ON GAMMA-P_S-OPERATIONS IN TOPOLOGICAL SPACES

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Abstrak

Topologi merupakan salah satu bidang tumpuan matematik. Kebelakangan ini, topologi menjadi komponen penting matematik gunaan kerana pelbagai kegunaannya dalam memahami masalah kehidupan sebenar. Konsep asas ruang topologi (X, τ) berkait rapat dengan set terbuka. Operasi pada τ telah dikaji oleh ramai penyelidik. Antara operasi yang telah dikaji ialah terbuka-γ, pra-terbuka-γ, semi-terbuka-γ, terbuka-γ-b, terbuka-γ-β dan terbuka-α-γ. Walau bagaimanapun, pentakrifan kelas set baharu operasi γ pada topologi τ dengan menggabungkan operasi sedia ada belum pernah diteroka. Sehubungan itu, kajian ini bertujuan mentakrif beberapa kelas set baharu, membina kelas fungsi baharu serta memperkenal jenis aksiom pemisahan dan ruang pemisahan baharu berasaskan set terbuka-γ. Kelas baharu yang terbina ialah set sekata-terbuka-γ dan terbuka-γ-P_S. Set peluaran-τ_γ-P_S, pedalaman-τ_γ-P_S, terbitan-τ_γ-P_S dan sempadan-τ_γ-P_S turut terbentuk hasil dari takrifan set terbuka-γ-P_S dan pelengkapnya. Set terbuka-γ-P_S dan pedalaman-τ_γ-P_S seterusnya digunakan untuk mentakrif kelas set baharu bagi set terbuka-γ-P_S yang dinamakan set tertutup teritlak-γ-P_S. Seterusnya, beberapa kelas fungsi baharu yang dikenali sebagai selanjar-γ-P_S, selanjar-(γ, β)-P_S dan tak terlerai-(γ, β)-P_S yang berasaskan set terbuka-γ-P_S diperkenalkan. Selanjutnya, beberapa jenis fungsi-γ-P_S yang lain seperti terbuka-β-P_S dan terbuka-(γ, β)-P_S dibina. Di samping itu, beberapa kelas aksiom pemisahan-γ-P_S baharu diperkenal dengan menggunakan set terbuka-γ-P_S dan pelengkapnya serta set tertutup teritlak-γ-P_S. Perhubungan dan sifat bagi setiap kelas set, fungsi-γ-P_S dan aksiom pemisahan-γ-P_S turut terbentuk. Kesimpulannya, kajian ini telah berjaya mentakrif beberapa kelas set baharu dengan menggunakan operasi γ pada topologi τ.

Kata kunci: Set terbuka sekata-γ, set -γ-P_S, fungsi -γ-P_S, aksiom pemisahan -γ-P_S, ruang-γ.

Abstract

Topology is one of the focus areas in mathematics. Recently, topology has become an important component in applied mathematics due to its vast applications in understanding real life problems. The basic concept of topological space (X, τ) deals with open sets. Operations on τ have been investigated by numerous researchers. Among these operations are γ-open, γ-preopen, γ-semiopen, γ-b-open, γ-β-open and α-γ-open which involve τ_γ-P_S-interior and τ_γ-P_S-closure. However, no one has attempted to define new class of set using operation γ on the topology τ by combining the existing operations. This study, therefore, aims to define new classes of sets, construct new classes of functions, and introduce new types of separation axioms and spaces using the concept of γ-open sets. The new classes developed are γ-regular-open and γ-P_S-open sets. By applying γ-P_S-open sets and their complements, the notions of τ_γ-P_S-closure, τ_γ-P_S-interior, τ_γ-P_S-derived set and τ_γ-P_S-boundary of a set are established. The notions of γ-P_S-open and τ_γ-P_S-closure sets are then used to define a new class of γ-P_S-open sets called γ-P_S-generalised closed sets. Moreover, several new classes of functions called γ-P_S-continuous, (γ, β)-P_S-continuous and (γ, β)-P_S-irresolute functions in term of γ-P_S-open sets are introduced. Furthermore, other types of γ-P_S -functions such as β-P_S-open and (γ, β)-P_S-open are constructed. In addition, some new classes of γ-P_S-separation axioms are established by using γ-P_S-open and its complement as well as γ-P_S-generalised closed sets. The relationships and properties of each class of sets, γ-P_S-functions and γ-P_S- separation axioms are also established. In conclusion, this study has succeeded in defining new classes of sets using operation γ on the topology τ.

Keywords: γ-regular-open set, γ-P_S- sets, γ-P_S- functions, γ-P_S- separation axioms, γ- spaces.

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Declaration Associated With This Thesis

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$P(X)$	Power set of the space X
τ_γ	The class of all γ-open subsets of a space X
τ_γ-$Cl(A)$	τ_γ-closure of a set A of X
τ_γ-$Int(A)$	τ_γ-interior of a set A of X
$\tau_{\alpha\text{-}\gamma}$	The family of all α-γ-open subsets of a space X
τ_γ-$PO(X)$	The class of all γ-preopen subsets of a space X
τ_γ-$SO(X)$	The class of all γ-semiopen subsets of a space X
τ_γ-$\beta O(X)$	The class of all γ-β-open subsets of a space X
τ_γ-$BO(X)$	The class of all γ-b-open subsets of a space X
τ_γ-$RO(X)$	The class of all γ-regular-open subsets of a space X
τ_γ-$PC(X)$	The class of all γ-preclosed subsets of a space X
τ_γ-$SC(X)$	The class of all γ-semiclosed subsets of a space X
τ_γ-$\beta C(X)$	The class of all γ-β-closed subsets of a space X
τ_γ-$BC(X)$	The class of all γ-b-closed subsets of a space X
τ_γ-$RC(X)$	The class of all γ-regular-closed subsets of a space X
τ_γ-$P_SO(X)$	The class of all γ-P_S-open subsets of a space X
τ_γ-$P_SC(X)$	The class of all γ-P_S-closed subsets of a space X
τ_γ-$pCl(A)$	τ_γ-preclosure of a set A of X
τ_γ-$sCl(A)$	τ_γ-semi-closure of a set A of X
$\tau_{\alpha\text{-}\gamma}$-$Cl(A)$	$\tau_{\alpha\text{-}\gamma}$-closure of a set A of X
τ_γ-$bCl(A)$	τ_γ-b-closure of a set A of X
τ_γ-$\beta Cl(A)$	τ_γ-β-closure of a set A of X
τ_γ-$P_SCl(A)$	τ_γ-P_S-closure of a set A of X
τ_γ-$pInt(A)$	τ_γ-preinterior of a set A of X
τ_γ-$sInt(A)$	τ_γ-semi-interior of a set A of X
$\tau_{\alpha\text{-}\gamma}$-$Int(A)$	$\tau_{\alpha\text{-}\gamma}$-interior of a set A of X

$\boldsymbol{\tau_\gamma}$**-*bInt*(*A*)**	τ_γ-b-interior of a set A of X
$\boldsymbol{\tau_\gamma}$**-*βInt*(*A*)**	τ_γ-β-interior of a set A of X
$\boldsymbol{\tau_\gamma}$**-$\boldsymbol{P_S}$*Int*(*A*)**	τ_γ-P_S-interior of a set A of X
$\boldsymbol{\tau_\gamma}$**-$\boldsymbol{P_S}$*D*(*A*)**	The set of all τ_γ-P_S-limit points of a set A of X
$\boldsymbol{\tau_\gamma}$**-$\boldsymbol{P_S}$*Bd*(*A*)**	τ_γ-P_S-boundary of a set A of X
$\boldsymbol{\tau_\gamma}$**-$\boldsymbol{P_S}$*GC*(*X*)**	The class of all γ-P_S-g-closed subsets of a space X
$\boldsymbol{\tau_\gamma}$**-$\boldsymbol{P_S}$*GO*(*X*)**	The class of all γ-P_S-g-open subsets of a space X

CHAPTER ONE

INTRODUCTION

1.1 Introduction

Topology is one of the major areas in mathematics. Recently, topology has become an important component of applied mathematics as many mathematicians and scientists employing concepts of topology to model and understand real-world structures and phenomena. The term topology, literally, means the study of position or location. In other words, topology is the study of shapes, including their properties, deformations applied to them, mappings between them, and configurations composed of them. The definition of a topology that used throughout this study is stated as follows.

Definition 1.1.1. (see Steen and Seebach, 1978) Let X be a nonempty set. A class τ of subsets of X is a topology on X if τ satisfies the following three conditions:

1. The empty set ϕ and the whole set X belong to τ.

2. The union of any finite or infinite number of sets in τ belongs to τ.

3. The intersection of any two (finite) number of sets in τ belongs to τ.

Members of τ are often called as open sets and the pair (X, τ) is called a topological space. A subset A of a topological space (X, τ) is closed if its complement is open. The closure of A is the intersection of all closed sets of X containing A. The interior of A

is the union of all open sets of X contained in A. Many researchers defined types of generalization open sets by using interior and closure operators in a topological space (X, τ) such as semiopen set (Levine, 1963), α-open set (Njastad, 1965), preopen set (Mashhour, Abd El-Monsef and El-Deeb, 1982), β-open set (Abd El-Monsef, El-Deeb and Mahmoud, 1983) which is equivalent to semi preopen set (Andrijevic, 1986) and b-open set (Andrijevic, 1996).

1.2 Research Background

In 1979, Kasahara was the first person who defined and investigated the concept of operations on τ. He used notation α as the operation on τ, that is, a function

$$\alpha\colon \tau \to P(X) \tag{1.1}$$

The equation (1.1) is called an operation on τ if $U \subseteq \alpha(U)$ satisfies for every $U \in \tau$. After the work of Kasahara, Jankovic (1983) defined the concept of operation-closures of α and investigated function with strongly closed graph. Ogata (1991) defined and investigated the concept of operation-open sets, that is, γ-open sets, and used it to investigate γ-T_i spaces, γ-regular space and characterized γ-T_i spaces using the notion of γ-open or γ-closed set for $i = 0, \frac{1}{2}, 1, 2$. He replaced α with γ in (1.1) as an operation on τ to avoid a confusion between the concept of α-open sets (Njastad, 1965) and one of operation α-open sets where the symbol α is operation as defined by Kasahara (1979).

In recent years, many concepts of operation γ in a topological space (X, τ) have been developed. Van An, Cuong and Maki (2008) developed an operation γ on $PO(X, \tau)$ to introduce the notion of pre-γ-open sets. Krishnan, Ganster and Balachandran (2007) developed an operation γ on $SO(X, \tau)$ to define the notion of semi-γ-open sets. Tahiliani (2011) developed an operation γ on $\beta O(X, \tau)$ to describe the notion of β-γ-open sets and Carpintero, Rajesh and Rosas (2012b) developed an operation γ on $BO(X, \tau)$ to define the notion of b-γ-open sets.

In the following subsection, γ- Sets, γ- Functions and γ- Separation axioms will be briefly discussed.

1.2.1 γ- Sets

Ogata (1991) introduced the notion of γ-open sets of a topological space (X, τ). He defined the complement of a γ-open subset of a space X as γ-closed. In addition, he also proved that the union of any collection of γ-open sets in a topological space (X, τ) is γ-open, but the intersection of any two γ-open sets in a space X need not be a γ-open set. Further study by Krishnan and Balachandran (2006a) defined two types of sets called γ-preopen and γ-semi preopen sets of a topological space (X, τ). Similarly they defined the complements of γ-preopen and γ-semi preopen sets of a space X as γ-preclosed and γ-semi preclosed respectively. They also proved that the union of any collection of γ-preopen (respectively, γ-semi preopen) sets is γ-preopen (respectively,

γ-semi preopen), but the intersection of any two γ-preopen (respectively, γ-semi preopen) sets need not be a γ-preopen (respectively, γ-semi preopen) set.

Subsequently, Krishnan and Balachandran (2006b) defined the notion of γ-semiopen sets. They defined the complement of a γ-semiopen set as γ-semiclosed. They also proved that the union of any collection of γ-semiopen sets is γ-semiopen, but the intersection of any two γ-semiopen sets need not be a γ-semiopen set. Kalaivani and Krishnan (2009) defined the notion of α-γ-open sets. The complement of an α-γ-open set of a space X is called α-γ-closed and they showed that the union of any collection of α-γ-open sets is α-γ-open. However, the intersection of any two α-γ-open sets need not be an α-γ-open set. In 2009, Basu, Afsan and Ghosh proposed the notion of γ-β-open sets. They defined the complement of a γ-β-open set of a space X as γ-β-closed and they verified that the union of any collection of γ-β-open sets is γ-β-open. The intersection of any two γ-β-open sets in a space X, however, need not be a γ-β-open set. Carpintero, Rajesh and Rosas (2012a) described the notion of γ-b-open sets of a topological space (X, τ). They stated that the complement of a γ-b-open subset of a space X is γ-b-closed and they showed that the union of any collection of γ-b-open sets in a topological space (X, τ) is γ-b-open, but the intersection of any two γ-b-open sets in a space X need not be a γ-b-open set.

Therefore, in general the class of all γ-open, γ-preopen, γ-semiopen, α-γ-open, γ-β-open, γ-semi preopen and γ-b-open sets of any topological space (X, τ) need not be a topology because it does not satisfy the Condition 3 in Definition 1.1.1. In this case, it is called supra topology. Details of γ-sets will be discussed in Section 2.1.

Ogata (1991) introduced and studied the notions of τ_γ-closure and γ-closure of a subset A of a space X with an operation γ on τ. Meanwhile, Krishnan and Balachandran (2006b) proposed and investigated the notion of τ_γ-interior of a set A of X with an operation γ on τ by using γ-open set. They also defined another two topological properties called τ_γ-semi-closure and τ_γ-semi-interior of a subset A of a space X by using γ-semiopen and γ-semiclosed sets.

Furthermore, Krishnan and Balachandran (2006a) investigated the notions of τ_γ-preclosure, τ_γ-preinterior, τ_γ-semi preclosure and τ_γ-semi preinterior using γ-preclosed, γ-preopen, γ-semi preclosed and γ-semi preopen sets respectively. Three years later, Kalaivani and Krishnan (2009) introduced $\tau_{\alpha-\gamma}$-closure and $\tau_{\alpha-\gamma}$-interior using α-γ-closed and α-γ-open sets. Later on, Basu, Afsan and Ghosh (2009) proposed τ_γ-β-closure and τ_γ-β-interior using γ-β-closed and γ-β-open sets. Carpintero, Rajesh and Rosas (2012a) defined τ_γ-b-closure and τ_γ-b-interior using γ-b-closed and γ-b-open sets. In addition, Ghosh (2012) introduced some topological properties called τ_γ-prederived set and τ_γ-preboundary of a set of a space X with an operation γ on τ by using γ-preopen set and τ_γ-preclosure. While Basu, Afsan and Ghosh (2009) defined τ_γ-derived set, τ_γ-boundary of a set, γ-β-derived set and γ-β-boundary of a set using γ-open set, τ_γ-closure, γ-β-open set and τ_γ-β-closure.

Ogata (1991), Krishnan and Balachandran (2006a), Kalaivani and Krishnan (2009), Basu, Afsan and Ghosh (2009) and Carpintero, Rajesh and Rosas (2012a) showed another set called γ-generalized closed, γ-pre-generalized closed, α-γ-generalized closed, γ-β-generalized closed and γ-b-generalized closed, respectively with an operation γ on τ via γ-open set and γ-closure, γ-preopen set and τ_γ-preclosure, α-γ-open set and $\tau_{\alpha-\gamma}$-closure, γ-β-open set and τ_γ-β-closure, and γ-b-open set and τ_γ-b-closure, respectively. Besides that, Basu, Afsan and Ghosh (2009) also stated the sets τ-γ-β-open and γ-γ-β-open with an operation γ on τ.

1.2.2 γ- Functions

Continuous functions which deal with operation γ on τ have attracted many researchers. Among them are Krishnan and Balachandran (2006a; 2006b), Carpintero, Rajesh and Rosas (2012a), Basu, Afsan and Ghosh (2009), Ghosh (2012), Kalaivani and Krishnan (2009) and Kalaivani, Kumar and Krishnan (2012).

Krishnan and Balachandran (2006a; 2006b) introduced types of mapping called (γ, β)-precontinuous in terms of γ-preopen sets and (γ, β)-semi-continuous in terms of γ-semiopen sets. While, Carpintero, Rajesh and Rosas (2012a) defined a mapping which is (γ, β)-b-continuous for γ-b-open sets. Basu, Afsan and Ghosh (2009) proposed more types of mappings such as γ-continuous, γ-β-continuous, γ'-β-open, γ'-β-closed,

γ'-closed, γ'-β-generalized closed and (γ, γ')-β-irresolute mappings where γ and γ' are operations. They investigated some properties from it and defined another two functions called τ-γ-β-continuous and γ-γ-β-continuous by using τ-γ-β-open and γ-γ-β-open sets respectively.

Ghosh (2012), Kalaivani and Krishnan (2012) and Kalaivani, Kumar and Krishnan (2012) introduced some other types of mappings called γ-precontinuity, γ-preopenness and γ-preclosedness; α-(γ, β)-continuous, α-(γ, β)-open and α-(γ, β)-closed; and α-γ-continuous, α-γ-irresolute, α-γ-open and α-γ-closed, respectively.

1.2.3 γ- Separation Axioms

Separation axioms is one of the most important and interesting concepts in topology. Ogata (1991) introduced the concept of γ-T_i for $i = 0, \frac{1}{2}, 1, 2$, and characterized γ-T_i spaces using the notion of γ-open and γ-closed sets. Krishnan and Balachandran (2006a) introduced the concept of γ-preT_i for $i = 0, \frac{1}{2}, 1, 2$, and characterized γ-preT_i spaces using the notion of γ-preclosed and γ-preopen sets. Ghosh (2012) proved that every topological space (X, τ) with an operation γ on τ is γ-pre$T_{\frac{1}{2}}$ as well as he studied two new separation axioms utilizing γ-preopen and γ-preclosed sets such as γ-pre-regularity and γ-pre-normality. He studied some other properties by using γ-preopen sets in a topological space (X, τ). Krishnan and Balachandran (2006b) introduced the concept of γ-semiT_i for $i = 0, \frac{1}{2}, 1, 2$, and characterized γ-semiT_i spaces using the notion of

γ-semiclosed and γ-semiopen sets. Furthermore, Basu, Afsan and Ghosh (2009) defined the concept of γ-β-T_i for $i = 0, \frac{1}{2}, 1, 2$, γ-β-regular and γ-β-normal spaces and examined some of its properties. They also proved that every topological space (X, τ) with an operation γ on τ is γ-β-$T_{\frac{1}{2}}$.

1.3 Problem Statement

The study of the γ-operation on τ, γ-open set, γ-functions and its separation axioms was initially introduced by Ogata (1991). This approach has led many researchers to introduce other operations on τ and their properties. However, to the best of our knowledge, no one has attempted to define γ-P_S-open sets for $\gamma\colon \tau \to P(X)$. Therefore, this study will theoretically develop γ-P_S-open sets as an extension of Ogata's work (1991) which lies strictly between the classes of γ-regular-open set and γ-preopen set, but there is no relation between γ-P_S-open set with γ-open, α-γ-open and γ-semiopen sets. In addition, some new types of γ-P_S- functions and γ-P_S- separation axioms by using γ-P_S-open sets will also be developed.

1.4 Research Objectives

The main objective of this study is to develop γ-P_S- operations in topological spaces. In order to accomplish the main objective, the following subobjectives must be considered.

1. To construct γ-regular-open set, γ-extremally disconnected, γ-locally indiscrete and γ-hyperconnected spaces.

2. To construct a new class of sets in a topological space (X, τ) with an operation γ on τ which is the γ-P_S-open set.

3. To develop new operations for γ-P_S-open set including τ_γ-P_S-interior, τ_γ-P_S-closure, τ_γ-P_S-derived set and τ_γ-P_S-boundary.

4. To define new classes of γ-P_S- functions such as γ-P_S-continuous, (γ, β)-P_S-continuous, (γ, β)-P_S-irresolute, β-P_S-open, β-P_S-closed, (γ, β)-P_S-open, (γ, β)-P_S-closed, $(\gamma, \beta P_S)$-open and $(\gamma, \beta P_S)$-closed and some characterizations of these functions.

5. To define new class of sets called γ-P_S-g-closed and to apply it to introduce γ-P_S-g-continuous, β-P_S-g-open and β-P_S-g-closed.

6. To establish some new γ-P_S- separation axioms utilizing γ-P_S-open sets such as γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$, γ-P_S-regularity, γ-P_S-normality, γ-P_S-R_0 and γ-P_S-R_1 spaces.

1.5 Scope of the research

This research is focuses on theoretical development of the γ-P_S-open set where γ-P_S-open set is a new class of sets in a topological space (X, τ) with an operation γ on τ. The construction of this new class comes from the parallelism of ideas and the variation of classes in the previous work. It also include τ_γ-P_S-closure, τ_γ-P_S-interior, τ_γ-P_S-derived set and τ_γ-P_S-boundary are notions of a space X. Further, γ-P_S-continuous,

(γ, β)-P_S-continuous, (γ, β)-P_S-closed and β-P_S-generalized closed are new functions as well as γ-P_S-T_i spaces for $i = 0, \frac{1}{2}, 1, 2$, γ-P_S-regularity and γ-P_S-normality are new separation axioms.

1.6 Significance of the study

The major contribution of this study is the new theoretical results on γ- operations which is γ-P_S-open set. This study also contributes to a greater challenge of finding new operation of topological spaces. This development is give a new insight in the field of topology.

1.7 Thesis Outline

There are seven chapters in this thesis. Chapter One provides the introduction to topology, research background, problem statement, research objectives, scope of the research, and significance of the study.

Chapter Two is divided into four sections. The first section is the introduction of some definitions of sets, γ- sets and γ- operations in topological space (X, τ). The second, third and fourth sections are devoted to the main definitions and results on γ- sets, γ- functions and γ- separation axioms respectively and will be used in the remainder of the work.

Chapter Three provides the preliminary work of this study. A new class of γ- sets called γ-regular-open sets in a topological space (X, τ) with its complement which is γ-regular-closed sets is defined. We also introduce some new classes of γ- spaces namely γ-extremally disconnected, γ-locally indiscrete and γ-hyperconnected spaces. Furthermore, some characterizations of these γ- spaces are studied. The results from this chapter will be used in the development in the next three chapters.

The main findings of this thesis are given in Chapters 4, 5 and 6. Chapter Four defines a new class of sets called γ-P_S-open sets in a topological space (X, τ), where γ is an operation on τ. This class of sets lies strictly between the classes of γ-regular-open and γ-preopen sets. Some basic relationships between γ-P_S-open set and other types of γ- sets are obtained. Moreover, the complement of γ-P_S-open set which is γ-P_S-closed is provided with some important properties. After that the notions of τ_γ-P_S-closure, τ_γ-P_S-interior, τ_γ-P_S-derived set and τ_γ-P_S-boundary of a set by using γ-P_S-open and γ-P_S-closed sets are studied. Later, we define another class of γ-P_S- sets called γ-P_S-generalized closed set by using γ-P_S-open set and τ_γ-P_S-closure operator. Finally, some results and their properties are discussed.

Chapter Five defines some new classes of γ-P_S- functions called γ-P_S-continuous, (γ, β)-P_S-continuous and (γ, β)-P_S-irresolute functions via γ-P_S-open sets. The relationships among them which include γ-P_S- functions and the other classes of functions are given. Furthermore, some properties and characterizations of these γ-P_S- functions are investigated. Moreover, some other classes of γ-P_S- functions namely

β-P_S-open, β-P_S-closed, (γ, β)-P_S-open, (γ, β)-P_S-closed, $(\gamma, \beta P_S)$-open and $(\gamma, \beta P_S)$-closed functions are introduced. Next γ-P_S-g-continuous, β-P_S-g-open and β-P_S-g-closed which are types of γ-P_S- functions as well as their properties are defined. A completely γ-continuous function has been defined by applying γ-regular-open set. Finally, some composition of these γ-P_S- functions are given.

In Chapter Six, we introduce some new types of γ-P_S- separation axioms called γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$ spaces by using γ-P_S-open, γ-P_S-closed and γ-P_S-g-closed sets. Some relationships between γ-P_S-T_i spaces and other γ- separation axioms are also given. By applying γ-P_S-open and γ-P_S-closed sets, many types of γ-P_S- separation axioms called γ-P_S-regular and γ-P_S-normal spaces are defined. In addition, some basic properties and preservation theorems of these γ-P_S- spaces are obtained. Later, γ-P_S-R_1 and γ-P_S-R_2 spaces as well as some of their properties are defined.

Finally, Chapter Seven concludes the whole thesis with a summary of the study and also provides suggestions for further researches in the field of general topology.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This chapter recalls all necessary backgrounds needed for the sets, functions, spaces and separation axioms in topological spaces. Also we will use these concepts throughout this thesis. Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms assumed unless explicitly stated. Let A be any subset of a topological space (X, τ), $Int(A)$ and $Cl(A)$ denotes the interior of A and the closure of A, respectively.

2.2 Preliminaries and Basic Definitions

In this section, we recall some definitions, preliminaries, results available in the literature to construct γ-P_S-open set.

