4.6.1.3)

Substituting (4.6.1.3) into (4.6.1.2) and simplifying, we have

$$\alpha_4(z)=1+\frac{z^2}{2}+\frac{3z}{2}$$
$$\alpha_5(z)=-3-4z-z^2$$
$$\alpha_6(z)=3+\frac{5z}{2}+\frac{z^2}{2}$$

$$\beta_0(z)=\frac{1}{79833600}(218460+3551z+53669z^2-59400z^4-34452z^5$$
$$+2310z^6+8382z^7+3300z^8+605z^9+55z^{10}+2z^{11}).$$

$$\beta_1(z)=\frac{1}{39916800}(107844-56816z-250041z^2+277200z^4+158136z^5$$
$$-12474z^6-38808z^7-14850z^8-26405z^9-231z^{10}-8z^{11}).$$

$$\beta_2(z)=\frac{1}{39916800}(-489984-8036z+1046606z^2-1164240z^4$$
$$-648648z^5+61908z^6+160908z^7+59235z^8+10120z^9$$
$$+847z^{10}+28z^{11}).$$

$$\beta_3(z)=\frac{1}{39916800}(1370028+206020z-2599817z^2+2910600z^4$$
$$+1563408z^5-188034z^6-393096z^7-136950z^8-22220z^9$$
$$-1771z^{10}-56z^{11}).$$

$$\beta_4(z)=\frac{1}{39916800}(-2343990-358285z+4450215z^2-4851000z^4$$
$$-2443980z^5+395010z^6+624690z^7+200475z^8+30525z^9$$
$$+2310z^{10}+70z^{11}).$$

$$\beta_5(z)=\frac{1}{39916800}(22270908+29694952z+4271333z^2+5821200z^4$$
$$+2544696z^5-621390z^6-662376z^7-189750z^8-26840z^9$$
$$-1925z^{10}-56z^{11}).$$

$$
\begin{aligned}
\beta_6(z) &= \frac{1}{39916800}(18775416 + 40666312z + 26010886z^2 - 5821200z^4 \\
&\quad -1380456z^5 + 675444z^6 + 453948z^7 + 113025z^8 + 14740z^9 \\
&\quad +1001z^{10} + 28z^{11}). \\
\beta_7(z) &= \frac{1}{39916800}(226644 + 3056996z + 7115229z^2 + 6652800z^3 \\
&\quad +2649240z^4 + 8448z^5 - 419958z^6 - 181368z^7 - 38610z^8 \\
&\quad -4620z^9 - 297z^{10} - 8z^{11}). \\
\beta_8(z) &= \frac{1}{79833600}(21714 - 139405z - 308891z^2 + 415800z^4 + 431244z^5 \\
&\quad +216678z^6 + 63822z^7 + 11550z^8 + 1265z^9 + 77z^{10} - 2z^{11}).
\end{aligned}
\tag{4.6.1.4}
$$

Evaluating (4.6.1.4) at the non-interpolating points .i.e, at z=-7, -6, -5, -4, 0 and 1 yields

$$
\begin{aligned}
&-302400y_{n+6} + 725760y_{n+5} - 453600y_{n+4} + 30240y_n = h^3(56f_{n+8} - 550f_{n+7} \\
&- 59f_{n+6} - 146842f_{n+5} - 250015f_{n+4} - 129986f_{n+3} - 62905f_{n+2} \\
&- 14206f_{n+1} - 293f_n).
\end{aligned}
\tag{4.6.1.5}
$$

$$
\begin{aligned}
&-725760y_{n+6} + 1814400y_{n+5} - 1209600y_{n+4} + 120960y_{n+1} = h^3(145f_{n+8} - 358f_{n+7} \\
&+ 400f_{n+6} - 353408f_{n+5} - 562750f_{n+4} - 228920f_{n+3} - 63608f_{n+2} + 80f_{n+1} \\
&- 107f_n).
\end{aligned}
\tag{4.6.1.6}
$$

$$
\begin{aligned}
&-453600y_{n+6} + 1209600y_{n+5} - 907200y_{n+4} + 151200y_{n+2} = h^3(88f_{n+8} - 874f_{n+7} \\
&+ 179f_{n+6} - 220118f_{n+5} - 316585f_{n+4} - 63598f_{n+3} - 4511f_{n+2} + 686f_{n+1} \\
&- 67f_n).
\end{aligned}
\tag{4.6.1.7}
$$

$$
\begin{aligned}
&-1209600y_{n+6} + 3628800y_{n+5} - 3628800y_{n+4} + 1209600y_{n+3} = h^3(331f_{n+8} \\
&- 3308f_{n+7} + 5048f_{n+6} - 596756f_{n+5} - 633170f_{n+4} + 29324f_{n+3} - 13712f_{n+2} \\
&+ 2932f_{n+1} - 289f_n).
\end{aligned}
\tag{4.6.1.8}
$$

$$
\begin{aligned}
&1209600y_{n+7} - 3628800y_{n+6} + 3628800y_{n+5} - 1209600y_{n+4} = h^3(329f_{n+8} \\
&+ 6868f_{n+7} + 568952f_{n+6} + 674876f_{n+5} - 71030f_{n+4} + 41516f_{n+3} \\
&- 14848f_{n+2} + 3268f_{n+1} - 331f_n).
\end{aligned}
\tag{4.6.1.9}
$$

$$
\begin{aligned}
&151200y_{n+8} - 907200y_{n+6} + 1209600y_{n+5} - 453600y_{n+4} = h^3(1352f_{n+8} \\
&+ 72214f_{n+7} + 301171f_{n+6} + 239018f_{n+5} - 16265f_{n+4} + 10258f_{n+3} \\
&- 3679f_{n+2} + 814f_{n+1} - 83f_n).
\end{aligned}
\tag{4.6.1.10}
$$

The first derivative of (4.6.1.4) gives

$$\alpha_4'(z) = z + \frac{3}{2}$$
$$\alpha_5'(z) = -4 - 2z$$
$$\alpha_6'(z) = z + \frac{5}{2}$$

$$\beta_0'(z) = \frac{1}{79833600}(3551 + 107338z - 237600z^3 - 172260z^4 + 13860z^5 + 58674z^6 + 26400z^7 + 5445z^8 + 550z^9 + 22z^{10}).$$

$$\beta_1'(z) = \frac{1}{39916800}(-56816 - 500082z + 1108800z^3 + 790680z^4 - 74844z^5 - 27165z^6 - 118800z^7 - 23760z^8 - 2310z^9 - 88z^{10}).$$

$$\beta_2'(z) = \frac{1}{39916800}(-8036 + 2093212z - 4656960z^3 - 3243240z^4 + 371448z^5 + 1126356z^6 + 473880z^7 + 91080z^8 + 8470z^9 + 308z^{10}).$$

$$\beta_3'(z) = \frac{1}{39916800}(206020 - 5199634z + 11642400z^3 + 7817040z^4 - 1128204z^5 - 2751672z^6 - 1095600z^7 - 199980z^8 - 17710z^9 - 616z^{10}).$$

$$\beta_4'(z) = \frac{1}{39916800}(-358285 + 8900430z - 19404000z^3 - 12219900z^4 + 2370060z^5 + 4372830z^6 + 1603800z^7 + 274725z^8 + 23100z^9 + 770z^{10}).$$

$$\beta_5'(z) = \frac{1}{39916800}(29694952 + 8542666z + 23284800z^3 + 12723480z^4 - 3728340z^5 - 4636632z^6 - 1518000z^7 - 241560z^8 - 19250z^9 - 616z^{10}).$$

$$\beta_6'(z) = \frac{1}{39916800}(40666312 + 52021772z - 23284800z^3 - 6902280z^4 + 4052664z^5 + 3177636z^6 + 904200z^7 + 132660z^8 + 10010z^9 + 308z^{10}). \qquad (4.6.1.11)$$

$$\beta_7'(z) = \frac{1}{39916800}(3056996 + 14230458z + 19958400z^2 + 10596960z^3 + 42240z^4 - 2519748z^5 - 1269576z^6 - 308880z^7 - 41580z^8 - 2970z^9 - 88z^{10}).$$

$$\beta_8'(z) = \frac{1}{79833600}(-139405 - 617782z + 1663200z^3 + 2156220z^4 + 1300068z^5 + 446754z^6 + 92400z^7 + 11385z^8 + 770z^9 - 22z^{10}).$$

Evaluating (4.6.1.11) at all the grid points. That is, at z = -7, -6, -5, -4, -3, -2,-1, 0 and 1 gives

$$19958400hy'_{n} + 89812800y_{n+6} - 199584000y_{n+5} + 109771200y_{n+4} = h^3(-56392f_{n+8} + 525650f_{n+7} - 1454687f_{n+6} + 47251694f_{n+5} + 76360205f_{n+4} + 64336102f_{n+3} + 36115835f_{n+2} + 21844442f_{n+1} + 1230751f_{n}). \quad (4.6.1.12)$$

$$79833600hy'_{n+1} + 279417600y_{n+6} - 638668800y_{n+5} + 359251200y_{n+4} = h^3(-25225f_{n+8} + 261904f_{n+7} - 1080320f_{n+6} + 133424216f_{n+5} + 250018750f_{n+4} + 151080800f_{n+3} + 83667176f_{n+2} + 5982520f_{n+1} - 127261f_{n}). \quad (4.6.1.13)$$

$$19958400hy'_{n+2} + 49896000y_{n+6} - 119750400y_{n+5} + 69854400y_{n+4} = h^3(-10280f_{n+8} + 100478f_{n+7} - 30649f_{n+6} + 24325186f_{n+5} + 41077595f_{n+4} + 19048970f_{n+3} + 2126077f_{n+2} - 163786f_{n+1} + 12809f_{n}). \quad (4.6.1.14)$$

$$79833600hy'_{n+3} + 119750400y_{n+6} - 319334400y_{n+5} + 199584000y_{n+4} = h^3(-22453f_{n+8} + 224504f_{n+7} - 30416f_{n+6} + 58136960f_{n+5} + 82734710f_{n+4} + 5108248f_{n+3} + 314456f_{n+2} - 117904f_{n+1} + 13495f_{n}). \quad (4.6.1.15)$$

$$19958400hy'_{n+4} + 9979200y_{n+6} - 39916800y_{n+5} + 29937600y_{n+4} = h^3(-2536f_{n+8} + 24490f_{n+7} - 18835f_{n+6} + 4687766f_{n+5} + 2327785f_{n+4} - 477394f_{n+3} + 135935f_{n+2} - 26990f_{n+1} + 2579f_{n}). \quad (4.6.1.16)$$

$$79833600hy'_{n+5} - 39916800y_{n+6} + 39916800y_{n+4} = h^3(4193f_{n+8} - 63200f_{n+7} + 876448f_{n+6} + 11681624f_{n+5} + 843730f_{n+4} - 25808f_{n+3} - 16840f_{n+2} + 6232f_{n+1} - 779f_{n}). \quad (4.6.1.17)$$

$$19958400hy'_{n+6} - 29937600y_{n+6} + 39916800y_{n+5} - 9979200y_{n+4} = h^3(9080f_{n+8} - 124714f_{n+7} + 1685395f_{n+6} + 5488874f_{n+5} - 711625f_{n+4} + 434770f_{n+3} - 161791f_{n+2} + 36590f_{n+1} - 3779f_{n}). \quad (4.6.1.18)$$

$$79833600hy'_{n+7} - 199584000y_{n+6} + 319334400y_{n+5} - 119750400y_{n+4} = h^3(-139405f_{n+8} + 6113992f_{n+7} + 81332624f_{n+6} + 59389904f_{n+5} - 716570f_{n+4} + 412040f_{n+3} - 16072f_{n+2} - 18464f_{n+1} + 3551f_{n}). \quad (4.6.1.19)$$

$$9958400hy'_{n+8} - 69854400y_{n+6} + 119750400y_{n+5} = h^3(1228408f_{n+8} + 21871106f_{n+7} - 49896000y_{n+4} + 35389241f_{n+6} + 32050750f_{n+5} - 7218235f_{n+4} + 4636022f_{n+3} - 1871741f_{n+2} + 449354f_{n+1} - 48505f_{n}). \quad (4.6.1.20)$$

The second derivative of (4.6.1.4) gives

$$\alpha_4''(z) = 1$$
$$\alpha_5''(z) = -2$$
$$\alpha_6''(z) = 1$$

$$\beta_0''(z) = \frac{1}{79833600}(107338 - 712800z^2 - 689040z^3 + 69300z^4 + 352044z^5 + 184800z^6 + 43560z^7 + 4950z^8 + 220z^9).$$

$$\beta_1''(z) = \frac{1}{39916800}(-500082 + 3326400z^2 + 3162720z^3 - 374220z^4 - 1629936z^5 - 831600z^6 - 190080z^7 - 20790z^8 - 880z^9).$$

$$\beta_3''(z) = \frac{1}{39916800}(-5199634 + 34927200z^2 + 31268160z^3 - 5641020z^4 - 16510032z^5 - 7669200z^6 - 1599840z^7 - 159390z^8 - 6160z^9).$$

$$\beta_4''(z) = \frac{1}{39916800}(8900430 - 58212000z^2 - 48879600z^3 + 11850300z^4 + 26236980z^5 + 11226600z^6 + 2197800z^7 + 207900z^8 + 7700z^9).$$

$$\beta_5''(z) = \frac{1}{39916800}(8542666 + 69854400z^2 + 50893920z^3 - 18641700z^4 - 27819792z^5 - 10626000z^6 - 1932480z^7 - 173250z^8 - 6160z^9).$$

$$\beta_6''(z) = \frac{1}{39916800}(52021772 - 69854400z^2 - 27609120z^3 + 20263320z^4 + 19065816z^5 + 6329400z^6 + 1061280z^7 + 90090z^8 + 3080z^9).$$

$$\beta_7''(z) = \frac{1}{39916800}(14230458 + 39916800z + 31790880z^2 + 168960z^3 - 12598740z^4 - 7617456z^5 - 2162160z^6 - 332640z^7 - 26730z^8 - 880z^9).$$

$$\beta_8''(z) = \frac{1}{79833600}(-617782 + 4989600z^2 + 8624880z^3 + 6500340z^4 + 2680524z^5 + 646800z^6 + 91080z^7 + 6930z^8 - 220z^9). \qquad (4.6.1.21)$$

Equation (4.6.1.21) is evaluated at all the grid points. i.e, at z=-7, -6, -5, -4, -3, -2,-1, 0 and 1gives

$$362880h^2 y_n'' - 362880y_{n+6} + 725760y_{n+5} - 362880y_{n+4} = h^3(2920f_{n+8} - 26854f_{n+7} + 107493f_{n+6} + 437594f_{n+5} + 42497f_{n+4} - 807426f_{n+3} - 15065f_{n+2} - 576190f_{n+1} - 104181f_n). \qquad (4.6.1.22)$$

$$362880h^2 y_{n+1}'' - 362880y_{n+6} + 725760y_{n+5} - 362880y_{n+4} = h^3(-4753f_{n+8} + 44334f_{n+7} - 216284f_{n+6} - 1229602f_{n+5} - 4608150f_{n+4} - 2478902f_{n+3} - 4755244f_{n+2} - 1294806f_{n+1} + 28207f_n). \qquad (4.6.1.23)$$

$$
\begin{aligned}
&1814400h^2 y''_{n+2} - 1814400 y_{n+6} + 3628800 y_{n+5} = h^3 (1272 f_{n+8} - 12286 f_{n+7} \\
&+ 39001 f_{n+6} - 992322 f_{n+5} - 1645435 f_{n+4} - 2115722 f_{n+3} - 755229 f_{n+2} \\
&+ 40394 f_{n+1} - 2873 f_n).
\end{aligned}
\tag{4.6.1.24}
$$

$$
\begin{aligned}
&362880h^2 y''_{n+3} - 362880 y_{n+6} + 725760 y_{n+5} - 362880 y_{n+4} = h^3 (-689 f_{n+8} \\
&+ 7022 f_{n+7} - 64092 f_{n+6} - 1586786 f_{n+5} - 4123990 f_{n+4} - 1599606 f_{n+3} \\
&+ 127828 f_{n+2} - 18838 f_{n+1} + 1551 f_n).
\end{aligned}
\tag{4.6.1.25}
$$

$$
\begin{aligned}
&1814400h^2 y''_{n+4} - 1814400 y_{n+6} + 3628800 y_{n+5} - 1814400 y_{n+4} = h^3 (904 f_{n+8} \\
&- 9342 f_{n+7} + 31097 f_{n+6} - 1006274 f_{n+5} - 949755 f_{n+4} + 155126 f_{n+3} - 44093 f_{n+2} \\
&+ 8778 f_{n+1} - 841 f_n).
\end{aligned}
\tag{4.6.1.26}
$$

$$
\begin{aligned}
&725760h^2 y''_{n+5} - 725760 y_{n+6} + 1451520 y_{n+5} - 725760 y_{n+4} = h^3 (-285 f_{n+8} \\
&+ 3542 f_{n+7} - 30764 f_{n+6} - 20538 f_{n+5} + 64994 f_{n+4} - 23102 f_{n+3} \\
&+ 7620 f_{n+2} - 1630 f_{n+1} + 163 f_n).
\end{aligned}
\tag{4.6.1.27}
$$

$$
\begin{aligned}
&1814400h^2 y''_{n+6} - 1814400 y_{n+6} + 3628800 y_{n+5} - 1814400 y_{n+4} = h^3 (2936 f_{n+8} \\
&- 40958 f_{n+7} + 7422233 f_{n+6} + 1264574 f_{n+5} - 254075 f_{n+4} + 141174 f_{n+3} \\
&- 51997 f_{n+2} + 11722 f_{n+1} - 1209 f_n).
\end{aligned}
\tag{4.6.1.28}
$$

$$
\begin{aligned}
&3628800h^2 y''_{n+7} - 3628800 y_{n+6} + 7257600 y_{n+5} - 3628800 y_{n+4} = h^3 (-28081 f_{n+8} \\
&+ 1293678 f_{n+7} + 4729252 f_{n+6} + 776606 f_{n+5} + 809130 f_{n+4} - 472694 f_{n+3} \\
&+ 190292 f_{n+2} - 45462 f_{n+1} + 4879 f_n).
\end{aligned}
\tag{4.6.1.29}
$$

$$
\begin{aligned}
&1814400h^2 y''_{n+8} - 1814400 y_{n+6} + 3628800 y_{n+5} - 1814400 y_{n+4} = h^3 (520968 f_{n+8} \\
&+ 2880386 f_{n+7} + 62329 f_{n+6} + 3185982 f_{n+5} - 2111995 f_{n+4} + 1336822 f_{n+3} \\
&- 550461 f_{n+2} + 133706 f_{n+1} - 14537 f_n).
\end{aligned}
\tag{4.6.1.30}
$$

Joining equations (4.6.1.5) - (4.6.1.10), (4.6.1.12) and (4.6.1.22) to produce a block of the form (1.10)

$$
\begin{pmatrix}
0 & 0 & 0 & -453600 & 725760 & -302400 & 0 & 0 \\
120960 & 0 & 0 & -1209600 & 1814400 & -725760 & 0 & 0 \\
0 & 151200 & 0 & -907200 & 1209600 & -453600 & 0 & 0 \\
0 & 0 & 1209600 & -3628800 & 3628800 & -1209600 & 0 & 0 \\
0 & 0 & 0 & -1209600 & 3628800 & -3628800 & 1209600 & 0 \\
0 & 0 & 0 & -453600 & 1209600 & -907200 & 0 & 151200 \\
0 & 0 & 0 & 109771200 & -199584000 & 89812800 & 0 & 0 \\
0 & 0 & 0 & -362880 & 725760 & -362880 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ y_{n+8}
\end{pmatrix}
=
$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -30240 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-7} \\ y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -19958400 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y'_{n-7} \\ y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix} +$$

$$+ h^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -362880 \end{pmatrix} \begin{pmatrix} y''_{n-7} \\ y''_{n-6} \\ y''_{n-5} \\ y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix} +$$

$$h^3 \begin{pmatrix} -14206 & -62905 & -129986 & -250015 & -146842 & -59 & -550 & 56 \\ 80 & -63608 & -228920 & -562750 & -353408 & 400 & -358 & 145 \\ 686 & -4511 & -63598 & -316585 & -220118 & 179 & -874 & 88 \\ 2932 & -13712 & 29324 & -633170 & -596756 & 5048 & -3308 & 331 \\ 3268 & -14848 & 41516 & -71030 & 674876 & 568952 & 6868 & 329 \\ 814 & -3679 & 10258 & -16265 & 239018 & 301171 & 72214 & 1352 \\ 21844442 & 36115835 & 64336102 & 76360205 & 47251694 & -1454687 & 525650 & -56392 \\ -576190 & -15065 & -807426 & 42497 & -437594 & 107493 & -26854 & 2920 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \\ f_{n+8} \end{pmatrix} +$$

$$h^3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -293 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -107 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -67 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -289 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -331 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -83 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1230751 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -104181 \end{pmatrix} \begin{pmatrix} f_{n-7} \\ f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$

Multiplying the above equation by $(A^0)^{-1}$

$$
\begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{pmatrix}
\begin{pmatrix} y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4}\\ y_{n+5}\\ y_{n+6}\\ y_{n+7}\\ y_{n+8} \end{pmatrix}
=
\begin{pmatrix}
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1
\end{pmatrix}
\begin{pmatrix} y_{n-7}\\ y_{n-6}\\ y_{n-5}\\ y_{n-4}\\ y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_{n} \end{pmatrix}
+
$$

$$
h\begin{pmatrix}
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&3\\
0&0&0&0&0&0&0&4\\
0&0&0&0&0&0&0&5\\
0&0&0&0&0&0&0&6\\
0&0&0&0&0&0&0&7\\
0&0&0&0&0&0&0&8
\end{pmatrix}
\begin{pmatrix} y'_{n-7}\\ y'_{n-6}\\ y'_{n-5}\\ y'_{n-4}\\ y'_{n-3}\\ y'_{n-2}\\ y'_{n-1}\\ y'_{n} \end{pmatrix}
+h^2
\begin{pmatrix}
0&0&0&0&0&0&0&\frac{1}{2}\\
0&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&\frac{9}{2}\\
0&0&0&0&0&0&0&8\\
0&0&0&0&0&0&0&\frac{25}{2}\\
0&0&0&0&0&0&0&18\\
0&0&0&0&0&0&0&\frac{49}{2}\\
0&0&0&0&0&0&0&32
\end{pmatrix}
\begin{pmatrix} y''_{n-7}\\ y''_{n-6}\\ y''_{n-5}\\ y''_{n-4}\\ y''_{n-3}\\ y''_{n-2}\\ y''_{n-1}\\ y''_{n} \end{pmatrix}
+
$$

$$
+h^3
\begin{pmatrix}
\frac{409}{2408}&\frac{-483}{2060}&\frac{674}{2285}&\frac{-298}{1107}&\frac{353}{2081}&\frac{-106}{1515}&\frac{338}{19867}&\frac{-165}{89141}\\
\frac{393}{269}&\frac{-4423}{2977}&\frac{2812}{1495}&\frac{-488}{285}&\frac{1051}{976}&\frac{-1249}{2816}&\frac{621}{5764}&\frac{-179}{15283}\\
\frac{2479}{572}&\frac{-752}{237}&\frac{3550}{761}&\frac{-899}{211}&\frac{857}{319}&\frac{-553}{499}&\frac{1077}{3997}&\frac{-173}{5902}\\
\frac{7827}{890}&\frac{-1192}{247}&\frac{6471}{703}&\frac{-2368}{297}&\frac{3357}{667}&\frac{-1987}{957}&\frac{1299}{2573}&\frac{-256}{4661}\\
\frac{7571}{510}&\frac{-1193}{185}&\frac{2687}{168}&\frac{-4499}{365}&\frac{2727}{337}&\frac{-1091}{327}&\frac{943}{1162}&\frac{-460}{5209}\\
\frac{6183}{275}&\frac{-6183}{770}&\frac{3001}{120}&\frac{-1730}{103}&\frac{3324}{269}&\frac{-2691}{550}&\frac{459}{385}&\frac{-999}{7700}\\
\frac{5740}{181}&\frac{-4858}{507}&\frac{8236}{227}&\frac{-2959}{138}&\frac{4339}{236}&\frac{-2077}{331}&\frac{1289}{780}&\frac{-2402}{13423}\\
\frac{15098}{355}&\frac{-843}{76}&\frac{5426}{109}&\frac{-17315}{661}&\frac{10721}{411}&\frac{-4728}{685}&\frac{2425}{914}&\frac{-313}{1374}
\end{pmatrix}
\begin{pmatrix} f_{n+1}\\ f_{n+2}\\ f_{n+3}\\ f_{n+4}\\ f_{n+5}\\ f_{n+6}\\ f_{n+7}\\ f_{n+8} \end{pmatrix}
$$

$$+h^3\begin{pmatrix}0&0&0&0&0&0&0&\frac{37}{408}\\0&0&0&0&0&0&0&\frac{617}{1341}\\0&0&0&0&0&0&0&\frac{259}{232}\\0&0&0&0&0&0&0&\frac{4410}{2141}\\0&0&0&0&0&0&0&\frac{760}{231}\\0&0&0&0&0&0&0&\frac{7663}{1594}\\0&0&0&0&0&0&0&\frac{5481}{829}\\0&0&0&0&0&0&0&\frac{1523}{175}\end{pmatrix}\begin{pmatrix}f_{n-7}\\f_{n-6}\\f_{n-5}\\f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_n\end{pmatrix} \tag{4.6.1.31}$$

whose solution is

$$\begin{aligned}y_{n+1}=y_n+hy'_n+\frac{1}{2}h^2y''_n+\frac{h^3}{39916800}(&-73886f_{n+8}+679110f_{n+7}\\&-2792861f_{n+6}+6771082f_{n+5}-10745445f_{n+4}+11774146f_{n+3}\\&-9359135f_{n+2}+6779886f_{n+1}+3619903f_n).\end{aligned} \tag{4.6.1.32}$$

$$\begin{aligned}y_{n+2}=y_n+2hy'_n+2h^2y''_n+\frac{h^3}{623700}(&-7305f_{n+8}+67196f_{n+7}\\&-276634f_{n+6}+671628f_{n+5}-1067950f_{n+4}+1173140f_{n+3}\\&-926646f_{n+2}+911204f_{n+1}+286967f_n).\end{aligned} \tag{4.6.1.33}$$

$$\begin{aligned}y_{n+3}=y_n+3hy'_n+\frac{9}{2}h^2y''_n+\frac{3h^3}{492800}(&-4815f_{n+8}+44262f_{n+7}\\&-182043f_{n+6}+441306f_{n+5}-699885f_{n+4}+766290f_{n+3}\\&-521217f_{n+2}+711918f_{n+1}+183384f_n).\end{aligned} \tag{4.6.1.34}$$

$$\begin{aligned}y_{n+4}=y_n+4hy'_n+8h^2y''_n+\frac{h^3}{155925}(&-8564f_{n+8}+78720f_{n+7}\\&-323744f_{n+6}+784768f_{n+5}-1243200f_{n+4}+1435264f_{n+3}\\&-752480f_{n+2}+1371264f_{n+1}+321172f_n).\end{aligned} \tag{4.6.1.35}$$

$$\begin{aligned}y_{n+5}=y_n+5hy'_n+\frac{25}{2}h^2y''_n+\frac{h^3}{1596672}(&-141000f_{n+8}+1295750f_{n+7}\\&-5327125f_{n+6}+12920250f_{n+5}-19680625f_{n+4}+25537250f_{n+3}\\&-10296375f_{n+2}+23702750f_{n+1}+5253125f_n).\end{aligned} \tag{4.6.1.36}$$

$$y_{n+6} = y_n + 6hy'_n + 18h^2 y''_n + \frac{h^3}{15400}(-1998 f_{n+8} + 18360 f_{n+7} - 75348 f_{n+6} + 190296 f_{n+5} - 258660 f_{n+4} + 385128 f_{n+3} - 123660 f_{n+2} + 346248 f_{n+1} + 74034 f_n). \quad (4.6.1.37)$$

$$y_{n+7} = y_n + 7hy'_n + \frac{49}{2}h^2 y''_n + \frac{h^3}{5702400}(-1020425 f_{n+8} + 9423582 f_{n+7} - 35782103 f_{n+6} + 104842066 f_{n+5} - 122270925 f_{n+4} + 206894170 f_{n+3} - 54639557 f_{n+2} + 180838518 f_{n+1} + 37701874 f_n). \quad (4.6.1.38)$$

$$y_{n+8} = y_n + 8hy'_n + 32h^2 y''_n + \frac{8h^3}{155925}(-4440 f_{n+8} + 51712 f_{n+7} - 134528 f_{n+6} + 508416 f_{n+5} - 510560 f_{n+4} + 970240 f_{n+3} - 216192 f_{n+2} + 828928 f_{n+1} + 169624 f_n). \quad (4.6.1.39)$$

Substituting (4.6.1.35) - (4.6.1.37) into (4.6.1.13) – (4.6.1.20) to give the first derivative of the block

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{7257600}(-40187 f_{n+8} + 369744 f_{n+7} - 1522673 f_{n+6} + 3698922 f_{n+5} - 64888311 f_{n+4} + 6488191 f_{n+3} - 5225623 f_{n+2} + 4124231 f_{n+1}. + 1624505 f_n). \quad (4.6.1.40)$$

$$y'_{n+2} = y'_n + 2hy''_n + \frac{h^2}{113400}(-1563 f_{n+8} + 14368 f_{n+7} - 59092 f_{n+6} + 143232 f_{n+5} - 227030 f_{n+4} + 247328 f_{n+3} - 183708 f_{n+2} + 235072 f_{n+1} + 58193 f_n). \quad (4.6.1.41)$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{h^2}{89600}(-1935 f_{n+8} + 17784 f_{n+7} - 73128 f_{n+6} + 177264 f_{n+5} - 281430 f_{n+4} + 315000 f_{n+3} - 150624 f_{n+2} + 328608 f_{n+1} + 71661 f_n). \quad (4.6.1.42)$$

$$y'_{n+4} = y'_n + 4hy''_n + \frac{h^2}{28350}(-836 f_{n+8} + 7680 f_{n+7} - 31552 f_{n+6} + 76288 f_{n+5} - 118440 f_{n+4} + 160256 f_{n+3} - 46400 f_{n+2} + 148992 f_{n+1} + 30812 f_n). \quad (4.6.1.43)$$

$$y'_{n+5} = y'_n + 5hy''_n + \frac{h^2}{290304}(-10875 f_{n+8} + 100000 f_{n+7} - 412000 f_{n+6} + 1020600 f_{n+5} - 1283750 f_{n+4} + 2294000 f_{n+3} - 465000 f_{n+2}. + 1987000 f_{n+1} + 398825 f_n). \quad (4.6.1.44)$$

$$y'_{n+6} = y'_n + 6hy''_n + \frac{h^2}{1400}(-63f_{n+8} + 576f_{n+7} - 2268f_{n+6} + 7200f_{n+5} - 6390f_{n+4} + 14208f_{n+3} - 2196f_{n+2} + 11808f_{n+1} + 2325f_n). \quad (4.6.1.45)$$

$$y'_{n+7} = y'_n + 7hy''_n + \frac{7h^2}{1036800}(-8183f_{n+8} + 84168f_{n+7} - 145432f_{n+6} + 1009792f_{n+5} - 689430f_{n+4} + 1830248f_{n+3} - 225008f_{n+2} + 1484112f_{n+1} + 288533f_n). \quad (4.6.1.46)$$

$$y'_{n+8} = y'_n + 8hy''_n + \frac{h^2}{28350}(47104f_{n+7} - 14848f_{n+6} + 251904f_{n+5} - 145280f_{n+4} + 419840f_{n+3} - 44544f_{n+2} + 329728f_{n+1} + 63296f_n). \quad (4.6.1.47)$$

Substituting (4.6.1.35) - (4.6.1.37) into (4.6.1.23) – (4.6.1.30) to give the second derivative of the block

$$y''_{n+1} = y''_n + \frac{h}{1069200}(-10004f_{n+8} + 92186f_{n+7} - 380447f_{n+6} + 927046f_{n+5} - 1482974f_{n+4} + 1648632f_{n+3} - 1356711f_{n+2} + 1316197f_{n+1} + 315273f_n). \quad (4.6.1.48)$$

$$y''_{n+2} = y''_n + \frac{h}{113400}(-833f_{n+8} + 7624f_{n+7} - 31154f_{n+6} + 74728f_{n+5} - 116120f_{n+4} + 120088f_{n+3} - 42494f_{n+2} + 182584f_{n+1} + 32377f_n). \quad (4.6.1.49)$$

$$y''_{n+3} = y''_n + \frac{h}{44800}(-369f_{n+8} + 3402f_{n+7} - 14062f_{n+6} + 34434f_{n+5} - 56160f_{n+4} + 79934f_{n+3} + 3438f_{n+2} + 70902f_{n+1} + 12881f_n). \quad (4.6.1.50)$$

$$y''_{n+4} = y''_n + \frac{h}{28350}(-214f_{n+8} + 1952f_{n+7} - 7912f_{n+6} + 18464f_{n+5} - 18160f_{n+4} + 65504f_{n+3} + 488f_{n+2} + 45152f_{n+1} + 8126f_n). \quad (4.6.1.51)$$

$$y''_{n+5} = y''_n + \frac{h}{145152}(-1225f_{n+8} + 11450f_{n+7} - 49150f_{n+6} + 170930f_{n+5} - 4000f_{n+4} + 318350f_{n+3} + 7550f_{n+2} + 230150f_{n+1} + 41705f_n). \quad (4.6.1.52)$$

$$y''_{n+6} = y''_n + \frac{h}{1400}(-9f_{n+8} + 72f_{n+7} + 158f_{n+6} + 2664f_{n+5} - 360f_{n+4} + 3224f_{n+3} + 18f_{n+2} + 2232f_{n+1} + 401f_n). \quad (4.6.1.53)$$

$$y''_{n+7} = y''_n + \frac{h}{518400}(-8183f_{n+8} + 223174f_{n+7} + 522046f_{n+6} + 736078f_{n+5} + 54880f_{n+4} + 1085937f_{n+3} + 48706f_{n+2} + 816634f_{n+1} + 149527f_n). \quad (4.6.1.54)$$

$$y''_{n+8} = y''_n + \frac{h}{28350}(7912f_{n+8} + 47104f_{n+7} - 7424f_{n+6} + 83968f_{n+5} - 36320f_{n+4} + 83968f_{n+3} - 7424f_{n+2} + 47104f_{n+1} + 7912f_n). \quad (4.6.1.55)$$

4.6.2 Properties of Eight–Step Block Method for Third Order ODEs .

The establishment of order, zero-stability and region of absolute stability of eight–step block method for third order ODEs are considered in this section.

4.6.2.1 Order of Eight–Step Block Method for Third Order ODEs

In finding the order of the block method (4.6.1.32 – 4.6.1.39), the strategy stated in section 3.2.2.1 is used and it is shown below

$$\begin{pmatrix} \sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{h^m}{m!}y_n^{(m)}-\frac{3619903}{39916800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(39916800)(m!)}y_n^{(3+m)}\begin{pmatrix}6779886(1)^m-9359135(2)^m+11774146(3)^m\\-10745445(4)^m+6771082(5)^m-2792861(6)^m\\+679110(7)^m-73886(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(2h)^m}{m!}y_n^{(m)}-\frac{286967}{623700}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(623700)(m!)}y_n^{(3+m)}\begin{pmatrix}911204(1)^m-926646(2)^m+1173140(3)^m\\-1067950(4)^m+671628(5)^m-276634(6)^m\\+67196(7)^m-7305(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(3h)^m}{m!}y_n^{(m)}-\frac{183384}{492800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(492800)(m!)}y_n^{(3+m)}\begin{pmatrix}711918(1)^m-521217(2)^m+766290(3)^m\\-699885(4)^m+441306(5)^m-182043(6)^m\\+44262(7)^m-4815(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(4h)^m}{m!}y_n^{(m)}-\frac{321172}{155925}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(155925)(m!)}y_n^{(3+m)}\begin{pmatrix}137126(1)^m-752480(2)^m+1435264(3)^m\\-1243200(4)^m+784768(5)^m-323744(6)^m\\+78720(7)^m-8564(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(5h)^m}{m!}y_n^{(m)}-\frac{5253125}{1596672}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(1596672)(m!)}y_n^{(3+m)}\begin{pmatrix}23702750(1)^m-10296375(2)^m+25537250(3)^m\\-19680625(4)^m+12920250(5)^m-5327125(6)^m\\+1295750(7)^m-141000(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(6h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(6h)^m}{m!}y_n^{(m)}-\frac{74034}{15400}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(15400)(m!)}y_n^{(3+m)}\begin{pmatrix}346248(1)^m-123660(2)^m+385128(3)^m\\-258660(4)^m+190296(5)^m-75348(6)^m\\+18360(7)^m-1998(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(7h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(7h)^m}{m!}y_n^{(m)}-\frac{37701874}{5702400}h^2y_n''-\sum_{m=0}^{\infty}\frac{7h^{3+m}}{(5702400)(m!)}y_n^{(3+m)}\begin{pmatrix}180838518(1)^m-54639557(2)^m+206894170(3)^m\\-122270925(4)^m+104842066(5)^m-35782103(6)^m\\+9423582(7)^m-1020425(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(8h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(8h)^m}{m!}y_n^{(m)}-\frac{8(169624)}{155925}h^2y_n''-\sum_{m=0}^{\infty}\frac{8h^{3+m}}{(155925)(m!)}y_n^{(3+m)}\begin{pmatrix}828928(1)^m-216192(2)^m+970240(3)^m\\-510560(4)^m+508416(5)^m-134528(6)^m\\+51712(7)^m-4440(8)^m\end{pmatrix}
\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

Comparing the coefficients of h^m and y_n^m produces

$$C_0 = \begin{pmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \\ 5-5 \\ 6-6 \\ 7-7 \\ 8-8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{1}{2!} \\ \frac{(2)^2}{2!} - \frac{(2)^2}{2!} \\ \frac{(3)^2}{2!} - \frac{(3)^2}{2!} \\ \frac{(4)^2}{2!} - \frac{(4)^2}{2!} \\ \frac{(5)^2}{2!} - \frac{(5)^2}{2!} \\ \frac{(6)^2}{2!} - \frac{(6)^2}{2!} \\ \frac{(7)^2}{2!} - \frac{(7)^2}{2!} \\ \frac{(8)^2}{2!} - \frac{(8)^2}{2!} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3=\begin{pmatrix}
\frac{1}{3!}-\frac{3619903}{39916800}-\frac{1}{(39916800)(0!)}\begin{pmatrix}6779886(1)^0-9359135(2)^0+11774146(3)^0-10745445(4)^0\\+6771082(5)^0-2792861(6)^0+679110(7)^0-73886(8)^0\end{pmatrix}\\
\frac{(2)^3}{3!}-\frac{286967}{623700}-\frac{1}{(623700)(0!)}\begin{pmatrix}911204(1)^0-926646(2)^0+1173140(3)^0-1067950(4)^0\\+671628(5)^0-276634(6)^0+67196(7)^0-7305(8)^0\end{pmatrix}\\
\frac{(3)^3}{3!}-\frac{183384}{492800}-\frac{1}{(492800)(0!)}\begin{pmatrix}711918(1)^0-521217(2)^0+766290(3)^0-699885(4)^0+\\441306(5)^0-182043(6)^0+44262(7)^0-4815(8)^0\end{pmatrix}\\
\frac{(4)^3}{3!}-\frac{321172}{155925}-\frac{1}{(155925)(0!)}\begin{pmatrix}137126(1)^0-752480(2)^0+1435264(3)^0-1243200(4)^0\\+784768(5)^0-323744(6)^0+78720(7)^0-8564(8)^0\end{pmatrix}\\
\frac{(5)^3}{3!}-\frac{5253125}{1596672}-\frac{1}{(1596672)(0!)}\begin{pmatrix}23702750(1)^0-10296375(2)^0+25537250(3)^0-19680625(4)^0\\+12920250(5)^0-5327125(6)^0+1295750(7)^0-141000(8)^0\end{pmatrix}\\
\frac{(6)^3}{3!}-\frac{74034}{15400}-\frac{1}{(15400)(0!)}\begin{pmatrix}346248(1)^0-123660(2)^0+385128(3)^0-258660(4)^0+190296(5)^0\\-75348(6)^0+18360(7)^0-1998(8)^0\end{pmatrix}\\
\frac{(7)^3}{3!}-\frac{37701874}{5702400}-\frac{7}{(5702400)(0!)}\begin{pmatrix}180838518(1)^0-54639557(2)^0+206894170(3)^0-122270925(4)^0+\\104842066(5)^0-35782103(6)^0+9423582(7)^0-1020425(8)^0\end{pmatrix}\\
\frac{(8)^3}{3!}-\frac{8(169624)}{155925}-\frac{8}{(155925)(0!)}\begin{pmatrix}828928(1)^0-216192(2)^0+970240(3)\;-510560(4)^0+\\508416(5)^0-134528(6)^0+51712(7)^0-4440(8)^0\end{pmatrix}
\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_4=\begin{pmatrix}
\frac{1}{4!}-\frac{1}{(39916800)(1!)}\begin{pmatrix}6779886(1)^1-9359135(2)^1+11774146(3)^1-10745445(4)^1\\+6771082(5)^1-2792861(6)^1+679110(7)^1-73886(8)^1\end{pmatrix}\\
\frac{(2)^4}{4!}-\frac{1}{(623700)(1!)}\begin{pmatrix}911204(1)^1-926646(2)^1+1173140(3)^1-1067950(4)^1\\+671628(5)^1-276634(6)^1+67196(7)^1-7305(8)^1\end{pmatrix}\\
\frac{(3)^4}{4!}-\frac{1}{(492800)(1!)}\begin{pmatrix}711918(1)^1-521217(2)^1+766290(3)^1-699885(4)^1+\\441306(5)^1-182043(6)^1+44262(7)^1-4815(8)^1\end{pmatrix}\\
\frac{(4)^4}{4!}-\frac{1}{(155925)(1!)}\begin{pmatrix}137126(1)^1-752480(2)^1+1435264(3)^1-1243200(4)^1\\+784768(5)^1-323744(6)^1+78720(7)^1-8564(8)^1\end{pmatrix}\\
\frac{(5)^4}{4!}-\frac{1}{(1596672)(1!)}\begin{pmatrix}23702750(1)^1-10296375(2)^1+25537250(3)^1-19680625(4)^1\\+12920250(5)^1-5327125(6)^1+1295750(7)^1-141000(8)^1\end{pmatrix}\\
\frac{(6)^4}{4!}-\frac{1}{(15400)(1!)}\begin{pmatrix}346248(1)^1-123660(2)^1+385128(3)^1-258660(4)^1+190296(5)^1\\-75348(6)^1+18360(7)^1-1998(8)^1\end{pmatrix}\\
\frac{(7)^4}{4!}-\frac{7}{(5702400)(1!)}\begin{pmatrix}180838518(1)^1-54639557(2)^1+206894170(3)^1-122270925(4)^1+\\104842066(5)^1-35782103(6)^1+9423582(7)^1-1020425(8)^1\end{pmatrix}\\
\frac{(8)^4}{4!}-\frac{8}{(155925)(1!)}\begin{pmatrix}828928(1)^1-216192(2)^1+970240(3)^1-510560(4)^1+\\508416(5)^1-134528(6)^1+51712(7)^1-4440(8)^1\end{pmatrix}
\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$
C_5 = \begin{pmatrix}
\frac{1}{5!} - \frac{1}{(39916800)(2!)}\begin{pmatrix} 6779886(1)^2 - 9359135(2)^2 + 11774146(3)^2 - 10745445(4)^2 \\ + 6771082(5)^2 - 2792861(6)^2 + 679110(7)^2 - 73886(8)^2 \end{pmatrix} \\
\frac{(2)^5}{5!} - \frac{1}{(623700)(2!)}\begin{pmatrix} 911204(1)^2 - 926646(2)^2 + 1173140(3)^2 - 1067950(4)^2 \\ + 671628(5)^2 - 276634(6)^2 + 67196(7)^2 - 7305(8)^2 \end{pmatrix} \\
\frac{(3)^5}{5!} - \frac{1}{(492800)(2!)}\begin{pmatrix} 711918(1)^2 - 521217(2)^2 + 766290(3)^2 - 699885(4)^2 + \\ 441306(5)^2 - 182043(6)^2 + 44262(7)^2 - 4815(8)^2 \end{pmatrix} \\
\frac{(4)^5}{5!} - \frac{1}{(155925)(2!)}\begin{pmatrix} 137126(1)^2 - 752480(2)^2 + 1435264(3)^2 - 1243200(4)^2 \\ + 784768(5)^2 - 323744(6)^2 + 78720(7)^2 - 8564(8)^2 \end{pmatrix} \\
\frac{(5)^5}{5!} - \frac{1}{(1596672)(2!)}\begin{pmatrix} 23702750(1)^2 - 10296375(2)^2 + 25537250(3)^2 - 19680625(4)^2 \\ + 12920250(5)^2 - 5327125(6)^2 + 1295750(7)^2 - 141000(8)^2 \end{pmatrix} \\
\frac{(6)^5}{5!} - \frac{7}{(15400)(2!)}\begin{pmatrix} 346248(1)^2 - 123660(2)^2 + 385128(3)^2 - 258660(4)^2 + 190296(5)^2 \\ - 75348(6)^2 + 18360(7)^2 - 1998(8)^2 \end{pmatrix} \\
\frac{(7)^5}{5!} - \frac{1}{(5702400)(2!)}\begin{pmatrix} 180838518(1)^2 - 54639557(2)^2 + 206894170(3)^2 - 122270925(4)^2 + \\ 104842066(5)^2 - 35782103(6)^2 + 9423582(7)^2 - 1020425(8)^2 \end{pmatrix} \\
\frac{(8)^5}{5!} - \frac{8}{(155925)(2!)}\begin{pmatrix} 828928(1)^2 - 216192(2)^2 + 970240(3)^2 - 510560(4)^2 + \\ 508416(5)^2 - 134528(6)^2 + 51712(7)^2 - 4440(8)^2 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_6 = \begin{pmatrix}
\frac{1}{6!} - \frac{1}{(39916800)(3!)}\begin{pmatrix} 6779886(1)^3 - 9359135(2)^3 + 11774146(3)^3 - 10745445(4)^3 \\ + 6771082(5)^3 - 2792861(6)^3 + 679110(7)^3 - 73886(8)^3 \end{pmatrix} \\
\frac{(2)^6}{6!} - \frac{1}{(623700)(3!)}\begin{pmatrix} 911204(1)^3 - 926646(2)^3 + 1173140(3)^3 - 1067950(4)^3 \\ + 671628(5)^3 - 276634(6)^3 + 67196(7)^3 - 7305(8)^3 \end{pmatrix} \\
\frac{(3)^6}{6!} - \frac{1}{(492800)(3!)}\begin{pmatrix} 711918(1)^3 - 521217(2)^3 + 766290(3)^3 - 699885(4)^3 + \\ 441306(5)^3 - 182043(6)^3 + 44262(7)^3 - 4815(8)^3 \end{pmatrix} \\
\frac{(4)^6}{6!} - \frac{1}{(155925)(3!)}\begin{pmatrix} 137126(1)^3 - 752480(2)^3 + 1435264(3)^3 - 1243200(4)^3 \\ + 784768(5)^3 - 323744(6)^3 + 78720(7)^3 - 8564(8)^3 \end{pmatrix} \\
\frac{(5)^6}{6!} - \frac{1}{(1596672)(3!)}\begin{pmatrix} 23702750(1)^3 - 10296375(2)^3 + 25537250(3)^3 - 19680625(4)^3 \\ + 12920250(5)^3 - 5327125(6)^3 + 1295750(7)^3 - 141000(8)^3 \end{pmatrix} \\
\frac{(6)^6}{6!} - \frac{1}{(15400)(3!)}\begin{pmatrix} 346248(1)^3 - 123660(2)^3 + 385128(3)^3 - 258660(4)^3 + 190296(5)^3 \\ - 75348(6)^3 + 18360(7)^3 - 1998(8)^3 \end{pmatrix} \\
\frac{(7)^6}{6!} - \frac{7}{(5702400)(3!)}\begin{pmatrix} 180838518(1)^3 - 54639557(2)^3 + 206894170(3)^3 - 122270925(4)^3 + \\ 104842066(5)^3 - 35782103(6)^3 + 9423582(7)^3 - 1020425(8)^3 \end{pmatrix} \\
\frac{(8)^6}{6!} - \frac{8}{(155925)(3!)}\begin{pmatrix} 828928(1)^3 - 216192(2)^3 + 970240(3)^3 - 510560(4)^3 + \\ 508416(5)^3 - 134528(6)^3 + 51712(7)^3 - 4440(8)^3 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_7 = \begin{pmatrix}
\frac{1}{7!} - \frac{1}{(39916800)(4!)}\left(\begin{array}{l} 6779886(1)^4 - 9359135(2)^4 + 11774146(3)^4 - 10745445(4)^4 \\ + 6771082(5)^4 - 2792861(6)^4 + 679110(7)^4 - 73886(8)^4 \end{array}\right) \\
\frac{(2)^7}{7!} - \frac{1}{(623700)(4!)}\left(\begin{array}{l} 911204(1)^4 - 926646(2)^4 + 1173140(3)^4 - 1067950(4)^4 \\ + 671628(5)^4 - 276634(6)^4 + 67196(7)^4 - 7305(8)^4 \end{array}\right) \\
\frac{(3)^7}{7!} - \frac{1}{(492800)(4!)}\left(\begin{array}{l} 711918(1)^4 - 521217(2)^4 + 766290(3)^4 - 699885(4)^4 + \\ 441306(5)^4 - 182043(6)^4 + 44262(7)^4 - 4815(8)^4 \end{array}\right) \\
\frac{(4)^7}{7!} - \frac{1}{(155925)(4!)}\left(\begin{array}{l} 137126(1)^4 - 752480(2)^4 + 1435264(3)^4 - 1243200(4)^4 \\ + 784768(5)^4 - 323744(6)^4 + 78720(7)^4 - 8564(8)^4 \end{array}\right) \\
\frac{(5)^7}{7!} - \frac{1}{(1596672)(4!)}\left(\begin{array}{l} 23702750(1)^4 - 10296375(2)^4 + 25537250(3)^4 - 19680625(4)^4 \\ + 12920250(5)^4 - 5327125(6)^4 + 1295750(7)^4 - 141000(8)^4 \end{array}\right) \\
\frac{(6)^7}{7!} - \frac{1}{(15400)(4!)}\left(\begin{array}{l} 346248(1)^4 - 123660(2)^4 + 385128(3)^4 - 258660(4)^4 + 190296(5)^4 \\ - 75348(6)^4 + 18360(7)^4 - 1998(8)^4 \end{array}\right) \\
\frac{(7)^7}{7!} - \frac{7}{(5702400)(4!)}\left(\begin{array}{l} 180838518(1)^4 - 54639557(2)^4 + 206894170(3)^4 - 122270925(4)^4 + \\ 104842066(5)^4 - 35782103(6)^4 + 9423582(7)^4 - 1020425(8)^4 \end{array}\right) \\
\frac{(8)^7}{7!} - \frac{8}{(155925)(4!)}\left(\begin{array}{l} 828928(1)^4 - 216192(2)^4 + 970240(3)^4 - 510560(4)^4 + \\ 508416(5)^4 - 134528(6)^4 + 51712(7)^4 - 4440(8)^4 \end{array}\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_8 = \begin{pmatrix}
\frac{1}{8!} - \frac{1}{(39916800)(5!)}\left(\begin{array}{l} 6779886(1)^5 - 9359135(2)^5 + 11774146(3)^5 - 10745445(4)^5 \\ + 6771082(5)^5 - 2792861(6)^5 + 679110(7)^5 - 73886(8)^5 \end{array}\right) \\
\frac{(2)^8}{8!} - \frac{1}{(623700)(5!)}\left(\begin{array}{l} 911204(1)^5 - 926646(2)^5 + 1173140(3)^5 - 1067950(4)^5 \\ + 671628(5)^5 - 276634(6)^5 + 67196(7)^5 - 7305(8)^5 \end{array}\right) \\
\frac{(3)^8}{8!} - \frac{1}{(492800)(5!)}\left(\begin{array}{l} 711918(1)^5 - 521217(2)^5 + 766290(3)^5 - 699885(4)^5 + \\ 441306(5)^5 - 182043(6)^5 + 44262(7)^5 - 4815(8)^5 \end{array}\right) \\
\frac{(4)^8}{8!} - \frac{1}{(155925)(5!)}\left(\begin{array}{l} 137126(1)^5 - 752480(2)^5 + 1435264(3)^5 - 1243200(4)^5 \\ + 784768(5)^5 - 323744(6)^5 + 78720(7)^5 - 8564(8)^5 \end{array}\right) \\
\frac{(5)^8}{8!} - \frac{1}{(1596672)(5!)}\left(\begin{array}{l} 23702750(1)^5 - 10296375(2)^5 + 25537250(3)^5 - 19680625(4)^5 \\ + 12920250(5)^5 - 5327125(6)^5 + 1295750(7)^5 - 141000(8)^5 \end{array}\right) \\
\frac{(6)^8}{8!} - \frac{1}{(15400)(5!)}\left(\begin{array}{l} 346248(1)^5 - 123660(2)^5 + 385128(3)^5 - 258660(4)^5 + 190296(5)^5 \\ - 75348(6)^5 + 18360(7)^5 - 1998(8)^5 \end{array}\right) \\
\frac{(7)^8}{8!} - \frac{7}{(5702400)(5!)}\left(\begin{array}{l} 180838518(1)^5 - 54639557(2)^5 + 206894170(3)^5 - 122270925(4)^5 + \\ 104842066(5)^5 - 35782103(6)^5 + 9423582(7)^5 - 1020425(8)^5 \end{array}\right) \\
\frac{(8)^8}{8!} - \frac{8}{(155925)(5!)}\left(\begin{array}{l} 828928(1)^5 - 216192(2)^5 + 970240(3)^5 - 510560(4)^5 + \\ 508416(5)^5 - 134528(6)^5 + 51712(7)^5 - 4440(8)^5 \end{array}\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$C_9=\begin{pmatrix}
\frac{1}{9!}-\frac{1}{(39916800)(6!)}\begin{pmatrix}6779886(1)^6-9359135(2)^6+11774146(3)^6-10745445(4)^6\\+6771082(5)^6-2792861(6)^6+679110(7)^6-73886(8)^6\end{pmatrix}\\
\frac{(2)^9}{9!}-\frac{1}{(623700)(6!)}\begin{pmatrix}911204(1)^6-926646(2)^6+1173140(3)^6-1067950(4)^6\\+671628(5)^6-276634(6)^6+67196(7)^6-7305(8)^6\end{pmatrix}\\
\frac{(3)^9}{9!}-\frac{1}{(492800)(6!)}\begin{pmatrix}711918(1)^6-521217(2)^6+766290(3)^6-699885(4)^6+\\441306(5)^6-182043(6)^6+44262(7)^6-4815(8)^6\end{pmatrix}\\
\frac{(4)^9}{9!}-\frac{1}{(155925)(6!)}\begin{pmatrix}137126(1)^6-752480(2)^6+1435264(3)^6-1243200(4)^6\\+784768(5)^6-323744(6)^6+78720(7)^6-8564(8)^6\end{pmatrix}\\
\frac{(5)^9}{9!}-\frac{1}{(1596672)(6!)}\begin{pmatrix}23702750(1)^6-10296375(2)^6+25537250(3)^6-19680625(4)^6\\+12920250(5)^6-5327125(6)^6+1295750(7)^6-141000(8)^6\end{pmatrix}\\
\frac{(6)^9}{9!}-\frac{1}{(15400)(6!)}\begin{pmatrix}346248(1)^6-123660(2)^6+385128(3)^6-258660(4)^6+190296(5)^6\\-75348(6)^6+18360(7)^6-1998(8)^6\end{pmatrix}\\
\frac{(7)^9}{9!}-\frac{7}{(5702400)(6!)}\begin{pmatrix}180838518(1)^6-54639557(2)^6+206894170(3)^6-122270925(4)^6+\\104842066(5)^6-35782103(6)^6+9423582(7)^6-1020425(8)^6\end{pmatrix}\\
\frac{(8)^9}{9!}-\frac{8}{(155925)(6!)}\begin{pmatrix}828928(1)^6-216192(2)^6+970240(3)^6-510560(4)^6+\\508416(5)^6-134528(6)^6+51712(7)^6-4440(8)^6\end{pmatrix}
\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_{10}=\begin{pmatrix}
\frac{1}{10!}-\frac{1}{(39916800)(7!)}\begin{pmatrix}6779886(1)^7-9359135(2)^7+11774146(3)^7-10745445(4)^7\\+6771082(5)^7-2792861(6)^7+679110(7)^7-73886(8)^7\end{pmatrix}\\
\frac{(2)^{10}}{10!}-\frac{1}{(623700)(7!)}\begin{pmatrix}911204(1)^7-926646(2)^7+1173140(3)^7-1067950(4)^7\\+671628(5)^7-276634(6)^7+67196(7)^7-7305(8)^7\end{pmatrix}\\
\frac{(3)^{10}}{10!}-\frac{1}{(492800)(7!)}\begin{pmatrix}711918(1)^7-521217(2)^7+766290(3)^7-699885(4)^7+\\441306(5)^7-182043(6)^7+44262(7)^7-4815(8)^7\end{pmatrix}\\
\frac{(4)^{10}}{10!}-\frac{1}{(155925)(7!)}\begin{pmatrix}137126(1)^7-752480(2)^7+1435264(3)^7-1243200(4)^7\\+784768(5)^7-323744(6)^7+78720(7)^7-8564(8)^7\end{pmatrix}\\
\frac{(5)^{10}}{10!}-\frac{1}{(1596672)(7!)}\begin{pmatrix}23702750(1)^7-10296375(2)^7+25537250(3)^7-19680625(4)^7\\+12920250(5)^7-5327125(6)^7+1295750(7)^7-141000(8)^7\end{pmatrix}\\
\frac{(6)^{10}}{10!}-\frac{1}{(15400)(7!)}\begin{pmatrix}346248(1)^7-123660(2)^7+385128(3)^7-258660(4)^7+190296(5)^7\\-75348(6)^7+18360(7)^7-1998(8)^7\end{pmatrix}\\
\frac{(7)^{10}}{10!}-\frac{7}{(5702400)(7!)}\begin{pmatrix}180838518(1)^7-54639557(2)^7+206894170(3)^7-122270925(4)^7+\\104842066(5)^7-35782103(6)^7+9423582(7)^7-1020425(8)^7\end{pmatrix}\\
\frac{(8)^{10}}{10!}-\frac{8}{(155925)(7!)}\begin{pmatrix}828928(1)^7-216192(2)^7+970240(3)^7-510560(4)^7+\\508416(5)^7-134528(6)^7+51712(7)^7-4440(8)^7\end{pmatrix}
\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_{11}=\begin{pmatrix} \frac{1}{11!}-\frac{1}{(39916800)(8!)}\begin{pmatrix}6779886(1)^8-9359135(2)^8+11774146(3)^8-10745445(4)^8\\ +6771082(5)^8-2792861(6)^{\backslash 8}+679110(7)^8-73886(8)^8\end{pmatrix}\\ \frac{(2)^{11}}{11!}-\frac{1}{(623700)(8!)}\begin{pmatrix}911204(1)^8-926646(2)^8+1173140(3)^8-1067950(4)^8\\ +671628(5)^8-276634(6)^8+67196(7)^8-7305(8)^8\end{pmatrix}\\ \frac{(3)^{11}}{11!}-\frac{1}{(492800)(8!)}\begin{pmatrix}711918(1)^8-521217(2)^8+766290(3)^8-699885(4)^8+\\ 441306(5)^8-182043(6)^8+44262(7)^8-4815(8)^8\end{pmatrix}\\ \frac{(4)^{11}}{11!}-\frac{1}{(155925)(8!)}\begin{pmatrix}137126(1)^8-752480(2)^8+1435264(3)^8-1243200(4)^8\\ +784768(5)^8-323744(6)^8+78720(7)^8-8564(8)^8\end{pmatrix}\\ \frac{(5)^{11}}{11!}-\frac{1}{(1596672)(8!)}\begin{pmatrix}23702750(1)^8-10296375(2)^8+25537250(3)^8-19680625(4)^8\\ +12920250(5)^8-5327125(6)^8+1295750(7)^8-141000(8)^8\end{pmatrix}\\ \frac{(6)^{11}}{11!}-\frac{1}{(15400)(8!)}\begin{pmatrix}346248(1)^8-123660(2)^8+385128(3)^8-258660(4)^8+190296(5)^8\\ -75348(6)^8+18360(7)^8-1998(8)^8\end{pmatrix}\\ \frac{(7)^{11}}{11!}-\frac{7}{(5702400)(8!)}\begin{pmatrix}180838518(1)^8-54639557(2)^8+206894170(3)^8-122270925(4)^8+\\ 104842066(5)^8-35782103(6)^8+9423582(7)^8-1020425(8)^8\end{pmatrix}\\ \frac{(8)^{11}}{11!}-\frac{8}{(155925)(8!)}\begin{pmatrix}828928(1)^8-216192(2)^8+970240(3)^8-510560(4)^8+\\ 508416(5)^8-134528(6)^8+51712(7)^8-4440(8)^8\end{pmatrix} \end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_{12}=\begin{pmatrix} \frac{1}{12!}-\frac{1}{(39916800)(9!)}\begin{pmatrix}6779886(1)^9-9359135(2)^9+11774146(3)^9-10745445(4)^9\\ +6771082(5)^9-2792861(6)^9+679110(7)^9-73886(8)^0\end{pmatrix}\\ \frac{(2)^{12}}{12!}-\frac{1}{(623700)(9!)}\begin{pmatrix}911204(1)^9-926646(2)^9+1173140(3)^9-1067950(4)^9\\ +671628(5)^9-276634(6)^9+67196(7)^9-7305(8)^9\end{pmatrix}\\ \frac{(3)^{12}}{12!}-\frac{1}{(492800)(9!)}\begin{pmatrix}711918(1)^9-521217(2)^9+766290(3)^9-699885(4)^9+\\ 441306(5)^9-182043(6)^9+44262(7)^9-4815(8)^9\end{pmatrix}\\ \frac{(4)^{12}}{12!}-\frac{1}{(155925)(9!)}\begin{pmatrix}137126(1)^9-752480(2)^9+1435264(3)^9-1243200(4)^9\\ +784768(5)^9-323744(6)^9+78720(7)^9-8564(8)^9\end{pmatrix}\\ \frac{(5)^{12}}{12!}-\frac{1}{(159667)(9!)}\begin{pmatrix}23702750(1)^9-10296375(2)^9+25537250(3)^9-19680625(4)^9\\ +12920250(5)^9-5327125(6)^9+1295750(7)^9-141000(8)^9\end{pmatrix}\\ \frac{(6)^{12}}{12!}-\frac{1}{(15400)(9!)}\begin{pmatrix}346248(1)^9-123660(2)^9+385128(3)^9-258660(4)^9+190296(5)^9\\ -75348(6)^9+18360(7)^9-1998(8)^9\end{pmatrix}\\ \frac{(7)^{12}}{12!}-\frac{7}{(5702400)(9!)}\begin{pmatrix}180838518(1)^9-54639557(2)^9+206894170(3)^9-122270925(4)^9+\\ 104842066(5)^9-35782103(6)^9+9423582(7)^9-1020425(8)^9\end{pmatrix}\\ \frac{(8)^{12}}{12!}-\frac{8}{(155925)(9!)}\begin{pmatrix}828928(1)^9-216192(2)^9+970240(3)^9-510560(4)^9+\\ 508416(5)^9-134528(6)^9+51712(7)^9-4440(8)^9\end{pmatrix} \end{pmatrix}=\begin{pmatrix}\frac{9}{5669}\\ \frac{113}{11313}\\ \frac{155}{6179}\\ \frac{80}{1701}\\ \frac{551}{7278}\\ \frac{403}{3623}\\ \frac{846}{5513}\\ \frac{374}{1847}\end{pmatrix}$$

Hence, the block has order $(9,9,9,9,9,9,9,9)^T$ with error constants

$$\left(\frac{1}{630939},\frac{113}{11313},\frac{155}{6179},\frac{80}{1701},\frac{316}{4176},\frac{403}{3623},\frac{854}{5555},\frac{374}{1847}\right)^T.$$

4.6.2.2 Zero Stability of Eight–Step Block Method for Third Order ODEs.

Equation (3.2.2.2.1) is applied to eight-step block method (4.6.1.32 – 4.6.1.39) this gives

$$\det[rA^{(0)} - A^{(1)}] = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0$$

which implies $r = 0,0,0,0,0,0,0,1$. Hence, the method is zero stable.

4.6.2.3 Consistency and Convergence of Eight–Step Block Method for Third Order ODEs.

The block method (4.6.1.32 – 4.6.1.39) is consistent because it satisfies the conditions listed in Definition 1.4. Hence, it is also convergent because it is zero-stable and consistent.

4.6.2.4 Region of Absolute Stability of Eight–Step Block Method for Third Order ODEs.

Applying the equation (3.2.2.4.2) for eight-step block method (4.6.1.32 –4.6.1.39) we have

$$\bar{h}(\theta, h) = \frac{A - B}{C + D}$$

where

$$A=\begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{4i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{5i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{6i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{7i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{8i\theta} \end{pmatrix},\quad B=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C=\begin{pmatrix}
\frac{6779886}{39916800}e^{i\theta} & -\frac{9359135}{39916800}e^{2i\theta} & \frac{11774146}{39916800}e^{3i\theta} & -\frac{10745445}{39916800}e^{4i\theta} & \frac{6771082}{39916800}e^{5i\theta} & -\frac{2792861}{39916800}e^{6i\theta} & \frac{679110}{39916800}e^{7i\theta} & -\frac{73886}{39916800}e^{8i\theta} \\
\frac{911204}{623700}e^{i\theta} & -\frac{926646}{623700}e^{2i\theta} & \frac{1173140}{623700}e^{3i\theta} & -\frac{1067950}{623700}e^{4i\theta} & \frac{671628}{623700}e^{5i\theta} & -\frac{276634}{623700}e^{6i\theta} & \frac{67196}{623700}e^{7i\theta} & -\frac{7305}{623700}e^{8i\theta} \\
\frac{2135754}{492800}e^{i\theta} & -\frac{1563651}{492800}e^{2i\theta} & \frac{2298070}{492800}e^{3i\theta} & -\frac{2099655}{492800}e^{4i\theta} & \frac{1323918}{492800}e^{5i\theta} & -\frac{546129}{492800}e^{6i\theta} & \frac{132786}{492800}e^{7i\theta} & -\frac{14445}{492800}e^{8i\theta} \\
\frac{137126}{155925}e^{i\theta} & -\frac{752480}{155925}e^{2i\theta} & \frac{1435264}{155925}e^{3i\theta} & -\frac{1243200}{155925}e^{4i\theta} & \frac{784768}{155925}e^{5i\theta} & -\frac{323744}{155925}e^{6i\theta} & \frac{78720}{155925}e^{7i\theta} & -\frac{8564}{155925}e^{8i\theta} \\
\frac{23702750}{1596672}e^{i\theta} & -\frac{10296375}{1596672}e^{2i\theta} & \frac{25537250}{1596672}e^{3i\theta} & -\frac{19680625}{1596672}e^{4i\theta} & \frac{12920250}{1596672}e^{5i\theta} & -\frac{5327125}{1596672}e^{6i\theta} & \frac{1295750}{1596672}e^{7i\theta} & -\frac{141000}{1596672}e^{8i\theta} \\
\frac{346248}{15400}e^{i\theta} & -\frac{123660}{15400}e^{2i\theta} & \frac{385128}{15400}e^{3i\theta} & -\frac{258660}{15400}e^{4i\theta} & \frac{190296}{15400}e^{5i\theta} & -\frac{75348}{15400}e^{6i\theta} & \frac{18360}{15400}e^{7i\theta} & -\frac{1998}{15400}e^{8i\theta} \\
\frac{180838518}{5702400}e^{i\theta} & -\frac{54639557}{5702400}e^{2i\theta} & \frac{206894170}{5702400}e^{3i\theta} & -\frac{122270925}{5702400}e^{4i\theta} & \frac{104842066}{5702400}e^{5i\theta} & -\frac{35782103}{5702400}e^{6i\theta} & \frac{9423582}{5702400}e^{7i\theta} & -\frac{1020425}{5702400}e^{8i\theta} \\
\frac{6631424}{155925}e^{i\theta} & -\frac{1729536}{155925}e^{2i\theta} & \frac{7761920}{155925}e^{3i\theta} & -\frac{4084480}{155925}e^{4i\theta} & \frac{4067328}{155925}e^{5i\theta} & -\frac{1076224}{155925}e^{6i\theta} & \frac{413696}{155925}e^{7i\theta} & -\frac{35520}{155925}e^{8i\theta}
\end{pmatrix}$$

$$D=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3619903}{39916800} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{286967}{623700} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{550152}{492800} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{321172}{155925} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5253125}{1596672} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{74034}{15400} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1374548}{207900} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{37701874}{5702400}
\end{pmatrix}$$

The above matrix is simplified and equating the imaginary part to zero, this gives

$$\bar{h}(\theta,h)=\frac{(9.9378\mathrm{E}+155)\cos 8\theta-9.9378\mathrm{E}+155}{(1.4872\mathrm{E}+151)\cos 8\theta-3.4497\mathrm{E}+153}$$

Evaluating $\bar{h}(\theta,h)$ at intervals of θ of 30^{0}, the following tabulation are obtained

Table 4.5

Interval of Absolute Stability of Eight–Step Block Method for Third Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta,h)$	0	431.18	431.18	0	431.18	431.18	0

Hence, the interval of absolute stability is (0, 431.18). This is shown in the diagram below

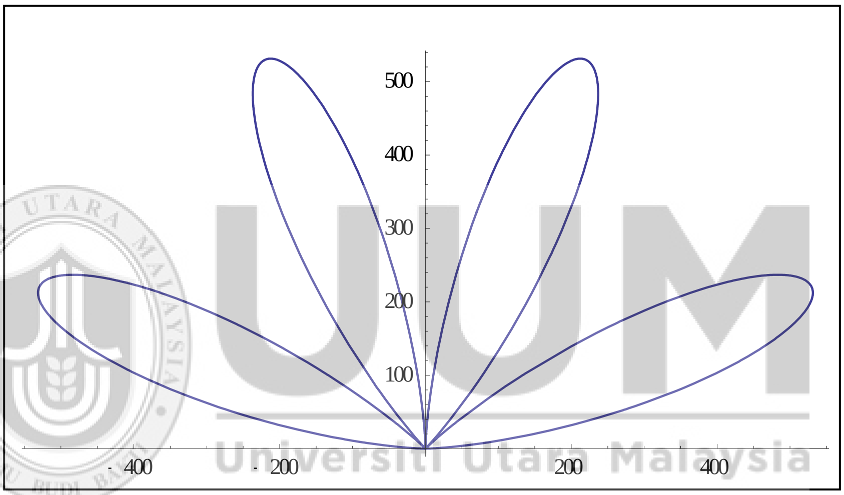

Figure 4.10. Region of absolute stability of eight–step block method for third order ODEs.