We shall start some basic definitions of sets by using the interior and the closure of a set A.

Definition 2.2.1. Let A be any subset of a topological space (X, τ). Then A is called:

1. Regular-open if $A = Int(Cl(A))$ (Steen and Seebach, 1978).

2. Preopen if $A \subseteq Int(Cl(A))$ (Mashhour, Abd El-Monsef and El-Deeb, 1982).

3. Semiopen if $A \subseteq Cl(Int(A))$ (Levine, 1963).

4. α-open if $A \subseteq Int(Cl(Int(A)))$ (Njastad, 1965).

5. b-open if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ (Andrijevic, 1996).

6. β-open if $A \subseteq Cl(Int(Cl((A)))$ (Abd El-Monsef, El-Deeb and Mahmoud, 1983).

The notion β-open set is also known as semi preopen (Andrijevic, 1986).

The complement of a regular-open set is said to be regular-closed (Steen and Seebach, 1978). Similarly the complement of a preopen set is said to be preclosed (El-Deeb, Hasanein, Mashhour and Noiri, 1983), the complement of α-open set is α-closed (Reilly and Vamanmurthy, 1985), the complement of semiopen set is semiclosed (Crossley and Hildebrand, 1971), the complement of b-open set is b-closed (Andrijevic, 1996), the complement of β-open set is β-closed (Abd El-Monsef, Mahmoud and Lashin, 1986) and the complement of semi preopen set is semi preclosed (Andrijevic, 1986). In other words, the complement for Definition 2.2.1 is given as follows.

Definition 2.2.2. A subset A of a space X is said to be:

1. Regular-closed if $A = Cl(Int(A))$ (Steen and Seebach, 1978).

2. Preclosed if $Cl(Int(A)) \subseteq A$ (El-Deeb, Hasanein, Mashhour and Noiri, 1983).

3. Semiclosed if $Int(Cl(A)) \subseteq A$ (Crossley and Hildebrand, 1971).

4. α-closed if $Cl(Int(Cl(A))) \subseteq A$ (Reilly and Vamanmurthy, 1985).

5. b-closed if $Int(Cl(A)) \cap Cl(Int(A)) \subseteq A$ (Andrijevic, 1996).

6. β-closed if $Int(Cl(Int(A))) \subseteq A$ (Abd El-Monsef, Mahmoud and Lashin, 1986).

The notion β-closed set is also known as semi preclosed (Andrijevic, 1986).

Definition 2.2.3. (Khalaf and Asaad, 2009) A preopen subset A of a topological space (X, τ) is said to be P_S-open if for each $x \in A$, there exists a semiclosed set F such that $x \in F \subseteq A$. The complement of a P_S-open set is said to be P_S-closed.

Lemma 2.2.4. (Khalaf and Asaad, 2009) A subset A of a topological space (X, τ) is P_S-open if and only if A is preopen set and A is a union of semiclosed sets.

There are many operations γ on the sets in Definition 2.2.1, defined as follows:

Definition 2.2.5. Let (X, τ) be a topological space. Then:

1. An operation γ on the topology τ on X is a function $\gamma\colon \tau \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U. A nonempty set A of X with an operation γ on τ is said to be γ-open if for each $x \in A$, there exists an open set U such that $x \in U$ and $\gamma(U) \subseteq A$ (Ogata, 1991).

2. An operation γ on $PO(X)$ is a function $\gamma\colon PO(X) \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in PO(X, \tau)$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U. A nonempty set A of X is said to be pre-γ-open if for each $x \in A$, there exists a preopen set U such that $x \in U$ and $\gamma(U) \subseteq A$ (Van An, Cuong and Maki, 2008).

3. An operation γ on $SO(X)$ is a function $\gamma\colon SO(X) \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in SO(X, \tau)$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U. A nonempty set A of X is said to be semi-γ-open if for each $x \in A$, there exists a semiopen set U such that $x \in U$ and $\gamma(U) \subseteq A$ (Krishnan, Ganster and Balachandran, 2007).

4. An operation γ on $BO(X)$ is a function $\gamma\colon BO(X) \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in BO(X, \tau)$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U. A nonempty set A of X is said to be b-γ-open if for each $x \in A$, there exists a b-open set U such that $x \in U$ and $\gamma(U) \subseteq A$ (Carpintero, Rajesh and Rosas, 2012b).

5. An operation γ on $\beta O(X)$ is a function $\gamma\colon \beta O(X) \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \beta O(X, \tau)$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U. A nonempty set A of X is said to be β-γ-open if for each $x \in A$, there exists a β-open set U such that $x \in U$ and $\gamma(U) \subseteq A$ (Tahiliani, 2011).

Throughout of our research, we only use the operation $\gamma\colon \tau \to P(X)$ as defined in Definition 2.2.5 (1).

In the remaining sections of this chapter, we review some properties involving sets, functions and separation axioms by using operation γ.

2.3 Properties of γ- Sets

In this section, we consider sets, operations on them and some relationships between them.

The operations defined by $\gamma(A) = A$, $\gamma(A) = X$, $\gamma(A) = Cl(A)$ and $\gamma(A) = Int(Cl(A))$ are examples of operation γ.

We frequently work with operation $\gamma\colon \tau \rightarrow P(X)$, particularly γ-open sets in a topological space (X, τ), defined as follows:

Definition 2.3.1. Ogata (1991) A subset A of a topological space (X, τ) is called a γ-open if for each $x \in A$, there exists an open set U such that $x \in U$ and $\gamma(U) \subseteq A$. The complement of a γ-open set is called a γ-closed.

It is obvious from Definition 2.3.1 that every γ-open set is open, this means that $\tau_\gamma \subseteq \tau$, where τ_γ denotes the set of all γ-open sets in a topological space (X, τ).

To simplify discussions as we proceed, we recall the following three definitions. They are the notions of τ_γ-closure, γ-closure and τ_γ-interior of a set.

Definition 2.3.2. (Ogata, 1991) Let A be any subset of a topological space (X, τ) and γ be an operaton on τ. Then the τ_γ-closure of A is defined as the intersection of all γ-closed sets containing A and it is denoted by τ_γ-$Cl(A)$. That is,

$$\tau_\gamma\text{-}Cl(A) = \cap\{F : A \subseteq F, X\backslash F \in \tau_\gamma\}$$

Definition 2.3.3. (Ogata, 1991) The point $x \in X$ is in the γ-closure of a set $A \subseteq X$ if $\gamma(U) \cap A \neq \phi$ for each open set U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by $Cl_\gamma(A)$.

Definition 2.3.4. (Krishnan and Balachandran, 2006b) Let A be any subset of a topological space (X, τ) and γ be an operaton on τ. Then the τ_γ-interior of A is defined as the union of all γ-open sets contained in A and it is denoted by τ_γ-$Int(A)$. That is, τ_γ-$Int(A) = \cup\{U : U \subseteq A$ and $U \in \tau_\gamma\}$.

The following two remarks follow directly from the definition of τ_γ-closure, γ-closure and τ_γ-interior.

Remark 2.3.5. Let (X, τ) be a topological space and let A be any subset of X, then:

$$\tau_\gamma\text{-}Int(A) \subseteq Int(A) \subseteq A \subseteq Cl(A) \subseteq Cl_\gamma(A) \subseteq \tau_\gamma\text{-}Cl(A)$$

Remark 2.3.6. For any subset A of a topological space (X, τ). Then:

1. A is γ-open if and only if τ_γ-$Int(A) = A$ (Krishnan and Balachandran, 2006b).

2. A is γ-closed if and only if τ_γ-$Cl(A) = A$ (Ogata, 1991).

If A is both γ-open and γ-closed set, then we say that A is γ-clopen which is defined as follows:

Definition 2.3.7. Let A be any subset of a topological space (X, τ) with an operation γ on τ. Then A is γ-clopen if and only if A is both γ-open and γ-closed. The complement of a γ-clopen set is also a γ-clopen set.

Now, the following definition with their complement are very important sets throughout this thesis which are used τ_γ-closure and τ_γ-interior.

Definition 2.3.8. Let (X, τ) be a topological space and γ be an operation on τ. A subset A of X is said to be:

1. γ-preopen if $A \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A))$ (Krishnan and Balachandran, 2006a).

2. γ-semiopen if $A \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(A))$ (Krishnan and Balachandran, 2006b).

3. α-γ-open if $A \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(A)))$ (Kalaivani and Krishnan, 2009).

4. γ-b-open if $A \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(A)) \cup \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A))$ (Carpintero, Rajesh and Rosas, 2012a).

5. γ-β-open if $A \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A)))$ (Basu, Afsan and Ghosh, 2009).

The notion γ-β-open set is also known as γ-semi preopen (Krishnan and Balachandran, 2006a).

The complement of a γ-preopen set is said to be γ-preclosed (Krishnan and Balachandran, 2006a). Similarly the complement of a γ-semiopen set is γ-semiclosed (Krishnan and Balachandran, 2006b), the complement of α-γ-open set is α-γ-closed (Kalaivani and Krishnan, 2009), the complement of γ-b-open set is γ-b-closed (Carpintero, Rajesh and Rosas, 2012a), the complement of γ-β-open set is γ-β-closed (Basu, Afsan and Ghosh, 2009) and the complement of γ-semi preopen set is γ-semi preclosed (Krishnan and Balachandran, 2006a). In other words, the complement for Definition 2.3.8 is given as follows.

Definition 2.3.9. A subset A of a space X is said to be:

1. γ-preclosed if τ_γ-$Cl(\tau_\gamma$-$Int(A)) \subseteq A$ (Krishnan and Balachandran, 2006a).

2. γ-semiclosed if τ_γ-$Int(\tau_\gamma$-$Cl(A)) \subseteq A$ (Krishnan and Balachandran, 2006b).

3. α-γ-closed if τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(A))) \subseteq A$ (Kalaivani and Krishnan, 2009).

4. γ-b-closed if τ_γ-$Int(\tau_\gamma$-$Cl(A)) \cap \tau_\gamma$-$Cl(\tau_\gamma$-$Int(A)) \subseteq A$ (Carpintero, Rajesh and Rosas, 2012a).

5. γ-β-closed if τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(A))) \subseteq A$ (Basu, Afsan and Ghosh, 2009).

The notion γ-β-closed set is also known as γ-semi preclosed (Krishnan and Balachandran, 2006a).

The observation from Definition 2.3.8, we have the following remark which provides the relationships between γ- sets.

Remark 2.3.10. Let (X, τ) be a topological space and γ be an operation on τ. Let A be any subset of X.

1. If A is γ-open, then A is γ-preopen and if A is γ-preopen, then A is γ-semi preopen (Krishnan and Balachandran, 2006a).

2. If A is γ-preopen, then A is γ-β-open (Ghosh, 2012).

3. If A is γ-preopen, then A is γ-b-open and if A is γ-b-open, then A is γ-β-open.

4. If A is γ-semiopen, then A is γ-b-open.

5. If A is γ-open, then A is γ-β-open and if A is γ-semiopen, then A is γ-β-open (Basu, Afsan and Ghosh, 2009).

6. If A is γ-open, then A is γ-semiopen (Krishnan and Balachandran, 2006b).

7. If A is γ-open, then A is α-γ-open and if A is α-γ-open, then A is γ-preopen, γ-semiopen and γ-semi preopen (Kalaivani and Krishnan, 2013).

8. If A is γ-open, then A is γ-b-open (Carpintero, Rajesh and Rosas, 2012a).

9. A is α-γ-open if and only if it is γ-preopen and γ-semiopen (Kalaivani and Krishnan, 2013).

From the observation of the previous study, the following figure follows directly from Remark 2.3.10.

$$\begin{tikzcd}
\gamma\text{-open} \arrow[r] & \alpha\text{-}\gamma\text{-open} \arrow[r] \arrow[d] & \gamma\text{-semiopen} \arrow[d] & \\
\gamma\text{-clopen} \arrow[u] & \gamma\text{-preopen} \arrow[r] & \gamma\text{-}b\text{-open} \arrow[r] & \gamma\text{-}\beta\text{-open}
\end{tikzcd}$$

Remark: $A \longrightarrow B$ represents A implies B

Figure 2.1. The relations between various types of γ-sets

Next, there are more notions of interior and closure can be defined by using the sets in the Definitions 2.3.8 and 2.3.9.

Definition 2.3.11. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then:

1. The τ_γ-preinterior of A is defined as the union of all γ-preopen sets of X contained in A and it is denoted by τ_γ-$pInt(A)$ and τ_γ-preclosure of A is defined as the intersection of all γ-preclosed sets of X containing A and it is denoted by τ_γ-$pCl(A)$ (Krishnan and Balachandran, 2006a).

2. The τ_γ-semi-interior of A is defined as the union of all γ-semiopen sets of X contained in A and it is denoted by τ_γ-$sInt(A)$ and τ_γ-semi-closure of A is defined as the intersection of all γ-semiclosed sets of X containing A and it is denoted by τ_γ-$sCl(A)$ (Krishnan and Balachandran, 2006b).

3. The $\tau_{\alpha-\gamma}$-interior of A is defined as the union of all α-γ-open sets of X contained in A and it is denoted by $\tau_{\alpha-\gamma}$-$Int(A)$ and $\tau_{\alpha-\gamma}$-closure of A is defined as the intersection of all α-γ-closed sets of X containing A and it is denoted by $\tau_{\alpha-\gamma}$-$Cl(A)$ (Kalaivani and Krishnan, 2009).

4. The τ_γ-b-interior of A is defined as the union of all γ-b-open sets of X contained in A and it is denoted by τ_γ-$bInt(A)$ and τ_γ-b-closure of A is defined as the intersection of all γ-b-closed sets of X containing A and it is denoted by τ_γ-$bCl(A)$ (Carpintero, Rajesh and Rosas, 2012a).

5. The τ_γ-β-interior of A is defined as the union of all γ-β-open sets of X contained in A and it is denoted by τ_γ-$\beta Int(A)$ and τ_γ-β-closure of A is defined as the intersection of all γ-β-closed sets of X containing A and it is denoted by τ_γ-$\beta Cl(A)$ (Basu, Afsan and Ghosh, 2009).

The following theorem follows directly from Definition 2.3.11.

Theorem 2.3.12. Let (X, τ) be a topological space and γ be an operation on τ. For any subset A of a space X, the following statements are true.

1. A is γ-preopen if and only if τ_γ-$pInt(A) = A$ and A is γ-preclosed if and only if τ_γ-$pCl(A) = A$ (Krishnan and Balachandran, 2006a).

2. A is γ-semiopen if and only if τ_γ-$sInt(A) = A$ and A is γ-semiclosed if and only if τ_γ-$sCl(A) = A$ (Krishnan and Balachandran, 2006b).

3. A is α-γ-open if and only if $\tau_{\alpha-\gamma}$-$Int(A) = A$ and A is α-γ-closed if and only if $\tau_{\alpha-\gamma}$-$Cl(A) = A$ (Kalaivani and Krishnan, 2009).

4. A is γ-b-open if and only if τ_γ-$bInt(A) = A$ and A is γ-b-closed if and only if τ_γ-$bCl(A) = A$ (Carpintero, Rajesh and Rosas, 2012a).

5. A is γ-β-open if and only if τ_γ-$\beta Int(A) = A$ and A is γ-β-closed if and only if τ_γ-$\beta Cl(A) = A$ (Basu, Afsan and Ghosh, 2009).

Clearly, by using Remark 2.3.10 and Definition 2.3.11, we lay down the following remark.

Remark 2.3.13. Let (X, τ) be a topological space and γ be an operation on τ. Let A be any subset of X. Then:

1. τ_γ-$Int(A) \subseteq \tau_{\alpha-\gamma}$-$Int(A) \subseteq \tau_\gamma$-$pInt(A) \subseteq \tau_\gamma$-$bInt(A) \subseteq \tau_\gamma$-$\beta Int(A) \subseteq A$.

2. τ_γ-$Int(A) \subseteq \tau_{\alpha-\gamma}$-$Int(A) \subseteq \tau_\gamma$-$sInt(A) \subseteq \tau_\gamma$-$bInt(A) \subseteq \tau_\gamma$-$\beta Int(A) \subseteq A$.

3. $A \subseteq \tau_\gamma$-$\beta Cl(A) \subseteq \tau_\gamma$-$bCl(A) \subseteq \tau_\gamma$-$pCl(A) \subseteq \tau_{\alpha-\gamma}$-$Cl(A) \subseteq \tau_\gamma$-$Cl(A)$.

4. $A \subseteq \tau_\gamma$-$\beta Cl(A) \subseteq \tau_\gamma$-$bCl(A) \subseteq \tau_\gamma$-$sCl(A) \subseteq \tau_{\alpha-\gamma}$-$Cl(A) \subseteq \tau_\gamma$-$Cl(A)$.

The following lemma provides some useful relationships involving τ_γ-interior, τ_γ-closure and complement.

Lemma 2.3.14. (Krishnan and Balachandran, 2006a) Let (X, τ) be a topological space and γ be an operation on τ. Let A be any subset of X. Then τ_γ-$Cl(A) = X \backslash \tau_\gamma$-$Int(X \backslash A)$ and τ_γ-$Int(A) = X \backslash \tau_\gamma$-$Cl(X \backslash A)$.

The next theorem presents another important properties of γ-open set and the sets defined in Definition 2.3.8.

Theorem 2.3.15. Let A be a subset of a topological space (X, τ) and γ be an operation on τ. Then:

1. $x \in \tau_\gamma$-$Cl(A)$ if and only if $A \cap U \neq \phi$ for every γ-open set U of X containing x (Ogata, 1991).

2. $x \in \tau_\gamma$-$pCl(A)$ if and only if $A \cap U \neq \phi$ for every γ-preopen set U of X containing x (Krishnan and Balachandran, 2006a).

3. $x \in \tau_\gamma$-$sCl(A)$ if and only if $A \cap U \neq \phi$ for every γ-semiopen set U of X containing x (Krishnan and Balachandran, 2006b).

4. $x \in \tau_{\alpha-\gamma}$-$Cl(A)$ if and only if $A \cap U \neq \phi$ for every α-γ-open set U of X containing x (Kalaivani and Krishnan, 2009).

5. $x \in \tau_\gamma$-$bCl(A)$ if and only if $A \cap U \neq \phi$ for every γ-b-open set U of X containing x (Carpintero, Rajesh and Rosas, 2012a).

6. $x \in \tau_\gamma$-$\beta Cl(A)$ if and only if $A \cap U \neq \phi$ for every γ-β-open set U of X containing x (Basu, Afsan and Ghosh, 2009).

A point x in X is a limit point of a set A if every open set of x intersects A in a point other than x. So the set of all limit points of A is a derived set of A. Therefore, by applying this property, many athors defind derived set by their sets. The following definition is the definitions of γ-derived set, γ-prederived set and γ-β-derived set.

Definition 2.3.16. Let (X, τ) be a topological space and γ be an operation on τ. Let A be any subset of X. Then:

1. The γ-derived set of A is defined as $\{x$: for every γ-open set U containing x, $U \cap A \setminus \{x\} \neq \phi\}$ and it is denoted by τ_γ-$D(A)$ (Basu, Afsan and Ghosh, 2009).

2. The γ-prederived set of A is defined as $\{x$: for every γ-preopen set U containing x, $U \cap A \setminus \{x\} \neq \phi\}$ and it is denoted by τ_γ-$pD(A)$ (Ghosh, 2012).

3. The γ-β-derived set of A is defined as $\{x$: for every γ-β-open set U containing x, $U \cap A \setminus \{x\} \neq \phi\}$ and it is denoted by τ_γ-$\beta D(A)$ (Basu, Afsan and Ghosh, 2009).

The boundary of a set is the set of all points which is belong to closure but does not belong to interior. Then by using the same idea, the following is the definition of γ-boundary, γ-preboundary and γ-β-boundary by using their notions of interior and closure.

Definition 2.3.17. Let (X, τ) be a topological space and γ be an operation on τ. Let A be any subset of X. Then:

1. The γ-boundary of A is defined as $\tau_\gamma\text{-}Cl(A) \cap \tau_\gamma\text{-}Cl(X \backslash A)$ and it is denoted by $\tau_\gamma\text{-}Bd(A)$ (Basu, Afsan and Ghosh, 2009).

2. The γ-preboundary of A is defined as $\tau_\gamma\text{-}pCl(A) \cap \tau_\gamma\text{-}pCl(X \backslash A)$ and it is denoted by $\tau_\gamma\text{-}pBd(A)$ (Ghosh, 2012).

3. The γ-β-boundary of A is defined as $\tau_\gamma\text{-}\beta Cl(A) \cap \tau_\gamma\text{-}\beta Cl(X \backslash A)$ and it is denoted by $\tau_\gamma\text{-}\beta Bd(A)$ (Basu, Afsan and Ghosh, 2009).

The subset A of a space X is dense if the closure of A is X, then we have the following definition.

Definition 2.3.18. (Carpintero, Rajesh and Rosas, 2012c) A subset D of a topological space (X, τ) is said to be:

1. γ-dense if $\tau_\gamma\text{-}Cl(D) = X$.

2. γ-semi-dense if $\tau_\gamma\text{-}sCl(D) = X$.

By using γ-open set with γ-closure, and also for the other types of sets in Definition 2.3.8 with their closure operators, we have the following definition.

Definition 2.3.19. Let (X, τ) be a topological space and γ be an operation on τ. A subset A of X is called:

1. γ-generalized closed (γ-g-closed) if $Cl_{\gamma}(A) \subseteq G$ whenever $A \subseteq G$ and G is a γ-open set in X (Ogata, 1991).

2. γ-pre-generalized closed (γ-pre$g$-closed) if τ_{γ}-$pCl(A) \subseteq G$ whenever $A \subseteq G$ and G is a γ-preopen set in X (Krishnan and Balachandran, 2006a).

3. γ-semi-generalized closed (γ-semi$g$-closed) if τ_{γ}-$sCl(A) \subseteq G$ whenever $A \subseteq G$ and G is a γ-semiopen set in X (Krishnan and Balachandran, 2006b).

4. α-γ-generalized closed (α-γ-g-closed) if $\tau_{\alpha-\gamma}$-$Cl(A) \subseteq G$ whenever $A \subseteq G$ and G is an α-γ-open set in X (Kalaivani and Krishnan, 2009).

5. γ-b-generalized closed (γ-b-g-closed) if τ_{γ}-$bCl(A) \subseteq G$ whenever $A \subseteq G$ and G is a γ-b-open set in X (Carpintero, Rajesh and Rosas, 2012a).

6. γ-β-generalized closed (γ-βg-closed) if τ_{γ}-$\beta Cl(A) \subseteq G$ whenever $A \subseteq G$ and G is a γ-β-open set in X (Basu, Afsan and Ghosh, 2009).

Some basic relationships between types of γ- closed sets and γ- generalized closed sets as defined in Definition 2.3.9 and Definition 2.3.19 respectively, and then we make them as a figure.

Remark 2.3.20. Every γ-closed set is γ-g-closed (Ogata, 1991). Similarly, every α-γ-closed set is α-γ-g-closed (Kalaivani and Krishnan, 2009), every γ-preclosed set is γ-pre$g$-closed (Krishnan and Balachandran, 2006a), every γ-semiclosed set is γ-semi$g$-closed (Krishnan and Balachandran, 2006b), every γ-b-closed set is γ-b-g-closed and every γ-β-closed set is γ-βg-closed.

From the observation of the previous study, the relations as shown in Figure 2.2 follow directly from Definition 2.3.9, Definition 2.3.19 and Remark 2.3.20.

$$
\begin{tikzcd}
\gamma\text{-pre}g\text{-closed} & \gamma\text{-preclosed} \arrow[l] \arrow[r] & \gamma\text{-}b\text{-closed} \arrow[r] \arrow[dr] & \gamma\text{-}b\text{-}g\text{-closed} \\
\gamma\text{-closed} \arrow[r] \arrow[d] & \alpha\text{-}\gamma\text{-closed} \arrow[r] \arrow[u] \arrow[d] & \gamma\text{-semiclosed} \arrow[u] \arrow[d] & \gamma\text{-}\beta\text{-closed} \arrow[d] \\
\gamma\text{-}g\text{-closed} & \alpha\text{-}\gamma\text{-}g\text{-closed} & \gamma\text{-semi}g\text{-closed} & \gamma\text{-}\beta g\text{-closed}
\end{tikzcd}
$$

Remark: $A \longrightarrow B$ represents A implies B

Figure 2.2. Relations among types of γ-closed sets and γ-g-closed sets

In the above figure, we notice that the notions of γ-g-closed, α-γ-g-closed, γ-semi$g$-closed, γ-pre$g$-closed, γ-b-g-closed and γ-βg-closed are independent. In addition, there is no relation between γ-semiclosed set and γ-preclosed set.

The complement of a γ-pre$g$-closed set is said to be γ-pre$g$-open (Ghosh, 2012). Similarly the complement of a γ-semi$g$-closed set is γ-semi$g$-open (Krishnan and Balachandran, 2006b), the complement of α-γ-g-closed set is α-γ-g-open (Kalaivani and Krishnan, 2009), the complement of γ-b-g-closed set is γ-b-g-open (Carpintero, Rajesh and Rosas, 2012a) and the complement of γ-βg-closed set is γ-β-open (Basu, Afsan and Ghosh, 2009). In other words, the complement for Definition 2.3.19 is given as follows.

Definition 2.3.21. A subset A of a topological space (X, τ) is said to be:

1. γ-pre$g$-open if $F \subseteq \tau_\gamma$-$pInt(A)$ whenever $F \subseteq A$ and F is a γ-preclosed set in X (Krishnan and Balachandran, 2006a).

2. γ-semi$g$-open if $F \subseteq \tau_\gamma$-$sInt(A)$ whenever $F \subseteq A$ and F is a γ-semiclosed set in X (Krishnan and Balachandran, 2006b).

3. α-γ-g-open if $F \subseteq \tau_{\alpha-\gamma}$-$Int(A)$ whenever $F \subseteq A$ and F is an α-γ-closed set in X (Kalaivani and Krishnan, 2009).

4. γ-b-g-open if $F \subseteq \tau_\gamma$-$bInt(A)$ whenever $F \subseteq A$ and F is a γ-b-closed set in X (Carpintero, Rajesh and Rosas, 2012a).

5. γ-βg-open if $F \subseteq \tau_\gamma$-$\beta Int(A)$ whenever $F \subseteq A$ and F is a γ-β-closed set in X (Basu, Afsan and Ghosh, 2009).

Each singleton set is either γ-preclosed or its complement is γ-pre$g$-closed. Similarly for the other sets. So the following theorem shows between types of γ- closed sets and γ-g- closed sets

Theorem 2.3.22. Let (X, τ) be any topological space and γ be an operation on τ. Then, for each element $x \in X$, the following statements are true:

1. Either the set $\{x\}$ is γ-closed or the set $X \backslash \{x\}$ is γ-g-closed (Ogata, 1991).

2. Either the set $\{x\}$ is α-γ-closed or the set $X \backslash \{x\}$ is α-γ-g-closed (Kalaivani and Krishnan, 2009).

3. Either the set $\{x\}$ is γ-preclosed or the set $X \backslash \{x\}$ is γ-pre$g$-closed (Krishnan and Balachandran, 2006a).

4. Either the set $\{x\}$ is γ-semiclosed or the set $X\backslash\{x\}$ is γ-semi$g$-closed (Krishnan and Balachandran, 2006b).

5. Either the set $\{x\}$ is γ-β-closed or the set $X\backslash\{x\}$ is γ-βg-closed (Basu, Afsan and Ghosh, 2009).

6. Either the set $\{x\}$ is γ-b-closed or the set $X\backslash\{x\}$ is γ-b-g-closed (Carpintero, Rajesh and Rosas, 2012a).

Using the notions of interior, other sets can also be defined.

Definition 2.3.23. (Basu, Afsan and Ghosh, 2009) A subset A of a topological space (X, τ) with an operation γ on τ is called γ-γ-β-open (respectively, τ-γ-β-open) if τ_γ-$Int(A) = \tau_\gamma$-$\beta Int(A)$ (respectively, $Int(A) = \tau_\gamma$-$\beta Int(A)$).

Since τ_γ is not a topology in general, but it becomes a topology when an operation γ is regular, defined as follows.

Definition 2.3.24. Ogata (1991): Let (X, τ) be any topological space. An operation γ on τ is said to be regular if for every open neighborhood U and V of each $x \in X$, there exists an open neighborhood W of x such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.

Ogata (1991) showed that when γ is a regular operation, then τ_γ is a topology on X. The following are some related results on regular operation γ.

Lemma 2.3.25. (Krishnan and Balachandran, 2006a) Let (X, τ) be a topological space and γ be a regular operation on τ. Then for every γ-open set U and every subset A of X, we have τ_γ-$Cl(A) \cap U \subseteq \tau_\gamma$-$Cl(A \cap U)$.

Theorem 2.3.26. (Krishnan and Balachandran, 2006a) Let A be any subset of a topological space (X, τ) and γ be a regular operation on τ. Then the following statements hold:

1. τ_γ-$pInt(A) = A \cap \tau_\gamma$-$Int(\tau_\gamma$-$Cl(A))$

2. τ_γ-$pCl(A) = A \cup \tau_\gamma$-$Cl(\tau_\gamma$-$Int(A))$

Theorem 2.3.27. (Krishnan and Balachandran, 2006b) Let A be any subset of a topological space (X, τ) and γ be a regular operation on τ. Then the following properties are true:

1. τ_γ-$sInt(A) = A \cap \tau_\gamma$-$Cl(\tau_\gamma$-$Int(A))$

2. τ_γ-$sCl(A) = A \cup \tau_\gamma$-$Int(\tau_\gamma$-$Cl(A))$

Theorem 2.3.28. (Carpintero, Rajesh and Rosas, 2012a) Let A be any subset of a topological space (X, τ) and γ be a regular operation on τ. Then the following statements hold:

1. τ_γ-$bInt(A) = \tau_\gamma$-$pInt(A) \cup \tau_\gamma$-$sInt(A)$

2. τ_γ-$bCl(A) = \tau_\gamma$-$pCl(A) \cap \tau_\gamma$-$sCl(A)$

Krishnan and Balachandran (2006a) showed that the intersection of any γ-open set and γ-preopen set is γ-preopen set in any topological space (X, τ), where γ is a regular operation on τ.

Theorem 2.3.29. (Krishnan and Balachandran, 2006a) Let G, H be any subsets of a topological space (X, τ) and γ be a regular operation on τ. If G is γ-open and H is γ-preopen, then $G \cap H$ is γ-preopen set.

In general, $\tau_\gamma \subseteq \tau$, but the converse is true when a space (X, τ) is γ-regular. So the definition of γ-regular space (X, τ) is defined as follows.

Definition 2.3.30. (Ogata, 1991) A topological space (X, τ) with an operation γ on τ is said to be γ-regular if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that $\gamma(U) \subseteq V$.

Then we can conclude from the previous definition that τ_γ-$Int(A)$ and $Int(A)$ (τ_γ and τ) are equivalent as shown in the next remark.

Remark 2.3.31. If a topological space (X, τ) is γ-regular, then $\tau_\gamma = \tau$ (Ogata, 1991) and hence τ_γ-$Int(A) = Int(A)$ (Krishnan, 2003).

In general, the concept of γ-preopen set and preopen set are independent, but they are equivalent in a γ-regular space. Similarly for a γ-semiopen set and semiopen set, α-γ-open set and α-open set, γ-b-open set and b-open set and, γ-β-open set and β-open set in which they are independent but equivalent in a γ-regular space.

Remark 2.3.32. Let (X, τ) be a topological space and γ be an operation on τ. Then:

1. The concept of γ-preopen set and preopen set are independent, while in a γ-regular space these concepts are equivalent (Krishnan and Balachandran, 2006a).

2. The concept of α-γ-open set and α-open set are independent, while in a γ-regular space these concepts are equivalent (Kalaivani and Krishnan, 2013).

3. The concept of γ-β-open set and β-open set are independent, while in a γ-regular space these concepts are equivalent (Basu, Afsan and Ghosh, 2009).

4. The concept of γ-semiopen set and semiopen set are independent, while in a γ-regular space these concepts are equivalent (Krishnan and Balachandran, 2006b).

5. The concept of γ-b-open set and b-open set are independent, while in a γ-regular space these concepts are equivalent (Carpintero, Rajesh and Rosas, 2012a).

2.4 Properties of γ- Functions

In this section, we review basic definitions and properties related to functions. Some types of functions were constructed using γ-open set and other types of sets in Definition 2.3.8. These functions will be used in Chapter Five.

Definition 2.4.1. Let (X, τ) and (Y, σ) be two topological spaces and let $\gamma\colon \tau \to P(X)$ and $\beta\colon \sigma \to P(Y)$ be operations on τ and σ respectively. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be:

1. (γ, β)-continuous if for each open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(\gamma(U)) \subseteq \beta(V)$ (Ogata, 1991).

2. (γ, β)-precontinuous if for each β-preopen set V of Y containing $f(x)$, there

exists a γ-preopen set U of X containing x such that $f(U) \subseteq V$ (Krishnan and Balachandran, 2006a).

3. (γ, β)-semi-continuous if for each β-semiopen set V of Y containing $f(x)$, there exists a γ-semiopen set U of X containing x such that $f(U) \subseteq V$ (Krishnan and Balachandran, 2006b).

4. α-(γ, β)-continuous if for each α-β-open set V of Y containing $f(x)$, there exists an α-γ-open set U of X containing x such that $f(U) \subseteq V$ (Kalaivani and Krishnan, 2012).

5. (γ, β)-b-continuous if for each β-b-open set V of Y containing $f(x)$, there exists a γ-b-open set U of X containing x such that $f(U) \subseteq V$ (Carpintero, Rajesh and Rosas, 2012a).

Definition 2.4.2. (Basu, Afsan and Ghosh, 2009) Let (X, τ) and (Y, σ) be two topological spaces and let $\gamma\colon \tau \to P(X)$ and $\gamma^{'}: \tau^{'} \to P(Y)$ be operations on τ and $\tau^{'}$ respectively. A function $f\colon (X, \tau) \to (Y, \tau^{'})$ is called $(\gamma, \gamma^{'})$-β-irresolute if for each $\gamma^{'}$-β-open set V of Y containing $f(x)$, there exists a γ-β-open set U of X containing x such that $f(U) \subseteq V$.

Definition 2.4.3. Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be:

1. γ-precontinuous if for each open set V of Y containing $f(x)$, there exists a γ-preopen set U of X containing x such that $f(U) \subseteq V$ (Ghosh, 2012).

2. γ-continuous if for each open set V of Y containing $f(x)$, there exists a γ-open set U of X containing x such that $f(U) \subseteq V$ (Basu, Afsan and Ghosh, 2009).

3. γ-β-continuous if for each open set V of Y containing $f(x)$, there exists a γ-β-open set U of X containing x such that $f(U) \subseteq V$ (Basu, Afsan and Ghosh, 2009).

Definition 2.4.4. Let (X, τ) and (Y, σ) be two topological spaces and let $\gamma\colon \tau \to P(X)$ and $\beta\colon \sigma \to P(Y)$ be operations on τ and σ respectively. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be:

1. (γ, β)-preclosed if for every γ-preclosed set F of X, $f(F)$ is β-preclosed set in Y (Krishnan and Balachandran, 2006a).

2. (γ, β)-semiclosed if for every γ-semiclosed set F of X, $f(F)$ is β-semiclosed set in Y (Krishnan and Balachandran, 2006b).

3. (γ, β)-b-closed if for every γ-b-closed set F of X, $f(F)$ is β-b-closed set in Y (Carpintero, Rajesh and Rosas, 2012a).

4. α-(γ, β)-closed if for every α-γ-closed set F of X, $f(F)$ is α-β-closed set in Y (Kalaivani and Krishnan, 2012).

5. α-(γ, β)-open if for every α-γ-open set F of X, $f(F)$ is α-β-open set in Y (Kalaivani and Krishnan, 2012).

Definition 2.4.5. Let $\gamma^{'}$ be an operation on $\tau^{'}$. A function $f\colon (X, \tau) \to (Y, \tau^{'})$ is called:

1. $\gamma^{'}$-preopen if for each open set G in X, $f(G)$ is $\gamma^{'}$-preopen set in Y (Ghosh, 2012).

2. $\gamma^{'}$-β-open if for each open set G in X, $f(G)$ is $\gamma^{'}$-β-open set in Y (Basu, Afsan and Ghosh, 2009).

3. $\gamma^{'}$-closed (respectively, $\gamma^{'}$-β-closed and $\gamma^{'}$-βg-closed) if for each closed set F in X, $f(F)$ is $\gamma^{'}$-closed (respectively, $\gamma^{'}$-β-closed and $\gamma^{'}$-βg-closed) set in Y (Basu, Afsan and Ghosh, 2009).

4. $\gamma^{'}$-preclosed (respectively, $\gamma^{'}$-pre$g$-closed) if for each closed set F in X, $f(F)$ is $\gamma^{'}$-preclosed (respectively, $\gamma^{'}$-pre$g$-closed) set in Y (Ghosh, 2012).

5. $\gamma^{'}$-pre-anti-continuous if the inverse image of each $\gamma^{'}$-preopen set in Y is open in X, or if the inverse image of each $\gamma^{'}$-preclosed set in Y is closed in X (Ghosh, 2012).

6. $\gamma^{'}$-β-anti-continuous if the inverse image of each $\gamma^{'}$-β-open set in Y is open in X, or if the inverse image of each $\gamma^{'}$-β-closed set in Y is closed in X (Basu, Afsan and Ghosh, 2009).

Definition 2.4.6. A function $f\colon (X,\tau) \to (Y,\tau^{'})$ with an operation γ on τ is called:

1. γ-pre-anti-open (respectively, γ-pre-anti-closed) if the image of each γ-preopen (respectively, γ-preclosed) set in X is open (respectively, closed) in Y (Ghosh, 2012).

2. γ-β-anti-open (respectively, γ-β-anti-closed) if the image of each γ-β-open (respectively, γ-β-closed) set in X is open (respectively, closed) in Y (Basu, Afsan and Ghosh, 2009).

Definition 2.4.7. (Kalaivani, Kumar and Krishnan, 2012) Let γ be an operation on both τ and σ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called:

1. α-γ-continuous if for every open set V in Y, $f^{-1}(V)$ is α-γ-open set in X.

2. α-γ-irresolute if for every α-γ-open set V in Y, $f^{-1}(V)$ is α-γ-open set in X.

3. α-γ-open if for every open set G in X, $f(G)$ is α-γ-open set in Y.

4. α-γ-closed if for every closed set F in X, $f(F)$ is α-γ-closed set in Y.

Definition 2.4.8. (Basu, Afsan and Ghosh, 2009) Let (X,τ) and $(Y,\tau^{'})$ be two topological spaces and γ be an operation on τ. A function $f\colon (X,\tau) \to (Y,\tau^{'})$ is called τ-γ-β-continuous (respectively, γ-γ-β-continuous) if for each open set V of Y, $f^{-1}(V)$ is τ-γ-β-open (respectively, γ-γ-β-open) set in X.

Definition 2.4.9. (Khalaf and Asaad, 2009) A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be P_S-continuous if the inverse image of each open set in Y is P_S-open in X.

Definition 2.4.10. (Arya and Gupta, 1974) A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be completely continuous if the inverse image of each open set in Y is regular-open in X.

Theorem 2.4.11. (Ghosh, 2012) Let (X,τ) and (Y,σ) be two topological spaces and γ be an operation on τ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is γ-precontinuous if for every open set V in Y, $f^{-1}(V)$ is γ-preopen set in X.

Theorem 2.4.12. (Basu, Afsan and Ghosh, 2009) Let (X,τ) and (Y,σ) be two topological spaces and γ be an operation on τ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is γ-continuous if for every open set V in Y, $f^{-1}(V)$ is γ-open set in X.

Theorem 2.4.13. (Long, 1986) Let $f\colon X \to Y$ be a function and $\{A_\lambda : \lambda \in \Lambda\}$ be an indexed family of subsets of Y. Then the induced function $f^{-1}\colon Y \to X$ has the following properties:

1. $f^{-1}(\bigcup(\{A_\lambda : \lambda \in \Lambda\})) = \bigcup(f^{-1}(\{A_\lambda : \lambda \in \Lambda\}))$.

2. $f^{-1}(\bigcap(\{A_\lambda : \lambda \in \Lambda\})) = \bigcap(f^{-1}(\{A_\lambda : \lambda \in \Lambda\}))$.

2.5 Properties of γ- Separation Axioms

Definition 2.5.1. (Ogata, 1991) A topological space (X, τ) with an operation γ on τ is said to be:

1. γ-T_0 if for each pair of distinct points x, y in X, there exists an open sets G such that either $x \in \gamma(G)$ and $y \notin \gamma(G)$ or $y \in \gamma(G)$ and $x \notin \gamma(G)$.

2. γ-T_1 if for each pair of distinct points x, y in X, there exist two open sets G and H such that $x \in \gamma(G)$ but $y \notin \gamma(G)$ and $y \in \gamma(H)$ but $x \notin \gamma(H)$.

3. γ-T_2 if for each pair of distinct points x, y in X, there exist two open sets G and H containing x and y respectively such that $\gamma(G) \cap \gamma(H) = \phi$.

4. γ-$T_{\frac{1}{2}}$ if every γ-g-closed set in X is γ-closed.

Definition 2.5.2. (Krishnan and Balachandran, 2006a) A topological space (X, τ) with an operation γ on τ is said to be:

1. γ-preT_0 if for each pair of distinct points x, y in X, there exists a γ-preopen set G such that either $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$.

2. γ-preT_1 if for each pair of distinct points x, y in X, there exist two γ-preopen sets G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.

3. γ-preT_2 if for each pair of distinct points x, y in X, there exist two γ-preopen sets G and H containing x and y respectively such that $G \cap H = \phi$.

4. γ-pre$T_{\frac{1}{2}}$ if every γ-pre$g$-closed set in X is γ-preclosed.

Similarly, (Krishnan and Balachandran, 2006b) and (Basu, Afsan and Ghosh, 2009) defined γ-semiT_i and γ-βT_i for $i = 0, \frac{1}{2}, 1, 2$ by using γ-semiopen set and γ-β-open set, respectively.

Definition 2.5.3. (Carpintero, Rajesh and Rosas, 2012a) A topological space (X, τ) with an operation γ on τ is said to be γ-b-$T_{\frac{1}{2}}$ if every γ-b-g-closed set in X is γ-b-closed.

Some properties of the above definitions are as follows:

Theorem 2.5.4. Let (X, τ) be a topological space and γ be an operation on τ. Then:

1. X is γ-preT_0 if and only if τ_γ-$pCl(\{x\}) \neq \tau_\gamma$-$pCl(\{y\})$, for every pair of distinct points x, y of X (Ghosh, 2012).

2. X is γ-semiT_0 if and only if τ_γ-$sCl(\{x\}) \neq \tau_\gamma$-$sCl(\{y\})$, for every pair of distinct points x, y of X (Krishnan and Balachandran, 2006b).

3. X is γ-βT_0 if and only if τ_γ-$\beta Cl(\{x\}) \neq \tau_\gamma$-$\beta Cl(\{y\})$, for every pair of distinct points x, y of X (Basu, Afsan and Ghosh, 2009).

Theorem 2.5.5. Let (X, τ) be a topological space and γ be an operation on τ. Then:

1. X is γ-preT_1 if and only if for any $x \in X$, the singleton set $\{x\}$ is γ-preclosed (Krishnan and Balachandran, 2006a).

2. X is γ-semiT_1 if and only if for any $x \in X$, the singleton set $\{x\}$ is γ-semiclosed (Krishnan and Balachandran, 2006b).

3. X is γ-βT_1 if and only if for any $x \in X$, the singleton set $\{x\}$ is γ-β-closed (Basu, Afsan and Ghosh, 2009).

Theorem 2.5.6. For any topological space (X, τ) with an operation γ on τ. Then:

1. X is γ-pre$T_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is γ-preclosed or γ-preopen (Krishnan and Balachandran, 2006a).

2. X is γ-semi$T_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is γ-semiclosed or γ-semiopen (Krishnan and Balachandran, 2006b).

3. X is γ-b-$T_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is γ-b-closed or γ-b-open (Carpintero, Rajesh and Rosas, 2012a).

4. X is γ-$\beta T_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is γ-β-closed or γ-β-open (Basu, Afsan and Ghosh, 2009).

Theorem 2.5.7. Let γ be an operation on τ, then:

1. Every topological space (X, τ) is γ-pre$T_{\frac{1}{2}}$ (Ghosh, 2012).

2. Every topological space (X, τ) is γ-$\beta T_{\frac{1}{2}}$ (Basu, Afsan and Ghosh, 2009).

Definition 2.5.8. (Ghosh, 2012) A topological space (X, τ) with an operation γ on τ is said to be:

1. γ-pre-regular if for each γ-preclosed set F of X not containing x, there exist disjoint γ-preopen sets U and V such that $x \in U$ and $F \subseteq V$.
2. γ-pre-normal if for each pair of disjoint γ-preclosed sets E, F of X, there exist disjoint γ-preopen sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Definition 2.5.9. (Basu, Afsan and Ghosh, 2009) A topological space (X, τ) with an operation γ on τ is said to be:

1. γ-β-regular if for each γ-β-closed set F of X not containing x, there exist disjoint γ-β-open sets U and V such that $x \in U$ and $F \subseteq V$.
2. γ-β-normal if for each pair of disjoint γ-β-closed sets E, F of X, there exist disjoint γ-β-open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Theorem 2.5.10. (Ghosh, 2012) Let (X, τ) be a topological space and γ be an operation on τ. Then the following conditions hold:

1. X is γ-pre-regular if and only if for each point x in X and each γ-preopen set U containing x, there exists a γ-preopen set V such that $x \in V \subseteq \tau_\gamma\text{-}pCl(V) \subseteq U$.
2. X is γ-pre-normal if and only if for each γ-preclosed set F in X and each γ-preopen set U containing F, there exists a γ-preopen set V containing F such that $\tau_\gamma\text{-}pCl(V) \subseteq U$.

Theorem 2.5.11. (Basu, Afsan and Ghosh, 2009) The following statements are equivalent for a topological space (X, τ) with an operation γ on τ:

1. X is γ-β-regular if and only if for each point x in X and each γ-β-open set U containing x, there exists a γ-β-open set V such that $x \in V \subseteq \tau_\gamma$-$\beta Cl(V) \subseteq U$.

2. X is γ-β-normal if and only if for each γ-β-closed set F in X and each γ-β-open set U containing F, there exists a γ-β-open set V containing F such that τ_γ-$\beta Cl(V) \subseteq U$.

CHAPTER THREE

γ-REGULAR-OPEN SETS AND γ- SPACES

3.1 Introduction

In this chapter, we construct a new class of γ- sets called γ-regular-open sets in a topological space (X, τ) with its complement which is γ-regular-closed sets. Also, we investigate some basic properties and results. In addition, we define some new classes of γ- spaces called γ-extremally disconnected, γ-locally indiscrete and γ-hyperconnected spaces by using γ-open and γ-closed sets. Finally, we investigate some characterizations and theorems of these γ- spaces.

3.2 γ-Regular-Open Sets

In this section, we introduce a new class of sets called γ-regular-open sets in a topological space (X, τ) with an operation γ on τ. This class of sets lies strictly between the classes of γ-clopen and γ-open sets.

Definition 3.2.1. A subset R of a topological space (X, τ) with an operation γ on τ is said to be γ-regular-open if $R = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$.

The complement of a γ-regular-open set is γ-regular-closed. Or equivalently, a subset R of a space X is said to be γ-regular-closed if $R = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))$.

The class of all γ-regular-open and γ-regular-closed subsets of a topological space (X, τ)

is denoted by τ_γ-$RO(X,\tau)$ or τ_γ-$RO(X)$ and τ_γ-$RC(X,\tau)$ or τ_γ-$RC(X)$, respectively.

Remark 3.2.2. It is clear from the definition that every γ-regular-open (respctively, γ-regular-closed) set is γ-open (respctively, γ-closed) and every γ-clopen set is both γ-regular-open and γ-regular-closed.

Converses of the above remark are not true. It can be seen from the following example.

Example 3.2.3. Let $X = \{a,b,c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$. Define an operation $\gamma\colon \tau \to P(X)$ by $\gamma(R) = R$ for every $R \in \tau$. Then $\tau_\gamma = \tau$ and hence τ_γ-$RO(X,\tau) = \{\phi, X, \{a\}, \{b\}\}$ and τ_γ-$RC(X,\tau) = \{\phi, X, \{a,c\}, \{b,c\}\}$. Then the set $\{a,b\}$ is γ-open, but it is not γ-regular-open. Also the sets $\{a\}$ and $\{a,c\}$ are γ-regular-open and γ-regular-closed, respectively, but they are not γ-clopen.

From the Definition 3.2.1, Definition 2.3.8 (1) and (2), Definition 2.3.9 (1) and (2) and Definition 2.3.7, we have the following remark.

Remark 3.2.4. Let R be any subset of a topological space (X,τ) and γ be an operation on τ. Then:

1. R is γ-clopen if and only if it is γ-regular-open and γ-regular-closed.

2. R is γ-regular-open if and only if it is γ-preopen and γ-semiclosed.

3. R is γ-regular-closed if and only if it is γ-semiopen and γ-preclosed.

By Definition 2.3.9, Figure 2.1, Remark 3.2.2 and Remark 3.2.4, we have the following relations as illustrated in Figure 3.1.

$$
\begin{tikzcd}
\gamma\text{-regular-open} \arrow[r] \arrow[ddr] & \gamma\text{-open} \arrow[r] & \alpha\text{-}\gamma\text{-open} \arrow[r] \arrow[dl] & \gamma\text{-preopen} \arrow[dl] \\
\gamma\text{-clopen} \arrow[u] \arrow[dd] & \gamma\text{-semiopen} \arrow[r] & \gamma\text{-}b\text{-open} \arrow[r] & \gamma\text{-}\beta\text{-open} \\
 & \gamma\text{-semiclosed} \arrow[r] & \gamma\text{-}b\text{-closed} \arrow[r] & \gamma\text{-}\beta\text{-closed} \\
\gamma\text{-regular-closed} \arrow[r] \arrow[uur] & \gamma\text{-closed} \arrow[r] & \alpha\text{-}\gamma\text{-closed} \arrow[r] \arrow[ul] & \gamma\text{-preclosed} \arrow[ul]
\end{tikzcd}
$$

Remark: $A \longrightarrow B$ represents A implies B

Figure 3.1. The relations between γ-regular-open set, γ-regular-closed set and various types of γ-sets

In the sequel, none of the implications that concerning γ-regular-open set and γ-regular-closed set in the above figure is reversible. It is notice that γ-open set lies strictly between the classes of γ-regular-open set and α-γ-open set, and α-γ-open set lies strictly between the classes of γ-open set and both γ-preopen set and γ-semiopen set. But γ-b-open set lies strictly between the classes of both γ-preopen set and γ-semiopen set, and γ-β-open set. Also, γ-preopen set is not comparable with γ-semiopen set. In the same manner we can find the relation between the types of γ- closed sets.