4.7 Comments on the Properties of the Block Methods for Third Order ODEs.

The discussion on the analysis of the block methods for third ODEs is considered in this section . The new block methods derived are zero-stable and having order k+1 where k is the step-length and these make the methods to be consistent and therefore convergent for any step-length k examined in this chapter. The interval of absolute stability of the new block methods becomes larger as even step-length k increases (refer to Tables 4.1, 4.3 and 4.5) while the interval of absolute stability decreases as odd step-length k increases (refer to Tables 4.2 and 4.4). The region of absolute stability of even step-length k faces up due to their positive stability function over $[0,\ \pi]$ while the stability region for odd step-length k is below the line because of their negative stability function over $[0,\ \pi]$.

4.8 Test Problems for Third Order ODEs

The accuracy of the new block methods is examined by solving third order ODEs displayed below. The same differential problems existing methods solved are also considered for the purpose of comparison in terms error.

Problem 12: $y''' + y' = 0,\ \ y(0) = 1, y'(0) = 1, y''(0) = 1, h = 0.1, x \in [0,1],$

Exact Solution: $y(x) = \sin x - \cos x + 2$

Problem 13: $y''' = -y',\ \ y(0) = 1, y'(0) = -1, y''(0) = 1, h = 0.1$

Exact Solution: $y(x) = e^{-x}$

Problem 14: $y''' - y'' + y' - y = 0,\ \ y(0) = 1, y'(0) = 0, y''(0) = -1, h = 0.01$

Exact Solution: $y(x) = \cos x$

Problem 15: $y''' + 4y' = x,\ \ y(0) = 0, y'(0) = 0, y''(0) = 1, h = 0.1,\ \ 0 \le x \le 1$

Exact Solution: $y(x) = \dfrac{3}{16}(1 - \cos 3x) + \dfrac{1}{8}x^2$

Problem 16: $y''' = e^x,\ \ y(0) = 3, y'(0) = 1, y''(0) = 5, h = 0.1$

Exact Solution: $y(x) = 2 + 2x^2 + e^x$

Problem17: $y''' + y'' + 3y' - 5y = 2 + 6x - 5x^2,\ \ y(0) = -1, y'(0) = 1, y''(0) = -3, h = 0.1$

Exact Solution: $y(x) = x^2 - e^x + e^{-x}\sin(2x)$

Problem18: $y''' = 3\sin x,\ \ y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1$

Exact Solution: $y(x) = 3\cos x + \dfrac{x^2}{2} - 2$

Problem 19: $y''' = 2y'' - 4,\ \ y(0) = 1, y'(0) = 2, y''(0) = 6,\ \ 0 \le x \le 1$

Exact Solution: $y(x) = x^2 + e^{2x}$

Problem 20: $y''' = y + 3e^x,\ \ y(0) = 0, y'(0) = 1, y''(0) = 2,\ \ 0 \le x \le 1$

Exact Solution: $y(x) = xe^x$

Problem 21: $y''' = 8y' - 3y - 4e^x,\ \ y(0) = 2, y'(0) = -2, y''(0) = 10,\ \ 0 \le x \le 1$

Exact Solution: $y(x) = e^x + e^{-3x}$

4.9 Numerical Results for Third Order ODEs.

The tables shown below are the numerical results when the new block methods with step-length $k = 4(1)8$ were applied to third order differential equations above. The generated numerical results are compared with the existing methods in terms of error.

Table 4.6

Comparison of the New Block Method k=4 with Block Method (Sagir, 2014) for Solving Problem 14

x-values	Exact Solution	Computed Solution	Error in new Method, k=4, h=0.01	Error in Sagir (2014), k=3 and 1 hybrid point, h=0.01
0.01	0.999950000416665260	0.99995000041666526	0.000000E-00	1.9990E-07
0.02	0.999800006666577760	0.99980000666657776	0.000000E-00	1.9560E-07
0.03	0.999550033748987540	0.99955003374898754	0.000000E-00	1.3651E-07
0.04	0.999200106660977920	0.99920010666097803	1.110223E-16	2.5210E-07
0.05	0.998750260394966280	0.99875026039506543	9.914292E-14	1.3039E-06
0.06	0.998200539935204190	0.99820053993548796	2.837730E-13	3.0280E-06
0.07	0.997551000253279590	0.99755100025383447	5.548895E-13	3.3453E-06
0.08	0.996801706302619440	0.99680170630353293	9.134915E-13	1.2405E-06
0.09	0.995952733011994270	0.99595273301345333	1.459055E-12	1.3290E-06
0.1	0.995004165278025710	0.99500416528020486	2.179146E-12	1.7180E-05

Table 4.7

Comparison of the New Block Method k=4 with Block Method (Adesanya et al., 2011) and Four-Point Implicit Method (Awoyemi et al., 2014) for Solving Problem 15.

x-values	Exact Solution	Computed Solution	Error in new Method, *k*=4, *h*=0.1	Error in Adesanya (2011), *k*=4, *h*=0.1	Error in Awoyemi et. al. (2014) *k*=4, *h*=0.1
0.1	0.004987516654767195	0.004987518309523811	1.654757E-09	1.189947E-11	1.1899E-11
0.2	0.019801063624459044	0.019801067214704843	3.590246E-09	3.042207E-09	3.0422E-09
0.3	0.043999572204435337	0.043999585078763913	1.287433E-08	7.779556E-08	7.7796E-08
0.4	0.076867491997406501	0.076867520274488232	2.827708E-08	7.746692E-07	1.5559E-07
0.5	0.117443317649723790	0.117443363840483320	4.619076E-08	4.59901E-06	3.0541E-07
0.6	0.164557921035623750	0.164557970996208670	4.996058E-08	6.478349E-06	4.6102E-07
0.7	0.216881160706204830	0.216881215486879250	5.478067E-08	5.783963E-06	3.138E-07
0.8	0.272974910431491690	0.272974971156164700	6.072467E-08	2.354715E-06	7.0374E-07
0.9	0.331350392754953820	0.331350460450782920	6.769583E-08	3.766592E-06	1.0177E-06
1.0	0.390527531852589200	0.390527607277850240	7.542526E-08	1.23312E-05	1.6528E-06

Table 4.8

Comparison of the New Block Method k=4 with Block Method (Adesanya et. al., 2012) and Numerical Method (Awoyemi et. al., 2006).

x-values	Exact Solution	Computed Solution	Error in new Method, k=4, h=0.1	Error in Adesanya et al.(2012),k= 4, h=0.1	Error in Awoyemi et al., (2006), k=4, h=0.1
0.1	1.10482925136880250	1.10482925139736100	2.855849E-11	1.54055E-09	1.189947E-11
0.2	1.21860275295381950	1.21860275313777770	1.839582E-10	9.84550E-09	3.042207E-09
0.3	1.34018371753573360	1.34018371798374990	4.480163E-10	2.36528E-08	7.779556E-08
0.4	1.46835734830576550	1.46835734911999990	8.142345E-10	4.32732E-08	7.749556E-07
0.5	1.60184297671383020	1.60184303501985360	5.830602E-08	3.90181E-08	3.398961E-06
0.6	1.73930685848535730	1.73930702719420860	1.687089E-07	6.97008E-08	9.501398E-06
0.7	1.87937549995320290	1.87937583086698700	3.309138E-07	5.20329E-08	1.756558E-06
0.8	2.02064938155235740	2.02064992484746100	5.432951E-07	1.35224E-07	2.745889E-05
0.9	2.16171694135681890	2.16171777294913920	8.315923E-07	4.74834E-07	3.888082E-05
1.0	2.30116867893975650	2.30116986981570060	1.190876E-06	1.06936E-06	5.137153E-05

Table 4.9

Comparison of the New Block Method k=5 with Block Method (Adesanya et al., 2011) for Solving Problem 15

x-values	Exact Solution	Computed Solution	Error in new Method, k=5, h=0.1	Error in Adesanya et al., (2011) k= 5, h=0.1
0.1	0.004987516654767195	0.004987516654761906	5.289172E-15	1.189944E-11
0.2	0.019801063624459044	0.019801063619047623	5.411421E-12	3.042207E-09
0.3	0.043999572204435337	0.043999571892857157	3.115782E-10	7.779556E-08
0.4	0.076867491997406501	0.076867486476190489	5.521216E-09	7.746692E-07
0.5	0.117443317649723790	0.117443266369047640	5.128068E-08	4.59901E-06
0.6	0.164557921035623690	0.164557882208255980	3.882737E-08	6.478349E-06
0.7	0.216881160706204810	0.216881139053495870	2.165271E-08	5.783963E-06
0.8	0.272974910431491690	0.272974949460567990	3.902908E-08	2.354715E-06
0.9	0.331350392754953820	0.331350450610851550	5.785590E-08	3.766592E-06
1.0	0.390527531852589200	0.390527606362710110	7.451012E-08	1.23312E-05

Table 4.10

Comparison of the New Block Method k=5 with Block Method (Olabode, 2009) for Solving Problem 16

x-values	Exact Solution	Computed Solution	Error in new Method, k=5 h=0.1	Error in Olabode (2009), k= 5, h=0.1
0.1	3.12517091807564770	3.12517091807901700	3.369305E-12	9.24352E-10
0.2	3.30140275816016970	3.30140275818177020	2.160050E-11	8.3983E-10
0.3	3.52985880757600330	3.52985880762933580	5.333245E-11	4.23997E-10
0.4	3.81182469764127060	3.81182469774115700	9.988632E-11	3.58729E-10
0.5	4.14872127070012820	4.14872127086002700	1.598988E-10	2.99872E-10
0.6	4.54211880039050890	4.54211880064164930	2.511404E-10	3.90509E-09
0.7	4.99375270747047660	4.99375270786662550	3.961489E-10	1.47048E-09
0.8	5.50554092849246860	5.50554092908515090	5.926823E-10	2.49247E-09
0.9	6.07960311115695000	6.07960311199986680	8.429168E-10	0.15695E-09
1.0	6.71828182845904550	6.71828182960364820	1.144603E-09	3.54096E-09
1.1	7.42416602394643380	7.42416602547347630	1.527042E-09	1.40536E-08
1.2	8.20011692273654800	8.20011692476393960	2.027392E-09	3.32635E-08

Table 4.11

Comparison of the New Block Method k=5 with Block Method (Olabode, 2007) for Solving Problem 17

x-values	Exact Solution	Computed Solution	Error in new Method, k=5, h=0.1	Error in Olabode (2007), k=5, h=0.1
0.1	-0.915407473756112530	-0.915407474745166150	9.890536E-10	6.408641E-07
0.2	-0.862573985499428990	-0.862573989084014610	3.584586E-09	1.511330E-05
0.3	-0.841561375114168730	-0.841561320252506960	5.486166E-08	6.364443E-05
0.4	-0.850966529765555760	-0.850965870569312280	6.591962E-07	1.675667E-04
0.5	-0.888343319155555420	-0.888343663001383340	3.438458E-07	3.507709E-04
0.6	-0.950604904717254560	-0.950605617264062340	7.125468E-07	6.410875E-04
0.7	-1.034392853932994700	-1.034393298381865300	4.444489E-07	1.071642E-03
0.8	-1.136403556878909500	-1.136403991293333200	4.344144E-07	1.682213E-03
0.9	-1.253666211231613500	-1.253666673510994000	4.622794E-07	2.520603E-03
1.0	-1.383769999219783400	-1.383771287025621100	1.287806E-06	3.644014E-03

Table 4.12

Comparison of the New Block Method k=6 with Block Predictor-Corrector Method (Olabode, 2013) for Solving Problem 13

x-values	Exact Solution	Computed Solution	Error in new Method, k=6, *h*=0.1	Error in Olabode (2013), *k*=6, *h*=0.1
0.1	0.904837418035959520	0.904837416666666620	1.617595E-13	1.66845548E-09
0.2	0.818730753077981820	0.818730753076952310	1.029510E-12	1.07247812E-08
0.3	0.740818220681717770	0.740818220679151600	2.566169E-12	1.45900969E-08
0.4	0.670320046035639330	0.670320046030800640	4.838685E-12	5.60600832E-08
0.5	0.606530659712633420	0.606530659704430540	8.202883E-12	1.07616918E-07
0.6	0.548811636094026390	0.548811636078357150	1.566924E-11	1.80036532E-07
0.7	0.496585303791409470	0.496585367879705970	6.483969E-08	3.31228689E-06
0.8	0.449328964117221560	0.449329145415281290	1.812981E-07	3.46704102E-04
0.9	0.406569659740599050	0.406570008976985970	3.492364E-07	2.27964863E-03
1.0	0.367879441171442330	0.367880009560811240	5.683894E-07	6.38022004E-03
1.1	0.332871083698079500	0.332871921996075790	8.382980E-07	1.26418315E-02
1.2	0.301194211912202030	0.301195370169472790	1.158257E-06	2.10623676E-02

Table 4.13

Comparison of the New Block Method k=6 with Block Method and Predictor-Corrector Method (Olabode, 2013) for Solving Problem 16

x-values	Exact Solution	Computed Solution	Error in new Method, k=6, h=0.1	Error in Olabode (2013) Block Method, k=6, h=0.1	Error in Olabode (2013) Predictor-Corrector Method Method k=6, h=0.1
0.1	3.125170918075647700	3.12517091807535910	2.886580E-13	9.24352E-10	1.403538619E-09
0.2	3.301402758160169700	3.30140275815833300	1.836753E-12	18.3983E-10	3.269138249E-08
0.3	3.529858807576003300	3.52985880757143060	4.572787E-12	24.2400E-10	1.395151714E-07
0.4	3.811824697641270600	3.81182469763270680	8.563816E-12	53.5873E-10	3.723331807E-07
0.5	4.148721270700128200	4.14872127068638810	1.374012E-11	7.00128E-10	7.869058836E-07
0.6	4.542118800390509700	4.54211880037033210	2.017764E-11	3.90509E-10	1.444874331E-06
0.7	4.993752707470477500	4.99375270744311630	2.736122E-11	6.52952E-09	2.414343780E-06
0.8	5.505540928492468600	5.50554092845571400	3.675460E-11	2.15075E-08	3.770586333E-06
0.9	6.079603111156950800	6.07960311110872720	4.822365E-11	3.88430E-08	5.596789306E-06
1.0	6.718281828459045500	6.71828182839715460	6.189094E-11	6.15410E-08	7.984910299E-05
1.1	7.424166023946433800	7.42416602386879900	7.763479E-11	9.00536E-08	1.103654222E-05
1.2	8.200116922736548000	8.20011692264096050	9.558754E-11	1.27263E-07	1.486399811E-05

Table 4.14

Comparison of the New Block Method k=6 with Block Method and Predictor-Corrector Method (Olablode, 2013) for Solving Problem 18

x-values	Exact Solution	Computed Solution	Error in new Method, k=6, h=0.1	Error in Olabode (2013) Block Method, k=6, h=0.1	Error in Olabode (2013) Predictor-Corrector Method k=6, h=0.1
0.1	0.990012495834077020	0.990012499999999960	6.370460E-13	1.65922E-10	4.172279744E-09
0.2	0.960199733523725120	0.960199733527778100	4.052980E-12	4.76275E-10	9.578546178E-08
0.3	0.911009467376818090	0.911009467386911350	1.009326E-11	6.23182E-10	3.991586710E-07
0.4	0.843182982008655380	0.843182982027559040	1.890366E-11	19.9134E-10	1.036864440E-06
0.5	0.757747685671118280	0.757747685701456340	3.033807E-11	3.28882E-10	2.128509889E-06
0.6	0.656006844729034810	0.656006844773587390	4.455258E-11	1.27096E-09	3.789539851E-06
0.7	0.539526561853465480	0.539526565317591910	5.987466E-11	4.84653E-09	6.130086676E-06
0.8	0.410120128041496560	0.410120128118615590	7.711903E-11	1.09585E-08	9.253867047E-06
0.9	0.269829904811993430	0.269829904908177540	9.618412E-11	2.0188E-08	1.325714643E-05
1.0	0.120906917604419300	0.120906917721584670	1.171654E-10	3.53956E-08	1.822777782E-05
1.1	-0.034211635723267797	-0.034211635583299711	1.399681E-10	5.66233E-08	2.424432295E-05
1.2	-0.192926736569979610	-0.192926736405285740	1.646939E-10	8.35700E-08	3.137526880E-05

Table 4.15

Comparison of the New Block Method k=7 with Block Method (Omar, 1999) in which Maximum Errors were considered for Solving Problem 19

h-values	New Method	Omar (1999)	Number of Steps	Error in new Method, k=7	Error in Omar (1999) k=8
10^{-2}	7-Step Method	S2PEB	54	3.018736E-02	1.64491E+08
		P2PEB	54	3.018736E-02	1.64491E+08
		S3PEB	39	2.308550E-03	7.14468E+11
		P3PEB	39	2.308550E-03	7.14468E+11
10^{-3}	7-Step Method	S2PEB	504	1.856659E-06	5.58385E-03
		P2PEB	504	1.856659E-06	5.58385E-03
		S3PEB	339	1.278984E-06	9.98913E-02
		P3PEB	339	1.278984E-06	9.98913E-02
10^{-4}	7-Step Method	S2PEB	5004	6.566552E-10	3.48576E-05
		P2PEB	5004	6.566552E-10	3.48576E-05
		S3PEB	3339	4.135359E-12	1.53942E-03
		P3PEB	3339	4.135359E-12	1.53942E-03
10^{-5}	7-Step Method	S2PEB	50004	6.523351E-10	3.92325E-05
		P2PEB	50004	6.523351E-10	3.92325E-05
		S3PEB	33339	3.865352E-12	2.22810E-05
		P3PEB	33339	3.865352E-12	2.22810E-05

Table 4.16

Comparison of the New Block Method k=7 with Block Method (Omar, 1999) in which Maximum Errors were considered for Solving Problem 20

h-values	New Method	Omar (1999)	Number of Steps	Error in new Method, k=7	Error in Omar (1999) k=8
10^{-2}	7-Step Method	S2PEB	54	4.097902E-04	7.63504E-03
		P2PEB	54	4.097902E-04	7.63504E-03
		S3PEB	39	1.440162E-03	7.64136E-03
		P3PEB	39	1.440162E-03	7.64136E-03
10^{-3}	7-Step Method	S2PEB	504	1.091941E-06	7.62628E-04
		P2PEB	504	1.091941E-06	7.62628E-04
		S3PEB	339	2.582187E-07	7.62629E-04
		P3PEB	339	2.582187E-07	7.62629E-04
10^{-4}	7-Step Method	S2PEB	5004	1.091607E-09	7.62546E-05
		P2PEB	5004	1.091607E-09	7.62546E-05
		S3PEB	3339	2.464766E-10	7.62546E-05
		P3PEB	3339	2.464766E-10	7.62546E-05
10^{-5}	7-Step Method	S2PEB	50004	8.158452E-11	7.62538E-06
		P2PEB	50004	8.158452E-11	7.62538E-06
		S3PEB	33339	1.298517E-11	7.62538E-06
		P3PEB	33339	1.298517E-11	7.62538E-06

Table 4.17

Comparison of the New Block Method k=8 with Block Method (Omar, 1999) in which Maximum Errors were considered for Solving Problem 19

h-values	New Method	Omar (1999)	Number of Steps	Error in new Method, k=8	Error in Omar (1999) k=8
10^{-2}	8-Step Method	S2PEB	54	1.267345E-02	1.64491E+08
		P2PEB	54	1.267345E-02	1.64491E+08
		S3PEB	39	8.063762E-04	7.14468E+11
		P3PEB	39	8.063762E-04	7.14468E+11
10^{-3}	8-Step Method	S2PEB	504	3.161755E-06	5.58385E-03
		P2PEB	504	3.161755E-06	5.58385E-03
		S3PEB	339	1.736923E-07	9.98913E-02
		P3PEB	339	1.736923E-07	9.98913E-02
10^{-4}	8-Step Method	S2PEB	5004	2.526122E-10	3.48576E-05
		P2PEB	5004	2.526122E-10	3.48576E-05
		S3PEB	3339	1.172396E-11	1.53942E-03
		P3PEB	3339	1.172396E-11	1.53942E-03
10^{-5}	8-Step Method	S2PEB	50004	5.236188E-09	3.92325E-05
		P2PEB	50004	5.236188E-09	3.92325E-05
		S3PEB	33339	2.625740E-10	2.22810E-05
		P3PEB	33339	2.625740E-10	2.22810E-05

Table 4.18

Comparison of the New Block Method k=8 with Block Method (Omar, 1999) whereby Maximum Errors were considered for Solving Problem 21

h-values	New Method	Omar (1999)	Number of Steps	Error in new Method, k=8	Error in Omar (1999) k=8
10^{-2}	8-Step Method	S2PEB	54	5.058909E-02	2.70995E-01
		P2PEB	54	5.058909E-02	2.70995E-01
		S3PEB	39	3.866111E-03	8.01861E-01
		P3PEB	39	3.866111E-03	8.01861E-01
10^{-3}	8-Step Method	S2PEB	504	3.565494E-07	1.19154E-02
		P2PEB	504	3.565494E-07	1.19154E-02
		S3PEB	339	1.674995E-08	1.25684E-02
		P3PEB	339	1.674995E-08	1.25684E-02
10^{-4}	8-Step Method	S2PEB	5004	1.723066E-11	1.17030E-03
		P2PEB	5004	1.723066E-11	1.17030E-03
		S3PEB	3339	2.646772E-13	1.17117E-03
		P3PEB	3339	2.646772E-13	1.17117E-03
10^{-5}	8-Step Method	S2PEB	50004	2.149037E-10	1.17019E-04
		P2PEB	50004	2.149037E-10	1.17019E-04
		S3PEB	33339	1.489830E-11	1.17020E-04
		P3PEB	33339	1.489830E-11	1.17020E-04

4.10 Comments on the Results

Due to the unavailability of third order non-linear ODEs in the existing literatures, the new block methods are only applied to linear ODEs. It is obvious in Table 4.6 that the results of the new block method k=4 for solving Problem 14 are more efficient in terms of error than Sagir (2014) k=3 plus one hybrid point. Similarly, the results in Tables 4.7 and 4.8 also show that the new method performed better than Adesanya et al. (2011), Awoyemi et al. (2014), Adesanya et al. (2012) and Awoyemi et al. (2006) when Problems 12 and 15 were solved.

Furthermore, the accuracy of the new block method k=5 in Tables 4.9, 4.10 and 4.11 is high when compared with Adesanya et al. (2011), Olabode (2009) and Olabode (2007) of the same step-length k for solving Problems 15, 16 and 17. The results of the new block method k=6 shown in Table 4.12 outperform Olabode (2013) block method k=6 for the solution of Problem 13. In the same way, Olabode (2013) predictor-corrector method k=6 is compared with the results of the new method in Tables 4.13 and 4.14, it is evident that the new method performed better for solving Problems 16 and 18.

Moreover, the results of the new block methods k=7 and 8 presented in Tables 4.15, 4.16, 4.17 and 4.18 are compared favourably in terms of accuracy than Omar (1999) k=8 whereby maximum errors were selected after solving Problems 19, 20 and 21.

4.11 Summary

This chapter contains the development of block methods with step-length k=4(1)8 using interpolation and collocation technique for the solution of third order initial value problems of ODEs. The properties of the methods are analyzed. The numerical results generated when the new methods were applied to solve third order problems are compared with the existing methods of the same step-length and they are found better in accuracy.

CHAPTER FIVE

DEVELOPING BLOCK METHODS FOR SOLVING FOURTH ORDER ODEs DIRECTLY

5.1 Introduction

The development of block methods with step-length $k = 5(1)8$ using interpolation and collocation technique for the direct solution of fourth order initial value problems of ODEs are considered in this chapter. These are shown below:

5.2 Five –Step Block Method for Fourth Order ODEs.

In this section, the derivation of five-step block method and the establishment of its properties are examined.

5.2.1 Derivation of Five–Step Block Method for Fourth Order ODEs.

Power series of the form

$$y(x) = \sum_{j=0}^{k+4} a_j x^j \tag{5.2.1.1}$$

is considered as an approximate solution to the general fourth order problem of the form

$$y^{iv} = f(x, y, y', y''); y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0, y'''(x_0) = y'''_0 \tag{5.2.1.2}$$

where in (5.2.1.1) k =5 is the step-length. The first, second, third and fourth derivatives of (5.2.1.1) give

$$y'(x) = \sum_{j=1}^{k+4} j a_j x^{j-1} \tag{5.2.1.3}$$

$$y''(x) = \sum_{j=2}^{k+4} j(j-1) a_j x^{j-2} \tag{5.2.1.4}$$

$$y'''(x) = \sum_{j=3}^{k+4} j(j-1)(j-2)a_j x^{j-3} \tag{5.2.1.5}$$

$$y^{iv}(x) = \sum_{j=4}^{k+4} j(j-1)(j-2)(j-3)a_j x^{j-4} = f(x, y, y', y'', y''') \tag{5.2.1.6}$$

Interpolating (5.2.1.1) at $x = x_{n+i}, i = 0(1)3$ and collocating (5.2.1.6) at the points $x = x_{n+i}, i = 0(1)5$ as explained in figure 5.1 below

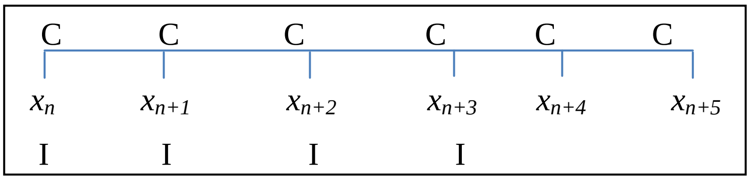

Figure 5.1. Five–step interpolation and collocation method for fourth order ODEs

This approach gives the following result

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 \\ 0 & 0 & 0 & 0 & 24 & 120x_n & 360x_n^2 & 840x_n^3 & 1680x_n^4 & 3024x_n^5 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+1} & 360x_{n+1}^2 & 840x_{n+1}^3 & 1680x_{n+1}^4 & 3024x_{n+1}^5 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+2} & 360x_{n+2}^2 & 840x_{n+2}^3 & 1680x_{n+2}^4 & 3024x_{n+2}^5 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+3} & 360x_{n+3}^2 & 840x_{n+3}^3 & 1680x_{n+3}^4 & 3024x_{n+3}^5 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+4} & 360x_{n+4}^2 & 840x_{n+4}^3 & 1680x_{n+4}^4 & 3024x_{n+4}^5 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+5} & 360x_{n+5}^2 & 840x_{n+5}^3 & 1680x_{n+5}^4 & 3024x_{n+5}^5 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} \tag{5.2.1.7}$$

In order to find the values *a's* in (5.2.1.7), Gaussian elimination method is employed.

The values of *a's* are below:

$$\begin{aligned} a_0 = \quad & y_n + \frac{x_n^4}{24} f_n + \frac{x_n^8}{40320h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) + \\ & \frac{x_n^3 h}{262880}(19151f_n + 73967f_{n+1} + 12610f_{n+2} + 14878f_{n+3} - 5549f_{n+4} + 883f_{n+5}) + \\ & \frac{x_n h^3}{100800}(937f_n + 18240f_{n+1} + 5990f_{n+2} + 140f_{n+3} - 135f_{n+4} + 28f_{n+5}) + \frac{x_n}{6h}(11y_n \end{aligned}$$

$$-18y_{n+1}+9y_{n+2}-2y_{n+3})+\frac{x_n^5}{7200h}(137f_n-300f_{n+1}+300f_{n+2}-200f_{n+3}$$

$$+75f_{n+4}-12f_{n+5})+\frac{x_n^2h^2}{120960}(4233f_n+43274f_{n+1}+5662f_{n+2}+3432f_{n+3}$$

$$-1391f_{n+4}+230f_{n+5})+\frac{x_n^3}{6h^3}(y_n-3y_{n+1}+3y_{n+2}-y_{n+3})+\frac{x_n^7}{20160h^3}(17f_n$$

$$-71f_{n+1}+118f_{n+2}-98f_{n+3}+41f_{n+4}-7f_{n+5})+\frac{x_n^2}{2h^2}(2y_n-5y_{n+1}+4y_{n+2}$$

$$-y_{n+3})+\frac{x_n^6}{8640h^2}(45f_n-154f_{n+1}+214f_{n+2}-156f_{n+3}+61f_{n+4}-10f_{n+5})$$

$$+\frac{x_n^9}{362880h^5}(f_n-5f_{n+1}+10f_{n+2}-10f_{n+3}+5f_{n+4}-f_{n+5}).$$

$$a_1=\frac{x_n^2h}{120960}(19151f_n+73967f_{n+1}+12610f_{n+2}+14878f_{n+3}-5549f_{n+4}+883f_{n+5})$$

$$-\frac{x_n^4}{1440h}(137f_n-300f_{n+1}+300f_{n+2}-200f_{n+3}+75f_{n+4}-12f_{n+5})-\frac{x_n^5}{1440h^2}(45f_n$$

$$-154f_{n+1}+214f_{n+2}-156f_{n+3}+61f_{n+4}-10f_{n+5})+\frac{x_n^6}{2880h^3}(17f_n-71f_{n+1}+118f_{n+2}$$

$$-98f_{n+3}+41f_{n+4}-7f_{n+5})-\frac{x_n^7}{5040h^4}(3f_n-14f_{n+1}+26f_{n+2}-24f_{n+3}+11f_{n+4}-2f_{n+5})$$

$$-\frac{x_n^8}{40320h^5}(f_n-5f_{n+1}+10f_{n+2}-10f_{n+3}+5f_{n+4}-f_{n+5})-\frac{h^3}{100800}(937f_n+18240f_{n+1}$$

$$+5990f_{n+2}+140f_{n+3}-135f_{n+4}+28f_{n+5})-\frac{x_n^2h^2}{60480}(4233f_n+43274f_{n+1}+5662f_{n+2}$$

$$+3432f_{n+3}-1391f_{n+4}+230f_{n+5})-\frac{1}{6h}(11y_n-18y_{n+1}+9y_{n+2}-2y_{n+3})-\frac{x_n}{2h^2}(2y_n$$

$$-5y_{n+1}+4y_{n+2}-y_{n+3})+\frac{x_n^2}{h^3}(y_n-3y_{n+1}+3y_{n+2}-y_{n+3}).$$

$$a_2=\frac{1}{2h^2}(2y_n-5y_{n+1}+4y_{n+2}-y_{n+3})+\frac{x_n^4}{576h^2}(45f_n-154f_{n+1}+214f_{n+2}-$$

$$156f_{n+3}+61f_{n+4}-10f_{n+5})+\frac{h^2}{120960}(4233f_n+43274f_{n+1}+5662f_{n+2}+3432f_{n+3}$$

$$-1391f_{n+4}+230f_{n+5})+\frac{x_n^2}{4}f_n+\frac{x_n^6}{1440h^4}(3f_n-14f_{n+1}+26f_{n+2}-24f_{n+3}+11f_{n+4}-$$

$$2f_{n+5})+\frac{x_n^3}{720h}(137f_n-300f_{n+1}+300f_{n+2}-200f_{n+3}+75f_{n+4}-12f_{n+5})+\frac{x_n}{2h^3}(y_n$$

$$-3y_{n+1}+3y_{n+2}-y_{n+3}).+\frac{x_n^5}{960h^3}(17f_n-71f_{n+1}+118f_{n+2}-98f_{n+3}+41f_{n+4}-$$

$$7f_{n+5}) + \frac{x_n^7}{10080h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5}) + \frac{x_n h}{120960}(19151f_n$$
$$+73967f_{n+1} + 12610f_{n+2} + 14878f_{n+3} - 5549f_{n+4} + 883f_{n+5})$$

$$a_3 = -\frac{h}{362880}(19151f_n + 73967f_{n+1} + 12610f_{n+2} + 14878f_{n+3} - 5549f_{n+4} +$$
$$883f_{n+5}) - \frac{x_n}{6}f_n - \frac{1}{2h^3}(y_n - 3y_{n+1} + 3y_{n+2} - y_{n+3}). + \frac{x_n^4}{576h^3}(17f_n - 71f_{n+1} +$$
$$118f_{n+2} - 98f_{n+3} + 41f_{n+4} - 7f_{n+5}) - \frac{x_n^5}{720h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} +$$
$$11f_{n+4} - 2f_{n+5}) - \frac{x_n^3}{432h^2}(45f_n - 154f_{n+1} + 214f_{n+2} - 156f_{n+3} + 61f_{n+4} - 10f_{n+5})$$
$$-\frac{x_n^2}{720h}(137f_n - 300f_{n+1} + 300f_{n+2} - 200f_{n+3} + 75f_{n+4} - 12f_{n+5}) - \frac{x_n^6}{4320h^5}(f_n$$
$$-5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5}).$$

$$a_4 = \frac{1}{24}f_n + \frac{x_n^4}{576h^4}(3f_n - 14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) + \frac{x_n^3}{576h^3}(17f_n$$
$$-71f_{n+1} + 118f_{n+2} - 98f_{n+3} + 41f_{n+4} - 7f_{n+5}) + \frac{x_n^2}{576h^2}(45f_n - 154f_{n+1} + 214f_{n+2}$$
$$-156f_{n+3} + 61f_{n+4} - 10f_{n+5}) + \frac{x_n}{1440h}(137f_n - 300f_{n+1} + 300f_{n+2} - 200f_{n+3} +$$
$$75f_{n+4} - 12f_{n+5}) + \frac{x_n^5}{2880h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5})$$

$$a_5 = -\frac{1}{7200h}(137f_n - 300f_{n+1} + 300f_{n+2} - 200f_{n+3} + 75f_{n+4} - 12f_{n+5}) - \frac{x_n^3}{720h^4}(3f_n$$
$$-14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) - \frac{x_n^2}{960h^3}(17f_n - 71f_{n+1} + 118f_{n+2} -$$
$$98f_{n+3} + 41f_{n+4} - 7f_{n+5}) - \frac{x_n}{1440h^2}(45f_n - 154f_{n+1} + 214f_{n+2} - 156f_{n+3} + 61f_{n+4} -$$
$$10f_{n+5}) - \frac{x_n^4}{2880h^5}(f_n - 5f_{n+1} + 10f_{n+2} - 10f_{n+3} + 5f_{n+4} - f_{n+5})$$

$$a_6 = \frac{1}{8640h^2}(45f_n - 154f_{n+1} + 214f_{n+2} - 156f_{n+3} + 61f_{n+4} - 10f_{n+5}) + \frac{x_n^2}{1440h^4}(3f_n$$
$$-14f_{n+1} + 26f_{n+2} - 24f_{n+3} + 11f_{n+4} - 2f_{n+5}) + \frac{x_n}{2880h^3}(17f_n - 71f_{n+1} + 118f_{n+2} -$$

$$98f_{n+3}+41f_{n+4}-7f_{n+5})+\frac{x_n^3}{4320h^5}(f_n-5f_{n+1}+10f_{n+2}-10f_{n+3}+5f_{n+4}-f_{n+5}).$$

$$a_7 = -\frac{1}{20160h^3}(17f_n-71f_{n+1}+118f_{n+2}-98f_{n+3}+41f_{n+4}-7f_{n+5})-\frac{x_n}{5040h^4}(3f_n -14f_{n+1}+26f_{n+2}-24f_{n+3}+11f_{n+4}-2f_{n+5})-\frac{x_n^2}{10080h^5}(f_n-5f_{n+1}+10f_{n+2}-10f_{n+3} +5f_{n+4}-f_{n+5}).$$

$$a_8 = \frac{1}{40320h^4}(3f_n-14f_{n+1}+26f_{n+2}-24f_{n+3}+11f_{n+4}-2f_{n+5})+\frac{x_n}{10080h^5}(f_n-5f_{n+1} +10f_{n+2}-10f_{n+3}+5f_{n+4}-f_{n+5}).$$

$$a_9 = \frac{1}{362880h^5}(f_n-5f_{n+1}+10f_{n+2}-10f_{n+3}+5f_{n+4}-f_{n+5}).$$

The values of *a's* are substituted into equation (5.2.1.1) and simplified, this gives a continuous linear multistep method of the form:

$$y(x)=\sum_{j=0}^{k-2}\alpha_j(x)y_{n+j}+h^4\sum_{j=0}^{k}\beta_j(x)f_{n+j} \qquad (5.2.1.8)$$

where $x = zh + x_n + 4h$ (5.2.1.9)

Substituting (5.2.1.9) into (5.2.1.8) and simplifying, we have

$$\alpha_0(z) = -1-\frac{11z}{6}-z^2-\frac{z^3}{6}$$

$$\alpha_1(z) = 4+7z+\frac{7z^2}{2}+\frac{z^3}{2}$$

$$\alpha_2(z) = -6-\frac{19z}{2}-4z^2-\frac{z^3}{2}$$

$$\alpha_3(z) = 4+\frac{13z}{2}+\frac{3z^2}{2}+\frac{z^3}{6}$$

$$\beta_0(z) = \frac{1}{1814400}(-2520-4026z-2685z^2-1675z^3+756z^5+210z^6 -90z^7-45z^8-5z^9)$$

$$\beta_1(z) = \frac{1}{1814400}(312480+567840z+312210z^2+60245z^3-5040z^5 -1260z^6+630z^7+270z^8+59z^9)$$

$$\beta_2(z)=\frac{1}{362880}(238896+461556z+269682z^2+44866z^3+3024z^5$$
$$+588z^6-396z^7-126z^8-10z^9).$$
$$\beta_3(z)=\frac{1}{362880}(62496+178728z+181968z^2+71138z^3+6048z^5$$
$$-168z^6+612z^7+144z^8+10z^9). \qquad (5.2.1.10)$$
$$\beta_4(z)=\frac{1}{362880}(-504+3054z+16335z^2+24365z^3+15120z^4+3276z^5$$
$$-630z^6-450z^7-81z^8-5z^9).$$
$$\beta_5(z)=\frac{1}{1814400}(-504z+3450z^2-4415z^3+3024z^5+2100z^6+630z^7$$
$$+90z^8+5z^9).$$

Evaluating (5.2.1.10) at the non-interpolating points .i.e, at z= 0 and 1 yields

$$720y_{n+4}-2880y_{n+3}+4320y_{n+2}-2880y_{n+1}+720y_n=h^4(-f_{n+4}+124f_{n+3}+474f_{n+2}$$
$$+124f_{n+1}-f_n). \qquad (5.2.1.11)$$

$$720y_{n+5}-7200y_{n+3}+14400y_{n+2}-10800y_{n+1}+5040y_n=h^4(-f_{n+5}+120f_{n+4}$$
$$+970f_{n+3}+2020f_{n+2}+495f_{n+1}+4f_n). \qquad (5.2.1.12)$$

The first derivative of (5.2.1.10) gives

$$\alpha_0'(z)=\frac{-11}{6}-2z-\frac{z^2}{2}$$
$$\alpha_1'(z)=7+7z+\frac{3z^2}{2}$$
$$\alpha_2'(z)=\frac{-19}{2}-8z-\frac{3z^2}{2}$$
$$\alpha_3'(z)=\frac{13}{3}+3z+\frac{3z^2}{6}$$
$$\beta_0'(z)=\frac{1}{604800}(-1342-1790z-1675z^2+1260z^4+420z^5-210z^6$$
$$-120z^7-15z^8).$$
$$\beta_1'(z)=\frac{1}{604800}(189280+208140z+60245z^2-8400z^4-2520z^5+$$
$$1470z^6+720z^7+75z^8).$$
$$\beta_2'(z)=\frac{1}{604800}(769260+898940z+224330z^2+25200z^4+5880z^5$$
$$-4620z^6-1680z^7-150z^8).$$

$$\beta_3'(z) = \frac{1}{604800}(297880 + 606560z + 355960z^2 - 50400z^4 - 1680z^5$$
$$+7140z^6 + 1920z^7 + 150z^8).$$
$$\beta_4'(z) = \frac{1}{604800}(5090 + 54450z + 121825z^2 + 100800z^3 + 27300z^4 \tag{5.2.1.13}$$
$$-6300z^5 - 5250z^6 - 1080z^7 - 75z^8).$$
$$\beta_5'(z) = \frac{1}{604800}(-168 - 2300z - 4415z^2 + 5040z^4 + 4200z^5 + 1470z^6$$
$$+240z^7 + 15z^8).$$

Evaluating (5.2.1.13) at all the grid points. That is, at z=-4, -3, -2,-1, 0 and1 gives

$$100800hy_n' - 33600y_{n+3} + 151200y_{n+2} - 302400y_{n+1} + 184800y_n = h^4(-28f_{n+5} + 135f_{n+4} - 140f_{n+3} - 5990f_{n+2} - 18240f_{n+1} - 937f_n). \tag{5.2.1.14}$$

$$302400hy_{n+1}' + 50400y_{n+3} - 302400y_{n+2} + 151200y_{n+1} + 100800y = h^4(-99f_{n+5} + 700f_{n+4} - 2290f_{n+3} + 12900f_{n+2} + 14045f_{n+1} - 56f_n). \tag{5.2.1.15}$$

$$302400hy_{n+2}' - 100800y_{n+3} - 151200y_{n+2} + 302400y_{n+1} - 50400y_n = h^4(106f_{n+5} - 735f_{n+4} + 2440f_{n+3} - 17650f_{n+2} - 9330f_{n+1} + 169f_n). \tag{5.2.1.16}$$

$$100800hy_{n+3}' - 184800y_{n+3} + 302400y_{n+2} - 151200y_{n+1} + 33600y_n = h^4(-33f_{n+5} + 170f_{n+4} + 610f_{n+3} + 18480f_{n+2} - 82f_{n+1} + 6055f_n). \tag{5.2.1.17}$$

$$302400hy_{n+4}' - 1310400y_{n+3} + 2872800y_{n+2} - 2116800y_{n+1} + 554400y_n = h^4(-84f_{n+5} + 2545f_{n+4} + 148940f_{n+3} + 384630f_{n+2} + 94640f_{n+1} - 671f_n). \tag{5.2.1.18}$$

$$302400hy_{n+5}' - 2368800y_{n+3} + 5745600y_{n+2} - 4687200y_{n+1} + 1310400y_n = h^4(2041f_{n+5} + 148380f_{n+4} + 608630f_{n+3} + 958580f_{n+2} + 224505f_{n+1} - 1736f_n). \tag{5.2.1.19}$$

The second derivative of (5.2.1.10) gives

$$\alpha_0''(z) = -2 - z$$
$$\alpha_1''(z) = 7 + 3z$$
$$\alpha_2''(z) = -8 - 3z$$
$$\alpha_3''(z) = 3 + z$$
$$\beta_0''(z) = \frac{1}{60480}(-179 - 335z + 504z^3 + 210z^4 - 126z^5 - 84z^6$$
$$-12z^7).$$
$$\beta_1''(z) = \frac{1}{60480}(20814 + 12049z - 3360z^3 - 1260z^4 + 882z^5$$
$$+504z^6 + 60z^7).$$

$$\beta_2''(z) = \frac{1}{60480}(89894 + 44866z + 10080z^3 + 2940z^4 - 2772z^5 - 1176z^6$$
$$-120z^7).$$
$$\beta_3''(z) = \frac{1}{60480}(60656 + 71138z - 20160z^3 - 840z^4 + 4284z^5 + 1344z^6$$
$$+120z^7). \tag{5.2.1.20}$$
$$\beta_4''(z) = \frac{1}{60480}(5445 + 24365z + 30240z^2 + 10920z^3 - 3150z^4 - 3150z^5$$
$$-756z^6 - 60z^7).$$
$$\beta_5''(z) = \frac{1}{60480}(-230 - 883z + 2016z^3 + 2100z^4 + 882z^5 + 165z^6 + 12z^7).$$

Evaluating (5.2.1.20) at all the grid points. i.e, at z=-4, -3, -2,-1, 0 and 1 produces

$$60480h^2 y_n'' + 60480y_{n+3} - 241920y_{n+2} + 30240y_{n+1} - 120960y_n = h^4(230f_{n+5} - 1391f_{n+4} + 3432f_{n+3} + 5662f_{n+2} + 43274f_{n+1} + 4233f_n). \tag{5.2.1.21}$$

$$60480h^2 y_{n+1}'' - 60480y_{n+2} + 120960y_{n+1} - 60480y_{n+1} = h^4(-11f_{n+5} + 66f_{n+4} - 154f_{n+3} + 8f_{n+2} - 4803f_{n+1} - 146f_n). \tag{5.2.1.22}$$

$$60480h^2 y_{n+2}'' - 60480y_{n+3} + 120960y_{n+2} - 60480y_{n+1} = h^4(11f_{n+4} - 212f_{n+3} - 4638f_{n+2} - 212f_{n+1} + 11f_n). \tag{5.2.1.23}$$

$$60480h^2 y_{n+3}'' - 120960y_{n+3} + 302400y_{n+2} - 241920y_{n+1} - 60480y_n = h^4(11f_{n+5} - 296f_{n+4} + 5778f_{n+3} + 39604f_{n+2} + 10427f_{n+1} - 84f_n). \tag{5.2.1.24}$$

$$60480h^2 y_{n+4}'' - 181440y_{n+3} + 483480y_{n+2} - 423360y_{n+1} + 120960y_n = h^4(-230f_{n+5} + 5445f_{n+4} + 60656f_{n+3} + 89894f_{n+2} + 20814f_{n+1} - 179f_n). \tag{5.2.1.25}$$

$$60480h^2 y_{n+5}'' - 241920y_{n+3} + 665280y_{n+2} - 604800y_{n+1} + 181440y_n = h^4(4065f_{n+5} + 63854f_{n+4} + 116542f_{n+3} + 143712f_{n+2} + 29689f_{n+1} - 22f_n). \tag{5.2.1.26}$$

The third derivative of (5.2.1.10) gives

$$\alpha_0'''(z) = -1$$
$$\alpha_1'''(z) = 3$$
$$\alpha_2'''(z) = -3$$
$$\alpha_3'''(z) = 1$$
$$\beta_0'''(z) = \frac{1}{60480}(-335 + 1512z^2 + 840z^3 - 630z^4 - 504z^5 - 84z^6)$$
$$\beta_1'''(z) = \frac{1}{60480}(12049 - 10080z^2 - 5040z^3 + 4410z^4 + 3024z^5 + 420z^6)$$

$$\beta_2'''(z) = \frac{1}{60480}(44866 + 30240z^2 + 11760z^3 - 13860z^4 - 7056z^5 - 840z^6)$$

$$\beta_3'''(z) = \frac{1}{60480}(71138 - 60480z^2 - 3360z^3 + 21420z^4 + 8064z^5 + 840z^6)$$

$$\beta_4'''(z) = \frac{1}{60480}(24365 + 60480z + 32760z^2 - 12600z^3 - 15750z^4 - 4536z^5 - 420z^6) \qquad (5.2.1.27)$$

$$\beta_5'''(z) = \frac{1}{60480}(-883 + 60480z^2 + 8400z^3 + 4410z^4 + 1008z^5 + 84z^6)$$

Equation (5.2.1.27) is evaluated at all the grid points. That is, at z= -4, -3, -2,-1, 0 and 1 gives

$$60480h^3 y_n''' - 60480y_{n+3} + 181440y_{n+2} - 181440y_{n+1} + 60480y_n = h^4(-883f_{n+5} + 5549f_{n+4} - 14878f_{n+3} + 12610f_{n+2} - 73967f_{n+1} - 19151f_n). \qquad (5.2.1.28)$$

$$60480h^3 y_{n+1}''' - 60480y_{n+3} + 181440y_{n+2} - 181440y_{n+1} + 60480y_n = h^4(251f_{n+5} - 1717f_{n+4} + 5366f_{n+3} - 20906f_{n+2} - 14033f_{n+1} + 799f_n). \qquad (5.2.1.29)$$

$$60480h^3 y_{n+2}''' - 60480y_{n+3} + 181440y_{n+2} - 181440y_{n+1} + 60480y_n = h^4(-211f_{n+5} + 1517f_{n+4} - 5470f_{n+3} + 22018f_{n+2} + 12721f_{n+1} - 335f_n). \qquad (5.2.1.30)$$

$$60480h^3 y_{n+3}''' - 60480y_{n+3} + 181440y_{n+2} - 181440y_{n+1} + 60480y_n = h^4(251f_{n+5} - 2389f_{n+4} + 28214f_{n+3} + 55702f_{n+2} + 8815f_{n+1} + 127f_n). \qquad (5.2.1.31)$$

$$60480h^3 y_{n+4}''' - 60480y_{n+3} + 181440y_{n+2} - 181440y_{n+1} + 60480y_n = h^4(-883f_{n+5} + 24365f_{n+4} + 71138f_{n+3} + 44866f_{n+2} + 12049f_{n+1} - 335f_n). \qquad (5.2.1.32)$$

$$60480h^3 y_{n+5}''' - 60480y_{n+3} + 181440y_{n+2} - 181440y_{n+1} + 60480y_n = h^4(19067f_{n+5} + 84299f_{n+4} + 37622f_{n+3} + 65110f_{n+2} + 4783f_{n+1} + 799f_n). \qquad (5.2.1.33)$$

Combining equations (5.2.1.11), (5.2.1.12), (5.2.1.14), (5.2.1.21) and (5.2.1.28) to form a block of the form (1.10).

$$\begin{pmatrix} -2880 & 4320 & -2880 & 720 & 0 \\ -10800 & 14400 & -7200 & 0 & 720 \\ -302400 & 151200 & -33600 & 0 & 0 \\ 302400 & -241920 & 60480 & 0 & 0 \\ -181440 & 181440 & -604800 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & -720 \\ 0 & 0 & 0 & 0 & -2880 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} +$$

$$h\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -100800 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-4}' \\ y_{n-3}' \\ y_{n-2}' \\ y_{n-1}' \\ y_n' \end{pmatrix} + h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -60480 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-4}'' \\ y_{n-3}'' \\ y_{n-2}'' \\ y_{n-1}'' \\ y_n'' \end{pmatrix} +$$

$$h^3\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-60480\end{pmatrix}\begin{pmatrix}y'''_{n-4}\\y'''_{n-3}\\y'''_{n-2}\\y'''_{n-1}\\y'''_{n}\end{pmatrix}+h^4\begin{pmatrix}124&474&124&-1&0\\495&2020&970&120&-1\\-18240&-5990&-140&135&-28\\43274&5662&3432&-1391&230\\-73967&12610&-14878&5549&-883\end{pmatrix}\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\end{pmatrix}$$

$$+h^4\begin{pmatrix}0&0&0&0&-1\\0&0&0&0&4\\0&0&0&0&-937\\0&0&0&0&4233\\0&0&0&0&-19151\end{pmatrix}\begin{pmatrix}f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_{n}\end{pmatrix}$$

The above equation is multiplied by the inverse of A^0

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\end{pmatrix}=\begin{pmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}y_{n-4}\\y_{n-3}\\y_{n-2}\\y_{n-1}\\y_{n}\end{pmatrix}+h\begin{pmatrix}0&0&0&0&1\\0&0&0&0&2\\0&0&0&0&3\\0&0&0&0&4\\0&0&0&0&5\end{pmatrix}\begin{pmatrix}y'_{n-4}\\y'_{n-3}\\y'_{n-2}\\y'_{n-1}\\y'_{n}\end{pmatrix}+$$

$$h^2\begin{pmatrix}0&0&0&0&\frac{1}{2}\\0&0&0&0&2\\0&0&0&0&\frac{9}{2}\\0&0&0&0&8\\0&0&0&0&\frac{25}{2}\end{pmatrix}\begin{pmatrix}y''_{n-4}\\y''_{n-3}\\y''_{n-2}\\y''_{n-1}\\y''_{n}\end{pmatrix}+h^3\begin{pmatrix}0&0&0&0&\frac{1}{6}\\0&0&0&0&\frac{4}{3}\\0&0&0&0&\frac{9}{2}\\0&0&0&0&\frac{32}{3}\\0&0&0&0&\frac{125}{6}\end{pmatrix}\begin{pmatrix}y'''_{n-4}\\y'''_{n-3}\\y'''_{n-2}\\y'''_{n-1}\\y'''_{n}\end{pmatrix}+$$

$$h^4\begin{pmatrix}\frac{185}{6844}&\frac{-106}{4789}&\frac{43}{3068}&\frac{-71}{13837}&\frac{8}{9881}\\\frac{1592}{2835}&\frac{-982}{2835}&\frac{88}{405}&\frac{-32}{405}&\frac{176}{14175}\\\frac{1809}{640}&\frac{-189}{160}&\frac{730}{853}&\frac{-351}{1120}&\frac{413}{8357}\\\frac{3328}{405}&\frac{-5888}{2835}&\frac{3355}{1429}&\frac{-795}{992}&\frac{256}{2025}\\\frac{3553}{196}&\frac{-856}{355}&\frac{1529}{265}&\frac{-691}{472}&\frac{242}{943}\end{pmatrix}\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\end{pmatrix}+h^4\begin{pmatrix}0&0&0&0&\frac{256}{9455}\\0&0&0&0&\frac{1409}{4684}\\0&0&0&0&\frac{1593}{1400}\\0&0&0&0&\frac{1889}{662}\\0&0&0&0&\frac{3233}{561}\end{pmatrix}\begin{pmatrix}f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_{n}\end{pmatrix} \quad (5.2.1.34)$$

which leads to

$$y_{n+1}=y_n+hy'_n+\frac{1}{2}h^2y''_n+\frac{1}{6}h^3y'''_n+\frac{h^4}{5443200}(4407f_{n+5}-27930f_{n+4}+76290f_{n+3} \quad (5.2.1.35)$$
$$-120480f_{n+2}+147135f_{n+1}+147378f_n).$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2 y''_n + \frac{4}{3}h^3 y'''_n + \frac{h^4}{907200}(11264 f_{n+5} - 71680 f_{n+4} + 197120 f_{n+3} - 314240 f_{n+2} + 509440 f_{n+1} + 272896 f_n). \quad (5.2.1.36)$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2}h^2 y''_n + \frac{9}{2}h^3 y'''_n + \frac{h^4}{67200}(3321 f_{n+5} - 21060 f_{n+4} + 57510 f_{n+3} - 79380 f_{n+2} + 189945 f_{n+1} + 76464 f_n). \quad (5.2.1.37)$$

$$y_{n+4} = y_n + 4hy'_n + 8h^2 y''_n + \frac{32}{3}h^3 y'''_n + \frac{h^4}{1814400}(229376 f_{n+5} - 1454080 f_{n+4} + 4259840 f_{n+3} - 3768320 f_{n+2} + 14909440 f_{n+1} + 5177344 f_n). \quad (5.2.1.38)$$

$$y_{n+5} = y_n + 5hy'_m + \frac{25}{2}h^2 y''_n + \frac{125}{6}h^3 y'''_n + \frac{h^4}{145152}(37250 f_{n+5} - 212500 f_{n+4} + 837500 f_{n+3} - 350000 f_{n+2} + 2631250 f_{n+1} + 836500 f_n). \quad (5.2.1.39)$$

Substituting (5.2.1.35) - (5.2.1.37) into (5.2.1.15) – (5.2.1.19) to give the first derivative of the block

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2 y'''_n + \frac{h^3}{241920}(834 f_{n+5} - 5298 f_{n+4} + 14532 f_{n+3} - 23172 f_{n+2} + 29850 f_{n+1} + 23574 f_n). \quad (5.2.1.40)$$

$$y'_{n+2} = y'_n + 2hy''_n + 2h^2 y'''_n + \frac{h^3}{1890}(42 f_{n+5} - 267 f_{n+4} + 732 f_{n+3} - 1140 f_{n+2} + 2202 f_{n+1} + 951 f_n). \quad (5.2.1.41)$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{9}{2}h^2 y'''_n + \frac{h^3}{13440}(729 f_{n+5} - 4617 f_{n+4} + 12690 f_{n+3} - 13122 f_{n+2} + 48357 f_{n+1} + 16443 f_n). \quad (5.2.1.42)$$

$$y'_{n+4} = y'_n + 4hy''_n + 8h^2 y'''_n + \frac{h^3}{945}(96 f_{n+5} - 601 f_{n+4} + 2112 f_{n+3} - 672 f_{n+2} + 7008 f_{n+1} + 2136 f_n). \quad (5.2.1.43)$$

$$y'_{n+5} = y'_n + 5hy''_n + \frac{25}{2}h^2 y'''_n + \frac{h^3}{24192}(4125 f_{n+5} - 13125 f_{n+4} + 116250 f_{n+3} + 3750 f_{n+2} + 305625 f_{n+1} + 87375 f_n). \quad (5.2.1.44)$$

Substituting (5.2.1.35) - (5.2.1.37) into (5.2.1.22) – (5.2.1.26) to give the second derivative of the block

$$y''_{n+1} = y''_n + hy'''_n + \frac{h^2}{20160}(214 f_{n+5} - 1364 f_{n+4} + 3764 f_{n+3} - 6088 f_{n+2} + 8630 f_{n+1} + 4924 f_n). \quad (5.2.1.45)$$

$$y''_{n+2} = y''_n + 2hy'''_n + \frac{h^2}{630}(16 f_{n+5} - 101 f_{n+4} + 272 f_{n+3} - 370 f_{n+2} + 1088 f_{n+1} + 355 f_n) \quad (5.2.1.46)$$

$$y''_{n+3} = y''_n + 3hy'''_n + \frac{h^2}{10080}(405 f_{n+5} - 2592 f_{n+4} + 7830 f_{n+3} - 648 f_{n+2} + 31509 f_{n+1} + 8856 f_n). \tag{5.2.1.47}$$

$$y''_{n+4} = y''_n + 4hy'''_n + \frac{h^2}{630}(32 f_{n+5} - 160 f_{n+4} + 1216 f_{n+3} + 352 f_{n+2} + 2848 f_{n+1} + 752 f_n). \tag{5.2.1.48}$$

$$y''_{n+5} = y''_n + 5hy'''_n + \frac{h^2}{10080}(1375 f_{n+5} + 6250 f_{n+4} + 31250 f_{n+3} + 12500 f_{n+2} + 59375 f_{n+1} + 1525 f_n). \tag{5.2.1.49}$$

Substituting (5.2.1.35) - (5.2.1.37) into (5.2.1.29) – (5.2.1.33) to give the third derivative of the block

$$y'''_{n+1} = y'''_n + \frac{h}{5760}(108 f_{n+5} - 692 f_{n+4} + 1928 f_{n+3} - 3192 f_{n+2} + 5708 f_{n+1} + 1900 f_n). \tag{5.2.1.50}$$

$$y'''_{n+2} = y'''_n + \frac{h}{360}(4 f_{n+5} - 24 f_{n+4} + 56 f_{n+3} + 56 f_{n+2} + 516 f_{n+1} + 112 f_n) \tag{5.2.1.51}$$

$$y'''_{n+3} = y'''_n = y'_n + \frac{h}{640}(12 f_{n+5} - 84 f_{n+4} + 456 f_{n+3} + 456 f_{n+2} + 876 f_{n+1} + 204 f_n) \tag{5.2.1.52}$$

$$y'''_{n+4} = y'''_n + \frac{h}{180}(56 f_{n+4} + 256 f_{n+3} + 96 f_{n+2} + 256 f_{n+1} + 56 f_n) \tag{5.2.1.53}$$

$$y'''_{n+5} = y''_n + \frac{h}{1728}(570 f_{n+5} + 2250 f_{n+4} + 1500 f_{n+3} + 1500 f_{n+2} + 2250 f_{n+1} + 570 f_n). \tag{5.2.1.54}$$

5.2.2 Properties of Five–Step Block Method for Fourth Order ODEs.

This section considers the order, zero-stability and region of absolute stability of five–step block method for fourth order ODEs.