In general, the concept of γ-regular-open set and regular-open set are independent but they are identical in a γ-regular space (X, τ). This is explained from Remark 3.2.5 to Remark 3.2.7.

Remark 3.2.5. The concept of γ-regular-open set and regular-open set are independent. It is shown in the following example.

Example 3.2.6. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$.

Let $\gamma\colon \tau \to P(X)$ be an operation defined as follows:

For every $R \in \tau$, then:

$$\gamma(R) = \begin{cases} R & \text{if } R = \{b\} \\ R \cup \{a\} & \text{if } R \neq \{b\} \end{cases}$$

Obviously, $\tau_\gamma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, τ_γ-$RO(X, \tau) = \{\phi, X, \{a\}, \{b\}\}$ and $RO(X, \tau)$ = $\{\phi, X, \{a\}, \{b, c\}\}$. So the set $\{b\}$ is γ-regular-open, but it is not regular-open. Also the set $\{b, c\}$ is regular-open, but it is not γ-regular-open.

Remark 3.2.7 follows directly from Remark 2.3.31.

Remark 3.2.7. If (X, τ) be a γ-regular space and γ be an operation on τ, then the concept of γ-regular-open set and regular-open set coincide.

The union and intersection of any two γ-regular-open (respectively, γ-regular-closed) sets need not be γ-regular-open (respectively, γ-regular-closed). It is shown in the following example.

Example 3.2.8. Let $X = \{a, b, c\}$ and τ be the discrete topology. Let $\gamma\colon \tau \to P(X)$ be an operation defined by:

For every $R \in \tau$,

$$\gamma(R) = \begin{cases} R & \text{if } R \neq \{c\} \\ X & \text{if } R = \{c\} \end{cases}$$

Clearly, $\tau_\gamma = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and τ_γ-$RO(X, \tau) = \tau_\gamma$-$RC(X, \tau) = \{\phi, X, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Then the sets $\{a\}$ and $\{b\}$ are γ-regular-open (respectively, γ-regular-closed), but $\{a\} \cup \{b\} = \{a, b\}$ is not γ-regular-open (respectively, not γ-regular-closed) set. For the other part, the sets $\{a, c\}$ and $\{b, c\}$ are γ-regular-open (respectively, γ-regular-closed) sets, but $\{a, c\} \cap \{b, c\} = \{c\}$ is not γ-regular-open (respectively, not γ-regular-closed) set.

It is shown from the above example that the class of all γ-regular-open sets of any topological space (X, τ) may not be a topology on X in general.

Theorem 3.2.9. Let (X, τ) be a topological space and γ be a regular operation on τ. Then:

1. The intersection of two γ-regular-open sets is γ-regular-open.

2. The union of two γ-regular-closed sets is γ-regular-closed.

Proof. Straightforward from Corollary 3.2.31 and Corollary 3.2.32 and the fact that every γ-regular-open set is γ-open and every γ-regular-closed set is γ-closed. $\square$

Now, we present some important relationships involving some γ- sets as we mentioned in Definitions 3.2.1, 2.3.8, and 2.3.9, by using τ_γ-closure and τ_γ-interior. This is explained from Theorem 3.2.10 to Remark 3.2.25.

Theorem 3.2.10. For any subset R of a topological space (X, τ) and γ be an operation on τ, then $R \in \tau_\gamma\text{-}\beta O(X)$ if and only if $\tau_\gamma\text{-}Cl(R) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))$.

Proof. Let $R \in \tau_\gamma\text{-}\beta O(X)$, then $R \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))$. So $\tau_\gamma\text{-}Cl(R) \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))))$ implies that $\tau_\gamma\text{-}Cl(R) \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))) \subseteq \tau_\gamma\text{-}Cl(R)$. Therefore, $\tau_\gamma\text{-}Cl(R) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))$.

Conversely, suppose that $\tau_\gamma\text{-}Cl(R) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))$. This implies that $\tau_\gamma\text{-}Cl(R) \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))$. Then $R \subseteq \tau_\gamma\text{-}Cl(R) \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))$. Hence $R \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))$. Therefore, $R \in \tau_\gamma\text{-}\beta O(X)$. □

Theorem 3.2.11. For any subset R of a topological space (X, τ) with an operation γ on τ, $R \in \tau_\gamma\text{-}SO(X)$ if and only if $\tau_\gamma\text{-}Cl(R) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))$.

Proof. The proof is similar to Theorem 3.2.10. □

Corollary 3.2.12. For any subset R of a topological space (X, τ), if $R \in \tau_\gamma\text{-}SO(X)$, then $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}RC(X)$.

Proof. The proof is immediate consequence of Theorem 3.2.11. □

The following corollary follows directly from Corollary 3.2.12 and Theorem 3.2.11 and using complements.

Corollary 3.2.13. For any subset R of a topological space (X, τ), the following statements hold:

1. $R \in \tau_\gamma\text{-}SC(X)$ if and only if $\tau_\gamma\text{-}Int(R) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$.

2. If $R \in \tau_\gamma\text{-}SC(X)$, then $\tau_\gamma\text{-}Int(R) \in \tau_\gamma\text{-}RO(X)$. $\square$

Theorem 3.2.14. Let R be any subset of a topological space (X, τ) and γ be an operation on τ. Then:

1. $R \in \tau_\gamma\text{-}\beta O(X)$ if and only if $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}RC(X)$.

2. $R \in \tau_\gamma\text{-}\beta O(X)$ if and only if $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}SO(X)$.

3. $R \in \tau_\gamma\text{-}\beta O(X)$ if and only if $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}\beta O(X)$.

4. $R \in \tau_\gamma\text{-}\beta O(X)$ if and only if $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}BO(X)$.

Proof. (1) The proof is immediate consequence of Theorem 3.2.10.

(2) Let $R \in \tau_\gamma\text{-}\beta O(X)$, then by (1), $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}RC(X) \subseteq \tau_\gamma\text{-}SO(X)$. So $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}SO(X)$. Conversely, let $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}SO(X)$. Then by Theorem 3.2.11, $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Cl(R)) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))$ which implies that $\tau_\gamma\text{-}Cl(R) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))$ and hence by Theorem 3.2.10, $R \in \tau_\gamma\text{-}\beta O(X)$.

(3) Let $R \in \tau_\gamma\text{-}\beta O(X)$, then by (2), $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}SO(X) \subseteq \tau_\gamma\text{-}\beta O(X)$. So $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}\beta O(X)$. Conversely, let $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}\beta O(X)$, by (2), $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Cl(R)) \in \tau_\gamma\text{-}SO(X)$. Since $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Cl(R)) = \tau_\gamma\text{-}Cl(R)$. Then $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}SO(X)$ and hence by (2), $R \in \tau_\gamma\text{-}\beta O(X)$.

(4) Let $R \in \tau_\gamma\text{-}\beta O(X)$, so by (2), $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}SO(X) \subseteq \tau_\gamma\text{-}BO(X)$. So $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}BO(X)$. Conversely, let $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}BO(X) \subseteq \tau_\gamma\text{-}\beta O(X)$, then $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}\beta O(X)$ and thus by (3), $R \in \tau_\gamma\text{-}\beta O(X)$. $\square$

From Theorem 3.2.10 and Theorem 3.2.14, we have the following corollary.

Corollary 3.2.15. Let R be any subset of a topological space (X, τ). Then:

1. $R \in \tau_\gamma\text{-}\beta C(X)$ if and only if $\tau_\gamma\text{-}Int(R) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)))$.

2. $R \in \tau_\gamma\text{-}\beta C(X)$ if and only if $\tau_\gamma\text{-}Int(R) \in \tau_\gamma\text{-}RO(X)$.

3. $R \in \tau_\gamma\text{-}\beta C(X)$ if and only if $\tau_\gamma\text{-}Int(R) \in \tau_\gamma\text{-}SC(X)$.

4. $R \in \tau_\gamma\text{-}\beta C(X)$ if and only if $\tau_\gamma\text{-}Int(R) \in \tau_\gamma\text{-}\beta C(X)$.

5. $R \in \tau_\gamma\text{-}\beta C(X)$ if and only if $\tau_\gamma\text{-}Int(R) \in \tau_\gamma\text{-}BC(X)$. □

From Theorem 3.2.14 (1) and Corollary 3.2.15 (2), we have the following remark.

Remark 3.2.16. Let R be any subset of a topological space (X, τ). Then:

1. If $R \in \tau_\gamma\text{-}RO(X)$, then $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}RC(X)$ and $\tau_\gamma\text{-}Int(R) \in \tau_\gamma\text{-}RO(X)$.

2. If $R \in \tau_\gamma\text{-}RC(X)$, then $\tau_\gamma\text{-}Cl(R) \in \tau_\gamma\text{-}RC(X)$ and $\tau_\gamma\text{-}Int(R) \in \tau_\gamma\text{-}RO(X)$.

Theorem 3.2.17. For any subset R of a topological space (X, τ), then

$$\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)))) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))$$

Proof. Since $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))) \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))$. Then $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)))) \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))$. On the other hand, since $\tau_\gamma\text{-}Int(R) \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))$ implies that $\tau_\gamma\text{-}Int(R) \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)))$ and hence $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)) \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))))$. Therefore, $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)))) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))$. □

Theorem 3.2.18. For any subset R of a topological space (X, τ), then

$$\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$$

Proof. The proof is similar to Theorem 3.2.17. □

It is obvious from Theorem 3.2.17 and Theorem 3.2.18 that τ_γ-$Cl(\tau_\gamma$-$Int(R))$ and τ_γ-$Int(\tau_\gamma$-$Cl(R))$ are γ-regular-closed and γ-regular-open sets, respectively.

Remark 3.2.19. For every γ-closed subset R of X, τ_γ-$Int(R)$ is γ-regular-open set.

Theorem 3.2.20. Let R be any subset of a topological space (X, τ) and γ be a regular operation on τ. Then $R \in \tau_\gamma$-$PO(X)$ if and only if τ_γ-$sCl(R) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$.

Proof. Let $R \in \tau_\gamma$-$PO(X)$, then $R \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)$ implies that τ_γ-$sCl(R) \subseteq \tau_\gamma$-$sCl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))$. Since τ_γ-$Int(\tau_\gamma$-$Cl(R)) \in \tau_\gamma$-$RO(X)$. But τ_γ-$RO(X) \subseteq \tau_\gamma$-$SC(X)$ in general, so τ_γ-$Int(\tau_\gamma$-$Cl(R)) \in \tau_\gamma$-$SC(X)$ and hence τ_γ-$sCl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$. So τ_γ-$sCl(R) \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$. On the other hand, by Theorem 2.3.27 (2), we have τ_γ-$sCl(R) = R \cup \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$. Hence τ_γ-$Int(\tau_\gamma$-$Cl(R)) \subseteq \tau_\gamma$-$sCl(R)$. Therefore, τ_γ-$sCl(R) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$.

Conversely, let τ_γ-$sCl(R) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$, then τ_γ-$sCl(R) \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$ and hence $R \subseteq \tau_\gamma$-$sCl(R) \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$ which implies that $R \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$. Therefore, $R \in \tau_\gamma$-$PO(X)$. □

Theorem 3.2.21. Let P and R be subsets of a topological space (X, τ) and γ be an operation on τ. Then P is γ-preopen if and only if there exists a γ-regular-open set R containing P such that τ_γ-$Cl(P) = \tau_\gamma$-$Cl(R)$.

Proof. Let P be a γ-preopen set, then $P \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P))$. Put $R = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P))$ is a γ-regular-open set containing P. Since P is γ-preopen, then P is γ-β-open. By Theorem 3.2.10, $\tau_\gamma\text{-}Cl(P) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P))) = \tau_\gamma\text{-}Cl(R)$.

Conversely, suppose R be a γ-regular-open set and P be any subset of X such that $P \subseteq R$ and $\tau_\gamma\text{-}Cl(P) = \tau_\gamma\text{-}Cl(R)$. Then $P \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$ and hence $P \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P))$. This means that P is γ-preopen set. This completes the proof. $\square$

Theorem 3.2.22. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then A is γ-preopen if and only if there exists a γ-open set U in X such that $A \subseteq U \subseteq \tau_\gamma\text{-}Cl(A)$.

Proof. Let A be any γ-preopen subset of X, then $A \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A)) \subseteq \tau_\gamma\text{-}Cl(A)$. If we take $U = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A))$ and $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A))$ is a γ-open set, then U is γ-open. Hence $A \subseteq U \subseteq \tau_\gamma\text{-}Cl(A)$.

Conversely, suppose that $A \subseteq U \subseteq \tau_\gamma\text{-}Cl(A)$, where U is γ-open set in X. Since $U \subseteq \tau_\gamma\text{-}Cl(A)$, then $\tau_\gamma\text{-}Int(U) \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A))$. Furthermore, U is γ-open implies that $\tau_\gamma\text{-}Int(U) = U$. Thus, $U \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A))$. But $A \subseteq U$. Therefore, $A \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A))$. Hence A is γ-preopen set in X. $\square$

Theorem 3.2.23. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then A is γ-semiopen if and only if there exists a γ-open set U in X such that $U \subseteq A \subseteq \tau_\gamma\text{-}Cl(U)$.

Proof. The proof is similar to Theorem 3.2.22. $\square$

Corollary 3.2.24. Let B be any subset of a topological space (X, τ) and γ be an operation on τ. Then B is γ-semiclosed if and only if there exists a γ-closed set F in X such that τ_γ-$Int(F) \subseteq B \subseteq F$.

Proof. The proof is directly from Theorem 3.2.23, and using complements. $\square$

Remark 3.2.25. If S is both γ-semiopen and γ-semiclosed subset of a topological space (X, τ) with an operation γ on τ and τ_γ-$Cl(\tau_\gamma$-$Int(S)) \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(S))$. Then S is both γ-regular-open and γ-regular-closed.

Next, the relationships between γ-regular-open set, γ-regular-closed set and γ- sets will be established.

Theorem 3.2.26. Let R be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following statements are equivalent:

1. R is γ-regular-open.

2. R is γ-open and γ-semiclosed.

3. R is α-γ-open and γ-semiclosed.

4. R is γ-preopen and γ-semiclosed.

5. R is γ-open and γ-β-closed.

6. R is α-γ-open and γ-β-closed.

Proof. (1) $\Rightarrow$ (2) Let R be γ-regular-open set. Since every γ-regular-open set is γ-open and every γ-regular-open set is γ-semiclosed. Then R is γ-open and γ-semiclosed.

(2) $\Rightarrow$ (3) Let R be γ-open and γ-semiclosed set. Since every γ-open set is α-γ-open. Then R is α-γ-open and γ-semiclosed.

(3) $\Rightarrow$ (4) Let R be α-γ-open and γ-semiclosed set. Since every α-γ-open set is γ-preopen. Then R is γ-preopen and γ-semiclosed.

(4) $\Rightarrow$ (5) Let R be γ-preopen and γ-semiclosed set. Then $R \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$ and $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \subseteq R$. Therefore, we have $R = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$. Then R is γ-regular-open set and hence it is γ-open. Since every γ-semiclosed set is γ-β-closed. Then R is γ-open and γ-β-closed.

(5) $\Rightarrow$ (6) It is obvious since every γ-open set is α-γ-open.

(6) $\Rightarrow$ (1) Let R be α-γ-open and γ-β-closed set. Then $R \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)))$ and $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))) \subseteq R$. Then $\tau_\gamma\text{-}Int(R) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))) = R$ and hence $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)))$ $= R$. Therefore, R is γ-regular-open set. $\square$

Theorem 3.2.27. Let S be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following statements are equivalent:

1. S is γ-regular-closed.

2. S is γ-closed and γ-semiopen.

3. S is α-γ-closed and γ-semiopen.

4. S is γ-preclosed and γ-semiopen.

5. S is γ-closed and γ-β-open.

6. S is α-γ-closed and γ-β-open.

Proof. Similar to Theorem 3.2.26 taking $R = X \backslash S$. $\square$

Theorem 3.2.28. Let R be any subset of a topological space (X, τ) and γ be an operation on τ. So the following properties of R are equivalent:

1. R is γ-clopen.

2. R is γ-regular-open and γ-regular-closed.

3. R is γ-open and α-γ-closed.

4. R is γ-open and γ-preclosed.

5. R is α-γ-open and γ-preclosed.

6. R is α-γ-open and γ-closed.

7. R is γ-preopen and γ-closed.

8. R is γ-preopen and α-γ-closed.

Proof. (1) $\Leftrightarrow$ (2) See Remark 3.2.4 (1).

The implications (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5), (6) $\Rightarrow$ (7) and (7) $\Rightarrow$ (8) are obvious (see Figure 3.1).

(5) $\Rightarrow$ (6) Let R be α-γ-open and γ-preclosed set. Then $R \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)))$ and $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))) \subseteq R$. This implies that $R =$

τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R)))$ and hence τ_γ-$Cl(R) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))))$. By Theorem 3.2.17, we get τ_γ-$Cl(R) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))$. Since τ_γ-$Cl(\tau_\gamma$-$Int(R))) \subseteq R$, then τ_γ-$Cl(R) \subseteq R$. But in general $R \subseteq \tau_\gamma$-$Cl(R)$. Then τ_γ-$Cl(R) = R$. It is obvious that R is γ-closed.

(8) $\Rightarrow$ (1) Let R be γ-preopen and α-γ-closed. Then $R \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$ and τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))) \subseteq R$. Therefore, we have τ_γ-$Cl(R) \subseteq \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))) \subseteq R$ and hence τ_γ-$Cl(R) \subseteq R$. But in general $R \subseteq \tau_\gamma$-$Cl(R)$. Then τ_γ-$Cl(R) = R$. It is clear that R is γ-closed. Since τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))) \subseteq R$ implies that τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))) \subseteq \tau_\gamma$-$Int(R)$. Then by Theorem 3.2.18, $R \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)) \subseteq \tau_\gamma$-$Int(R)$ and hence $R \subseteq \tau_\gamma$-$Int(R)$. But in general

τ_γ-$Int(R) \subseteq R$. Then τ_γ-$Int(R) = R$. This means that R is γ-open. Therefore, R is γ-clopen. □

Lemma 3.2.29. Let R and S be any two subsets of a topological space (X, τ) and γ be an operation on τ. Then:

$$\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S)) \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S))$$

Proof. The proof is obvious and hence it is omitted. □

The converse of the above Lemma 3.2.29 is true when the operation γ is regular operation on τ and if one of the set is γ-open in X, as shown by the following theorem.

Theorem 3.2.30. Let (X, τ) be a topological space and γ be a regular operation on τ. If R is γ-open subset of X and S is any subset of X. Then

$$\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S))$$

Proof. It is enough to prove

τ_γ-$Int(\tau_\gamma$-$Cl(R)) \cap \tau_\gamma$-$Int(\tau_\gamma$-$Cl(S)) \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R \cap S))$ since the converse is similar to Lemma 3.2.29. Since R is γ-open subset of a space X and γ is a regular operation on τ. Then by using Lemma 2.3.25, we have

$$\begin{aligned}
\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) &\subseteq \tau_\gamma\text{-}Int[\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S))] \\
&\subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl[R \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S))]) \\
&\subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl[R \cap \tau_\gamma\text{-}Cl(S)]) \\
&\subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S))
\end{aligned}$$

So τ_γ-$Int(\tau_\gamma$-$Cl(R)) \cap \tau_\gamma$-$Int(\tau_\gamma$-$Cl(S)) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R \cap S))$. This completes the proof. □

From Theorem 3.2.30, we have the following corollary.

Corollary 3.2.31. If R and S are γ-open subsets of a topological space (X, τ) and γ be a regular operation on τ, then

$$\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S))$$

Proof. Clear. □

Corollary 3.2.32. If E and F are γ-closed subsets of a topological space (X, τ) and γ be a regular operation on τ, then

$$\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(E)) \cup \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(F)) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(E \cup F))$$

Proof. The proof is similar to Corollary 3.2.31 taking $R = X\backslash E$ and $S = X\backslash F$. □

3.3 γ-Extremally Disconnected Spaces

In this section, we introduce a new space called γ-extremally disconnected, and to obtain several characterizations of γ-extremally disconnected spaces by utilizing γ-regular-open sets and γ-regular-closed sets.

Definition 3.3.1. A topological space (X, τ) with an operation γ on τ is said to be γ-extremally disconnected if the τ_γ-closure of every γ-open set of X is γ-open in X. Or equivalently, a space X is γ-extremally disconnected if the τ_γ-interior of every γ-closed set of X is γ-closed in X.

The following lemma follows directly from the Lemma 2.3.14 and it is useful in this section.

Lemma 3.3.2. Let R and S be any subsets of a topological space (X, τ) and γ be an operation on τ. If $R \cap S = \phi$, then $\tau_\gamma\text{-}Int(R) \cap \tau_\gamma\text{-}Cl(S) = \phi$ and $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Int(S) = \phi$.

Proof. Obvious. □

In the following theorem, a γ-extremally disconnected space X is equivalent to every two disjoint γ-open sets of X have disjoint τ_γ-closures.

Theorem 3.3.3. A space X is γ-extremally disconnected if and only if $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Cl(S) = \phi$ for every γ-open subsets R and S of X with $R \cap S = \phi$.

Proof. Suppose R and S are two γ-open subsets of a γ-extremally disconnected space X such that $R \cap S = \phi$. Then by Lemma 3.3.2, τ_γ-$Cl(R) \cap S = \phi$ which implies that τ_γ-$Int(\tau_\gamma$-$Cl(R)) \cap \tau_\gamma$-$Cl(S) = \phi$ and hence τ_γ-$Cl(R) \cap \tau_\gamma$-$Cl(S) = \phi$.

Conversely, let O be any γ-open subset of a space X, then $X \backslash O$ is γ-closed set and hence τ_γ-$Int(X \backslash O)$ is γ-open set such that $O \cap \tau_\gamma$-$Int(X \backslash O) = \phi$. Then by hypothesis, we have τ_γ-$Cl(O) \cap \tau_\gamma$-$Cl(\tau_\gamma$-$Int(X \backslash O)) = \phi$ which implies that τ_γ-$Cl(O) \cap \tau_\gamma$-$Cl(X \backslash \tau_\gamma$-$Cl(O)) = \phi$ and hence τ_γ-$Cl(O) \cap X \backslash \tau_\gamma$-$Int(\tau_\gamma$-$Cl(O)) = \phi$. This means that τ_γ-$Cl(O) \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(O))$. Since τ_γ-$Int(\tau_\gamma$-$Cl(O)) \subseteq \tau_\gamma$-$Cl(O)$ in general. Then τ_γ-$Cl(O) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(O))$. So τ_γ-$Cl(O)$ is γ-open set in X. Therefore, X is γ-extremally disconnected space. □

Some fundamental characterizations of γ-extremally disconnected spaces are given in the following theorems.

Theorem 3.3.4. Let (X, τ) be a topological space and γ be a regular operation on τ. Then the following properties are equivalent:

1. X is γ-extremally disconnected.
2. τ_γ-$Cl(R) \cap \tau_\gamma$-$Cl(S) = \tau_\gamma$-$Cl(R \cap S)$ for every γ-open subsets R and S of X.
3. τ_γ-$Cl(R) \cap \tau_\gamma$-$Cl(S) = \tau_\gamma$-$Cl(R \cap S)$ for every γ-regular-open subsets R and S of X.
4. τ_γ-$Int(E) \cup \tau_\gamma$-$Int(F) = \tau_\gamma$-$Int(E \cup F)$ for every γ-regular-closed subsets E and F of X.

5. τ_γ-$Int(E) \cup \tau_\gamma$-$Int(F) = \tau_\gamma$-$Int(E \cup F)$ for every γ-closed subsets E and F of X.

Proof. (1) $\Rightarrow$ (2) Let R and S be any two γ-open subsets of a γ-extremally disconnected space X. Then by Corollary 3.2.31,

$$\begin{aligned}\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Cl(S) &= \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) \\ &= \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S)) = \tau_\gamma\text{-}Cl(R \cap S)\end{aligned}$$

(2) $\Rightarrow$ (3) Is clear since every γ-regular-open set is γ-open.