5.2.2.1 Order of Five–Step Block Method for Fourth Order ODEs.

In finding the order of the block method (5.2.1.35 – 5.2.1.39), the same strategy mentioned in section 3.2.2.1 is used as displayed below

$$
\begin{pmatrix}
\sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{h^m}{m!}y_n^{(m)}-\frac{147378}{5443200}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(5443200)(m!)}y_n^{(4+m)}\begin{pmatrix}147135(1)^m-120480(2)^m\\+76290(3)^m-27930(4)^m\\+4407(5)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{(2h)^m}{m!}y_n^{(m)}-\frac{272896}{907200}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(907200)(m!)}y_n^{(4+m)}\begin{pmatrix}509440(1)^m-314240(2)^m\\+197120(3)^m-71680(4)^m\\+11264(5)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^{m}-\sum_{m=0}^{3}\frac{(3h)^m}{m!}y_n^{(m)}-\frac{76464}{67200}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(67200)(m!)}y_n^{(4+m)}\begin{pmatrix}189945(1)^m-79380(2)^m\\+57510(3)^m-21060(4)^m\\+3321(5)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^{m}-\sum_{m=0}^{3}\frac{(4h)^m}{m!}y_n^{(m)}-\frac{5177344}{1814400}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(1814400)(m!)}y_n^{(4+m)}\begin{pmatrix}14909440(1)^m-3768320(2)^m\\+4259840(3)^m-1454080(4)^m\\+229376(5)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^{m}-\sum_{m=0}^{3}\frac{(5h)^m}{m!}y_n^{(m)}-\frac{836500}{145152}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(145152)(m!)}y_n^{(4+m)}\begin{pmatrix}2631250(1)^m-350000(2)^m\\+837500(3)^m-212500(4)^m\\+37250(5)^m\end{pmatrix}
\end{pmatrix}
=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}
$$

The coefficients of h^m and y_n^m are compared and this give

$$
C_0=\begin{pmatrix}1-1\\1-1\\1-1\\1-1\\1-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}
$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \\ 5-5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{1}{2!} \\ \frac{(2)^2}{2!} - \frac{(2)^2}{2!} \\ \frac{(3)^2}{2!} - \frac{(3)^2}{2!} \\ \frac{(4)^2}{2!} - \frac{(4)^2}{2!} \\ \frac{(5)^2}{2!} - \frac{(5)^2}{2!} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} \frac{1}{3!} - \frac{1}{3!} \\ \frac{(2)^3}{3!} - \frac{(2)^3}{3!} \\ \frac{(3)^3}{3!} - \frac{(3)^3}{3!} \\ \frac{(4)^3}{3!} - \frac{(4)^3}{3!} \\ \frac{(5)^3}{3!} - \frac{(5)^3}{3!} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_4 = \begin{pmatrix} \frac{1}{4!} - \frac{147378}{5443200} - \frac{1}{(5443200)(0!)} \begin{pmatrix} 147135(1)^0 - 120480(2)^0 + 76290(3)^0 - 27930(4)^0 + \\ 4407(5)^0 \end{pmatrix} \\ \frac{(2)^4}{4!} - \frac{272896}{907200} - \frac{1}{(907200)(0!)} \begin{pmatrix} 509440(1)^0 - 314240(2)^0 + 197120(3)^0 - 71680(4)^0 \\ + 11264(5)^0 \end{pmatrix} \\ \frac{(3)^4}{4!} - \frac{76464}{67200} - \frac{1}{(67200)(0!)} \left(189945(1)^0 - 79380(2)^0 + 57510(3)^0 - 21060(4)^0 + 3321(5)^0\right) \\ \frac{(4)^4}{4!} - \frac{5177344}{1814400} - \frac{1}{(1814400)(0!)} \begin{pmatrix} 14909440(1)^0 - 3768320(2)^0 + 4259840(3)^0 - 1454080(4)^0 \\ + 229376(5)^0 \end{pmatrix} \\ \frac{(5)^4}{4!} - \frac{836500}{145152} - \frac{1}{(145152)(0!)} \begin{pmatrix} 2631250(1)^0 - 350000(2)^0 + 837500(3)^0 - 212500(4)^0 \\ + 37250(5)^0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_5=\begin{pmatrix} \frac{1}{5!}-\frac{1}{(5443200)(1!)}\begin{pmatrix}147135(1)^1-120480(2)^1+76290(3)^1-27930(4)^1+\\4407(5)^1\end{pmatrix}\\ \frac{(2)^5}{5!}-\frac{1}{(907200)(1!)}\begin{pmatrix}509440(1)^1-314240(2)^1+197120(3)^1-71680(4)^1\\+11264(5)^1\end{pmatrix}\\ \frac{(3)^5}{5!}-\frac{1}{(67200)(1!)}\left(189945(1)^1-79380(2)^1+57510(3)^1-21060(4)^1+3321(5)^1\right)\\ \frac{(4)^5}{5!}-\frac{1}{(1814400)(1!)}\begin{pmatrix}14909440(1)^1-3768320(2)^1+4259840(3)^1-1454080(4)^1\\+229376(5)^1\end{pmatrix}\\ \frac{(5)^5}{5!}-\frac{1}{(145152)(1!)}\begin{pmatrix}2631250(1)^1-350000(2)^1+837500(3)^1-212500(4)^1\\+37250(5)^1\end{pmatrix}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$C_6=\begin{pmatrix} \frac{1}{6!}-\frac{1}{(5443200)(2!)}\begin{pmatrix}147135(1)^2-120480(2)^2+76290(3)^2-27930(4)^2+\\4407(5)^2\end{pmatrix}\\ \frac{(2)^6}{6!}-\frac{1}{(907200)(2!)}\begin{pmatrix}509440(1)^2-314240(2)^2+197120(3)^2-71680(4)^2\\+11264(5)^2\end{pmatrix}\\ \frac{(3)^6}{6!}-\frac{1}{(67200)(2!)}\left(189945(1)^2-79380(2)^2+57510(3)^2-21060(4)^2+3321(5)^2\right)\\ \frac{(4)^6}{6!}-\frac{1}{(1814400)(2!)}\begin{pmatrix}14909440(1)^2-3768320(2)^2+4259840(3)^2-1454080(4)^2\\+229376(5)^2\end{pmatrix}\\ \frac{(5)^6}{6!}-\frac{1}{(145152)(2!)}\begin{pmatrix}2631250(1)^2-350000(2)^2+837500(3)^2-212500(4)^2\\+37250(5)^2\end{pmatrix}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$C_7=\begin{pmatrix} \frac{1}{7!}-\frac{1}{(5443200)(3!)}\begin{pmatrix}147135(1)^3-120480(2)^3+76290(3)^3-27930(4)^3+\\4407(5)^3\end{pmatrix}\\ \frac{(2)^7}{7!}-\frac{1}{(907200)(3!)}\begin{pmatrix}509440(1)^3-314240(2)^3+197120(3)^3-71680(4)^3\\+11264(5)^3\end{pmatrix}\\ \frac{(3)^7}{7!}-\frac{1}{(67200)(3!)}\left(189945(1)^3-79380(2)^3+57510(3)^3-21060(4)^3+3321(5)^3\right)\\ \frac{(4)^7}{7!}-\frac{1}{(1814400)(3!)}\begin{pmatrix}14909440(1)^3-3768320(2)^3+4259840(3)^3-1454080(4)^3\\+229376(5)^3\end{pmatrix}\\ \frac{(5)^7}{7!}-\frac{1}{(145152)(3!)}\begin{pmatrix}2631250(1)^3-350000(2)^3+837500(3)^3-212500(4)^3\\+37250(5)^3\end{pmatrix}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$C_8 = \begin{pmatrix} \frac{1}{8!} - \frac{1}{(5443200)(4!)}\begin{pmatrix} 147135(1)^4 - 120480(2)^4 + 76290(3)^4 - 27930(4)^4 + \\ 4407(5)^4 \end{pmatrix} \\ \frac{(2)^8}{8!} - \frac{1}{(907200)(4!)}\begin{pmatrix} 509440(1)^4 - 314240(2)^4 + 197120(3)^4 - 71680(4)^4 \\ +11264(5)^4 \end{pmatrix} \\ \frac{(3)^8}{8!} - \frac{1}{(67200)(4!)}\left(189945(1)^4 - 79380(2)^4 + 57510(3)^4 - 21060(4)^4 + 3321(5)^4\right) \\ \frac{(4)^8}{8!} - \frac{1}{(1814400)(4!)}\begin{pmatrix} 14909440(1)^4 - 3768320(2)^4 + 4259840(3)^4 - 1454080(4)^4 \\ +229376(5)^4 \end{pmatrix} \\ \frac{(5)^8}{8!} - \frac{1}{(145152)(4!)}\begin{pmatrix} 2631250(1)^4 - 350000(2)^4 + 837500(3)^4 - 212500(4)^4 \\ +37250(5)^4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_9 = \begin{pmatrix} \frac{1}{9!} - \frac{1}{(5443200)(5!)}\begin{pmatrix} 147135(1)^5 - 120480(2)^5 + 76290(3)^5 - 27930(4) \ + \\ 4407(5)^5 \end{pmatrix} \\ \frac{(2)^9}{9!} - \frac{1}{(907200)(5!)}\begin{pmatrix} 509440(1)^5 - 314240(2)^5 + 197120(3)^5 - 71680(4)^5 \\ +11264(5)^5 \end{pmatrix} \\ \frac{(3)^9}{9!} - \frac{1}{(67200)(5!)}\left(189945(1)^5 - 79380(2)^5 + 57510(3)^5 - 21060(4)^5 + 3321(5)^5\right) \\ \frac{(4)^9}{9!} - \frac{1}{(1814400)(5!)}\begin{pmatrix} 14909440(1)^5 - 3768320(2)^5 + 4259840(3)^5 - 1454080(4)^5 \\ +229376(5)^5 \end{pmatrix} \\ \frac{(5)^9}{9!} - \frac{1}{(145152(5!)}\begin{pmatrix} 2631250(1)^5 - 350000(2)^5 + 837500(3)^5 - 212500(4)^5 \\ +37250(5)^5 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_{10} = \begin{pmatrix} \frac{1}{10!} - \frac{1}{(5443200)(6!)}\begin{pmatrix} 147135(1)^6 - 120480(2)^6 + 76290(3)^6 - 27930(4)^6 + \\ 4407(5)^0 \end{pmatrix} \\ \frac{(2)^{10}}{10!} - \frac{1}{(907200)(6!)}\begin{pmatrix} 509440(1)^6 - 314240(2)^6 + 197120(3)^6 - 71680(4)^6 \\ +11264(5)^6 \end{pmatrix} \\ \frac{(3)^{10}}{10!} - \frac{1}{(67200)(6!)}\left(189945(1)^6 - 79380(2)^6 + 57510(3)^6 - 21060(4)^6 + 3321(5)^6\right) \\ \frac{(4)^{10}}{10!} - \frac{1}{(1814400)(6!)}\begin{pmatrix} 14909440(1)^6 - 3768320(2)^6 + 4259840(3)^6 - 1454080(4)^6 \\ +229376(5)^6 \end{pmatrix} \\ \frac{(5)^{10}}{10!} - \frac{1}{(145152)(6!)}\begin{pmatrix} 2631250(1)^6 - 350000(2)^6 + 837500(3)^6 - 212500(4)^6 \\ +37250(5)^6 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{-17}{26556} \\ \frac{-137}{14175} \\ \frac{-571}{14727} \\ \frac{-415}{4178} \\ \frac{-358}{1769} \end{pmatrix}$$

Hence, the block is having an order $(6,6,6,6,6)^T$ with error constants

$$\left(\frac{-17}{26556}, \frac{-137}{14175}, \frac{-571}{14727}, \frac{-415}{4178}, \frac{-358}{1769}\right)^T$$

5.2.2.2 Zero Stability of Five–Step Block Method for Fourth Order ODEs.

Applying the equation (3.2.2.2.1) to five-step block method (5.2.1.35 – 5.2.1.39) we have

$$\det[rA^{(0)} - A^{(1)}] = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0$$

This implies $r = 0,0,0,0,1$. Hence, the method is zero stable.

5.2.2.3 Consistency and Convergence of Five–Step Block Method for Fourth Order ODEs.

The conditions stated in Definition 1.4 are satisfied on block method (5.2.1.35 – 5.2.1.39) and this makes the method to be consistent. Hence, since the method is zero-stable and consistent, it is therefore convergent.

5.2.2.4 Region of Absolute Stability of Five–Step Block Method for Fourth Order ODEs.

Equation (3.2.2.4.2) is applied to five-step block method (5.2.1.35 – 5.2.1.39), we have

$$\bar{h}(\theta,h)=\frac{\begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 & 0 \\ 0 & 0 & 0 & e^{4i\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{5i\theta} \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}{\begin{pmatrix} \frac{147135}{5443200}e^{i\theta} & -\frac{120480}{5443200}e^{2i\theta} & \frac{76290}{5443200}e^{3i\theta} & -\frac{27930}{5443200}e^{4i\theta} & \frac{4407}{5443200}e^{5i\theta} \\ \frac{509440}{907200}e^{i\theta} & -\frac{314240}{907200}e^{2i\theta} & \frac{197120}{907200}e^{3i\theta} & -\frac{71680}{907200}e^{4i\theta} & \frac{11264}{907200}e^{5i\theta} \\ \frac{189945}{67200}e^{i\theta} & -\frac{79380}{67200}e^{2i\theta} & \frac{57510}{67200}e^{3i\theta} & -\frac{21060}{67200}e^{4i\theta} & \frac{3321}{67200}e^{5i\theta} \\ \frac{465920}{56700}e^{i\theta} & -\frac{117760}{56700}e^{2i\theta} & \frac{133120}{56700}e^{3i\theta} & -\frac{45440}{56700}e^{4i\theta} & \frac{7168}{56700}e^{5i\theta} \\ \frac{2631250}{145152}e^{i\theta} & -\frac{350000}{145152}e^{2i\theta} & \frac{837500}{145152}e^{3i\theta} & -\frac{212500}{145152}e^{4i\theta} & \frac{37250}{145152}e^{5i\theta} \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{147378}{5443200} \\ 0 & 0 & 0 & 0 & \frac{272896}{907200} \\ 0 & 0 & 0 & 0 & \frac{76464}{67200} \\ 0 & 0 & 0 & 0 & \frac{161792}{56700} \\ 0 & 0 & 0 & 0 & \frac{836500}{145152} \end{pmatrix}}$$

Simplifying the above equation and equating the imaginary part to zero we have

$$\bar{h}(\theta,h)=\frac{(3.8442E+58)\cos 5\theta-(3.8442E+58)}{(4.3242E+52)\cos 5\theta+(1.1433E+55)}$$

Evaluating $\bar{h}(\theta,h)$ at intervals of θ of 30^0 gives results as tabulated in Table 5.1.

Table 5.1

Interval of Absolute Stability of Five-Step Block Method for Fourth Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta,h)$	0	-6294.81	-1677.99	-3362.33	-5053.05	-449.00	-6750.19

Therefore, the interval of absolute stability is (-6750.19, 0). This is shown in the diagram below

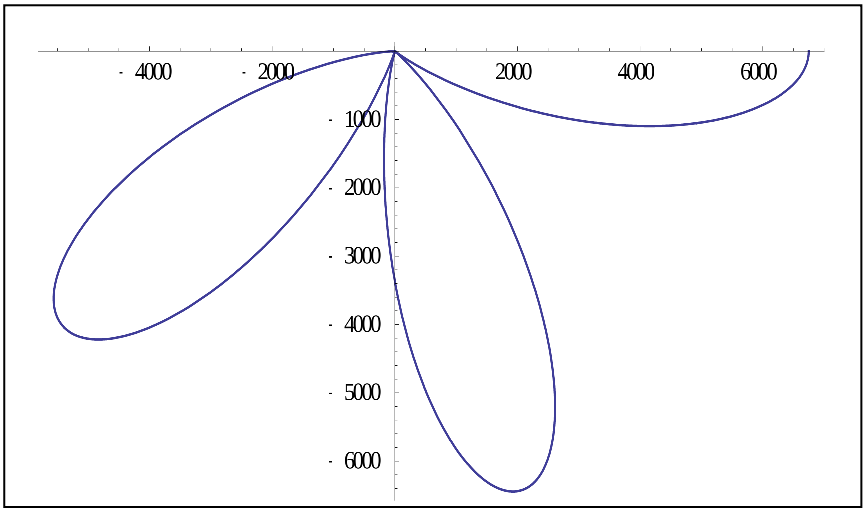

Figure 5.2. Region of absolute stability of five–step block method for fourth order ODEs.

5.3 Six –Step Block Method for Fourth Order ODEs.

This section considers the derivation of six-step block method and establishment of its properties for fourth order ODEs.

5.3.1 Derivation of Six–Step Block Method for Fourth Order ODEs.

Power series of the form (5.2.1.1) is considered as an approximate solution to the general fourth order problem of the form (5.2.1.2) where in (5.2.1.1) k =6 is the step-length. The first, second and third derivatives of (5.2.1.1) are given in (5.2.1.3), (5.2.1.4), (5.2.1.5) and (5.2.1.6).

Equation (5.2.1.1) is interpolated at $x = x_{n+i}, i = 1(1)4$ and (5.2.1.6) is collocated at $x = x_{n+i}, i = 0(1)6$. The diagram is shown below

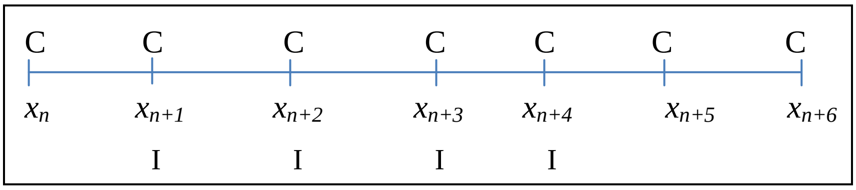

Figure 5.3. Six–step interpolation and collocation method for fourth order ODEs

This approach yields the result obtained below

$$AX = B \tag{5.3.1.1}$$

where

$$A = \begin{pmatrix}
1 & x_{n+1} & x^2_{n+1`} & x^3_{n+1} & x^4_{n+1} & x^5_{n+1} & x^6_{n+1} & x^7_{n+1} & x^8_{n+1} & x^9_{n+1} & x^{10}_{n+!} \\
1 & x_{n+2} & x^2_{n+2} & x^3_{n+2} & x^4_{n+2} & x^5_{n+2} & x^6_{n+2} & x^7_{n+2} & x^8_{n+2} & x^9_{n+2} & x^{10}_{n+2} \\
1 & x_{n+3} & x^2_{n+3} & x^3_{n+3} & x^4_{n+3} & x^5_{n+3} & x^6_{n+3} & x^7_{n+3} & x^8_{n+3} & x^9_{n+3} & x^{10}_{n+3} \\
1 & x_{n+4} & x^2_{n+4} & x^3_{n+4} & x^4_{n+4} & x^5_{n+4} & x^6_{n+4} & x^7_{n+4} & x^8_{n+4} & x^9_{n+4} & x^{10}_{n+4} \\
0 & 0 & 0 & 0 & 24 & 120x_n & 360x^2_n & 840x^3_n & 1680x^4_n & 3024x^5_n & 5040x^6_n \\
0 & 0 & 0 & 0 & 24 & 120x_{n+1} & 360x^2_{n+1} & 840x^3_{n+1} & 1680x^4_{n+1} & 3024x^5_{n+1} & 5040x^6_{n+1} \\
0 & 0 & 0 & 0 & 24 & 120x_{n+2} & 360x^2_{n+2} & 840x^3_{n+2} & 1680x^4_{n+2} & 3024x^5_{n+2} & 5040x^6_{n+2} \\
0 & 0 & 0 & 0 & 24 & 120x_{n+3} & 360x^2_{n+3} & 840x^3_{n+3} & 1680x^4_{n+3} & 3024x^5_{n+3} & 5040x^6_{n+3} \\
0 & 0 & 0 & 0 & 24 & 120x_{n+4} & 360x^2_{n+4} & 840x^3_{n+4} & 1680x^4_{n+4} & 3024x^5_{n+4} & 5040x^6_{n+4} \\
0 & 0 & 0 & 0 & 24 & 120x_{n+5} & 360x^2_{n+5} & 840x^3_{n+5} & 1680x^4_{n+5} & 3024x^5_{n+5} & 5040x^6_{n+5} \\
0 & 0 & 0 & 0 & 24 & 120x_{n+6} & 360x^2_{n+6} & 840x^3_{n+6} & 1680x^4_{n+6} & 3024x^5_{n+6} & 5040x^6_{n+6}
\end{pmatrix}$$

$$X = \left(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\right)^T$$

$$B = \left(y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}\right)^T$$

In order to find the values of a's in (5.3.1.1) the use of Gaussian elimination method is applied and this produces

$$\begin{aligned}
a_0 = \quad & 4y_{n+1} - 6y_{n+2} + 4y_{n+3} - y_{n+4} + \frac{h^4}{15120}(-16f_n + 2574f_{n+1} + 10029f_{n+2} + 2504f_{n+3} \\
& +54f_{n+4} - 30f_{n+5} + 5f_{n+6}) + \frac{x_n^4}{24}f_n + \frac{x_n^3 h}{3628800}(183795f_n + 885080f_{n+1} + 168935f_{n+2} + \\
& 390440f_{n+3} - 159455f_{n+4} + 50080f_{n+5} + 6875f_{n+6}) + \frac{x_n h^3}{3628800}(25432f_n + 1796760f_{n+1}
\end{aligned}$$

$$
\begin{aligned}
&+4609500 f_{n+2} + 1132000 f_{n+3} - 4632 f_{n+5} + 940 f_{n+6}) + \frac{x_n^{10}}{3628800h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} \\
&-20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) + \frac{x_n^7}{40320h^3}(49 f_n - 232 f_{n+1} + 461 f_{n+2} - 496 f_{n+3} + \\
&307 f_{n+4} - 104 f_{n+5} + 15 f_{n+6}) + \frac{x_n^3}{6h^3}(y_{n+1} - 3y_{n+2} + 3y_{n+3} - y_{n+4}) + \frac{x_n^2}{2h^2}(3y_{n+1} - 8y_{n+2} \\
&+7y_{n+3} - 2y_{n+4}) + \frac{x_n^6}{129600h^2}(812 f_n - 3132 f_{n+1} + 5265 f_{n+2} - 5080 f_{n+3} + 2970 f_{n+4} \\
&-972 f_{n+5} + 137 f_{n+6}) + \frac{x_n^2 h^2}{604800}(19523 f_n + 325342 f_{n+1} + 414440 f_{n+2} + 137360 f_{n+3} - \\
&19825 f_{n+4} + 5962 f_{n+5} - 802 f_{n+6}) + \frac{x_n^5}{7200h}(147 f_n - 360 f_{n+1} + 450 f_{n+2} - 400 f_{n+3} + \\
&225 f_{n+4} - 72 f_{n+5} + 10 f_{n+6}) + \frac{x_n}{6h}(26y_{n+1} - 57y_{n+2} + 42y_{n+3} - 11y_{n+4}) + \frac{x_n^9}{725760h^5}(7 f_n \\
&-40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} + 5 f_{n+6}) + \frac{x_n^8}{241920h^4}(35 f_n - 186 f_{n+1} + \\
&411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} - 114 f_{n+5} + 17 f_{n+6})
\end{aligned}
$$

$$
\begin{aligned}
a_1 = \; & \frac{h^3}{3628800}(25432 f_n + 1796760 f_{n+1} + 4609500 f_{n+2} + 1132000 f_{n+3} - 4632 f_{n+5} + \\
&940 f_{n+6}) + \frac{1}{6h}(26y_{n+1} - 57y_{n+2} + 42y_{n+3} - 11y_{n+4}) + \frac{x_n^4}{1440h}(147 f_n - 360 f_{n+1} + \\
&450 f_{n+2} - 400 f_{n+3} + 225 f_{n+4} - 72 f_{n+5} + 10 f_{n+6}) - \frac{x_n^3}{6} f_n + \frac{x_n^2 h}{1209600}(183795 f_n + \\
&885080 f_{n+1} + 168935 f_{n+2} + 390440 f_{n+3} - 159455 f_{n+4} + 50080 f_{n+5} + 6875 f_{n+6}) + \\
&\frac{x_n h^2}{302400}(19523 f_n + 325342 f_{n+1} + 414440 f_{n+2} + 137360 f_{n+3} - 19825 f_{n+4} + 5962 f_{n+5} \\
&-802 f_{n+6}) - \frac{x_n^9}{362880h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) - \\
&\frac{x_n^2}{h^3}(y_{n+1} - 3y_{n+2} + 3y_{n+3} - y_{n+4}) - \frac{x_n^6}{5760h^3}(49 f_n - 232 f_{n+1} + 461 f_{n+2} - 496 f_{n+3} + \\
&307 f_{n+4} - 104 f_{n+5} + 15 f_{n+6}) - \frac{x_n}{h^2}(3y_{n+1} - 8y_{n+2} + 7y_{n+3} - 2y_{n+4}) - \frac{x_n^5}{21600h^2}(812 f_n \\
&-3132 f_{n+1} + 5265 f_{n+2} - 5080 f_{n+3} + 2970 f_{n+4} - 972 f_{n+5} + 137 f_{n+6}) - \frac{x_n^8}{80640h^5}(7 f_n \\
&-40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} + 5 f_{n+6}) - \frac{x_n^7}{30240h^4}(35 f_n - 186 f_{n+1} \\
&+411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} - 114 f_{n+5} + 17 f_{n+6})
\end{aligned}
$$

$$a_2 = \frac{1}{2h^2}(3y_{n+1} - 8y_{n+2} + 7y_{n+3} - 2y_{n+4}) + \frac{x_n^4}{8640h^2}(812f_n - 3132f_{n+1} + 5265f_{n+2} - 5080f_{n+3} + 2970f_{n+4} - 972f_{n+5} + 137f_{n+6}) + \frac{h^2}{604800}(19523f_n + 325342f_{n+1} + 414440f_{n+2} + 137360f_{n+3} - 19825f_{n+4} + 5962f_{n+5} - 802f_{n+6}) + \frac{x_n^2}{4}f_n + \frac{x_n^8}{80640h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6}) + \frac{x_n h}{1209600}(183795f_n + 885080f_{n+1} + 168935f_{n+2} + 390440f_{n+3} - 159455f_{n+4} + 50080f_{n+5} - 6875f_{n+6}) + \frac{x_n}{2h^3}(y_{n+1} - 3y_{n+2} + 3y_{n+3} - y_{n+4}) + \frac{x_n^5}{1920h^3}(49f_n - 232f_{n+1} + 461f_{n+2} - 496f_{n+3} + 307f_{n+4} - 104f_{n+5} + 15f_{n+6}) + \frac{x_n^7}{20160h^5}(7f_n - 40f_{n+1} + 95f_{n+2} - 120f_{n+3} + 85f_{n+4} - 32f_{n+5} + 5f_{n+6}) + \frac{x_n^6}{8640h^4}(35f_n - 186f_{n+1} + 411f_{n+2} - 484f_{n+3} + 321f_{n+4} - 114f_{n+5} + 17f_{n+6}) + \frac{x_n^3}{720h}(147f_n - 360f_{n+1} + 450f_{n+2} - 400f_{n+3} + 225f_{n+4} - 72f_{n+5} + 10f_{n+6}).$$

$$a_3 = -\frac{h}{3628800}(183795f_n + 885080f_{n+1} + 168935f_{n+2} + 390440f_{n+3} - 159455f_{n+4} + 50080f_{n+5} - 6875f_{n+6}) - \frac{1}{6h^3}(y_{n+1} - 3y_{n+2} + 3y_{n+3} - y_{n+4}) + \frac{x_n^4}{1152h^3}(49f_n - 232f_{n+1} + 461f_{n+2} - 496f_{n+3} + 307f_{n+4} - 104f_{n+5} + 15f_{n+6}) - \frac{x_n}{6}f_n - \frac{x_n^7}{30240h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6}) - \frac{x_n^6}{8640h^5}(7f_n - 40f_{n+1} + 95f_{n+2} - 120f_{n+3} + 85f_{n+4} - 32f_{n+5} + 5f_{n+6}) - \frac{x_n^5}{4320h^4}(35f_n - 186f_{n+1} + 411f_{n+2} - 484f_{n+3} + 321f_{n+4} \; 2970f_{n+4} - 972f_{n+5} + 137f_{n+6}).$$

$$a_4 = \frac{1}{24}f_n + \frac{x_n^6}{17280h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6}) + \frac{x_n}{1440h}(147f_n - 360f_{n+1} + 450f_{n+2} - 400f_{n+3} + 225f_{n+4} - 72f_{n+5} + 10f_{n+6}) + \frac{x_n^5}{5760h^5}(7f_n - 40f_{n+1} + 95f_{n+2} - 120f_{n+3} + 85f_{n+4} - 32f_{n+5} + 5f_{n+6}) + \frac{x_n^4}{3456h^4}(35f_n - 186f_{n+1} + 411f_{n+2} - 484f_{n+3} + 321f_{n+4} - 114f_{n+5} + 17f_{n+6}) + \frac{x_n^3}{1152h^3}(49f_n - 232f_{n+1} + 461f_{n+2} - 496f_{n+3} + 307f_{n+4} - 104f_{n+5} + 15f_{n+6})$$

$$+\frac{x_n^2}{8640h^2}(812 f_n - 3132 f_{n+1} + 5265 f_{n+2} - 5080 f_{n+3} + 2970 f_{n+4} - 972 f_{n+5} + 137 f_{n+6}).$$

$$a_5 = -\frac{1}{7200h}(147 f_n - 360 f_{n+1} + 450 f_{n+2} - 400 f_{n+3} + 225 f_{n+4} - 72 f_{n+5} + 10 f_{n+6}) - \frac{x_n^5}{14400h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) - \frac{x_n}{21600h^2}(812 f_n - 3132 f_{n+1} + 5265 f_{n+2} - 5080 f_{n+3} + 2970 f_{n+4} - 972 f_{n+5} + 137 f_{n+6}) - \frac{x_n^4}{5760h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} + 5 f_{n+6}) - \frac{x_n^3}{4320h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} - 114 f_{n+5} + 17 f_{n+6}) - \frac{x_n^2}{1920h^3}(49 f_n - 232 f_{n+1} + 461 f_{n+2} - 496 f_{n+3} + 307 f_{n+4} - 104 f_{n+5} + 15 f_{n+6}).$$

$$a_6 = \frac{1}{129600h^2}(812 f_n - 3132 f_{n+1} + 5265 f_{n+2} - 5080 f_{n+3} + 2970 f_{n+4} - 972 f_{n+5} + 137 f_{n+6}) + \frac{x_n^4}{17280h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) + \frac{x_n}{5760h^3}(49 f_n - 232 f_{n+1} + 461 f_{n+2} - 496 f_{n+3} + 307 f_{n+4} - 104 f_{n+5} + 15 f_{n+6}) + \frac{x_n^3}{8640h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} + 5 f_{n+6}) + \frac{x_n^2}{8640h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} - 114 f_{n+5} + 17 f_{n+6}).$$

$$a_7 = -\frac{1}{40320h^3}(49 f_n - 232 f_{n+1} + 461 f_{n+2} - 496 f_{n+3} + 307 f_{n+4} - 104 f_{n+5} + 15 f_{n+6}) - \frac{x_n^3}{30240h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) - \frac{x_n}{30240h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} - 114 f_{n+5} + 17 f_{n+6}) - \frac{x_n^2}{20160h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} + 5 f_{n+6}).$$

$$a_8 = \frac{1}{241920h^4}(35 f_n - 186 f_{n+1} + 411 f_{n+2} - 484 f_{n+3} + 321 f_{n+4} - 114 f_{n+5} + 17 f_{n+6}) + \frac{x_n^2}{80640h^6}(f_n - 6 f_{n+1} + 15 f_{n+2} - 20 f_{n+3} + 15 f_{n+4} - 6 f_{n+5} + f_{n+6}) + \frac{x_n}{80640h^5}(7 f_n - 40 f_{n+1} + 95 f_{n+2} - 120 f_{n+3} + 85 f_{n+4} - 32 f_{n+5} + 5 f_{n+6})$$

$$a_9 = \frac{1}{725760h^5}(7f_n - 40f_{n+1} + 95f_{n+2} - 120f_{n+3} + 85f_{n+4} - 32f_{n+5} + 5f_{n+6}) -$$
$$\frac{x_n}{362880h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6})$$

$$a_{10} = \frac{1}{3628800h^6}(f_n - 6f_{n+1} + 15f_{n+2} - 20f_{n+3} + 15f_{n+4} - 6f_{n+5} + f_{n+6})$$

Substituting the values of a's into equation (5.2.1.1) and simplifying, this gives a continuous linear multistep method of the form:

$$y(x) = \sum_{j=1}^{k-2} \alpha_j(x) y_{n+j} + h^4 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (5.3.1.2)$$

where $x = zh + x_n + 5h$ (5.3.1.3)

Equation (5.3.1.3) is substituted into (5.3.1.2) and simplified to give

$$\alpha_1(z) = -1 - \frac{11z}{6} - z^2 - \frac{z^3}{6}$$
$$\alpha_2(z) = 4 + 7z + \frac{7z^2}{2} + \frac{z^3}{2}$$
$$\alpha_3(z) = -6 - \frac{19z}{2} - 4z^2 - \frac{z^3}{2}$$
$$\alpha_4(z) = 4 + \frac{13z}{3} + \frac{3z^2}{2} + \frac{z^3}{6}$$
$$\beta_0(z) = \frac{1}{3628800}(1200 + 1984z + 2088z^2 + 1955z^3 - 1008z^5 - 364z^6 + 90z^7 + 75z^8 + 15z^9 + z^{10}).$$
$$\beta_1(z) = \frac{1}{3628800}(-12240 - 19740z - 17898z^2 - 15080z^3 + 7560z^5 + 2604z^6 - 720z^7 - 540z^8 - 100z^9 - 6z^{10}).$$
$$\beta_2(z) = \frac{1}{3628800}(642960 + 1164900z + 655740z^2 + 149815z^3 - 25200z^5 - 7980z^6 + 2610z^7 + 1665z^8 + 275z^9 + 15z^{10}).$$
$$\beta_3(z) = \frac{1}{3628800}(2364960 + 4576600z + 2655060z^2 + 409560z^3 + 50400z^5 + 13160z^6 - 5760z^7 - 2760z^8 - 400z^9 - 20z^{10}).$$

$$\beta_4(z) = \frac{1}{3628800}(642960 + 1816500z + 1851000z^2 + 740705z^3 - 75600z^5$$
$$-7140z^6 + 7470z^7 + 2565z^8 + 325z^9 + 15z^{10}).$$
$$\beta_5(z) = \frac{1}{3628800}(-12240 + 18852z + 150822z^2 + 231920z^3 + 151200z^4 + \qquad (5.3.1.4)$$
$$38808z^5 - 4116z^6 - 5040z^7 - 1260z^8 - 140z^9 - 6z^{10}).$$
$$\beta_6(z) = \frac{1}{3628800}(1200 + 940z - 4812z^2 - 6875z^3 + 5040z^5 + 3836z^6 + 1350z^7$$
$$+255z^8 + 25z^9 + z^{10}).$$

Evaluating (5.3.1.4) at the non-interpolating points. That is, at z=-5, 0 and 1 gives

$$15120y_{n+4} - 60480y_{n+3} + 90720y_{n+2} - 60480y_{n+1} + 15120y_n = h^4(5f_{n+6}$$
$$-30f_{n+5} + 54f_{n+4} + 2504f_{n+3} + 10029f_{n+2} + 2574f_{n+1} - 16f_n). \qquad (5.3.1.5)$$

$$15120y_{n+5} - 60480y_{n+4} + 90720y_{n+3} - 60480y_{n+2} + 15120y_{n+1} = h^4(5f_{n+6}$$
$$-51f_{n+5} + 2679f_{n+4} + 9854f_{n+3} + 2679f_{n+2} - 51f_{n+1} + 5f_n). \qquad (5.3.1.6)$$

$$15120y_{n+6} - 151200y_{n+4} + 302400y_{n+3} - 226800y_{n+2} + 60480y_{n+1} = h^4(4f_{n+6}$$
$$+2370_{n+5} + 20745f_{n+4} + 41920f_{n+3} + 10770f_{n+2} - 234f_{n+1} + 25f_n). \qquad (5.3.1.7)$$

The first derivative of (5.1.3.4) gives

$$\alpha_1'(z) = \frac{-11}{6} - 2z - \frac{z^2}{2}$$
$$\alpha_2'(z) = 7 + 7z + \frac{3z^2}{2}$$
$$\alpha_3'(z) = \frac{-19}{2} - 8z - \frac{3z^2}{2}$$
$$\alpha_4'(z) = \frac{13}{3} + 3z + \frac{3z^2}{6}$$
$$\beta_0'(z) = \frac{1}{3628800}(1984 + 4176z + 5865z^2 - 5040z^4 - 2184z^5 + 630z^6$$
$$+600z^7 + 135z^8 + 10z^9).$$
$$\beta_1'(z) = \frac{1}{3628800}(-19740 - 35796z - 45240z^2 + 37800z^4 + 15624z^5$$
$$-5040z^6 - 4320z^7 - 900z^8 - 60z^9).$$
$$\beta_2'(z) = \frac{1}{3628800}(1164900 + 1311480z + 449445z^2 - 126000z^4 -$$
$$47880z^5 + 18270z^6 + 13320z^7 + 2475z^8 + 150z^9).$$
$$\beta_3'(z) = \frac{1}{3628800}(4576600 + 5310120z + 1228680z^2 + 252000z^4$$
$$+78960z^5 - 40320z^6 - 22080z^7 - 3600z^8 - 200z^9).$$

$$\beta_4'(z) = \frac{1}{3628800}(1816500 + 3702000z + 2222115z^2 - 378000z^4$$
$$-42840z^5 + 52290z^6 + 20520z^7 + 2925z^8 + 150z^9).$$
$$\beta_5'(z) = \frac{1}{3628800}(18852 + 301644z + 695760z^2 + 604800z^3 + 194040z^4 \tag{5.3.1.8}$$
$$-24696z^5 - 35280z^6 - 10080z^7 - 1260z^8 - 60z^9).$$
$$\beta_6'(z) = \frac{1}{3628800}(940 - 9624z - 20625z^2 + 25200z^4 + 23016z^5 +$$
$$9450z^6 + 2040z^7 + 225z^8 + 10z^9).$$

Equation (5.3.1.8) is evaluated at all the grid points. i.e, at z=-5, -4, -3, -2, -1, 0 and 1 produces

$$907200hy_n' - 1663200y_{n+4} + 6350400y_{n+3} - 8618400y_{n+2} + 3931200y_{n+1} =$$
$$h^4(-235f_{n+6} + 1158f_{n+5} - 283000f_{n+3} - 1152375f_{n+2} - 449190f_{n+1} \tag{5.3.1.9}$$
$$-6358f_n).$$

$$100800hy_{n+1}' - 3360y_{n+4} + 15120y_{n+3} - 30240y_{n+2} + 18480y_{n+1} = h^4(7f_{n+6} \tag{5.3.1.10}$$
$$-75f_{n+5} + 385f_{n+4} - 6690f_{n+3} - 17715f_{n+2} - 1147f_{n+1} + 35f_n).$$

$$907200hy_{n+2}' + 151200y_{n+4} - 907200y_{n+3} + 453600y_{n+2} + 302400y_{n+1} =$$
$$h^4(-134f_{n+6} + 1122f_{n+5} - 4425f_{n+4} + 35440f_{n+3} + 44580f_{n+2} - 1146f_{n+1} \tag{5.3.1.11}$$
$$+163f_n).$$

$$907200hy_{n+3}' + 302400y_{n+4} + 453600y_{n+3} - 907200y_{n+2} + 151200y_{n+1} =$$
$$h^4(134f_{n+6} - 1101f_{n+5} + 3960f_{n+4} - 49270f_{n+3} - 30750f_{n+2} + 1611f_{n+1} \tag{5.3.1.12}$$
$$-184f_n).$$

$$10080hy_{n+4}' - 18480y_{n+4} + 30240y_{n+3} - 15120y_{n+2} + 3360y_{n+1} = h^4(-7f_{n+6} \tag{5.3.1.13}$$
$$+14f_{n+5} + 1000f_{n+4} + 17960f_{n+3} + 6445f_{n+2} - 238f_{n+1} + 26f_n).$$

$$907200hy_{n+5}' + 3931200y_{n+4} + 8618400y_{n+3} - 6350400y_{n+2} + 1663200y_{n+1} =$$
$$h^4(235f_{n+6} + 4713f_{n+5} + 454125f_{n+4} + 1144150f_{n+3} + 291225f_{n+2} - 4935f_{n+1} \tag{5.3.1.14}$$
$$+487f_n).$$

$$907200hy_{n+6}' - 7106400y_{n+4} + 17236800y_{n+3} - 14061600y_{n+2} + 3931200y_{n+1}$$
$$= h^4(7658f_{n+6} + 435930f_{n+5} + 1848915f_{n+4} + 2845040f_{n+3} + 696540f_{n+2} \tag{5.3.1.15}$$
$$-14418f_{n+1} + 1535f_n).$$

The second derivative of (5.3.1.4) gives

$$
\begin{aligned}
\alpha_1''(z) &= -2 - z \\
\alpha_2''(z) &= 7 + 3z \\
\alpha_3''(z) &= -8 - 3z \\
\alpha_4''(z) &= 3 + z \\
\beta_0''(z) &= \frac{1}{604800}(696 + 1955z - 3360z^3 - 1820z^4 + 630z^5 + 700z^6 + 180z^7 \\
&\qquad + 15z^8). \\
\beta_1''(z) &= \frac{1}{604800}(-5966 - 15080z + 25200z^3 + 13020z^4 - 5040z^5 - 5040z^6 \\
&\qquad - 1200z^7 - 90z^8). \\
\beta_2''(z) &= \frac{1}{604800}(218580 + 149815z - 84000z^3 - 39900z^4 + 18270z^5 + \\
&\qquad 15540z^6 + 3300z^7 + 225z^8). \\
\beta_3''(z) &= \frac{1}{604800}(885020 + 409560z + 168000z^3 + 65800z^4 - 40320z^5 - \\
&\qquad 25760z^6 - 4800z^7 - 300z^8). \\
\beta_4''(z) &= \frac{1}{604800}(617000 + 740705z - 252000z^3 - 35700z^4 + 52290z^5 \\
&\qquad + 23940z^6 + 3900z^7 + 225z^8). \\
\beta_5''(z) &= \frac{1}{604800}(50274 + 231920z + 302400z^2 + 129360z^3 - 20580z^4 \\
&\qquad - 35280z^5 - 11760z^6 - 1680z^7 - 90z^8). \\
\beta_6''(z) &= \frac{1}{604800}(-1604 - 6875z + 16800z^3 + 19180z^4 + 9450z^5 + 2380z^6 \\
&\qquad + 300z^7 + 15z^8).
\end{aligned}
\tag{5.3.1.16}
$$

Evaluating (5.3.1.16) at all the grid points. That is, at z=-5, -4, -3, -2,-1, 0 and 1 gives

$$
\begin{aligned}
&302400h^2 y_n'' + 604800y_{n+4} - 2116800y_{n+3} + 2419200y_{n+2} - 907200y_{n+1} = \\
&h^4(-802f_{n+6} + 5962f_{n+5} - 19825f_{n+4} + 137360f_{n+3} + 414440f_{n+2} \\
&+ 325342f_{n+1} + 19523f_n).
\end{aligned}
\tag{5.3.1.17}
$$

$$
\begin{aligned}
&302400h^2 y_{n+1}'' + 302400y_{n+4} - 1209600y_{n+3} + 1512000y_{n+2} - 604800y_{n+1} = \\
&h^4(148f_{n+6} - 943f_{n+5} + 2130f_{n+4} + 48350f_{n+3} + 201340f_{n+2} \\
&+ 27177f_{n+1} - 1002f_n).
\end{aligned}
\tag{5.3.1.18}
$$

$$
\begin{aligned}
&302400h^2 y_{n+2}'' - 302400y_{n+3} + 604800y_{n+2} - 302400y_{n+1} = h^4(-7f_{n+6} \\
&+ 42f_{n+5} - 50f_{n+4} - 920f_{n+3} - 23295f_{n+2} - 1018f_{n+1} + 48f_n).
\end{aligned}
\tag{5.3.1.19}
$$

$$302400h^2 y''_{n+3} - 302400y_{n+4} + 604800y_{n+3} - 302400y_{n+2} - 604800y_{n+1} = h^4(-7f_{n+6} + 97f_{n+5} - 1165f_{n+4} - 7f_n). \quad (5.3.1.20)$$

$$302400h^2 y''_{n+4} - 604800y_{n+4} + 1512000y_{n+3} - 120960y_{n+2} + 302400y_{n+1} = h^4(148f_{n+6} - 2038f_{n+5} + 30285f_{n+4} + 196160f_{n+3} + 53530f_{n+2} - 978f_{n+1} + 93f_n). \quad (5.3.1.21)$$

$$302400h^2 y''_{n+5} - 907200y_{n+4} + 2419200y_{n+3} - 2116800y_{n+2} + 604800y_{n+1} = h^4(-802f_{n+6} + 25137f_{n+5} + 308500f_{n+4} + 442510f_{n+3} + 109290f_{n+2} - 2983f_{n+1} + 348f_n). \quad (5.3.1.22)$$

$$302400h^2 y''_{n+6} - 1209600y_{n+4} + 3326400y_{n+3} - 3024000y_{n+2} + 907200y_{n+1} = h^4(19823f_{n+6} + 322282f_{n+5} + 575180f_{n+4} + 728600f_{n+3} + 140915f_{n+2} + 2902f_{n+1} - 502f_n). \quad (5.3.1.23)$$

The third derivative of (5.3.1.4) gives

$$\alpha_1'''(z) = -1$$
$$\alpha_2'''(z) = 3$$
$$\alpha_3'''(z) = -3$$
$$\alpha_4'''(z) = 1$$

$$\beta_0'''(z) = \frac{1}{120960}(391 - 2016z^2 - 1456z^3 + 630z^4 + 840z^5 + 252z^6 + 24z^7).$$

$$\beta_1'''(z) = \frac{1}{120960}(-3016 + 15120z^2 + 10416z^3 - 5040z^4 - 6048z^5 - 1680z^6 - 144z^7).$$

$$\beta_2'''(z) = \frac{1}{120960}(29963 - 50400z^2 - 31920z^3 + 18270z^4 + 18648z^5 + 4620z^6 + 360z^7).$$

$$\beta_3'''(z) = \frac{1}{120960}(81912 + 100800z^2 + 52640z^3 - 40320z^4 - 30912z^5 - 6720z^6 - 480z^7).$$

$$\beta_4'''(z) = \frac{1}{120960}(148141 - 151200z^2 - 28560z^3 + 52290z^4 + 28728z^5 + 5460z^6 + 360z^7).$$

$$\beta_5'''(z) = \frac{1}{120960}(46384 + 120960z + 77616z^2 - 16464z^3 - 35280z^4 - 14112z^5 - 2352z^6 - 144z^7).$$

$$\beta_6'''(z) = \frac{1}{120960}(-1375 + 100800z^2 + 15344z^3 + 9450z^4 + 2856z^5 + 420z^6 + 24z^7). \tag{5.3.1.24}$$

Equation (5.3.1.24) is evaluated at all the grid points. i.e, at z=-5, -4, -3, -2,-1, 0 and 1 yields

$$120960h^3 y_n''' - 120960y_{n+4} + 362880y_{n+3} - 362880y_{n+2} + 120960y_{n+1} = h^4(1375f_{n+6} - 10016f_{n+5} + 31891f_{n+4} - 78088f_{n+3} - 33787f_{n+2} - 177016f_{n+1} - 36759f_n). \tag{5.3.1.25}$$

$$120960h^3 y_{n+1}''' - 120960y_{n+4} + 362880y_{n+3} - 362880y_{n+2} + 120960y_{n+1} = h^4(-351f_{n+6} + 2608f_{n+5} - 8531f_{n+4} - 3080f_{n+3} - 126709f_{n+2} - 46792f_{n+1} + 1415f_n). \tag{5.3.1.26}$$

$$120960h^3 y_{n+2}''' - 120960y_{n+4} + 362880y_{n+3} - 362880y_{n+2} + 120960y_{n+1} = h^4(191f_{n+6} - 1568f_{n+5} + 6067f_{n+4} - 35592f_{n+3} - 32731f_{n+2} + 3464f_{n+1} - 311f_n). \tag{5.3.1.27}$$

$$120960h^3 y_{n+3}''' - 120960y_{n+4} + 362880y_{n+3} - 362880y_{n+2} + 120960y_{n+1} = h^4(-191f_{n+6} + 1648f_{n+5} - 7475f_{n+4} + 39416f_{n+3} + 28907f_{n+2} - 2056f_{n+1} + 231f_n). \tag{5.3.1.28}$$

$$120960h^3 y_{n+4}''' - 120960y_{n+4} + 362880y_{n+3} - 362880y_{n+2} + 120960y_{n+1} = h^4(351f_{n+6} - 3872f_{n+5} + 54163f_{n+4} + 114424f_{n+3} + 15365f_{n+2} + 1160f_{n+1} - 151f_n). \tag{5.3.1.29}$$

$$120960h^3 y_{n+5}''' - 120960y_{n+4} + 362880y_{n+3} - 362880y_{n+2} + 120960y_{n+1} = h^4(-1375f_{n+6} + 46384f_{n+5} + 148141f_{n+4} + 81912f_{n+3} + 29963f_{n+2} - 3016f_{n+1} + 391f_n). \tag{5.3.1.30}$$

$$120960h^3 y_{n+6}''' - 120960y_{n+4} + 362880y_{n+3} - 362880y_{n+2} + 120960y_{n+1} = h^4(36799f_{n+6} + 176608f_{n+5} + 55219f_{n+4} + 156920f_{n+3} - 10459f_{n+2} + 9608f_{n+1} - 1335f_n). \tag{5.3.1.31}$$

Joining equations (5.3.1.5) - (5.3.1.7), (5.3.1.9), (5.3.1.17) and (5.3.1.25) to form a block (1.10). This produces

$$\begin{pmatrix} -60480 & 90720 & -60480 & 15120 & 0 & 0 \\ 15120 & -60480 & 90720 & -60480 & 15120 & 0 \\ 60480 & -226800 & 302400 & -151200 & 0 & 15120 \\ 3931200 & -8618400 & 6350400 & -1663200 & 0 & 0 \\ -907200 & 2419200 & -2116800 & 604800 & 0 & 0 \\ 120960 & -362880 & 362880 & -120960 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -15120 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -907200 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_{n} \end{pmatrix} +$$

$$h^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -302400 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y''_{n-5} \\ y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_{n} \end{pmatrix} + h^3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -120960 \end{pmatrix} \begin{pmatrix} y'''_{n-5} \\ y'''_{n-4} \\ y'''_{n-3} \\ y'''_{n-2} \\ y'''_{n-1} \\ y'''_{n} \end{pmatrix}$$

$$+ h^4 \begin{pmatrix} 2574 & 10029 & 2504 & 54 & -30 & 5 \\ -51 & 2679 & 9854 & 2679 & -51 & 5 \\ -234 & 10770 & 41920 & 20745 & 2370 & 4 \\ -449190 & -1152375 & -283000 & 0 & 1158 & -235 \\ 325342 & 414440 & 137360 & -19825 & 5962 & -802 \\ -177016 & -33787 & -78088 & 31891 & -10016 & 1375 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix}$$

$$+ h^4 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -16 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 25 \\ 0 & 0 & 0 & 0 & 0 & -6358 \\ 0 & 0 & 0 & 0 & 0 & 19523 \\ 0 & 0 & 0 & 0 & 0 & -36759 \end{pmatrix} \begin{pmatrix} f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}$$

Multiplying the above equation by $(A^0)^{-1}$

$$
\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}
\begin{pmatrix} y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4}\\ y_{n+5}\\ y_{n+6} \end{pmatrix}
=
\begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{pmatrix}
\begin{pmatrix} y_{n-5}\\ y_{n-4}\\ y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_{n} \end{pmatrix}
+h
\begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&2\\ 0&0&0&0&0&3\\ 0&0&0&0&0&4\\ 0&0&0&0&0&5\\ 0&0&0&0&0&6 \end{pmatrix}
\begin{pmatrix} y'_{n-5}\\ y'_{n-4}\\ y'_{n-3}\\ y'_{n-2}\\ y'_{n-1}\\ y'_{n} \end{pmatrix}
+
$$

$$
h^2
\begin{pmatrix} 0&0&0&0&0&\frac{1}{2}\\ 0&0&0&0&0&2\\ 0&0&0&0&0&\frac{9}{2}\\ 0&0&0&0&0&8\\ 0&0&0&0&0&\frac{25}{2}\\ 0&0&0&0&0&18 \end{pmatrix}
\begin{pmatrix} y''_{n-5}\\ y''_{n-4}\\ y''_{n-3}\\ y''_{n-2}\\ y''_{n-1}\\ y''_{n} \end{pmatrix}
+h^3
\begin{pmatrix} 0&0&0&0&0&\frac{1}{6}\\ 0&0&0&0&0&\frac{4}{3}\\ 0&0&0&0&0&\frac{9}{2}\\ 0&0&0&0&0&\frac{32}{3}\\ 0&0&0&0&0&\frac{125}{6}\\ 0&0&0&0&0&36 \end{pmatrix}
\begin{pmatrix} y'''_{n-5}\\ y'''_{n-4}\\ y'''_{n-3}\\ y'''_{n-2}\\ y'''_{n-1}\\ y'''_{n} \end{pmatrix}
+
$$

$$
h^4
\begin{pmatrix}
\frac{74}{2397} & \frac{-157}{4947} & \frac{181}{6749} & \frac{-196}{13303} & \frac{110}{23653} & \frac{-17}{26556}\\
\frac{1135}{1832} & \frac{-199}{405} & \frac{388}{945} & \frac{-127}{567} & \frac{998}{14175} & \frac{-137}{14175}\\
\frac{1447}{473} & \frac{-721}{409} & \frac{261}{160} & \frac{-196}{219} & \frac{3159}{11200} & \frac{-571}{14727}\\
\frac{5050}{573} & \frac{-667}{187} & \frac{1361}{314} & \frac{-928}{405} & \frac{2048}{2835} & \frac{-415}{4178}\\
\frac{11431}{591} & \frac{-1743}{320} & \frac{7039}{717} & \frac{-5539}{1231} & \frac{3661}{2489} & \frac{-358}{1769}\\
\frac{6318}{175} & \frac{-243}{35} & \frac{684}{35} & \frac{-243}{35} & \frac{486}{175} & \frac{-9}{25}
\end{pmatrix}
\begin{pmatrix} f_{n+1}\\ f_{n+2}\\ f_{n+3}\\ f_{n+4}\\ f_{n+5}\\ f_{n+6} \end{pmatrix}
+
$$

$$
h^4
\begin{pmatrix} 0&0&0&0&0&\frac{93}{3518}\\ 0&0&0&0&0&\frac{513}{1762}\\ 0&0&0&0&0&\frac{721}{656}\\ 0&0&0&0&0&\frac{997}{362}\\ 0&0&0&0&0&\frac{1607}{289}\\ 0&0&0&0&0&\frac{1719}{175} \end{pmatrix}
\begin{pmatrix} f_{n-5}\\ f_{n-4}\\ f_{n-3}\\ f_{n-2}\\ f_{n-1}\\ f_{n} \end{pmatrix}
\qquad (5.3.1.32)
$$

which gives

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2 y''_n + \frac{1}{6}h^3 y'''_n + \frac{h^4}{3628800}(-2323 f_{n+6} + 16876 f_{n+5} - 53465 f_{n+4} + 97320 f_{n+3} - 115165 f_{n+2} + 112028 f_{n+1} + 95929 f_n). \tag{5.3.1.33}$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2 y''_n + \frac{4}{3}h^3 y'''_n + \frac{h^4}{14175}(-137 f_{n+6} + 998 f_{n+5} - 3175 f_{n+4} + 5820 f_{n+3} - 6965 f_{n+2} + 8782 f_{n+1} + 4127 f_n). \tag{5.3.1.34}$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2}h^2 y''_n + \frac{9}{2}h^3 y'''_n + \frac{h^4}{44800}(-1737 f_{n+6} + 12636 f_{n+5} - 40095 f_{n+4} + 73080 f_{n+3} - 78975 f_{n+2} + 137052 f_{n+1} + 49239 f_n). \tag{5.3.1.35}$$

$$y_{n+4} = y_n + 4hy'_n + 8h^2 y''_n + \frac{32}{3}h^3 y'''_n + \frac{h^4}{28350}(-2816 f_{n+6} + 20480 f_{n+5} - 64960 f_{n+4} + 122880 f_{n+3} - 101120 f_{n+2} + 249856 f_{n+1} + 78080 f_n). \tag{5.3.1.36}$$

$$y_{n+5} = y_n + 5hy'_n + \frac{25}{2}h^2 y''_n + \frac{125}{6}h^3 y'''_n + \frac{h^4}{145152}(-29375 f_{n+6} + 213500 f_{n+5} - 653125 f_{n+4} + 1425000 f_{n+3} - 790625 f_{n+2} + 2807500 f_{n+1} + 807125 f_n). \tag{5.3.1.37}$$

$$y_{n+6} = y_n + 6hy'_n + 18h^2 y''_n + 36h^3 y'''_n + \frac{h^4}{3628800}(-126 f_{n+6} + 972 f_{n+5} - 2430 f_{n+4} + 6840 f_{n+3} - 2430 f_{n+2} + 12636 f_{n+1} + 3438 f_n). \tag{5.3.1.38}$$

Substituting (5.3.1.33) - (5.3.1.36) into (5.3.1.10) – (5.3.1.15) to give the first derivative of the block

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2 y'''_n + \frac{h^3}{3628800}(-9809 f_{n+6} + 71364 f_{n+5} - 226605 f_{n+4} + 414160 f_{n+3} - 494715 f_{n+2} + 506604 f_{n+1} + 343801 f_n). \tag{5.3.1.39}$$

$$y'_{n+2} = y_n + 2hy''_n + 2h^2 y'''_n + \frac{h^3}{28350}(-491 f_{n+6} + 3576 f_{n+5} - 11370 f_{n+4} + 20800 f_{n+3} - 24465 f_{n+2} + 35976 f_{n+1} + 13774 f_n). \tag{5.3.1.40}$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{9}{2}h^2 y'''_n + \frac{h^3}{44800}(-1917 f_{n+6} + 13932 f_{n+5} - 44145 f_{n+4} + 80640 f_{n+3} - 72495 f_{n+2} + 172692 f_{n+1} + 52893 f_n). \tag{5.3.1.41}$$

$$y'_{n+4} = y'_n + 4hy''_n + 8h^2 y'''_n + \frac{h^3}{28350}(-2272 f_{n+6} + 16512 f_{n+5} - 52080 f_{n+4} + 108800 f_{n+3} - 54240 f_{n+2} + 223872 f_{n+1} + 61808 f_n). \tag{5.3.1.42}$$

$$y'_{n+5} = y'_n + 5hy''_n + \frac{25}{2}h^2 y'''_n + \frac{h^3}{145152}(-18625 f_{n+6} + 136500 f_{n+5} - 358125 f_{n+4} + 1070000 f_{n+3} - 256875 f_{n+2} + 1945500 f_{n+1} + 505625 f_n). \tag{5.3.1.43}$$

$$y'_{n+6} = y'_n + 6hy''_n + 18h^2 y'''_n + \frac{h^3}{700}(-126 f_{n+6} + 1296 f_{n+5} - 1620 f_{n+4} + 8640 f_{n+3} - 810 f_{n+2} + 14256 f_{n+1} + 3564 f_n). \quad (5.3.1.44)$$

Substituting (5.3.1.33) - (5.3.1.36) into (5.3.1.18) – (5.3.1.23) to give the second derivative of the block

$$y''_{n+1} = y''_n + hy'''_n + \frac{h^2}{120960}(-995 f_{n+6} + 7254 f_{n+5} - 23109 f_{n+4} + 42484 f_{n+3} - 51453 f_{n+2} + 57750 f_{n+1} + 28549 f_n). \quad (5.3.1.45)$$

$$y''_{n+2} = y''_n + 2hy'''_n + \frac{h^2}{120960}(-2432 f_{n+6} + 17664 f_{n+5} - 55872 f_{n+4} + 100864 f_{n+3} - 107520 f_{n+2} + 223488 f_{n+1} + 65728 f_n). \quad (5.3.1.46)$$

$$y''_{n+3} = y''_n + 3hy'''_n + \frac{h^2}{4480}(-141 f_{n+6} + 1026 f_{n+5} - 3267 f_{n+4} + 6300 f_{n+3} - 2403 f_{n+2} + 14850 f_{n+1} + 3795 f_n). \quad (5.3.1.47)$$

$$y''_{n+4} = y''_n + 4hy'''_n + \frac{h^2}{945}(-40 f_{n+6} + 288 f_{n+5} - 840 f_{n+4} + 2624 f_{n+3} - 72 f_{n+2} + 4512 f_{n+1} + 1088 f_n). \quad (5.3.1.48)$$

$$y''_{n+5} = y''_n + 5hy'''_n + \frac{h^2}{24192}(-1375 f_{n+6} + 11550 f_{n+5} - 5625 f_{n+4} + 102500 f_{n+3} + 9375 f_{n+2} + 150750 f_{n+1} + 35225 f_n). \quad (5.3.1.49)$$

$$y''_{n+6} = y''_n + 6hy'''_n + \frac{h^2}{140}(216 f_{n+5} + 54 f_{n+4} + 816 f_{n+3} + 108 f_{n+2} + 1080 f_{n+1} + 246 f_n). \quad (5.3.1.50)$$

Substituting (5.3.1.33) - (5.3.1.36) into (5.3.1.26) – (5.3.1.31) to give the third derivative of the block

$$y'''_{n+1} = y'''_n + \frac{h}{60480}(-863 f_{n+6} + 6312 f_{n+5} - 20211 f_{n+4} + 37504 f_{n+3} - 46461 f_{n+2} + 65112 f_{n+1} + 19087 f_n). \quad (5.3.1.51)$$

$$y'''_{n+2} = y'''_n + \frac{h}{3780}(-37 f_{n+6} + 264 f_{n+5} - 807 f_{n+4} + 1328 f_{n+3} + 33 f_{n+2} + 5640 f_{n+1} + 1139 f_n). \quad (5.3.1.52)$$

$$y'''_{n+3} = y'''_n + \frac{h}{2240}(-29 f_{n+6} + 216 f_{n+5} - 729 f_{n+4} + 2176 f_{n+3} + 1161 f_{n+2} + 3240 f_{n+1} + 685 f_n). \quad (5.3.1.53)$$

$$y'''_{n+4} = y'''_n + \frac{h}{1890}(-16 f_{n+6} + 96 f_{n+5} + 348 f_{n+4} + 3008 f_{n+3} + 768 f_{n+2} + 572 f_n). \qquad (5.3.1.54)$$

$$y'''_{n+5} = y'''_n + \frac{h}{12096}(-275 f_{n+6} + 5640_{n+5} + 11625 f_{n+4} + 16000 f_{n+3} + 6375 f_{n+2} + 17400 f_{n+1} + 3715 f_n). \qquad (5.3.1.55)$$

$$y'''_{n+6} = y'''_n + \frac{h}{140}(41 f_{n+6} + 216 f_{n+5} + 27 f_{n+4} + 272 f_{n+3} + 27 f_{n+2} + 216 f_{n+1} + 41 f_n). \qquad (5.3.1.56)$$

5.3.2 Properties of Six–Step Block Method for Fourth Order ODEs.

In this segment, the order, zero-stability and region of absolute stability of six–step block method for fourth order ODEs are examined.

5.3.2.1 Order of Six–Step Block Method for Fourth Order ODEs.

The approach discussed in section 3.2.2.1 is applied in finding the order of the block method (5.3.1.33 – 5.3.1.38) as shown below

$$
\begin{pmatrix}
\sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{h^m}{m!}y_n^{(m)}-\frac{95929}{3628800}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(3628800)(m!)}y_n^{(4+m)}\begin{pmatrix}112028(1)^m-115165(2)^m\\+97320(3)^m-53465(4)^m\\+16876(5)^m-2323(6)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{(2h)^m}{m!}y_n^{(m)}-\frac{4127}{14175}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(14175)(m!)}y_n^{(4+m)}\begin{pmatrix}8782(1)^m-6965(2)^m\\+5820(3)^m-3175(4)^m\\+998(5)^m-137(6)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^{m}-\sum_{m=0}^{3}\frac{(3h)^m}{m!}y_n^{(m)}-\frac{49239}{44800}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(44800)(m!)}y_n^{(4+m)}\begin{pmatrix}137052(1)^m-78975(2)^m\\+73080(3)^m-40095(4)^m\\+12636(5)^m-1737(6)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^{m}-\sum_{m=0}^{3}\frac{(4h)^m}{m!}y_n^{(m)}-\frac{78080}{28350}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(28350)(m!)}y_n^{(4+m)}\begin{pmatrix}249856(1)^m-101120(2)^m\\+122880(3)^m-64960(4)^m\\+20480(5)^m-2816(6)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^{m}-\sum_{m=0}^{3}\frac{(5h)^m}{m!}y_n^{(m)}-\frac{807125}{145152}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(145152)(m!)}y_n^{(4+m)}\begin{pmatrix}2807500(1)^m-790625(2)^m\\+1425000(3)^m-653125(4)^m\\+213500(5)^m-29375(6)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(6h)^m}{m!}y_n^{m}-\sum_{m=0}^{3}\frac{(6h)^m}{m!}y_n^{(m)}-\frac{3438}{3628800}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(3628800)(m!)}y_n^{(4+m)}\begin{pmatrix}12636(1)^m-2430(2)^m\\+6840(3)^m-2430(4)^m\\+972(5)^m-126(6)^m\end{pmatrix}
\end{pmatrix}
=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}
$$

Comparing the coefficients of h^m and y_n^m yields

$$C_0 = \begin{pmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \\ 5-5 \\ 6-6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{1}{2!} \\ \frac{(2)^2}{2!} - \frac{(2)^2}{2!} \\ \frac{(3)^2}{2!} - \frac{(3)^2}{2!} \\ \frac{(4)^2}{2!} - \frac{(4)^2}{2!} \\ \frac{(5)^2}{2!} - \frac{(5)^2}{2!} \\ \frac{(6)^2}{2!} - \frac{(6)^2}{2!} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} \frac{1}{3!} - \frac{1}{3!} \\ \frac{(2)^3}{3!} - \frac{(2)^3}{3!} \\ \frac{(3)^3}{3!} - \frac{(3)^3}{3!} \\ \frac{(4)^3}{3!} - \frac{(4)^3}{3!} \\ \frac{(5)^3}{3!} - \frac{(5)^3}{3!} \\ \frac{(6)^3}{3!} - \frac{(6)^3}{3!} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
C_4 = \begin{pmatrix}
\frac{1}{4!} - \frac{95929}{3628800} - \frac{1}{(3628800)(0!)}\begin{pmatrix} 112028(1)^0 - 115165(2)^0 + 97320(3)^0 - 53465(4)^0 + \\ 16876(5)^0 - 2323(6)^0 \end{pmatrix} \\
\frac{(2)^4}{4!} - \frac{4127}{14175} - \frac{1}{(14175)(0!)}\left(8782(1)^0 - 6965(2)^0 + 5820(3)^0 - 3175(4)^0 + 998(5)^0 - 137(6)^0\right) \\
\frac{(3)^4}{4!} - \frac{49239}{44800} - \frac{1}{(44800)(0!)}\begin{pmatrix} 137052(1)^0 - 78975(2)^0 + 73080(3)^0 - 40095(4)^0 \\ + 12636(5)^0 - 1737(6)^0 \end{pmatrix} \\
\frac{(4)^4}{4!} - \frac{78080}{28350} - \frac{1}{(28350)(0!)}\begin{pmatrix} 249856(1)^0 - 101120(2)^0 + 122880(3)^0 - 64960(4)^0 \\ + 20480(5)^0 - 2816(6)^0 \end{pmatrix} \\
\frac{(5)^4}{4!} - \frac{807125}{145152} - \frac{1}{(145152)(0!)}\begin{pmatrix} 2807500(1)^0 - 790625(2)^0 + 1425000(3)^0 - 653125(4)^0 \\ + 213500(5)^0 - 29375(6)^0 \end{pmatrix} \\
\frac{(6)^4}{4!} - \frac{3438}{3628800} - \frac{1}{(3628800)(0!)}\left(12636(1)^0 - 2430(2)^0 + 6840(3)^0 - 2430(4)^0 + 972(5)^0 - 126(6)^0\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_5 = \begin{pmatrix}
\frac{1}{5!} - \frac{1}{(3628800)(1!)}\begin{pmatrix} 112028(1)^1 - 115165(2)^1 + 97320(3)^1 - 53465(4)^1 + 16876(5)^1 \\ - 2323(6)^1 \end{pmatrix} \\
\frac{(2)^5}{5!} - \frac{1}{(14175)(1!)}\left(8782(1)^1 - 6965(2)^1 + 5820(3)^1 - 3175(4)^1 + 998(5)^1 - 137(6)^1\right) \\
\frac{(3)^5}{5!} - \frac{1}{(44800)(1!)}\begin{pmatrix} 137052(1)^1 - 78975(2)^1 + 73080(3)^1 - 40095(4)^1 \\ + 12636(5)^1 - 1737(6)^1 \end{pmatrix} \\
\frac{(4)^5}{5!} - \frac{1}{(28350)(1!)}\begin{pmatrix} 249856(1)^1 - 101120(2)^1 + 122880(3)^1 - 64960(4)^1 \\ + 20480(5)^1 - 2816(6)^1 \end{pmatrix} \\
\frac{(5)^5}{5!} - \frac{1}{(145152)(1!)}\begin{pmatrix} 2807500(1)^1 - 790625(2)^1 + 1425000(3)^1 - 653125(4)^1 \\ + 213500(5)^1 - 29375(6)^1 \end{pmatrix} \\
\frac{(6)^5}{5!} - \frac{1}{(3628800)(1!)}\left(12636(1)^1 - 2430(2)^1 + 6840(3)^1 - 2430(4)^1 + 972(5)^1 - 126(6)^1\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_6 = \begin{pmatrix}
\frac{1}{6!} - \frac{1}{(3628800)(2!)}\begin{pmatrix} 112028(1)^2 - 115165(2)^2 + 97320(3)^2 - 53465(4)^2 + 16876(5)^2 \\ - 2323(6)^2 \end{pmatrix} \\
\frac{(2)^6}{6!} - \frac{1}{(14175)(2!)}\left(8782(1)^2 - 6965(2)^2 + 5820(3)^2 - 3175(4)^2 + 998(5)^2 - 137(6)^2\right) \\
\frac{(3)^6}{6!} - \frac{1}{(44800)(2!)}\begin{pmatrix} 137052(1)^2 - 78975(2)^2 + 73080(3)^2 - 40095(4)^2 \\ + 12636(5)^2 - 1737(6)^2 \end{pmatrix} \\
\frac{(4)^6}{6!} - \frac{1}{(28350)(2!)}\begin{pmatrix} 249856(1)^2 - 101120(2)^2 + 122880(3)^2 - 64960(4)^2 \\ + 20480(5)^2 - 2816(6)^2 \end{pmatrix} \\
\frac{(5)^6}{6!} - \frac{1}{(145152)(2!)}\begin{pmatrix} 2807500(1)^2 - 790625(2)^2 + 1425000(3)^2 - 653125(4)^2 \\ + 213500(5)^2 - 29375(6)^2 \end{pmatrix} \\
\frac{(6)^6}{6!} - \frac{1}{(3628800)(2!)}\left(12636(1)^2 - 2430(2)^2 + 6840(3)^2 - 2430(4)^2 + 972(5)^2 - 126(6)^2\right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$C_7 = \begin{pmatrix} \frac{1}{7!} - \frac{1}{(3628800)(3!)}\begin{pmatrix} 112028(1)^3 - 115165(2)^3 + 97320(3)^3 - 53465(4)^3 + 16876(5)^3 \\ -2323(6)^3 \end{pmatrix} \\ \frac{(2)^7}{7!} - \frac{1}{(14175)(3!)}\left(8782(1)^3 - 6965(2)^3 + 5820(3)^3 - 3175(4)^3 + 998(5)^3 - 137(6)^3\right) \\ \frac{(3)^7}{7!} - \frac{1}{(44800)(3!)}\begin{pmatrix} 137052(1)^3 - 78975(2)^3 + 73080(3)^3 - 40095(4)^3 \\ +12636(5)^3 - 1737(6)^3 \end{pmatrix} \\ \frac{(4)^7}{7!} - \frac{1}{(28350)(3!)}\begin{pmatrix} 249856(1)^3 - 101120(2)^3 + 122880(3)^3 - 64960(4)^3 \\ +20480(5)^3 - 2816(6)^3 \end{pmatrix} \\ \frac{(5)^7}{7!} - \frac{1}{(145152)(3!)}\begin{pmatrix} 2807500(1)^3 - 790625(2)^3 + 1425000(3)^3 - 653125(4)^3 \\ +213500(5)^3 - 29375(6)^3 \end{pmatrix} \\ \frac{(6)^7}{7!} - \frac{1}{(3628800)(3!)}\left(12636(1)^3 - 2430(2)^3 + 6840(3)^3 - 2430(4)^3 + 972(5)^3 - 126(6)^3\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_8 = \begin{pmatrix} \frac{1}{8!} - \frac{1}{(3628800)(4!)}\begin{pmatrix} 112028(1)^4 - 115165(2)^4 + 97320(3)^4 - 53465(4)^4 + 16876(5)^4 \\ -2323(6)^4 \end{pmatrix} \\ \frac{(2)^8}{8!} - \frac{1}{(14175)(4!)}\left(8782(1)^4 - 6965(2)^4 + 5820(3)^4 - 3175(4)^4 + 998(5)^4 - 137(6)^4\right) \\ \frac{(3)^8}{8!} - \frac{1}{(44800)(4!)}\begin{pmatrix} 137052(1)^4 - 78975(2)^4 + 73080(3)^4 - 40095(4)^4 \\ +12636(5)^4 - 1737(6)^4 \end{pmatrix} \\ \frac{(4)^8}{8!} - \frac{1}{(28350)(4!)}\begin{pmatrix} 249856(1)^4 - 101120(2)^4 + 122880(3)^4 - 64960(4)^4 \\ +20480(5)^4 - 2816(6)^4 \end{pmatrix} \\ \frac{(5)^8}{8!} - \frac{1}{(145152)(4!)}\begin{pmatrix} 2807500(1)^4 - 790625(2)^4 + 1425000(3)^4 - 653125(4)^4 \\ +213500(5)^4 - 29375(6)^4 \end{pmatrix} \\ \frac{(6)^8}{8!} - \frac{1}{(3628800)(4!)}\left(12636(1)^4 - 2430(2)^4 + 6840(3)^4 - 2430(4)^4 + 972(5)^4 - 126(6)^4\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_9 = \begin{pmatrix} \frac{1}{9!} - \frac{1}{(3628800)(5!)}\begin{pmatrix} 112028(1)^5 - 115165(2)^5 + 97320(3)^5 - 53465(4)^5 + 16876(5)^5 \\ -2323(6)^5 \end{pmatrix} \\ \frac{(2)^9}{9!} - \frac{1}{(14175)(5!)}\left(8782(1)^5 - 6965(2)^5 + 5820(3)^5 - 3175(4)^5 + 998(5)^5 - 137(6)^5\right) \\ \frac{(3)^9}{9!} - \frac{1}{(44800)(5!)}\begin{pmatrix} 137052(1)^5 - 78975(2)^5 + 73080(3)^5 - 40095(4)^5 \\ +12636(5)^5 - 1737(6)^5 \end{pmatrix} \\ \frac{(4)^9}{9!} - \frac{1}{(28350)(5!)}\begin{pmatrix} 249856(1)^5 - 101120(2)^5 + 122880(3)^5 - 64960(4)^5 \\ +20480(5)^5 - 2816(6)^5 \end{pmatrix} \\ \frac{(5)^9}{9!} - \frac{1}{(145152)(5!)}\begin{pmatrix} 2807500(1)^5 - 790625(2)^5 + 1425000(3)^5 - 653125(4)^5 \\ +213500(5)^5 - 29375(6)^5 \end{pmatrix} \\ \frac{(6)^9}{9!} - \frac{1}{(3628800)(5!)}\left(12636(1)^5 - 2430(2)^5 + 6840(3)^5 - 2430(4)^5 + 972(5)^5 - 126(6)^5\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_{10}=\begin{pmatrix} \dfrac{1}{10!}-\dfrac{1}{(3628800)(6!)}\begin{pmatrix}112028(1)^6-115165(2)^6+97320(3)^6-53465(4)^6+16876(5)^6\\ -2323(6)^6\end{pmatrix} \\ \dfrac{(2)^{10}}{10!}-\dfrac{1}{(14175)(6!)}\left(8782(1)^6-6965(2)^6+5820(3)^6-3175(4)^6+998(5)^6-137(6)^6\right) \\ \dfrac{(3)^{10}}{10!}-\dfrac{1}{(44800)(6!)}\begin{pmatrix}137052(1)^6-78975(2)^6+73080(3)^6-40095(4)^6\\ +12636(5)^6-1737(6)^6\end{pmatrix} \\ \dfrac{(4)^{10}}{10!}-\dfrac{1}{(28350)(6!)}\begin{pmatrix}249856(1)^6-101120(2)^6+122880(3)^6-64960(4)^6\\ +20480(5)^6-2816(6)^6\end{pmatrix} \\ \dfrac{(5)^{10}}{10!}-\dfrac{1}{(145152)(6!)}\begin{pmatrix}2807500(1)^6-790625(2)^6+1425000(3)^6-653125(4)^6\\ +213500(5)^6-29375(6)^6\end{pmatrix} \\ \dfrac{(6)^{10}}{10!}-\dfrac{1}{(3628800)(6!)}\left(12636(1)^6-2430(2)^6+6840(3)^6-2430(4)^6+972(5)^6-126(6)^6\right) \end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_{11}=\begin{pmatrix} \dfrac{1}{11!}-\dfrac{1}{(3628800)(7!)}\begin{pmatrix}112028(1)^7-115165(2)^7+97320(3)^7-53465(4)^7+16876(5)^7\\ -2323(6)^7\end{pmatrix} \\ \dfrac{(2)^{11}}{11!}-\dfrac{1}{(14175)(7!)}\left(8782(1)^7-6965(2)^7+5820(3)^7-3175(4)^7+998(5)^7-137(6)^7\right) \\ \dfrac{(3)^{11}}{11!}-\dfrac{1}{(44800)(7!)}\begin{pmatrix}137052(1)^7-78975(2)^7+73080(3)^7-40095(4)^7\\ +12636(5)^7-1737(6)^7\end{pmatrix} \\ \dfrac{(4)^{11}}{11!}-\dfrac{1}{(28350)(7!)}\begin{pmatrix}249856(1)^7-101120(2)^7+122880(3)^7-64960(4)^7\\ +20480(5)^7-2816(6)^7\end{pmatrix} \\ \dfrac{(5)^{11}}{11!}-\dfrac{1}{(145152)(7!)}\begin{pmatrix}2807500(1)^7-790625(2)^7+1425000(3)^7-653125(4)^7\\ +213500(5)^7-29375(6)^7\end{pmatrix} \\ \dfrac{(6)^{11}}{11!}-\dfrac{1}{(3628800)(7!)}\left(12636(1)^7-2430(2)^7+6840(3)^7-2430(4)^7+972(5)^7-126(6)^7\right) \end{pmatrix}=\begin{pmatrix}\dfrac{39}{74183}\\ \dfrac{109}{13912}\\ \dfrac{243}{7700}\\ \dfrac{361}{4457}\\ \dfrac{430}{2599}\\ \dfrac{81}{275}\end{pmatrix}$$

Hence, the block has order $(7,7,7,7,7,7)^T$ with error constants

$$\left(\frac{39}{74183},\frac{109}{13912},\frac{243}{7700},\frac{361}{4457},\frac{430}{2599},\frac{81}{275}\right)^T$$

5.3.2.2 Zero Stability of Six–Step Block Method for Fourth Order ODEs.

Equation (3.2.2.2.1) is applied to six-step block method (5.3.1.33 – 5.3.1.38). This gives

$$\det[rA^{(0)} - A^{(1)}] = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0$$

whis implies $r = 0,0,0,0,0,1$. Hence, the method is zero stable.

5.3.2.3 Consistency and Convergence of Six–Step Block Method for Fourth Order ODEs.

The block method (5.3.1.33 – 5.3.1.38) is consistent because it fulfills the conditions highlighted in Definition 1.4. Hence, it is also convergent since it is zero-stable and consistent.

5.3.2.4 Region of Absolute Stability of Six–Step Block Method for Fourth Order ODEs.

Equation (3.2.2.4.2) is applied to six-step block (5.3.1.33 – 5.3.1.38) we have

$$\bar{h}(\theta, h) = \frac{A - B}{C + D}$$

where

$$A = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{4i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{5i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{6i\theta} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C=\begin{pmatrix} \frac{112028}{3628800}e^{i\theta} & -\frac{115165}{3628800}e^{2i\theta} & \frac{97320}{3628800}e^{3i\theta} & -\frac{53465}{3628800}e^{4i\theta} & \frac{16876}{3628800}e^{5i\theta} & -\frac{2323}{3628800}e^{6i\theta} \\ \frac{8782}{14175}e^{i\theta} & -\frac{6965}{14175}e^{2i\theta} & \frac{5820}{14175}e^{3i\theta} & -\frac{3175}{14175}e^{4i\theta} & \frac{998}{14175}e^{5i\theta} & -\frac{137}{14175}e^{6i\theta} \\ \frac{137052}{44800}e^{i\theta} & -\frac{78975}{44800}e^{2i\theta} & \frac{73080}{44800}e^{3i\theta} & -\frac{40095}{44800}e^{4i\theta} & \frac{12636}{44800}e^{5i\theta} & -\frac{1737}{44800}e^{6i\theta} \\ \frac{249856}{28350}e^{i\theta} & -\frac{101120}{28350}e^{2i\theta} & \frac{122880}{28350}e^{3i\theta} & -\frac{64960}{28350}e^{4i\theta} & \frac{20480}{28350}e^{5i\theta} & -\frac{2816}{28350}e^{6i\theta} \\ \frac{2807500}{145152}e^{i\theta} & -\frac{790625}{145152}e^{2i\theta} & \frac{1425000}{145152}e^{3i\theta} & -\frac{653125}{145152}e^{4i\theta} & \frac{213500}{145152}e^{5i\theta} & -\frac{29375}{145152}e^{6i\theta} \\ \frac{12636}{350}e^{i\theta} & -\frac{2430}{350}e^{2i\theta} & \frac{6840}{350}e^{3i\theta} & -\frac{2430}{350}e^{4i\theta} & \frac{972}{350}e^{5i\theta} & -\frac{126}{350}e^{6i\theta} \end{pmatrix}$$

$$D=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{95929}{3628800} \\ 0 & 0 & 0 & 0 & 0 & \frac{4127}{14175} \\ 0 & 0 & 0 & 0 & 0 & \frac{49239}{44800} \\ 0 & 0 & 0 & 0 & 0 & \frac{78080}{28350} \\ 0 & 0 & 0 & 0 & 0 & \frac{807125}{145152} \\ 0 & 0 & 0 & 0 & 0 & \frac{3438}{350} \end{pmatrix}$$

The matrix above is simplified, after finding the determinant and equating the imaginary part to zero we have

$$\bar{h}(\theta,h)=\frac{(3.3376\mathrm{E}+86)\cos 6\theta-(3.3376\mathrm{E}+86)}{(9.6573E+79)\cos 6\theta-(4.2077\mathrm{E}+82)}$$

The value of $\bar{h}(\theta,h)$ is evaluated at the intervals of θ of 30^{0} and this produced the results tabulated in Table 5.2.

Table 5.2

Interval of Absolute Stability of Six–Step Block Method for Fourth Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta,h)$	0	15828.05	0	15828.05	0	15828.05	0

Hence, the interval of absolute stability is (0, 15828.05). This is shown in the diagram below

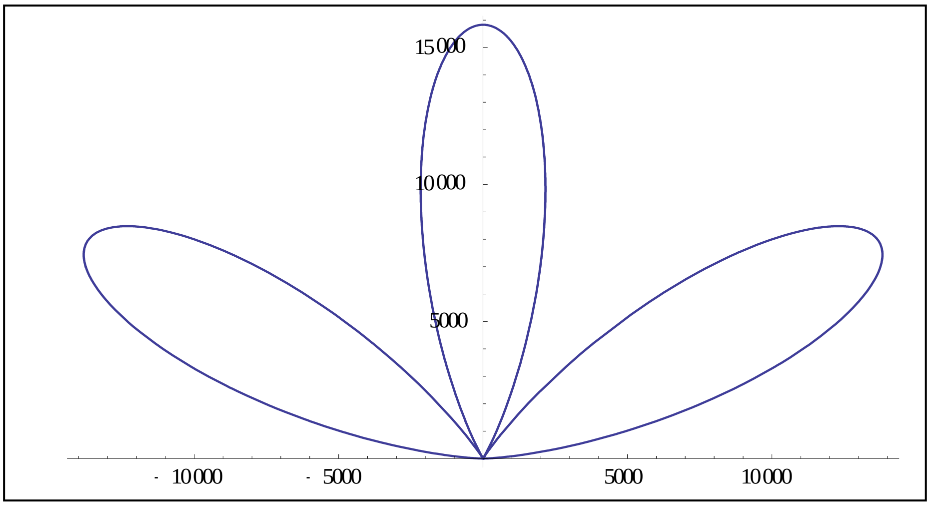

Figure 5.4. Region of absolute stability of six–step block method for fourth order ODEs.

5.4 Seven–Step Block Method for Fourth Order ODEs.

This section contains the derivation of seven–step block method for solving fourth order ODEs. It also includes the properties of the block method.

5.4.1 Derivation of Seven–Step Block Method for Fourth Order ODEs.

The power series of the form (5.2.1.1) is considered as an approximate solution to the general fourth order problem of the form (5.2.1.2) where in (5.2.1.1) k =7. The first, second and third derivatives of (5.2.1.1) are given in (5.2.1.3), (5.2.1.4), (5.2.1.5) and (5.2.1.6).

Interpolating equation (5.2.1.1) at the points $x = x_{n+i}, i = 2(1)5$ and collocating (5.2.1.6) at $x = x_{n+i}, i = 0(1)7$. The diagram is demonstrated below

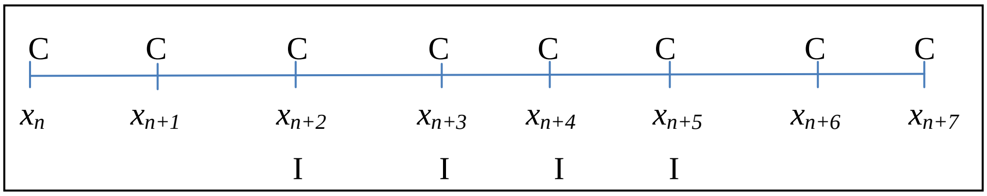

Figure 5.5. Seven–step interpolation and collocation method for fourth order ODEs.

This technique produces

$$AX = B \tag{5.4.1.1}$$

where

$$X = \left(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}\right)^T$$

$$B = \left(y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}, f_{n+7}\right)^T$$

$$A = \begin{pmatrix}
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 & x_{n+2}^{10} & x_{n+2}^{11} \\
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 & x_{n+3}^{10} & x_{n+3}^{11} \\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 & x_{n+4}^{10} & x_{n+4}^{11} \\
1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 & x_{n+5}^9 & x_{n+6}^{10} & x_{n+7}^{11} \\
0 & 0 & 0 & 0 & 24 & 120x_n & 360x_n^2 & 840x_n^3 & 1680x_n^4 & 3024x_n^5 & 5040x_n^6 & 7920x_n^7 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+1} & 360x_{n+1}^2 & 840x_{n+1}^3 & 1680x_{n+1}^4 & 3024x_{n+1}^5 & 5040x_{n+1}^6 & 7920x_{n+1}^7 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+2} & 360x_{n+2}^2 & 840x_{n+2}^3 & 1680x_{n+2}^4 & 3024x_{n+2}^5 & 5040x_{n+2}^6 & 7920x_{n+2}^7 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+3} & 360x_{n+3}^2 & 840x_{n+3}^3 & 1680x_{n+3}^4 & 3024x_{n+3}^5 & 5040x_{n+3}^6 & 7920x_{n+3}^7 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+4} & 360x_{n+4}^2 & 840x_{n+4}^3 & 1680x_{n+4}^4 & 3024x_{n+4}^5 & 5040x_{n+4}^6 & 7920x_{n+4}^7 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+5} & 360x_{n+5}^2 & 840x_{n+5}^3 & 1680x_{n+5}^4 & 3024x_{n+5}^5 & 5040x_{n+5}^6 & 7920x_{n+5}^7 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+6} & 360x_{n+6}^2 & 840x_{n+6}^3 & 1680x_{n+6}^4 & 3024x_{n+6}^5 & 5040x_{n+6}^6 & 7920x_{n+6}^7 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+7} & 360x_{n+7}^2 & 840x_{n+7}^3 & 1680x_{n+7}^4 & 3024x_{n+7}^5 & 5040x_{n+7}^6 & 7920x_{n+7}^7
\end{pmatrix}$$

Gaussian elimination method is applied in finding the values of *a's* in (5.4.1.1). This gives

$$a_0 = \quad 10\, y_{n+2} - 20\, y_{n+3} + 15\, y_{n+4} - 4\, y_{n+5} + \frac{h^4}{15120}\,(9\, f_n + 2335\, f_{n+1} + 20850\, f_{n+2} +$$

$$41745\, f_{n+3} + 10945\, f_{n+4} + 339\, f_{n+5} + 60\, f_{n+6} - 5\, f_{n+7}) + \frac{h^3 x_n}{9979200}\,(86853\, f_n +$$

$$4776925\, f_{n+1} + 20392980\, f_{n+2} + 31203915\, f_{n+3} + 7753465\, f_{n+4} - 213513\, f_{n+5} +$$

$$35190\, f_{n+6} - 2615\, f_{n+7}) + \frac{x_n^4}{24} f_n + \frac{x_n^{11}}{39916800h^7} (f_{n-} 7\, f_{n+1} + 21\, f_{n+2} - 35\, f_{n+3} +$$

$$35\, f_{n+4} - 21\, f_{n+5} + 7\, f_{n+6} - f_{n+7}) + \frac{x_n}{6h}(47\, y_{n+2} - 114\, y_{n+3} + 93\, y_{n+4} - 26\, y_{n+5}) +$$

$$\frac{x_n^4}{50400h} (1089\, f_n - 2940\, f_{n+1} + 4410\, f_{n+2} - 4900\, f_{n+3} + 3675\, f_{n+4} - 1764\, f_{n+5}$$

$$+490\, f_{n+6} - 60\, f_{n+7}) + \frac{x_n^2 h^2}{604800}(19162\, f_n + 326909\, f_{n+1} + 561299\, f_{n+2} +$$

$$751735\, f_{n+3} + 117780\, f_{n+4} + 16783\, f_{n+5} - 5129\, f_{n+6} + 661\, f_{n+7}) + \frac{x_n^{10}}{3628800h^6} (4\, f_n$$

$$-27\, f_{n+1} + 78\, f_{n+2} - 125\, f_{n+3} + 120\, f_{n+4} - 69\, f_{n+5} + 22\, f_{n+6} - 3\, f_{n+7}) + \frac{x_n^8}{291420h^4} (56\, f_n$$

$$-333\, f_{n+1} + 852\, f_{n+2} - 1219\, f_{n+3} + 1056\, f_{n+4} - 555\, f_{n+5} + 164\, f_{n+6} - 21\, f_{n+7}) +$$

$$\frac{x_n^9}{2177289h^5} (46\, f_n - 295\, f_{n+1} + 810\, f_{n+2} - 1235\, f_{n+3} + 1130\, f_{n+4} - 621\, f_{n+5} + 190\, f_{n+6}$$

$$-25\, f_{n+7}) + \frac{x_n^3}{6h^3}(y_{n+2} - 3y_{n+3} + 3y_{n+4} - y_{n+5}) + \frac{x_n^7}{604800h^3}(967\, f_n - 5104\, f_{n+1} +$$

$$11787\, f_{n+2} - 15560\, f_{n+3} + 12725\, f_{n+4} - 6432\, f_{n+5} + 1849\, f_{n+6} - 232\, f_{n+7})$$

$$a_1 = \frac{h^3}{9979200} (86853\, f_n + 4776925\, f_{n+1} + 20392980\, f_{n+2} + 31203915\, f_{n+3} +$$

$$7753465\, f_{n+4} - 213513\, f_{n+5} + 35190\, f_{n+6} - 2615\, f_{n+7}) - \frac{x_n^3}{6} f_n - \frac{x_n^4}{10080h}(1089\, f_n -$$

$$2940\, f_{n+1} + 4410\, f_{n+2} - 4900\, f_{n+3} + 3675\, f_{n+4} - 1764\, f_{n+5} + 490\, f_{n+6} - 60\, f_{n+7}) -$$

$$\frac{x_n^2 h}{3628800} (535111\, f_n + 2767238\, f_{n+1} + 473931\, f_{n+2} + 2944390\, f_{n+3} - 747475\, f_{n+4}$$

$$+498474\, f_{n+5} - 138143\, f_{n+6} + 16874\, f_{n+7}) - \frac{x_n h^2}{302400}(19162\, f_n + 326909\, f_{n+1}$$

$$+561299\, f_{n+2} + 751735\, f_{n+3} + 117780\, f_{n+4} + 16783\, f_{n+5} - 5129\, f_{n+6} + 661\, f_{n+7})$$

$$-\frac{x_n^{10}}{3628800h^7} (f_{n-} 7\, f_{n+1} + 21\, f_{n+2} - 35\, f_{n+3} + 35\, f_{n+4} - 21\, f_{n+5} + 7\, f_{n+6} - f_{n+7}) -$$

$$+\frac{x_n^2}{2h^2} (4\, y_{n+2} - 11y_{n+3} + 10y_{n+4} - 3y_{n+5}) + \frac{x_n^6}{129600h^2}(938\, f_n - 4014\, f_{n+1} + 7911\, f_{n+2}$$

$$-9490\, f_{n+3} + 7380\, f_{n+4} - 3618\, f_{n+5} + 1019\, f_{n+6} - 126\, f_{n+7}) + \frac{x_n^3}{6h^3} (y_{n+2} - 3y_{n+3} +$$

$$3y_{n+4} - y_{n+5}) + \frac{x_n^7}{604800h^3}(967\, f_n - 5104\, f_{n+1} + 11787\, f_{n+2} - 15560\, f_{n+3} + 12725\, f_{n+4}$$

$$-6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7}) + \frac{x_n^5}{50400h} (1089 f_n - 2940 f_{n+1} + 4410 f_{n+2} -$$

$$4900 f_{n+3} + 3675 f_{n+4} - 1764 f_{n+5} + 490 f_{n+6} - 60 f_{n+7}) + \frac{x_n^3 h}{10886400} (535111 f_n +$$

$$2767238 f_{n+1} + 473931 f_{n+2} + 2944390 f_{n+3} - 747475 f_{n+4} + 498474 f_{n+5} -$$

$$138143 f_{n+6} - +16874 f_{n+7}) + \frac{x_n^2}{2h^2} (4 y_{n+2} - 11 y_{n+3} + 10 y_{n+4} - 3 y_{n+5})$$

$$\frac{x_n}{h^2} (4 y_{n+2} - 11 y_{n+3} + 10 y_{n+4} - 3 y_{n+5}) - \frac{x_n^5}{21600h^2} (938 f_n - 4014 f_{n+1} + 7911 f_{n+2}$$

$$-9490 f_{n+3} + 7380 f_{n+4} - 3618 f_{n+5} + 1019 f_{n+6} - 126 f_{n+7}) - \frac{x_n^9}{3628800h^6} (4 f_n -$$

$$27 f_{n+1} + 78 f_{n+2} - 125 f_{n+3} + 120 f_{n+4} - 69 f_{n+5} + 22 f_{n+6} - 3 f_{n+7}) - \frac{x_n^7}{30240h^4} (56 f_n$$

$$-333 f_{n+1} + 852 f_{n+2} - 1219 f_{n+3} + 1056 f_{n+4} - 555 f_{n+5} + 164 f_{n+6} - 21 f_{n+7}) -$$

$$\frac{x_n^8}{241920h^5} (46 f_n - 295 f_{n+1} + 810 f_{n+2} - 1235 f_{n+3} + 1130 f_{n+4} - 621 f_{n+5} + 190 f_{n+6}$$

$$-25 f_{n+7}) - \frac{x_n^2}{2h^3} (y_{n+2} - 3y_{n+3} + 3y_{n+4} - y_{n+5}) - \frac{x_n^6}{86400h^3} (967 f_n - 5104 f_{n+1} +$$

$$11787 f_{n+2} - 15560 f_{n+3} + 12725 f_{n+4} - 6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7})$$

$$a_2 = \frac{h^2}{604800} (19162 f_n + 326909 f_{n+1} + 561299 f_{n+2} + 751735 f_{n+3} + 117780 f_{n+4}$$

$$+16783 f_{n+5} - 5129 f_{n+6} + 661 f_{n+7}) + \frac{x_n^2}{4} f_n + \frac{1}{2h^2} (4 y_{n+2} - 11 y_{n+3} + 10 y_{n+4} - 3 y_{n+5})$$

$$+ \frac{x_n^4}{8640h^2} (938 f_n - 4014 f_{n+1} + 7911 f_{n+2} - 9490 f_{n+3} + 7380 f_{n+4} - 3618 f_{n+5} +$$

$$1019 f_{n+6} - 126 f_{n+7}) + \frac{x_n^9}{725760h^7} (f_{n-} + 7 f_{n+1} + 21 f_{n+2} - 35 f_{n+3} + 35 f_{n+4} - 21 f_{n+5}$$

$$+7 f_{n+6} - f_{n+7}) + \frac{x_n}{2h^3} (y_{n+2} - 3y_{n+3} + 3y_{n+4} - y_{n+5}) + \frac{x_n^5}{28800h^3} (967 f_n - 5104 f_{n+1}$$

$$+11787 f_{n+2} - 15560 f_{n+3} + 12725 f_{n+4} - 6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7}) +$$

$$\frac{x_n^8}{80640h^6} (4 f_n - 27 f_{n+1} + 78 f_{n+2} - 125 f_{n+3} + 120 f_{n+4} - 69 f_{n+5} + 22 f_{n+6} - 3 f_{n+7})$$

$$+ \frac{x_n^6}{8640h^4} (56 f_n - 333 f_{n+1} + 852 f_{n+2} - 1219 f_{n+3} + 1056 f_{n+4} - 555 f_{n+5} + 164 f_{n+6}$$

$$-21 f_{n+7}) + \frac{x_n^7}{60480h^5} (46 f_n - 295 f_{n+1} + 810 f_{n+2} - 1235 f_{n+3} + 1130 f_{n+4} - 621 f_{n+5}$$

$$+190 f_{n+6} - 25 f_{n+7}) + \frac{x_n^3}{5040h} (1089 f_n - 2940 f_{n+1} + 4410 f_{n+2} - 4900 f_{n+3} + 3675 f_{n+4}$$

$$-1764 f_{n+5} + 490 f_{n+6} - 60 f_{n+7}) + \frac{x_n h}{3628800} (535111 f_n + 2767238 f_{n+1} + 473931 f_{n+2}$$
$$+2944390 f_{n+3} - 747475 f_{n+4} + 498474 f_{n+5} - 138143 f_{n+6} + 16874 f_{n+7})$$

$$a_3 = \frac{h}{10886400} (535111 f_n + 2767238 f_{n+1} + 473931 f_{n+2} + 2944390 f_{n+3} -$$
$$747475 f_{n+4} + 498474 f_{n+5} - 138143 f_{n+6} + 16874 f_{n+7}) - \frac{x_n}{6} f_n - \frac{1}{6} (y_{n+2} - 3y_{n+3} +$$
$$3y_{n+4} - y_{n+5}) - \frac{x_n^4}{17280h^3}(967 f_n - 5104 f_{n+1} + 11787 f_{n+2} - 15560 f_{n+3} + 12725 f_{n+4}$$
$$-6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7}) - \frac{x_n^8}{725760h^7} (f_{n\text{-}} 7 f_{n+1} + 21 f_{n+2} - 35 f_{n+3} + 35 f_{n+4}$$
$$-21 f_{n+5} + 7 f_{n+6} - f_{n+7}) - \frac{x_n^7}{30240h^6} (4 f_n - 27 f_{n+1} + 78 f_{n+2} - 125 f_{n+3} + 120 f_{n+4}$$
$$-69 f_{n+5} + 22 f_{n+6} - 3 f_{n+7}) \frac{h^2}{604800}(19162 f_n + 326909 f_{n+1} + 561299 f_{n+2} + 751735 f_{n+3}$$
$$+117780 f_{n+4} + 16783 f_{n+5} - 5129 f_{n+6} + 661 f_{n+7}) + + \frac{1}{2h^2}(4 y_{n+2} - 11y_{n+3} + 10y_{n+4}$$
$$-3y_{n+5}) + \frac{x_n^5}{28800h^3}(967 f_n - 5104 f_{n+1} + 11787 f_{n+2} - 15560 f_{n+3} + 12725 f_{n+4} -$$
$$6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7}) - \frac{x_n^4}{17280h^3} (967 f_n - 5104 f_{n+1} + 11787 f_{n+2} -$$
$$15560 f_{n+3} + 12725 f_{n+4} - 6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7}) - \frac{x_n^6}{25920h^5} (46 f_n - 295 f_{n+1}$$
$$+810 f_{n+2} - 1235 f_{n+3} + 1130 f_{n+4} - 621 f_{n+5} + 190 f_{n+6} - 25 f_{n+7}) - \frac{x_n^2}{5040h}(1089 f_n -$$
$$2940 f_{n+1} + 4410 f_{n+2} - 4900 f_{n+3} + 3675 f_{n+4} - 1764 f_{n+5} + 490 f_{n+6} - 60 f_{n+7}) -$$
$$\frac{x_n^3}{6480h^2}(938 f_n - 4014 f_{n+1} + 7911 f_{n+2} - 9490 f_{n+3} + 7380 f_{n+4} - 3618 f_{n+5} +$$
$$1019 f_{n+6} - 126 f_{n+7}).$$

$$a_4 = \frac{x_n}{10080h}(1089 f_n - 2940 f_{n+1} + 4410 f_{n+2} - 4900 f_{n+3} + 3675 f_{n+4} - 1764 f_{n+5} +$$
$$490 f_{n+6} - 60 f_{n+7}) + \frac{1}{24} f_n + \frac{x_n^7}{120960h^7} (f_{n\text{-}} 7 f_{n+1} + 21 f_{n+2} - 35 f_{n+3} + 35 f_{n+4} - 21 f_{n+5}$$
$$+7 f_{n+6} - f_{n+7}) + \frac{x_n^6}{17280h^6} (4 f_n - 27 f_{n+1} + 78 f_{n+2} - 125 f_{n+3} + 120 f_{n+4} - 69 f_{n+5} +$$
$$22 f_{n+6} - 3 f_{n+7}) + \frac{x_n^4}{3456h^4} (56 f_n \text{-} 333 f_{n+1} + 852 f_{n+2} - 1219 f_{n+3} + 1056 f_{n+4} - 555 f_{n+5}$$

$$+164 f_{n+6} - 21 f_{n+7}) + \frac{x_n^5}{17280h^5} (46 f_n - 295 f_{n+1} + 810 f_{n+2} - 1235 f_{n+3} + 1130 f_{n+4} - 621 f_{n+5} + 190 f_{n+6} - 25 f_{n+7}) + \frac{x_n^2}{8640h^2} (938 f_n - 4014 f_{n+1} + 7911 f_{n+2} - 9490 f_{n+3} +$$

$$7380 f_{n+4} - 3618 f_{n+5} + 1019 f_{n+6} - 126 f_{n+7}) + \frac{x_n^3}{17280h^3} (967 f_n - 5104 f_{n+1} + 11787 f_{n+2} - 15560 f_{n+3} + 12725 f_{n+4} - 6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7}).$$

$$a_5 = -\frac{1}{50400h}(1089 f_n - 2940 f_{n+1} + 4410 f_{n+2} - 4900 f_{n+3} + 3675 f_{n+4} - 1764 f_{n+5} + 490 f_{n+6} - 60 f_{n+7}) - \frac{x_n}{21600h^2}(938 f_n - 4014 f_{n+1} + 7911 f_{n+2} - 9490 f_{n+3} + 7380 f_{n+4} - 3618 f_{n+5} + 1019 f_{n+6} - 126 f_{n+7}) - \frac{x_n^6}{86400h^7} (f_{n} - 7 f_{n+1} + 21 f_{n+2} - 35 f_{n+3} + 35 f_{n+4} - 21 f_{n+5} + 7 f_{n+6} - f_{n+7}) - \frac{x_n^5}{14400h^6} (4 f_n - 27 f_{n+1} + 78 f_{n+2} - 125 f_{n+3} + 120 f_{n+4} - 69 f_{n+5} + 22 f_{n+6} - 3 f_{n+7}) - \frac{x_n^3}{4320h^4} (56 f_n - 333 f_{n+1} + 852 f_{n+2} - 1219 f_{n+3} + 1056 f_{n+4} - 555 f_{n+5} + 164 f_{n+6} - 21 f_{n+7}) - \frac{x_n^4}{17280h^5} (46 f_n - 295 f_{n+1} + 810 f_{n+2} - 1235 f_{n+3} + 1130 f_{n+4} - 621 f_{n+5} + 190 f_{n+6} - 25 f_{n+7}) - \frac{x_n^2}{28800h^3} (967 f_n - 5104 f_{n+1} + 11787 f_{n+2} - 15560 f_{n+3} + 12725 f_{n+4} - 6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7})$$

$$a_6 = \frac{1}{129600h}(938 f_n - 4014 f_{n+1} + 7911 f_{n+2} - 9490 f_{n+3} + 7380 f_{n+4} - 3618 f_{n+5} + 1019 f_{n+6} - 126 f_{n+7}) + \frac{x_n}{86400h^3} (967 f_n - 5104 f_{n+1} + 11787 f_{n+2} - 15560 f_{n+3} + 12725 f_{n+4} - 6432 f_{n+5} + 1849 f_{n+6} - 232 f_{n+7}) + \frac{x_n^5}{86400h^7} (f_n - 7 f_{n+1} + 21 f_{n+2} - 35 f_{n+3} + 35 f_{n+4} - 21 f_{n+5} + 7 f_{n+6} - f_{n+7}) + \frac{x_n^4}{17280h^6} (4 f_n - 27 f_{n+1} + 78 f_{n+2} - 125 f_{n+3} + 120 f_{n+4} - 69 f_{n+5} + 22 f_{n+6} - 3 f_{n+7}) + \frac{x_n^2}{8640h^4} (56 f_n - 333 f_{n+1} + 852 f_{n+2} - 1219 f_{n+3} + 1056 f_{n+4} - 555 f_{n+5} + 164 f_{n+6} - 21 f_{n+7}) + \frac{x_n^3}{25920h^5} (46 f_n - 295 f_{n+1} + 810 f_{n+2} - 1235 f_{n+3} + 1130 f_{n+4} - 621 f_{n+5} + 190 f_{n+6} - 25 f_{n+7})$$

$$a_7 = -\frac{1}{604800h^3}(967f_n - 5104f_{n+1} + 11787f_{n+2} - 15560f_{n+3} + 12725f_{n+4} - 6432f_{n+5} + 1849f_{n+6} - 232f_{n+7}) - \frac{x_n}{30240h^4}(56f_n - 333f_{n+1} + 852f_{n+2} - 1219f_{n+3} + 1056f_{n+4} - 555f_{n+5} + 164f_{n+6} - 21f_{n+7}) - \frac{x_n^4}{120960h^7}(f_n - 7f_{n+1} + 21f_{n+2} - 35f_{n+3} + 35f_{n+4} - 21f_{n+5} + 7f_{n+6} - f_{n+7}) - \frac{x_n^3}{30240h^6}(4f_n - 27f_{n+1} + 78f_{n+2} - 125f_{n+3} + 120f_{n+4} - 69f_{n+5} + 22f_{n+6} - 3f_{n+7}) - \frac{x_n^2}{60480h^5}(46f_n - 295f_{n+1} + 810f_{n+2} - 1235f_{n+3} + 1130f_{n+4} - 621f_{n+5} + 190f_{n+6} - 25f_{n+7})$$

$$a_8 = \frac{1}{241920h^4}(56f_n - 333f_{n+1} + 852f_{n+2} - 1219f_{n+3} + 1056f_{n+4} - 555f_{n+5} + 164f_{n+6} - 21f_{n+7}) + \frac{x_n}{241920h^5}(46f_n - 295f_{n+1} + 810f_{n+2} - 1235f_{n+3} + 1130f_{n+4} - 621f_{n+5} + 190f_{n+6} - 25f_{n+7}) + \frac{x_n^3}{241920h^7}(f_n - 7f_{n+1} + 21f_{n+2} - 35f_{n+3} + 35f_{n+4} - 21f_{n+5} + 7f_{n+6} - f_{n+7}) + \frac{x_n^2}{80640h^6}(4f_n - 27f_{n+1} + 78f_{n+2} - 125f_{n+3} + 120f_{n+4} - 69f_{n+5} + 22f_{n+6} - 3f_{n+7})$$

$$a_9 = -\frac{1}{2177280h^5}(46f_n - 295f_{n+1} + 810f_{n+2} - 1235f_{n+3} + 1130f_{n+4} - 621f_{n+5} + 190f_{n+6} - 25f_{n+7}) - \frac{x_n}{362880h^6}(4f_n - 27f_{n+1} + 78f_{n+2} - 125f_{n+3} + 120f_{n+4} - 69f_{n+5} + 22f_{n+6} - 3f_{n+7}) - \frac{x_n^2}{725760h^7}(f_n - 7f_{n+1} + 21f_{n+2} - 35f_{n+3} + 35f_{n+4} - 21f_{n+5} + 7f_{n+6} - f_{n+7})$$

$$a_{10} = \frac{1}{3628800h^6}(4f_n - 27f_{n+1} + 78f_{n+2} - 125f_{n+3} + 120f_{n+4} - 69f_{n+5} + 22f_{n+6} - 3f_{n+7}) + \frac{x_n}{3628800h^7}(f_n - 7f_{n+1} + 21f_{n+2} - 35f_{n+3} + 35f_{n+4} - 21f_{n+5} + 7f_{n+6} - f_{n+7})$$

$$a_{11} = -\frac{1}{39916800h^7}(f_n - 7f_{n+1} + 21f_{n+2} - 35f_{n+3} + 35f_{n+4} - 21f_{n+5} + 7f_{n+6} - f_{n+7})$$

The values of *a's* are substituted into equation (5.2.1.1) and simplified, this gives a continuous linear multistep method of the form:

$$y(x)=\sum_{j=2}^{k-2}\alpha_j(x)y_{n+j}+h^4\sum_{j=0}^{k}\beta_j(x)f_{n+j} \tag{5.4.1.2}$$

where $x = zh + x_n + 6h$ (5.4.1.3)

Substituting (5.4.1.3) into (5.4.1.2) and on simplifying we have as follows

$$\alpha_2(z) = -1-\frac{11z}{6}-z^2-\frac{z^3}{6}$$
$$\alpha_3(z) = 4+7z+\frac{7z^2}{2}+\frac{z^3}{2}$$
$$\alpha_4(z) = -6-\frac{19z}{2}-4z^2-\frac{z^3}{2}$$
$$\alpha_5(z) = 4+\frac{13z}{3}+\frac{3z^2}{2}+\frac{z^3}{6}$$
$$\beta_0(z)=\frac{1}{119750400}(-360z-27918z^2-41261z^3+23760z^5 +10164z^6-1386z^7-1980z^8-550z^9-66z^{10}-3z^{11}).$$
$$\beta_1(z)=\frac{1}{119750400}(66804z+264330z^2+353342z^3-199584z^5 -83160z^6+12672z^7+16335z^8+4345z^9+495z^{10}+21z^{11}).$$
$$\beta_2(z)=\frac{1}{13305600}(-44880-73220z-130768z^2-151569z^3 +83160z^5+33264z^6-5874z^7-6600z^8-1650z^9-176z^{10} -7z^{11}).$$
$$\beta_3(z)=\frac{1}{23950080}(4243536+7690860z+4523310z^2+1277606z^3 -332640z^5-123816z^6+26928z^7+24849z^8+5665z^9+ 561z^{10}+21z^{11}).$$
$$\beta_4(z)=\frac{1}{23950080}(15608736+30203040z+17327970z^2+2414269z^3 +498960z^5+158004z^6-47718z^7-32076z^8-6490z^9 -594z^{10}-21z^{11}).$$
$$\beta_5(z)=\frac{1}{13305600}(2357520+6661340z+6852142z^2+2812194z^3 -332640z^5-49896z^6+30624z^7+14025z^8+2475z^9+209z^{10} +7z^{11}).$$
$$\beta_6(z)=\frac{1}{119750400}(-403920+619596z+4781700z^2+7364533z^3 +4989600z^4+1446984z^5-64680z^6-176022z^7-55440z^8 -8470z^9-660z^{10}-21z^{11}). \tag{5.4.1.4}$$

$$\beta_7(z)=\frac{1}{119750400}\ (39600+31380z-130878z^2-185614z^3+142560z^5$$
$$+116424z^6+45936z^7+10395z^8+1375z^9+99z^{10}+3z^{11}).$$

Evaluating (5.4.1.4) at the non-interpolating points .that is, at z=-6, -5. 0 and 1 yields

$$60480\,y_{n+5}-226800y_{n+4}+302400y_{n+3}+151200y_n-151200y_{n+2}=(9\,f_n$$
$$2335f_{n+1}+20850f_{n+2}+41745f_{n+3}+10945f_{n+4}-339f_{n+5}+60f_{n+6}-5f_{n+7}). \qquad (5.4.1.5)$$

$$15120\,y_{n+5}-60480y_{n+4}+90720y_{n+3}-60480y_{n+2}+15120y_{n+1}=(5\,f_n$$
$$-51f_{n+1}+2679f_{n+2}+9854f_{n+3}+2679f_{n+4}-51f_{n+5}+5f_{n+6}). \qquad (5.4.1.6)$$

$$15120y_{n+6}-60480y_{n+5}+90720y_{n+4}-60480y_{n+3}+15120y_{n+2}=(5f_{n+1}$$
$$-51f_{n+2}+2679f_{n+3}+9854f_{n+4}+2679f_{n+5}-51f_{n+6}+5f_{n+7}). \qquad (5.4.1.7)$$

$$151200\,y_{n+7}-151200y_{n+5}+302400y_{n+4}-226800y_{n+3}+60480y_{n+2}=(5\,f_n$$
$$+60f_{n+1}-339f_{n+2}+10945f_{n+3}+41745f_{n+4}+20850f_{n+5}+2335f_{n+6}+9f_{n+7}). \qquad (5.4.1.8)$$

The first derivative of (5.4.1.4) gives

$$\alpha_2'(z)=\frac{-11}{6}-2z-\frac{z^2}{2}$$
$$\alpha_3'(z)=7+7z+\frac{3z^2}{2}$$
$$\alpha_4'(z)=\frac{-19}{2}-8z-\frac{3z^2}{2}$$
$$\alpha_5'(z)=\frac{13}{3}+3z+\frac{3z^2}{6}$$
$$\beta_0'(z)=\frac{1}{39916800}\ (-120-18612z-41261z^2+39600z^4+20328z^5$$
$$-3234z^6-5280z^7-1650z^8-220z^9-11z^{10}).$$
$$\beta_1'(z)=\frac{1}{39916800}(22268+176220\,z+353342z^2-332640z^4$$
$$-166320z^2+29568z^6+43560z^7+13035z^8+1650z^9+77z^{10}).$$
$$\beta_2'(z)=\frac{1}{13305600}(-73220-261536\,z-454707z^2+415800z^4-199584z^5$$
$$-41118z^6-52800z^7-14850z^8-1760z^9-77z^{10}).$$

$$\beta_3'(z)=\frac{1}{7983360}(2563620+3015540z+1277606z^2-554400z^4-247632z^5$$
$$+62832z^6+66264z^7+16995z^8+1870z^9+77z^{10}).$$
$$\beta_4'(z)=\frac{1}{7983360}(10067680+11551980z+2414269z^2+831600z^4+316008z^5$$
$$-111342z^6-85536z^7-19470z^8-1980z^9-77z^{10}).$$
$$\beta_5'(z)=\frac{1}{13305600}(6661340+13704284z+8436582z^2-1663200z^4$$
$$-299376z^5+214368z^6+112200z^7+22275z^8+2090z^9$$
$$+77z^{10}).$$
$$\beta_6'(z)=\frac{h^3}{39916800}(206532+3187800z+7364533z^2+6652800z^3 \qquad (5.4.1.9)$$
$$+2411640z^4-129360z^5-410718z^6-147840z^7-25410z^8$$
$$-2200z^9-77z^{10}).$$
$$\beta_7'(z)=\frac{h^3}{39916800}(10460-87252z-185614z^2+237600z^4+232848z^5$$
$$+107184z^6+27720z^7+4125z^8+330z^9+11z^{10}).$$

Evaluating (5.4.1.9) at all the grid points. i.e, at z=-6, -5, -4, -3, -2,-1, 0 and 1 gives

$$9979200hy_n'-43243200y_{n+5}+154677600y_{n+4}-189604800y_{n+3}$$
$$+78170400y_{n+2}=h^4(-86853f_n-4776925f_{n+1}-20392980f_{n+2} \qquad (5.4.1.10)$$
$$-31203915f_{n+3}-7753465f_{n+4}+213513f_{n+5}-35190f_{n+6}+2615f_{n+7}).$$

$$9979200hy_{n+1}'-18295200y_{n+5}+69854400y_{n+4}-94802400y_{n+3}$$
$$+43243200y_{n+2}=h^4(-2615f_n-51633f_{n+1}-4996005f_{n+2} \qquad (5.4.1.11)$$
$$-12584600f_{n+3}-3204525f_{n+4}+54915f_{n+5}-5567f_{n+6}+30f_{n+7}).$$

$$3326400hy_{n+2}'-1108800y_{n+5}+4989600y_{n+4}-9979200y_{n+3}+6098400y_{n+2}$$
$$=h^4(-10f_n+1225f_{n+1}-38061f_{n+2}-584245f_{n+3}-221120f_{n+4}+12915f_{n+5} \qquad (5.4.1.12)$$
$$-2545f_{n+6}+241f_{n+7}).$$

$$9979200hy_{n+3}'+1663200y_{n+5}-9979200y_{n+4}+4989600y_{n+3}+3326400y_{n+2}$$
$$=h^4(-723f_n+6854f_{n+1}-27789f_{n+2}+515685f_{n+3}+364535f_{n+4} \qquad (5.4.1.13)$$
$$-33492f_{n+5}+7281f_{n+6}-751f_{n+7}).$$

$$9979200hy_{n+4}'-3326400y_{n+5}-4989600y_{n+4}+9979200y_{n+3}-1663200y_{n+2}$$
$$=h^4(751f_n-7281f_{n+1}+33492f_{n+2}-364535f_{n+3}-515685f_{n+4}+27789f_{n+5} \qquad (5.4.1.14)$$
$$-6854f_{n+6}+723f_{n+7}).$$

$$3326400hy'_{n+5} - 6098400y_{n+5} + 9979200\,y_{n+4} - 4989600\,y_{n+3} + 1108800y_{n+2}$$
$$= h^4(-241f_n + 2545f_{n+1} - 12915f_{n+2} + 221120f_{n+3} + 584245f_{n+4} + 38061f_{n+5} \quad (5.4.1.15)$$
$$-1225f_{n+6} + 10f_{n+7}).$$

$$3326400hy'_{n+6} - 43243200y_{n+5} + 94802400y_{n+4} - 69854400y_{n+3}$$
$$+18295200y_{n+2} = h^4(-30f_n + 5567f_{n+1} - 54915f_{n+2} + 3204525f_{n+3} \quad (5.4.1.16)$$
$$+12584600f_{n+4} + 4996005f_{n+5} + 51633f_{n+6} + 2615f_{n+7}).$$

$$3326400hy'_{n+7} - 78170400y_{n+5} + 189604800y_{n+4} - 154677600y_{n+3}$$
$$+43243200y_{n+2} = h^4(-2615f_n + 35190f_{n+1} - 213513f_{n+2} + \quad (5.4.1.17)$$
$$7753465f_{n+3} + 31203915f_{n+4} + 20392980f_{n+5} + 4776925f_{n+6}$$
$$+86853f_{n+7}).$$

The second derivative of (5.4.1.4) gives

$$\alpha_2''(z) = -2 - z$$
$$\alpha_3''(z) = 7 + 3z$$
$$\alpha_4''(z) = -8 - 3z$$
$$\alpha_5''(z) = 3 + z$$

$$\beta_0''(z) = \frac{1}{1814400}(-846 - 3751z + 7200z^3 + 4620z^4 - 882z^5 - 1680z^6$$
$$-600z^7 - 90z^8 - 5z^9).$$

$$\beta_1''(z) = \frac{1}{1814400}(8010 + 32122z - 60480z^3 - 37800z^4 + 8064z^5$$
$$+13860z^6 + 4740z^7 + 675z^8 + 35z^9).$$

$$\beta_2''(z) = \frac{1}{604800}(-11888 - 41337z + 75600z^3 + 45360z^4 - 11214z^5$$
$$-16800z^6 - 5400z^7 - 720z^8 - 35z^9).$$

$$\beta_3''(z) = \frac{1}{362880}(137070 + 116146z - 100800z^3 - 56280z^4 + 17136z^5$$
$$+21084z^6 + 6180z^7 + 765z^8 + 35z^9).$$

$$\beta_4''(z) = \frac{1}{362880}(525090 + 219479z + 151200z^3 + 71820z^4 - 30366z^5$$
$$-27216z^6 - 7080z^7 - 810z^8 - 35z^9).$$

$$\beta_5''(z) = \frac{1}{604800}(622922 + 766962z - 302400z^3 - 68040z^4 + 58464z^5$$
$$+35700z^6 + 8100z^7 + 855z^8 + 35z^9).$$

$$\beta_6''(z) = \frac{1}{1814400}(144900 + 669503z + 907200z^2 + 438480z^3 - 29400z^4$$
$$-112014z^5 - 47040z^6 - 9240z^7 - 900z^8 - 35z^9).$$

$$\beta_7''(z) = \frac{1}{1814400}(-3966 - 16874z + 43200z^3 + 52920z^4 + 29232z^5 + 8820z^6 + 1500z^7 + 135z^8 + 5z^9). \quad (5.4.1.18)$$

Equation (5.4.1.18) is evaluated at all the grid points. That is, at z=-6, -5, -4, -3, -2,-1, 0 and 1 produces.

$$302400h^2 y_n'' + 907200y_{n+5} - 3024000y_{n+4} + 3326400y_{n+3} - 1209600y_{n+2} = h^4(19162f_n + 326909f_{n+1} + 561299f_{n+2} + 751735f_{n+3} + 117780f_{n+4} + 661f_{n+5} - 5129f_{n+6} + 19523f_{n+7}). \quad (5.4.1.19)$$

$$302400h^2 y_{n+1}'' + 604800y_{n+5} - 2116800y_{n+4} + 2419200y_{n+3} - 907200y_{n+2} = h^4(-661f_n + 24150f_{n+1} + 311461f_{n+2} + 437575f_{n+3} + 114225f_{n+4} - 5944f_{n+5} + 1335f_{n+6} - 141f_{n+7}). \quad (5.4.1.20)$$

$$302400h^2 y_{n+2}'' + 302400y_{n+5} - 1209600y_{n+4} + 1512000y_{n+3} - 604800y_{n+2} = h^4(141f_n - 1989f_{n+1} + 30138f_{n+2} + 196405f_{n+3} + 53285f_{n+4} - 831f_{n+5} + 44f_{n+6} + 7f_{n+7}). \quad (5.4.1.21)$$

$$302400h^2 y_{n+3}'' - 302400y_{n+4} + 604800y_{n+3} - 302400y_{n+2} = h^4(-7f_n + 97f_{n+1} - 1165f_{n+2} - 23050f_{n+3} - 1165f_{n+4} + 97f_{n+5} - 7f_{n+6}). \quad (5.4.1.22)$$

$$302400h^2 y_{n+4}'' - 302400y_{n+5} + 604800y_{n+4} - 302400y_{n+3} = h^4(-7f_{n+1} + 97f_{n+2} - 1165f_{n+3} - 23050f_{n+4} - 1165f_{n+5} + 97f_{n+6} - 7f_{n+7}). \quad (5.4.1.23)$$

$$302400h^2 y_{n+5}'' + 604800y_{n+5} - 1512000y_{n+4} + 1209600y_{n+3} - 302400y_{n+2} = h^4(7f_n + 44f_{n+1} - 831f_{n+2} + 53285f_{n+3} + 196405f_{n+4} + 30138f_{n+5} - 1989f_{n+6} + 141f_{n+7}). \quad (5.4.1.24)$$

$$302400h^2 y_{n+6}'' - 907200y_{n+5} + 2419200y_{n+4} - 2116800y_{n+3} + 604800y_{n+2} = h^4(-141f_n + 1335f_{n+1} - 5944f_{n+2} + 114225f_{n+3} + 437575f_{n+4} + 311461f_{n+5} + 24150f_{n+6} - 661f_{n+7}). \quad (5.4.1.25)$$

$$302400h^2 y_{n+7}'' - 1209600y_{n+5} + 3326400y_{n+4} - 3024000y_{n+3} + 907200y_{n+2} = h^4(661f_n - 5129f_{n+1} + 16783f_{n+2} + 117780f_{n+3} + 751735f_{n+4} + 561299f_{n+5} + 326909f_{n+6} + 19162f_{n+7}). \quad (5.4.1.26)$$

The third derivative of (4.1.4.4) gives

$$\alpha_2'''(z) = -1$$
$$\alpha_3'''(z) = 3$$

$$\alpha_4'''(z) = -3$$
$$\alpha_5'''(z) = 1$$

$$\beta_0'''(z) = \frac{1}{1814400}(-3751 + 21600z^2 + 18480z^3 - 4410z^4 - 10080z^5$$
$$-4200z^6 - 720z^7 - 45z^8).$$

$$\beta_1'''(z) = \frac{1}{1814400}(\ 32122 - 181440z^2 - 151200z^3 + 40320z^4$$
$$+83160z^5 + 33180z^6 + 5400z^7 + 315z^8).$$

$$\beta_2'''(z) = \frac{1}{1814400}(-124011 + 680400z^2 + 544320z^3 - 168210z^4$$
$$-302400z^5 - 113400z^6 - 17280z^7 - 945z^8).$$

$$\beta_3'''(z) = \frac{1}{1814400}(580730 - 1512000z^2 - 1125600z^3 + 428400z^4$$
$$+632520z^5 + 216300z^6 + 30600z^7 + 1575z^8).$$

$$\beta_4'''(z) = \frac{1}{1814400}\ (1097395 + 2268000z^2 + 1436400z^3 - 759150z^4$$
$$-816480z^5 - 247800z^6 - 32400z^7 - 1575z^8).$$

$$\beta_5'''(z) = \frac{1}{1814400}(2300886 - 2721600z^2 - 816480z^3 + 876960z^4$$
$$+642600z^5 + 170100z^6 + 20520z^7 + 945z^8).$$

$$\beta_6'''(z) = \frac{1}{1814400}(669503 + 1814400z + 1315440z^2 - 117600z^3$$
$$-560070z^4 - 282240z^5 - 64680z^6 - 7200z^7 - 315z^8).$$

$$\beta_7'''(z) = \frac{1}{1814400}(-16874 + 129600z^2 + 211680z^3 + 146160z^4 \qquad (5.4.1.27)$$
$$+52920z^5 + 10500z^6 + 1080z^7 + 45z^8).$$

Evaluating (5.4.1.27) at all the grid points. i.e, at z=-6, -5, -4, -3, -2,-1, 0 and 1 yields

$$1814400h^3y_n''' - 1814400y_{n+5} + 5443200y_{n+4} - 5443200y_{n+3} + 1814400y_{n+2}$$
$$= h^4(-535111f_n - 2767238f_{n+1} - 473931f_{n+2} - 2944390f_{n+3} + 747475f_{n+4} \qquad (5.4.1.28)$$
$$-498474f_{n+5} + 138143f_{n+6} - 16874f_{n+7}).$$

$$1814400h^3y_{n+1}''' - 1814400y_{n+5} + 5443200y_{n+4} - 5443200y_{n+3} + 1814400y_{n+2}$$
$$= h^4(16874f_n - 669503f_{n+1} - 2300886f_{n+2} - 1097395f_{n+3} - 580730f_{n+4} \qquad (5.4.1.29)$$
$$+124011f_{n+5} - 32122f_{n+6} + 3751f_{n+7}).$$

$$1814400h^3 y'''_{n+2} - 1814400 y_{n+5} + 5443200 y_{n+4} - 5443200 y_{n+3} + 1814400 y_{n+2}$$
$$= h^4(-3751 f_n + 47482 f_{n+1} - 780651 f_{n+2} - 1769350 f_{n+3} - 77485 f_{n+4}$$
$$- 49194 f_{n+5} + 12863 f_{n+6} - 1514 f_{n+7}). \qquad (5.4.1.30)$$

$$1814400h^3 y'''_{n+3} - 1814400 y_{n+5} + 5443200 y_{n+4} - 5443200 y_{n+3} + 1814400 y_{n+2}$$
$$= h^4(1514 f_n - 15263 f_{n+1} + 83754 f_{n+2} - 543955 f_{n+3} - 480890 f_{n+4} \qquad (5.4.1.31)$$
$$+ 59211 f_{n+5} - 12922 f_{n+6} + 1315 f_{n+7}).$$

$$1814400h^3 y'''_{n+4} - 1814400 y_{n+5} + 5443200 y_{n+4} - 5443200 y_{n+3} + 1814400 y_{n+2}$$
$$= h^4(-1351 f_n + 12922 f_{n+1} - 59211 f_{n+2} + 480890 f_{n+3} + 543955 f_{n+4} - 83754 f_{n+5} \qquad (5.4.1.32)$$
$$+ 15263 f_{n+6} - 1514 f_{n+7}).$$

$$1814400h^3 y'''_{n+5} - 1814400 y_{n+5} + 5443200 y_{n+4} - 5443200 y_{n+3} + 1814400 y_{n+2}$$
$$= h^4(1514 f_n - 12863 f_{n+1} + 49194 f_{n+2} + 177485 f_{n+3} + 1769350 f_{n+4} \qquad (5.4.1.33)$$
$$+ 780651 f_{n+5} - 47482 f_{n+6} + 3715 f_{n+7}).$$

$$1814400h^3 y'''_{n+6} - 1814400 y_{n+5} + 5443200 y_{n+4} - 5443200 y_{n+3} + 1814400 y_{n+2}$$
$$= h^4(-3751 f_n + 32122 f_{n+1} - 124011 f_{n+2} + 580730 f_{n+3} + 1097395 f_{n+4} \qquad (5.4.1.34)$$
$$+ 2300886 f_{n+5} + 669503 f_{n+6} - 16874 f_{n+7}).$$

$$1814400h^3 y'''_{n+7} - 1814400 y_{n+5} + 5443200 y_{n+4} - 5443200 y_{n+3} + 1814400 y_{n+2}$$
$$= h^4(16874 f_n - 138143 f_{n+1} + 498474 f_{n+2} - 747475 f_{n+3} + 2944390 f_{n+4} \qquad (5.4.1.35)$$
$$+ 473931 f_{n+5} + 2767238 f_{n+6} + 535111 f_{n+7}).$$

Combining equations (5.4.1.5) - (5.4.1.8), (5.4.1.10), (5.4.1.19) and (5.4.1.28) to form a block of the form (1.10) as follows

$$\begin{pmatrix} 0 & -151200 & 302400 & -226800 & 60480 & 0 & 0 \\ 15120 & -60480 & 90720 & -60480 & 15120 & 0 & 0 \\ 0 & 15120 & -60480 & 90720 & -60480 & 15120 & 0 \\ 0 & 60480 & -226800 & 302400 & -151200 & 0 & 151200 \\ 0 & 78170400 & -189604800 & 154677600 & -43243200 & 0 & 0 \\ 0 & -1209600 & 3326400 & -3024000 & 907200 & 0 & 0 \\ 0 & 1814400 & -5443200 & 5443200 & 1814400 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \end{pmatrix} =$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -151200 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix}
+ h
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -9979200 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_{n} \end{pmatrix}
+
$$

$$
h^2
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -302400 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} y''_{n-6} \\ y''_{n-5} \\ y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_{n} \end{pmatrix}
+ h^3
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1814400
\end{pmatrix}
\begin{pmatrix} y'''_{n-6} \\ y'''_{n-5} \\ y'''_{n-4} \\ y'''_{n-3} \\ y'''_{n-2} \\ y'''_{n-1} \\ y'''_{n} \end{pmatrix}
+
$$

$$
h^4
\begin{pmatrix}
2335 & 20850 & 41745 & 10945 & -339 & 60 & -5 \\
-51 & 2679 & 9854 & 2679 & -51 & 5 & 0 \\
5 & -51 & 2679 & 9854 & 2679 & -51 & 5 \\
60 & -339 & 10945 & 41745 & 20850 & 2335 & 9 \\
-4776925 & -20392980 & -31203915 & -7753465 & 213513 & -35190 & 2615 \\
326909 & 561299 & 751735 & 117780 & 16783 & -5129 & 661 \\
-2767238 & -473931 & -2944390 & 747475 & -498474 & 138143 & -16874
\end{pmatrix}
\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \end{pmatrix}
$$

$$
+ h^4
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -5 \\
0 & 0 & 0 & 0 & 0 & 0 & -86853 \\
0 & 0 & 0 & 0 & 0 & 0 & 19162 \\
0 & 0 & 0 & 0 & 0 & 0 & -535111
\end{pmatrix}
\begin{pmatrix} f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}
$$

Multiplying the equation above by the inverse of A^0

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix} +$$

$$h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_{n} \end{pmatrix} + h^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{49}{2} \end{pmatrix} \begin{pmatrix} y''_{n-6} \\ y''_{n-5} \\ y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_{n} \end{pmatrix}$$

$$+ h^3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{32}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{125}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 36 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{343}{6} \end{pmatrix} \begin{pmatrix} y'''_{n-6} \\ y'''_{n-5} \\ y'''_{n-4} \\ y'''_{n-3} \\ y'''_{n-2} \\ y'''_{n-1} \\ y'''_{n} \end{pmatrix} +$$

$$h^4 \begin{pmatrix} \frac{86}{2489} & -\frac{167}{3904} & \frac{131}{2897} & -\frac{349}{10533} & \frac{108}{6883} & -\frac{79}{18286} & \frac{39}{74183} \\ \frac{1456}{2159} & -\frac{974}{1485} & \frac{969}{1415} & -\frac{974}{1955} & \frac{39}{166} & -\frac{321}{4976} & \frac{109}{13912} \\ \frac{1253}{382} & -\frac{1727}{712} & \frac{963}{352} & -\frac{4193}{2097} & \frac{1249}{1322} & -\frac{1891}{7282} & \frac{243}{7700} \\ \frac{3133}{334} & -\frac{964}{183} & \frac{4151}{579} & -\frac{4183}{816} & \frac{2307}{952} & -\frac{611}{917} & \frac{361}{4457} \\ \frac{149670}{7301} & -\frac{1927}{216} & \frac{1951}{125} & -\frac{8685}{844} & \frac{1627}{329} & -\frac{2019}{1484} & \frac{430}{2599} \\ \frac{3244}{85} & -\frac{2967}{226} & \frac{11493}{385} & -\frac{6642}{385} & \frac{4553}{508} & -\frac{666}{275} & \frac{81}{275} \\ \frac{9841}{154} & -\frac{9014}{511} & \frac{3827}{74} & -\frac{3817}{149} & \frac{1201}{77} & -\frac{1832}{487} & \frac{795}{16776} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \end{pmatrix} +$$

$$h^4\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{89}{3435} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1073}{3805} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1407}{1318} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2887}{1080} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1543}{286} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{18342}{1925} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{13093}{852} \end{pmatrix}\begin{pmatrix} f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \tag{5.4.1.36}$$

This leads to

$$\begin{aligned} y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2 y''_n + \frac{1}{6}h^3 y'''_n + \frac{h^4}{119750400}&(62956 f_{n+7} - 517351 f_{n+6} \\ &+ 1878984 f_{n+5} - 3967805 f_{n+4} + 5415020 f_{n+3} - 5122521 f_{n+2} \\ &+ 4137616 f_{n+1} + 3102701 f_n). \end{aligned} \tag{5.4.1.37}$$

$$\begin{aligned} y_{n+2} = y_n + 2hy'_n + 2h^2 y''_n + \frac{4}{3}h^3 y'''_n + \frac{h^4}{467775}&(3665 f_{n+7} - 30176 f_{n+6} \\ &+ 109899 f_{n+5} - 233050 f_{n+4} + 320335 f_{n+3} - 306810 f_{n+2} \\ &+ 315461 f_{n+1} + 132526 f_n). \end{aligned} \tag{5.4.1.38}$$

$$\begin{aligned} y_{n+3} = y_n + 3hy'_n + \frac{9}{2}h^2 y''_n + \frac{9}{2}h^3 y'''_n + \frac{h^4}{492800}&(15552 f_{n+7} - 127971 f_{n+6} \\ &+ 465588 f_{n+5} - 985365 f_{n+4} + 1348200 f_{n+3} - 1195317 f_{n+2} \\ &+ 1616436 f_{n+1} + 526077 f_n). \end{aligned} \tag{5.4.1.39}$$

$$\begin{aligned} y_{n+4} = y_n + 4hy'_n + 8h^2 y''_n + \frac{32}{3}h^3 y'''_n + \frac{h^4}{467775}&(9472 f_{n+7} - 77920 f_{n+6} \\ &+ 283392 f_{n+5} - 599480 f_{n+4} + 838400 f_{n+3} - 616032 f_{n+2} \\ &+ 1096960 f_{n+1} + 312608 f_n). \end{aligned} \tag{5.4.1.40}$$

$$\begin{aligned} y_{n+5} = y_n + 5hy'_n + \frac{25}{2}h^2 y''_n + \frac{125}{6}h^3 y'''_n + \frac{h^4}{4790016}&(792500 f_{n+7} - 6516875 f_{n+6} \\ &+ 23688000 f_{n+5} - 49290625 f_{n+4} + 74762500 f_{n+3} - 42733125 f_{n+2} \\ &+ 98195000 f_{n+1} + 25842625 f_n). \end{aligned} \tag{5.4.1.41}$$

$$y_{n+6} = y_n + 6hy'_n + 18h^2 y''_n + 36h^3 y'''_n + \frac{h^4}{11550}(3402 f_{n+7} - 27972 f_{n+6}$$
$$+103518 f_{n+5} - 199260 f_{n+4} + 344790 f_{n+3} - 151632 f_{n+2} \tag{5.4.1.42}$$
$$+440802 f_{n+1} + 110052 f_n).$$

$$y_{n+7} = y_n + 7hy'_n + \frac{49}{2}h^2 y''_n + \frac{343}{6}h^3 y'''_n + \frac{h^4}{17107200}(8163400 f_{n+7} - 64354003 f_{n+6}$$
$$+266827932 f_{n+5} - 438242525 f_{n+4} + 884720480 f_{n+3} - 301769685 f_{n+2} \tag{5.4.1.43}$$
$$+1093194508 f_{n+1} + 262892693 f_n).$$

Substituting (5.4.1.38) - (5.4.1.41) into (5.4.1.11) – (5.4.1.17) to give the first derivative of the block

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2 y'''_n + \frac{h^3}{3628800}(8002 f_{n+7} - 65823 f_{n+6} + 239406 f_{n+5} \tag{5.4.1.44}$$
$$-506675 f_{n+4} + 694230 f_{n+3} - 662757 f_{n+2} + 562618 f_{n+1} + 335799 f_n).$$

$$y'_{n+2} = y'_n + 2hy''_n + 2h^2 y'''_n + \frac{h^3}{28350}(398 f_{n+7} - 3277 f_{n+6} + 11934 f_{n+5} \tag{5.4.1.45}$$
$$-25300 f_{n+4} + 34730 f_{n+3} - 32823 f_{n+2} + 38762 f_{n+1} + 13376 f_n).$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{9}{2}h^2 y'''_n + \frac{h^3}{44800}(1566 f_{n+7} - 12879 f_{n+6} + 46818 f_{n+5} \tag{5.4.1.46}$$
$$-98955 f_{n+4} + 135450 f_{n+3} - 105381 f_{n+2} + 183654 f_{n+1} + 51327 f_n).$$

$$y'_{n+4} = y'_n + 4hy''_n + 8h^2 y'''_n + \frac{h^3}{14175}(928 f_{n+7} - 7632 f_{n+6} + 27744 f_{n+5} \tag{5.4.1.47}$$
$$-58520 f_{n+4} + 86880 f_{n+3} - 46608 f_{n+2} + 118432 f_{n+1} + 29976 f_n).$$

$$y'_{n+5} = y'_n + 5hy''_n + \frac{25}{2}h^2 y'''_n + \frac{5h^3}{145152}(3050 f_{n+7} - 25075 f_{n+6} + 91350 f_{n+5} \tag{5.4.1.48}$$
$$-178375 f_{n+4} + 320750 f_{n+3} - 115425 f_{n+2} + 410450 f_{n+1} + 98075 f_n).$$

$$y'_{n+6} = y'_n + 6hy''_n + 18h^2 y'''_n + \frac{h^3}{1400}(216 f_{n+7} - 1764 f_{n+6} + 7125 f_{n+5} \tag{5.4.1.49}$$
$$-10800 f_{n+4} + 24840 f_{n+3} - 6156 f_{n+2} + 30024 f_{n+1} + 6912 f_n)$$

$$y'_{n+7} = y'_n + 7hy''_n + \frac{49}{2}h^2 y'''_n + \frac{7h^3}{518400}(16366 f_{n+7} - 93639 f_{n+6} + 619458 f_{n+5} \tag{5.4.1.50}$$
$$-660275 f_{n+4} + 1944810 f_{n+3} - 338541 f_{n+2} + 2242534 f_{n+1} + 502887 f_n).$$

Substituting (5.4.1.38) - (5.4.1.41) into (5.4.1.20) – (5.4.1.26) to give the second derivative of the block

$$y''_{n+1} = y''_n + hy'''_n + \frac{h^2}{1814400}(12062 f_{n+7} - 99359 f_{n+6} + 36113 f_{n+5}$$
$$- 768805 f_{n+4} + 1059430 f_{n+3} - 1025096 f_{n+2} + 950684 f_{n+1} \qquad (5.4.1.51)$$
$$+ 416173 f_n).$$

$$y''_{n+2} = y''_n + 2hy'''_n + \frac{h^2}{28350}(466 f_{n+7} - 3832 f_{n+6} + 13926 f_{n+5}$$
$$- 29405 f_{n+4} + 39950 f_{n+3} - 34986 f_{n+2} + 55642 f_{n+1} \qquad (5.4.1.52)$$
$$+ 14939 f_n).$$

$$y''_{n+3} = y''_n + 3hy'''_n + \frac{h^2}{604800}(15552 f_{n+7} - 127899 f_{n+6} + 465102 f_{n+5}$$
$$- 985365 f_{n+4} + 1394820 f_{n+3} - 650997 f_{n+2} + 2113614 f_{n+1} \qquad (5.4.1.53)$$
$$+ 496773 f_n).$$

$$y''_{n+4} = y''_n + 4hy'''_n + \frac{h^2}{14175}(496 f_{n+7} - 4072 f_{n+6} + 14736 f_{n+5}$$
$$- 29960 f_{n+4} + 56720 f_{n+3} - 11496 f_{n+2} + 71152 f_{n+1} \qquad (5.4.1.54)$$
$$+ 15824 f_n).$$

$$y''_{n+5} = y''_n + 5hy'''_n + \frac{h^2}{72576}(3250 f_{n+7} - 26875 f_{n+6} + 102900 f_{n+5}$$
$$- 130625 f_{n+4} + 421250 f_{n+3} - 40125 f_{n+2} + 475000 f_{n+1} \qquad (5.4.1.55)$$
$$+ 102425 f_n).$$

$$y''_{n+6} = y''_n + 6hy'''_n + \frac{h^2}{350}(18 f_{n+7} - 126 f_{n+6} + 918 f_{n+5} - 495 f_{n+4} \qquad (5.4.1.56)$$
$$+ 2670 f_{n+3} - 108 f_{n+2} + 2826 f_{n+1} + 597 f_n).$$

$$y''_{n+7} = y''_n + 7hy'''_n + \frac{h^2}{43182}(5453 f_{n+7} + 24400 f_{n+6} + 160800 f_{n+5}$$
$$- 34000 f_{n+4} + 400000 f_{n+3} + 1200 f_{n+2} + 413600 f_{n+1} + \qquad (5.4.1.57)$$
$$86506 f_n).$$

Substituting (5.4.1.38) - (5.4.1.41) into (5.4.1.29) – (5.4.1.35) to give the third derivative of the block

$$y'''_{n+1} = y'''_n + \frac{h}{120960}(1375 f_{n+7} - 11351 f_{n+6} + 41499 f_{n+5} - 88547 f_{n+4} \qquad (5.4.1.58)$$
$$+ 123133 f_{n+3} - 121797 f_{n+2} + 139849 f_{n+1} + 36799 f_n).$$

$$y'''_{n+2} = y'''_n + \frac{h}{18900}(160 f_{n+7} - 1305 f_{n+6} + 4680 f_{n+5} - 9635 f_{n+4} + \qquad (5.4.1.59)$$
$$12240 f_{n+3} - 3195 f_{n+2} + 29320 f_{n+1} + 5535 f_n).$$

$$y'''_{n+3} = y'''_{n} + \frac{h}{22400}(225 f_{n+7} - 1865 f_{n+6} + 6885 f_{n+5} - 15165 f_{n+4} + 29635 f_{n+3} + 6885 f_{n+2} + 33975 f_{n+1} + 6625 f_{n}). \tag{5.4.1.60}$$

$$y'''_{n+4} = y'''_{n} + \frac{h}{945}(8 f_{n+7} - 64 f_{n+6} + 216 f_{n+5} - 106 f_{n+4} + 1784 f_{n+3} + 216 f_{n+2} + 1448 f_{n+1} + 278 f_{n}). \tag{5.4.1.61}$$

$$y'''_{n+5} = y'''_{n} + \frac{h}{24192}(275 f_{n+7} - 2475 f_{n+6} + 17055 f_{n+5} + 13625 f_{n+4} + 41625 f_{n+3} + 6975 f_{n+2} + 36725 f_{n+1} + 7155 f_{n}). \tag{5.4.1.62}$$

$$y'''_{n+6} = y'''_{n} + \frac{h}{140}(41 f_{n+6} + 216 f_{n+5} + 27 f_{n+4} + 272 f_{n+3} + 27 f_{n+2} + 216 f_{n+1} + 41 f_{n}). \tag{5.4.1.63}$$

$$y'''_{n+7} = y'''_{n} + \frac{h}{17280}(5257 f_{n+7} + 25039 f_{n+6} + 9261 f_{n+5} + 20923 f_{n+4} + 20923 f_{n+3} + 9261 f_{n+2} + 25039 f_{n+1} + 5257 f_{n}). \tag{5.4.1.64}$$

5.4.2 Properties of Seven–Step Block Method for Fourth Order ODEs.

The establishment of order, zero-stability and region of absolute stability of seven–step block method for fourth order ODEs are considered in this section.