(3) $\Leftrightarrow$ (4) Let E and F be two γ-regular-closed subsets of X. Then $X \backslash E$ and $X \backslash F$ are γ-regular-open sets. By (3) and Lemma 2.3.14, we have

$$\begin{aligned}&\tau_\gamma\text{-}Cl(X \backslash E) \cap \tau_\gamma\text{-}Cl(X \backslash F) = \tau_\gamma\text{-}Cl(X \backslash E \cap X \backslash F) \\ \Leftrightarrow\ & X \backslash \tau_\gamma\text{-}Int(E) \cap X \backslash \tau_\gamma\text{-}Int(F) = \tau_\gamma\text{-}Cl(X \backslash (E \cup F)) \\ \Leftrightarrow\ & X \backslash (\tau_\gamma\text{-}Int(E) \cup \tau_\gamma\text{-}Int(F)) = X \backslash \tau_\gamma\text{-}Int(E \cup F) \\ \Leftrightarrow\ & \tau_\gamma\text{-}Int(E) \cup \tau_\gamma\text{-}Int(F) = \tau_\gamma\text{-}Int(E \cup F).\end{aligned}$$

(4) $\Rightarrow$ (5) Let E and F be two γ-closed subsets of X. Then τ_γ-$Cl(\tau_\gamma$-$Int(E))$ and τ_γ-$Cl(\tau_\gamma$-$Int(F))$ are γ-regular-closed sets. Then by (4) and Corollary 3.2.32, we get τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(E))) \cup \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(F))) =$ τ_γ-$Int[\tau_\gamma$-$Cl(\tau_\gamma$-$Int(E)) \cup \tau_\gamma$-$Cl(\tau_\gamma$-$Int(F))]$ and hence τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(E))) \cup \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(F))) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(E \cup F)))$. Since E and F are γ-closed subsets of X. Then by Theorem 3.2.18, we obtain τ_γ-$Int(\tau_\gamma$-$Cl(E)) \cup \tau_\gamma$-$Int(\tau_\gamma$-$Cl(F)) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(E \cup F))$. This implies that τ_γ-$Int(E) \cup \tau_\gamma$-$Int(F) = \tau_\gamma$-$Int(E \cup F)$.

(5) $\Leftrightarrow$ (2) The proof is similar to (3) $\Leftrightarrow$ (4).

(2) $\Rightarrow$ (1) Let U be any γ-open subset of a space X, then $X\backslash U$ is γ-closed set and hence τ_γ-$Int(X\backslash U)$ is γ-open set. Then by (2), we have τ_γ-$Cl(U) \cap \tau_\gamma$-$Cl(\tau_\gamma$-$Int(X\backslash U)) = \tau_\gamma$-$Cl(U \cap \tau_\gamma$-$Int(X\backslash U))$ which implies that τ_γ-$Cl(U) \cap \tau_\gamma$-$Cl(X\backslash\tau_\gamma$-$Cl(U)) = \tau_\gamma$-$Cl(\phi)$ since $U \cap \tau_\gamma$-$Int(X\backslash U) = \phi$. Hence τ_γ-$Cl(U) \cap X\backslash\tau_\gamma$-$Int(\tau_\gamma$-$Cl(U)) = \phi$. This means that τ_γ-$Cl(U) \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(U))$. Since τ_γ-$Int(\tau_\gamma$-$Cl(U)) \subseteq \tau_\gamma$-$Cl(U)$ in general. Then τ_γ-$Cl(U) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(U))$. So τ_γ-$Cl(U)$ is γ-open set in X. Therefore, a space X is γ-extremally disconnected. $\square$

Corollary 3.3.5. Let (X, τ) be a topological space and γ be a regular operation on τ. Then the following statements are equivalent:

1. X is γ-extremally disconnected.
2. τ_γ-$Cl(R) \cap \tau_\gamma$-$Cl(S) = \phi$ for every γ-open subsets R and S of X with $R \cap S = \phi$.
3. τ_γ-$Cl(R) \cap \tau_\gamma$-$Cl(S) = \phi$ for every γ-regular-open subsets R and S of X with $R \cap S = \phi$.

Proof. (1) $\Leftrightarrow$ (2) See Theorem 3.3.3.

(2) $\Rightarrow$ (3) Since every γ-regular-open set is γ-open. Then the proof is clear.

(3) $\Rightarrow$ (2) Let R and S be any two γ-open subsets of a space X such that $R \cap S = \phi$. Then by Lemma 3.3.2, $R \cap \tau_\gamma$-$Cl(S) = \phi$ implies that τ_γ-$Cl(R) \cap \tau_\gamma$-$Int(\tau_\gamma$-$Cl(S)) = \phi$ and hence τ_γ-$Int(\tau_\gamma$-$Cl(R)) \cap \tau_\gamma$-$Int(\tau_\gamma$-$Cl(S)) = \phi$. Since τ_γ-$Int(\tau_\gamma$-$Cl(R))$ and τ_γ-$Int(\tau_\gamma$-$Cl(S))$ are two γ-regular-open sets. Then by (3), we obtain τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))) \cap \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(S))) = \phi$. Since

τ_γ-$Int(\tau_\gamma$-$Cl(R))$ and τ_γ-$Int(\tau_\gamma$-$Cl(S))$ are two γ-regular-open sets, then τ_γ-$Int(\tau_\gamma$-$Cl(R))$ and τ_γ-$Int(\tau_\gamma$-$Cl(S))$ are two γ-β-open sets. So by Theorem 3.2.10, we get τ_γ-$Cl(R) \cap \tau_\gamma$-$Cl(S) = \phi$. This completes the proof. □

Corollary 3.3.6. A space X is γ-extremally disconnected if and only if τ_γ-$Cl(R) \cap \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(S))) = \phi$ for every γ-open subset R and every subset S of X with $R \cap S = \phi$.

Proof. See Corollary 3.3.5, since R and τ_γ-$Int(\tau_\gamma$-$Cl(S))$ are two γ-open subsets of X such that $R \cap \tau_\gamma$-$Int(\tau_\gamma$-$Cl(S)) = \phi$. □

Theorem 3.3.7. Let (X, τ) be a topological space and γ be a regular operation on τ. Then X is γ-extremally disconnected if and only if τ_γ-$Int(\tau_\gamma$-$Cl(R)) \cup \tau_\gamma$-$Int(\tau_\gamma$-$Cl(S)) =$ τ_γ-$Int(\tau_\gamma$-$Cl(R \cup S))$ for every γ-open subsets R and S of X.

Proof. Let (X, τ) be a γ-extremally disconnected space and let R and S be any two γ-open subsets of X. Then τ_γ-$Cl(R)$ and τ_γ-$Cl(S)$ are γ-closed subsets of X. So by Theorem 3.3.4 (5), we have

$$\begin{aligned}\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cup \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) &= \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R) \cup \tau_\gamma\text{-}Cl(S)) \\ &= \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cup S)).\end{aligned}$$

Conversely, let E and F be two γ-closed subsets of X. Then τ_γ-$Int(E)$ and τ_γ-$Int(F)$ are γ-open subsets of X. So by hypothesis,

τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(E))) \cup \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(F))) =$

τ_γ-$Int(\tau_\gamma$-$Cl[\tau_\gamma$-$Int(E) \cup \tau_\gamma$-$Int(F)]) = \tau_\gamma$-$Int[\tau_\gamma$-$Cl(\tau_\gamma$-$Int(E)) \cup \tau_\gamma$-$Cl(\tau_\gamma$-$Int(F))]$.

Since E and F are γ-closed subsets of X. Then by Corollary 3.2.32,

$\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(E))) \cup \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(F))) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(E \cup F)))$ and hence by Theorem 3.2.18, $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(E)) \cup \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(F)) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(E \cup F))$ which implies that $\tau_\gamma\text{-}Int(E) \cup \tau_\gamma\text{-}Int(F) = \tau_\gamma\text{-}Int(E \cup F)$. Therefore, by Theorem 3.3.4 (5), X is γ-extremally disconnected space. □

Theorem 3.3.8. Let (X, τ) be a topological space and γ be a regular operation on τ. Then X is γ-extremally disconnected if and only if $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(E)) \cap \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(F)) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(E \cap F))$ for every γ-closed subsets E and F of X.

Proof. Similar to Theorem 3.3.7 taking $R = X \backslash E$ and $S = X \backslash F$. □

Since the relation between γ-regular-open and γ-regular-closed sets are independent, but they are equivalent when a space X is γ-extremally disconnected in the following remark shows.

Remark 3.3.9. A space X is γ-extremally disconnected if and only if $\tau_\gamma\text{-}RO(X) = \tau_\gamma\text{-}RC(X)$.

The following remark follows directly from Theorem 3.2.9 and Remark 3.3.9.

Remark 3.3.10. Let (X, τ) be a topological space and γ be a regular operation on τ. So the following conditions are equivalent:

1. X is γ-extremally disconnected.
2. $R_1 \cap R_2$ is γ-regular-closed for all γ-regular-closed subsets R_1 and R_2 of X.
3. $R_1 \cup R_2$ is γ-regular-open for all γ-regular-open subsets R_1 and R_2 of X.

Theorem 3.3.11. The following statements are equivalent for any topological space (X, τ).

1. X is γ-extremally disconnected.
2. Every γ-regular-closed subset of X is γ-open in X.
3. Every γ-regular-closed subset of X is α-γ-open in X.
4. Every γ-regular-closed subset of X is γ-preopen in X.
5. Every γ-semiopen subset of X is α-γ-open in X.
6. Every γ-semiclosed subset of X is α-γ-closed in X.
7. Every γ-semiclosed subset of X is γ-preclosed in X.
8. Every γ-semiopen subset of X is γ-preopen in X.
9. Every γ-β-open subset of X is γ-preopen in X.
10. Every γ-β-closed subset of X is γ-preclosed in X.
11. Every γ-b-closed subset of X is γ-preclosed in X.
12. Every γ-b-open subset of X is γ-preopen in X.
13. Every γ-regular-open subset of X is γ-preclosed in X.
14. Every γ-regular-open subset of X is γ-closed in X.
15. Every γ-regular-open subset of X is α-γ-closed in X.

Proof. (1) $\Rightarrow$ (2) Let R be any γ-regular-closed subset of a γ-extremally disconnected space X. Then $R = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))$. Since R is γ-regular-closed set, then it is γ-closed and hence $R = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)) = \tau_\gamma\text{-}Int(R)$. Therefore, R is γ-open set in X.

The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear since every γ-open set is α-γ-open and every α-γ-open set is γ-preopen.

(4) $\Rightarrow$ (5) Let S be a γ-semiopen set. So $S \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(S))$. Since $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(S))$ is γ-regular-closed set. Then by (4), $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(S))$ is γ-preopen and hence $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(S)) \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(S)))) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(S)))$. So $S \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(S)))$. Therefore, S is α-γ-open set.

The implications (5) $\Leftrightarrow$ (6), (6) $\Rightarrow$ (7), (7) $\Leftrightarrow$ (8), (9) $\Leftrightarrow$ (10), (10) $\Rightarrow$ (11), (11) $\Leftrightarrow$ (12) and (14) $\Rightarrow$ (15) are obvious.

(8) $\Rightarrow$ (9) Let G be a γ-β-open set. Then by Theorem 3.2.14 (2), $\tau_\gamma\text{-}Cl(G)$ is γ-semiopen set. So by (8), $\tau_\gamma\text{-}Cl(G)$ is γ-preopen set. So $\tau_\gamma\text{-}Cl(G) \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Cl(G))) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(G))$ and hence $G \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(G))$. Therefore, G is γ-preopen set in X.

(12) $\Rightarrow$ (13) Let H be a γ-regular-open set. Then H is γ-β-open set. By Theorem 3.2.14 (4), $\tau_\gamma\text{-}Cl(H)$ is γ-b-open set. Thus by (12), $\tau_\gamma\text{-}Cl(H)$ is γ-preopen. So $\tau_\gamma\text{-}Cl(H) \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Cl(H))) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(H))$. Since H is γ-regular-open set. Hence $\tau_\gamma\text{-}Cl(H) \subseteq H$. Since $H \subseteq \tau_\gamma\text{-}Cl(H)$. Then $\tau_\gamma\text{-}Cl(H) = H$. This means that H is γ-closed and hence it is γ-preclosed.

(13) $\Rightarrow$ (14) Let U be a γ-regular-open set. Then by (13), U is γ-preclosed set. So τ_γ-$Cl(\tau_\gamma$-$Int(U)) \subseteq U$. Since U is γ-regular-open set, then U is γ-open. Hence τ_γ-$Cl(U) \subseteq U$. But in general $U \subseteq \tau_\gamma$-$Cl(U)$. Therefore, τ_γ-$Cl(U) = U$. This means that U is γ-closed.

(15) $\Rightarrow$ (1) Let V be any γ-open set of X. Then τ_γ-$Int(\tau_\gamma$-$Cl(V))$ is γ-regular-open set. By (15), τ_γ-$Int(\tau_\gamma$-$Cl(V))$ is α-γ-closed. So

τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(V))))) \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(V))$. By Theorem 3.2.17, we get τ_γ-$Cl(V) \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(V))$. But τ_γ-$Int(\tau_\gamma$-$Cl(V)) \subseteq \tau_\gamma$-$Cl(V)$ in general. Then τ_γ-$Cl(V) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(V))$ and hence τ_γ-$Cl(V)$ is γ-open set of X. Therefore, X is γ-extremally disconnected space. □

Theorem 3.3.12. The following conditions are equivalent for any topological space (X, τ).

1. X is γ-extremally disconnected.
2. The τ_γ-closure of every γ-β-open set of X is γ-regular-open in X.
3. The τ_γ-closure of every γ-b-open set of X is γ-regular-open in X.
4. The τ_γ-closure of every γ-semiopen set of X is γ-regular-open in X.
5. The τ_γ-closure of every α-γ-open set of X is γ-regular-open in X.
6. The τ_γ-closure of every γ-open set of X is γ-regular-open in X.
7. The τ_γ-closure of every γ-regular-open set of X is γ-regular-open in X.
8. The τ_γ-closure of every γ-preopen set of X is γ-regular-open in X.

Proof. (1) $\Rightarrow$ (2) Let R be a γ-β-open subset of a γ-extremally disconnected space X. Then by (1) and Theorem 3.2.10, we have $\tau_\gamma\text{-}Cl(R) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$ implies that $\tau_\gamma\text{-}Cl(R) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Cl(R)))$. Hence $\tau_\gamma\text{-}Cl(R)$ is γ-regular-open set in X.

The implications (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7) are clear.

(7) $\Rightarrow$ (8) Let P be any γ-preopen set of X, then $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P))$ is γ-regular-open. By (7), $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P)))$ is γ-regular-open set. So $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P))) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P)))))$. Since every γ-preopen set is γ-β-open. Then by Theorem 3.2.10 and Theorem 3.2.18, we have $\tau_\gamma\text{-}Cl(P) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P)) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Cl(P)))$. Hence $\tau_\gamma\text{-}Cl(P)$ is γ-regular-open set in X.

(8) $\Rightarrow$ (1) Let S be a γ-open set of X. Then S is γ-preopen and by (8), $\tau_\gamma\text{-}Cl(S)$ is γ-regular-open set in X. Then $\tau_\gamma\text{-}Cl(S)$ is γ-open. Therefore, X is γ-extremally disconnected space. □

Remark 3.3.13. γ-regular-open set in Theorem 3.3.12 can be replaced by γ-regular-closed set according to Remark 3.3.9.

3.4 γ-Locally Indiscrete and γ-Hyperconnected Spaces

In this section, we introduce new types of spaces called γ-locally indiscrete and γ-hyperconnected. We establish some properties and characterizations of these spaces.

Definition 3.4.1. A topoplogical space (X, τ) with an operation γ on τ is said to be:

1. γ-locally indiscrete if every γ-open subset of X is γ-closed, or every γ-closed

subset of X is γ-open.

2. γ-hyperconnected if every nonempty γ-open subset of X is γ-dense.

The relation between γ-extremally disconnected, γ-hyperconnected and γ-locally indiscrete spaces are given in the following theorem.

Theorem 3.4.2. Let (X, τ) be a topological space and γ be an operation on τ. Then the following hold:

1. If X is γ-locally indiscrete, then X is γ-extremally disconnected.

2. If X is γ-hyperconnected, then X is γ-extremally disconnected.

Proof. Follows from their definitions. □

Some characterizations of γ-locally indiscrete space are as follows.

Theorem 3.4.3. If (X, τ) is γ-locally indiscrete space, then the following statements are true:

1. Every γ-semiopen subset of X is γ-open and hence it is γ-closed.

2. Every γ-semiclosed subset of X is γ-closed and hence it is γ-open.

3. Every γ-open subset of X is γ-regular-open and hence it is γ-regular-closed.

4. Every γ-closed subset of X is γ-regular-closed and hence it is γ-regular-open.

5. Every γ-semiopen subset of X is γ-regular-open and hence it is γ-regular-closed.

6. Every γ-semiclosed subset of X is γ-regular-closed and hence it is γ-regular-open.

7. Every γ-β-open subset of X is γ-preopen.

8. Every γ-β-closed subset of X is γ-preclosed.

Proof. (1) Let S be any γ-semiopen subset of a γ-locally indiscrete space (X, τ), then $S \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(S))$. Since $\tau_\gamma\text{-}Int(S)$ is γ-open subset of X, then it is γ-closed. So $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(S)) = \tau_\gamma\text{-}Int(S)$ implies that $S \subseteq \tau_\gamma\text{-}Int(S)$. But $\tau_\gamma\text{-}Int(S) \subseteq S$. Then $S = \tau_\gamma\text{-}Int(S)$, this means that S is γ-open. Since a space X is γ-locally indiscrete, then S is γ-closed.

(2) The proof is similar to part (1).

(3) Let O be any γ-open subset of a γ-locally indiscrete space (X, τ). Since every γ-open set is γ-closed, then $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(O)) = O$. This implies that O is a γ-regular-closed set.

(4) The proof is similar to part (3).

(5) Follows directly from (1) and (3).

(6) Follows directly from (2) and (4).

(7) Let P be a γ-β-open subset of a γ-locally indiscrete space (X, τ). Then $P \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P)))$. Since $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P))$ is γ-open set and hence it is γ-closed in γ-locally indiscrete space X. Thus, $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P))) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P))$. Hence $P \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(P))$. So P is γ-preopen set in X.

(8) The proof is similar to part (7). □

From Theorem 3.4.3, we have the following corollary.

Corollary 3.4.4. If (X, τ) is γ-locally indiscrete space, then:

1. τ_γ-$RO(X) = \tau_\gamma = \tau_{\alpha-\gamma} = \tau_\gamma$-$SO(X)$.

2. τ_γ-$PO(X) = \tau_\gamma$-$BO(X) = \tau_\gamma$-$\beta O(X)$. □

When a space (X, τ) is γ-hyperconnected, then τ_γ-$RO(X)$ becomes indiscrete topology as shown in the following theorem.

Theorem 3.4.5. A topological space (X, τ) is γ-hyperconnected if and only if τ_γ-$RO(X) = \{\phi, X\}$.

Proof. In general ϕ and X are γ-regular-open subsets of a γ-hyperconnected space X. Let R be any nonempty proper subset of X which is γ-regular-open. Then R is γ-open set. Since X is γ-hyperconnected space. So τ_γ-$Int(\tau_\gamma$-$Cl(R)) = \tau_\gamma$-$Int(X) = X$ and hence R is γ-regular-open set in X. Contradiction. Therefore, τ_γ-$RO(X) = \{\phi, X\}$.

Conversely, suppose that τ_γ-$RO(X) = \{\phi, X\}$ and let S be any nonempty γ-open subset of X. Then S is γ-β-open set. By Theorem 3.2.10, τ_γ-$Cl(S) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(S)))$. Since τ_γ-$Int(\tau_\gamma$-$Cl(S))$ is γ-regular-open set and S is nonempty γ-open set. Then τ_γ-$Int(\tau_\gamma$-$Cl(S))$ should be X. Therefore, τ_γ-$Cl(S) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(S))) = \tau_\gamma$-$Cl(X) = X$. Hence a space X is γ-hyperconnected. □

Corollary 3.4.6. A space (X, τ) is γ-hyperconnected if and only if τ_γ-$RC(X) = \{\phi, X\}$.

□

Theorem 3.4.7. If a space (X, τ) is γ-hyperconnected and γ be a regular operation on τ, then every nonempty γ-preopen subset of X is γ-semi-dense. That is, if a space (X, τ) is γ-hyperconnected with a regular operation γ on τ, then τ_γ-$sCl(P) = X$ for every nonempty γ-preopen set P of X.

Proof. Let P be any nonempty γ-preopen subset of a γ-hyperconnected space X. By Theorem 3.2.18 and Theorem 3.2.20, we have τ_γ-$sCl(P) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(P)) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))))$. Since τ_γ-$Int(\tau_\gamma$-$Cl(P))$ is a nonempty γ-open set and X is γ-hyperconnected space. Then τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))) = X$ and hence τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P)))) = \tau_\gamma$-$Int(X) = X$. Thus τ_γ-$sCl(P) = X$. This completes the proof. $\square$

3.5 Conclusion

This chapter discusses γ-regular-open set, γ-regular-closed set, γ-extremally disconnected space, γ-locally indiscrete space and γ-hyperconnected space which will be used to construct γ-P_S- sets, γ-P_S- functions and γ-P_S- separation axioms in Chapters 4-6.

CHAPTER FOUR

γ-P_S- SETS

4.1 Introduction

In this chapter, γ-P_S-open sets will be constructed by using γ-preopen and γ-semiclosed sets which as have been discussed in Chapter 2. Some operations and properties that involved this γ-P_S-open sets will be established.

4.2 γ-P_S-Open Sets

This section begins with the construction of a new class of sets called γ-P_S-open set.

Definition 4.2.1. A γ-preopen subset A of a topological space (X, τ) is called γ-P_S-open if for each $x \in A$, there exists a γ-semiclosed set F such that $x \in F \subseteq A$.

The class of all γ-P_S-open subsets of a topological space (X, τ) is denoted by τ_γ-$P_SO(X)$ or τ_γ-$P_SO(X, \tau)$.

Theorem 4.2.2. Let A be a subset of a topological space (X, τ) and γ be an operation on τ. Then A is γ-P_S-open set if and only if A is γ-preopen set and A is a union of γ-semiclosed sets.

Proof. The proof is directly from Definition 4.2.1. □

The following is an example of the class of all γ-P_S-open sets and some other types of γ- sets.

Example 4.2.3. Let a space $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Define an operation $\gamma \colon \tau \to P(X)$ as follows:

For every $A \in \tau$

$$\gamma(A) = \begin{cases} A & \text{if } a \in A \\ Cl(A) & \text{if } a \notin A \end{cases}$$

Clearly, $\tau_\gamma = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}\}$.

So we can easily find the following classes of types of γ- sets:

τ_γ-$RO(X) = \{\phi, X, \{a\}, \{b, c\}\}$;

τ_γ-$SO(X) = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}\} = \tau_{\alpha-\gamma}$;

τ_γ-$SC(X) = \{\phi, X, \{a\}, \{c\}, \{b, c\}\}$;

τ_γ-$PO(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} = \tau_\gamma$-$BO(X) = \tau_\gamma$-$\beta O(X)$;

τ_γ-$P_SO(X) = \{\phi, X, \{a\}, \{a, c\}, \{b, c\}\}$.

The union of any class of γ-P_S-open sets in any topological space (X, τ) is also a γ-P_S-open set as shown in the next theorem.

Theorem 4.2.4. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a class of γ-P_S-open sets in a topological space (X, τ). Then $\bigcup_{\lambda\in\Lambda}\{A_\lambda\}$ is also γ-P_S-open set in (X, τ).

Proof. Let $x \in \{A_\lambda\}_{\lambda\in\Lambda}$. Then $x \in A_\lambda$ for some $\lambda \in \Lambda$. Since A_λ is γ-P_S-open set in X and hence it is γ-preopen set. Since any union of γ-preopen set is γ-preopen. So $\bigcup_{\lambda\in\Lambda}\{A_\lambda\}$ is γ-preopen set. Let $x \in \bigcup_{\lambda\in\Lambda}\{A_\lambda\}$, there exist $\lambda \in \Lambda$ such that $x \in A_\lambda$. Since A_λ is γ-P_S-open set for each λ, there exists a γ-semiclosed set F such that $x \in F \subseteq A_\lambda \subseteq \bigcup_{\lambda\in\Lambda}\{A_\lambda\}$ implies that $x \in F \subseteq \bigcup_{\lambda\in\Lambda}\{A_\lambda\}$. Therefore, $\bigcup_{\lambda\in\Lambda}\{A_\lambda\}$ is γ-P_S-open set. $\square$

The intersection of any two γ-P_S-open sets in (X, τ) may not be a γ-P_S-open set as demonstrated in the following example.

Example 4.2.5. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Then the sets $\{a, c\}$ and $\{b, c\}$ are γ-P_S-open, but the intersection $\{a, c\} \cap \{b, c\} = \{c\}$ is not a γ-P_S-open set since $\{c\} \notin \tau_\gamma$-$P_SO(X)$.

It is clear from the above example that the class of all γ-P_S-open sets of any topological space (X, τ) need not be a topology on X in general. Therefore, it is called a supratopology on X.

The following theorem shows that the class of all γ-P_S-open sets will be a topology on X whenever the class of all γ-preopen sets is a topology on X.