5.4.2.1 Order of Seven–Step Block Method for Fourth Order ODEs.

The method used in section 3.2.2.1 is applied in finding the order of block method (5.4.1.37 – 5.4.1.43) as shown below

$$
\begin{pmatrix}
\sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{h^m}{m!}y_n^{(m)}-\frac{335799}{3628800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(3628800)(m!)}y_n^{(3+m)}\begin{pmatrix}562618(1)^m-662757(2)^m\\+694230(3)^m-506675(4)^m\\+239406(5)^m-65823(6)^m\\+8002(7)^m.\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(2h)^m}{m!}y_n^{(m)}-\frac{13376}{28350}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(28350)(m!)}y_n^{(3+m)}\begin{pmatrix}38762(1)^m-32823(2)^m\\+34730(3)^m-25300(4)^m\\+11934(5)^m-3277(6)^m\\+398(7)^m.\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(3h)^m}{m!}y_n^{(m)}-\frac{51327}{44800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(44800)(m!)}y_n^{(3+m)}\begin{pmatrix}183654(1)^m-105381(2)^m\\+135450(3)^m-98955(4)^m\\+46818(5)^m-12879(6)^m\\+1566(7)^m.\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(4h)^m}{m!}y_n^{(m)}-\frac{29976}{14175}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(14175)(m!)}y_n^{(3+m)}\begin{pmatrix}118432(1)^m-46608(2)^m\\+86880(3)^m-58520(4)^m\\+27744(5)^m-7632(6)^m\\+928(7)^m.\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(5h)^m}{m!}y_n^{(m)}-\frac{98075}{145152}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(145152)(m!)}y_n^{(3+m)}\begin{pmatrix}410450(1)^m-115425(2)^m\\+320750(3)^m-178375(4)^m\\+91350(5)^m-25075(6)^m\\+3050(7)^m.\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(6h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(6h)^m}{m!}y_n^{(m)}-\frac{6912}{1400}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(1400)(m!)}y_n^{(3+m)}\begin{pmatrix}30024(1)^m-6156(2)^m\\+24840(3)^m-10800(4)^m\\+7128(5)^m-1764(6)^m\\+216(7)^m.\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(7h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(7h)^m}{m!}y_n^{(m)}-\frac{7(502887)}{518400}h^2y_n''-\sum_{m=0}^{\infty}\frac{7h^{3+m}}{(518400*m!)}y_n^{(3+m)}\begin{pmatrix}2242534(1)^m-338541(2)^m\\+1944810(3)^m-660275(4)^m\\+619458(5)^m-93639(6)^m\\+16366(7)^m.\end{pmatrix}
\end{pmatrix}
=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\end{pmatrix}
$$

Comparing the coefficients of h^m and y_n^m gives

$$C_0 = \begin{pmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \\ 5-5 \\ 6-6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{1}{2!} \\ \frac{(2)^2}{2!} - \frac{(2)^2}{2!} \\ \frac{(3)^2}{2!} - \frac{(3)^2}{2!} \\ \frac{(4)^2}{2!} - \frac{(4)^2}{2!} \\ \frac{(5)^2}{2!} - \frac{(5)^2}{2!} \\ \frac{(6)^2}{2!} - \frac{(6)^2}{2!} \\ \frac{(7)^2}{2!} - \frac{(7)^2}{2!} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
C_3 = \begin{pmatrix}
\frac{1}{3!} - \frac{335799}{3628800} - \frac{1}{(3628800)(0!)} \left(\begin{array}{l} 562618(1)^0 - 662757(2)^0 + 694230(3)^0 \\ - 506675(4)^0 + 239406(5)^0 - 65823(6)^0 \\ + 8002(7)^0. \end{array} \right) \\
\frac{(2)^3}{3!} - \frac{13376}{28350} - \frac{1}{(28350)(0!)} \left(\begin{array}{l} 38762(1)^0 - 32823(2)^0 + 34730(3)^0 - 25300(4)^0 \\ + 11934(5)^0 - 3277(6)^0 + 398(7)^0 \end{array} \right) \\
\frac{(3)^3}{3!} - \frac{51327}{44800} - \frac{1}{(44800)(0!)} \left(\begin{array}{l} 183654(1)^0 - 105381(2)^0 + 135450(3)^0 - 98955(4)^0 \\ + 46818(5)^0 - 12879(6)^0 + 1566(7)^0 \end{array} \right) \\
\frac{(4)^3}{3!} - \frac{29976}{14175} - \frac{1}{(14175)(0!)} \left(\begin{array}{l} 118432(1)^0 - 46608(2)^0 + 86880(3)^0 - 58520(4)^0 \\ + 27744(5)^0 - 7632(6)^0 + 928(7)^0 \end{array} \right) \\
\frac{(5)^3}{3!} - \frac{98075}{145152} - \frac{1}{(145152)(0!)} \left(\begin{array}{l} 410450(1)^0 - 115425(2)^0 + 320750(3)^0 - 178375(4)^0 \\ + 91350(5)^0 - 25075(6)^0 + 3050(7)^0 \end{array} \right) \\
\frac{(6)^3}{3!} - \frac{6912}{1400} - \frac{1}{(1400)(0!)} \left(\begin{array}{l} 30024(1)^0 - 6156(2)^0 + 24840(3)^0 - 10800(4)^0 + 7128(5)^0 \\ - 1764(6)^0 + 216(7)^0 \end{array} \right) \\
\frac{(7)^3}{3!} - \frac{7(502887)}{518400} - \frac{7}{(518400)(0!)} \left(\begin{array}{l} 2242534(1)^0 - 338541(2)^0 + 1944810(3)^0 \\ - 660275(4)^0 + 619458(5)^0 - 93639(6)^0 \\ + 16366(7)^0 \end{array} \right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_4 = \begin{pmatrix}
\frac{1}{4!} - \frac{1}{(3628800)(1!)} \left(\begin{array}{l} 562618(1)^1 - 662757(2)^1 + 694230(3)^1 - 506675(4)^1 + \\ 239406(5)^1 - 65823(6)^1 + 8002(7)^1 \end{array} \right) \\
\frac{(2)^4}{4!} - \frac{1}{(28350)(1!)} \left(\begin{array}{l} 38762(1)^1 - 32823(2)^1 + 34730(3)^1 - 25300(4)^1 + 11934(5)^1 \\ - 3277(6)^1 + 398(7)^1 \end{array} \right) \\
\frac{(3)^4}{4!} - \frac{1}{(44800)(1!)} \left(\begin{array}{l} 183654(1)^1 - 105381(2)^1 + 135450(3)^1 - 98955(4)^1 + 46818(5)^1 \\ - 12879(6)^1 + 1566(7)^1 \end{array} \right) \\
\frac{(4)^4}{4!} - \frac{1}{(14175)(1!)} \left(\begin{array}{l} 118432(1)^1 - 46608(2)^1 + 86880(3)^1 - 58520(4)^1 + 27744(5)^1 - \\ 7632(6)^1 + 928(7)^1 \end{array} \right) \\
\frac{(5)^4}{4!} - \frac{1}{(145152)(1!)} \left(\begin{array}{l} 410450(1)^1 - 115425(2)^1 + 320750(3)^1 - 178375(4)^1 + 91350(5)^1 \\ - 25075(6)^1 + 3050(7)^1 \end{array} \right) \\
\frac{(6)^4}{4!} - \frac{1}{(1400)(1!)} \left(\begin{array}{l} 30024(1)^1 - 6156(2)^1 + 24840(3)^1 - 10800(4)^1 + 7128(5)^1 - 1764(6)^1 \\ + 216(7)^1 \end{array} \right) \\
\frac{(7)^4}{4!} - \frac{7}{(518400)(1!)} \left(\begin{array}{l} 2242534(1)^1 - 338541(2)^1 + 1944810(3)^1 - 660275(4)^1 + \\ 619458(5)^1 - 93639(6)^1 + 16366(7)^1 \end{array} \right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$C_5=\begin{pmatrix}
\frac{1}{5!}-\frac{1}{(3628800)(2!)}\begin{pmatrix}562618(1)^2-662757(2)^2+694230(3)^2-506675(4)^2+\\239406(5)^2-65823(6)^2+8002(7)^2\end{pmatrix}\\
\frac{(2)^5}{5!}-\frac{1}{(28350(2!)}\begin{pmatrix}38762(1)^2-32823(2)^2+34730(3)^2-25300(4)^2+11934(5)^2\\-3277(6)^2+398(7)^2\end{pmatrix}\\
\frac{(3)^5}{5!}-\frac{1}{(44800)(2!)}\begin{pmatrix}183654(1)^2-105381(2)^2+135450(3)^2-98955(4)^2+46818(5)^2\\-12879(6)^2+1566(7)^2\end{pmatrix}\\
\frac{(4)^5}{5!}-\frac{1}{(14175)(2!)}\begin{pmatrix}118432(1)^2-46608(2)^2+86880(3)^2-58520(4)^2+27744(5)^2-\\7632(6)^2+928(7)^2\end{pmatrix}\\
\frac{(5)^5}{5!}-\frac{1}{(145152)(2!)}\begin{pmatrix}410450(1)^2-115425(2)^2+320750(3)^2-178375(4)^2+91350(5)^2\\-25075(6)^2+3050(7)^2\end{pmatrix}\\
\frac{(6)^5}{5!}-\frac{1}{(1400)(2!)}\begin{pmatrix}30024(1)^2-6156(2)^2+24840(3)^2-10800(4)^2+7128(5)^2-1764(6)^2\\+216(7)^2\end{pmatrix}\\
\frac{(7)^5}{5!}-\frac{7}{(518400)(2!)}\begin{pmatrix}2242534(1)^2-338541(2)^2+1944810(3)^2-660275(4)^2+\\619458(5)^2-93639(6)^2+16366(7)^2\end{pmatrix}
\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_6=\begin{pmatrix}
\frac{1}{6!}-\frac{1}{(3628800)(3!)}\begin{pmatrix}562618(1)^3-662757(2)^3+694230(3)^3-506675(4)^3+\\239406(5)^3-65823(6)^3+8002(7)^3\end{pmatrix}\\
\frac{(2)^6}{6!}-\frac{1}{(28350)(3!)}\begin{pmatrix}38762(1)^3-32823(2)^3+34730(3)^3-25300(4)^3+11934(5)^3\\-3277(6)^3+398(7)^3\end{pmatrix}\\
\frac{(3)^6}{6!}-\frac{1}{(44800)(3!)}\begin{pmatrix}183654(1)^3-105381(2)^3+135450(3)^3-98955(4)^3+46818(5)^3\\-12879(6)^3+1566(7)^3\end{pmatrix}\\
\frac{(4)^6}{6!}-\frac{1}{(14175)(3!)}\begin{pmatrix}118432(1)^3-46608(2)^3+86880(3)^3-58520(4)^3+27744(5)^3-\\7632(6)^3+928(7)^3\end{pmatrix}\\
\frac{(5)^6}{6!}-\frac{1}{(145152)(3!)}\begin{pmatrix}410450(1)^3-115425(2)^3+320750(3)^3-178375(4)^3+91350(5)^3\\-25075(6)^3+3050(7)^3\end{pmatrix}\\
\frac{(6)^6}{6!}-\frac{1}{(1400)(3!)}\begin{pmatrix}30024(1)^3-6156(2)^3+24840(3)^3-10800(4)^3+7128(5)^3-1764(6)^3\\+216(7)^3\end{pmatrix}\\
\frac{(7)^6}{6!}-\frac{7}{(518400)(3!)}\begin{pmatrix}2242534(1)^3-338541(2)^3+1944810(3)^3-660275(4)^3+\\619458(5)^3-93639(6)^3+16366(7)^3\end{pmatrix}
\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$
C_7 = \begin{pmatrix}
\frac{1}{7!} - \frac{1}{(3628800)(4!)} \begin{pmatrix} 562618(1)^4 - 662757(2)^4 + 694230(3)^4 - 506675(4)^4 + \\ 239406(5)^4 - 65823(6)^4 + 8002(7)^4 \end{pmatrix} \\
\frac{(2)^7}{7!} - \frac{1}{(28350)(4!)} \begin{pmatrix} 38762(1)^4 - 32823(2)^4 + 34730(3)^4 - 25300(4)^4 + 11934(5)^4 \\ - 3277(6)^4 + 398(7)^4 \end{pmatrix} \\
\frac{(3)^7}{7!} - \frac{1}{(44800)(4!)} \begin{pmatrix} 183654(1)^4 - 105381(2)^4 + 135450(3)^4 - 98955(4)^4 + 46818(5)^4 \\ - 12879(6)^4 + 1566(7)^4 \end{pmatrix} \\
\frac{(4)^7}{7!} - \frac{1}{(14175)(4!)} \begin{pmatrix} 118432(1)^4 - 46608(2)^4 + 86880(3)^4 - 58520(4)^4 + 27744(5)^4 - \\ 7632(6)^4 + 928(7)^4 \end{pmatrix} \\
\frac{(5)^7}{7!} - \frac{1}{(145152)(4!)} \begin{pmatrix} 410450(1)^4 - 115425(2)^4 + 320750(3)^4 - 178375(4)^4 + 91350(5)^4 \\ - 25075(6)^4 + 3050(7)^4 \end{pmatrix} \\
\frac{(6)^7}{7!} - \frac{1}{(1400)(4!)} \begin{pmatrix} 30024(1)^4 - 6156(2)^4 + 24840(3)^4 - 10800(4)^4 + 7128(5)^4 - 1764(6)^4 \\ + 216(7)^4 \end{pmatrix} \\
\frac{(7)^7}{7!} - \frac{7}{(518400)(4!)} \begin{pmatrix} 2242534(1)^4 - 338541(2)^4 + 1944810(3)^4 - 660275(4)^4 + \\ 619458(5)^4 - 93639(6)^4 + 16366(7)^4 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_8 = \begin{pmatrix}
\frac{1}{8!} - \frac{1}{(3628800)(5!)} \begin{pmatrix} 562618(1)^5 - 662757(2)^5 + 694230(3)^5 - 506675(4)^5 + \\ 239406(5)^5 - 65823(6)^5 + 8002(7)^5 \end{pmatrix} \\
\frac{(2)^8}{8!} - \frac{1}{(28350)(5!)} \begin{pmatrix} 38762(1)^5 - 32823(2)^5 + 34730(3)^5 - 25300(4)^5 + 11934(5)^5 \\ - 3277(6)^5 + 398(7)^5 \end{pmatrix} \\
\frac{(3)^8}{8!} - \frac{1}{(44800)(5!)} \begin{pmatrix} 183654(1)^5 - 105381(2)^5 + 135450(3)^5 - 98955(4)^5 + 46818(5)^5 \\ - 12879(6)^5 + 1566(7)^5 \end{pmatrix} \\
\frac{(4)^8}{8!} - \frac{1}{(14175)(5!)} \begin{pmatrix} 118432(1)^5 - 46608(2)^5 + 86880(3)^5 - 58520(4)^5 + 27744(5)^5 - \\ 7632(6)^5 + 928(7)^5 \end{pmatrix} \\
\frac{(5)^8}{8!} - \frac{1}{(145152)(5!)} \begin{pmatrix} 410450(1)^5 - 115425(2)^5 + 320750(3)^5 - 178375(4)^5 + 91350(5)^5 \\ - 25075(6)^5 + 3050(7)^5 \end{pmatrix} \\
\frac{(6)^8}{8!} - \frac{1}{(1400)(5!)} \begin{pmatrix} 30024(1)^5 - 6156(2)^5 + 24840(3)^5 - 10800(4)^5 + 7128(5)^5 - 1764(6)^5 \\ + 216(7)^5 \end{pmatrix} \\
\frac{(7)^8}{8!} - \frac{7}{(518400)(5!)} \begin{pmatrix} 2242534(1)^5 - 338541(2)^5 + 1944810(3)^5 - 660275(4)^5 + \\ 619458(5)^5 - 93639(6)^5 + 16366(7)^5 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$C_9 = \begin{pmatrix} \frac{1}{9!} - \frac{1}{(3628800)(6!)}\begin{pmatrix} 562618(1)^6 - 662757(2)^6 + 694230(3)^6 - 506675(4)^6 + \\ 239406(5)^6 - 65823(6)^6 + 8002(7)^6 \end{pmatrix} \\ \frac{(2)^9}{9!} - \frac{1}{(28350)(6!)}\begin{pmatrix} 38762(1)^6 - 32823(2)^6 + 34730(3)^6 - 25300(4)^6 + 11934(5)^6 \\ -3277(6)^6 + 398(7)^6 \end{pmatrix} \\ \frac{(3)^9}{9!} - \frac{1}{(44800)(6!)}\begin{pmatrix} 183654(1)^6 - 105381(2)^6 + 135450(3)^6 - 98955(4)^6 + 46818(5)^6 \\ -12879(6)^6 + 1566(7)^6 \end{pmatrix} \\ \frac{(4)^9}{9!} - \frac{1}{(14175)(6!)}\begin{pmatrix} 118432(1)^6 - 46608(2)^6 + 86880(3)^6 - 58520(4)^6 + 27744(5)^6 - \\ 7632(6)^6 + 928(7)^6 \end{pmatrix} \\ \frac{(5)^9}{9!} - \frac{1}{(145152)(6!)}\begin{pmatrix} 410450(1)^6 - 115425(2)^6 + 320750(3)^6 - 178375(4)^6 + 91350(5)^6 \\ -25075(6)^6 + 3050(7)^6 \end{pmatrix} \\ \frac{(6)^9}{9!} - \frac{1}{(1400)(6!)}\begin{pmatrix} 30024(1)^6 - 6156(2)^6 + 24840(3)^6 - 10800(4)^6 + 7128(5)^6 - 1764(6)^6 \\ +216(7)^6 \end{pmatrix} \\ \frac{(7)^9}{9!} - \frac{7}{(518400)(6!)}\begin{pmatrix} 2242534(1)^6 - 338541(2)^6 + 1944810(3)^6 - 660275(4)^6 + \\ 619458(5)^6 - 93639(6)^6 + 16366(7)^6 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_{10} = \begin{pmatrix} \frac{1}{10!} - \frac{1}{(3628800)(7!)}\begin{pmatrix} 562618(1)^7 - 662757(2)^7 + 694230(3)^7 - 506675(4)^7 + \\ 239406(5)^7 - 65823(6)^7 + 8002(7)^7 \end{pmatrix} \\ \frac{(2)^{10}}{10!} - \frac{1}{(28350)(7!)}\begin{pmatrix} 38762(1)^7 - 32823(2)^7 + 34730(3)^7 - 25300(4)^7 + 11934(5)^7 \\ -3277(6)^7 + 398(7)^7 \end{pmatrix} \\ \frac{(3)^{10}}{10!} - \frac{1}{(44800)(7!)}\begin{pmatrix} 183654(1)^7 - 105381(2)^7 + 135450(3)^7 - 98955(4)^7 + 46818(5)^7 \\ -12879(6)^7 + 1566(7)^7 \end{pmatrix} \\ \frac{(4)^{10}}{10!} - \frac{1}{(14175)(7!)}\begin{pmatrix} 118432(1)^7 - 46608(2)^7 + 86880(3)^7 - 58520(4)^7 + 27744(5)^7 - \\ 7632(6)^7 + 928(7)^7 \end{pmatrix} \\ \frac{(5)^{10}}{10!} - \frac{1}{(145152)(7!)}\begin{pmatrix} 410450(1)^7 - 115425(2)^7 + 320750(3)^7 - 178375(4)^7 + 91350(5)^7 \\ -25075(6)^7 + 3050(7)^7 \end{pmatrix} \\ \frac{(6)^{10}}{10!} - \frac{1}{(1400)(7!)}\begin{pmatrix} 30024(1)^7 - 6156(2)^7 + 24840(3)^7 - 10800(4)^7 + 7128(5)^7 - 1764(6)^7 \\ +216(7)^7 \end{pmatrix} \\ \frac{(7)^{10}}{10!} - \frac{7}{(518400)(7!)}\begin{pmatrix} 2242534(1)^7 - 338541(2)^7 + 1944810(3)^7 - 660275(4)^7 + \\ 619458(5)^7 - 93639(6)^7 + 16366(7)^7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
C_{11}=\begin{pmatrix}
\frac{1}{11!}-\frac{1}{(3628800)(8!)}\begin{pmatrix}562618(1)^8-662757(2)^8+694230(3)^8-506675(4)^8\\+239406(5)^8-65823(6)^8+8002(7)^8.\end{pmatrix}\\
\frac{(2)^{11}}{11!}-\frac{1}{(28350)(8!)}\begin{pmatrix}38762(1)^8-32823(2)^8+34730(3)^8-25300(4)^8+\\11934(5)^8-3277(6)^8+398(7)^8.\end{pmatrix}\\
\frac{(3)^{11}}{11!}-\frac{1}{(44800)(8!)}\begin{pmatrix}183654(1)^8-105381(2)^8+135450(3)^8-98955(4)^8\\+46818(5)^8-12879(6)^8+1566(7)^8.\end{pmatrix}\\
\frac{(4)^{11}}{11!}-\frac{1}{(14175)(8!)}\begin{pmatrix}118432(1)^8-46608(2)^8+86880(3)^8-58520(4)^8\\+27744(5)^8-7632(6)^8+928(7)^8.\end{pmatrix}\\
\frac{(5)^{11}}{11!}-\frac{1}{(145152)(8!)}\begin{pmatrix}410450(1)^8-115425(2)^8+320750(3)^8-178375(4)\\+91350(5)^8-25075(6)^8+3050(7)^8.\end{pmatrix}\\
\frac{(6)^{11}}{11!}-\frac{1}{(1400)(8!)}\begin{pmatrix}30024(1)^8-6156(2)^8+24840(3)^8-10800(4)^8+7128(5)^8\\-1764(6)^8+216(7)^8.\end{pmatrix}\\
\frac{(7)^{11}}{11!}-\frac{7}{(518400)(8!)}\begin{pmatrix}2242534(1)^8-338541(2)^8+1944810(3)^8-660275(4)^8+\\619458(5)^8-93639(6)^8+16366(7)^8.\end{pmatrix}
\end{pmatrix}
=\begin{pmatrix}\frac{-165}{89141}\\\frac{-179}{15283}\\\frac{-2889}{98560}\\\frac{-256}{4661}\\\frac{-460}{5209}\\\frac{-999}{7700}\\\frac{-2402}{13423}\end{pmatrix}
$$

Hence, the block has order $(8,8,8,8,8,8,8)^T$ with error constants

$$\left(\frac{-165}{89141},\frac{-179}{15283},\frac{-2889}{98560},\frac{-256}{4661},\frac{-460}{5209},\frac{-999}{7700},\frac{2402}{13423}\right)^T$$

5.4.2.2 Zero Stability of Seven–Step Block Method for Fourth Order ODEs.

Applying the equation (3.2.2.2.1) to seven-step block method (5.4.1.37 –5.4.1.43) we have

$$
\det[rA^{(0)}-A^{(1)}]=\left|r\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\end{pmatrix}\right|=0
$$

which implies $r=0,0,0,0,0,0,1$. Hence, the method is zero stable.

5.4.2.3 Consistency and Convergence of Seven–Step Block Method for Fourth Order ODEs.

The conditions highlighted in Definition 1.4 are satisfied on block method (5.4.1.37 – 5.4.1.43). This implies that the method is consistent. Hence, it is also convergent because it is zero-stable and consistent.

5.4.2.4 Region of Absolute Stability of Seven–Step Block Method for Fourth Order ODEs.

Equation (3.2.2.4.2) is applied to seven-step block (5.4.1.37 – 5.4.1.43), to produce

$$\bar{h}(\theta,h)=\frac{A-B}{C+D}$$

where

$$A=\begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{4i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{5i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{6i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{7i\theta} \end{pmatrix},\quad B=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C=\begin{pmatrix} \frac{562618}{3628800}e^{i\theta} & -\frac{662757}{3628800}e^{2i\theta} & \frac{694230}{3628800}e^{3i\theta} & -\frac{506675}{3628800}e^{4i\theta} & \frac{239406}{3628800}e^{5i\theta} & -\frac{65823}{3628800}e^{6i\theta} & \frac{8002}{3628800}e^{7i\theta} \\ \frac{38762}{28350}e^{i\theta} & -\frac{32823}{28350}e^{2i\theta} & \frac{34730}{28350}e^{3i\theta} & -\frac{25300}{28350}e^{4i\theta} & \frac{11934}{28350}e^{5i\theta} & -\frac{3277}{28350}e^{6i\theta} & \frac{398}{28350}e^{7i\theta} \\ \frac{183654}{44800}e^{i\theta} & -\frac{105381}{44800}e^{2i\theta} & \frac{135450}{44800}e^{3i\theta} & -\frac{98955}{44800}e^{4i\theta} & \frac{46818}{44800}e^{5i\theta} & -\frac{12879}{44800}e^{6i\theta} & \frac{1566}{44800}e^{7i\theta} \\ \frac{118432}{14175}e^{i\theta} & -\frac{46608}{14175}e^{2i\theta} & \frac{86880}{14175}e^{3i\theta} & -\frac{58520}{14175}e^{4i\theta} & \frac{27744}{14175}e^{5i\theta} & -\frac{7632}{14175}e^{6i\theta} & \frac{928}{14175}e^{7i\theta} \\ \frac{2052250}{145152}e^{i\theta} & -\frac{577125}{145152}e^{2i\theta} & \frac{1603750}{145152}e^{3i\theta} & -\frac{891875}{145152}e^{4i\theta} & \frac{456750}{145152}e^{5i\theta} & -\frac{125375}{145152}e^{6i\theta} & \frac{15250}{145152}e^{7i\theta} \\ \frac{30024}{700}e^{i\theta} & -\frac{6156}{700}e^{2i\theta} & \frac{24840}{700}e^{3i\theta} & -\frac{10800}{700}e^{4i\theta} & \frac{7128}{700}e^{5i\theta} & -\frac{1764}{700}e^{6i\theta} & \frac{216}{700}e^{7i\theta} \\ \frac{15697738}{518400}e^{i\theta} & \frac{2369787}{518400}e^{2i\theta} & \frac{13613670}{518400}e^{4i\theta} & -\frac{4621925}{518400}e^{5i\theta} & \frac{4336206}{518400}e^{5i\theta} & \frac{655473}{518400}e^{6i\theta} & \frac{114562}{518400}e^{7i\theta} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{335799}{3628800} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{13376}{28350} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{51327}{44800} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{29976}{14175} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{490375}{145152} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6912}{700} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3520209}{518400} \end{pmatrix}$$

Simplifying the above matrix and equating the imaginary part to zero we have

$$\bar{h}(\theta,h) = \frac{= 1.4775\text{E} + 113\cos 7\theta - 1.4775\text{E} + 113}{4.2753\text{E} + 108\cos 7\theta + 7.0527\text{E} + 110}$$

Evaluating $\bar{h}(\theta,h)$ at intervals of θ of 30^0, the following tabulation are obtained.

Table 5.3

Interval of Absolute Stability of Seven–Step Block Method for Third Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta,h)$	0	-29.39	-7.33	-15.04	-23.16	-1.93	-31.74

Therefore, the interval of absolute stability is (-31.74, 0). This is demonstrated in the diagram below

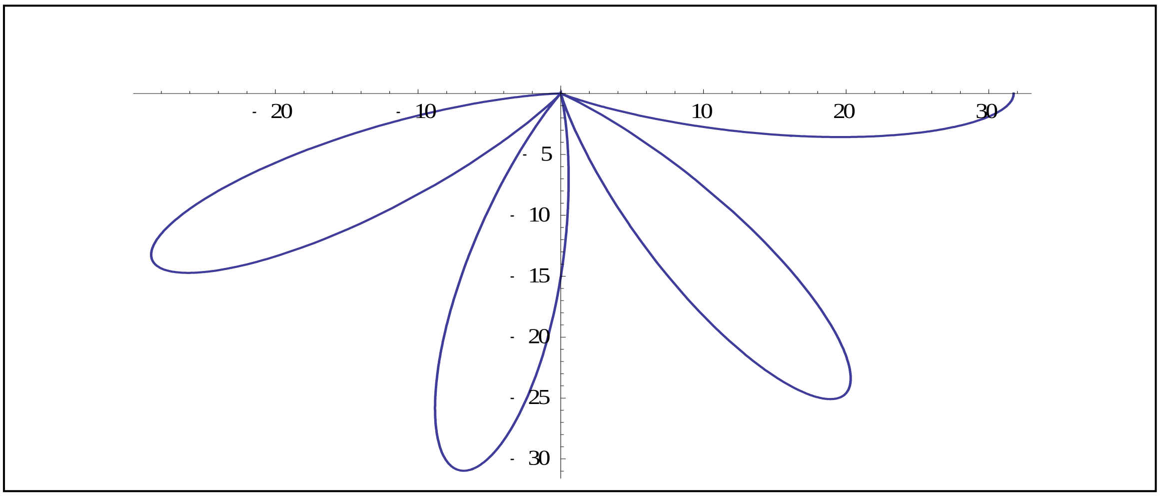

Figure 5.6. Region of absolute stability of Seven–step block method for fourth order ODEs.

5.5 Eight–Step Block Method for Fourth Order ODEs.

This section contains the derivation of eight–step block method for solving fourth order ODEs. It also includes the properties of the block method.

5.5.1 Derivation of Eight–Step Block Method for Fourth Order ODEs.

Power series of the form (5.2.1.1) is considered as an approximate solution to the general fourth order problem of the form (5.2.1.2) where in (5.2.1.1) the step-length k=8 is used. The first, second and third derivatives of (5.2.1.1) are given in (5.2.1.3), (5.2.1.4), (5.2.1.5) and (5.2.1.6).

Interpolating equation (5.2.1.1) at $x = x_{n+i}, i = 3(1)6$ and collocating (5.2.1.6) at $x = x_{n+i}, i = 0(1)8$ as represented in the Figure 5.7 below

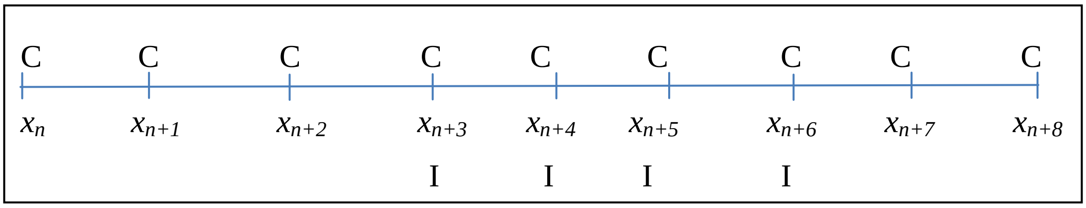

Figure 5.7. Eight –step interpolation and collocation method for fourth order ODEs.

This produces the result in matrix below

$$AX = B \tag{5.5.1.1}$$

where

$$A=\begin{pmatrix}
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 & x_{n+3}^{10} & x_{n+3}^{11} & x_{n+3}^{12}\\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 & x_{n+4}^{10} & x_{n+4}^{11} & x_{n+4}^{12}\\
1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 & x_{n+5}^9 & x_{n+5}^{10} & x_{n+5}^{11} & x_{n+5}^{12}\\
1 & x_{n+6} & x_{n+6}^2 & x_{n+6}^3 & x_{n+6}^4 & x_{n+6}^5 & x_{n+6}^6 & x_{n+6}^7 & x_{n+6}^8 & x_{n+6}^9 & x_{n+6}^{10} & x_{n+6}^{11} & x_{n+6}^{12}\\
0 & 0 & 0 & 0 & 24 & 120x_n & 360x_n^2 & 840x_n^3 & 1680x_n^4 & 3024x_n^5 & 5040x_n^6 & 7920x_n^7 & 11880x_n^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+1} & 360x_{n+1}^2 & 840x_{n+1}^3 & 1680x_{n+1}^4 & 3024x_{n+1}^5 & 5040x_{n+1}^6 & 7920x_{n+1}^7 & 11880x_{n+1}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+2} & 360x_{n+2}^2 & 840x_{n+2}^3 & 1680x_{n+2}^4 & 3024x_{n+2}^5 & 5040x_{n+2}^6 & 7920x_{n+2}^7 & 11880x_{n+2}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+3} & 360x_{n+3}^2 & 840x_{n+3}^3 & 1680x_{n+3}^4 & 3024x_{n+3}^5 & 5040x_{n+3}^6 & 7920x_{n+3}^7 & 11880x_{n+3}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+4} & 360x_{n+4}^2 & 840x_{n+4}^3 & 1680x_{n+4}^4 & 3024x_{n+4}^5 & 5040x_{n+4}^6 & 7920x_{n+4}^7 & 11880x_{n+4}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+5} & 360x_{n+5}^2 & 840x_{n+5}^3 & 1680x_{n+5}^4 & 3024x_{n+5}^5 & 5040x_{n+5}^6 & 7920x_{n+5}^7 & 11880x_{n+5}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+6} & 360x_{n+6}^2 & 840x_{n+6}^3 & 1680x_{n+6}^4 & 3024x_{n+6}^5 & 5040x_{n+6}^6 & 7920x_{n+6}^7 & 11880x_{n+6}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+7} & 360x_{n+7}^2 & 840x_{n+7}^3 & 1680x_{n+7}^4 & 3024x_{n+7}^5 & 5040x_{n+7}^6 & 7920x_{n+7}^7 & 11880x_{n+7}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+8} & 360x_{n+8}^2 & 840x_{n+8}^3 & 1680x_{n+8}^4 & 3024x_{n+8}^5 & 5040x_{n+8}^6 & 7920x_{n+8}^7 & 11880x_{n+8}^8
\end{pmatrix}$$

$$X = \left(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\right)^T$$

$$B = \left(y_{n+3}, y_{n+4}, y_{n+5}, y_{n+6}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}, f_{n+7}, f_{n+8}\right)^T$$

In order to find the values *a's* in (5.5.1.1), Gaussian elimination method is employed.

The values of *a's* are below:

$$a_0 = \frac{x_n^{12}}{479001600h^8}(f_n - 8f_{n+1} + 28f_{n+2} - 56f_{n+3} + 70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7}$$
$$+f_{n+8}) + \frac{x_n^{11}}{79833600h^7}(9f_n - 70f_{n+1} + 238f_{n+2} - 462f_{n+3} + 560f_{n+4} - 434f_{n+5} +$$

$$
\begin{aligned}
&210 f_{n+6} - 58 f_{n+7} + 7 f_{n+8}) + \frac{x_n^{10}}{14515200h^6}(39 f_n - 292 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3} + \\
&2090 f_{n+4} - 1564 f_{n+5} + 732 f_{n+6} - 196 f_{n+7} + 23 f_{n+8}) + \frac{x_n^{9}}{2177280h^5}(81 f_n - 575 f_{n+1} \\
&+1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4} - 2581 f_{n+5} + 1170 f_{n+6} - 305 f_{n+7} + 35 f_{n+8}) + \\
&\frac{x_n^{8}}{9676800h^4}(3207 f_n - 21056 f_{n+1} + 61156 f_{n+2} - 102912 f_{n+3} + 109930 f_{n+4} - \\
&6352 f_{n+5} + 33636 f_{n+6} - 8576 f_{n+7} + 967 f_{n+8}) + \frac{x_n^{7}}{1209600h^3}(2403 f_n - 13960 f_{n+1} + \\
&36706 f_{n+2} - 57384 f_{n+3} + 58280 f_{n+4} - 39128 f_{n+5} + 16830 f_{n+6} - 4216 f_{n+7} + 469 f_{n+8}) \\
&+ \frac{x_n^{6}}{3628800h^2}(29531 f_n - 138528 f_{n+1} + 312984 f_{n+2} - 448672 f_{n+3} + 435330 f_{n+4} - \\
&284256 f_{n+5} + 120008 f_{n+6} - 29664 f_{n+7} + 3267 f_{n+8}) + \frac{x_n^{5}}{100800h}(2283 f_n - 6720 f_{n+1} + \\
&11760 f_{n+2} - 15680 f_{n+3} + 14700 f_{n+4} - 9408 f_{n+5} + 3920 f_{n+6} - 960 f_{n+7} + 105 f_{n+8}) + \\
&\frac{x_n^4}{24} f_n + \frac{x_n^3 h}{43545600}(2082753 f_n + 11532880 f_{n+1} + 255896 f_{n+2} + 16294176 f_{n+3} - \\
&2298350 f_{n+4} + 6510512 f_{n+5} - 2192400 f_{n+6} + 531424 f_{n+7} - 57691 f_{n+8}) + \\
&\frac{x_n^2 h^2}{958003200}(29248987 f_n + 527286424 f_{n+1} + 851733508 f_{n+2} + 1592033896 f_{n+3} \\
&+1358008930 f_{n+4} + 427869928 f_{n+5} - 45488444 f_{n+6} + 10509592 f_{n+7} - \\
&1103621 f_{n+8}) + \frac{x_n^3}{6h^3}(y_{n+3} - 3y_{n+4} + 3y_{n+5} - y_{n+6}) + \frac{x_n^2}{2h^2}(5y_{n+3} - 14y_{n+4} + 13y_{n+5} \\
&-4y_{n+6}) + \frac{x_n h^3}{53222400}(436331 f_n + 25829880 f_{n+1} + 106603540 f_{n+2} + 241795400 f_{n+3} \\
&+311177490 f_{n+4} + 74235784 f_{n+5} - 1971340 f_{n+6} + 339000 f_{n+7} - 26885 f_{n+8}) + \\
&\frac{x_n}{6h}(74y_{n+3} - 189y_{n+4} + 162y_{n+5} - 47y_{n+6}) + 20y_{n+3} - 45y_{n+4} + 36y_{n+5} - 10y_{n+6} + \\
&\frac{h^4}{241920}(19 f_n + 39160 f_{n+1} + 321940 f_{n+2} + 1103560 f_{n+3} + 1743010 f_{n+4} + 430216 f_{n+5} \\
&-10700 f_{n+6} + 1720 f_{n+7} - 125 f_{n+8})
\end{aligned}
$$

$$
\begin{aligned}
a_1 = \; & -\frac{x_n^{11}}{39916800h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} \\
&+f_{n+8}) - \frac{x_n^{10}}{7257600h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + 210 f_{n+6} \\
&-58 f_{n+7} + 7 f_{n+8}) - \frac{x_n^{9}}{1451520h^6}(39 f_n - 292 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3} + 2090 f_{n+4} - \\
&1564 f_{n+5} + 732 f_{n+6} - 196 f_{n+7} + 23 f_{n+8}) - \frac{x_n^{8}}{241920h^5}(81 f_n - 575 f_{n+1} + 1790 f_{n+2} -
\end{aligned}
$$

$$3195 f_{n+3} + 3580 f_{n+4} - 2581 f_{n+5} + 1170 f_{n+6} - 305 f_{n+7} + 35 f_{n+8}) - \frac{x_n^7}{1209600 h^4}(3207 f_n$$
$$-21056 f_{n+1} + 61156 f_{n+2} - 102912 f_{n+3} + 109930 f_{n+4} - 76352 f_{n+5} + 33636 f_{n+6} - 8576 f_{n+7}$$
$$+967 f_{n+8}) - \frac{x_n^{11}}{39916800 h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7}$$
$$\frac{x_n^4}{20160 h}(2283 f_n - 6720 f_{n+1} + 11760 f_{n+2} - 15680 f_{n+3} + 14700 f_{n+4} - 9408 f_{n+5} + 3920 f_{n+6}$$
$$-960 f_{n+7} + 105 f_{n+8}) - \frac{x_n^3}{6} f_n - \frac{x_n^2 h}{14515200 h^8}(2082753 f_n + 11532880 f_{n+1} + 255896 f_{n+2}$$
$$+16294176 f_{n+3} - 2298350 f_{n+4} + 6510512 f_{n+5} - 2192400 f_{n+6} + 531424 f_{n+7} -$$
$$57691 f_{n+8}) - \frac{x_n^2}{2h^3}(y_{n+3} - 3y_{n+4} + 3y_{n+5} - y_{n+6}) - \frac{x_n h^2}{479001600}(29248987 f_n +$$
$$29248987 f_{n+1} + 851733508 f_{n+2} + 1592033896 f_{n+3} + 1358008930 f_{n+4} + 427869928 f_{n+5}$$
$$-45488444 f_{n+6} + 10509592 f_{n+7} - 1103621 f_{n+8}) - \frac{x_n}{h^2}(5y_{n+3} - 14 y_{n+4} + 13 y_{n+5} - 4 y_{n+6})$$
$$-\frac{h^3}{53222400}(436331 f_n + 25829880 f_{n+1} + 106603540 f_{n+2} + 241795400 f_{n+3} +$$
$$311177490 f_{n+4} + 74235784 f_{n+5} - 1971340 f_{n+6} + 339000 f_{n+7} - 26885 f_{n+8})$$
$$-\frac{1}{6h}(74 y_{n+3} - 189 y_{n+4} + 162 y_{n+5} - 47 y_{n+6})$$

$$a_2 = \frac{x_n^{10}}{7257600 h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8})$$
$$+\frac{x_n^9}{1451520 h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + 210 f_{n+6} - 58 f_{n+7}$$
$$+7 f_{n+8}) + \frac{x_n^8}{322560 h^6}(39 f_n - 292 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3} + 2090 f_{n+4} - 1564 f_{n+5} +$$
$$732 f_{n+6} - 196 f_{n+7} + 23 f_{n+8}) + \frac{x_n^7}{60480 h^5}(81 f_n - 575 f_{n+1} + 1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4}$$
$$-2581 f_{n+5} + 1170 f_{n+6} - 305 f_{n+7} + 35 f_{n+8}) + \frac{x_n^6}{345600 h^4}(3207 f_n - 21056 f_{n+1} + 61156 f_{n+2}$$
$$-102912 f_{n+3} + 109930 f_{n+4} - 76352 f_{n+5} + 33636 f_{n+6} - 8576 f_{n+7} + 967 f_{n+8}) +$$
$$\frac{x_n^5}{57600 h^3}(2403 f_n - 13960 f_{n+1} + 36706 f_{n+2} - 57384 f_{n+3} + 58280 f_{n+4} - 39128 f_{n+5} +$$
$$16830 f_{n+6} - 4216 f_{n+7} + 469 f_{n+8}) + \frac{x_n^4}{241920 h^2}(29531 f_n - 138528 f_{n+1} + 312984 f_{n+2} -$$
$$448672 f_{n+3} + 435330 f_{n+4} - 284256 f_{n+5} + 120008 f_{n+6} - 29664 f_{n+7} + 3267 f_{n+8}) +$$
$$\frac{x_n^3}{10080 h}(2283 f_n - 6720 f_{n+1} + 11760 f_{n+2} - 15680 f_{n+3} + 14700 f_{n+4} - 9408 f_{n+5} + 3920 f_{n+6}$$

$$-960 f_{n+7} + 105 f_{n+8}) + \frac{x_n^2}{4} f_n + \frac{x_n}{2h^3}(y_{n+3} - 3y_{n+4} + 3y_{n+5} - y_{n+6}) +$$
$$\frac{x_n h}{14515200}(2082753 f_n + 11532880 f_{n+1} + 255896 f_{n+2} + 16294176 f_{n+3}$$
$$- 2298350 f_{n+4} + 6510512 f_{n+5} - 2192400 f_{n+6} + 531424 f_{n+7} - 57691 f_{n+8})$$
$$+ \frac{1}{2h^2}(5y_{n+3} - 14y_{n+4} + 13y_{n+5} - 4y_{n+6}) + \frac{h^2}{958003200}(29248987 f_n +$$
$$527286424 f_{n+1} + 851733508 f_{n+2} + 1592033896 f_{n+3} + 1358008930 f_{n+4} +$$
$$427869928 f_{n+5} - 45488444 f_{n+6} + 10509592 f_{n+7} - 1103621 f_{n+8}).$$

$$a_3 = -\frac{x_n^9}{2177280h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7}$$
$$+ f_{n+8}) - \frac{x_n^8}{483840h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + 210 f_{n+6}$$
$$-58 f_{n+7} + 7 f_{n+8}) - \frac{x_n^7}{120960h^6}(39 f_n - 292 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3} + 2090 f_{n+4} -$$
$$1564 f_{n+5} + 732 f_{n+6} - 196 f_{n+7} + 23 f_{n+8}) - \frac{x_n^6}{25920h^5}(81 f_n - 575 f_{n+1} + 1790 f_{n+2} -$$
$$3195 f_{n+3} + 3580 f_{n+4} - 2581 f_{n+5} + 1170 f_{n+6} - 305 f_{n+7} + 35 f_{n+8}) - \frac{x_n^5}{172800h^4}(3207 f_n$$
$$-21056 f_{n+1} + 61156 f_{n+2} - 102912 f_{n+3} + 109930 f_{n+4} - 76352 f_{n+5} + 33636 f_{n+6} -$$
$$8576 f_{n+7} + 967 f_{n+8}) - \frac{x_n^4}{34560h^3}(2403 f_n - 13960 f_{n+1} + 36706 f_{n+2} - 57384 f_{n+3} +$$
$$58280 f_{n+4} - 39128 f_{n+5} + 16830 f_{n+6} - 4216 f_{n+7} + 469 f_{n+8}) - \frac{x_n^3}{181440h^2}(29531 f_n$$
$$-138528 f_{n+1} + 312984 f_{n+2} - 448672 f_{n+3} + 435330 f_{n+4} - 284256 f_{n+5} + 120008 f_{n+6}$$
$$-29664 f_{n+7} + 3267 f_{n+8}) - \frac{x_n^2}{10080h}(2283 f_n - 6720 f_{n+1} + 11760 f_{n+2} - 15680 f_{n+3}$$
$$+14700 f_{n+4} - 9408 f_{n+5} + 3920 f_{n+6} - 960 f_{n+7} + 105 f_{n+8}) - \frac{h}{43545600}(2082753 f_n$$
$$+11532880 f_{n+1} + 255896 f_{n+2} + 16294176 f_{n+3} - 2298350 f_{n+4} + 6510512 f_{n+5} -$$
$$2192400 f_{n+6} + 531424 f_{n+7} - 57691 f_{n+8}) - \frac{x_n}{6} f_n - \frac{1}{6h^3}(y_{n+3} - 3y_{n+4} + 3y_{n+5} - y_{n+6})$$

$$a_4 = \frac{x_n^8}{967680h^8}(f_n - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8})$$
$$+ \frac{x_n^7}{241920h^7}(9 f_n - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + 10 f_{n+6} - 58 f_{n+7}$$

$$+7f_{n+8})+\frac{x_n^6}{69120h^6}(39f_n-292f_{n+1}+956f_{n+2}-1788f_{n+3}+2090f_{n+4}-1564f_{n+5}$$

$$+732f_{n+6}-196f_{n+7}+23f_{n+8})+\frac{x_n^5}{17280h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}$$

$$+3580f_{n+4}-2581f_{n+5}+1170f_{n+6}-305f_{n+7}+35f_{n+8})+\frac{x_n^4}{138240h^4}(3207f_n-21056f_{n+1}$$

$$+61156f_{n+2}-102912f_{n+3}+109930f_{n+4}-6352f_{n+5}+33636f_{n+6}-8576f_{n+7}+967f_{n+8})+$$

$$\frac{x_n^3}{34560h^3}(2403f_n-13960f_{n+1}+36706f_{n+2}-57384f_{n+3}+58280f_{n+4}-39128f_{n+5}+$$

$$6830f_{n+6}-4216f_{n+7}+469f_{n+8})+\frac{x_n^2}{241920h^2}(29531f_n-138528f_{n+1}+312984f_{n+2}-$$

$$448672f_{n+3}+435330f_{n+4}-284256f_{n+5}+120008f_{n+6}-29664f_{n+7}+3267f_{n+8})+$$

$$\frac{x_n}{20160h}(2283f_n-6720f_{n+1}+11760f_{n+2}-15680f_{n+3}+14700f_{n+4}-9408f_{n+5}+3920f_{n+6}$$

$$-960f_{n+7}+105f_{n+8})+\frac{1}{24}f_n$$

$$a_5=-\frac{x_n^7}{604800h^8}(f_n-8f_{n+1}+28f_{n+2}-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})$$

$$-\frac{x_n^6}{172800h^7}(9f_n-70f_{n+1}+238f_{n+2}-462f_{n+3}+560f_{n+4}-434f_{n+5}+10f_{n+6}-58f_{n+7}$$

$$+7f_{n+8})-\frac{x_n^5}{57600h^6}(39f_n-292f_{n+1}+956f_{n+2}-1788f_{n+3}+2090f_{n+4}-1564f_{n+5}+$$

$$732f_{n+6}196f_{n+7}+23f_{n+8})-\frac{x_n^4}{17280h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}$$

$$-2581f_{n+5}+1170f_{n+6}-305f_{n+7}+35f_{n+8})-\frac{x_n^3}{172800h^4}(3207f_n-21056f_{n+1}+61156f_{n+2}$$

$$-102912f_{n+3}+109930f_{n+4}-6352f_{n+5}+33636f_{n+6}-8576f_{n+7}+967f_{n+8})-$$

$$\frac{x_n^2}{57600h^3}(2403f_n-13960f_{n+1}+36706f_{n+2}-57384f_{n+3}+58280f_{n+4}-39128f_{n+5}+$$

$$6830f_{n+6}-4216f_{n+7}+469f_{n+8})-\frac{x_n}{604800h^2}(29531f_n-138528f_{n+1}+312984f_{n+2}-$$

$$448672f_{n+3}+435330f_{n+4}-284256f_{n+5}+120008f_{n+6}-29664f_{n+7}+3267f_{n+8})-$$

$$\frac{1}{100800h}(2283f_n-6720f_{n+1}+11760f_{n+2}-15680f_{n+3}+14700f_{n+4}-9408f_{n+5}+3920f_{n+6}$$

$$-960f_{n+7}+105f_{n+8})$$

$$a_6 = \frac{x_n^6}{518400h^8}(f_n - 8f_{n+1} + 28f_{n+2} - 56f_{n+3} + 70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} + f_{n+8})$$
$$+\frac{x_n^5}{172800h^7}(9f_n - 70f_{n+1} + 238f_{n+2} - 462f_{n+3} + 560f_{n+4} - 434f_{n+5} + 10f_{n+6} - 58f_{n+7}$$
$$+7f_{n+8}) + \frac{x_n^4}{69120h^6}(39f_n - 292f_{n+1} + 956f_{n+2} - 1788f_{n+3} + 2090f_{n+4} - 1564f_{n+5} +$$
$$732f_{n+6}196f_{n+7} + 23f_{n+8}) + \frac{x_n^3}{25920h^5}(81f_n - 575f_{n+1} + 1790f_{n+2} - 3195f_{n+3} + 3580f_{n+4}$$
$$-2581f_{n+5} + 1170f_{n+6} - 305f_{n+7} + 35f_{n+8}) + \frac{x_n^2}{345600h^4}(3207f_n - 21056f_{n+1} + 61156f_{n+2}$$
$$-102912f_{n+3} + 109930f_{n+4} - 6352f_{n+5} + 33636f_{n+6} - 8576f_{n+7} + 967f_{n+8}) +$$
$$\frac{x_n}{172800h^3}(2403f_n - 13960f_{n+1} + 36706f_{n+2} - 57384f_{n+3} + 58280f_{n+4} - 39128f_{n+5} +$$
$$6830f_{n+6} - 4216f_{n+7} + 469f_{n+8}) + \frac{1}{3628800h^2}(29531f_n - 138528f_{n+1} + 312984f_{n+2} -$$
$$448672f_{n+3} + 435330f_{n+4} - 284256f_{n+5} + 120008f_{n+6} - 29664f_{n+7} + 3267f_{n+8})$$

$$a_7 = -\frac{x_n^5}{604800h^8}(f_n - 8f_{n+1} + 28f_{n+2} - 56f_{n+3} + 70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} +$$
$$f_{n+8}) - \frac{x_n^4}{241920h^7}(9f_n - 70f_{n+1} + 238f_{n+2} - 462f_{n+3} + 560f_{n+4} - 434f_{n+5} + 10f_{n+6} -$$
$$58f_{n+7} + 7f_{n+8}) - \frac{x_n^3}{120960h^6}(39f_n - 292f_{n+1} + 956f_{n+2} - 1788f_{n+3} + 2090f_{n+4} - 1564f_{n+5}$$
$$+732f_{n+6}196f_{n+7} + 23f_{n+8}) - \frac{x_n^2}{60480h^5}(81f_n - 575f_{n+1} + 1790f_{n+2} - 3195f_{n+3} + 3580f_{n+4}$$
$$-2581f_{n+5} + 1170f_{n+6} - 305f_{n+7} + 35f_{n+8}) - \frac{x_n}{120600h^4}(3207f_n - 21056f_{n+1} + 61156f_{n+2}$$
$$-102912f_{n+3} + 109930f_{n+4} - 6352f_{n+5} + 33636f_{n+6} - 8576f_{n+7} + 967f_{n+8}) -$$
$$\frac{1}{1209600h^3}(2403f_n - 13960f_{n+1} + 36706f_{n+2} - 57384f_{n+3} + 58280f_{n+4} - 39128f_{n+5}$$
$$6830f_{n+6} - 4216f_{n+7} + 469f_{n+8})$$

$$a_8 = \frac{x_n^4}{967680h^8}(f_n - 8f_{n+1} + 28f_{n+2} - 56f_{n+3} + 70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} + f_{n+8})$$
$$+\frac{x_n^3}{483840h^7}(9f_n - 70f_{n+1} + 238f_{n+2} - 462f_{n+3} + 560f_{n+4} - 434f_{n+5} + 10f_{n+6} - 58f_{n+7}$$
$$+7f_{n+8}) + \frac{x_n^2}{322560h^6}(39f_n - 292f_{n+1} + 956f_{n+2} - 1788f_{n+3} + 2090f_{n+4} - 1564f_{n+5} +$$

$$732 f_{n+6} 196 f_{n+7} + 23 f_{n+8}) + \frac{x_n}{241920h^5}(81 f_n \; - 575 f_{n+1} + 1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4}$$
$$- 2581 f_{n+5} + 1170 f_{n+6} - 305 f_{n+7} + 35 f_{n+8}) + \frac{1}{9676800h^4}(3207 f_n \; - 21056 f_{n+1} + 61156 f_{n+2}$$
$$- 102912 f_{n+3} + 109930 f_{n+4} - 6352 f_{n+5} + 33636 f_{n+6} - 8576 f_{n+7} + 967 f_{n+8})$$

$$a_9 = \; -\frac{x_n^3}{2177280h^8}(f_n \; - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8})$$
$$-\frac{x_n^2}{1451520h^7}(9 f_n \; - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + 10 f_{n+6} - 58 f_{n+7}$$
$$+ 7 f_{n+8}) - \frac{x_n}{1451520h^6}(39 f_n \; - 292 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3} + 2090 f_{n+4} - 1564 f_{n+5} +$$
$$732 f_{n+6} 196 f_{n+7} + 23 f_{n+8}) - \frac{1}{2177280h^5}(81 f_n \; - 575 f_{n+1} + 1790 f_{n+2} - 3195 f_{n+3} + 3580 f_{n+4}$$
$$- 2581 f_{n+5} + 1170 f_{n+6} - 305 f_{n+7} + 35 f_{n+8})$$

$$a_{10} = \; \frac{x_n^2}{7257600h^8}(f_n \; - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8})$$
$$+\frac{x_n}{7257600h^7}(9 f_n \; - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + 10 f_{n+6} - 58 f_{n+7}$$
$$+ 7 f_{n+8}) + \frac{1}{14515200h^6}(39 f_n \; - 292 f_{n+1} + 956 f_{n+2} - 1788 f_{n+3} + 2090 f_{n+4} - 1564 f_{n+5} +$$
$$732 f_{n+6} + 196 f_{n+7} + 23 f_{n+8})$$

$$a_{11} = \; -\frac{x_n}{39916800h^8}(f_n \; - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8})$$
$$-\frac{1}{79833600h^7}(9 f_n \; - 70 f_{n+1} + 238 f_{n+2} - 462 f_{n+3} + 560 f_{n+4} - 434 f_{n+5} + 10 f_{n+6} - 58 f_{n+7}$$
$$+ 7 f_{n+8})$$

$$a_{12} = \; -\frac{x_n}{479001600h^8}(f_n \; - 8 f_{n+1} + 28 f_{n+2} - 56 f_{n+3} + 70 f_{n+4} - 56 f_{n+5} + 28 f_{n+6} - 8 f_{n+7} + f_{n+8})$$

Substituting the values of a's into equation (5.2.1.1) and simplifying, this gives a continuous linear multistep method of the form:

$$y(x) = \sum_{j=3}^{k-2} \alpha_j(x) y_{n+j} + h^4 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (5.5.1.2)$$

where $x = zh + x_n + 7h$ (5.5.1.3)

Equation (5.5.1.3) is substituted into (5.5.1.2) and simplified as follows

$$\alpha_3(z) = -1 - \frac{11z}{6} - z^2 - \frac{z^3}{6}$$

$$\alpha_4(z) = 4 + 7z + \frac{7z^2}{2} + \frac{z^3}{2}$$

$$\alpha_5(z) = -6 - \frac{19z}{2} - 4z^2 - \frac{z^3}{2}$$

$$\alpha_6(z) = 4 + \frac{13z}{3} + \frac{3z^2}{2} + \frac{z^3}{6}$$

$$\beta_0(z) = \frac{1}{958003200}\ (\text{-}54120 + 76510z + 132823\,z^2 - 233750z^3 - 142560z^5$$
$$-68904z^6 + 3960z^7 + 12573z^8 + 4400z^9 + 726z^{10} +$$
$$60z^{11} + 2z^{12}).$$

$$\beta_1(z) = \frac{1}{119750400}\ (54120 + 76150z - 160741\,z^2 - 275011z^3 - 166320z^5$$
$$-79068z^6 - 5346z^7 - 14553z^8 - 4950z^9 - 792z^{10} -$$
$$63z^{11} - 2z^{12}).$$

$$\beta_2(z) = \frac{1}{239500800}\ (-299640 - 401962z + 1458421z^2 + 2342934z^3$$
$$-1397088z^5 - 648648\,z^6 + 53064z^7 + 120681z^8 +$$
$$39490z^9 + 6072z^{10} + 462z^{11} + 14z^{12}).$$

$$\beta_3(z) = \frac{1}{119750400}\ (-25080 - 123410z - 2106673z^2 - 3000371z^3$$
$$-1746360z^5 + 781704\,z^6 - 80586\,z^7 - 147411z^8$$
$$-45650z^9 - 6666z^{10} - 483z^{11} - 14z^{12}).$$

$$\beta_4(z) = \frac{1}{95800320}\ (16595304 + 30227870z + 19023001z^2 + 6746674z^3$$
$$-2328480z^5 - 977592\,z^6 + 135432z^7 + 187407z^8$$
$$+53460z^9 + 7326z^{10} + 504z^{11} + 14z^{12}).$$

$$\beta_5(z) = \frac{1}{119750400}\ (78422520 + 151550770z + 85710089z^2 + 10435095z^3$$
$$+3492720z^5 + 1272348\,z^6 - 266310\,z^7 - 248391z^8 -$$
$$63250z^9 - 8052z^{10} - 525z^{11} - 14z^{12}).$$

$$\beta_6(z) = \frac{1}{239500800}(42056520 + 119368550t + 124268317z^2 + 52255742z^3$$
$$-6985440z^5 - 1380456\,z^6 + 578952z^7 + 340461z^8 +$$
$$75350z^9 + 8844z^{10} + 546z^{11} + 14z^{12}).$$

$$\beta_7(z) = \frac{h^4}{119750400}\ (-349800 + 696106z + 4648877z^2 + 71307831z^3 +$$
$$4989600\, z^4 + 1589544z^5 + 4224\, z^6 - 179982\, z^7$$
$$-68013z^8 - 12870z^9 - 1386z^{10} - 81z^{11} - 2z^{12}).$$

$$\beta_8(z) = \frac{h^4}{958003200}\ (262680 + 174530z - 914201z^2 - 1251162z^3$$
$$+997920z^5 + 862488\, z^6 + 371448z^7 + 95733z^8 \tag{5.5.1.4}$$
$$+15400z^9 + 1518z^{10} + 84z^{11} + 2z^{12}).$$

Evaluating (5.5.1.4) at the non-interpolating points .i.e, at z= -7, -6, -5, 0 and 1 yields

$$2419200y_{n+6} - 8709120y_{n+5} + 10886400y_{n+4} - 4838400y_{n+3} + 241920y_n$$
$$= (19\, f_n + 39160\, f_{n+1} + 321940\, f_{n+2} + 1103560\, f_{n+3} + 1743010\, f_{n+4} \tag{5.5.1.5}$$
$$+ 430216\, f_{n+5} - 10700\, f_{n+6} + 1720\, f_{n+7} - 125\, f_{n+8}).$$

$$2903040y_{n+6} - 10886400y_{n+5} + 14515200\, y_{n+4} - 7257600y_{n+3} + 725760y_{n+1}$$
$$= (35\, f_n + 152\, f_{n+1} + 113060\, f_{n+2} + 998840\, f_{n+3} + 2006210\, f_{n+4} + 523400\, f_{n+5} \tag{5.5.1.6}$$
$$-15292\, f_{n+6} + 2600\, f_{n+7} - 205\, f_{n+8}).$$

$$725760y_{n+6} - 2903040y_{n+5} + 4354560y_{n+4} - 2903040y_{n+3} + 725760y_{n+2}$$
$$= (-41\, f_n + 568\, f_{n+1} - 3596\, f_{n+2} + 130888\, f_{n+3} + 470122\, f_{n+4} + 130888\, f_{n+5} \tag{5.5.1.7}$$
$$-3596\, f_{n+6} + 568\, f_{n+7} - 41\, f_{n+8}).$$

$$725760y_{n+7} - 2903040y_{n+6} + 4354560\, y_{n+5} - 2903040y_{n+4} + 725760y_{n+3}$$
$$= (-41\, f_n + 328\, f_{n+1} - 908\, f_{n+2} - 152\, f_{n+3} + 125722\, f_{n+4} + 475288\, f_{n+5} \tag{5.5.1.8}$$
$$+127444\, f_{n+6} - 2120\, f_{n+7} + 199\, f_{n+8}).$$

$$725760y_{n+8} - 7257600y_{n+6} + 14515200\, y_{n+5} - 10886400y_{n+4} + 2903040y_{n+3}$$
$$= (-35\, f_n - 520\, f_{n+1} + 3860\, f_{n+2} - 18232\, f_{n+3} + 527810\, f_{n+4} + 2001800\, f_{n+5} \tag{5.5.1.9}$$
$$+1001780\, f_{n+6} + 111800\, f_{n+7} + 467\, f_{n+8}).$$