Theorem 4.2.6. Suppose the class of all γ-preopen sets in a topological space (X, τ) forms a topology on X. If A and B are γ-P_S-open subsets of X, then $A \cap B$ is γ-P_S-open set and hence the class of all γ-P_S-open sets forms a topology on X.

Proof. Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since A and B are both γ-P_S-open sets in X, then A and B are both γ-preopen sets in X. Then there exists γ-semiclosed sets F and E in X such that $x \in F \subseteq A$ and $x \in E \subseteq B$, respectively. This implies that $x \in F \cap E \subseteq A \cap B$, since any intersection of γ-semiclosed sets is γ-semiclosed set, then $F \cap E$ is γ-semiclosed set. Since the class of all γ-preopen sets in X forms a topology on X, then the intersection of two γ-preopen sets is also γ-preopen set. Thus,

$A \cap B$ is γ-preopen set. Hence by Definition 4.2.1, $A \cap B$ is γ-P_S-open set. Therefore, the class of all γ-P_S-open sets forms a topology on X. □

The following is an example in which the intersection of two γ-preopen sets is not γ-preopen, but the class of γ-P_S-open sets is still a topology on X.

Example 4.2.7. Consider the space $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Define an operation $\gamma \colon \tau \to P(X)$ by $\gamma(A) = A$ for all $A \in \tau$. Then $\tau_\gamma = \tau$.

So we can easily find the following classes of τ_γ-$P_SO(X)$, τ_γ-$PO(X)$, τ_γ-$SO(X)$ and τ_γ-$SC(X)$:

τ_γ-$SO(X) = \tau_\gamma$;

τ_γ-$SC(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$;

τ_γ-$PO(X) = P(X) \backslash \{\{b\}, \{a, b\}\}$;

τ_γ-$P_SO(X) = \{\phi, X, \{a\}, \{b, c, d\}\}$;

Here, $\{b, c\}$ and $\{b, d\}$ are γ-preopen sets, but $\{b, c\} \cap \{b, d\} = \{b\}$ is not a γ-preopen set. Therefore, the intersection of two γ-preopen sets need not be γ-preopen, but the class of γ-P_S-open sets is still a topology on X.

Theorem 4.2.8. Let A and B be two subsets of a topological space (X, τ). Then A is γ-P_S-open set if and only if for each $x \in A$, there exists a γ-P_S-open set B such that $x \in B \subseteq A$.

Proof. Suppose that A is γ-P_S-open set in the space (X, τ). Then for each $x \in A$, put $B = A$ is a γ-P_S-open set such that $x \in B \subseteq A$.

Conversely, suppose that for each $x \in A$, there exists a γ-P_S-open set B such that $x \in B \subseteq A$. Thus $A = \bigcup_{x \in A} B_x$ where $B_x \in \tau_\gamma$-$P_SO(X)$. Therefore, by Theorem $4.2.4$, A is γ-P_S-open set in X. $\square$

The following Remark $4.2.9$ (1) shows a relation between γ-P_S-open set and γ-preopen set which is directly follows from Definition $4.2.1$. While Remark $4.2.9$ (2) follows from Remark $3.2.4$ (2) and Definition $4.2.1$.

Remark 4.2.9. Let (X, τ) be a topological space and γ be an operation on τ. Then the following conditions are true:

1. Every γ-P_S-open subset of a space X is γ-preopen.

2. Every γ-regular-open subset of a space X is γ-P_S-open.

Converses of the above remark are not true which can be seen in the following example.

Example 4.2.10. Let (X, τ) be a topological space and γ be an operation on τ as in Example $4.2.3$. Then $\{b\}$ is γ-preopen set but not a γ-P_S-open set. Also the set $\{a, c\}$ is γ-P_S-open set but not γ-regular-open set.

Since every γ-preopen set is γ-b-open and every γ-b-open set is γ-β-open, then by using Remark $4.2.9$ (1), we have the following Remark $4.2.11$.

Remark 4.2.11. Every γ-P_S-open subset of a space X is γ-b-open and hence it is γ-β-open.

The relations between γ-P_S-open set and various types of γ- sets as results from Remark 4.2.9, Remark 4.2.11 and Figure 3.1 is described in Figure 4.1.

$$
\begin{tikzcd}
\gamma\text{-regular-open} \arrow[r] \arrow[d] & \gamma\text{-open} \arrow[r] & \alpha\text{-}\gamma\text{-open} \arrow[r] \arrow[d] & \gamma\text{-semiopen} \arrow[d] & \\
\gamma\text{-}P_S\text{-open} \arrow[rr] & & \gamma\text{-preopen} \arrow[r] & \gamma\text{-}b\text{-open} \arrow[r] & \gamma\text{-}\beta\text{-open}
\end{tikzcd}
$$

Remark: $A \longrightarrow B$ represents A implies B

Figure 4.1. The relations between γ-P_S-open set and various types of γ- sets

In the sequel, none of the implications that concerning γ-P_S-open set in the above figure is reversible. It is noticed that γ-P_S-open set lies strictly between the classes of γ-regular-open set and γ-preopen set. Moreover, the relation between γ-P_S-open set and γ-open set are independent. Similarly the relation between γ-P_S-open set and α-γ-open set are independent, and as well as the relation between γ-P_S-open set and γ-semiopen set are independent which is explained in the following example.

Example 4.2.12. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Then $\{a, c\}$ is γ-P_S-open set but $\{a, c\}$ is not γ-semiopen set and hence $\{a, c\}$ is not α-γ-open set and it is not γ-open set. Also the set $\{a, b\}$ is γ-open, α-γ-open and γ-semiopen set, but $\{a, b\}$ is not γ-P_S-open set, because $\{a, b\} \notin \tau_\gamma$-$P_SO(X)$.

Some important relationships between γ-P_S-open set and other types of γ- sets as mentioned in Theorem 3.2.26 and Theorem 3.2.28 are obtained.

Theorem 4.2.13. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following statements are equivalent:

1. A is γ-regular-open.

2. A is γ-P_S-open and γ-semiclosed.

3. A is γ-open and γ-semiclosed.

4. A is α-γ-open and γ-semiclosed.

5. A is γ-preopen and γ-semiclosed.

6. A is γ-open and γ-β-closed.

7. A is α-γ-open and γ-β-closed.

Proof. It is enough to proof (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) since the proof of the other implications follows directly from Theorem 3.2.26.

(1) $\Rightarrow$ (2) Let A be any γ-regular-open set, then by Remark 4.2.9 (2) and Remark 3.2.4 (2), A is γ-P_S-open and γ-semiclosed set, respectively.

(2) $\Rightarrow$ (3) Let A be any γ-P_S-open set, then by Remark 4.2.9 (1), A is γ-preopen set and since A is γ-semiclosed set. Then by Remark 3.2.4 (2), A is γ-regular-open set and hence it is γ-open set. □

Theorem 4.2.14. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following conditions are equivalent:

1. A is γ-clopen.

2. A is γ-regular-open and γ-regular-closed.

3. A is γ-open and α-γ-closed.

4. A is γ-open and γ-preclosed.

5. A is α-γ-open and γ-preclosed.

6. A is α-γ-open and γ-closed.

7. A is γ-P_S-open and γ-closed.

8. A is γ-preopen and γ-closed.

9. A is γ-preopen and α-γ-closed.

Proof. It is enough to proof (6) $\Rightarrow$ (7) and (7) $\Rightarrow$ (8) since the proof of the other implications follows directly from Theorem 3.2.28.

(6) $\Rightarrow$ (7) Let A be any α-γ-open and γ-closed set, then A is γ-preopen set and γ-semiclosed set and hence by Remark 3.2.4 (2), A is γ-regular-open set and hence by Remark 4.2.9 (2), A is γ-P_S-open set.

(7) $\Rightarrow$ (8) It is obvious since every γ-P_S-open set is γ-preopen. □

The concept of γ-P_S-open set and P_S-open set are independent, however in a γ-regular space they are equivalent. This is explained from Remark 4.2.15 to Theorem 4.2.18.

Remark 4.2.15. The concept of γ-P_S-open set and P_S-open set are independent. It is shown in Example 4.2.16 and Example 4.2.17.

Example 4.2.16. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. The class of all P_S-open sets in X is $P_SO(X) = \{\phi, \{a\}, \{b, c\}, X\}$. Here the set $\{a, c\}$ is γ-P_S-open set but it is not P_S-open set since $\{a, c\} \notin P_SO(X)$.

Example 4.2.17. Let a space $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a, b\}, \{c\}, X\}$. Then $P_SO(X) = \tau$. Define an operation γ on τ by:

For every $A \in \tau$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{c\} \\ X & \text{if } A \neq \{c\} \end{cases}$$

Obviously, $\tau_\gamma = \{\phi, X, \{c\}\}$ and τ_γ-$P_SO(X) = \{\phi, X\}$. Then the set $\{c\}$ is P_S-open, but $\{c\}$ is not γ-P_S-open set.

Theorem 4.2.18. If (X, τ) be a γ-regular space and γ be an operation on τ, then the concept of γ-P_S-open set and P_S-open set are equivalent.

Proof. Since (X, τ) is γ-regular space, then by Remark 2.3.32 (1), the concept of γ-preopen set and preopen set are equivalent, and by Remark 2.3.32 (4), the concept of γ-semiopen set and semiopen set are equivalent and hence the concept of γ-semiclosed set and semiclosed set are equivalent. So by Lemma 2.2.4 and Theorem 4.2.2, the concept of γ-P_S-open set and P_S-open set are equivalent. □

Next, recall the construction of γ-locally indiscrete space (X, τ) in Section 3.4 and Figure 4.1. The concept of γ-P_S-open set and γ-open set are independent, but they are identical in a γ-locally indiscrete space (X, τ). This is explained in Theorem 4.2.19.

Theorem 4.2.19. Let (X, τ) be a topological space and γ be an operation on τ. If (X, τ) is γ-locally indiscrete, then the concept of γ-P_S-open set and γ-open set are identical (That is, τ_γ-$P_SO(X) = \tau_\gamma$).

Proof. Let U be any γ-P_S-open subset of a γ-locally indiscrete space X, then U is γ-preopen set and for each $x \in U$, there exists a γ-semiclosed set F in X containing x such that $F \subseteq U$. By Theorem 3.4.3 (2), F is γ-open. It follows that U is γ-open. Hence τ_γ-$P_SO(X) \subseteq \tau_\gamma$. On the other hand, Let V be any γ-open subset of a γ-locally indiscrete space X. Since every γ-open set is γ-closed, then τ_γ-$Int(\tau_\gamma$-$Cl(V)) = V$. This implies that V is a γ-regular-open set and hence by Remark 4.2.9 (2), V is γ-P_S-open set. Then $\tau_\gamma \subseteq \tau_\gamma$-$P_SO(X)$. Therefore, τ_γ-$P_SO(X) = \tau_\gamma$. This completes the proof. $\square$

The following example shows that the converse of the Theorem 4.2.19 is not true.

Example 4.2.20. Consider the space $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define an operation γ on τ by $\gamma(A) = A$ for all $A \in \tau$. Thus, $\tau_\gamma = \tau$.

So we can easily find the following classes of τ_γ-$PO(X)$, τ_γ-$SO(X)$, τ_γ-$SC(X)$ and τ_γ-$P_SO(X)$:

τ_γ-$PO(X) = \tau_\gamma$;

τ_γ-$SO(X) = P(X)\backslash\{d\}$ and τ_γ-$SC(X) = P(X)\backslash\{a, b, c\}$;

τ_γ-$P_SO(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Then τ_γ-$P_SO(X) = \tau_\gamma$. But the space X is not γ-locally indiscrete, because $\{a\}$ is a γ-open set, but it is not a γ-closed set.

From Theorem 4.2.19 and Corollary 3.4.4 (1), we have the following corollary.

Corollary 4.2.21. If (X,τ) is γ-locally indiscrete space, then:

$$\tau_\gamma\text{-}RO(X) = \tau_\gamma = \tau_\gamma\text{-}P_SO(X) = \tau_{\alpha-\gamma} = \tau_\gamma\text{-}SO(X). \qquad \square$$

Lemma 4.2.22. If (X,τ) is γ-locally indiscrete space, then γ is a regular operation.

Proof. The proof is obvious and hence it is omitted. $\square$

Theorem 4.2.23. If a space (X,τ) is γ-locally indiscrete, then τ_γ-$P_SO(X,\tau)$ is a topology on X.

Proof. Directly follows from Theorem 4.2.19 and Lemma 4.2.22. $\square$

As we mentioned in Remark 4.2.9 (1), every γ-P_S-open set is γ-preopen, but the converse is true when a topological space (X,τ) is γ-semiT_1. This is explained in the next theorem.

Theorem 4.2.24. Let (X,τ) be a topological space and γ be an operation on τ. If (X,τ) is γ-semiT_1, then the concept of γ-P_S-open set and γ-preopen set are identical (That is, τ_γ-$P_SO(X) = \tau_\gamma$-$PO(X)$).

Proof. Let G be any γ-preopen subset of a space X. If $G = \phi$, then G is γ-P_S-open set. If $G \neq \phi$, then for each $x \in G$. Since a space X is γ-semiT_1, then by Theorem 2.5.5 (2), $\{x\}$ is γ-semiclosed set and hence $x \in \{x\} \subseteq G$. Therefore, G is γ-P_S-open set. Hence τ_γ-$PO(X) \subseteq \tau_\gamma$-$P_SO(X)$. Since every γ-P_S-open set is γ-preopen. Then τ_γ-$P_SO(X) \subseteq \tau_\gamma$-$PO(X)$. Therefore, τ_γ-$P_SO(X) = \tau_\gamma$-$PO(X)$. $\square$

Notice that, when a space (X, τ) is γ-hyperconnected (refer to Section 3.4), then τ_γ-$P_SO(X)$ becomes indiscrete topology as shown in the following theorem.

Theorem 4.2.25. Let (X, τ) be a topological space and γ be a regular operation on τ. A topological space (X, τ) is γ-hyperconnected if and only if τ_γ-$P_SO(X) = \{\phi, X\}$.

Proof. Suppose that (X, τ) be a γ-hyperconnected space and suppose, if possible, that G be any γ-P_S-open subset of X. If $G = \phi$, then G is γ-P_S-open set of X. If $G \neq \phi$. Then for each $x \in G$, there exists a γ-semiclosed set F such that $x \in F \subseteq G$. Since F is γ-semiclosed set of (X, τ), then τ_γ-$Int(\tau_\gamma$-$Cl(F)) \subseteq F$ implies that τ_γ-$sCl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(F))) \subseteq \tau_\gamma$-$sCl(F) = F$. Since X is γ-hyperconnected space, then by Theorem 3.4.7, τ_γ-$sCl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(F))) = X \subseteq F$ since τ_γ-$Int(\tau_\gamma$-$Cl(F))$ is a nonempty γ-preopen set of X. This implies that $F = X$. Since $F = X \subseteq G$, then $G = X$. Therefore, G must be equal to X, which completes the proof.

Conversely, Suppose that τ_γ-$P_SO(X) = \{\phi, X\}$. Since τ_γ-$RO(X) \subseteq \tau_\gamma$-$P_SO(X)$ in general, then τ_γ-$RO(X) = \{\phi, X\}$. By Theorem 3.4.5, we have (X, τ) is a γ-hyperconnected space. □

Theorem 4.2.25 and Theorem 3.4.5 contribute the following corollary.

Corollary 4.2.26. Let γ be a regular operation on τ. A topological space (X, τ) is γ-hyperconnected if and only if τ_γ-$RO(X) = \tau_\gamma$-$P_SO(X) = \{\phi, X\}$. □

Theorem 4.2.27. The following conditions are equivalent for any topological space (X, τ).

1. (X, τ) is γ-extremally disconnected.

2. The τ_γ-closure of every γ-P_S-open set of X is γ-regular-open in X.

Proof. (1) $\Rightarrow$ (2) Let (X, τ) be any γ-extremally disconnected space and let A be any γ-P_S-open set of X, then A is γ-preopen set. By Theorem 3.3.12 (8), τ_γ-$Cl(A)$ is γ-regular-open in X.

(2) $\Rightarrow$ (1) Let A be any γ-regular-open set in X, then A is γ-P_S-open set of X. By hypothesis, τ_γ-$Cl(A)$ is γ-regular-open set in X. By Theorem 3.3.12 (7), (X, τ) is γ-extremally disconnected space. □

Since every γ-regular-open set is γ-P_S-open, then by using Theorem 4.2.27 and Theorem 3.3.12, we have the following theorem.

Theorem 4.2.28. Let (X, τ) be any γ-extremally disconnected space, then the following statements hold:

1. The τ_γ-closure of every γ-β-open set of X is γ-P_S-open in X.
2. The τ_γ-closure of every γ-b-open set of X is γ-P_S-open in X.
3. The τ_γ-closure of every γ-semiopen set of X is γ-P_S-open in X.
4. The τ_γ-closure of every α-γ-open set of X is γ-P_S-open in X.
5. The τ_γ-closure of every γ-open set of X is γ-P_S-open in X.
6. The τ_γ-closure of every γ-regular-open set of X is γ-P_S-open in X.
7. The τ_γ-closure of every γ-P_S-open set of X is γ-P_S-open in X.

8. The τ_γ-closure of every γ-preopen set of X is γ-P_S-open in X.

Remark 4.2.29. For any topological space (X,τ) with an operation γ on τ, we have:

1. If $\tau_\gamma,\ \tau_{\alpha-\gamma},\ \tau_\gamma$-$SO(X)$ and τ_γ-$PO(X)$ are indiscrete topologies, then τ_γ-$P_SO(X)$ is also indiscrete topology.

2. If τ_γ-$P_SO(X)$ is discrete topology, then τ_γ-$PO(X)$ is also discrete topology.

3. τ_γ is discrete topology if and only if τ_γ-$P_SO(X)$ is discrete topology.

The converse of part (1) and (2) of Remark 4.2.29 are not true in general as it is shown in the following two examples:

Example 4.2.30. Let (X,τ) be a topological space and γ be an operation on τ as in Example 4.2.17. Then τ_γ-$P_SO(X)$ is an indiscrete topology, but τ_γ, $\tau_{\alpha-\gamma}$, τ_γ-$SO(X)$ and τ_γ-$PO(X)$ are not indiscrete topologies.

Example 4.2.31. Consider the space $X = \{a,b,c\}$ with the topology $\tau = \{\phi,\ X,\ \{a,b\},\ \{c\}\}$. Define an operation γ on τ by $\gamma(A) = A$ for all $A \in \tau$. Then $\tau_\gamma = \tau$. Hence τ_γ-$PO(X)$ is discrete topology. Since (X,τ) is γ-locally indiscrete space, then by Theorem 4.2.19, τ_γ-$P_SO(X) = \tau_\gamma$. Therefore, τ_γ-$P_SO(X)$ is not discrete topology.

In general, the relation between γ-P_S-open and γ-semiclosed sets are independent, but if a singleton set is γ-P_S-open, then it is γ-semiclosed which is directly follows from Definition 4.2.1.

Remark 4.2.32. Let (X,τ) be a topological space and γ be an operation on τ. For each element $x \in X$, if the set $\{x\}$ is γ-P_S-open, then $\{x\}$ is γ-semiclosed.

Remark 4.2.33. Let (X, τ) be a topological space and γ be an operation on τ. For each element $x \in X$, the set $\{x\}$ is γ-P_S-open if and only if $\{x\}$ is γ-regular-open.

The intersection of any α-γ-open set and γ-preopen set is γ-preopen set in any topological space (X, τ), where γ is a regular operation on τ.

Theorem 4.2.34. Let G, H be any subsets of a topological space (X, τ) and γ be a regular operation on τ. If G is α-γ-open and H is γ-preopen, then $G \cap H$ is γ-preopen set in (X, τ).

Proof. Since G is α-γ-open set and H is γ-preopen set. Then $G \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(G)))$ and $H \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(H))$. Hence

$$
\begin{aligned}
G \cap H &\subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(G))) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(H)) \\
&= \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(G))) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(H))) \\
&= \tau_\gamma\text{-}Int[\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(G)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(H))] \text{ (by Lemma 2.3.25)} \\
&\subseteq \tau_\gamma\text{-}Int[\tau_\gamma\text{-}Cl[\tau_\gamma\text{-}Int(G) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(H))]] \\
&\subseteq \tau_\gamma\text{-}Int[\tau_\gamma\text{-}Cl[\tau_\gamma\text{-}Int(G) \cap \tau_\gamma\text{-}Cl(H)]] \text{ (by Lemma 2.3.25)} \\
&\subseteq \tau_\gamma\text{-}Int[\tau_\gamma\text{-}Cl[\tau_\gamma\text{-}Cl[\tau_\gamma\text{-}Int(G) \cap H]]] \\
&= \tau_\gamma\text{-}Int[\tau_\gamma\text{-}Cl[\tau_\gamma\text{-}Int(G) \cap H]] \\
&\subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(G \cap H))
\end{aligned}
$$

Therefore, $G \cap H$ is γ-preopen set. $\square$

The intersection of any γ-P_S-open set and γ-regular-open set is γ-P_S-open set in any topological space (X, τ), where γ is a regular operation on τ.

Theorem 4.2.35. Let G, H be any subsets of a topological space (X, τ) and γ be a regular operation on τ. If G is γ-P_S-open and H is γ-regular-open, then $G \cap H$ is γ-P_S-open set in (X, τ).

Proof. Since G is γ-P_S-open set in (X, τ), then G is γ-preopen set and $G = \bigcup F_\lambda$ where F_λ is γ-semiclosed set for each λ. Then $G \cap H = \bigcup F_\lambda \cap H = \bigcup (F_\lambda \cap H)$. Since H is γ-regular-open set, then H is α-γ-open, by Theorem $4.2.34$, $G \cap H$ is γ-preopen set. Again since H is γ-regular-open set, then H is γ-semiclosed set and hence $F_\lambda \cap H$ is γ-semiclosed for each λ. Therefore, by Theorem $4.2.2$, $G \cap H$ is γ-P_S-open set in (X, τ). □

So far, we have been talking exclusively about γ-P_S-open sets. Now it is time to look at the complementary concept and define γ-P_S-closed sets.

4.3 γ-P_S-Closed Sets

We will introduce a new class of sets called γ-P_S-closed which is the complement of γ-P_S-open set in a topological space (X, τ) as discussed in previous Section 4.2.

Definition 4.3.1. A subset B of a topological space (X, τ) with an opeation γ on τ is called γ-P_S-closed if $X \backslash B$ is γ-P_S-open. The class of all γ-P_S-closed subsets of a topological space (X, τ) is denoted by τ_γ-$P_SC(X)$ or τ_γ-$P_SC(X, \tau)$.

The following theorem is follows directly from Definition $4.3.1$.

Theorem 4.3.2. Let B be a subset of a topological space (X, τ) and γ be an operation on τ. Then B is γ-P_S-closed if and only if B is γ-preclosed set and B is an intersection of γ-semiopen sets. $\square$

Any intersection of γ-P_S-closed sets in any topological space (X, τ) is also γ-P_S-closed set as the next corollary shows.

Corollary 4.3.3. Let $\{B_\lambda\}_{\lambda \in \Lambda}$ be a class of γ-P_S-closed subsets of a topological space (X, τ). Then $\bigcap_{\lambda \in \Lambda}\{B_\lambda\}$ is also γ-P_S-closed set in (X, τ).

Proof. The proof is follows from Theorem 4.2.4. $\square$

But the union of any two γ-P_S-closed sets in a topological space (X, τ) may not be a γ-P_S-closed set as shown in the following example.

Example 4.3.4. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Then the class of γ-P_S-closed sets in (X, τ) is:

τ_γ-$P_SC(X) = \{X, \phi, \{a\}, \{b\}, \{b, c\}\}$. Here, $\{a\}$ and $\{b\}$ are γ-P_S-closed sets, but the union $\{a\} \cup \{b\} = \{a, b\}$ is not a γ-P_S-closed set since $\{a, b\} \notin \tau_\gamma$-$P_SC(X)$.

Remark 4.3.5. Every γ-regular-closed set of a space X is γ-P_S-closed and every γ-P_S-closed set of a space X is γ-preclosed.

In the next corollary, the proof is similar to Theorem 4.2.13 taking $A = X \backslash B$.

Corollary 4.3.6. Let B be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following conditions are equivalent:

1. B is γ-regular-closed.

2. B is γ-P_S-closed and γ-semiopen.

3. B is γ-closed and γ-semiopen.

4. B is α-γ-closed and γ-semiopen.

5. B is γ-preclosed and γ-semiopen.

6. B is γ-closed and γ-β-open.

7. B is α-γ-closed and γ-β-open. $\square$

In the next corollary, the proof is similar to Theorem 4.2.14 taking $A = X\backslash B$.

Corollary 4.3.7. Let B be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following properties are equivalent:

1. B is γ-clopen.

2. B is γ-regular-open and γ-regular-closed.

3. B is γ-open and α-γ-closed.

4. B is γ-open and γ-preclosed.

5. B is α-γ-open and γ-preclosed.

6. B is α-γ-open and γ-closed.

7. B is γ-P_S-open and γ-closed.

8. B is γ-P_S-closed and γ-open.

9. B is γ-preopen and γ-closed.

10. B is γ-preopen and α-γ-closed. $\square$

Remark 4.3.8. Let (X, τ) be a topological space and γ be an operation on τ. If a subset B of X is both γ-open and γ-closed, then B is both γ-P_S-open and γ-P_S-closed.