The first derivative of (5.5.1.4) gives

$$\alpha_3'(z) = \frac{-11}{6} - 2z - \frac{z^2}{2}$$
$$\alpha_4'(z) = 7 + 7z + \frac{3z^2}{2}$$
$$\alpha_5'(z) = \frac{-19}{2} - 8z - \frac{3z^2}{2}$$
$$\alpha_6'(z) = \frac{13}{3} + 3z + \frac{3z^2}{6}$$

$$\beta_0'(z) = \frac{1}{479001600}\ (-38255 + 132823z + 350625z^2 - 356400z^4 - 206712z^5$$
$$+13860z^6 + 50292z^7 + 19800z^8 + 3630z^9 + 330z^{10} + 12\,z^{11}).$$

$$\beta_1'(z) = \frac{1}{119750400}\ (76150 - 321482z - 825033z^2 + 831600z^4$$
$$+474408z^5 - 37422z^6 - 116424z^7 - 44550z^8 - 7920z^9$$
$$-693z^{10} - 24z^{11}).$$

$$\beta_2'(z) = \frac{1}{119750400}\ (-200981 + 1458421z + 3514401z^2 - 3492720z^4$$
$$-1945944z^5 + 185724z^6 + 482724z^7 + 177705z^8 + 30360z^9$$
$$+2541z^{10} + 84z^{11}).$$

$$\beta_3'(z) = \frac{1}{119750400}(-123410 - 4213346z - 9001113z^2 + 8731800z^4$$
$$+4690224z^5 - 564102z^6 - 1179288z^7 - 410850z^8 - 66660z^9$$
$$-5313z^{10} - 168z^{11}).$$

$$\beta_4'(z) = \frac{1}{47900160}\ (15113935 + 19023001z + 10120011z^2 - 5821200z^4$$
$$-2932776z^5 + 474012z^6 + 749628z^7 + 240570z^8 + 36630z^9$$
$$+2772z^{10} + 84z^{11}).$$

$$\beta_5'(z) = \frac{h^3}{119750400}\ (151550770 + 171420178z + 31305285z^2$$
$$+17463600z^4 + 7634088z^5 - 1864170z^6 - 1987128z^7$$
$$-569250z^8 - 80520z^9 - 5775z^{10} - 168z^{11}).$$

$$\beta_6'(z) = \frac{h^3}{119750400}\ (59684275 + 124268317z + 78383613z^2$$
$$-17463600z^4 + 7634088z^5 + 2026332z^6 + 1361844z^7$$
$$+339075z^8 + 44220z^9 + 3003z^{10} + 84z^{11}).$$

$$\beta_7'(z) = \frac{h^3}{119750400}\ (696106 + 9297754z + 21392349z^2 + 19958400z^3$$
$$+7947720z^4 + 25344z^5 - 1259874z^6 - 544104z^7 - 115830z^8 \qquad (5.5.1.10)$$
$$-13860z^9 - 891z^{10} - 24z^{11}).$$

$$\beta_8'(z) = \frac{h^3}{479001600}\ (87265 - 914201z - 1876743z^2 + 2494800z^4$$
$$+2587464z^5 + 1300068z^6 + 382932z^7 + 69300z^8 + 7590z^9$$
$$+462z^{10} + 12z^{11}).$$

Evaluating (5.5.1.10) at all the grid points. That is, at z= -7, -6, -5, -4, -3, -2,-1, 0 and 1 yields

$$
\begin{aligned}
&53222400hy'_n - 416908800y_{n+6} + 1437004800y_{n+5} - 1676505600\,y_{n+4} \\
&+ 656409600\,y_{n+3} = h^4(-436331f_n - 25829880f_{n+1} - 106603540f_{n+2} \\
&- 241795400f_{n+3} - 311177490f_{n+4} - 74235784f_{n+5} + 1971340f_{n+6} \\
&- 339000f_{n+7} + 26885f_{n+8}).
\end{aligned} \tag{5.5.1.11}
$$

$$
\begin{aligned}
&479001600hy'_{n+1} - 2075673600y_{n+6} + 7424524800y_{n+5} - 9101030400y_{n+4} \\
&+ 3752179200y_{n+3} = h^4(-4408904f_n + 29995f_{n+1} - 228452540f_{n+2} \\
&- 980542760f_{n+3} - 1495688270f_{n+4} - 373846040f_{n+5} + 11088484f_{n+6} \\
&- 1929080f_{n+7} + 155515f_{n+8}).
\end{aligned} \tag{5.5.1.12}
$$

$$
\begin{aligned}
&479001600hy'_{n+2} - 878169600\,y_{n+6} + 3353011200y_{n+5} - 4550515200y_{n+4} \\
&+ 2075673600y_{n+3} = h^4(38255f_n - 431560f_{n+1} - 1407244f_{n+2} \\
&- 241950520f_{n+3} - 601382950f_{n+4} - 155959480f_{n+5} + 3707060f_{n+6} \\
&- 573256f_{n+7} + 39695f_{n+8}).
\end{aligned} \tag{5.5.1.13}
$$

$$
\begin{aligned}
&159667200hy'_{n+3} - 53222400y_{n+6} + 239500800y_{n+5} - 479001600y_{n+4} \\
&+ 292723200y_{n+3} = h^4(-3305f_n + 25960f_{n+1} - 33740f_{n+2} - 1641848f_{n+3} \\
&- 28275110f_{n+4} - 10428680f_{n+5} + 527380f_{n+6} - 95720f_{n+7} + 8263f_{n+8}).
\end{aligned} \tag{5.5.1.14}
$$

$$
\begin{aligned}
&479001600hy'_{n+4} + 79833600y_{n+6} - 479001600y_{n+5} + 239500800y_{n+4} \\
&+ 159667200y_{n+3} = h^4(15769f_n - 160856f_{n+1} + 770524f_{n+2} \\
&- 2216936f_{n+3} + 25856710f_{n+4} + 16614616f_{n+5} - 1166084f_{n+6} \\
&+ 223336f_{n+7} - 20279f_{n+8}).
\end{aligned} \tag{5.5.1.15}
$$

$$
\begin{aligned}
&479001600hy'_{n+5} - 159667200y_{n+6} - 239500800y_{n+5} + 479001600y_{n+4} \\
&- 79833600y_{n+3} = h^4(-15769f_n + 162200f_{n+1} - 791020f_{n+2} \\
&+ 2490680f_{n+3} - 18601510f_{n+4} - 23869816f_{n+5} + 892340f_{n+6} \\
&- 202840f_{n+7} + 18935f_{n+8}).
\end{aligned} \tag{5.5.1.16}
$$

$$
\begin{aligned}
&159667200hy'_{n+6} - 292723200y_{n+6} + 479001600y_{n+5} - 239500800y_{n+4} \\
&+ 53222400y_{n+3} = h^4(3305f_n - 38008f_{n+1} + 214700f_{n+2} - 805000f_{n+3} \\
&+ 10845110f_{n+4} + 27858680f_{n+5} + 1919468f_{n+6} - 85240f_{n+7} \\
&+ 3785f_{n+8}).
\end{aligned} \tag{5.5.1.17}
$$

$$
\begin{aligned}
&479001600hy'_{n+7} - 2075673600y_{n+6} + 4550515200y_{n+5} - 3353011200y_{n+4} \\
&+ 878169600y_{n+3} = h^4(-38255f_n + 304600f_{n+1} - 803924f_{n+2} - 493640f_{n+3} \\
&+ 151139350f_{n+4} + 606203080f_{n+5} + 238737100f_{n+6} + 2784424f_{n+7} \\
&+ 87265f_{n+8}).
\end{aligned} \tag{5.5.1.18}
$$

$$
\begin{aligned}
&479001600hy'_{n+8} - 3752179200y_{n+6} + 9101030400y_{n+5} - 7424524800y_{n+4} \\
&+ 2075673600y_{n+3} = h^4(-29995f_n + 114440f_{n+1} + 849260f_{n+2} \\
&- 8568904f_{n+3} + 370066670f_{n+4} + 1499467640f_{n+5} + 978023180f_{n+6} \\
&+ 229532360f_{n+7} + 4138949f_{n+8}).
\end{aligned} \tag{5.5.1.19}
$$

The second derivative of (5.5.1.4) gives

$$\alpha_3''(z) = -2 - z$$
$$\alpha_4''(z) = 7 + 3z$$
$$\alpha_5''(z) = -8 - 3z$$
$$\alpha_6''(z) = 3 + z$$

$$\begin{aligned}
\beta_0''(z) &= \frac{1}{479001600}(132823 + 701250z - 1425600z^3 - 1033560z^4 \\
&\quad +83160z^5 + 352044z^6 + 158400z^7 + 32670z^8 + 3300z^9 \\
&\quad +132z^{10}) \\
\beta_1''(z) &= \frac{1}{59875200}(-160741 - 825033z + 1663200z^3 + 1186020z^4 \\
&\quad -112266z^5 - 407484z^6 - 178200z^7 - 35640z^8 - 3465z^9 \\
&\quad -132z^{10}). \\
\beta_2''(z) &= \frac{1}{119750400}(1458421 + 7028802z - 13970880z^3 - 9729720z^4 \\
&\quad +1114344z^5 + 3379068z^6 + 1421640z^7 + 273240z^8 \\
&\quad +25410z^9 + 924z^{10}). \\
\beta_3''(z) &= \frac{1}{59875200}(-2106673 - 9001113z + 17463600z^3 + 11725560z^4 \\
&\quad -1692306z^5 - 4127508z^6 - 1643400z^7 - 299970z^8 \\
&\quad -26565z^9 - 924z^{10}). \\
\beta_4''(z) &= \frac{1}{47900160}(19023001 + 20240022z - 23284800z^3 - 14663880z^4 \\
&\quad +2844072z^5 + 5247396z^6 + 1924560z^7 + 329670z^8 + \\
&\quad 27720z^9 + 924z^{10}). \qquad (5.5.1.20) \\
\beta_5''(z) &= \frac{1}{59875200}(85710089 + 31305285z + 34927200z^3 + 19085220z^4 \\
&\quad -5592510z^5 - 6954948z^6 - 2277000z^7 - 362340z^8 - \\
&\quad 28875z^9 - 924z^{10}). \\
\beta_6''(z) &= \frac{1}{119750400}(124268317 + 156767226z - 69854400z^3 - 20706840z^4 \\
&\quad +12157992z^5 + 9532908z^6 + 2712600z^7 + 397980z^8 + 30030z^9 \\
&\quad +924z^{10}). \\
\beta_7''(z) &= \frac{1}{59875200}(4648877 + 21392349z + 29937600z^2 + 15895440z^3 + \\
&\quad 63360z^4 - 3779622z^5 - 1904364z^6 - 463320z^7 - 62370z^8 \\
&\quad -4455z^9 - 132z^{10}).
\end{aligned}$$

$$\beta_8''(z) = \frac{1}{479001600}(-914201 - 3753486z + 9979200z^3 + 12937320z^4$$
$$+7800408z^5 + 2680524z^6 + 554400z^7 + 68310\, z^8$$
$$+4620z^9 + 132z^{10}).$$

Evaluating (5.5.1.20) at all the grid points. That is, at z=-7, -6, -5, -4, -3, -2,-1,0 and 1 produces

$$\begin{aligned}&479001600h^2 y_n'' + 1916006400 y_{n+6} - 6227020800\, y_{n+5} + 6706022400 y_{n+4}\\ &= h^4(29248987 f_n + 527286424 f_{n+1} + 851733508 f_{n+2} + 1592033896 f_{n+3}\\ &- \ 2395008000 y_{n+3} + 1358008930 f_{n+4} + 427869928 f_{n+5}\\ &- 45488444 f_{n+6} + 10509592 f_{n+7} - 1103621 f_{n+8}).\end{aligned} \tag{5.5.1.21}$$

$$\begin{aligned}&479001600h^2 y_{n+1}'' + 1437004800 y_{n+6} - 4790016000 y_{n+5} + 5269017600 y_{n+4}\\ &= h^4(-995381 f_n + 38315656 f_{n+1} + 489953188 f_{n+2} + 944838952 f_{n+3}\\ &- 1916006400 y_{n+3} + 1121071570 f_{n+4} + 242304856 f_{n+5}\\ &- 1286396 f_{n+6} - 161288 f_{n+7} + 51643 f_{n+8}).\end{aligned} \tag{5.5.1.22}$$

$$\begin{aligned}&479001600h^2 y_{n+2}'' + 958003200 y_{n+6} - 3353011200\, y_{n+5} + 3832012800\, y_{n+4}\\ &= h^4(132823 f_n + 2109608 f_{n+1} + 41972644 f_{n+2} + 485916136 f_{n+3}\\ &- \ 1437004800 y_{n+3} + 702416410 f_{n+4} + 173494312 f_{n+5} - 5696252 f_{n+6}\\ &+ 1052056 f_{n+7} - 90521 f_{n+8}).\end{aligned} \tag{5.5.1.23}$$

$$\begin{aligned}&479001600h^2 y_{n+3}'' + 479001600\, y_{n+6} - 1916006400 y_{n+5} + 2395008000 y_{n+4}\\ &= h^4(-36401 f_n + 514552 f_{n+1} - 4169804 f_{n+2} + 49777048 f_{n+3}\\ &+ 308557450 f_{n+4} + 86441896 f_{n+5} - 2335532 f_{n+6} + 360904 f_{n+7}\\ &- 25313 f_{n+8}).\end{aligned} \tag{5.5.1.24}$$

$$\begin{aligned}&479001600h^2 y_{n+4}'' - 479001600 y_{n+5} + 958003200 y_{n+4} - 479001600\, y_{n+3}\\ &= h^4(-1747 f_n + 25064 f_{n+1} - 202564 f_{n+2} + 1943192 f_{n+3} + 36388910 f_{n+4}\\ &+ 1943192 f_{n+5} - 202564 f_{n+6} + 25064 f_{n+7} - 1747 f_{n+8}).\end{aligned} \tag{5.5.1.25}$$

$$\begin{aligned}&479001600h^2 y_{n+5}'' - 479001600 y_{n+6} + 958003200 y_{n+5} - 479001600\, y_{n+4}\\ &= h^4(1747 f_n - 13976 f_{n+1} + 37828 f_{n+2} + 55816 f_{n+3} - 1723070 f_{n+4}\\ &- 36609032 f_{n+5} - 1796444 f_{n+6} + 139672 f_{n+7} - 9341 f_{n+8}).\end{aligned} \tag{5.5.1.26}$$

$$\begin{aligned}&479001600h^2 y_{n+6}'' - 958003200\, y_{n+6} + 2395008000 y_{n+5} - 1916006400 y_{n+4}\\ &= h^4(-36401 f_n + 302296 f_{n+1} - 949532 f_{n+2} + 722152 f_{n+3}\\ &+ 479001600 y_{n+3} + 81855370 f_{n+4} + 313143976 f_{n+5}\\ &+ 46719364 f_{n+6} - 2859368 f_{n+7} + 186943 f_{n+8}).\end{aligned} \tag{5.5.1.27}$$

$$\begin{aligned}&479001600h^2 y_{n+7}'' - 1437004800 y_{n+6} + 3832012800 y_{n+5} - 3353011200 y_{n+4}\\ &+ 958003200 y_{n+3} = h^4(132823 f_n - 1285928 f_{n+1} + 5833684 f_{n+2} - 16853384 f_{n+3}\\ &+ 190230010 f_{n+4} + 685680712 f_{n+5} + 497073268 f_{n+6} + 37191016 f_{n+7}\\ &- 914201 f_{n+8}).\end{aligned} \tag{5.5.1.28}$$

$$479001600h^2 y''_{n+8} - 1916006400 y_{n+6} + 5269017600 y_{n+5} - 4790016000 y_{n+4}$$
$$+1437004800 y_{n+3} = h^4(-995381 f_n + 9010072 f_{n+1} - 35995004 f_{n+2} \qquad (5.5.1.29)$$
$$+82325608 f_{n+3} + 116886850 f_{n+4} + 1246489576 f_{n+5} + 861226948 f_{n+6}$$
$$+525786904 f_{n+7} + 29357227 f_{n+8}).$$

The third derivative of (5.5.1.4) gives

$$\alpha'''_3(z) = -1$$
$$\alpha'''_4(z) = 3$$
$$\alpha'''_5(z) = -3$$
$$\alpha'''_6(z) = 1$$
$$\beta'''_0(z) = \frac{1}{7257600}(10625 - 64800 z^2 - 62640z^3 + 6300z^4 + 32004z^5 + 16800z^6 + 3960z^7 + 450z^8 + 20z^9).$$
$$\beta'''_1(z) = \frac{1}{1814400}(-25001 + 151200 z^2 + 143760z^3 - 17010z^4 - 74088z^5 - 37800z^6 - 8640z^7 - 945z^8 - 40z^9).$$
$$\beta'''_2(z) = \frac{1}{1814400}(106497 - 635040 z^2 - 589680z^3 + 84420z^4 + 307188z^5 + 150780z^6 + 33120z^7 + 3465z^8 + 140z^9).$$
$$\beta'''_3(z) = \frac{1}{1814400}(-272761 + 1587600 z^2 + 1421280z^3 - 256410z^4 - 750456z^5 - 348600z^6 - 72720z^7 - 7245z^8 - 280z^9).$$
$$\beta'''_4(z) = \frac{1}{725760}(306667 - 1058400 z^2 - 888720z^3 + 215460 z^4 + 477036z^5 + 204120z^6 + 39960z^7 + 3780z^8 + 140z^9).$$
$$\beta'''_5(z) = \frac{1}{1814400}(948645 + 3175200 z^2 + 2313360z^3 - 847350z^4 - 1264536z^5 - 483000z^6 - 87840z^7 - 7875z^8 - 280z^9).$$
$$\beta'''_6(z) = \frac{1}{1814400}(2375261 - 3175200 z^2 - 1254960z^3 + 921060 z^4 + 866628z^5 + 287700z^6 + 48240z^7 + 4095z^8 + 140z^9).$$
$$\beta'''_7(z) = \frac{1}{1814400}(648253 + 1814400z + 1445040z^2 + 7680 z^3 - 572670z^4 - 346248z^5 - 98280z^6 - 15120z^7 - 1215z^8 - 40z^9). \qquad (5.5.1.30)$$
$$\beta'''_8(z) = \frac{1}{7257600}(-56871 + 453600z^2 + 784080z^3 + 590940z^4 + 243684z^5 + 58800z^6 + 8280z^7 + 630z^8 + 20z^9).$$

Equation (5.5.1.30) is evaluated at all the grid points. i.e, at z = -7, -6, -5, -4, -3, -2, -1, 0 and 1 gives

$$1814400h^3 y'''_n - 7257600\, y_{n+6} + 21772800 y_{n+5} - 21772800 y_{n+4} + 7257600\, y_{n+3}$$
$$= h^4(-2082753 f_n - 11532880 f_{n+1} - 255896 f_{n+2} - 16294176 f_{n+3} + 2298350 f_{n+4} \quad (5.5.1.31)$$
$$-6510512 f_{n+5} + 2192400 f_{n+6} - 531424 f_{n+7} + 57691 f_{n+8}).$$

$$1814400h^3 y'''_{n+1} - 7257600\, y_{n+6} + 21772800 y_{n+5} - 21772800 y_{n+4} + 7257600\, y_{n+3}$$
$$= h^4(57281 f_n - 2598692 f_{n+1} - 9465084 f_{n+2} - 5103460 f_{n+3} - 7767890 f_{n+4} \quad (5.5.1.32)$$
$$-217836 f_{n+5} - 390028 f_{n+6} + 94324 f_{n+7} - 10215 f_{n+8}).$$

$$1814400h^3 y'''_{n+2} - 7257600\, y_{n+6} + 21772800 y_{n+5} - 21772800 y_{n+4} + 7257600\, y_{n+3}$$
$$= h^4(-10625 f_n + 152496 f_{n+1} - 2975512 f_{n+2} - 8608544 f_{n+3} - 5133330 f_{n+4} \quad (5.5.1.33)$$
$$-1727920 f_{n+5} + 198544 f_{n+6} - 43488 f_{n+7} + 4379 f_{n+8}).$$

$$1814400h^3 y'''_{n+3} - 7257600\, y_{n+6} + 21772800 y_{n+5} - 21772800 y_{n+4} + 7257600\, y_{n+3}$$
$$= h^4(3969 f_n - 46756 f_{n+1} + 301060 f_{n+2} - 3344868 f_{n+3} - 6799570 f_{n+4} \quad (5.1.5.34)$$
$$-932204 f_{n+5} - 85644 f_{n+6} + 19700 f_{n+7} - 2087 f_{n+8}).$$

$$1814400h^3 y'''_{n+4} - 7257600\, y_{n+6} + 21772800 y_{n+5} - 21772800 y_{n+4} + 7257600\, y_{n+3}$$
$$= h^4(-2497 f_n + 26032 f_{n+1} - 130968 f_{n+2} + 474848 f_{n+3} - 2350610 f_{n+4} \quad (5.5.1.35)$$
$$-1783728 f_{n+5} + 166928 f_{n+6} - 31712 f_{n+7} + 2907 f_{n+8}).$$

$$1814400h^3 y'''_{n+5} - 7257600\, y_{n+6} + 21772800 y_{n+5} - 21772800 y_{n+4} + 7257600\, y_{n+3}$$
$$= h^4(2497 f_n - 25380 f_{n+1} + 121604 f_{n+2} - 376676 f_{n+3} + 2098350 f_{n+4} \quad (5.5.1.36)$$
$$+2035988 f_{n+5} - 265100 f_{n+6} + 41076 f_{n+7} - 3559 f_{n+8}).$$

$$1814400h^3 y'''_{n+6} - 7257600\, y_{n+6} + 21772800 y_{n+5} - 21772800 y_{n+4} + 7257600\, y_{n+3}$$
$$= h^4(-3969 f_n + 37808 f_{n+1} - 162584 f_{n+2} + 419040 f_{n+3} + 432110 f_{n+4} \quad (5.5.1.37)$$
$$+7299664 f_{n+5} + 3011472 f_{n+6} - 158176 f_{n+7} + 11035 f_{n+8}).$$

$$1814400h^3 y'''_{n+7} - 7257600\, y_{n+6} + 21772800 y_{n+5} - 21772800 y_{n+4} + 7257600\, y_{n+3}$$
$$= h^4(10625 f_n - 100004 f_{n+1} + 425988 f_{n+2} - 1091044 f_{n+3} + 3066670 f_{n+4} \quad (5.5.1.38)$$
$$+3794580 f_{n+5} + 9501044 f_{n+6} + 2593012 f_{n+7} - 56871 f_{n+8}).$$

$$1814400h^3 y'''_{n+8} - 7257600\, y_{n+6} + 21772800 y_{n+5} - 21772800 y_{n+4} + 7257600\, y_{n+3}$$
$$= h^4(-57281 f_n + 525744 f_{n+1} - 2156440 f_{n+2} + 5201632 f_{n+3} - 6999570 f_{n+4} \quad (5.5.1.39)$$
$$+14985296 f_{n+5} + 291856 f_{n+6} + 11527200 f_{n+7} + 2083163 f_{n+8}).$$

Joining equations (5.5.1.5) - (5.5.1.9), (5.5.1.11), (5.5.1.21) and (5.5.1.31) to give a block of the form (1.10) as follows

$$
\begin{pmatrix}
0 & 0 & -4838400 & 10886400 & -8709120 & 2419200 & 0 & 0 \\
725760 & 0 & -7257600 & 14515200 & -10886400 & 2903040 & 0 & 0 \\
0 & 725760 & -2903040 & 4354560 & -2903040 & 725760 & 0 & 0 \\
0 & 0 & 725760 & -2903040 & 4354560 & -2903040 & 725760 & 0 \\
0 & 0 & 2903040 & -10886400 & 14515200 & -7257600 & 0 & 725760 \\
0 & 0 & 656409600 & -1676505600 & 1437004800 & -416908800 & 0 & 0 \\
0 & 0 & -2395008000 & 6706022400 & -6227020800 & 1916006400 & 0 & 0 \\
0 & 0 & 7257600 & -21772800 & 21772800 & -7257600 & 0 & 0
\end{pmatrix}
\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ y_{n+8} \end{pmatrix}
$$

$$
=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -241920 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} y_{n-7} \\ y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix}
+h\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -53222400 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} y'_{n-7} \\ y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_{n} \end{pmatrix}
+
$$

$$
h^{2}\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -479001600 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} y''_{n-7} \\ y''_{n-6} \\ y''_{n-5} \\ y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_{n} \end{pmatrix}
+h^{3}\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1814400
\end{pmatrix}
\begin{pmatrix} y'''_{n-7} \\ y'''_{n-6} \\ y'''_{n-5} \\ y'''_{n-4} \\ y'''_{n-3} \\ y'''_{n-2} \\ y'''_{n-1} \\ y'''_{n} \end{pmatrix}
$$

$$+h^4\begin{pmatrix} 39160 & 321940 & 1103560 & 1743010 & 430216 & -10700 & 1720 & -125 \\ 152 & 113060 & 998840 & 2006210 & 523400 & -15292 & 2600 & -205 \\ 568 & -3596 & 130888 & 470122 & 130888 & -3596 & 568 & -41 \\ 328 & -908 & -152 & 125722 & 475288 & 127444 & -2120 & 199 \\ -520 & 3860 & -18232 & 527810 & 2001800 & 1001780 & 111800 & 467 \\ -25829880 & 106603540 & -241795400 & -311177490 & -74235784 & 1971340 & -339000 & 26885 \\ 527286424 & 851733508 & 1592033896 & 1358008930 & 427869928 & -45488444 & 10509592 & -1103621 \\ -11532880 & -255896 & -16294176 & 2298350 & -6510512 & 2192400 & -531424 & 57691 \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \\ f_{n+8} \end{pmatrix}$$

$$+h^4\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 19 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -41 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -41 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -35 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -436331 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 29248987 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2082753 \end{pmatrix}\begin{pmatrix} f_{n-7} \\ f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}$$

The above equation is multiplied by the inverse of A^0

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ y_{n+8} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n-7} \\ y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix} +$$

$$h\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{pmatrix}\begin{pmatrix} y'_{n-7} \\ y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_{n} \end{pmatrix} + h^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{49}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 \end{pmatrix}\begin{pmatrix} y''_{n-7} \\ y''_{n-6} \\ y''_{n-5} \\ y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_{n} \end{pmatrix}$$

$$
+h^{3}\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{125}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{343}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{256}{3}
\end{pmatrix}
\begin{pmatrix}
y'''_{n-7} \\ y'''_{n-6} \\ y'''_{n-5} \\ y'''_{n-4} \\ y'''_{n-3} \\ y'''_{n-2} \\ y'''_{n-1} \\ y'''_{n}
\end{pmatrix} +
$$

$$
h^{4}\begin{pmatrix}
\frac{269}{7060} & \frac{-875}{15851} & \frac{416}{5937} & \frac{-123}{1916} & \frac{4358}{107497} & \frac{-32}{1911} & \frac{101}{24781} & \frac{-13}{29296} \\
\frac{944}{1299} & \frac{-417}{497} & \frac{1831}{1742} & \frac{-1915}{2003} & \frac{1767}{2939} & \frac{-712}{2875} & \frac{82}{1363} & \frac{-44}{6727} \\
\frac{1407}{403} & \frac{-1133}{358} & \frac{29698}{7047} & \frac{-8661}{2251} & \frac{2021}{834} & \frac{-926}{927} & \frac{596}{2455} & \frac{-73}{2765} \\
\frac{14458}{1457} & \frac{-6831}{953} & \frac{1799}{164} & \frac{-800}{81} & \frac{1531}{246} & \frac{-5795}{2258} & \frac{632}{1013} & \frac{-53}{781} \\
\frac{5381}{249} & \frac{-1268}{99} & \frac{1777}{76} & \frac{-5502}{275} & \frac{3167}{249} & \frac{-488}{93} & \frac{1327}{1040} & \frac{-103}{742} \\
\frac{10999}{274} & \frac{-7599}{379} & \frac{27703}{634} & \frac{-3974}{115} & \frac{3649}{160} & \frac{-2420}{259} & \frac{793}{349} & \frac{-332}{1343} \\
\frac{8053}{120} & \frac{-1443}{50} & \frac{10901}{147} & \frac{-18086}{337} & \frac{5059}{133} & \frac{-3311}{221} & \frac{20455}{5554} & \frac{-335}{836} \\
\frac{22901}{220} & \frac{-9524}{243} & \frac{7721}{166} & \frac{-33226}{429} & \frac{18572}{309} & \frac{-8104}{375} & \frac{5905}{1028} & \frac{-1122}{1847}
\end{pmatrix}
\begin{pmatrix}
f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \\ f_{n+8}
\end{pmatrix} +
$$

$$
h^{4}\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{97}{3809} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1145}{4137} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{481}{462} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2957}{1135} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4389}{835} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2014}{217} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{449}{30} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{34646}{1533}
\end{pmatrix}
\begin{pmatrix}
f_{n-7} \\ f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n}
\end{pmatrix} \qquad (5.5.1.40)
$$

which gives

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2 y''_n + \frac{1}{6}h^3 y'''_n + \frac{h^4}{958003200}(-425111 f_{n+8} + 3904536 f_{n+7} - 16041916 f_{n+6} + 38838088 f_{n+5} - 61500210 f_{n+4} + 67126376 f_{n+3} - 52883276 f_{n+2} + 36501816 f_{n+1} + 24396497 f_n). \quad (5.5.1.41)$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2 y''_n + \frac{4}{3}h^3 y'''_n + \frac{h^4}{3742200}(-24477 f_{n+8} + 225136 f_{n+7} - 926764 f_{n+6} + 2249904 f_{n+5} - 3577790 f_{n+4} + 3933392 f_{n+3} - 3139836 f_{n+2} + 2719504 f_{n+1} + 1035731 f_n). \quad (5.5.1.42)$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2}h^2 y''_n + \frac{9}{2}h^3 y'''_n + \frac{h^4}{3942400}(-104085 f_{n+8} + 957096 f_{n+7} - 3938148 f_{n+6} + 9553464 f_{n+5} - 15168870 f_{n+4} + 16614360 f_{n+3} - 12476916 f_{n+2} + 13764168 f_{n+1} + 4104531 f_n). \quad (5.5.1.43)$$

$$y_{n+4} = y_n + 4hy'_n + 8h^2 y''_n + \frac{32}{3}h^3 y'''_n + \frac{h^4}{467775}(-7936 f_{n+8} + 72960 f_{n+7} - 300128 f_{n+6} + 727808 f_{n+5} - 1155000 f_{n+4} + 1282816 f_{n+3} - 838240 f_{n+2} + 1160448 f_{n+1} + 304672 f_n). \quad (5.5.1.44)$$

$$y_{n+5} = y_n + 5hy'_n + \frac{25}{2}h^2 y''_n + \frac{125}{6}h^3 y'''_n + \frac{h^4}{38320128}(-5319375 f_{n+8} + 48895000 f_{n+7} - 201077500 f_{n+6} + 487389000 f_{n+5} - 766681250 f_{n+4} + 895985000 f_{n+3} - 490807500 f_{n+2} + 828115000 f_{n+1} + 201421625 f_n). \quad (5.5.1.45)$$

$$y_{n+6} = y_n + 6hy'_n + 18h^2 y''_n + 36h^3 y'''_n + \frac{6h^4}{30800}(-1269 f_{n+8} + 11664 f_{n+7} - 47964 f_{n+6} + 117072 f_{n+5} - 177390 f_{n+4} + 224304 f_{n+3} - 102924 f_{n+2} + 206064 f_{n+1} + 47643 f_n). \quad (5.5.1.46)$$

$$y_{n+7} = y_n + 7hy'_n + \frac{49}{2}h^2 y''_n + \frac{343}{6}h^3 y'''_n + \frac{h^4}{136857600}(-54841241 f_{n+8} + 504037128 f_{n+7} - 2050386772 f_{n+6} + 5205732952 f_{n+5} - 7344827070 f_{n+4} + 10148873336 f_{n+3} - 3949712228 f_{n+2} + 9184285992 f_{n+1} + 2048300303 f_n). \quad (5.5.1.47)$$

$$y_{n+8} = y_n + 8hy'_n + 32h^2 y''_n + \frac{256}{3}h^3 y'''_n + \frac{h^4}{467775}(-284160 f_{n+8} + 2686976 f_{n+7} - 10108928 f_{n+6} + 28114944 f_{n+5} - 36229120 f_{n+4} + 54722560 f_{n+3} - 18333696 f_{n+2} + 48693248 f_{n+1} + 10571776 f_n). \quad (5.5.1.48)$$

Substituting (5.5.1.43) - (5.5.1.46) into (5.5.1.12) – (5.5.1.19) to give the first derivative of the block

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2 y'''_n + \frac{h^3}{39916800}(-73886 f_{n+8} + 679110 f_{n+7} - 2792861 f_{n+6} + 6771082 f_{n+5} - 10745445 f_{n+4} + 11774146 f_{n+3} - 9359135 f_{n+2} + 6779886 f_{n+1} + 3619903 f_n). \tag{5.5.1.49}$$

$$y'_{n+2} = y'_n + 2hy''_n + 2h^2 y'''_n + \frac{h^3}{623700}(-7305 f_{n+8} + 67196 f_{n+7} - 276634 f_{n+6} + 671628 f_{n+5} - 1067950 f_{n+4} + 1173140 f_{n+3} - 926646 f_{n+2} + 911204 f_{n+1} + 286967 f_n). \tag{5.5.1.50}$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{9}{2}h^2 y'''_n + \frac{3h^3}{492800}(-4815 f_{n+8} + 44262 f_{n+7} - 182043 f_{n+6} + 441306 f_{n+5} - 699885 f_{n+4} + 766290 f_{n+3} - 521217 f_{n+2} + 711918 f_{n+1} + 183384 f_n). \tag{5.5.1.51}$$

$$y'_{n+4} = y'_n + 4hy''_n + 8h^2 y'''_n + \frac{h^3}{155925}(-8564 f_{n+8} + 78720 f_{n+7} - 323744 f_{n+6} + 784768 f_{n+5} - 1243200 f_{n+4} + 1435264 f_{n+3} - 752480 f_{n+2} + 1371264 f_{n+1} + 321172 f_n). \tag{5.5.1.52}$$

$$y'_{n+5} = y'_n + 5hy''_n + \frac{25}{2}h^2 y'''_n + \frac{h^3}{1596672}(-141000 f_{n+8} + 1295750 f_{n+7} - 5327125 f_{n+6} + 12920250 f_{n+5} - 19680625 f_{n+4} + 25537250 f_{n+3} - 10296375 f_{n+2} + 23702750 f_{n+1} + 5253125 f_n). \tag{5.5.1.53}$$

$$y'_{n+6} = y'_n + 6hy''_n + 18h^2 y'''_n + \frac{h^3}{15400}(-1998 f_{n+8} + 18360 f_{n+7} - 75348 f_{n+6} + 190296 f_{n+5} - 258660 f_{n+4} + 385128 f_{n+3} - 123660 f_{n+2} + 346248 f_{n+1} + 74034 f_n). \tag{5.5.1.54}$$

$$y'_{n+7} = y'_n + 7hy''_n + \frac{49}{2}h^2 y'''_n + \frac{h^3}{5702400}(-1020425 f_{n+8} + 9423582 f_{n+7} - 35782103 f_{n+6} + 104842066 f_{n+5} - 122270925 f_{n+4} + 206894170 f_{n+3} - 54639557 f_{n+2} + 180838518 f_{n+1} + 37701874 f_n). \tag{5.5.1.55}$$

$$y'_{n+8} = y'_n + 8hy''_n + 32h^2 y''' + \frac{8h^3}{155925}(-4440 f_{n+8} + 51712 f_{n+7} - 134528 f_{n+6} + 508416 f_{n+5} - 510560 f_{n+4} + 970240 f_{n+3} - 216192 f_{n+2} + 828928 f_{n+1} + 169624 f_n). \tag{5.5.1.56}$$

Substituting (5.5.1.43) - (5.5.1.46) into (5.5.1.22) – (5.5.1.29) to give the second derivative of the block

$$y''_{n+1} = y''_n + hy'''_n + \frac{h^2}{7257600}(-40187 f_{n+8} + 369744 f_{n+7} - 1522673 f_{n+6} + 3698922 f_{n+5} - 64888311 f_{n+4} + 6488191 f_{n+3} - 5225623 f_{n+2} + 4124231 f_{n+1}. + 1624505 f_n). \quad (5.5.1.57)$$

$$y''_{n+2} = y''_n + 2hy'''_n + \frac{h^2}{113400}(-1563 f_{n+8} + 14368 f_{n+7} - 59092 f_{n+6} + 143232 f_{n+5} - 227030 f_{n+4} + 247328 f_{n+3} - 183708 f_{n+2} + 235072 f_{n+1}. + 58193 f_n). \quad (5.5.1.58)$$

$$y''_{n+3} = y''_n + 3hy'''_n + \frac{h^2}{89600}(-1935 f_{n+8} + 17784 f_{n+7} - 73128 f_{n+6} + 177264 f_{n+5} - 281430 f_{n+4} + 315000 f_{n+3} - 150624 f_{n+2} + 328608 f_{n+1} + 71661 f_n). \quad (5.5.1.59)$$

$$y''_{n+4} = y''_n + 4hy'''_n + \frac{h^2}{28350}(-836 f_{n+8} + 7680 f_{n+7} - 31552 f_{n+6} + 76288 f_{n+5} - 118440 f_{n+4} + 160256 f_{n+3} - 46400 f_{n+2} + 148992 f_{n+1}. + 30812 f_n). \quad (5.5.1.60)$$

$$y''_{n+5} = y''_n + 5hy'''_n + \frac{h^2}{290304}(-10875 f_{n+8} + 100000 f_{n+7} - 412000 f_{n+6} + 1020600 f_{n+5} - 1283750 f_{n+4} + 2294000 f_{n+3} - 465000 f_{n+2} + 1987000 f_{n+1}. + 398825 f_n). \quad (5.5.1.61)$$

$$y''_{n+6} = y''_n + 6hy'''_n + \frac{h^2}{1400}(-63 f_{n+8} + 576 f_{n+7} - 2268 f_{n+6} + 7200 f_{n+5} - 6390 f_{n+4} + 14208 f_{n+3} - 2196 f_{n+2} + 11808 f_{n+1} + 2325 f_n). \quad (5.5.1.62)$$

$$y''_{n+7} = y''_n + 7hy'''_n + \frac{7h^2}{1036800}(-8183 f_{n+8} + 84168 f_{n+7} - 145432 f_{n+6} + 1009792 f_{n+5} - 689430 f_{n+4} + 1830248 f_{n+3} - 225008 f_{n+2} + 1484112 f_{n+1} + 288533 f_n). \quad (5.5.1.63)$$

$$y''_{n+8} = y''_n + 8hy'''_n + \frac{h^2}{28350}(47104 f_{n+7} - 14848 f_{n+6} + 251904 f_{n+5} - 145280 f_{n+4} + 419840 f_{n+3} - 44544 f_{n+2} + 329728 f_{n+1} + 63296 f_n). \quad (5.5.1.64)$$

Substituting (5.5.1.43) - (5.5.1.46) into (5.5.1.32) – (5.5.1.39) to give the third derivative of the block

$$y'''_{n+1} = y'''_n + \frac{h}{1069200}(-10004 f_{n+8} + 92186 f_{n+7} - 380447 f_{n+6} + 927046 f_{n+5} - 1482974 f_{n+4} + 1648632 f_{n+3} - 1356711 f_{n+2} + 1316197 f_{n+1} + 315273 f_n). \quad (5.5.1.65)$$

$$y'''_{n+2} = y'''_n + \frac{h}{113400}(-833 f_{n+8} + 7624 f_{n+7} - 31154 f_{n+6} + 74728 f_{n+5} - 116120 f_{n+4} + 120088 f_{n+3} - 42494 f_{n+2} + 182584 f_{n+1} + 32377 f_n). \tag{5.5.1.66}$$

$$y'''_{n+3} = y'''_n + \frac{h}{44800}(-369 f_{n+8} + 3402 f_{n+7} - 14062 f_{n+6} + 34434 f_{n+5} - 56160 f_{n+4} + 79934 f_{n+3} + 3438 f_{n+2} + 70902 f_{n+1} + 12881 f_n). \tag{5.5.1.67}$$

$$y'''_{n+4} = y'''_n + \frac{h}{28350}(-214 f_{n+8} + 1952 f_{n+7} - 7912 f_{n+6} + 18464 f_{n+5} - 18160 f_{n+4} + 65504 f_{n+3} + 488 f_{n+2} + 45152 f_{n+1} + 8126 f_n). \tag{5.5.1.68}$$

$$y'''_{n+5} = y'''_n + \frac{h}{145152}(-1225 f_{n+8} + 11450 f_{n+7} - 49150 f_{n+6} + 170930 f_{n+5} - 4000 f_{n+4} + 318350 f_{n+3} + 7550 f_{n+2} + 230150 f_{n+1} + 41705 f_n). \tag{5.5.1.69}$$

$$y'''_{n+6} = y'''_n + \frac{h}{1400}(-9 f_{n+8} + 72 f_{n+7} + 158 f_{n+6} + 2664 f_{n+5} - 360 f_{n+4} + 3224 f_{n+3} + 18 f_{n+2} + 2232 f_{n+1} + 401 f_n). \tag{5.5.1.70}$$

$$y'''_{n+7} = y'''_n + \frac{h}{518400}(-8183 f_{n+8} + 223174 f_{n+7} + 522046 f_{n+6} + 736078 f_{n+5} + 54880 f_{n+4} + 1085937 f_{n+3} + 48706 f_{n+2} + 816634 f_{n+1} + 149527 f_n). \tag{5.5.1.71}$$

$$y'''_{n+8} = y'''_n + \frac{h}{28350}(7912 f_{n+8} + 47104 f_{n+7} - 7424 f_{n+6} + 83968 f_{n+5} - 36320 f_{n+4} + 83968 f_{n+3} - 7424 f_{n+2} + 47104 f_{n+1} + 7912 f_n). \tag{5.5.1.72}$$

5.5.2 Properties of Eight–Step Block Method for Fourth Order ODEs.

This section establishes the order, zero-stability and region of absolute stability of eight–step block method for fourth order ODEs.

5.5.2.1 Order of Eight–Step Block Method for Fourth Order ODEs.

In finding the order of the block method (5.5.1.41 – 5.5.1.48), the same strategy mentioned in section 3.2.2.1 is used as displayed below

$$
\begin{pmatrix}
\sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{h^m}{m!}y_n^{(m)}-\frac{3619903}{39916800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(39916800)(m!)}y_n^{(3+m)}\begin{pmatrix}6779886(1)^m-9359135(2)^m\\+11774146(3)^m-10745445(4)^m\\+6771082(5)^m-2792861(6)^m\\+679110(7)^m-73886(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(2h)^m}{m!}y_n^{(m)}-\frac{286967}{623700}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(623700)(m!)}y_n^{(3+m)}\begin{pmatrix}911204(1)^m-926646(2)^m\\+1173140(3)^m-1067950(4)^m+\\671628(5)^m-276634(6)^m\\+67196(7)^m-7305(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(3h)^m}{m!}y_n^{(m)}-\frac{183384}{492800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(492800)(m!)}y_n^{(3+m)}\begin{pmatrix}711918(1)^m-521217(2)^m+\\766290(3)^m-699885(4)^m\\+441306(5)^m-182043(6)^m\\+44262(7)^m-4815(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(4h)^m}{m!}y_n^{(m)}-\frac{321172}{155925}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(155925)(m!)}y_n^{(3+m)}\begin{pmatrix}137126(1)^m-752480(2)^m\\+1435264(3)^m-1243200(4)^m\\+784768(5)^m-323744(6)^m\\+78720(7)^m-8564(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(5h)^m}{m!}y_n^{(m)}-\frac{5253125}{1596672}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(1596672)(m!)}y_n^{(3+m)}\begin{pmatrix}23702750(1)^m-10296375(2)^m\\+25537250(3)^m-19680625(4)^m\\+12920250(5)^m-5327125(6)^m\\+1295750(7)^m-141000(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(6h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(6h)^m}{m!}y_n^{(m)}-\frac{74034}{15400}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(15400)(m!)}y_n^{(3+m)}\begin{pmatrix}346248(1)^m-123660(2)^m+\\385128(3)^m-258660(4)^m+\\190296(5)^m-75348(6)^m\\+18360(7)^m-1998(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(7h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(7h)^m}{m!}y_n^{(m)}-\frac{37701874}{5702400}h^2y_n''-\sum_{m=0}^{\infty}\frac{7h^{3+m}}{(5702400)(m!)}y_n^{(3+m)}\begin{pmatrix}180838518(1)^m-54639557(2)^m\\+206894170(3)^m-122270925(4)^m\\+104842066(5)^m-35782103(6)^m\\+9423582(7)^m-1020425(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(8h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(8h)^m}{m!}y_n^{(m)}-\frac{8(169624)}{155925}h^2y_n''-\sum_{m=0}^{\infty}\frac{8h^{3+m}}{(155925)(m!)}y_n^{(3+m)}\begin{pmatrix}828928(1)^m-216192(2)^m+\\970240(3)^m-510560(4)^m+\\508416(5)^m-134528(6)^m\\+51712(7)^m-4440(8)^m\end{pmatrix}
\end{pmatrix}
=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}
$$

Comparing the coefficients of h^m and y_n^m produces

$$
C_0=\begin{pmatrix}1-1\\1-1\\1-1\\1-1\\1-1\\1-1\\1-1\\1-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}
$$

$$C_1 = \begin{pmatrix} 1-1 \\ 2-2 \\ 3-3 \\ 4-4 \\ 5-5 \\ 6-6 \\ 7-7 \\ 8-8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \frac{1}{2!} - \frac{1}{2!} \\ \frac{(2)^2}{2!} - \frac{(2)^2}{2!} \\ \frac{(3)^2}{2!} - \frac{(3)^2}{2!} \\ \frac{(4)^2}{2!} - \frac{(4)^2}{2!} \\ \frac{(5)^2}{2!} - \frac{(5)^2}{2!} \\ \frac{(6)^2}{2!} - \frac{(6)^2}{2!} \\ \frac{(7)^2}{2!} - \frac{(7)^2}{2!} \\ \frac{(8)^2}{2!} - \frac{(8)^2}{2!} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
C_3 = \begin{pmatrix}
\frac{1}{3!} - \frac{3619903}{39916800} - \frac{1}{(39916800)(0!)} \begin{pmatrix} 6779886(1)^0 - 9359135(2)^0 + 11774146(3)^{0^0} \\ -10745445(4) + 6771082(5)^0 - 2792861(6)^0 \\ + 679110(7)^0 - 73886(8)^0 \end{pmatrix} \\
\frac{(2)^3}{3!} - \frac{286967}{623700} - \frac{1}{(623700)(0!)} \begin{pmatrix} 911204(1)^0 - 926646(2)^0 + 1173140(3)^0 \\ -1067950(4)^0 + 671628(5)^0 - 276634(6)^0 \\ + 67196(7)^0 - 7305(8)^0 \end{pmatrix} \\
\frac{(3)^3}{3!} - \frac{183384}{492800} - \frac{1}{(492800)(0!)} \begin{pmatrix} 711918(1)^0 - 521217(2)^0 + 766290(3)^0 \\ - 699885(4)^0 + 41306(5)^0 - 182043(6)^0 \\ + 44262(7)^0 - 4815(8)^0 \end{pmatrix} \\
\frac{(4)^3}{3!} - \frac{321172}{155925} - \frac{1}{(155925)(0!)} \begin{pmatrix} 137126(1)^0 - 752480(2)^0 + 1435264(3)^0 \\ -1243200(4)^0 + 784768(5)^0 - 323744(6)^0 \\ + 78720(7)^0 - 8564(8)^0 \end{pmatrix} \\
\frac{(5)^3}{3!} - \frac{5253125}{1596672} - \frac{1}{(1596672)(0!)} \begin{pmatrix} 23702750(1)^0 - 10296375(2)^0 + 25537250(3)^0 \\ -19680625(4)^0 + 12920250(5)^0 - 5327125(6)^0 \\ + 1295750(7)^0 - 141000(8)^0 \end{pmatrix} \\
\frac{(6)^3}{3!} - \frac{74034}{15400} - \frac{1}{(15400)(0!)} \begin{pmatrix} 346248(1)^0 - 123660(2)^0 + 385128(3)^0 - 258660(4)^0 \\ + 190296(5)^0 - 75348(6)^0 + 18360(7)^0 - 1998(8)^0 \end{pmatrix} \\
\frac{(7)^3}{3!} - \frac{37701874}{5702400} - \frac{1}{(5702400)(0!)} \begin{pmatrix} 180838518(1)^0 - 54639557(2)^0 + 206894170(3)^0 \\ -122270925(4)^0 + 104842066(5)^0 - 35782103(6)^0 \\ + 9423582(7)^0 - 1020425(8)^0 \end{pmatrix} \\
\frac{(8)^3}{3!} - \frac{8(169624)}{155925} - \frac{1}{(155925)(0!)} \begin{pmatrix} 828928(1)^0 - 216192(2)^0 + 970240(3) \;\; - 510560(4)^0 \\ + 508416(5)^0 - 134528(6)^0 + 51712(7)^0 - 4440(8)^0 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$C_4 = \begin{pmatrix} \dfrac{1}{4!} - \dfrac{1}{(39916800)(1!)}\begin{pmatrix} 6779886(1)^1 - 9359135(2)^1 + 11774146(3)^1 \\ -10745445(4)^1 + 6771082(5)^1 - 2792861(6)^1 \\ +679110(7)^1 - 73886(8)^1 \end{pmatrix} \\ \dfrac{(2)^4}{4!} - \dfrac{1}{(623700)(1!)}\begin{pmatrix} 911204(1)^1 - 926646(2)^1 + 1173140(3)^1 - \\ 1067950(4)^1 + 671628(5)^1 - 276634(6)^1 + \\ 67196(7)^1 - 7305(8)^1 \end{pmatrix} \\ \dfrac{(3)^4}{4!} - \dfrac{1}{(492800)(1!)}\begin{pmatrix} 711918(1)^1 - 521217(2)^1 + 766290(3)^1 \\ -699885(4)^1 + 441306(5)^1 - 182043(6)^1 \\ +44262(7)^1 - 4815(8)^1 \end{pmatrix} \\ \dfrac{(4)^4}{4!} - \dfrac{1}{(155925)(1!)}\begin{pmatrix} 137126(1)^1 - 752480(2)^1 + 1435264(3)^1 \\ -1243200(4)^1 + 784768(5)^1 - 323744(6)^1 \\ +78720(7)^1 - 8564(8)^1 \end{pmatrix} \\ \dfrac{(5)^4}{4!} - \dfrac{1}{(1596672)(1!)}\begin{pmatrix} 23702750(1)^1 - 10296375(2)^1 + 25537250(3)^1 \\ -19680625(4)^1 + 12920250(5)^1 - 5327125(6)^1 \\ +1295750(7)^1 - 141000(8)^1 \end{pmatrix} \\ \dfrac{(6)^4}{4!} - \dfrac{1}{(15400)(1!)}\begin{pmatrix} 346248(1)^1 - 123660(2)^1 + 385128(3)^1 - 258660(4)^1 \\ +190296(5)^1 - 75348(6)^1 + 18360(7)^1 - 1998(8)^1 \end{pmatrix} \\ \dfrac{(7)^4}{4!} - \dfrac{1}{(5702400)(1!)}\begin{pmatrix} 180838518(1)^1 - 54639557(2)^1 + 206894170(3)^1 \\ -122270925(4)^1 + 104842066(5)^1 - 35782103(6)^1 \\ +9423582(7)^1 - 1020425(8)^1 \end{pmatrix} \\ \dfrac{(8)^4}{4!} - \dfrac{1}{(155925)(1!)}\begin{pmatrix} 828928(1)^1 - 216192(2)^1 + 970240(3)^1 - 510560(4)^1 + \\ 508416(5)^1 - 134528(6)^1 + 51712(7)^1 - 4440(8)^1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
C_5 = \begin{pmatrix}
\dfrac{1}{5!} - \dfrac{1}{(39916800)(2!)} \left(\begin{array}{l} 6779886(1)^2 - 9359135(2)^2 + 11774146(3)^2 \\ - 10745445(4)^2 + 6771082(5)^2 - 2792861(6)^2 \\ + 679110(7)^2 - 73886(8)^2 \end{array} \right) \\
\dfrac{(2)^5}{5!} - \dfrac{1}{(623700)(2!)} \left(\begin{array}{l} 911204(1)^2 - 926646(2)^2 + 1173140(3)^2 - \\ 1067950(4)^2 + 671628(5)^2 - 276634(6)^2 \\ + 67196(7)^2 - 7305(8)^2 \end{array} \right) \\
\dfrac{(3)^5}{5!} - \dfrac{1}{(492800)(2!)} \left(\begin{array}{l} 711918(1)^2 - 521217(2)^2 + 766290(3)^2 \\ - 699885(4)^2 + 441306(5)^2 - 182043(6)^2 \\ + 44262(7)^2 - 4815(8)^2 \end{array} \right) \\
\dfrac{(4)^5}{5!} - \dfrac{1}{(155925)(2!)} \left(\begin{array}{l} 137126(1)^2 - 752480(2)^2 + 1435264(3)^2 \\ - 1243200(4)^2 + 784768(5)^2 - 323744(6)^2 \\ + 78720(7)^2 - 8564(8)^2 \end{array} \right) \\
\dfrac{(5)^5}{5!} - \dfrac{1}{(1596672)(2!)} \left(\begin{array}{l} 23702750(1)^2 - 10296375(2)^2 + 25537250(3)^2 \\ - 19680625(4)^2 + 12920250(5)^2 - 5327125(6)^2 \\ + 1295750(7)^2 - 141000(8)^2 \end{array} \right) \\
\dfrac{(6)^5}{5!} - \dfrac{1}{(15400)(2!)} \left(\begin{array}{l} 346248(1)^2 - 123660(2)^2 + 385128(3)^2 - 258660(4)^2 \\ + 190296(5)^2 - 75348(6)^2 + 18360(7)^2 - 1998(8)^2 \end{array} \right) \\
\dfrac{(7)^5}{5!} - \dfrac{1}{(5702400)(2!)} \left(\begin{array}{l} 180838518(1)^2 - 54639557(2)^2 + 206894170(3)^2 \\ - 122270925(4)^2 + 104842066(5)^2 - 35782103(6)^2 \\ + 9423582(7)^2 - 1020425(8)^2 \end{array} \right) \\
\dfrac{(8)^5}{5!} - \dfrac{1}{(155925)(2!)} \left(\begin{array}{l} 828928(1)^2 - 216192(2)^2 + 970240(3)^2 \\ - 510560(4)^2 + 508416(5)^2 - 134528(6)^2 \\ + 51712(7)^2 - 4440(8)^2 \end{array} \right)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_6 = \begin{pmatrix}
\dfrac{1}{6!} - \dfrac{1}{(39916800)(3!)} \begin{pmatrix} 6779886(1)^3 - 9359135(2)^3 + 11774146(3)^3 \\ -10745445(4)^3 + 6771082(5)^3 - 2792861(6)^3 \\ +679110(7)^3 - 73886(8)^3 \end{pmatrix} \\
\dfrac{(2)^6}{6!} - \dfrac{1}{(623700)(3!)} \begin{pmatrix} 911204(1)^3 - 926646(2)^3 + 1173140(3)^3 \\ -1067950(4)^3 + 671628(5)^3 - 276634(6)^3 \\ +67196(7)^3 - 7305(8)^3 \end{pmatrix} \\
\dfrac{(3)^6}{6!} - \dfrac{1}{(492800)(3!)} \begin{pmatrix} 711918(1)^3 - 521217(2)^3 + 766290(3)^3 \\ -699885(4)^3 + 441306(5)^3 - 182043(6)^3 \\ +44262(7)^3 - 4815(8)^3 \end{pmatrix} \\
\dfrac{(4)^6}{6!} - \dfrac{1}{(155925)(3!)} \begin{pmatrix} 137126(1)^3 - 752480(2)^3 + 1435264(3)^3 \\ -1243200(4)^3 + 784768(5)^3 - 323744(6)^3 \\ +78720(7)^3 - 8564(8)^3 \end{pmatrix} \\
\dfrac{(5)^6}{6!} - \dfrac{1}{(1596672)(3!)} \begin{pmatrix} 23702750(1)^3 - 10296375(2)^3 + 25537250(3)^3 \\ -19680625(4)^3 + 12920250(5)^3 - 5327125(6)^3 \\ +1295750(7)^3 - 141000(8)^3 \end{pmatrix} \\
\dfrac{(6)^6}{6!} - \dfrac{1}{(15400)(3!)} \begin{pmatrix} 346248(1)^3 - 123660(2)^3 + 385128(3)^3 - 258660(4)^3 \\ +190296(5)^3 - 75348(6)^3 + 18360(7)^3 - 1998(8)^3 \end{pmatrix} \\
\dfrac{(7)^6}{6!} - \dfrac{1}{(5702400)(3!)} \begin{pmatrix} 180838518(1)^3 - 54639557(2)^3 + 206894170(3)^3 \\ -122270925(4)^3 + 104842066(5)^3 - 35782103(6)^3 \\ +9423582(7)^3 - 1020425(8)^3 \end{pmatrix} \\
\dfrac{(8)^6}{6!} - \dfrac{1}{(155925)(3!)} \begin{pmatrix} 828928(1)^3 - 216192(2)^3 + 970240(3)^3 \\ -510560(4)^3 + 508416(5)^3 - 134528(6)^3 \\ +51712(7)^3 - 4440(8)^3 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_7 = \begin{pmatrix}
\dfrac{1}{7!} - \dfrac{1}{(39916800)(4!)}\begin{pmatrix} 6779886(1)^4 - 9359135(2)^4 + 11774146(3)^4 \\ -10745445(4)^4 + 6771082(5)^4 - 2792861(6)^4 \\ +679110(7)^4 - 73886(8)^4 \end{pmatrix} \\
\dfrac{(2)^7}{7!} - \dfrac{1}{(623700)(4!)}\begin{pmatrix} 911204(1)^4 - 926646(2)^4 + 1173140(3)^4 \\ -1067950(4)^4 + 671628(5)^4 - 276634(6)^4 \\ +67196(7)^4 - 7305(8)^4 \end{pmatrix} \\
\dfrac{(3)^7}{7!} - \dfrac{1}{(492800)(4!)}\begin{pmatrix} 711918(1)^4 - 521217(2)^4 + 766290(3)^4 \\ -699885(4)^4 + 441306(5)^4 - 182043(6)^4 \\ +44262(7)^4 - 4815(8)^4 \end{pmatrix} \\
\dfrac{(4)^7}{7!} - \dfrac{1}{(155925)(4!)}\begin{pmatrix} 137126(1)^4 - 752480(2)^4 + 1435264(3)^4 \\ -1243200(4)^4 + 784768(5)^4 - 323744(6)^4 \\ +78720(7)^4 - 8564(8)^4 \end{pmatrix} \\
\dfrac{(5)^7}{7!} - \dfrac{1}{(1596672)(4!)}\begin{pmatrix} 23702750(1)^4 - 10296375(2)^4 + 25537250(3)^4 \\ -19680625(4)^4 + 12920250(5)^4 - 5327125(6)^4 \\ +1295750(7)^4 - 141000(8)^4 \end{pmatrix} \\
\dfrac{(6)^7}{7!} - \dfrac{1}{(15400)(4!)}\begin{pmatrix} 346248(1)^4 - 123660(2)^4 + 385128(3)^4 - 258660(4)^4 \\ +190296(5)^4 - 75348(6)^4 + 18360(7)^4 - 1998(8)^4 \end{pmatrix} \\
\dfrac{(7)^7}{7!} - \dfrac{1}{(5702400)(4!)}\begin{pmatrix} 180838518(1)^4 - 54639557(2)^4 + 206894170(3)^4 \\ -122270925(4)^4 + 104842066(5)^4 - 35782103(6)^4 \\ +9423582(7)^4 - 1020425(8)^4 \end{pmatrix} \\
\dfrac{(8)^7}{7!} - \dfrac{1}{(155925)(4!)}\begin{pmatrix} 828928(1)^4 - 216192(2)^4 + 970240(3)^4 \\ -510560(4)^4 + 508416(5)^4 - 134528(6)^4 + 51712(7)^4 \\ -4440(8)^4 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_8 = \begin{pmatrix}
\frac{1}{8!} - \frac{1}{(39916800)(5!)}\begin{pmatrix} 6779886(1)^5 - 9359135(2)^5 + 11774146(3)^5 \\ -10745445(4)^5 + 6771082(5)^5 - 2792861(6)^5 \\ +679110(7)^5 - 73886(8)^5 \end{pmatrix} \\
\frac{(2)^8}{8!} - \frac{1}{(623700)(5!)}\begin{pmatrix} 911204(1)^5 - 926646(2)^5 + 1173140(3)^5 \\ -1067950(4)^5 + 671628(5)^5 - 276634(6)^5 \\ +67196(7)^5 - 7305(8)^5 \end{pmatrix} \\
\frac{(3)^8}{8!} - \frac{1}{(492800)(5!)}\begin{pmatrix} 711918(1)^5 - 521217(2)^5 + 766290(3)^5 \\ -699885(4)^5 + 441306(5)^5 - 182043(6)^5 \\ +44262(7)^5 - 4815(8)^5 \end{pmatrix} \\
\frac{(4)^8}{8!} - \frac{1}{(155925)(5!)}\begin{pmatrix} 137126(1)^5 - 752480(2)^5 + 1435264(3)^5 \\ -1243200(4)^5 + 784768(5)^5 - 323744(6)^5 \\ +78720(7)^5 - 8564(8)^5 \end{pmatrix} \\
\frac{(5)^8}{8!} - \frac{1}{(1596672)(5!)}\begin{pmatrix} 23702750(1)^5 - 10296375(2)^5 + 25537250(3)^5 \\ -19680625(4)^5 + 12920250(5)^5 - 5327125(6)^5 \\ +1295750(7)^5 - 141000(8)^5 \end{pmatrix} \\
\frac{(6)^8}{8!} - \frac{1}{(15400)(5!)}\begin{pmatrix} 346248(1)^5 - 123660(2)^5 + 385128(3)^5 - 258660(4)^5 \\ +190296(5)^5 - 75348(6)^5 + 18360(7)^5 - 1998(8)^5 \end{pmatrix} \\
\frac{(7)^8}{8!} - \frac{1}{(5702400)(5!)}\begin{pmatrix} 180838518(1)^5 - 54639557(2)^5 + 206894170(3)^5 \\ -122270925(4)^5 + 104842066(5)^5 - 35782103(6)^5 \\ +9423582(7)^5 - 1020425(8)^5 \end{pmatrix} \\
\frac{(8)^8}{8!} - \frac{1}{(155925)(5!)}\begin{pmatrix} 828928(1)^5 - 216192(2)^5 + 970240(3)^5 - 510560(4)^5 + \\ 508416(5)^5 - 134528(6)^5 + 51712(7)^5 - 4440(8)^5 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_9 = \begin{pmatrix}
\dfrac{1}{9!} - \dfrac{1}{(39916800)(6!)}\begin{pmatrix} 6779886(1)^6 - 9359135(2)^6 + 11774146(3)^6 \\ -10745445(4)^6 + 6771082(5)^6 - 2792861(6)^6 \\ +679110(7)^6 - 73886(8)^6 \end{pmatrix} \\
\dfrac{(2)^9}{9!} - \dfrac{1}{(623700)(6!)}\begin{pmatrix} 911204(1)^6 - 926646(2)^6 + 1173140(3)^6 \\ -1067950(4)^6 + 671628(5)^6 - 276634(6)^6 \\ +67196(7)^6 - 7305(8)^6 \end{pmatrix} \\
\dfrac{(3)^9}{9!} - \dfrac{1}{(492800)(6!)}\begin{pmatrix} 711918(1)^6 - 521217(2)^6 + 766290(3)^6 \\ -699885(4)^6 + 441306(5)^6 - 182043(6)^6 \\ +44262(7)^6 - 4815(8)^6 \end{pmatrix} \\
\dfrac{(4)^9}{9!} - \dfrac{1}{(155925)(6!)}\begin{pmatrix} 137126(1)^6 - 752480(2)^6 + 1435264(3)^6 \\ -1243200(4)^6 + 784768(5)^6 - 323744(6)^6 \\ +78720(7)^6 - 8564(8)^6 \end{pmatrix} \\
\dfrac{(5)^9}{9!} - \dfrac{1}{(1596672)(6!)}\begin{pmatrix} 23702750(1)^6 - 10296375(2)^6 + 25537250(3)^6 \\ -19680625(4)^6 + 12920250(5)^6 - 5327125(6)^6 \\ +1295750(7)^6 - 141000(8)^6 \end{pmatrix} \\
\dfrac{(6)^9}{9!} - \dfrac{1}{(15400)(6!)}\begin{pmatrix} 346248(1)^6 - 123660(2)^6 + 385128(3)^6 - 258660(4)^6 \\ +190296(5)^6 - 75348(6)^6 + 18360(7)^6 - 1998(8)^6 \end{pmatrix} \\
\dfrac{(7)^9}{9!} - \dfrac{1}{(5702400)(6!)}\begin{pmatrix} 180838518(1)^6 - 54639557(2)^6 + 206894170(3)^6 \\ -122270925(4)^6 + 104842066(5)^6 - 35782103(6)^6 \\ +9423582(7)^6 - 1020425(8)^6 \end{pmatrix} \\
\dfrac{(8)^9}{9!} - \dfrac{1}{(155925)(6!)}\begin{pmatrix} 828928(1)^6 - 216192(2)^6 + 970240(3)^6 - 510560(4)^6 \\ +508416(5)^6 - 134528(6)^6 + 51712(7)^6 - 4440(8)^6 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_{10} = \begin{pmatrix}
\frac{1}{10!} - \frac{1}{(39916800)(7!)}\begin{pmatrix} 6779886(1)^7 - 9359135(2)^7 + 11774146(3)^7 \\ -10745445(4)^7 + 6771082(5)^7 - 2792861(6)^7 \\ +679110(7)^7 - 73886(8)^7 \end{pmatrix} \\
\frac{(2)^{10}}{10!} - \frac{1}{(623700)(7!)}\begin{pmatrix} 911204(1)^7 - 926646(2)^7 + 1173140(3)^7 \\ -1067950(4)^7 + 671628(5)^7 - 276634(6)^7 \\ +67196(7)^7 - 7305(8)^7 \end{pmatrix} \\
\frac{(3)^{10}}{10!} - \frac{1}{(492800)(7!)}\begin{pmatrix} 711918(1)^7 - 521217(2)^7 + 766290(3)^7 \\ -699885(4)^7 + 441306(5)^7 - 182043(6)^7 \\ +44262(7)^7 - 4815(8)^7 \end{pmatrix} \\
\frac{(4)^{10}}{10!} - \frac{1}{(155925)(7!)}\begin{pmatrix} 137126(1)^7 - 752480(2)^7 + 1435264(3)^7 \\ -1243200(4)^7 + 784768(5)^7 - 323744(6)^7 \\ +78720(7)^7 - 8564(8)^7 \end{pmatrix} \\
\frac{(5)^{10}}{10!} - \frac{1}{(1596672)(7!)}\begin{pmatrix} 23702750(1)^7 - 10296375(2)^7 + 25537250(3)^7 \\ -19680625(4)^7 + 12920250(5)^7 - 5327125(6)^7 \\ +1295750(7)^7 - 141000(8)^7 \end{pmatrix} \\
\frac{(6)^{10}}{10!} - \frac{1}{(15400)(7!)}\begin{pmatrix} 346248(1)^7 - 123660(2)^7 + 385128(3)^7 - 258660(4)^7 \\ +190296(5)^7 - 75348(6)^7 + 18360(7)^7 - 1998(8)^7 \end{pmatrix} \\
\frac{(7)^{10}}{10!} - \frac{1}{(5702400)(7!)}\begin{pmatrix} 180838518(1)^7 - 54639557(2)^7 + 206894170(3)^7 \\ -122270925(4)^7 + 104842066(5)^7 - 35782103(6)^7 \\ +9423582(7)^7 - 1020425(8)^7 \end{pmatrix} \\
\frac{(8)^{10}}{10!} - \frac{1}{(155925)(7!)}\begin{pmatrix} 828928(1)^7 - 216192(2)^7 + 970240(3)^7 - 510560(4)^7 + \\ 508416(5)^7 - 134528(6)^7 + 51712(7)^7 - 4440(8)^7 \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_{11} = \begin{pmatrix}
\dfrac{1}{11!} - \dfrac{1}{(39916800)(8!)} \begin{pmatrix} 6779886(1)^{8} - 9359135(2)^{8} + 11774146(3)^{8} \\ -10745445(4)^{8} + 6771082(5)^{8} - 2792861(6)^{\backslash 8} \\ + 679110(7)^{8} - 73886(8)^{8} \end{pmatrix} \\
\dfrac{(2)^{11}}{11!} - \dfrac{1}{(623700)(8!)} \begin{pmatrix} 911204(1)^{8} - 926646(2)^{8} + 1173140(3)^{8} - 1067950(4)^{8} \\ + 671628(5)^{8} - 276634(6)^{8} + 67196(7)^{8} - 7305(8)^{8} \end{pmatrix} \\
\dfrac{(3)^{11}}{11!} - \dfrac{1}{(492800)(8!)} \begin{pmatrix} 711918(1)^{8} - 521217(2)^{8} + 766290(3)^{8} - 699885(4)^{8} + \\ 441306(5)^{8} - 182043(6)^{8} + 44262(7)^{8} - 4815(8)^{8} \end{pmatrix} \\
\dfrac{(4)^{11}}{11!} - \dfrac{1}{(155925)(8!)} \begin{pmatrix} 137126(1)^{8} - 752480(2)^{8} + 1435264(3)^{8} - 1243200(4)^{8} \\ + 784768(5)^{8} - 323744(6)^{8} + 78720(7)^{8} - 8564(8)^{8} \end{pmatrix} \\
\dfrac{(5)^{11}}{11!} - \dfrac{1}{(1596672)(8!)} \begin{pmatrix} 23702750(1)^{8} - 10296375(2)^{8} + 25537250(3)^{8} \\ -19680625(4)^{8} + 12920250(5)^{8} - 5327125(6)^{8} \\ + 1295750(7)^{8} - 141000(8)^{8} \end{pmatrix} \\
\dfrac{(6)^{11}}{11!} - \dfrac{1}{(15400)(8!)} \begin{pmatrix} 346248(1)^{8} - 123660(2)^{8} + 385128(3)^{8} - 258660(4)^{8} \\ + 190296(5)^{8} - 75348(6)^{8} + 18360(7)^{8} - 1998(8)^{8} \end{pmatrix} \\
\dfrac{(7)^{11}}{11!} - \dfrac{1}{(5702400)(8!)} \begin{pmatrix} 180838518(1)^{8} - 54639557(2)^{8} + 206894170(3)^{8} \\ -122270925(4)^{8} + 104842066(5)^{8} - 35782103(6)^{8} \\ + 9423582(7)^{8} - 1020425(8)^{8} \end{pmatrix} \\
\dfrac{(8)^{11}}{11!} - \dfrac{1}{(155925)(8!)} \begin{pmatrix} 828928(1)^{8} - 216192(2)^{8} + 970240(3)^{8} - 510560(4)^{8} + \\ 508416(5)^{8} - 134528(6)^{8} + 51712(7)^{8} - 4440(8)^{8} \end{pmatrix}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C_{12}=\begin{pmatrix}
\frac{1}{12!}-\frac{1}{(39916800)(9!)}\begin{pmatrix}6779886(1)^9-9359135(2)^9+11774146(3)^9\\-10745445(4)^9+6771082(5)^9-2792861(6)^9\\+679110(7)^9-73886(8)^0\end{pmatrix}\\
\frac{(2)^{12}}{12!}-\frac{1}{(623700)(9!)}\begin{pmatrix}911204(1)^9-926646(2)^9+1173140(3)^9-1067950(4)^9\\+671628(5)^9-276634(6)^9+67196(7)^9-7305(8)^9\end{pmatrix}\\
\frac{(3)^{12}}{12!}-\frac{1}{(492800)(9!)}\begin{pmatrix}711918(1)^9-521217(2)^9+766290(3)^9-699885(4)^9+\\441306(5)^9-182043(6)^9+44262(7)^9-4815(8)^9\end{pmatrix}\\
\frac{(4)^{12}}{12!}-\frac{1}{(155925)(9!)}\begin{pmatrix}137126(1)^9-752480(2)^9+1435264(3)^9-1243200(4)^9\\+784768(5)^9-323744(6)^9+78720(7)^9-8564(8)^9\end{pmatrix}\\
\frac{(5)^{12}}{12!}-\frac{1}{(1596672)(9!)}\begin{pmatrix}23702750(1)^9-10296375(2)^9+25537250(3)^9\\-19680625(4)^9+12920250(5)^9-5327125(6)^9\\+1295750(7)^9-141000(8)^9\end{pmatrix}\\
\frac{(6)^{12}}{12!}-\frac{1}{(15400)(9!)}\begin{pmatrix}346248(1)^9-123660(2)^9+385128(3)^9-258660(4)^9\\+190296(5)^9-75348(6)^9+18360(7)^9-1998(8)^9\end{pmatrix}\\
\frac{(7)^{12}}{12!}-\frac{1}{(5702400)(9!)}\begin{pmatrix}180838518(1)^9-54639557(2)^9+206894170(3)^9\\-122270925(4)^9+104842066(5)^9-35782103(6)^9\\+9423582(7)^9-1020425(8)^9\end{pmatrix}\\
\frac{(8)^{12}}{12!}-\frac{1}{(155925)(9!)}\begin{pmatrix}828928(1)^9-216192(2)^9+970240(3)^9-510560(4)^9\\+508416(5)^9-134528(6)^9+51712(7)^9-4440(8)^9\end{pmatrix}
\end{pmatrix}
=\begin{pmatrix}\frac{9}{5669}\\ \frac{113}{11313}\\ \frac{155}{6179}\\ \frac{80}{1701}\\ \frac{551}{7278}\\ \frac{403}{3623}\\ \frac{846}{5513}\\ \frac{374}{1847}\end{pmatrix}
$$

Hence, the block is having an order $(9,9,9,9,9,9,9,9)^T$ with error constants

$$\left(\frac{1}{630939},\frac{113}{11313},\frac{155}{6179},\frac{80}{1701},\frac{316}{4176},\frac{403}{3623},\frac{854}{5555},\frac{374}{1847}\right)^T$$

5.5.2.2 Zero Stability of Eight–Step Block Method for Fourth Order ODEs.

Equation (3.2.2.2.1) is applied to eight step block method (5.5.1.41 – 5.5.1.48), we have

$$
\det[rA^{(0)}-A^{(1)}]=\left| r\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{pmatrix}\right|=0
$$

which implies $r = 0,0,0,0,0,0,0,1$. Hence, the method is zero stable.

5.5.2.3 Consistency and Convergence of Eight–Step Block Method for Fourth Order ODEs

The block method (5.5.1.41–5.5.1.48) is consistent because the conditions stated in Definition 1.4 are satisfied. Hence, it is also convergent since it is zero-stable and consistent.

5.5.2.4 Region of Absolute Stability of Eight–Step Block Method for Fourth Order ODEs.

Applying the equation (3.2.2.4.2) to eight-step block method (5.5.1.41 –5.5.1.48) we have

$$\bar{h}(\theta, h) = \frac{A - B}{C + D}$$

where

$$A = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{3i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{4i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{5i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{6i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{7i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{8i\theta} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{36501816}{958003200}e^{i\theta} & -\frac{52883276}{958003200}e^{2i\theta} & \frac{67126376}{958003200}e^{3i\theta} & -\frac{61500210}{958003200}e^{4i\theta} & \frac{38838088}{958003200}e^{5i\theta} & -\frac{16041916}{958003200}e^{6i\theta} & \frac{3904536}{958003200}e^{7i\theta} & -\frac{425111}{958003200}e^{8i\theta} \\ \frac{2719504}{3742200}e^{i\theta} & -\frac{3139836}{3742200}e^{2i\theta} & \frac{3933392}{3742200}e^{3i\theta} & -\frac{3577790}{3742200}e^{4i\theta} & \frac{2249904}{3742200}e^{5i\theta} & -\frac{926764}{3742200}e^{6i\theta} & \frac{225136}{3742200}e^{7i\theta} & -\frac{24477}{3742200}e^{8i\theta} \\ \frac{13764168}{3942400}e^{i\theta} & -\frac{12476916}{3942400}e^{2i\theta} & \frac{16614360}{3942400}e^{3i\theta} & -\frac{15168870}{3942400}e^{4i\theta} & \frac{9553464}{3942400}e^{5i\theta} & \frac{3938148}{3942400}e^{6i\theta} & \frac{957096}{3942400}e^{7i\theta} & -\frac{104085}{3942400}e^{8i\theta} \\ \frac{4641792}{467775}e^{i\theta} & -\frac{3352960}{467775}e^{2i\theta} & \frac{5131264}{467775}e^{3i\theta} & -\frac{4620000}{467775}e^{4i\theta} & \frac{2911232}{467775}e^{5i\theta} & -\frac{1200512}{467775}e^{6i\theta} & \frac{291840}{467775}e^{7i\theta} & -\frac{31744}{467775}e^{8i\theta} \\ \frac{828115000}{38320128}e^{i\theta} & -\frac{490807500}{38320128}e^{2i\theta} & \frac{895985000}{38320128}e^{3i\theta} & -\frac{766681250}{38320128}e^{4i\theta} & \frac{487389000}{38320128}e^{5i\theta} & -\frac{201077500}{38320128}e^{6i\theta} & \frac{48895000}{38320128}e^{7i\theta} & -\frac{5319375}{38320128}e^{8i\theta} \\ \frac{1236384}{30800}e^{i\theta} & -\frac{617544}{30800}e^{2i\theta} & \frac{1345824}{30800}e^{3i\theta} & -\frac{1064340}{30800}e^{4i\theta} & \frac{702432}{30800}e^{5i\theta} & -\frac{287784}{30800}e^{6i\theta} & \frac{69984}{30800}e^{7i\theta} & -\frac{7614}{30800}e^{8i\theta} \\ \frac{9184285992}{136857600}e^{i\theta} & -\frac{3949712228}{136857600}e^{2i\theta} & \frac{10148873336}{136857600}e^{3i\theta} & -\frac{7344827070}{136857600}e^{4i\theta} & \frac{5205732952}{136857600}e^{5i\theta} & -\frac{2050386772}{136857600}e^{6i\theta} & \frac{504037128}{136857600}e^{7i\theta} & -\frac{54841241}{136857600}e^{8i\theta} \\ \frac{48693248}{467775}e^{i\theta} & -\frac{18333696}{467775}e^{2i\theta} & \frac{54722560}{467775}e^{3i\theta} & -\frac{36229120}{467775}e^{4i\theta} & \frac{28114944}{467775}e^{5i\theta} & -\frac{10108928}{467775}e^{6i\theta} & \frac{2686976}{467775}e^{7i\theta} & -\frac{284160}{467775}e^{8i\theta} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{24396497}{958003200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1035731}{3742200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4104531}{3942400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1218688}{467775} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{201421625}{38320128} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{285858}{30800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2048300303}{136857600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10571776}{467775} \end{pmatrix}$$

The matrix above is simplified, after finding the determinant and equating the

$$\bar{h}(\theta, h) = \frac{(1.8858\text{E}+171)\cos 8\theta - (1.8858\text{E}+171)}{(5.6926\text{E}+163)\cos 8\theta - (8.4513\text{E}+166)}$$

The value of $\bar{h}(\theta, h)$ is evaluated at the intervals of θ of 30^0 and this produced the results tabulated in Table 5.4.

Table 5.4

Interval of Absolute Stability of Eight–Step Block Method for Fourth Order ODEs

θ	0	30	60	90	120	150	180
$\bar{h}(\theta, h)$	0	33458.67	33458.67	0	33458.67	33458.67	0

Therefore, the Interval of absolute stability is (0, 33458.67). This is shown in the diagram below

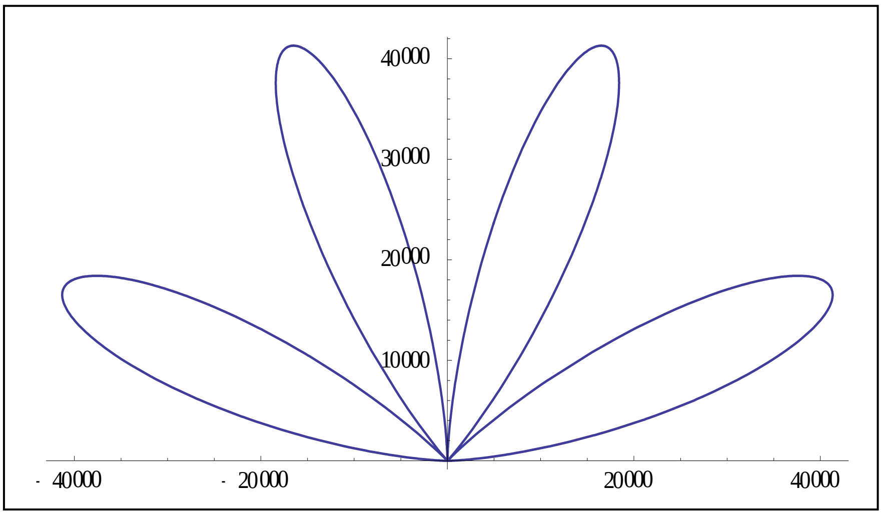

Figure 5.8. Region of absolute stability of eight–step block method for fourth order ODEs.

5.6 Comments on the Properties of the Block Methods for Fourth Order ODEs.

This segment discusses the analysis of the properties of the new block methods for fourth order ODEs. The order of the developed block methods for each step-length k is k+1 and they are all zero-stable. The new block methods are convergent because they are consistent and zero-stable. The interval of absolute stability for odd step-length k reduces as the value of k increases (refer to Tables 5.1 and 5.3) while the interval of absolute stability becomes larger for even step-length k as k increases (refer to Tables 5.2 and 5.4). The region of absolute stability of even step-length k is beyond the line as a result of their positive stability function over $[0, \pi]$ while absolute stability region for odd step-length k faces downward because of their negative function over $[0, \pi]$.

5.7 Test Problems for Fourth Order ODEs

The following fourth order initial values problems are considered in order to test the accuracy of the block methods. The methods are applied to the same differential problems existing methods solved for the purpose of comparison in terms error.

Problem 22: $y^{iv} + y'' = 0, \quad 0 \le x \le \frac{\pi}{2}, y(0) = 0, y'(0) = \frac{-1.1}{72 - 50\pi},$

$y''(0) = \frac{1}{144 - 100\pi},\ y'''(0) = \frac{1.2}{144 - 100\pi}, h = \frac{1}{320}$

Exact Solution: $y(x) = \frac{(1 - x - \cos x - 1.2\sin x)}{144 - 100\pi}$

Problem 23: $y^{iv} = (y')^2 - yy''' - 4x^2 + e^x(1 - 4x + x^2), \quad 0 \le x \le 1$

$y(0) = 1, y'(0) = 1,\ y''(0) = 3, y'''(0) = 1, \quad h = \frac{1}{320}$

Exact Solution: $y(x) = x^2 + e^x$

Problem 24: $y^{iv} = (x^4 + 14x^3 + 49x^2 + 32x - 12)e^x, y(0) = y'(0) = 0,$

$y''(0) = 2, y'''(0) = -6, \quad 0 \le x \le 1$

Exact Solution: $y(x) = x^2 e^x (x - 1)^2$

Problem 25: $y^{iv} = x, \ \ y(0) = 0, y'(0) = 1, y''(0) = y'''(0) = 0$

Exact Solution: $y(x) = \frac{x^5}{120} + x$

Problem 26: $y^{iv} = y \ \ y(0) = y'(0) = y''(0) = y'''(0) = 1, \ 0 \le x \le 1$

Exact Solution: $y(x) = e^x$

5.8 Numerical Results for Fourth Order ODEs.

The tables displayed below are the numerical results when the new block methods with step-length $k = 5(1)8$ were applied to fourth order differential equations above. The comparison of the generated numerical results is made with the existing methods in terms of error.

Table 5.5

Comparison of the New Block Method k=5 with Predictor-Corrector Method (Kayode , 2008a) and Predictor-Corrector Method (Kayode , 2008b) for Solving Problem 22

x-values	Exact Solution	Computed Solution	Error in new Method, k=5, h=1/320	Error in Kayode (2008a), k= 5, h=1/320	Error in Kayode, (2008b), k=5, h=1/320
0.103125	0.00130079958936715	0.00130079958936715	2.168404E-19	0.49873300E-15	0.48355417E-16
0.206250	0.00253177370019563	0.00253177370019563	1.734723E-18	0.67654216E-15	0.13933299E-15
0.306250	0.00365247897888499	0.00365247897888499	2.168404E-18	0.31350790E-14	0.66893539E-15
0.406250	0.00469595323180485	0.00469595323180484	1.127570E-17	0.94360283E-14	0.20129384E-14
0.506250	0.00565764236059347	0.00565764236059345	2.081668E-17	0.22116857E-13	0.46736053E-14
0.603125	0.00650775460803452	0.00650775460803448	3.989864E-17	0.43379363E-13	0.91874598E-14
0.703125	0.00729831476763851	0.00729831476763843	7.285839E-17	0.77870869E-13	0.16069038E-13
0.803125	0.00799852022272896	0.00799852022272883	1.249001E-16	0.12863495E-12	0.25407974E-13
0.903125	0.00860724670330246	0.00860724670330227	1.994932E-16	0.19927116E-12	0.38108926E-13
1.003125	0.00912428396703006	0.00912428396702977	2.914335E-16	0.29323245E-12	0.54051538E-13

Table 5.6

Comparison of the New Block Method k=5 with Predictor-Corrector Method (Kayode , 2008a) and Predictor-Corrector Method (Kayode , 2008b) for Solving Problem 23

x-values	Exact Solution	Computed Solution	Error in new Method, k=5, h=1/320	Error in Kayode (2008a), k= 5, h=1/320	Error in Kayode, (2008b), k=5, h=1/320
0.103125	1.119264744787591900	1.119264744787593600	1.776357E-15	4.6842974E-11	8.1714635E-12
0.206250	1.271599493198048500	1.271599493198084100	3.552714E-14	2.0687163E-10	1.4680923E-10
0.306250	1.452110907065013100	1.452110907065195200	1.820766E-13	9.0421981E-10	7.8813045E-10
0.406250	1.666216862500122800	1.666216862500711700	5.888623E-13	2.9137981E-09	2.6795135E-09
0.506250	1.915347109920916500	1.915347109922390400	1.473932E-12	7.5114031E-09	7.0615016E-09
0.603125	2.191581593606204900	2.191581593609278000	3.073097E-12	1.6231339E-08	1.5438411E-08
0.703125	2.514440293333696500	2.514440293339570900	5.874412E-12	3.2237145E-08	3.0900764E-08
0.803125	2.877516387746607200	2.877516387756956700	1.034950E-11	5.8891835E-08	5.6759556E-08
0.903125	3.282936158805099100	3.282936158822240500	1.714140E-11	1.0079927E-07	9.7549608E-08
1.003125	3.733049511495175400	3.733049511522218600	2.103739E-11	1.6373663E-07	1.5897083E-07

Table 5.7

Comparison of the New Block Method k=5 with Block Method (Omar & Suleiman, 2004) for Solving Problem 24 in which Maximum Errors were considered.

h-values	New Method	Omar and Suleiman (2004)	Number of Steps	Error in new Method, k=5	Error in Omar and Suleiman (2004), k=5
10^{-2}	5-Step Method	S2PEB	53	1.069793E-10	3.55338E-03
		P2PEB	53	1.069793E-10	3.55338E-03
		S3PEB	36	1.476508E-11	3.99807E-03
		P3PEB	36	1.476508E-11	3.99807E-03
10^{-3}	5-Step Method	S2PEB	503	1.429612E-11	3.40664E-04
		P2PEB	503	1.429612E-11	3.40664E-04
		S3PEB	336	4.858336E-13	3.41092E-04
		P3PEB	336	4.858336E-13	3.41092E-04
10^{-4}	5-Step Method	S2PEB	5003	4.152412E-11	3.40305E-05
		P2PEB	5003	4.152412E-11	3.40305E-05
		S3PEB	3336	3.176126E-12	3.40310E-05
		P3PEB	3336	3.176126E-12	3.40310E-05
10^{-5}	5-Step Method	S2PEB	50003	1.903970E-10	3.40280E-06
		P2PEB	50003	1.903970E-10	3.40280E-06
		S3PEB	33336	4.471712E-11	3.40280E-06
		P3PEB	33336	4.471712E-11	3.40280E-06

Table 5.8

Comparison of the New Block Method k=6 with Block Method (Olabode, 2009) and Block Method (Mohammed, 2010) for Solving Problem 25

x-values	Exact Solution	Computed Solution	Error in new Method, k=6, h=0.1	Error in Olabode (2009), k= 6, h=0.1	Error in Mohammed, (2010), k=6, h=0.1
0.1	0.100000083333333340	0.100000083333333340	0.000000E+00	1.66666667E-10	7.000000024E-10
0.2	0.200002666666666690	0.200002666666666690	0.000000E+00	3.33333305E-10	8.999999912E-10
0.3	0.300020250000000040	0.300020250000000040	0.000000E+00	5.99999994E-10	2.599999993E-09
0.4	0.400085333333333350	0.400085333333333350	0.000000E+00	7.66666675E-10	5.100000033E-09
0.5	0.500260416666666650	0.500260416666666650	0.000000E+00	9.33333300E-10	7.799999979E-09
0.6	0.600648000000000070	0.600648000000000070	0.000000E+00	1.10000009E-09	1.180000000E-08
0.7	0.701400583333333440	0.701400583333333440	0.000000E+00	1.27166666E-09	1.240000003E-08
0.8	0.802730666666666700	0.802730666666666810	1.110223E-16	1.45333334E-09	1.410000006E-08
0.9	0.904920750000000160	0.904920750000000050	1.110223E-16	1.64999991E-09	1.880000000E-08
1.0	1.008333333333333300	1.008333333333333300	0.000000E+00	1.87666660E-09	2.600000015E-08
1.1	1.113420916666666600	1.113420916666666400	2.220446E-16	2.13333329E-09	……………..
1.2	1.220736000000000300	1.220736000000000000	2.220446E-16	2.42999998E-09	……………..

Table 5.9

Comparison of the New Block Methods k=6, 7 and 8 with Block Method (Omar, 1999) where the selection of Maximum Errors were made for Solving Problem 24

h-values	New Method			Omar (1999)	Number of Steps	Error in new Method, *k*=6	Error in new Method, *k*=7	Error in new Method, *k*=8	Error in Omar (1999), *k*=8
10^{-2}	6-Step Method	7-Step Method	8-Step Method	S2PEB	54	1.637090E-10	3.547029E-11	1.728040E-11	1.00778E-02
				P2PEB	54	1.637090E-10	3.547029E-11	1.728040E-11	1.00778E-02
				S3PEB	39	1.023182E-11	1.250555E-12	7.958079E-13	1.00778E-02
				P3PEB	39	1.023182E-11	1.250555E-12	7.958079E-13	1.00778E-02
10^{-3}	6-Step Method	7-Step Method	8-Step Method	S2PEB	504	3.069545E-10	1.141416E-10	8.185452E-12	1.00778E-03
				P2PEB	504	3.069545E-10	1.141416E-10	8.185452E-12	1.00778E-02
				S3PEB	339	4.646949E-12	1.762146E-12	3.410605E-13	1.00778E-03
				P3PEB	339	4.646949E-12	1.762146E-12	3.410605E-13	1.00778E-02
10^{-4}	6-Step Method	7-Step Method	8-Step Method	S2PEB	5004	1.089120E-08	1.439730E-09	1.100034E-09	1.00008E-04
				P2PEB	5004	1.089120E-08	1.439730E-09	1.100034E-09	1.00008E-04
				S3PEB	3339	3.821299E-11	3.311129E-12	3.997513E-11	1.00008E-04
				P3PEB	3339	3.821299E-11	3.311129E-12	3.997513E-11	1.00008E-04
10^{-5}	6-Step Method	7-Step Method	8-Step Method	S2PEB	50004	8.426014E-09	2.466095E-09	1.035278E-09	1.00001E-05
				P2PEB	50004	8.426014E-09	2.466095E-09	1.035278E-09	1.00001E-05
				S3PEB	33339	3.341256E-10	9.124790E-11	3.454659E-11	1.00001E-05
				P3PEB	33339	3.341256E-10	9.124790E-11	3.454659E-11	1.00001E-05

Table 5.10

Comparison of the New Block Methods k=6, 7 and 8 with Block Method (Omar, 1999) whereby Maximum Errors were considered for Solving Problem 26

h-values	New Method			Omar in (1999)	Number of Steps	Error in new Method, *k*=6	Error in new Method, *k*=7	Error in new Method, *k*=8	Error in Omar (1999), *k*=8
10^{-2}	6-Step Method	7-Step Method	8-Step Method	S2PEB	54	2.158558E-10	3.725589E-11	1.193712E-11	8.37112E-04
				P2PEB	54	2.158558E-10	3.725589E-11	1.193712E-11	8.37112E-04
				S3PEB	39	3.996803E-11	1.074216E-11	2.199130E-12	8.37105E-04
				P3PEB	39	3.996803E-11	1.074216E-11	2.199130E-12	8.37105E-04
10^{-3}	6-Step Method	7-Step Method	8-Step Method	S2PEB	504	1.861622E-12	3.765876E-13	2.131628E-14	8.34604E-05
				P2PEB	504	1.861622E-12	3.765876E-13	2.131628E-14	8.34604E-05
				S3PEB	339	1.261213E-13	6.750156E-14	1.776357E-14	8.34604E-05
				P3PEB	339	1.261213E-13	6.750156E-14	1.776357E-14	8.34604E-05
10^{-4}	6-Step Method	7-Step Method	8-Step Method	S2PEB	5004	8.114398E-12	7.297274E-12	8.427037E-12	8.34353E-06
				P2PEB	5004	8.114398E-12	7.297274E-12	8.427037E-12	8.34353E-06
				S3PEB	3339	1.236344E-12	2.362555E-13	1.243450E-13	8.34353E-06
				P3PEB	3339	1.236344E-12	2.362555E-13	1.243450E-13	8.34353E-06
10^{-5}	6-Step Method	7-Step Method	8-Step Method	S2PEB	50004	8.299850E-11	2.202682E-11	1.938183E-11	8.34326E-07
				P2PEB	50004	8.299850E-11	2.202682E-11	1.938183E-11	8.34326E-07
				S3PEB	33339	1.555733E-11	4.297007E-12	1.257483E-12	8.34330E-07
				P3PEB	33339	1.555733E-11	4.297007E-12	1.257483E-12	8.34330E-07

5.9 Comments on the Results

The new block methods only examined one non-linear ODEs due to the limited exiting methods on it whereas four linear ODEs are tested. The numerical results of the new block methods k=5 shown in Table 5.5 and 5.6 for solving Problems 22 and 23 are found better that Kayode (2008a) k=5. Correspondingly, in Table 5.7 the new method also produced better accuracy than Omar (2004) k=5 when Problem 24 was solved whereby maximum errors were selected. In addition, the results generated from the new block method k=6 after solving Problem 25 are more efficient in terms of error than Olabode (2009) and Mohammed (2010). This can be seen in Table 5.8.

In Tables 5.9 and 5.10, the numerical results of the new block methods k=6, 7 and 8 are compared with each other because of few literatures on numerical methods with step-length k=6, 7 and 8 for fourth order ODEs. It is apparent that the accuracy of new block method k=8 is high compared with the new block methods k=6 and 7. It is also observed that new block method k=7 is better in accuracy than new block method k=6. This implies that accuracy of a method increases when the step-length k increases. However, the three new block methods in Tables 5.9 and 5.10 have better accuracy in terms of error than Omar (1999) k=8.

5.10 Summary

In this chapter, block methods with step-length k=5(1)8 for solving fourth order initial value problems of ODEs are derived via multistep collocation approach. The properties of the methods are verified. The application of the new block methods to solving fourth order problems is examined.The numerical results generated are compared with the existing methods and thereby claim superiority over the existing methods in terms of error (see Tables 5.5–5.10) .

CHAPTER SIX

CONCLUSION AND AREA OF FURTHER RESEARCH

6.1 Conclusion

In this research work, block methods for solving higher order initial value problems of ODEs without reducing them to its equivalent system of first order have been developed.

Conventionally, solving higher order initial value problems through the process of reduction has been found having some drawbacks which include computational burden, a lot of human effort and complications in writing the computer program which affects the accuracy of the method in terms of error. The introduction of direct methods was to overcome the setbacks encountered in reduction method. One of these direct methods is predictor-corrector method which at one point at a time approximates the numerical solution of ODEs and this leads to burden of computing that reduces the accuracy of the method in terms of error.

In order to proffer solution to these setbacks, block method was introduced to simultaneously provide approximate solution to ODEs. Furthermore, in order to bring improvement on the existing numerical methods, this research work presents the development of new block methods for the direct solution of higher order ODEs. The method of interpolation and collocation is adopted in developing the methods. The power series approximate solution is used as interpolation polynomials while its highest derivative is used as a collocation equation. The points of interpolation based on the order of differential equation are made at the points prior to the last two points while collocation points are chosen at all grid points within the interval of integration.

The interpolation and collocation equations are combined in a matrix form and Gaussian elimination method is applied to find the values of the unknown variables. These variables are substituted into the basis function to provide a continuous implicit scheme. Then, a transformation which includes the step-length is used and substituted into the continuous implicit scheme in order to find the coefficients of *y* and *f* functions. The continuous implicit scheme is evaluated at the non-interpolating points to give the discrete schemes while its derivatives based on the other of differential equation are evaluated at all the grid points to produce the derivatives of the discrete schemes.

These discrete schemes and its derivatives at x_n are combined together in a matrix form and matrix inversion is applied. This produces the block methods. The properties of the block methods which include: zero-stability, order, error constant, consistency, convergence and region of absolute stability are established. The order of the new block methods with step-length k=(d+1)(1)8 is k+1 where d is the order of differential equation. These block methods are consistent because the order of each block method is greater than one. The developed block methods are also found to be zero-stable and thereby converged.

Furthermore, it is observed that for odd step-length k, the interval of absolute stability for second order ODEs becomes larger as k increases as well as when k is even. These are shown in Tables 3.1, 3.3, 3.5 (odd step-length k) and Tables 3.2, 3.4, 3.6 (even step-length k). The interval of absolute stability for third order ODEs becomes larger as even step-length k increases (refer to Tables 4.1, 4.3 and 4.5) while the absolute stability interval decreases as odd step-length k increases (refer to Tables 4.2 and 4.4). likewise, the interval of absolute stability for odd step-length k of fourth order ODEs reduces as the value of k increases (refer to Tables 5.1 and 5.3) whereas the interval of absolute stability becomes larger for even step-length k as k increases (refer to Tables 5.2 and 5.4). The new block methods were applied to solve several second, third and fourth order initial value problems of ODEs using a Matlab codes and the results generated are better when comparison was made with the existing methods (refer to Tables 3.7 – 3.24, 4.6 – 4.18, 5.5 – 5.10)

6.2 Areas for Further Research

This research work considered the development of block methods with step-length $k = (d+1)(1)8$ where d is the order of differential equation for solving second, third and fourth order initial value problems of ODEs. Further researchers can extend the work by developing a generalized block method that will cater for any step-length k for solving dth order initial value problems of ODEs directly using interpolation and collocation strategy.

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Appendix A

MATLAB CODE OF THREE–STEP BLOCK METHOD FOR SOLVING SECOND ORDER ODES.

```
% programme to solve second order O .D. E with step length of three
clear
clc
% y'is represented by z
%syms x0 x y z g f0 y0 z0
%g (x,y,z)=x*z^2;
x0=0;  y0=1; z0=1/2; h=0.003125;
disp ('x - value      exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=x0*z0^2;
for i=1:3;
x(i)=x0+i*h;
y(i)=y0 + (h^3*i^3*(2*x0^2*z0^3 + z0^2))/6 + h*i*z0 +
(h^2*i^2*x0*z0^2)/2;; taylor series expansion of yn+i
z(i)=z0 + (h^2*i^2*(2*x0^2*z0^3 + z0^2))/2 + h*i*x0*z0^2;  %  [where
z=y’ is the derivative of the taylor series expansion of yn+i
f(i)=x(i)*z(i)^2;
end
yp1=z0+(h/72)*(27*f0+57*f(1)-15*f(2)+3f(3));
yp2=z0+(h/9)*(3f0+12*f(1)+3f(2));
yp3=z0+(h/24)*(9*f0+27*f(1)+27*f(2)+9*f(3));
fr1=x(1)*yp1^2;
fr2=x(2)*yp2^2;
fr3=x(3)*yp3^2;
yr1=y0+(h*z0)+((h^2)/1080)*(291*f0+342*fr1-117*fr2+24*fr3);
m1=toc;
err1=abs(1+1/2*log((2+x(1))/(2-x(1)))-yr1);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n', x(1),
1+1/2*log((2+x(1))/(2-x(1))),yr1, err1)
yr2=y0+(2*h*z0)+((h^2)/270)*(168*f0+396*fr1-36*fr2+12*fr3);
m2=toc;
err2=abs(1+1/2*log((2+x(2))/(2-x(2)))-yr2);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(2),1+1/2*log((2+x(2))/(2-x(2))) , yr2, err2)
yr3=y0+(3*h*z0)+((h^2)/120)*(117*f0+324*fr1+81*fr2+18*fr3);
m3=toc;
err3=abs(1+1/2*log((2+x(3))/(2-x(3)))-yr3);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n', x(3),
1+1/2*log((2+x(3))/(2-x(3))), yr3, err3)
x0=x(3); y0=yr3; z0=yp3;
end
```

Appendix B

MATLAB CODE OF FOUR–STEP BLOCK METHOD FOR SOLVING SECOND ORDER ODES.

```
% programme to solve second order O .D. E with step length of four
clear
clc
% y'is represented by z
%syms x0 x y z g f0 y0 z0
%g (x,y,z)=100y;
x0=0;  y0=1; z0=-10; h=0.01;
disp ('x - value     exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=100*y0;
for i=1:4;
x(i)=x0+i*h;
y(i)=(250*z0*h^5*i^5)/3 + (1250*y0*h^4*i^4)/3 + (50*z0*h^3*i^3)/3 +
50*y0*h^2*i^2 + z0*h*i + y0;% taylor series expansion of yn+i
z(i)=(1250*z0*h^4*i^4)/3 + (5000*y0*h^3*i^3)/3 + 50*z0*h^2*i^2 +
100*y0*h*i + z0; %  [where z=y' is the derivative of the taylor
series expansion of yn+i
f(i)=100*y(i);
end
yp1=z0+(h/720)*(251*f0+646*f(1)-264*f(2)+106*f(3)-19*f(4));
yp2=z0+(h/90)*(29*f0+124*f(1)+24*f(2)+4*f(3)-f(4));
yp3=z0+(h/160)*(54*f0+204*f(1)+144*f(2)+84*f(3)-6*f(4));
yp4=z0+(h/45)*(14*f0+64*f(1)+24*f(2)+64*f(3)+14*f(4));

yrr1=y0+(h*z0)+((h^2)/4320)*(1101*f0+1620*f(1)-846*f(2)+348*f(3)-
63*f(4));
yrr2=y0+(2*h*z0)+((h^2)/270)*(159*f0+432*f(1)-90*f(2)+48*f(3)-
9*f(4));
yrr3=y0+(3*h*z0)+((h^2)/480)*(441*f0+1404*f(1)+162*f(2)+180*f(3)-
27*f(4));
yrr4=y0+(4*h*z0)+((h^2)/135)*(168*f0+576*f(1)+144*f(2)+192*f(3));

fr1=100*yrr1;
fr2=100*yrr2;
fr3=100*yrr3;
fr4=100*yrr4;
yr1=y0+(h*z0)+((h^2)/4320)*(1101*f0+1620*fr1-846*fr2+348*fr3-
63*fr4);
m1=toc;
err1=abs(exp(-10*x(1))-yr1);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(1),exp(-
10*x(1)),yr1, err1)
yr2=y0+(2*h*z0)+((h^2)/270)*(159*f0+432*fr1-90*fr2+48*fr3-9*fr4);
m2=toc;
err2=abs(exp(-10*x(2))-yr2);
```

```
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(2),exp(-
10*x(2)),yr2, err2)
yr3=y0+(3*h*z0)+((h^2)/480)*(441*f0+1404*fr1+162*fr2+180*fr3-
27*fr4);
m3=toc;
err3=abs(exp(-10*x(3))-yr3);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(3),exp(-
10*x(3)),yr3, err3)
yr4=y0+(4*h*z0)+((h^2)/135)*(168*f0+576*fr1+144*fr2+192*fr3);
m4=toc;
err4=abs(exp(-10*x(4))-yr4);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(4),exp(-
10*x(4)),yr4, err4)
x0=x(4); y0=yr4; z0=yp4;
end
```

Appendix C

MATLAB CODE OF FIVE–STEP BLOCK METHOD FOR SOLVING SECOND ORDER ODES.

```
% programme to solve second order O .D. E with step length of five
clear
%clc
% y'is represented by z
%syms x0 x y z g f0 y0 z0
%g (x,y,z)=z;
x0=0;  y0=0; z0=-1; h=0.1;
disp ('x - value      exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=z0;
for i=1:5;
x(i)=x0+i*h;
y(i)=(z0*h^5*i^5)/120 + (z0*h^4*i^4)/24 + (z0*h^3*i^3)/6 +
(z0*h^2*i^2)/2 + z0*h*i + y0;
z(i)=(z0*h^4*i^4)/24 + (z0*h^3*i^3)/6 + (z0*h^2*i^2)/2 + z0*h*i +
z0;
f(i)=z(i);
end
yp1=z0+(h/1440)*(475*f0+1427*f(1)-798*f(2)+482*f(3)-
173*f(4)+27*f(5));
yp2=z0+(h/90)*(28*f0+129*f(1)+14*f(2)+14*f(3)-6*f(4)+f(5));
yp3=z0+(h/160)*(51*f0+219*f(1)+114*f(2)+114*f(3)-21*f(4)+3*f(5));
yp4=z0+(h/45)*(14*f0+64*f(1)+24*f(2)+64*f(3)+14*f(4));
yp5=z0+(h/288)*(95*f0+375*f(1)+250*f(2)+250*f(3)+375*f(4)+95*f(5));
fr1=yp1;
fr2=yp2;
fr3=yp3;
fr4=yp4;
fr5=yp5;
yr1=y0+(h*z0)+(h^2/20160)*(4924*f0+8630*fr1-6088*fr2+3764*fr3-
1364*fr4+214*fr5);
m1=toc;
err1=abs((1-(exp(x(1))))-yr1);
fprintf('%2.7f      %3.18f      %3.18f      %1.6e \n', x(1),(1-
(exp(x(1)))),yr1, err1)
yr2=y0+(2*h*z0)+((h^2)/630)*(355*f0+1088*fr1-370*fr2+272*fr3-
101*fr4+16*fr5);
m2=toc;
err2=abs((1-(exp(x(2))))-yr2);
fprintf('%2.7f      %3.18f      %3.18f      %1.6e \n', x(2),(1-
(exp(x(2)))),yr2,err2)
yr3=y0+(3*h*z0)+((h^2)/10080)*(8856*f0+31509*fr1-648*fr2+7830*fr3-
2592*fr4+405*fr5);
m3=toc;
err3=abs((1-(exp(x(3))))-yr3);
fprintf('%2.7f      %3.18f      %3.18f      %1.6e \n', x(3),(1-
(exp(x(3)))),yr3,err3)
```

```
yr4=y0+(4*h*z0)+((h^2)/630)*(752*f0+2848*fr1+352*fr2+1216*fr3-
160*fr4+32*fr5);
m4=toc;
err4=abs((1-(exp(x(4))))-yr4);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(4),(1-
(exp(x(4)))),yr4,err4)
yr5=y0+(5*h*z0)+((h^2)/10080)*(15250*f0+59375*fr1+12500*fr2+31250*fr
3+6250*fr4+1375*fr5);
m5=toc;
err5=abs((1-(exp(x(5))))-yr5);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(5),(1-
(exp(x(5)))),yr5, err5)
x0=x(5); y0=yr5; z0=yp5;
end
```

Appendix D

MATLAB CODE OF SIX–STEP BLOCK METHOD FOR SOLVING SECOND ORDER ODES.

```
% programme to solve second order O .D. E with step length of six
clear
clc
% y'is represented by z
%syms x0 x y z g f0 y0 z0
%g (x,y,z)=x*z^2;
x0=0;  y0=1; z0=1/2; h=0.1;
disp ('x - value      exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=x0*z0^2;
for i=1:6;
x(i)=x0+i*h;
y(i)=y0+(i*h)*z0+((i*h)^2/2)*(x0*z0^2)+((i*h)^3/6)*(z0^2+2*x0^2*z0^3
)+((i*h)^4/24)*(6*x0*z0^3+6*x0^3*z0^4)+((i*h)^5/120)*(36*x0^2*z0^4+2
4*x0^4*z0^5+6*z0^6);  % taylor series expansion of yn+i
z(i)=z0+(i*h)*(x0*z0^2)+((i*h)^2/2)*(z0^2+2*x0^2*z0^3)+((i*h)^3/6)*(
6*x0*z0^3+6*x0^3*z0^4)+((i*h)^4/24)*(36*x0^2*z0^4+24*x0^4*z0^5+6*z0^
3);  %  [where z=y' is the derivative of the taylor series
expansion of yn+i
f(i)=x(i)*z(i)^2;
end
yp1=z0+(h/60480)*(19087*f0+65112*f(1)-46461*f(2)+37504*f(3)-
20211*f(4)+6312*f(5)-863*f(6));
yp2=z0+(h/3780)*(1139*f0+5640*f(1)+33*f(2)+1328*f(3)-
807*f(4)+264*f(5)-37*f(6));
yp3=z0+(h/2240)*(685*f0+3240*f(1)+1161*f(2)+2176*f(3)-
729*f(4)+216*f(5)-29*f(6));
yp4=z0+(h/945)*(286*f0+1392*f(1)+384*f(2)+1504*f(3)+174*f(4)+48*f(5)
-8*f(6));
yp5=z0+(h/12096)*(3715*f0+17400*f(1)+6375*f(2)+16000*f(3)+11625*f(4)
+5640*f(5)-275*f(6));
yp6=z0+(h/140)*(41*f0+216*f(1)+27*f(2)+272*f(3)+27*f(4)+216*f(5)+41*
f(6));
fr1=x(1)*yp1^2;
fr2=x(2)*yp2^2;
fr3=x(3)*yp3^2;
fr4=x(4)*yp4^2;
fr5=x(5)*yp5^2;
fr6=x(6)*yp6^2;
yr1=y0+(h*z0)+((h^2)/120960)*(28549*f0+57750*fr1-
51453*fr2+42484*fr3-23109*fr4+7254*fr5-995*fr6);
m1=toc;
err1=abs(1+1/2*log((2+x(1))/(2-x(1)))-yr1);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n', x(1),
1+1/2*log((2+x(1))/(2-x(1))),yr1, err1)
yr2=y0+(2*h*z0)+((h^2)/1890)*(1027*f0+3492*fr1-1680*fr2+1576*fr3-
873*fr4+276*fr5-38*fr6);
```

```
m2=toc;
err2=abs(1+1/2*log((2+x(2))/(2-x(2)))-yr2);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(2),1+1/2*log((2+x(2))/(2-x(2))) , yr2, err2)
yr3=y0+(3*h*z0)+((h^2)/4480)*(3795*f0+14850*fr1-2403*fr2+6300*fr3-
3267*fr4+1026*fr5-141*fr6);
m3=toc;
err3=abs(1+1/2*log((2+x(3))/(2-x(3)))-yr3);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n', x(3),
1+1/2*log((2+x(3))/(2-x(3))), yr3, err3)
yr4=y0+(4*h*z0)+((h^2)/945)*(1088*f0+4512*fr1-72*fr2+2624*fr3-
840*fr4+288*fr5-40*fr6);
m4=toc;
err4=abs(1+1/2*log((2+x(4))/(2-x(4)))-yr4);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(4),1+1/2*log((2+x(4))/(2-x(4))), yr4, err4)
yr5=y0+(5*h*z0)+((h^2)/24192)*(35225*f0+150750*fr1+9375*fr2+102500*f
r3-5625*fr4+11550*fr5-1375*fr6);
m5=toc;
err5=abs(1+1/2*log((2+x(5))/(2-x(5)))-yr5);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n', x(5),
1+1/2*log((2+x(5))/(2-x(5))), yr5, err5)
yr6=y0+(6*h*z0)+((h^2)/70)*(123*f0+540*fr1+54*fr2+408*fr3+27*fr4+108
*fr5);
m6=toc;
err6=abs(1+1/2*log((2+x(6))/(2-x(6)))-yr6);      ]
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n', x(6),
1+1/2*log((2+x(6))/(2-x(6))), yr6, err6)
x0=x(6); y0=yr6; z0=yp6;
end
```

Appendix E

MATLAB CODE OF SEVEN–STEP BLOCK METHOD FOR SOLVING SECOND ORDER ODES.

```
% programme to solve second order O .D. E with step length of seven
clear
clc
% y'is represented by z
%syms x0 x y z g f0 y0 z0
%g (x,y,z)=100y;
x0=0;  y0=1; z0=-10; h=0.01;
disp ('x - value      exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=100*y0;
for i=1:7;
x(i)=x0+i*h;
y(i)=(12500*y0*h^6*i^6)/9 + (250*z0*h^5*i^5)/3 + (1250*y0*h^4*i^4)/3
+ (50*z0*h^3*i^3)/3 + 50*y0*h^2*i^2 + z0*h*i + y0; taylor series
expansion of yn+i
z(i)=(25000*y0*h^5*i^5)/3 + (1250*z0*h^4*i^4)/3 +
(5000*y0*h^3*i^3)/3 + 50*z0*h^2*i^2 + 100*y0*h*i + z0; %  [where
z=y' is the derivative of the taylor series expansion of yn+i
f(i)=100*y(i);
end
yp1=z0+(h/120960)*(36799*f0+139849*f(1)-121797*f(2)+123133*f(3)-
88547*f(4)+41499*f(5)-11351*f(6)+1375*f(7));
yp2=z0+(h/18900)*(5535*f0+29320*f(1)-3195*f(2)+12240*f(3)-
9635*f(4)+4680*f(5)-1305*f(6)+160*f(7));
yp3=z0+(h/22400)*(6625*f0+33975*f(1)+6885*f(2)+29635*f(3)-
15165*f(4)+6885*f(5)-1865*f(6)+225*f(7));
yp4=z0+(h/945)*(278*f0+1448*f(1)+216*f(2)+1784*f(3)-
106*f(4)+216*f(5)-64*f(6)+8*f(7));
yp5=z0+(h/24192)*(7155*f0+36725*f(1)+6975*f(2)+41625*f(3)+13625*f(4)
+17055*f(5)-2475*f(6)+275*f(7));
yp6=z0+(h/140)*(41*f0+216*f(1)+27*f(2)+272*f(3)+27*f(4)+216*f(5)+41*
f(6));
yp7=z0+(h/17280)*(5257*f0+25039*f(1)+9261*f(2)+20923*f(3)+20923*f(4)
+9261*f(5)+25039*f(6)+5257*f(7));

yrr1=y0+(h*z0)+((h^2)/3628800)*(832346*f0+1901368*f(1)-
2050192*f(2)+2118860*f(3)-1537610*f(4)+724225*f(5)-
198718*f(6)+24124*f(7));
yrr2=y0+(2*h*z0)+((h^2)/28350)*(14939*f0+55642*f(1)-
34986*f(2)+39950*f(3)-29405*f(4)+13926*f(5)-3832*f(6)+466*f(7));
yrr3=y0+(3*h*z0)+((h^2)/604800)*(496773*f0+2113614*f(1)-
650997*f(2)+1394820*f(3)-985365*f(4)+465102*f(5)-
127899*f(6)+15552*f(7));
yrr4=y0+(4*h*z0)+((h^2)/14175)*(15824*f0+71152*f(1)-
11496*f(2)+56720*f(3)-29960*f(4)+14736*f(5)-4072*f(6)+496*f(7));
```

```
yrr5=y0+(5*h*z0)+((h^2)/72576)*(102425*f0+475000*f(1)-
40125*f(2)+421250*f(3)-130625*f(4)+102900*f(5)-
26875*f(6)+3250*f(7));
yrr6=y0+(6*h*z0)+((h^2)/350)*(597*f0+2826*f(1)-108*f(2)+2670*f(3)-
495*f(4)+918*f(5)-126*f(6)+18*f(7));
yrr7=y0+(7*h*z0)+((h^2)/43182)*(86506*f0+413600*f(1)+1200*f(2)+40000
0*f(3)-34000*f(4)+160800*f(5)+24400*f(6)+5453*f(7));

fr1=100*yrr1;
fr2=100*yrr2;
fr3=100*yrr3;
fr4=100*yrr4;
fr5=100*yrr5;
fr6=100*yrr6;
fr7=100*yrr7;
yr1=y0+(h*z0)+((h^2)/1344780)*(308455*f0+704619*fr1-
759771*fr2+785218*fr3-569816*fr4+268387*fr5-73642*fr6+8940*fr7);
m1=toc;
err1=abs(exp(-10*x(1))-yr1);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(1),exp(-
10*x(1)),yr1, err1)
yr2=y0+(2*h*z0)+((h^2)/28350)*(14939*f0+55642*fr1-
34986*fr2+39950*fr3-29405*fr4+13926*fr5-3832*fr6+466*fr7);
m2=toc;
err2=abs(exp(-10*x(2))-yr2);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(2),exp(-
10*x(2)),yr2, err2)
yr3=y0+(3*h*z0)+((h^2)/604800)*(496773*f0+2113614*fr1-
650997*fr2+1394820*fr3-985365*fr4+465102*fr5-127899*fr6+15552*fr7);
m3=toc;
err3=abs(exp(-10*x(3))-yr3);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(3),exp(-
10*x(3)),yr3, err3)
yr4=y0+(4*h*z0)+((h^2)/14175)*(15824*f0+71152*fr1-
11496*fr2+56720*fr3-29960*fr4+14736*fr5-4072*fr6+496*fr7);
m4=toc;
err4=abs(exp(-10*x(4))-yr4);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(4),exp(-
10*x(4)),yr4, err4)
yr5=y0+(5*h*z0)+((h^2)/72576)*(102425*f0+475000*fr1-
40125*fr2+421250*fr3-130625*fr4+102900*fr5-26875*fr6+3250*fr7);
m5=toc;
err5=abs(exp(-10*x(5))-yr5);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(5),exp(-
10*x(5)),yr5, err5)
yr6=y0+(6*h*z0)+((h^2)/350)*(597*f0+2826*fr1-108*fr2+2670*fr3-
495*fr4+918*fr5-126*fr6+18*fr7);
m6=toc;
err6=abs(exp(-10*x(6))-yr6);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(6),exp(-
10*x(6)),yr6, err6)
yr7=y0+(7*h*z0)+((h^2)/43182)*(86506*f0+413600*fr1+1200*fr2+400000*f
r3-34000*fr4+160800*fr5+24400*fr6+5453*fr7);
m7=toc;
err7=abs(exp(-10*x(7))-yr7);
```

```
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n', x(7),exp(-
10*x(7)),yr7, err7)
x0=x(7); y0=yr7; z0=yp7;
end
```

Appendix F

MATLAB CODE OF EIGHT–STEP BLOCK METHOD FOR SOLVING SECOND ORDER ODES.

```
% programme to solve second order O .D. E with step length of eight
clear
clc
% y'is represented by z
%syms x0 x y z g f0 y0 z0
%g (x,y,z)=y'+2exp(x)(x+1);
x0=0;  y0=1; z0=1; h=0.01;
disp ('x - value     exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=z0+2*exp(x0)*(x0+1);
for i=1:8;
x(i)=x0+i*h;
y(i)=y0 + (h^3*i^3*(z0 + 2*exp(x0) + 4*exp(x0)*(x0 + 1)))/6 +
(h^4*i^4*(z0 + 6*exp(x0) + 6*exp(x0)*(x0 + 1)))/24 + (h^5*i^5*(z0 +
12*exp(x0) + 8*exp(x0)*(x0 + 1)))/120 + (h^6*i^6*(z0 + 12*exp(x0) +
8*exp(x0)*(x0 + 1)))/720 + h*i*z0 + (h^2*i^2*(z0 + 2*exp(x0)*(x0 +
1)))/2;% taylor series expansion of yn+i
z(i)=z0 + (h^2*i^2*(z0 + 2*exp(x0) + 4*exp(x0)*(x0 + 1)))/2 +
(h^3*i^3*(z0 + 6*exp(x0) + 6*exp(x0)*(x0 + 1)))/6 + (h^4*i^4*(z0 +
12*exp(x0) + 8*exp(x0)*(x0 + 1)))/24 + (h^5*i^5*(z0 + 12*exp(x0) +
8*exp(x0)*(x0 + 1)))/120 + h*i*(z0 + 2*exp(x0)*(x0 + 1%  [where z=y'
is the derivative of the taylor series expansion of yn+i
f(i)=z(i)+2*exp(x(i))*(x(i)+1);
end
yp1=z0+(h/1069200)*(315273*f0+1316197*f(1)-
1356711*f(2)+1648632*f(3)-1482974*f(4)+927046*f(5)-
380447*f(6)+92186*f(7)-10004*f(8));
yp2=z0+(h/113400)*(32377*f0+182584*f(1)-42494*f(2)+120088*f(3)-
116120*f(4)+74728*f(5)-31154*f(6)+7624*f(7)-833*f(8));
yp3=z0+(h/44800)*(12881*f0+70902*f(1)+3438*f(2)+79934*f(3)-
56160*f(4)+34434*f(5)-14062*f(6)+3402*f(7)-369*f(8));
yp4=z0+(h/28350)*(8126*f0+45152*f(1)+488*f(2)+65504*f(3)-
18160*f(4)+18464*f(5)-7912*f(6)+1952*f(7)-214*f(8));
yp5=z0+(h/145152)*(41705*f0+230150*f(1)+7550*f(2)+318350*f(3)-
4000*f(4)+170930*f(5)-49150*f(6)+11450*f(7)-1225*f(8));
yp6=z0+(h/1400)*(401*f0+2232*f(1)+18*f(2)+3224*f(3)-
360*f(4)+2664*f(5)+158*f(6)+72*f(7)-9*f(8));
yp7=z0+(h/518400)*(149527*f0+816634*f(1)+48706*f(2)+1085937*f(3)+548
80*f(4)+736078*f(5)+522046*f(6)+223174*f(7)-8183*f(8));
yp8=z0+(h/28350)*(7912*f0+47104*f(1)-7424*f(2)+83968*f(3)-
36320*f(4)+83968*f(5)-7424*f(6)+47104*f(7)+7912*f(8));
fr1=yp1+2*exp(x(1))*(x(1)+1);
fr2=yp2+2*exp(x(2))*(x(2)+1);
fr3=yp3+2*exp(x(3))*(x(3)+1);
fr4=yp4+2*exp(x(4))*(x(4)+1);
fr5=yp5+2*exp(x(5))*(x(5)+1);
fr6=yp6+2*exp(x(6))*(x(6)+1);
```

```
fr7=yp7+2*exp(x(7))*(x(7)+1);
fr8=yp8+2*exp(x(8))*(x(8)+1);
yr1=y0+(h*z0)+((h^2)/7257600)*(1624505*f0+4124232*fr1-
5225624*fr2+6488192*fr3-5888310*fr4+3698920*fr5-
1522672*fr6+369744*fr7-40187*fr8);
m1=toc;
err1=abs((((x(1))^2+1)*exp(x(1)))-yr1);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(1),
(((x(1))^2+1)*exp(x(1))),yr1, err1)
yr2=y0+(2*h*z0)+((h^2)/113400)*(58193*f0+235072*fr1-
183708*fr2+247328*fr3-227030*fr4+143232*fr5-59092*fr6+14368*fr7-
1563*fr8);
m2=toc;
err2=abs((((x(2))^2+1)*exp(x(2)))-yr2);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(2),(((x(2))^2+1)*exp(x(2))),yr2,err2)
yr3=y0+(3*h*z0)+((h^2)/89600)*(71661*f0+328608*fr1-
150624*fr2+315000*fr3-281430*fr4+177264*fr5-73128*fr6+17784*fr7-
1935*fr8);
m3=toc;
err3=abs((((x(3))^2+1)*exp(x(3)))-yr3);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(3),
(((x(3))^2+1)*exp(x(3))), yr3, err3)
yr4=y0+(4*h*z0)+((h^2)/28350)*(30812*f0+148992*fr1-
46400*fr2+160256*fr3-118440*fr4+76288*fr5-31552*fr6+7680*fr7-
836*fr8);
m4=toc;
err4=abs((((x(4))^2+1)*exp(x(4)))-yr4);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(4),(((x(4))^2+1)*exp(x(4))),yr4,err4)
yr5=y0+(5*h*z0)+((h^2)/290304)*(398825*f0+1987000*fr1-
465000*fr2+2294000*fr3-1283750*fr4+1020600*fr5-
412000*fr6+100000*fr7-10875*fr8);
m5=toc;
err5=abs((((x(5))^2+1)*exp(x(5)))-yr5);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(5),(((x(5))^2+1)*exp(x(5))), yr5, err5)
yr6=y0+(6*h*z0)+((h^2)/1400)*(2325*f0+11808*fr1-2196*fr2+14208*fr3-
6390*fr4+7200*fr5-2268*fr6+576*fr7-63*fr8);
m6=toc;
err6=abs((((x(6))^2+1)*exp(x(6)))-yr6);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(6),(((x(6))^2+1)*exp(x(6))), yr6, err6)
yr7=y0+(7*h*z0)+((7*h^2)/1036800)*(288533*f0+1484112*fr1-
225008*fr2+1830248*fr3-689430*fr4+1009792*fr5-145432*fr6+84168*fr7-
8183*fr8);
m7=toc;
err7=abs((((x(7))^2+1)*exp(x(7)))-yr7);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(7),(((x(7))^2+1)*exp(x(7))), yr7, err7)
yr8=y0+(8*h*z0)+((h^2)/28350)*(63296*f0+329728*fr1-
44544*fr2+419840*fr3-145280*fr4+251904*fr5-14848*fr6+47104*fr7);
m8=toc;
err8=abs((((x(8))^2+1)*exp(x(8)))-yr8);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(8),(((x(8))^2+1)*exp(x(8))), yr8, err8)
```

```
x0=x(8); y0=yr8; z0=yp8;
end
```

Appendix G

MATLAB CODE OF FOUR–STEP BLOCK METHOD FOR SOLVING THIRD ORDER ODES.

```
% programme to solve third order O .D. E with step length of four
clear
clc
% y'is represented by z
%y" is represented by v
%syms x0 x y z g f0 y0 z0 v0
%g (x,y,z)=v-z+y;
x0=0;  y0=1; z0=0; v0=-1; h=0.01;
disp ('x - value      exact - solution       computed - solution
error')
tic
for j=0:h:1;
f0=v0-z0+y0;
for i=1:4;
x(i)=x0+i*h;
y(i)=y0 + (h^3*i^3*(v0 + y0 - z0))/6 + (h^2*i^2*v0)/2 +
(h^4*i^4*y0)/24 + h*i*z0;
z(i)=z0 + (h^2*i^2*(v0 + y0 - z0))/2 + (h^3*i^3*y0)/6 + h*i*v0;
v(i)=(y0*h^2*i^2)/2 + (v0 + y0 - z0)*h*i + v0;
f(i)=v(i)-z(i)+y(i);
end
yj1=y0+(h*z0)+((1/2)*h^2*v0)+((h^3)/10080)*(1017*f0+1070*f(1)-
618*f(2)+258*f(3)-47*f(4));
yj2=y0+(2*h*z0)+(2*h^2*v0)+((h^3)/630)*(331*f0+664*f(1)-
240*f(2)+104*f(3)-19*f(4));
yj3=y0+(3*h*z0)+((9/2)*h^2*v0)+((h^3)/1120)*(1431*f0+3726*f(1)-
486*f(2)+450*f(3)-81*f(4));
yj4=y0+(4*h*z0)+(8*h^2*v0)+((h^3)/2205)*(5208*f0+15232*f(1)+672*f(2)
+2688*f(3)-280*f(4));

yp1=z0+(h*v0)+((h^2)/1440)*(367*f0+540*f(1)-282*f(2)+116*f(3)-
21*f(4));
yp2=z0+(2*h*v0)+((h^2)/270)*(159*f0+432*f(1)-90*f(2)+48*f(3)-
9*f(4));
yp3=z0+(3*h*v0)+((h^2)/160)*(147*f0+468*f(1)+54*f(2)+60*f(3)-
9*f(4));
yp4=z0+(4*h*v0)+((h^2)/45)*(56*f0+192*f(1)+48*f(2)+64*f(3));

ypp1=v0+(h/720)*(251*f0+646*f(1)-264*f(2)+106*f(3)-19*f(4));
ypp2=v0+(h/90)*(29*f0+124*f(1)+24*f(2)+4*f(3)-f(4));
ypp3=v0+(h/80)*(27*f0+102*f(1)+72*f(2)+42*f(3)-3*f(4));
ypp4=v0+(h/45)*(14*f0+64*f(1)+24*f(2)+64*f(3)+14*f(4));
fr1=ypp1-yp1+yj1;
fr2=ypp2-yp2+yj2;
fr3=ypp3-yp3+yj3;
fr4=ypp4-yp4+yj4;
yr1=y0+(h*z0)+((1/2)*h^2*v0)+((h^3)/30240)*(3051*f0+3210*fr1-
1854*fr2+774*fr3-141*fr4);
```

```
m1=toc;
err1=abs(cos(x(1))-yr1);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(1),cos(x(1)),yr1, err1)
yr2=y0+(2*h*z0)+(2*h^2*v0)+((h^3)/1890)*(993*f0+1992*fr1-
720*fr2+312*fr3-57*fr4);
m2=toc;
err2=abs(cos(x(2))-yr2);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(2),cos(x(2)),yr2, err2)
yr3=y0+(3*h*z0)+((9/2)*h^2*v0)+((h^3)/3360)*(4293*f0+11178*fr1-
1458*fr2+1350*fr3-243*fr4);
m3=toc;
err3=abs(cos(x(3))-yr3);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(3),cos(x(3)),yr3, err3)
yr4=y0+(4*h*z0)+(8*h^2*v0)+((h^3)/2205)*(5208*f0+15232*fr1+672*fr2+2
688*fr3-280*fr4);
m4=toc;
err4=abs(cos(x(4))-yr4);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(4),cos(x(4)),yr4, err4)
x0=x(4);y0=yr4;z0=yp4;v0=ypp4;
end
```

Appendix H

MATLAB CODE OF FIVE–STEP BLOCK METHOD FOR SOLVING THIRD ORDER ODES.

```
% programme to solve third order O .D. E with step length of five
clear
clc
% y'is represented by z
%syms x0 x y z g f0 y0 z0
%g (x,y,z)=x-4z;
x0=0;  y0=0; z0=0; v0=1; h=0.1;
disp ('x - value      exact - solution       computed - solution
error')
tic
for j=0:h:1;
f0=x0-4*z0;
for i=1:5;
x(i)=x0+i*h;
y(i)=y0-(h^4*i^4*(4*v0 - 1))/24 + (h^2*i^2*v0)/2 + (h^3*i^3*(x0 -
4*z0))/6 + h*i*z0;
z(i)=z0-(h^3*i^3*(4*.v0 - 1))/6 + (h^2*i^2*(x0 - 4*z0))/2 + h*i*v0;
v(i)=v0-(h^2*i^2*(4*v0 - 1))/2 + h*i*(x0 - 4*z0);
f(i)=x(i)-(4*z(i));
end
ypp1=v0+(h/1440)*(475*f0+1427*f(1)-798*f(2)+482*f(3)-
173*f(4)+27*f(5));
ypp2=v0+(h/90)*(28*f0+129*f(1)+14*f(2)+14*f(3)-6*f(4)+f(5));
ypp3=v0+(h/160)*(51*f0+219*f(1)+114*f(2)+114*f(3)-21*f(4)+3*f(5));
ypp4=v0+(h/45)*(14*f0+64*f(1)+24*f(2)+64*f(3)+14*f(4));
ypp5=v0+(h/288)*(95*f0+375*f(1)+250*f(2)+250*f(3)+375*f(4)+95*f(5));
yp1=z0+(h*v0)+(h^2/20160)*(4924*f0+8630*f(1)-6088*f(2)+3764*f(3)-
1364*f(4)+214*f(5));
yp2=z0+(2*h*v0)+((h^2)/630)*(355*f0+1088*f(1)-370*f(2)+272*f(3)-
101*f(4)+16*f(5)).;
yp3=z0+(3*h*v0)+((h^2)/10080)*(8856*f0+31509*f(1)-
648*f(2)+7830*f(3)-2592*f(4)+405*f(5));
yp4=z0+(4*h*v0)+((h^2)/630)*(752*f0+2848*f(1)+352*f(2)+1216*f(3)-
160*f(4)+32*f(5));
yp5=z0+(5*h*v0)+((h^2)/10080)*(15250*f0+59375*f(1)+12500*f(2)+31250*
f(3)+6250*f(4)+1375*f(5));
fr1=x(1)-(4*yp1);
fr2=x(2)-(4*yp2);
fr3=x(3)-(4*yp3);
fr4=x(4)-(4*yp4);
fr5=x(5)-(4*yp5);
yr1=y0+(h*z0)+(1/2*h^2*v0)+(h^3/241920)*(23574*f0+29850*fr1-
23172*fr2+14532*fr3-5298*fr4+834*fr5);
m1=toc;
err1=abs(((3/16)*(1-cos(2*x(1))))+((1/8)*x(1)^2)-yr1);
fprintf('%2.7f      %3.18f      %3.18f      %1.6e \n',
x(1),((3/16)*(1-cos(2*x(1))))+((1/8)*x(1)^2),yr1, err1)
yr2=y0+(2*h*z0)+(2*h^2*v0)+(h^3/1890)*(951*f0+2202*fr1-
1140*fr2+732*fr3-267*fr4+42*fr5);
```

```
m2=toc;
err2=abs(((3/16)*(1-cos(2*x(2))))+((1/8)*x(2)^2)-yr2);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(2),((3/16)*(1-cos(2*x(2))))+((1/8)*x(2)^2), yr2, err2)
yr3=y0+(3*h*z0)+(9/2*h^2*v0)+(h^3/13440)*(16443*f0+48357*fr1-
13122*fr2+12690*fr3-4617*fr4+729*fr5);
m3=toc;
err3=abs(((3/16)*(1-cos(2*x(3))))+((1/8)*x(3)^2)-yr3);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(3),((3/16)*(1-cos(2*x(3))))+((1/8)*x(3)^2),yr3, err3)
yr4=y0+(4*h*z0)+(8*h^2*v0)+(h^3/945)*(2136*f0+7008*fr1-
672*fr2+2112*fr3-600*fr4+96*fr5);
m4=toc;
err4=abs(((3/16)*(1-cos(2*x(4))))+((1/8)*x(4)^2)-yr4);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',x(4),((3/16)*(1-
cos(2*x(4))))+((1/8)*x(4)^2),yr4, err4)
yr5=y0+(5*h*z0)+(25/2*h^2*v0)+(h^3/72576)*(262125*f0+916875*fr1+1125
0*fr2+348750*fr3-39375*fr4+12375*fr5);
m5=toc;
err5=abs(((3/16)*(1-cos(2*x(5))))+((1/8)*x(5)^2)-yr5.);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',x(5),((3/16)*(1-
cos(2*x(5))))+((1/8)*x(5)^2),yr5, err5)
x0=x(5); y0=yr5; z0=yp5; v0=ypp5;
end
```