Corollary 4.3.9. If (X, τ) is γ-locally indiscrete space, then the concept of γ-P_S-closed set and γ-closed set are identical.

Proof. The proof is directly from Theorem 4.2.19, and using complements. $\square$

Theorem 4.3.10. If a space (X, τ) is γ-locally indiscrete, then every γ-P_S-open subset of X is γ-P_S-closed.

Proof. Let (X, τ) be a γ-locally indiscrete space and let G be any γ-P_S-open set in X. Then by Theorem 4.2.19, G is γ-open set in X. Since X is γ-locally indiscrete, then G is γ-closed set and hence by Corollary 4.3.9, G is γ-P_S-closed set. $\square$

From Theorem 4.3.10, we have the following corollary.

Corollary 4.3.11. If a space (X, τ) is γ-locally indiscrete, then every γ-P_S-closed subset of X is γ-P_S-open. $\square$

We can notice from Theorem 4.3.10 and Corollary 4.3.11 that if a space (X, τ) is γ-locally indiscrete, then τ_γ-$P_SO(X, \tau) = \tau_\gamma$-$P_SC(X, \tau)$, but the converse does not true in general as shown from the following example.

Example 4.3.12. Considering the space (X, τ) as defined in Example 4.2.17. Then τ_γ-$P_SO(X, \tau) = \tau_\gamma$-$P_SC(X, \tau)$, but (X, τ) is not a γ-locally indiscrete space since the set $\{c\}$ is γ-open, but it is not γ-closed.

The following corollary follows directly from Theorem 4.2.24, and using complements.

Corollary 4.3.13. If (X, τ) is γ-semiT_1 space, then the concept of γ-P_S-closed set and γ-preclosed set are identical (That is, τ_γ-$P_SC(X) = \tau_\gamma$-$PC(X)$). □

Corollary 4.3.14. If (X, τ) is γ-regular space, then the concept of γ-P_S-closed set and P_S-closed set are identical.

Proof. The proof is directly from Theorem 4.2.18, and using complements. □

Corollary 4.3.15. Let γ be a regular operation on τ. A topological space (X, τ) is γ-hyperconnected if and only if τ_γ-$P_SC(X) = \tau_\gamma$-$RC(X) = \{\phi, X\}$.

Proof. Follows from Corollary 4.2.26. □

Remark 4.3.16. We notice from Corollary 4.2.26 and Corollary 4.3.15 that the class of γ-P_S-open, γ-P_S-closed, γ-regular-open and γ-regular-closed sets in a γ-hyperconnected space (X, τ) are indiscrete topologies.

Corollary 4.3.17. Let (X, τ) be a topological space and γ be an operation on τ. A space (X, τ) is γ-extremally disconnected if and only if the τ_γ-interior of every γ-P_S-closed set of X is γ-regular-closed in X.

Proof. Similar to Theorem 4.2.27. □

Corollary 4.3.18. Let (X, τ) be any γ-extremally disconnected space, then the following statements hold:

1. The τ_γ-interior of every γ-β-closed set of X is γ-P_S-closed in X.
2. The τ_γ-interior of every γ-b-closed set of X is γ-P_S-closed in X.
3. The τ_γ-interior of every γ-semiclosed set of X is γ-P_S-closed in X.
4. The τ_γ-interior of every α-γ-closed set of X is γ-P_S-closed in X.
5. The τ_γ-interior of every γ-closed set of X is γ-P_S-closed in X.
6. The τ_γ-interior of every γ-regular-closed set of X is γ-P_S-closed in X.
7. The τ_γ-interior of every γ-P_S-closed set of X is γ-P_S-closed in X.
8. The τ_γ-interior of every γ-preclosed set of X is γ-P_S-closed in X.

Proof. The proof follows directly from Theorem 4.2.28. □

Theorem 4.3.19. Let (X, τ) be a topological space and γ be an operation on τ. For each element $x \in X$, if the set $X\backslash\{x\}$ is γ-P_S-closed, then $X\backslash\{x\}$ is γ-semiopen.

Proof. Let $X\backslash\{x\}$ be a γ-P_S-closed set in (X, τ), then $\{x\}$ is γ-P_S-open set and hence by Remark 4.2.32, $\{x\}$ is γ-semiclosed set. So $X\backslash\{x\}$ is γ-semiopen set in X. □

Corollary 4.3.20. Let (X, τ) be a topological space and γ be an operation on τ. For each element $x \in X$, the set $X\backslash\{x\}$ is γ-P_S-closed if and only if $X\backslash\{x\}$ is γ-regular-closed.

Proof. Similar to Remark 4.2.33. □

Corollary 4.3.21. Let E, F be any subsets of a topological space (X, τ) and γ be a regular operation on τ. If E is α-γ-closed and F is γ-preclosed, then $E \cup F$ is γ-preclosed set in (X, τ).

Proof. The proof is directly from Theorem 4.2.34, and using complements. $\square$

Corollary 4.3.22. Let E, F be any subsets of a topological space (X, τ) and γ be a regular operation on τ. If E is γ-P_S-closed and F is γ-regular-closed, then $E \cup F$ is γ-P_S-closed set in (X, τ).

Proof. The proof is directly from Theorem 4.2.35, and using complements. $\square$

4.4 γ-P_S- Operations

We begin with the definition of γ-P_S-limit point of a set A.

Definition 4.4.1. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. A point $x \in X$ is said to be γ-P_S-limit point of A if for evey γ-P_S-open set U containing x, $U \cap (A \backslash \{x\}) \neq \phi$. The set of all γ-P_S-limit points of A is called a γ-P_S-derived set of A and it is denoted by τ_γ-$P_S D(A)$.

From the Definition 4.4.1, notice that a γ-P_S-limit point x of any set A may or may not lie in the set A. Notice also that in every topology, the point x is not a γ-P_S-limit point of the set $\{x\}$.

Theorem 4.4.2. Let (X, τ) be a topological space and γ be an operation on τ. If there exists a γ-semiclosed set F of X containing x such that $F \cap (A \backslash \{x\}) \neq \phi$ for every subset A of X, then a point $x \in X$ is γ-P_S-limit point of A.

Proof. Let U be any γ-P_S-open set containing x. Then by Definition $4.2.1$, for each $x \in U$ and U is γ-preopen set in X, there exists a γ-semiclosed set F of X containing x such that $F \subseteq U$. By hypothesis, we have $F \cap (A \backslash \{x\}) \neq \phi$ and hence $U \cap (A \backslash \{x\}) \neq \phi$. Therefore, a point $x \in X$ is γ-P_S-limit point of A. □

Some basic properties of γ-P_S-derived set are mentioned in the Theorem $4.4.3$.

Theorem 4.4.3. Let A and B be subsets of a topological space (X, τ) and γ be an operation on τ. Then the following properties hold:

1. τ_γ-$P_SD(\phi) = \phi$.

2. If $A \subseteq B$, then τ_γ-$P_SD(A) \subseteq \tau_\gamma$-$P_SD(B)$.

3. τ_γ-$P_SD(A \cap B) \subseteq \tau_\gamma$-$P_SD(A) \cap \tau_\gamma$-$P_SD(B)$.

4. τ_γ-$P_SD(A) \cup \tau_\gamma$-$P_SD(B) \subseteq \tau_\gamma$-$P_SD(A \cup B)$.

Proof. The proof of part (2) follows directly from Definition $4.4.1$, and the proof of both (3) and (4) follow from part (2). □

The inclusions in (3) and (4) in Theorem $4.4.3$ cannot be replaced by equality in general, as it can be seen from the following two examples.

Example 4.4.4. Let (X, τ) be a topological space and γ be an operation on τ as in Example $4.2.7$. If we take $A = \{b\}$ and $B = \{c\}$. Then τ_γ-$P_SD(A) = \{c, d\}$ and

τ_γ-$P_SD(B) = \{b, d\}$. Thus, τ_γ-$P_SD(A) \cap \tau_\gamma$-$P_SD(B) = \{d\}$ and τ_γ-$P_SD(A \cap B) = \tau_\gamma$-$P_SD(\phi) = \phi$. This implies that τ_γ-$P_SD(A) \cap \tau_\gamma$-$P_SD(B) \neq \tau_\gamma$-$P_SD(A \cap B)$.

Example 4.4.5. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. If we take $A = \{a\}$ and $B = \{b\}$. Then τ_γ-$P_SD(A) = \phi$ and τ_γ-$P_SD(B) = \phi$. Thus, τ_γ-$P_SD(A) \cup \tau_\gamma$-$P_SD(B) = \phi$ and τ_γ-$P_SD(A \cup B) = \tau_\gamma$-$P_SD(\{a, b\}) = \{c\}$. It follows that τ_γ-$P_SD(A) \cup \tau_\gamma$-$P_SD(B) \neq \tau_\gamma$-$P_SD(A \cup B)$.

Since every γ-P_S-open set is γ-preopen. Then we have the following Remark 4.4.6.

Remark 4.4.6. For any subset A of a topological space (X, τ) with an operation γ on τ. τ_γ-$pD(A) \subseteq \tau_\gamma$-$P_SD(A)$.

The following example shows that the converse of Remark 4.4.6 is not true in general.

Example 4.4.7. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. If we take $A = \{a, c\}$. Then τ_γ-$P_SD(A) = \{b\}$ and τ_γ-$pD(A) = \phi$. So τ_γ-$P_SD(A) \nsubseteq \tau_\gamma$-$pD(A)$.

In the above example, we can conclude that τ_γ-$P_SD(A)$ and τ_γ-$D(A)$ are independent. Let $A = \{b, c\}$. Then τ_γ-$P_SD(A) = \{b\}$ and τ_γ-$D(A) = \{c\}$. Hence τ_γ-$P_SD(A) \neq \tau_\gamma$-$D(A)$.

Here are some other properties about the γ-P_S-derived set.

Theorem 4.4.8. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then we have the following properties:

1. $x \in \tau_\gamma\text{-}P_SD(A)$ implies $x \in \tau_\gamma\text{-}P_SD(A\backslash\{x\})$

2. $\tau_\gamma\text{-}P_SD(\tau_\gamma\text{-}P_SD(A))\backslash A \subseteq \tau_\gamma\text{-}P_SD(A)$.

3. $\tau_\gamma\text{-}P_SD(A \cup \tau_\gamma\text{-}P_SD(A)) \subseteq A \cup \tau_\gamma\text{-}P_SD(A)$.

Proof. (1) Clear.

(2) Let $x \in \tau_\gamma\text{-}P_SD(\tau_\gamma\text{-}P_SD(A))\backslash A$ and U is a $\gamma\text{-}P_S$-open set containing x. Then $U \cap (\tau_\gamma\text{-}P_SD(A)\backslash\{x\}) \neq \phi$. Let $y \in U \cap (\tau_\gamma\text{-}P_SD(A)\backslash\{x\})$. Then, since $y \in \tau_\gamma\text{-}P_SD(A)$ and $y \in U$, $U \cap (A\backslash\{y\}) \neq \phi$. Let $z \in U \cap (A\backslash\{y\})$. Hence $z \neq x$ for $z \in A$ and $x \in A$. So $U \cap (A\backslash\{x\}) \neq \phi$. Therefore, $x \in \tau_\gamma\text{-}P_SD(A)$.

(3) Let $x \in \tau_\gamma\text{-}P_SD(A \cup \tau_\gamma\text{-}P_SD(A))$. If $x \in A$, the result is clear. So, let $x \in \tau_\gamma\text{-}P_SD(A \cup \tau_\gamma\text{-}P_SD(A))\backslash A$. Then, for $\gamma\text{-}P_S$-open set U containing x, $U \cap (A \cup \tau_\gamma\text{-}P_SD(A))\backslash\{x\} \neq \phi$. Thus, $U \cap (A\backslash\{x\}) \neq \phi$ or $U \cap (\tau_\gamma\text{-}P_SD(A)\backslash\{x\}) \neq \phi$. Now, it follows similarly from the part (2) that $U \cap (A\backslash\{x\}) \neq \phi$. Hence, $x \in \tau_\gamma\text{-}P_SD(A)$. Therefore, in any case, $\tau_\gamma\text{-}P_SD(A \cup \tau_\gamma\text{-}P_SD(A)) \subseteq A \cup \tau_\gamma\text{-}P_SD(A)$. □

Now, we will define one of the most important operator on $\gamma\text{-}P_S$-open set. This set is known as the $\tau_\gamma\text{-}P_S$-closure of A.

Definition 4.4.9. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. The $\tau_\gamma\text{-}P_S$-closure of A is defined as the intersection of all $\gamma\text{-}P_S$-closed sets of X containing A and it is denoted by $\tau_\gamma\text{-}P_SCl(A)$. That is,

$$\tau_\gamma\text{-}P_SCl(A) = \bigcap\{F : A \subseteq F,\ X\backslash F \text{ is } \gamma\text{-}P_S\text{-open set in } X\}.$$

Some important properties of τ_γ-P_S-closure of a set will be given in Lemma 4.4.10, Theorem 4.4.11 and Theorem 4.4.13.

Lemma 4.4.10. The following statements are true for any subset A of a topological space (X, τ) with an operation γ on τ.

1. τ_γ-$P_SCl(\phi) = \phi$ and τ_γ-$P_SCl(X) = X$.

2. $A \subseteq \tau_\gamma$-$P_SCl(A)$.

3. τ_γ-$P_SCl(A)$ is the smallest γ-P_S-closed set containing A.

4. τ_γ-$P_SCl(A)$ is γ-P_S-closed set in X.

5. A is γ-P_S-closed if and only if τ_γ-$P_SCl(A) = A$.

6. τ_γ-$P_SCl(\tau_\gamma$-$P_SCl(A)) = \tau_\gamma$-$P_SCl(A)$.

Proof. Straightforward. □

Theorem 4.4.11. Let A and B be subsets of a topological space (X, τ) and γ be an operation on τ. Then the following conditions hold:

1. If τ_γ-$P_SCl(A) \cap \tau_\gamma$-$P_SCl(B) = \phi$, then $A \cap B = \phi$.

2. If $A \subseteq B$, then τ_γ-$P_SCl(A) \subseteq \tau_\gamma$-$P_SCl(B)$.

3. τ_γ-$P_SCl(A \cap B) \subseteq \tau_\gamma$-$P_SCl(A) \cap \tau_\gamma$-$P_SCl(B)$.

4. τ_γ-$P_SCl(A) \cup \tau_\gamma$-$P_SCl(B) \subseteq \tau_\gamma$-$P_SCl(A \cup B)$.

Proof. The proof of parts (1) and (2) follow directly from Lemma 4.4.10 (2) and Definition 4.4.9, respectively, and the proof of both (3) and (4) follow from part (2). □

The inclusions in (3) and (4) in Theorem 4.4.11 cannot be replaced by equality in general, as it can be seen from the following example.

Example 4.4.12. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Let $A = \{a\}$, $B = \{b\}$ and $C = \{c\}$. Then τ_γ-$P_SCl(B \cap C) = \tau_\gamma$-$P_SCl(\phi) = \phi$, but τ_γ-$P_SCl(B) \cap \tau_\gamma$-$P_SCl(C) = B \cap \{b, c\} = B$. Therefore, τ_γ-$P_SCl(B \cap C) \neq \tau_\gamma$-$P_SCl(B) \cap \tau_\gamma$-$P_SCl(C)$.

Also τ_γ-$P_SCl(A \cup B) = \tau_\gamma$-$P_SCl(\{a\} \cup \{b\}) = \tau_\gamma$-$P_SCl\{a, b\} = X$, but τ_γ-$P_SCl(A) \cup \tau_\gamma$-$P_SCl(B) = \tau_\gamma$-$P_SCl\{a\} \cup \tau_\gamma$-$P_SCl\{a\} = \{a\} \cup \{b\} = \{a, b\}$. Therefore, τ_γ-$P_SCl(A \cup B) \neq \tau_\gamma$-$P_SCl(A) \cup \tau_\gamma$-$P_SCl(B)$.

Theorem 4.4.13. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then $x \in \tau_\gamma$-$P_SCl(A)$ if and only if $A \cap U \neq \phi$ for every γ-P_S-open set U of X containing x.

Proof. Let $x \in \tau_\gamma$-$P_SCl(A)$ and let $A \cap U = \phi$ for some γ-P_S-open set U of X containing x. Then $A \subseteq X \backslash U$ and $X \backslash U$ is γ-P_S-closed set in X. So τ_γ-$P_SCl(A) \subseteq X \backslash U$. Thus, $x \in X \backslash U$. This is a contrtadiction. Hence $A \cap U \neq \phi$ for every γ-P_S-open set U of X containing x.

Conversely, suppose that $x \notin \tau_\gamma$-$P_SCl(A)$. So there exists a γ-P_S-closed set F such that $A \subseteq F$ and $x \notin F$. Then $X \backslash F$ is a γ-P_S-open set such that $x \in X \backslash F$ and $A \cap (X \backslash F) = \phi$. Contradiction of hypothesis. Therefore, $x \in \tau_\gamma$-$P_SCl(A)$. □

Definition $4.2.1$ and Theorem $4.4.13$ lead to the following corollary.

Corollary 4.4.14. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. If $A \cap F \neq \phi$ for every γ-semiclosed set F of X such that $x \in F$. Then $x \in \tau_\gamma$-$P_SCl(A)$.

Proof. Let U be any γ-P_S-open set of X such that $x \in U$. Then by Definition $4.2.1$, there exists a γ-semiclosed set F of X containing x such that $F \subseteq U$. By hypothesis, we have $A \cap F \neq \phi$ and hence $A \cap U \neq \phi$. Therefore, by Theorem $4.4.13$, $x \in \tau_\gamma$-$P_SCl(A)$. □

From Definition $4.4.1$ and Theorem $4.4.13$, we have the following corollary.

Corollary 4.4.15. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then τ_γ-$P_SD(A) \subseteq \tau_\gamma$-$P_SCl(A)$.

Proof. Obvious. □

Theorem 4.4.16. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then A is γ-P_S-closed if and only if A contains the set of its γ-P_S-limit points.

Proof. Suppose that A is γ-P_S-closed subset of a space (X, τ) and let $x \notin A$, then $x \in X \backslash A$ and $X \backslash A$ is γ-P_S-open set in X such that $A \cap X \backslash A = \phi$. This means that $x \notin \tau_\gamma$-$P_SD(A)$. Hence τ_γ-$P_SD(A) \subseteq A$.

Conversely, assume that A contains the set of its γ-P_S-limit points (That is, τ_γ-$P_SD(A) \subseteq A$). To show that A is γ-P_S-closed (or $X \backslash A$ is γ-P_S-open) set in X. Let $x \in X \backslash A$, then $x \notin A$. By assumption that there exists a γ-P_S-open set U_x of X containing x such that $A \cap U_x = \phi$. That is, $U_x \subseteq X \backslash A$ and hence $X \backslash A = \bigcup_{x \in X \backslash A} U_x$.

So $X \backslash A$ is a union of γ-P_S-open sets and by Theorem 4.2.4, $X \backslash A$ is γ-P_S-open. Consequently, A is γ-P_S-closed set in X. □

From Theorem 4.4.16, we have the next corollary.

Corollary 4.4.17. Let A and F be subsets of a topological space (X, τ) and γ be an operation on τ. If A is γ-P_S-closed such that $A \subseteq F$, then τ_γ-$P_S D(A) \subseteq F$. □

The following theorem shows that the set A together with all of its γ-P_S-limit points is γ-P_S-closed set.

Theorem 4.4.18. For any subset A of a topological space (X, τ) with an operation γ on τ. $A \cup \tau_\gamma$-$P_S D(A)$ is a γ-P_S-closed set in X.

Proof. Let $x \notin A \cup \tau_\gamma$-$P_S D(A)$. Then $x \notin A$ and $x \notin \tau_\gamma$-$P_S D(A)$. Since $x \notin \tau_\gamma$-$P_S D(A)$, there exists a γ-P_S-open set $G_x \subseteq X$ of x which contains no point of A other than x but $x \notin A$. So G_x contains no point of A, which implies $G_x \subseteq X \backslash A$. Again, G_x is a γ-P_S-open set of each of its points. But as G_x does not contain any point of A, no point of G_x can be a γ-P_S-limit point of A. Therefore, no point of G_x can belong to τ_γ-$P_S D(A)$. This implies that $G_x \subseteq X \backslash \tau_\gamma$-$P_S D(A)$. Hence, it follows that $x \in G_x \subseteq X \backslash A \cap X \backslash \tau_\gamma$-$P_S D(A) = X \backslash (A \cup \tau_\gamma$-$P_S D(A))$. Therefore, $A \cup \tau_\gamma$-$P_S D(A)$ is γ-P_S-closed. □

γ-P_S-limit points provide us with an easy means to find the τ_γ-P_S-closure of a set A.

Theorem 4.4.19. Let (X, τ) be a topological space and γ be an operation on τ. Let A be a subset of a space X. Then τ_γ-$P_S Cl(A) = A \cup \tau_\gamma$-$P_S D(A)$.

Proof. Since $A \subseteq A \cup \tau_\gamma\text{-}P_S D(A)$. Then by Theorem 4.4.18,

$$\tau_\gamma\text{-}P_S Cl(A) \subseteq A \cup \tau_\gamma\text{-}P_S D(A).$$

On the other hand, since $A \subseteq \tau_\gamma\text{-}P_S Cl(A)$ in general and by Corollary 4.4.15, $\tau_\gamma\text{-}P_S D(A) \subseteq \tau_\gamma\text{-}P_S Cl(A)$. So we have $A \cup \tau_\gamma\text{-}P_S D(A) \subseteq \tau_\gamma\text{-}P_S Cl(A)$. Therefore, in both cases, we obtain that $\tau_\gamma\text{-}P_S Cl(A) = A \cup \tau_\gamma\text{-}P_S D(A)$. □

Since every γ-P_S-closed set is γ-preclosed, then τ_γ-preclosure of a set is τ_γ-P_S-closure of that set and hence by using Remark 2.3.13 (3), we have the following remark.

Remark 4.4.20. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then $A \subseteq \tau_\gamma\text{-}\beta Cl(A) \subseteq \tau_\gamma\text{-}bCl(A) \subseteq \tau_\gamma\text{-}pCl(A) \subseteq \tau_\gamma\text{-}P_S Cl(A)$.

From Remark 4.4.20 and since $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(A)) \subseteq \tau_\gamma\text{-}pCl(A)$, we have the following remark.

Remark 4.4.21. Let $A \subseteq (X, \tau)$, $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(A)) \subseteq \tau_\gamma\text{-}P_S Cl(A)$.

Definition 4.4.22. A subset N of a topological space (X, τ) is called a γ-P_S-neighbourhood of a point $x \in X$, if there exists a γ-P_S-open set U in X such that $x \in U \subseteq N$.

The following remark follows directly from Definition 4.2.1 and Definition 4.4.22.

Remark 4.4.23. Let A be a subset of a topological space (X, τ). If A is γ-P_S-neighbourhood of a point $x \in X$, then there exists a γ-semicloed set F in X such that $x \in F \subseteq A$.

The following is a relation between γ-P_S-open set and γ-P_S-neighbourhood of a point $x \in X$.

Theorem 4.4.24. Let $U \subseteq (X, \tau)$ is γ-P_S-open if and only if it is a γ-P_S-neighbourhood of each of its points.

Proof. Let U be any γ-P_S-open set in (X, τ). Then by Definition 4.4.22, it is clear that U is a γ-P_S-neighbourhood of each of its points, since for every $x \in U$, $x \in U \subseteq U$ and $U \in \tau_\gamma$-$P_SO(X)$.

Conversely, suppose U is a γ-P_S-neighbourhood of each of its points. Then for each $x \in U$, there exists a γ-P_S-open set V_x containing x such that $V_x \subseteq U$. Then $U = \bigcup_{x \in U} V_x$. Since each V_x is γ-P_S-open. It follows from Theorem 4.2.4 that U is γ-P_S-open set in X. □

Remark 4.4.25. Let A and B be subsets of a topological space (X, τ). If A is γ-P_S-neighbourhood of every element in X and $A \subseteq B$, then B is also γ-P_S-neighbourhood of that element.

Remark 4.4.26. Since every γ-P_S-open set is γ-preopen, then every γ-P_S-neighbourhood of a point is γ-preneighbourhood of that point.

Now, we will define one of the most important operator on γ-P_S-open set. This set is known as the τ_γ-P_S-interior of A.

Definition 4.4.27. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. The τ_γ-P_S-interior of A is defined as the union of all γ-P_S-open sets of X contained in A and it is denoted by τ_γ-$P_SInt(A)$. That is,

$$\tau_\gamma\text{-}P_SInt(A) = \bigcup\{U : U \text{ is a } \gamma\text{-}P_S\text{-open set in } X \text{ and } U \subseteq A\}.$$

Some important properties of τ_γ-P_S-interior of a set will be given in Lemma 4.4.28 and Theorem 4.4.29.