Appendix I

MATLAB CODE OF SIX–STEP BLOCK METHOD FOR SOLVING THIRD ORDER ODES.

```
% programme to solve third order O .D. E with step length of six
clear
clc
% y'is represented by z
%syms x0 x y z g f0 y0 z0
%g (x,y,z)=-y;
x0=0;  y0=1; z0=-1; v0=1; h=0.1;
disp ('x - value      exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=-y0;
for i=1:6;
x(i)=x0+i*h;
y(i)=- (v0*h^5*i^5)/120 - (z0*h^4*i^4)/24 - (y0*h^3*i^3)/6 +
(v0*h^2*i^2)/2 + z0*h*i + y0;
z(i)=z0-(h^4*i^4*v0)/24 - (h^2*i^2*y0)/2 - (h^3*i^3*z0)/6 + h*i*v0;
v(i)=v0-(h^3*i^3*v0)/6 - (h^2*i^2*z0)/2 - h*i*y0;
f(i)=-y(i);
end
yj1=y0+(h*z0)+((1/2)*h^2*v0)+((h^3)/3628800)*(343801*f0+506604*f(1)-
494715*f(2)+414160*f(3)-226605*f(4)+71364*f(5)-9809*f(6));
yj2=y0+(2*h*z0)+(2*h^2*v0)+((h^3)/28350)*(13774*f0+35976*f(1)-
24465*f(2)+20800*f(3)-11370*f(4)+3576*f(5)-491*f(6));
yj3=y0+(3*h*z0)+((9/2)*h^2*v0)+((h^3)/44800)*(52893*f0+172692*f(1)-
72495*f(2)+80640*f(3)-44145*f(4)+13932*f(5)-1917*f(6));
yj4=y0+(4*h*z0)+(8*h^2*v0)+((h^3)/28350)*(61808*f0+223872*f(1)-
54240*f(2)+108800*f(3)-52080*f(4)+16512*f(5)-2272*f(6));
yj5=y0+(5*h*z0)+((25/2)*h^2*v0)+((h^3)/145152)*(505625*f0+1945500*f(
1)-256875*f(2)+1070000*f(3)-358125*f(4)+136500*f(5)-18625*f(6));
yj6=y0+(6*h*z0)+(18*h^2*v0)+((h^3)/700)*(3564*f0+14256*f(1)-
810*f(2)+8640*f(3)-1620*f(4)+1296*f(5)-126*f(6));

yp1=z0+(h*v0)+((h^2)/120960)*(28549*f0+57750*f(1)-
51453*f(2)+42484*f(3)-23109*f(4)+7254*f(5)-995*f(6));
yp2=z0+(2*h*v0)+((h^2)/120960)*(65728*f0+223488*f(1)-
107520*f(2)+100864*f(3)-55872*f(4)+17664*f(5)-2432*f(6));
yp3=z0+(3*h*v0)+((h^2)/4480)*(3795*f0+14850*f(1)-
2403*f(2)+6300*f(3)-3267*f(4)+1026*f(5)-141*f(6));
yp4=z0+(4*h*v0)+((h^2)/945)*(1088*f0+4512*f(1)-72*f(2)+2624*f(3)-
840*f(4)+288*f(5)-40*f(6));
yp5=z0+(5*h*v0)+((h^2)/120960)*(176125*f0+753751*f(1)+46875*f(2)+512
500*f(3)-28125*f(4)+57750*f(5)-6875*f(6));
yp6=z0+(6*h*v0)+((h^2)/140)*(246*f0+1080*f(1)+108*f(2)+816*f(3)+54*f
(4)+216*f(5));

ypp1=v0+(h/60480)*(19087*f0+65112*f(1)-46461*f(2)+37504*f(3)-
20211*f(4)+6312*f(5)-863*f(6));
```

```
ypp2=v0+(h/3780)*(1139*f0+5640*f(1)+33*f(2)+1328*f(3)-
807*f(4)+264*f(5)-37*f(6));
ypp3=v0+(h/2240)*(685*f0+3240*f(1)+1161*f(2)+2176*f(3)-
729*f(4)+216*f(5)-29*f(6));
ypp4=v0+(h/945)*(286*f0+1392*f(1)+384*f(2)+1504*f(3)+174*f(4)+48*f(5
)-8*f(6));
ypp5=v0+(h/12096)*(3715*f0+17400*f(1)+6375*f(2)+16000*f(3)+11625*f(4
)+5640*f(5)-275*f(6));
ypp6=v0+(h/140)*(41*f0+216*f(1)+27*f(2)+272*f(3)+27*f(4)+216*f(5)+41
*f(6));
fr1=-yj1;
fr2=-yj2;
fr3=-yj3;
fr4=-yj4;
fr5=-yj5;
fr6=-yj6;
yr1=y0+(h*z0)+((1/2)*h^2*v0)+((h^3)/3628800)*(343801*f0+506604*fr1-
494715*fr2+414160*fr3-226605*fr4+71364*fr5-9809*fr6);
m1=toc;
err1=abs((exp(-x(1)))-yr1);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(1),(exp(-
x(1))),y(1), err1)
yr2=y0+(2*h*z0)+(2*h^2*v0)+((h^3)/28350)*(13774*f0+35976*fr1-
24465*fr2+20800*fr3-11370*fr4+3576*fr5-491*fr6);
m2=toc;
err2=abs((exp(-x(2)))-yr2);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(2),(exp(-
x(2))),yr2, err2)
yr3=y0+(3*h*z0)+((9/2)*h^2*v0)+((h^3)/44800)*(52893*f0+172692*fr1-
72495*fr2+80640*fr3-44145*fr4+13932*fr5-1917*fr6);
m3=toc;
err3=abs((exp(-x(3)))-yr3);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n', x(3),(exp(-
x(3))),yr3, err3)
yr4=y0+(4*h*z0)+(8*h^2*v0)+((h^3)/28350)*(61808*f0+223872*fr1-
54240*fr2+108800*fr3-52080*fr4+16512*fr5-2272*fr6);
m4=toc;
err4=abs((exp(-x(4)))-yr4);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',x(4),(exp(-
x(4))),yr4, err4)
yr5=y0+(5*h*z0)+((25/2)*h^2*v0)+((h^3)/145152)*(505625*f0+1945500*fr
1-256875*fr2+1070000*fr3-358125*fr4+136500*fr5-18625*fr6);
m5=toc;
err5=abs((exp(-x(5)))-yr5);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',x(5),(exp(-
x(5))),yr5, err5)
yr6=y0+(6*h*z0)+(18*h^2*v0)+((h^3)/700)*(3564*f0+14256*fr1-
810*fr2+8640*fr3-1620*fr4+1296*fr5-126*fr6);
m6=toc;
err6=abs((exp(-x(6)))-yr6);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',x(6),(exp(-
x(6))),yr6, err6)
x0=x(6); y0=yr6; z0=yp6; v0=ypp6;
end
```

Appendix J

MATLAB CODE OF SEVEN–STEP BLOCK METHOD FOR SOLVING THIRD ORDER ODES.

```
% programme to solve third order O .D. E with step length of seven
clear
clc
% y'is represented by z
%syms x0 x y z g f0 y0 z0
%g (x,y,z)=2y"-4;
x0=0;  y0=1; z0=2; v0=6; h=0.01;
disp ('x - value     exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=2*v0-4;
for i=1:7;
x(i)=x0+i*h;
y(i)=y0 + (h^3*i^3*(2*v0 - 4))/6 + (h^4*i^4*(4*v0 - 8))/24 +
(h^5*i^5*(8*v0 - 16))/120 + (h^2*i^2*v0)/2 + h*i*z0;
z(i)=z0 + (h^2*i^2*(2*v0 - 4))/2 + (h^3*i^3*(4*v0 - 8))/6 +
(h^4*i^4*(8*v0 - 16))/24 + h*i*v0;
v(i)=v0 + (h^2*i^2*(4*v0 - 8))/2 + (h^3*i^3*(8*v0 - 16))/6 +
h*i*(2*v0 - 4);
f(i)=2*v(i)-4;
end
yj1=y0+(h*z0)+(1/2*h^2*v0)+((h^3)/3628800)*(335799*f0+562618*f(1)-
662757*f(2)+694230*f(3)-506675*f(4)+239406*f(5)-
65823*f(6)+8002*f(7));
yj2=y0+(2*h*z0)+(2*h^2*v0)+((h^3)/28350)*(13376*f0+38762*f(1)-
32823*f(2)+34730*f(3)-25300*f(4)+11934*f(5)-3277*f(6)+398*f(7));
yj3=y0+(3*h*z0)+((9/2)*h^2*v0)+((h^3)/44800)*(51327*f0+183654*f(1)-
105381*f(2)+135450*f(3)-98955*f(4)+46818*f(5)-12879*f(6)+1565*f(7));
yj4=y0+(4*h*z0)+(8*h^2*v0)+((h^3)/14175)*(29976*f0+118432*f(1)-
46608*f(2)+86880*f(3)-58520*f(4)+27744*f(5)-7632*f(6)+928*f(7));
yj5=y0+(5*h*z0)+((25/2)*h^2*v0)+((5*h^3)/145152)*(98075*f0+410450*f(
1)-115425*f(2)+320750*f(3)-178375*f(4)+91350*f(5)-
25075*f(6)+3050*f(7));
yj6=y0+(6*h*z0)+(18*h^2*v0)+((h^3)/1400)*(6912*f0+30024*f(1)-
6156*f(2)+24840*f(3)-10800*f(4)+7128*f(5)-1764*f(6)+216*f(7));
yj7=y0+(7*h*z0)+((49/2)*h^2*v0)+((7*h^3)/518400)*(502887*f0+2242534*
f(1)-338541*f(2)+1944810*f(3)-660275*f(4)+619458*f(5)-
93639*f(6)+16366*f(7));

ypp1=v0+(h/120960)*(36799*f0+139849*f(1)-121797*f(2)+123133*f(3)-
88547*f(4)+41499*f(5)-11351*f(6)+1375*f(7));
ypp2=v0+(h/18900)*(5535*f0+29320*f(1)-3195*f(2)+12240*f(3)-
9635*f(4)+4680*f(5)-1305*f(6)+160*f(7));
ypp3=v0+(h/22400)*(6625*f0+33975*f(1)+6885*f(2)+29635*f(3)-
15165*f(4)+6885*f(5)-1865*f(6)+225*f(7));
ypp4=v0+(h/945)*(278*f0+1448*f(1)+216*f(2)+1784*f(3)-
106*f(4)+216*f(5)-64*f(6)+8*f(7));
```

```
ypp5=v0+(h/24192)*(7155*f0+36725*f(1)+6975*f(2)+41625*f(3)+13625*f(4
)+17055*f(5)-2475*f(6)+275*f(7));
ypp6=v0+(h/140)*(41*f0+216*f(1)+27*f(2)+272*f(3)+27*f(4)+216*f(5)+41
*f(6));
ypp7=v0+(h/17280)*(5257*f0+25039*f(1)+9261*f(2)+20923*f(3)+20923*f(4
)+9261*f(5)+25039*f(6)+5257*f(7));

yp1=z0+(h*v0)+((h^2)/3628800)*(832346*f0+1901368*f(1)-
2050192*f(2)+2118860*f(3)-1537610*f(4)+724225*f(5)-
198718*f(6)+24124*f(7));
yp2=z0+(2*h*v0)+((h^2)/28350)*(14939*f0+55642*f(1)-
34986*f(2)+39950*f(3)-29405*f(4)+13926*f(5)-3832*f(6)+466*f(7));
yp3=z0+(3*h*v0)+((h^2)/604800)*(496773*f0+2113614*f(1)-
650997*f(2)+1394820*f(3)-985365*f(4)+465102*f(5)-
127899*f(6)+15552*f(7));
yp4=z0+(4*h*v0)+((h^2)/14175)*(15824*f0+71152*f(1)-
11496*f(2)+56720*f(3)-29960*f(4)+14736*f(5)-4072*f(6)+496*f(7));
yp5=z0+(5*h*v0)+((h^2)/72576)*(102425*f0+475000*f(1)-
40125*f(2)+421250*f(3)-130625*f(4)+102900*f(5)-
26875*f(6)+3250*f(7));
yp6=z0+(6*h*v0)+((h^2)/350)*(597*f0+2826*f(1)-108*f(2)+2670*f(3)-
495*f(4)+918*f(5)-126*f(6)+18*f(7));
yp7=z0+(7*h*v0)+((h^2)/259200)*(519253*f0+2482634*f(1)+7203*f(2)+240
1000*f(3)-204085*f(4)+965202*f(5)+146461*f(6)+32732*f(7));
fr1=2*ypp1-4;
fr2=2*ypp2-4;
fr3=2*ypp3-4;
fr4=2*ypp4-4;
fr5=2*ypp5-4;
fr6=2*ypp6-4;
fr7=2*ypp7-4;
yr1=y0+(h*z0)+(1/2*h^2*v0)+((h^3)/3628800)*(335799*f0+562618*fr1-
662757*fr2+694230*fr3-506675*fr4+239406*fr5-65823*fr6+8002*fr7);
err1=abs((x(1)^2+exp(2*x(1)))-yr1);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(1),(x(1)^2+exp(2*x(1))),yr1, err1)
yr2=y0+(2*h*z0)+(2*h^2*v0)+((h^3)/28350)*(13376*f0+38762*fr1-
32823*fr2+34730*fr3-25300*fr4+11934*fr5-3277*fr6+398*fr7);
m2=toc;
err2=abs((x(2)^2+exp(2*x(2)))-yr2);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(2),(x(2)^2+exp(2*x(2))),yr2, err2)
yr3=y0+(3*h*z0)+((9/2)*h^2*v0)+((h^3)/44800)*(51327*f0+183654*fr1-
105381*fr2+135450*fr3-98955*fr4+46818*fr5-12879*fr6+1566*fr7);
m3=toc;
err3=abs((x(3)^2+exp(2*x(3)))-yr3);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(3),(x(3)^2+exp(2*x(3))),yr3, err3)
yr4=y0+(4*h*z0)+(8*h^2*v0)+((h^3)/14175)*(29976*f0+118432*fr1-
46608*fr2+86880*fr3-58520*fr4+27744*fr5-7632*fr6+928*fr7);
m4=toc;
err4=abs((x(4)^2+exp(2*x(4)))-yr4);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e
\n',x(4),(x(4)^2+exp(2*x(4))),yr4, err4)
yr5=y0+(5*h*z0)+((25/2)*h^2*v0)+((5*h^3)/145152)*(98075*f0+410450*fr
1-115425*fr2+320750*fr3-178375*fr4+91350*fr5-25075*fr6+3050*fr7);
```

```
m5=toc;
err5=abs((x(5)^2+exp(2*x(5)))-yr5);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e
\n',x(5),(x(5)^2+exp(2*x(5))),yr5, err5)
yr6=y0+(6*h*z0)+(18*h^2*v0)+((h^3)/1400)*(6912*f0+30024*fr1-
6156*fr2+24840*fr3-10800*fr4+7128*fr5-1764*fr6+216*fr7);
m6=toc;
err6=abs((x(6)^2+exp(2*x(6)))-yr6);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e
\n',x(6),(x(6)^2+exp(2*x(6))),yr6, err6)
yr7=y0+(7*h*z0)+((49/2)*h^2*v0)+((7*h^3)/518400)*(502887*f0+2242534*
fr1-338541*fr2+1944810*fr3-660275*fr4+619458*fr5-
93639*fr6+16366*fr7);
m7=toc;
err7=abs((x(7)^2+exp(2*x(7)))-yr7);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e
\n',x(7),(x(7)^2+exp(2*x(7))),yr7, err7)
x0=x(7); y0=yr7; z0=yp7; v0=ypp7;
end
```

Appendix K

MATLAB CODE OF EIGHT–STEP BLOCK METHOD FOR SOLVING THIRD ORDER ODES.

```
% programme to solve third order O .D. E with step length of Eight
clear
clc
% y'is represented by z
%syms x0 x y z g f0 y0 z0
%g (x,y,z)=y"'=8y'-3y-4exp(x);
x0=0;  y0=2; z0=-2; v0=10; h=0.001;
disp ('x - value     exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=8*z0-3*y0-4*exp(x0);
for i=1:8;
x(i)=x0+i*h;
y(i)=y0 - (h^5*i^5*(3*v0 + 24*y0 - 64*z0 + 36*exp(x0)))/120 -
(h^4*i^4*(3*z0 - 8*v0 + 4*exp(x0)))/24 - (h^3*i^3*(3*y0 - 8*z0 +
4*exp(x0)))/6 + (h^2*i^2*v0)/2 + h*i*z0;
z(i)=z0 - (h^4*i^4*(3*v0 + 24*y0 - 64*z0 + 36*exp(x0)))/24 -
(h^3*i^3*(3*z0 - 8*v0 + 4*exp(x0)))/6 - (h^2*i^2*(3*y0 - 8*z0 +
4*exp(x0)))/2 + h*i*v0;
v(i)=v0 - (h^3*i^3*(3*v0 + 24*y0 - 64*z0 + 36*exp(x0)))/6 -
(h^2*i^2*(3*z0 - 8*v0 + 4*exp(x0)))/2 - h*i*(3*y0 - 8*z0 +
4*exp(x0));
f(i)=8*z(i)-3*y(i)-4*exp(x(i));
end
yp1=v0+(h/1069200)*(315273*f0+1316197*f(1)-
1356711*f(2)+1648632*f(3)-1482974*f(4)+927046*f(5)-
380447*f(6)+92186*f(7)-10004*f(8));
yp2=v0+(h/113400)*(32377*f0+182584*f(1)-42494*f(2)+120088*f(3)-
116120*f(4)+74728*f(5)-31154*f(6)+7624*f(7)-833*f(8));
yp3=v0+(h/44800)*(12881*f0+70902*f(1)+3438*f(2)+79934*f(3)-
56160*f(4)+34434*f(5)-14062*f(6)+3402*f(7)-369*f(8));
yp4=v0+(h/28350)*(8126*f0+45152*f(1)+488*f(2)+65504*f(3)-
18160*f(4)+18464*f(5)-7912*f(6)+1952*f(7)-214*f(8));
yp5=v0+(h/145152)*(41705*f0+230150*f(1)+7550*f(2)+318350*f(3)-
4000*f(4)+170930*f(5)-49150*f(6)+11450*f(7)-1225*f(8));
yp6=v0+(h/1400)*(401*f0+2232*f(1)+18*f(2)+3224*f(3)-
360*f(4)+2664*f(5)+158*f(6)+72*f(7)-9*f(8));
yp7=v0+(h/518400)*(149527*f0+816634*f(1)+48706*f(2)+1085937*f(3)+548
80*f(4)+736078*f(5)+522046*f(6)+223174*f(7)-8183*f(8));
yp8=v0+(h/28350)*(7912*f0+47104*f(1)-7424*f(2)+83968*f(3)-
36320*f(4)+83968*f(5)-7424*f(6)+47104*f(7)+7912*f(8));

yj1=z0+(h*v0)+(h^2/(7257600))*(1624505*f0+4124232*f(1)-
5225624*f(2)+6488192*f(3)-5888310*f(4)+3698920*f(5)-
15522672*f(6)+369744*f(7)-40187*f(8));
yj2=z0+(2*h*v0)+(h^2/(113400))*(58193*f0+235072*f(1)-
183708*f(2)+247328*f(3)-227030*f(4)+143232*f(5)-
59092*f(6)+14368*f(7)-1563*f(8));
```

```
yj3=z0+(3*h*v0)+(h^2/(89600))*(71661*f0+328608*f(1)-
150624*f(2)+315000*f(3)-281430*f(4)+177264*f(5)-
73128*f(6)+17784*f(7)-1935*f(8));
yj4=z0+(4*h*v0)+(h^2/(28350))*(30812*f0+148992*f(1)-
46400*f(2)+160256*f(3)-118440*f(4)+76288*f(5)-31552*f(6)+7680*f(7)-
836*f(8));
yj5=z0+(5*h*v0)+(h^2/(290304))*(398825*f0+1987000*f(1)-
465000*f(2)+2294000*f(3)-1283750*f(4)+1020600*f(5)-
412000*f(6)+100000*f(7)-10875*f(8));
yj6=z0+(6*h*v0)+(h^2/(1400))*(2325*f0+11808*f(1)-
2196*f(2)+14208*f(3)-6390*f(4)+7200*f(5)-2268*f(6)+576*f(7)-
63*f(8));
yj7=z0+(7*h*v0)+(7*h^2/(1036800))*(288533*f0+1484112*f(1)-
225008*f(2)+1830248*f(3)-689430*f(4)+1009792*f(5)-
145432*f(6)+84168*f(7)-8183*f(8));
yj8=z0+(8*h*v0)+(h^2/(28350))*(63296*f0+329728*f(1)-
44544*f(2)+419840*f(3)-145280*f(4)+251904*f(5)-
14848*f(6)+47104*f(7));

yrr1=y0+(h*z0)+(1/2)*(h^2*v0)+(h^3/(39916800))*(3619903*f0+6779886*f
(1)-9359135*f(2)+11774146*f(3)-10745445*f(4)+6771082*f(5)-
2792861*f(6)+679110*f(7)-73886*f(8));
yrr2=y0+(2*h*z0)+(2*h^2*v0)+(h^3/(623700))*(286967*f0+911204*f(1)-
926646*f(2)+1173140*f(3)-1067950*f(4)+671628*f(5)-
276634*f(6)+67196*f(7)-7305*f(8));
yrr3=y0+(3*h*z0)+((9/2)*h^2*v0)+(3*h^3/(492800))*(183384*f0+711918*f
(1)-521217*f(2)+766290*f(3)-699885*f(4)+441306*f(5)-
182043*f(6)+44262*f(7)-4815*f(8));
yrr4=y0+(4*h*z0)+(8*h^2*v0)+(h^3/(155925))*(321172*f0+1371264*f(1)-
752480*f(2)+1435264*f(3)-1243200*f(4)+784768*f(5)-
323744*f(6)+78720*f(7)-8564*f(8));
yrr5=y0+(5*h*z0)+((25/2)*h^2*v0)+(h^3/(1596672))*(5253125*f0+2370275
0*f(1)-10296375*f(2)+25537250*f(3)-19680625*f(4)+12920250*f(5)-
5327125*f(6)+1295750*f(7)-141000*f(8));
yrr6=y0+(6*h*z0)+(18*h^2*v0)+(h^3/(15400))*(74034*f0+346248*f(1)-
123660*f(2)+385128*f(3)-258660*f(4)+190296*f(5)-
75348*f(6)+18360*f(7)-1998*f(8));
yrr7=y0+(7*h*z0)+((49/2)*h^2*v0)+(h^3/(5702400))*(37701874*f0+180838
518*f(1)-54639557*f(2)+206894170*f(3)-122270925*f(4)+104842066*f(5)-
35782103*f(6)+9423582*f(7)-1020425*f(8));
yrr8=y0+(8*h*z0)+(32*h^2*v0)+(8*h^3/(155925))*(169624*f0+828928*f(1)
-216192*f(2)+970240*f(3)-510560*f(4)+508416*f(5)-
134528*f(6)+51712*f(7)-4440*f(8));

fr1=8*yj1-3*yrr1-4*exp(x(1));
fr2=8*yj2-3*yrr2-4*exp(x(2));
fr3=8*yj3-3*yrr3-4*exp(x(3));
fr4=8*yj4-3*yrr4-4*exp(x(4));
fr5=8*yj5-3*yrr5-4*exp(x(5));
fr6=8*yj6-3*yrr6-4*exp(x(6));
fr7=8*yj7-3*yrr7-4*exp(x(7));
fr8=8*yj8-3*yrr8-4*exp(x(8));
yr1=y0+(h*z0)+(1/2)*(h^2*v0)+(h^3/(39916800))*(3619903*f0+6779886*fr
1-9359135*fr2+11774146*fr3-10745445*fr4+6771082*fr5-
2792861*fr6+679110*fr7-73886*fr8);
m1=toc;
```

```
err1=abs((exp(x(1))+exp(-3*x(1)))-yr1);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(1),(exp(x(1))+exp(-3*x(1))),yr1, err1)
yr2=y0+(2*h*z0)+(2*h^2*v0)+(h^3/(623700))*(286967*f0+911204*fr1-
926646*fr2+1173140*fr3-1067950*fr4+671628*fr5-276634*fr6+67196*fr7-
7305*fr8);
m2=toc;
err2=abs((exp(x(2))+exp(-3*x(2)))-yr2);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(2),(exp(x(2))+exp(-3*x(2))),yr2, err2)
yr3=y0+(3*h*z0)+((9/2)*h^2*v0)+(3*h^3/(492800))*(183384*f0+711918*fr
1-521217*fr2+766290*fr3-699885*fr4+441306*fr5-182043*fr6+44262*fr7-
4815*fr8);
m3=toc;
err3=abs((exp(x(3))+exp(-3*x(3)))-yr3);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e \n',
x(3),(exp(x(3))+exp(-3*x(3))),yr3, err3)
yr4=y0+(4*h*z0)+(8*h^2*v0)+(h^3/(155925))*(321172*f0+1371264*fr1-
752480*fr2+1435264*fr3-1243200*fr4+784768*fr5-323744*fr6+78720*fr7-
8564*fr8);
m4=toc;
err4=abs((exp(x(4))+exp(-3*x(4)))-yr4);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(4),(exp(x(4))+exp(-3*x(4))),yr4, err4)
yr5=y0+(5*h*z0)+((25/2)*h^2*v0)+(h^3/(1596672))*(5253125*f0+23702750
*fr1-10296375*fr2+25537250*fr3-19680625*fr4+12920250*fr5-
5327125*fr6+1295750*fr7-141000*fr8);
m5=toc;
err5=abs((exp(x(5))+exp(-3*x(5)))-yr5);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(5),(exp(x(5))+exp(-3*x(5))),yr5, err5)
yr6=y0+(6*h*z0)+(18*h^2*v0)+(h^3/(15400))*(74034*f0+346248*fr1-
123660*fr2+385128*fr3-258660*fr4+190296*fr5-75348*fr6+18360*fr7-
1998*fr8);
m6=toc;
err6=abs((exp(x(6))+exp(-3*x(6)))-yr6);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(6),(exp(x(6))+exp(-3*x(6))),yr6, err6)
yr7=y0+(7*h*z0)+((49/2)*h^2*v0)+(h^3/(5702400))*(37701874*f0+1808385
18*fr1-54639557*fr2+206894170*fr3-122270925*fr4+104842066*fr5-
35782103*fr6+9423582*fr7-1020425*fr8);
m7=toc;
err7=abs((exp(x(7))+exp(-3*x(7)))-yr7);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(7),(exp(x(7))+exp(-3*x(7))),yr7, err7)
yr8=y0+(8*h*z0)+(32*h^2*v0)+(8*h^3/(155925))*(169624*f0+828928*fr1-
216192*fr2+970240*fr3-510560*fr4+508416*fr5-134528*fr6+51712*fr7-
4440*fr8);
m8=toc;
err8=abs((exp(x(8))+exp(-3*x(8)))-yr8);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(8),(exp(x(8))+exp(-3*x(8))),yr8, err8)
x0=x(8); y0=yr8; z0=yj8; v0=yp8;
end
```

Appendix L

MATLAB CODE OF FIVE–STEP BLOCK METHOD FOR SOLVING FOURTH ORDER ODES.

```
% programme to solve fourth order O .D. E with step length of five
clear
clc
% y'is represented by z
%y" is represented by v
%syms x0 x y z g f0 y0 z0 v0 w0
%g (x,y,z)+y"=0;
x0=0;  y0=0; z0=(-1.1/(72-50*pi)); v0=(1/(144-100*pi));w0=(1.2/(144-100*pi)); h=1/320;
disp ('x - value     exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=-v0;
for i=1:5;
x(i)=x0+i*h;
y(i)=(v0*h^6*i^6)/720 - (w0*h^5*i^5)/120 - (v0*h^4*i^4)/24 +
(w0*h^3*i^3)/6 + (v0*h^2*i^2)/2 + z0*h*i + y0;
z(i)=(v0*h^5*i^5)/120 - (w0*h^4*i^4)/24 - (v0*h^3*i^3)/6 +
(w0*h^2*i^2)/2 + v0*h*i + z0;
v(i)=(v0*h^4*i^4)/24 - (w0*h^3*i^3)/6 - (v0*h^2*i^2)/2 + w0*h*i +
v0;
w(i)=(v0*h^3*i^3)/6 - (w0*h^2*i^2)/2 - v0*h*i + w0;
f(i)=-v(i);
end
yp1=z0+(h*v0)+(1/2*h^2*w0)+(h^3/241920)*(23574*f0+29850*f(1)-
23172*f(2)+14532*f(3)-5298*f(4)+834*f(5));
yp2=z0+(2*h*v0)+(2*h^2*w0)+(h^3/1890)*(951*f0+2202*f(1)-
1140*f(2)+732*f(3)-267*f(4)+42*f(5));
yp3=z0+(3*h*v0)+(9/2*h^2*w0)+(h^3/13440)*(16443*f0+48357*f(1)-
13122*f(2)+12690*f(3)-4617*f(4)+729*f(5));
yp4=z0+(4*h*v0)+(8*h^2*w0)+(h^3/945)*(2136*f0+7008*f(1)-
672*f(2)+2112*f(3)-600*f(4)+96*f(5));
yp5=z0+(5*h*v0)+(25/2*h^2*w0)+(h^3/24192)*(87375*f0+305625*f(1)+3750
*f(2)+116250*f(3)-13125*f(4)+4125*f(5));

ypp1=v0+(h*w0)+(h^2/20160)*(4924*f0+8630*f(1)-6088*f(2)+3764*f(3)-
1364*f(4)+214*f(5));
ypp2=v0+(2*h*w0)+((h^2)/630)*(355*f0+1088*f(1)-370*f(2)+272*f(3)-
101*f(4)+16*f(5));
ypp3=v0+(3*h*w0)+((h^2)/10080)*(8856*f0+31509*f(1)-
648*f(2)+7830*f(3)-2592*f(4)+405*f(5));
ypp4=v0+(4*h*w0)+((h^2)/630)*(752*f0+2848*f(1)+352*f(2)+1216*f(3)-
160*f(4)+32*f(5));
ypp5=v0+(5*h*w0)+((h^2)/10080)*(15250*f0+59375*f(1)+12500*f(2)+31250
*f(3)+6250*f(4)+1375*f(5));
```

```
yppp1=w0+(h/5760)*(1900*f0+5708*f(1)-3192*f(2)+1928*f(3)-
692*f(4)+108*f(5));
yppp2=w0+(h/360)*(112*f0+516*f(1)+56*f(2)+56*f(3)-24*f(4)+4*f(5));
yppp3=w0+(h/640)*(51*f0+219*f(1)+114*f(2)+114*f(3)-21*f(4)+3*f(5));
yppp4=w0+(h/180)*(56*f0+256*f(1)+96*f(2)+256*f(3)+56*f(4));
yppp5=w0+(h/1728)*(570*f0+2250*f(1)+1500*f(2)+1500*f(3)+2250*f(4)+57
0*f(5));

fr1=-ypp1;
fr2=-ypp2;
fr3=-ypp3;
fr4=-ypp4;
fr5=-ypp5;
yr1=y0+(h*z0)+(1/2*h^2*v0)+(1/6*h^3*w0)+(h^4/5443200)*(147378*f0+147
135*fr1-120480*fr2+76290*fr3-27930*fr4+4407*fr5);
m1=toc;
err1=abs((1-x(1)-cos(x(1))-1.2*sin(x(1)))/(144-100*pi)-yr1);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n', x(1),(1-x(1)-
cos(x(1))-1.2*sin(x(1)))/(144-100*pi),yr1, err1)
yr2=y0+(2*h*z0)+(2*h^2*v0)+((4/3)*h^3*w0)+(h^4/907200)*(272896*f0+50
9440*fr1-314240*fr2+197120*fr3-71680*fr4+11264*fr5);
m2=toc;
err2=abs((1-x(2)-cos(x(2))-1.2*sin(x(2)))/(144-100*pi)-yr2);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n', x(2),(1-x(2)-
cos(x(2))-1.2*sin(x(2)))/(144-100*pi),yr2, err2)
yr3=y0+(3*h*z0)+(9/2*h^2*v0)+(9/2*h^3*w0)+(h^4/67200)*(76464*f0+1899
45*fr1-79380*fr2+57510*fr3-21060*fr4+3321*fr5);
m3=toc;
err3=abs((1-x(3)-cos(x(3))-1.2*sin(x(3)))/(144-100*pi)-yr3);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n', x(3),(1-x(3)-
cos(x(3))-1.2*sin(x(3)))/(144-100*pi),yr3, err3)
yr4=y0+(4*h*z0)+(8*h^2*v0)+((32/3)*h^3*w0)+(h^4/1814400)*(5177344*f0
+14909440*fr1-3768320*fr2+4259840*fr3-1454080*fr4+229376*fr5);
m4=toc;
err4=abs((1-x(4)-cos(x(4))-1.2*sin(x(4)))/(144-100*pi)-yr4);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',x(4),(1-x(4)-
cos(x(4))-1.2*sin(x(4)))/(144-100*pi),yr4, err4)
yr5=y0+(5*h*z0)+(25/2*h^2*v0)+(125/6*h^3*w0)+(h^4/145152)*(836500*f0
+2631250*fr1-350000*fr2+837500*fr3-212500*fr4+37250*fr5);
m5=toc;
err5=abs((1-x(5)-cos(x(5))-1.2*sin(x(5)))/(144-100*pi)-yr5);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',x(5),(1-x(5)-
cos(x(5))-1.2*sin(x(5)))/(144-100*pi),yr5, err5)
x0=x(5); y0=yr5; z0=yp5; v0=ypp5;w0=yppp5;
end
```

Appendix M

MATLAB CODE OF SIX–STEP BLOCK METHOD FOR SOLVING FOURTH ORDER ODES.

```
% programme to solve fourth order O .D. E with step length of six
clear
%clc
% y'is represented by z
%y" is represented by v
%syms x0 x y z g f0 y0 z0 v0 w0
%g (x,y,z)=x;
x0=0;  y0=0; z0=1; v0=0;w0=0; h=0.1;
disp ('x - value     exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=x0;
for i=1:6;
x(i)=x0+i*h;
y(i)=(h^5*i^5)/120 + (x0*h^4*i^4)/24 + (w0*h^3*i^3)/6 +
(v0*h^2*i^2)/2 + z0*h*i + y0;
z(i)=(h^4*i^4)/24 + (x0*h^3*i^3)/6 + (w0*h^2*i^2)/2 + v0*h*i + z0;
v(i)=(h^3*i^3)/6 + (x0*h^2*i^2)/2 + w0*h*i + v0;
w(i)=(h^2*i^2)/2 + x0*h*i + w0;
f(i)=x(i);
end
yp1=z0+(h*v0)+((1/2)*h^2*w0)+((h^3)/3628800)*(343801*f0+506604*f(1)-
494715*f(2)+414160*f(3)-226605*f(4)+71364*f(5)-9809*f(6));
yp2=z0+(2*h*v0)+(2*h^2*w0)+((h^3)/28350)*(13774*f0+35976*f(1)-
24465*f(2)+20800*f(3)-11370*f(4)+3576*f(5)-491*f(6));
yp3=z0+(3*h*v0)+((9/2)*h^2*w0)+((h^3)/44800)*(52893*f0+172692*f(1)-
72495*f(2)+80640*f(3)-44145*f(4)+13932*f(5)-1917*f(6));
yp4=z0+(4*h*v0)+(8*h^2*w0)+((h^3)/28350)*(61808*f0+223872*f(1)-
54240*f(2)+108800*f(3)-52080*f(4)+16512*f(5)-2272*f(6));
yp5=z0+(5*h*v0)+((25/2)*h^2*w0)+((h^3)/145152)*(505625*f0+1945500*f(
1)-256875*f(2)+1070000*f(3)-358125*f(4)+136500*f(5)-18625*f(6));
yp6=z0+(6*h*v0)+(18*h^2*w0)+((h^3)/700)*(3564*f0+14256*f(1)-
810*f(2)+8640*f(3)-1620*f(4)+1296*f(5)-126*f(6));

ypp1=v0+(h*w0)+((h^2)/120960)*(28549*f0+57750*f(1)-
51453*f(2)+42484*f(3)-23109*f(4)+7254*f(5)-995*f(6));
ypp2=v0+(2*h*w0)+((h^2)/120960)*(65728*f0+223488*f(1)-
107520*f(2)+100864*f(3)-55872*f(4)+17664*f(5)-2432*f(6));
ypp3=v0+(3*h*w0)+((h^2)/4480)*(3795*f0+14850*f(1)-
2403*f(2)+6300*f(3)-3267*f(4)+1026*f(5)-141*f(6));
ypp4=v0+(4*h*w0)+((h^2)/945)*(1088*f0+4512*f(1)-72*f(2)+2624*f(3)-
840*f(4)+288*f(5)-40*f(6));
ypp5=v0+(5*h*w0)+((h^2)/24192)*(35225*f0+150750*f(1)+9375*f(2)+10250
0*f(3)-5625*f(4)+11550*f(5)-1375*f(6));
ypp6=v0+(6*h*w0)+((h^2)/140)*(246*f0+1080*f(1)+108*f(2)+816*f(3)+54*
f(4)+216*f(5));
```

```
yppp1=w0+(h/60480)*(19087*f0+65112*f(1)-46461*f(2)+37504*f(3)-
20211*f(4)+6312*f(5)-863*f(6));
yppp2=w0+(h/3780)*(1139*f0+5640*f(1)+33*f(2)+1328*f(3)-
807*f(4)+264*f(5)-37*f(6));
yppp3=w0+(h/2240)*(685*f0+3240*f(1)+1161*f(2)+2176*f(3)-
729*f(4)+216*f(5)-29*f(6));
yppp4=w0+(h/945)*(286*f0+1392*f(1)+384*f(2)+1504*f(3)+174*f(4)+48*f(
5)-8*f(6));
yppp5=w0+(h/12096)*(3715*f0+17400*f(1)+6375*f(2)+16000*f(3)+11625*f(
4)+5640*f(5)-275*f(6));
yppp6=w0+(h/140)*(41*f0+216*f(1)+27*f(2)+272*f(3)+27*f(4)+216*f(5)+4
1*f(6));
fr1=x(1);
fr2=x(2);
fr3=x(3);
fr4=x(4);
fr5=x(5);
fr6=x(6);
yr1=y0+(h*z0)+((1/2)*h^2*v0)+((1/6)*h^3*w0)+((h^4)/3628800)*(95929*f
0+112028*fr1-115165*fr2+97320*fr3-53465*fr4+16876*fr5-2323*fr6);
m1=toc;
err1=abs(((x(1)^5/120)+x(1))-yr1);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(1),((x(1)^5/120)+x(1)),yr1, err1)
yr2=y0+(2*h*z0)+(2*h^2*v0)+((4/3)*h^3*w0)+((h^4)/14175)*(4127*f0+878
2*fr1-6965*fr2+5820*fr3-3175*fr4+998*fr5-137*fr6);
m2=toc;
err2=abs(((x(2)^5/120)+x(2))-yr2);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(2),((x(2)^5/120)+x(2)),yr2, err2)
yr3=y0+(3*h*z0)+((9/2)*h^2*v0)+((9/2)*h^3*w0)+((h^4)/44800)*(49239*f
0+137052*fr1-78975*fr2+73080*fr3-40095*fr4+12636*fr5-1737*fr6);
m3=toc;
err3=abs(((x(3)^5/120)+x(3))-yr3);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(3),((x(3)^5/120)+x(3)),yr3, err3)
yr4=y0+(4*h*z0)+(8*h^2*v0)+((32/3)*h^3*w0)+((h^4)/28350)*(78080*f0+2
49856*fr1-101120*fr2+122880*fr3-64960*fr4+20480*fr5-2816*fr6);
m4=toc;
err4=abs(((x(4)^5/120)+x(4))-yr4);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e
\n',x(4),((x(4)^5/120)+x(4)),yr4, err4)
yr5=y0+(5*h*z0)+((25/2)*h^2*v0)+((125/6)*h^3*w0)+((h^4)/145152)*(807
125*f0+2807500*fr1-790625*fr2+1425000*fr3-653125*fr4+213500*fr5-
29375*fr6);
m5=toc;
err5=abs(((x(5)^5/120)+x(5))-yr5);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e
\n',x(5),((x(5)^5/120)+x(5)),yr5, err5)
yr6=y0+(6*h*z0)+(18*h^2*v0)+(36*h^3*w0)+((h^4)/350)*(3438*f0+12636*fr1-
2430*fr2+6840*fr3-2430*fr4+972*fr5-126*fr6);
m6=toc;
err6=abs(((x(6)^5/120)+x(6))-yr6);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e
\n',x(6),((x(6)^5/120)+x(6)),yr6, err6)
x0=x(6); y0=yr6; z0=yp6; v0=ypp6;w0=yppp6;
end
```

Appendix N

MATLAB CODE OF SEVEN–STEP BLOCK METHOD FOR SOLVING FOURTH ORDER ODES.

```
% programme to solve fourth order O .D. E with step length of seven
clear
%clc
% y'is represented by z
%y" is represented by v
%syms x0 x y z g f0 y0 z0 v0 w0
%g (x,y,z)=(x^4+14*x^3+49*x^2+32*x-12)*exp(x);
x0=0;  y0=0; z0=0; v0=2;w0=-6; h=0.01;
disp ('x - value      exact - solution      computed - solution
error')
tic
for j=0:h:1;
f0=(x0^4+14*x0^3+49*x0^2+32*x0-12)*exp(x0);
for i=1:7;
x(i)=x0+i*h;
y(i)=y0 + (h^2*i^2*v0)/2 + (h^3*i^3*w0)/6 +
(h^6*i^6*(exp(x0)*(12*x0^2 + 84*x0 + 98) + exp(x0)*(x0^4 + 14*x0^3 +
49*x0^2 + 32*x0 - 12) + 2*exp(x0)*(4*x0^3 + 42*x0^2 + 98*x0 +
32)))/720 + h*i*z0 + (h^5*i^5*(exp(x0)*(x0^4 + 14*x0^3 + 49*x0^2 +
32*x0 - 12) + exp(x0)*(4*x0^3 + 42*x0^2 + 98*x0 + 32)))/120 +
(h^4*i^4*exp(x0)*(x0^4 + 14*x0^3 + 49*x0^2 + 32*x0 - 12))/24;
z(i)=z0 + (h^2*i^2*w0)/2 + (h^5*i^5*(exp(x0)*(12*x0^2 + 84*x0 + 98)
+ exp(x0)*(x0^4 + 14*x0^3 + 49*x0^2 + 32*x0 - 12) +
2*exp(x0)*(4*x0^3 + 42*x0^2 + 98*x0 + 32)))/120 + h*i*v0 +
(h^4*i^4*(exp(x0)*(x0^4 + 14*x0^3 + 49*x0^2 + 32*x0 - 12) +
exp(x0)*(4*x0^3 + 42*x0^2 + 98*x0 + 32)))/24 +
(h^3*i^3*exp(x0)*(x0^4 + 14*x0^3 + 49*x0^2 + 32*x0 - 12))/6;
v(i)=v0 + (h^4*i^4*(exp(x0)*(12*x0^2 + 84*x0 + 98) + exp(x0)*(x0^4 +
14*x0^3 + 49*x0^2 + 32*x0 - 12) + 2*exp(x0)*(4*x0^3 + 42*x0^2 +
98*x0 + 32)))/24 + h*i*w0 + (h^3*i^3*(exp(x0)*(x0^4 + 14*x0^3 +
49*x0^2 + 32*x0 - 12) + exp(x0)*(4*x0^3 + 42*x0^2 + 98*x0 + 32)))/6
+ (h^2*i^2*exp(x0)*(x0^4 + 14*x0^3 + 49*x0^2 + 32*x0 - 12))/2;
w(i)=w0 + (h^3*i^3*(exp(x0)*(12*x0^2 + 84*x0 + 98) + exp(x0)*(x0^4 +
14*x0^3 + 49*x0^2 + 32*x0 - 12) + 2*exp(x0)*(4*x0^3 + 42*x0^2 +
98*x0 + 32)))/6 + (h^2*i^2*(exp(x0)*(x0^4 + 14*x0^3 + 49*x0^2 +
32*x0 - 12) + exp(x0)*(4*x0^3 + 42*x0^2 + 98*x0 + 32)))/2 +
h*i*exp(x0)*(x0^4 + 14*x0^3 + 49*x0^2 + 32*x0 - 12);
f(i)=(x(i)^4+14*x(i)^3+49*x(i)^2+32*x(i)-12)*exp(x(i));
end
yp1=z0+(h*v0)+(1/2*h^2*w0)+((h^3)/3628800)*(335799*f0+562618*f(1)-
662757*f(2)+694230*f(3)-506675*f(4)+239406*f(5)-
65823*f(6)+8002*f(7));
yp2=z0+(2*h*v0)+(2*h^2*w0)+((h^3)/28350)*(13376*f0+38762*f(1)-
32823*f(2)+34730*f(3)-25300*f(4)+11934*f(5)-3277*f(6)+398*f(7));
yp3=z0+(3*h*v0)+((9/2)*h^2*w0)+((h^3)/44800)*(51327*f0+183654*f(1)-
105381*f(2)+135450*f(3)-98955*f(4)+46818*f(5)-12879*f(6)+1565*f(7));
yp4=z0+(4*h*v0)+(8*h^2*w0)+((h^3)/14175)*(29976*f0+118432*f(1)-
46608*f(2)+86880*f(3)-58520*f(4)+27744*f(5)-7632*f(6)+928*f(7));
```

```
yp5=z0+(5*h*v0)+((25/2)*h^2*w0)+((5*h^3)/145152)*(98075*f0+410450*f(
1)-115425*f(2)+320750*f(3)-178375*f(4)+91350*f(5)-
25075*f(6)+3050*f(7));
yp6=z0+(6*h*v0)+(18*h^2*w0)+((h^3)/1400)*(6912*f0+30024*f(1)-
6156*f(2)+24840*f(3)-10800*f(4)+7128*f(5)-1764*f(6)+216*f(7));
yp7=z0+(7*h*v0)+((49/2)*h^2*w0)+((7*h^3)/518400)*(502887*f0+2242534*
f(1)-338541*f(2)+1944810*f(3)-660275*f(4)+619458*f(5)-
93639*f(6)+16366*f(7));


ypp1=v0+(h*w0)+((h^2)/1814400)*(416173*f0+950684*f(1)-
1025097*f(2)+1059430*f(3)-768805*f(4)+362112*f(5)-
99359*f(6)+12062*f(7));
ypp2=v0+(2*h*w0)+((h^2)/28350)*(14939*f0+55642*f(1)-
34986*f(2)+39950*f(3)-29405*f(4)+13926*f(5)-3832*f(6)+466*f(7));
ypp3=v0+(3*h*w0)+((h^2)/604800)*(496773*f0+2113614*f(1)-
650997*f(2)+1394820*f(3)-985365*f(4)+465102*f(5)-
127899*f(6)+15552*f(7));
ypp4=v0+(4*h*w0)+((h^2)/14175)*(15824*f0+71152*f(1)-
11496*f(2)+56720*f(3)-29960*f(4)+14736*f(5)-4072*f(6)+496*f(7));
ypp5=v0+(5*h*w0)+((h^2)/72576)*(102425*f0+475000*f(1)-
40125*f(2)+421250*f(3)-130625*f(4)+102900*f(5)-
26875*f(6)+3250*f(7));
ypp6=v0+(6*h*w0)+((h^2)/350)*(597*f0+2826*f(1)-108*f(2)+2670*f(3)-
495*f(4)+918*f(5)-126*f(6)+18*f(7));
ypp7=v0+(7*h*w0)+((h^2)/259200)*(519253*f0+2482634*f(1)+7203*f(2)+24
01000*f(3)-204085*f(4)+965202*f(5)+146461*f(6)+32732*f(7));


yppp1=w0+(h/120960)*(36799*f0+139849*f(1)-121797*f(2)+123133*f(3)-
88547*f(4)+41499*f(5)-11351*f(6)+1375*f(7));
yppp2=w0+(h/18900)*(5535*f0+29320*f(1)-3195*f(2)+12240*f(3)-
9635*f(4)+4680*f(5)-1305*f(6)+160*f(7));
yppp3=w0+(h/22400)*(6625*f0+33975*f(1)+6885*f(2)+29635*f(3)-
15165*f(4)+6885*f(5)-1865*f(6)+225*f(7));
yppp4=w0+(h/945)*(278*f0+1448*f(1)+216*f(2)+1784*f(3)-
106*f(4)+216*f(5)-64*f(6)+8*f(7));
yppp5=w0+(h/24192)*(7155*f0+36725*f(1)+6975*f(2)+41625*f(3)+13625*f(
4)+17055*f(5)-2475*f(6)+275*f(7));
yppp6=w0+(h/140)*(41*f0+216*f(1)+27*f(2)+272*f(3)+27*f(4)+216*f(5)+4
1*f(6));
yppp7=w0+(h/17280)*(5257*f0+25039*f(1)+9261*f(2)+20923*f(3)+20923*f(
4)+9261*f(5)+25039*f(6)+5257*f(7));

yrr1=y0+(h*z0)+(1/2*h^2*v0)+(1/6*h^3*w0)+((h^4)/119750400)*(3102701*
f0+4137616*f(1)-5122521*f(2)+5415020*f(3)-3967805*f(4)+1878984*f(5)-
517351*f(6)+62956*f(7));
yrr2=y0+(2*h*z0)+(2*h^2*v0)+(4/3*h^3*w0)+((h^4)/467775)*(132526*f0+3
15461*f(1)-306810*f(2)+320335*f(3)-233050*f(4)+109899*f(5)-
30176*f(6)+3665*f(7));
yrr3=y0+(3*h*z0)+((9/2)*h^2*v0)+(9/2*h^3*w0)+((h^4)/492800)*(526077*
f0+1616436*f(1)-1195317*f(2)+1348200*f(3)-985365*f(4)+465588*f(5)-
127971*f(6)+15552*f(7));
```

```
yrr4=y0+(4*h*z0)+(8*h^2*v0)+(32/3*h^3*w0)+((4*h^4)/467775)*(312608*f
0+1096960*f(1)-616032*f(2)+838400*f(3)-599480*f(4)+283392*f(5)-
77920*f(6)+9472*f(7));
yrr5=y0+(5*h*z0)+((25/2)*h^2*v0)+(125/6*h^3*w0)+((h^4)/4790016)*(258
42625*f0+98195000*f(1)-42733125*f(2)+74762500*f(3)-
49290625*f(4)+23688000*f(5)-6516875*f(6)+792500*f(7));
yrr6=y0+(6*h*z0)+(18*h^2*v0)+(36*h^3*w0)+((h^4)/11550)*(110052*f0+44
0802*f(1)-151632*f(2)+344790*f(3)-199260*f(4)+103518*f(5)-
27972*f(6)+3402*f(7));
yrr7=y0+(7*h*z0)+((49/2)*h^2*v0)+(343/6*h^3*w0)+((h^3)/17107200)*(26
2892693*f0+1093194508*f(1)-301769685*f(2)+884720480*f(3)-
438242525*f(4)+266827932*f(5)-64354003*f(6)+8163400*f(7));

fr1=(x(1)^4+14*x(1)^3+49*x(1)^2+32*x(1)-12)*exp(x(1));
fr2=(x(2)^4+14*x(2)^3+49*x(2)^2+32*x(2)-12)*exp(x(2));
fr3=(x(3)^4+14*x(3)^3+49*x(3)^2+32*x(3)-12)*exp(x(3));
fr4=(x(4)^4+14*x(4)^3+49*x(4)^2+32*x(4)-12)*exp(x(4));
fr5=(x(5)^4+14*x(5)^3+49*x(5)^2+32*x(5)-12)*exp(x(5));
fr6=(x(6)^4+14*x(6)^3+49*x(6)^2+32*x(6)-12)*exp(x(6));
fr7=(x(7)^4+14*x(7)^3+49*x(7)^2+32*x(7)-12)*exp(x(7));

yr1=y0+(h*z0)+(1/2*h^2*v0)+(1/6*h^3*w0)+((h^4)/119750400)*(3102701*f
0+4137616*fr1-5122521*fr2+5415020*fr3-3967805*fr4+1878984*fr5-
517351*fr6+62956*fr7);
m1=toc;
err1=abs((x(1)^2*((1-x(1))^2)*exp(x(1)))-yr1);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(1),(x(1)^2*((1-x(1))^2)*exp(x(1))),yr1, err1)
yr2=y0+(2*h*z0)+(2*h^2*v0)+(4/3*h^3*w0)+((h^4)/467775)*(132526*f0+31
5461*fr1-306810*fr2+320335*fr3-233050*fr4+109899*fr5-
30176*fr6+3665*fr7);
m2=toc;
err2=abs((x(2)^2*((1-x(2))^2)*exp(x(2)))-yr2);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(2),(x(2)^2*((1-x(2))^2)*exp(x(2))),yr2, err2)
yr3=y0+(3*h*z0)+((9/2)*h^2*v0)+(9/2*h^3*w0)+((h^4)/492800)*(526077*f
0+1616436*fr1-1195317*fr2+1348200*fr3-985365*fr4+465588*fr5-
127971*fr6+15552*fr7);
m3=toc;
err3=abs((x(3)^2*((1-x(3))^2)*exp(x(3)))-yr3);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(3),(x(3)^2*((1-x(3))^2)*exp(x(3))),yr3, err3)
yr4=y0+(4*h*z0)+(8*h^2*v0)+(32/3*h^3*w0)+((4*h^4)/467775)*(312608*f0
+1096960*fr1-616032*fr2+838400*fr3-599480*fr4+283392*fr5-
77920*fr6+9472*fr7);
m4=toc;
err4=abs((x(4)^2*((1-x(4))^2)*exp(x(4)))-yr4);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e
\n',x(4),(x(4)^2*((1-x(4))^2)*exp(x(4))),yr4, err4)
yr5=y0+(5*h*z0)+((25/2)*h^2*v0)+(125/6*h^3*w0)+((h^4)/4790016)*(2584
2625*f0+98195000*fr1-42733125*fr2+74762500*fr3-
49290625*fr4+23688000*fr5-6516875*fr6+792500*fr7);
m5=toc;
err5=abs((x(5)^2*((1-x(5))^2)*exp(x(5)))-yr5);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e
\n',x(5),(x(5)^2*((1-x(5))^2)*exp(x(5))),yr5, err5)
```

```
yr6=y0+(6*h*z0)+(18*h^2*v0)+(36*h^3*w0)+((h^4)/11550)*(110052*f0+440
802*fr1-151632*fr2+344790*fr3-199260*fr4+103518*fr5-
27972*fr6+3402*fr7);
m6=toc;
err6=abs((x(6)^2*((1-x(6))^2)*exp(x(6)))-yr6);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(6),(x(6)^2*((1-x(6))^2)*exp(x(6))),yr6, err6)
yr7=y0+(7*h*z0)+((49/2)*h^2*v0)+(343/6*h^3*w0)+((h^4)/17107200)*(262
892693*f0+1093194508*fr1-301769685*fr2+884720480*fr3-
438242525*fr4+266827932*fr5-64354003*fr6+8163400*fr7);
m7=toc;
err7=abs((x(7)^2*((1-x(7))^2)*exp(x(7)))-yr7);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(7),(x(7)^2*((1-x(7))^2)*exp(x(7))),yr7, err7)
x0=x(7); y0=yr7; z0=yp7; v0=ypp7;w0=yppp7;
end
```

Appendix O

MATLAB CODE OF EIGHT–STEP BLOCK METHOD FOR SOLVING FOURTH ORDER ODES.

```
% programme to solve fourth order O .D. E with step length of Eight
clear
%clc
% y'is represented by z
%y" is represented by v
%syms x0 x y z g f0 y0 z0 v0 w0
%g (x,y,z)=y;
x0=0;  y0=1; z0=1; v0=1;w0=1; h=0.01;
disp ('x - value     exact - solution      computed - solution
error')
tic
for j=0:h:1
f0=y0;
for i=1:8;
x(i)=x0+i*h;
y(i)=(v0*h^6*i^6)/720 + (z0*h^5*i^5)/120 + (y0*h^4*i^4)/24 +
(w0*h^3*i^3)/6 + (v0*h^2*i^2)/2 + z0*h*i + y0;
z(i)=(v0*h^5*i^5)/120 + (z0*h^4*i^4)/24 + (y0*h^3*i^3)/6 +
(w0*h^2*i^2)/2 + v0*h*i + z0;
v(i)=(v0*h^4*i^4)/24 + (z0*h^3*i^3)/6 + (y0*h^2*i^2)/2 + w0*h*i +
v0;
w(i)=(v0*h^3*i^3)/6 + (z0*h^2*i^2)/2 + y0*h*i + w0;
f(i)=y(i);
end
yp1=z0+(h*v0)+((1/2)*h^2*w0)+(h^3/19958400)*(1809953*f0+3389944*f(1)
-4679567*f(2)+5887073*f(3)-5372722*f(4)+3385543*f(5)-
1396429*f(6)+339555*f(7)-36943*f(8));
yp2=z0+(2*h*v0)+(2*h^2*w0)+(h^3/623700)*(286967*f0+911204*f(1)-
926646*f(2)+1173140*f(3)-1067950*f(4)+671628*f(5)-
276634*f(6)+67196*f(7)-7305*f(8));
yp3=z0+(3*h*v0)+((9/2)*h^2*w0)+(3*h^3/492800)*(183384*f0+711918*f(1)
-521217*f(2)+766290*f(3)-699885*f(4)+441306*f(5)-
182043*f(6)+44262*f(7)-4815*f(8));
yp4=z0+(4*h*v0)+(8*h^2*w0)+(h^3/155925)*(321172*f0+1371264*f(1)-
752480*f(2)+1435264*f(3)-1243200*f(4)+784768*f(5)-
323744*f(6)+78720*f(7)-8564*f(8));
yp5=z0+(5*h*v0)+((25/2)*h^2*w0)+(5*h^3/798336)*(525312*f0+2370275*f(
1)-1029638*f(2)+2553725*f(3)-1968062*f(4)+1292025*f(5)-
532712*f(6)+129575*f(7)-14100*f(8));
yp6=z0+(6*h*v0)+(18*h^2*w0)+(h^3/15400)*(74034*f0+346248*f(1)-
123660*f(2)+385128*f(3)-258660*f(4)+190296*f(5)-
75348*f(6)+18360*f(7)-1998*f(8));
yp7=z0+(7*h*v0)+((49/2)*h^2*w0)+(h^3/207900)*(1374548*f0+6593072*f(1
)-1992067*f(2)+7543015*f(3)-4457798*f(4)+3822365*f(5)-
1304557*f(6)+343568*f(7)-37203*f(8));
yp8=z0+(8*h*v0)+(32*h^2*w0)+(8*h^3/155925)*(169624*f0+828928*f(1)-
216192*f(2)+970240*f(3)-510560*f(4)+508416*f(5)-
134528*f(6)+51712*f(7)-4440*f(8));
```

```
ypp1=v0+(h*w0)+((h^2)/7257600)*(1624505*f0+4124231*f(1)-
5225623*f(2)+6488191*f(3)-5888311*f(4)+3698922*f(5)-
1522673*f(6)+369744*f(7)-40187*f(8));
ypp2=v0+(2*h*w0)+((h^2)/113400)*(58193*f0+235072*f(1)-
183708*f(2)+247328*f(3)-227030*f(4)+143232*f(5)-
59092*f(6)+14368*f(7)-1563*f(8));
ypp3=v0+(3*h*w0)+((h^2)/89600)*(71661*f0+328608*f(1)-
150624*f(2)+315000*f(3)-281430*f(4)+177264*f(5)-
73128*f(6)+17784*f(7)-1935*f(8));
ypp4=v0+(4*h*w0)+((h^2)/28350)*(30812*f0+148992*f(1)-
46400*f(2)+160256*f(3)-118440*f(4)+76288*f(5)-31552*f(6)+7680*f(7)-
836*f(8));
ypp5=v0+(5*h*w0)+((h^2)/290304)*(398825*f0+1987000*f(1)-
465000*f(2)+2294000*f(3)-1283750*f(4)+1020600*f(5)-
412000*f(6)+100000*f(7)-10875*f(8));
ypp6=v0+(6*h*w0)+((h^2)/1400)*(2325*f0+11808*f(1)-
2196*f(2)+14208*f(3)-6390*f(4)+7200*f(5)-2268*f(6)+576*f(7)-
63*f(8));
ypp7=v0+(7*h*w0)+((7*h^2)/1036800)*(288533*f0+1484112*f(1)-
225008*f(2)+1830248*f(3)-689430*f(4)+1009792*f(5)-
145432*f(6)+84168*f(7)-8183*f(8));
ypp8=v0+(8*h*w0)+((h^2)/28350)*(63296*f0+329728*f(1)-
44544*f(2)+419840*f(3)-145280*f(4)+251904*f(5)-
14848*f(6)+47104*f(7));

yppp1=w0+(h/1069200)*(315273*f0+1316197*f(1)-
1356711*f(2)+1648632*f(3)-1482974*f(4)+927046*f(5)-
380447*f(6)+92186*f(7)-10004*f(8));
yppp2=w0+(h/113400)*(32377*f0+182584*f(1)-42494*f(2)+120088*f(3)-
116120*f(4)+74728*f(5)-31154*f(6)+7624*f(7)-833*f(8));
yppp3=w0+(h/44800)*(12881*f0+70902*f(1)+3438*f(2)+79934*f(3)-
56160*f(4)+34434*f(5)-14062*f(6)+3402*f(7)-369*f(8));
yppp4=w0+(h/28350)*(8126*f0+45152*f(1)+488*f(2)+65504*f(3)-
18160*f(4)+18464*f(5)-7912*f(6)+1952*f(7)-214*f(8));
yppp5=w0+(h/145152)*(41705*f0+230150*f(1)+7550*f(2)+318350*f(3)-
4000*f(4)+170930*f(5)-49150*f(6)+11450*f(7)-1225*f(8));
yppp6=w0+(h/1400)*(401*f0+2232*f(1)+18*f(2)+3224*f(3)-
360*f(4)+2664*f(5)+158*f(6)+72*f(7)-9*f(8));
yppp7=w0+(h/518400)*(149527*f0+816634*f(1)+48706*f(2)+1085937*f(3)+5
4880*f(4)+736078*f(5)+522046*f(6)+223174*f(7)-8183*f(8));
yppp8=w0+(h/28350)*(7912*f0+47104*f(1)-7424*f(2)+83968*f(3)-
36320*f(4)+83968*f(5)-7424*f(6)+47104*f(7)+7912*f(8));

yrr1=y0+(h*z0)+((1/2)*h^2*v0)+((1/6)*h^3*w0)+(h^4/958003200)*(243964
97*f0+36501816*f(1)-52883276*f(2)+67126376*f(3)-
61500210*f(4)+38838088*f(5)-16041916*f(6)+3904536*f(7)-425111*f(8));
yrr2=y0+(2*h*z0)+(2*h^2*v0)+((4/3)*h^3*w0)+(h^4/3742200)*(1035731*f0
+2719504*f(1)-3139836*f(2)+3933392*f(3)-3577790*f(4)+2249904*f(5)-
926764*f(6)+225136*f(7)-24477*f(8));
yrr3=y0+(3*h*z0)+((9/2)*h^2*v0)+((9/2)*h^3*w0)+(h^4/3942400)*(410453
1*f0+13764168*f(1)-12476916*f(2)+16614360*f(3)-
15168870*f(4)+9553464*f(5)-3938148*f(6)+957096*f(7)-104085*f(8));
yrr4=y0+(4*h*z0)+(8*h^2*v0)+((32/3)*h^3*w0)+(4*h^4/467775)*(304672*f
0+1160448*f(1)-838240*f(2)+1282816*f(3)-1155000*f(4)+727808*f(5)-
300128*f(6)+72960*f(7)-7936*f(8));
```

```
yrr5=y0+(5*h*z0)+((25/2)*h^2*v0)+((125/6)*h^3*w0)+(h^4/38320128)*(20
1421625*f0+828115000*f(1)-490807500*f(2)+895985000*f(3)-
766681250*f(4)+487389000*f(5)-201077500*f(6)+48895000*f(7)-
5319375*f(8));
yrr6=y0+(6*h*z0)+(18*h^2*v0)+(36*h^3*w0)+(6*h^4/30800)*(47643*f0+206
064*f(1)-102924*f(2)+224304*f(3)-177390*f(4)+117072*f(5)-
47964*f(6)+11664*f(7)-1269*f(8));
yrr7=y0+(7*h*z0)+((49/2)*h^2*v0)+((343/6)*h^3*w0)+(h^4/136857600)*(2
048300303*f0+9184285992*f(1)-3949712228*f(2)+10148873336*f(3)-
7344827070*f(4)+5205732952*f(5)-2050386772*f(6)+504037128*f(7)-
54841241*f(8));
yrr8=y0+(8*h*z0)+(32*h^2*v0)+((256/3)*h^3*w0)+(h^4/467775)*(10571776
*f0+48693248*f(1)-18333696*f(2)+54722560*f(3)-
36229120*f(4)+28114944*f(5)-10108928*f(6)+2686976*f(7)-284160*f(8));
fr1=yrr1;
fr2=yrr2;
fr3=yrr3;
fr4=yrr4;
fr5=yrr5;
fr6=yrr6;
fr7=yrr7;
fr8=yrr8;
yr1=y0+(h*z0)+((1/2)*h^2*v0)+((1/6)*h^3*w0)+(h^4/958003200)*(2439649
7*f0+36501816*fr1-52883276*fr2+67126376*fr3-
61500210*fr4+38838088*fr5-16041916*fr6+3904536*fr7-425111*fr8);
m1=toc;
err1=abs(exp(x(1))-yr1);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(1),exp(x(1)),yr1, err1)
yr2=y0+(2*h*z0)+(2*h^2*v0)+((4/3)*h^3*w0)+(h^4/3742200)*(1035731*f0+
2719504*fr1-3139836*fr2+3933392*fr3-3577790*fr4+2249904*fr5-
926764*fr6+225136*fr7-24477*fr8);
m2=toc;
err2=abs(exp(x(2))-yr2);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(2),exp(x(2)),yr2, err2)
yr3=y0+(3*h*z0)+((9/2)*h^2*v0)+((9/2)*h^3*w0)+(h^4/3942400)*(4104531
*f0+13764168*fr1-12476916*fr2+16614360*fr3-15168870*fr4+9553464*fr5-
3938148*fr6+957096*fr7-104085*fr8);
m3=toc;
err3=abs(exp(x(3))-yr3);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e \n',
x(3),exp(x(3)),yr3, err3)
yr4=y0+(4*h*z0)+(8*h^2*v0)+((32/3)*h^3*w0)+(4*h^4/467775)*(304672*f0
+1160448*fr1-838240*fr2+1282816*fr3-1155000*fr4+727808*fr5-
300128*fr6+72960*fr7-7936*fr8);
m4=toc;
err4=abs(exp(x(4))-yr4);
fprintf('%2.7f      %3.18f       %3.18f      %1.6e
\n',x(4),exp(x(4)),yr4, err4)
yr5=y0+(5*h*z0)+((25/2)*h^2*v0)+((125/6)*h^3*w0)+(h^4/38320128)*(201
421625*f0+828115000*fr1-490807500*fr2+895985000*fr3-
766681250*fr4+487389000*fr5-201077500*fr6+48895000*fr7-5319375*fr8);
m5=toc;
err5=abs(exp(x(5))-yr5);
```

```
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(5),exp(x(5)),yr5, err5)
yr6=y0+(6*h*z0)+(18*h^2*v0)+(36*h^3*w0)+(6*h^4/30800)*(47643*f0+2060
64*fr1-102924*fr2+224304*fr3-177390*fr4+117072*fr5-
47964*fr6+11664*fr7-1269*fr8);
m6=toc;
err6=abs(exp(x(6))-yr6);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(6),exp(x(6)),yr6, err6)
yr7=y0+(7*h*z0)+((49/2)*h^2*v0)+((343/6)*h^3*w0)+(h^4/136857600)*(20
48300303*f0+9184285992*fr1-3949712228*fr2+10148873336*fr3-
7344827070*fr4+5205732952*fr5-2050386772*fr6+504037128*fr7-
54841241*fr8);
m7=toc;
err7=abs(exp(x(7))-yr7);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(7),exp(x(7)),yr7, err7)
yr8=y0+(8*h*z0)+(32*h^2*v0)+((256/3)*h^3*w0)+(h^4/467775)*(10571776*
f0+48693248*fr1-18333696*fr2+54722560*fr3-36229120*fr4+28114944*fr5-
10108928*fr6+2686976*fr7-284160*fr8);
m8=toc;
err8=abs(exp(x(8))-yr8);
fprintf('%2.7f     %3.18f      %3.18f     %1.6e
\n',x(8),exp(x(8)),yr8, err8)
x0=x(8); y0=yr8; z0=yp8; v0=ypp8;w0=yppp8;
end
```