Lemma 4.4.28. Let A be subset of a topological space (X, τ) and γ be an operation on τ. Then the following statements are true:

1. τ_γ-$P_S Int(\phi) = \phi$ and τ_γ-$P_S Int(X) = X$.
2. τ_γ-$P_S Int(A) \subseteq A$.
3. τ_γ-$P_S Int(A)$ is the largest γ-P_S-open set contained in A.
4. τ_γ-$P_S Int(A)$ is γ-P_S-open set in X.
5. A is γ-P_S-open if and only if τ_γ-$P_S Int(A) = A$.
6. τ_γ-$P_S Int(\tau_\gamma$-$P_S Int(A)) = \tau_\gamma$-$P_S Int(A)$.

Proof. Straightforward. □

Theorem 4.4.29. The following properties hold for any subsets A and B of a topological space (X, τ) with an operation γ on τ.

1. If $A \cap B = \phi$, then τ_γ-$P_S Int(A) \cap \tau_\gamma$-$P_S Int(B) = \phi$.
2. If $A \subseteq B$, then τ_γ-$P_S Int(A) \subseteq \tau_\gamma$-$P_S Int(B)$.
3. τ_γ-$P_S Int(A \cap B) \subseteq \tau_\gamma$-$P_S Int(A) \cap \tau_\gamma$-$P_S Int(B)$.
4. τ_γ-$P_S Int(A) \cup \tau_\gamma$-$P_S Int(B) \subseteq \tau_\gamma$-$P_S Int(A \cup B)$.

Proof. The proof of parts (1) and (2) follow directly from Lemma 4.4.28 (2) and Definition 4.4.27, respectively, and the proof of both (3) and (4) follow from part (2). $\square$

The inclusions in (3) and (4) in Theorem 4.4.29 cannot be replaced by equality in general, as it is shown in the following example.

Example 4.4.30. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Let $A = \{b\}$ and $B = \{c\}$. Then τ_γ-$P_SInt(A) \cup \tau_\gamma$-$P_SInt(B) = \phi \cup \phi = \phi$, but τ_γ-$P_SInt(A \cup B) = \tau_\gamma$-$P_SInt\{b, c\} = \{b, c\}$. Hence τ_γ-$P_SInt(A) \cup \tau_\gamma$-$P_SInt(B) \neq \tau_\gamma$-$P_SInt(A \cup B)$.

Again, if we take $C = \{a, c\}$ and $D = \{b, c\}$. Then τ_γ-$P_SInt(C \cap D) = \tau_\gamma$-$P_SInt\{c\} = \phi$, but τ_γ-$P_SInt(C) \cap \tau_\gamma$-$P_SInt(D) = C \cap D = \{c\}$. So τ_γ-$P_SInt(C \cap D) \neq \tau_\gamma$-$P_SInt(C) \cap \tau_\gamma$-$P_SInt(D)$.

Since every γ-P_S-open set is γ-preopen, then every τ_γ-P_S-interior of a set is τ_γ-preinterior of that set and hence by using Remark 2.3.13 (1), we have the following remark.

Remark 4.4.31. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then τ_γ-$P_SInt(A) \subseteq \tau_\gamma$-$pInt(A) \subseteq \tau_\gamma$-$bInt(A) \subseteq \tau_\gamma$-$\beta Int(A) \subseteq A$.

γ-P_S-limit points provide us to find the τ_γ-P_S-interior of a set A.

Theorem 4.4.32. Let (X, τ) be a topological space and γ be an operation on τ. Let A be a subset of a space X. Then τ_γ-$P_SInt(A) = A \backslash \tau_\gamma$-$P_SD(X \backslash A)$.

Proof. Let $x \in A \backslash \tau_\gamma$-$P_SD(X \backslash A)$, then $x \notin \tau_\gamma$-$P_SD(X \backslash A)$ and hence there exists a γ-P_S-open set U_x containing x such that $U_x \cap (X \backslash A) = \phi$. That is, $x \in U_x \subseteq A$ and hence $A = \bigcup_{x \in A} U_x$. So A is a union of γ-P_S-open sets and hence by Theorem 4.2.4, A is γ-P_S-open set in X containing x. Then by Lemma 4.4.28 (5), $x \in \tau_\gamma$-$P_SInt(A)$. Thus, $A \backslash \tau_\gamma$-$P_SD(X \backslash A) \subseteq \tau_\gamma$-$P_SInt(A)$.

On the other hand, if $x \in \tau_\gamma$-$P_SInt(A) \subseteq A$, then $x \notin \tau_\gamma$-$P_SD(X \backslash A)$ since τ_γ-$P_SInt(A)$ is γ-P_S-open set and τ_γ-$P_SInt(A) \cap (X \backslash A) = \phi$. So $x \in A \backslash \tau_\gamma$-$P_SD(X \backslash A)$. This implies that τ_γ-$P_SInt(A) \subseteq A \backslash \tau_\gamma$-$P_SD(X \backslash A)$. Therefore, in both cases, we obtain that τ_γ-$P_SInt(A) = A \backslash \tau_\gamma$-$P_SD(X \backslash A)$. □

Some relations between τ_γ-P_S-interior and τ_γ-P_S-closure of any set are shown in the Theorem 4.4.33 and Corollary 4.4.34.

Theorem 4.4.33. Let (X, τ) be a topological space and γ be an operation on τ. For any subset A of a space X. The following statements are true:

1. τ_γ-$P_SInt(X \backslash A) = X \backslash \tau_\gamma$-$P_SCl(A)$.

2. τ_γ-$P_SCl(X \backslash A) = X \backslash \tau_\gamma$-$P_SInt(A)$.

3. τ_γ-$P_SInt(A) = X \backslash \tau_\gamma$-$P_SCl(X \backslash A)$.

4. τ_γ-$P_SCl(A) = X \backslash \tau_\gamma$-$P_SInt(X \backslash A)$.

Proof. (1) By using Theorem 4.4.19 and Theorem 4.4.32, we obtain

$$\tau_\gamma\text{-}P_SInt(X \backslash A) = (X \backslash A) \backslash \tau_\gamma\text{-}P_SD(A)$$
$$= X \backslash A \cap X \backslash \tau_\gamma\text{-}P_SD(A) = X \backslash (A \cup \tau_\gamma\text{-}P_SD(A)) = X \backslash \tau_\gamma\text{-}P_SCl(A).$$

The proof of parts (2), (3) and (4) are similar to the part (1) and hence it is omitted. □

Corollary 4.4.34. Let A and B be subsets of a topological space (X, τ) and γ be an operation on τ. If $A \cap B = \phi$, then τ_γ-$P_SInt(A) \cap \tau_\gamma$-$P_SCl(B) = \phi$ and τ_γ-$P_SCl(A) \cap \tau_\gamma$-$P_SInt(B) = \phi$.

Proof. Directly follows from Theorem 4.4.33. □

Remark 4.4.35. For any subset $A \subseteq (X, \tau)$ and γ be a regular operation on τ. Then:

1. If $A \in \tau_\gamma$-$P_SO(X)$, then τ_γ-$pInt(A) \in \tau_\gamma$-$P_SO(X)$ and τ_γ-$sCl(A) \in \tau_\gamma$-$P_SO(X)$.
2. If $A \in \tau_\gamma$-$P_SC(X)$, then τ_γ-$pCl(A) \in \tau_\gamma$-$P_SC(X)$ and τ_γ-$sInt(A) \in \tau_\gamma$-$P_SC(X)$.

The following remark is straightforward from Remark 4.4.20 and Remark 4.4.31.

Remark 4.4.36. The following conditions are true for any subset A of a topological space (X, τ) with an operation γ on τ.

1. τ_γ-$P_SInt(\tau_\gamma$-$pInt(A)) = \tau_\gamma$-$pInt(\tau_\gamma$-$P_SInt(A)) = \tau_\gamma$-$P_SInt(A)$.
2. τ_γ-$P_SInt(\tau_\gamma$-$bInt(A)) = \tau_\gamma$-$bInt(\tau_\gamma$-$P_SInt(A)) = \tau_\gamma$-$P_SInt(A)$.
3. τ_γ-$P_SInt(\tau_\gamma$-$\beta Int(A)) = \tau_\gamma$-$\beta Int(\tau_\gamma$-$P_SInt(A)) = \tau_\gamma$-$P_SInt(A)$.
4. τ_γ-$P_SCl(\tau_\gamma$-$pCl(A)) = \tau_\gamma$-$pCl(\tau_\gamma$-$P_SCl(A)) = \tau_\gamma$-$P_SCl(A)$.
5. τ_γ-$P_SCl(\tau_\gamma$-$bCl(A)) = \tau_\gamma$-$bCl(\tau_\gamma$-$P_SCl(A)) = \tau_\gamma$-$P_SCl(A)$.
6. τ_γ-$P_SCl(\tau_\gamma$-$\beta Cl(A)) = \tau_\gamma$-$\beta Cl(\tau_\gamma$-$P_SCl(A)) = \tau_\gamma$-$P_SCl(A)$.

Next, other sets such as γ-γ-P_S-open, τ-γ-P_S-open and γ-P_S-boundary can be defined using the notions of τ_γ-P_S-interior or τ_γ-P_S-closure. Then the relations among them will be obtained.

Definition 4.4.37. A subset A of a topological space (X, τ) with an operation γ on τ is called γ-γ-P_S-open (respectively, τ-γ-P_S-open) if τ_γ-$Int(A) = \tau_\gamma$-$P_S Int(A)$ (respectively, $Int(A) = \tau_\gamma$-$P_S Int(A)$).

The relation between the sets in Definition 4.4.37, γ-open set and γ-P_S-open set are shown in the next two theorems.

Theorem 4.4.38. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following statements are equivalent:

1. A is γ-γ-P_S-open and γ-P_S-open.

2. A is γ-γ-P_S-open and γ-open.

3. A is γ-P_S-open and γ-open.

Proof. Follows from Definition 4.4.37, Remark 2.3.6 (1) and Lemma 4.4.28 (5). □

Theorem 4.4.39. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following statements are equivalent:

1. A is τ-γ-P_S-open and γ-P_S-open.

2. A is τ-γ-P_S-open and open.

3. A is γ-P_S-open and open.

Proof. Follows from Definition 4.4.37 and Lemma 4.4.28 (5). □

Theorem 4.4.40. In a γ-regular space (X, τ), then the concept of γ-γ-P_S-open set and τ-γ-P_S-open set are equivalent.

Proof. The proof follows from Definition 4.4.37 and Remark 2.3.31. □

Definition 4.4.41. Let A be any subset of a topological space (X,τ) and γ be an operation on τ. Then the γ-P_S-boundary of A is defined as τ_γ-$P_SCl(A)\backslash\tau_\gamma$-$P_SInt(A)$ and it is denoted by τ_γ-$P_SBd(A)$.

Theorem 4.4.42. Let A be a subset of a topological space (X,τ) and γ be an operation on τ. Then the following conditions are true:

1. τ_γ-$P_SBd(A) = \tau_\gamma$-$P_SCl(A) \cap \tau_\gamma$-$P_SCl(X\backslash A)$
2. τ_γ-$P_SBd(A) = \tau_\gamma$-$P_SBd(X\backslash A)$
3. τ_γ-$P_SBd(A) \subseteq \tau_\gamma$-$P_SCl(A)$
4. τ_γ-$P_SCl(A) = \tau_\gamma$-$P_SInt(A) \cup \tau_\gamma$-$P_SBd(A)$.
5. τ_γ-$P_SInt(A) \cap \tau_\gamma$-$P_SBd(A) = \phi$.
6. τ_γ-$P_SBd(\tau_\gamma$-$P_SInt(A)) \subseteq \tau_\gamma$-$P_SBd(A)$.
7. τ_γ-$P_SBd(\tau_\gamma$-$P_SCl(A)) \subseteq \tau_\gamma$-$P_SBd(A)$.
8. τ_γ-$P_SBd(\tau_\gamma$-$P_SBd(A)) \subseteq \tau_\gamma$-$P_SBd(A)$.
9. τ_γ-$P_SInt(A) = A\backslash\tau_\gamma$-$P_SBd(A)$.

Proof. Straightforward. □

Theorem 4.4.43. Let A be a subset of a topological space (X,τ) and γ be an operation on τ. Then the following statements are true:

1. τ_γ-$P_SBd(A)$ is γ-P_S-closed.

2. A is γ-P_S-open if and only if $A \cap \tau_\gamma$-$P_SBd(A) = \phi$ (That is, τ_γ-$P_SBd(A) \subseteq (X \backslash A)$).

3. A is γ-P_S-closed if and only if τ_γ-$P_SBd(A) \subseteq A$.

Proof. (1) By using Theorem 4.4.42 (1), Theorem 4.4.11 (3) and Lemma 4.4.10 (5), we obtain that

$$\begin{aligned}\tau_\gamma\text{-}P_SCl(\tau_\gamma\text{-}P_SBd(A)) &= \tau_\gamma\text{-}P_SCl(\tau_\gamma\text{-}P_SCl(A) \cap \tau_\gamma\text{-}P_SCl(X\backslash A))\\ &\subseteq \tau_\gamma\text{-}P_SCl(\tau_\gamma\text{-}P_SCl(A)) \cap \tau_\gamma\text{-}P_SCl(\tau_\gamma\text{-}P_SCl(X\backslash A))\\ &\subseteq \tau_\gamma\text{-}P_SCl(A) \cap \tau_\gamma\text{-}P_SCl(X\backslash A)\\ &= \tau_\gamma\text{-}P_SBd(A)\end{aligned}$$

Therefore, τ_γ-$P_SBd(A)$ is γ-P_S-closed set in (X, τ).

(2) Assume that A is γ-P_S-open set in X. Then τ_γ-$P_SBd(A) = \tau_\gamma$-$P_SCl(A)\backslash\tau_\gamma$-$P_SInt(A) = \tau_\gamma$-$P_SCl(A)\backslash A$ and hence $A \cap \tau_\gamma$-$P_SBd(A)$ $= A \cap \tau_\gamma$-$P_SCl(A)\backslash A = \phi$. This implies that τ_γ-$P_SBd(A) \subseteq (X\backslash A)$.

Conversely, let $A \cap \tau_\gamma$-$P_SBd(A) = \phi$. Then by Theorem 4.4.42 (1), $A \cap \tau_\gamma$-$P_SCl(A) \cap \tau_\gamma$-$P_SCl(X\backslash A) = \phi$ implies that $A \cap \tau_\gamma$-$P_SCl(X\backslash A) = \phi$ and hence τ_γ-$P_SCl(X\backslash A) \subseteq X\backslash A$. But always $X\backslash A \subseteq \tau_\gamma$-$P_SCl(X\backslash A)$. It follows that τ_γ-$P_SCl(X\backslash A) = X\backslash A$. Therefore, $X\backslash A$ is γ-P_S-closed set in X and hence A is γ-P_S-open.

(3) Let A be any γ-P_S-closed set in X. Then

$$\tau_\gamma\text{-}P_SBd(A) = \tau_\gamma\text{-}P_SCl(A)\backslash\tau_\gamma\text{-}P_SInt(A) = A\backslash\tau_\gamma\text{-}P_SInt(A) \subseteq A$$

Conversely, assume that τ_γ-$P_SBd(A) \subseteq A$. Then τ_γ-$P_SBd(A) \cap X\backslash A = \phi$ and by Theorem 4.4.42 (1), we get τ_γ-$P_SCl(A) \cap \tau_\gamma$-$P_SCl(X\backslash A) \cap X\backslash A = \phi$ implies that τ_γ-$P_SCl(A) \cap X\backslash A = \phi$. Hence τ_γ-$P_SCl(A) \subseteq A$. Since $A \subseteq \tau_\gamma$-$P_SCl(A)$ in general. So $A = \tau_\gamma$-$P_SCl(A)$. This means that A is γ-P_S-closed set in X. □

Theorem 4.4.44. For any set $A \subseteq (X, \tau)$, we have:

1. τ_γ-$P_SInt(\tau_\gamma$-$P_SCl(A)\backslash A) = \phi$.
2. τ_γ-$P_SInt(A\backslash\tau_\gamma$-$P_SInt(A)) = \phi$.
3. τ_γ-$P_SCl(A\backslash\tau_\gamma$-$P_SInt(A)) \subseteq \tau_\gamma$-$P_SBd(A)$.
4. τ_γ-$P_SInt(\tau_\gamma$-$P_SCl(A)\backslash A) \subseteq \tau_\gamma$-$P_SBd(A)$.

Proof. Obvious. □

Theorem 4.4.45. The following properties hold for any subsets A and B of a topological space (X, τ) with an operation γ on τ.

1. If τ_γ-$P_SCl(A) \cap \tau_\gamma$-$P_SCl(B) = \phi$, then τ_γ-$P_SBd(A) \cap \tau_\gamma$-$P_SBd(B) = \phi$.
2. If $A \cap B = \phi$, then τ_γ-$P_SInt(A) \cap \tau_\gamma$-$P_SBd(B) = \phi$.
3. If $A \cap B = \phi$ and A is γ-P_S-open, then $A \cap \tau_\gamma$-$P_SBd(B) = \phi$.
4. If $A \subseteq B$ and B is γ-P_S-closed, then τ_γ-$P_SBd(A) \subseteq B$.

Proof. (1) The proof follows directly from Theorem 4.4.42 (3).

(2) Let $A \cap B = \phi$, then by Corollary 4.4.34, τ_γ-$P_SInt(A) \cap \tau_\gamma$-$P_SCl(B) = \phi$ and hence by Theorem 4.4.42 (3), we get τ_γ-$P_SInt(A) \cap \tau_\gamma$-$P_SBd(B) = \phi$.

(3) Follows directly from the part (2).

(4) Let $A \subseteq B$ and B is γ-P_S-closed set in (X, τ), then $A \cap X \backslash B = \phi$ and $X \backslash B$ is γ-P_S-open set in X. So by the part (3), we have τ_γ-$P_SBd(A) \cap X \backslash B = \phi$. This implies that τ_γ-$P_SBd(A) \subseteq B$. □

From Theorem 4.4.11 (4), Theorem 4.4.42 (1) and Theorem 4.4.43 (1), we have the following corollary.

Corollary 4.4.46. For any subset A of a topological space (X, τ) and γ be an operation on τ. Then τ_γ-$P_SBd(\tau_\gamma$-$P_SBd(\tau_\gamma$-$P_SBd(A))) = \tau_\gamma$-$P_SBd(\tau_\gamma$-$P_SBd(A))$. □

Remark 4.4.47. Let $A \subseteq X$, we have τ_γ-$P_SBd(A) = \phi$ if and only if A is both γ-P_S-open and γ-P_S-closed.

For any subsets A and B of a space (X, τ). Then $A \subseteq B$ does not imply that either τ_γ-$P_SBd(A) \subseteq \tau_\gamma$-$P_SBd(B)$ or τ_γ-$P_SBd(B) \subseteq \tau_\gamma$-$P_SBd(A)$, as it can be seen from the following example.

Example 4.4.48. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Let $A = \{b\}$ and $B = \{b, c\}$. Then τ_γ-$P_SBd(A) = A$ and τ_γ-$P_SBd(B) = \phi$. This shows that τ_γ-$P_SBd(A) \nsubseteq \tau_\gamma$-$P_SBd(B)$. Also if we take $A = \{b\}$ and $B = \{a, b\}$. Then τ_γ-$P_SBd(A) = A$ and τ_γ-$P_SBd(B) = \{b, c\}$. This shows that τ_γ-$P_SBd(B) \nsubseteq \tau_\gamma$-$P_SBd(A)$.

By using Remark $4.4.20$ and Remark $4.4.31$, we have the following remark.

Remark 4.4.49. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then τ_γ-$pBd(A) \subseteq \tau_\gamma$-$P_SBd(A)$.

The following example shows that the converse of the above remark is not true in general.

Example 4.4.50. Considering the space (X, τ) as defined in Example $4.2.3$. Let $A = \{a, b\}$. Then τ_γ-$P_SBd(A) = \{b, c\}$ and τ_γ-$pBd(A) = \{c\}$. Hence τ_γ-$P_SBd(A) \nsubseteq \tau_\gamma$-$pBd(A)$.

In the next example, γ-P_S-boundary of a set and γ-boundary of that set are independent each other.

Example 4.4.51. Considering the space (X, τ) given in Example $4.2.7$. If we take $A = \{a, c, d\}$. Then τ_γ-$P_SBd(A) = \{b, c, d\}$ and τ_γ-$Bd(A) = \{b\}$. Hence τ_γ-$P_SBd(A) \nsubseteq \tau_\gamma$-$Bd(A)$. Again let $A = \{b, d\}$. Then τ_γ-$P_SBd(A) = \phi$ and τ_γ-$Bd(A) = \{b, c, d\}$. Hence τ_γ-$Bd(A) \nsubseteq \tau_\gamma$-$P_SBd(A)$.

Corollary $4.4.52$ relates with τ_γ-$P_SO(X) = \tau_\gamma$ as have been explained in Theorem $4.2.19$.

Corollary 4.4.52. Let (X, τ) be a γ-locally indiscrete space and $A \subseteq X$, then the following are true:

1. τ_γ-$P_SCl(A) = \tau_\gamma$-$Cl(A)$.

2. τ_γ-$P_SInt(A) = \tau_\gamma$-$Int(A)$.

3. τ_γ-$P_SD(A) = \tau_\gamma$-$D(A)$.

4. τ_γ-$P_SBd(A) = \tau_\gamma$-$Bd(A)$. □

Corollary 4.4.53. If a space (X, τ) is γ-locally indiscrete, then

$$\tau_\gamma\text{-}P_SCl(A) = \tau_\gamma\text{-}P_SInt(A) \text{ for any subset } A \text{ of } X$$

Proof. Follows directly from Theorem 4.3.10 and Corollary 4.3.11. □

Similarly, for Corollary 4.4.54 will relate with τ_γ-$P_SO(X) = \tau_\gamma$-$PO(X)$ as we explained in Theorem 4.2.24.

Corollary 4.4.54. Let (X, τ) be a γ-semiT_1 space and $A \subseteq X$, then the following are true:

1. τ_γ-$P_SCl(A) = \tau_\gamma$-$pCl(A)$.

2. τ_γ-$P_SInt(A) = \tau_\gamma$-$pInt(A)$.

3. τ_γ-$P_SD(A) = \tau_\gamma$-$pD(A)$.

4. τ_γ-$P_SBd(A) = \tau_\gamma$-$pBd(A)$. □

Theorem 4.4.55. For any subset A in (X, τ). If A is γ-semiopen, then τ_γ-$P_SCl(A) = \tau_\gamma$-$pCl(A)$.

Proof. Let $x \notin \tau_\gamma$-$pCl(A)$, then by Theorem 2.3.15 (2), there exists a γ-preopen set U containing x such that $A \cap U = \phi$. By Lemma 3.3.2, τ_γ-$Int(A) \cap \tau_\gamma$-$Cl(U) = \phi$ implies that τ_γ-$Cl(\tau_\gamma$-$Int(A)) \cap \tau_\gamma$-$Int(\tau_\gamma$-$Cl(U)) = \phi$. Since A is γ-semiopen, then $A \cap \tau_\gamma$-$Int(\tau_\gamma$-$Cl(U)) = \phi$. Since U is γ-preopen set containing x, then $x \in$

τ_γ-$Int(\tau_\gamma$-$Cl(U))$ and τ_γ-$Int(\tau_\gamma$-$Cl(U))$ is γ-P_S-open set. So by Theorem 4.4.13, $x \notin \tau_\gamma$-$P_SCl(A)$. Hence τ_γ-$P_SCl(A) \subseteq \tau_\gamma$-$pCl(A)$. By Remark 4.4.20, τ_γ-$pCl(A) \subseteq \tau_\gamma$-$P_SCl(A)$. Then τ_γ-$P_SCl(A) = \tau_\gamma$-$pCl(A)$. □

Similar to Theorem 4.4.55, we can show that τ_γ-$pCl(A) = \tau_\gamma$-$Cl(A) = \tau_{\alpha-\gamma}$-$Cl(A)$ for every γ-semiopen set A in (X, τ) in the next corollary.

Corollary 4.4.56. For each γ-semiopen A in (X, τ), we have

$$\tau_\gamma\text{-}P_SCl(A) = \tau_\gamma\text{-}pCl(A) = \tau_\gamma\text{-}Cl(A) = \tau_{\alpha-\gamma}\text{-}Cl(A). \qquad \square$$

The next corollary follows directly from Corollary 4.4.56 and using complements.

Corollary 4.4.57. For each γ-semiclosed A in (X, τ), we have

$$\tau_\gamma\text{-}P_SInt(A) = \tau_\gamma\text{-}pInt(A) = \tau_\gamma\text{-}Int(A) = \tau_{\alpha-\gamma}\text{-}Int(A). \qquad \square$$

Theorem 4.4.58. For any subset A in (X, τ). If A is γ-β-open, then τ_γ-$Cl(A) = \tau_{\alpha-\gamma}$-$Cl(A)$.

Proof. Let $x \notin \tau_{\alpha-\gamma}$-$Cl(A)$, then by Theorem 2.3.15 (4), there exists an α-γ-open set U containing x such that $A \cap U = \phi$. By Lemma 3.3.2, we have τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(A))) \cap \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(U))) = \phi$. Since A is γ-β-open, then $A \cap \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(U)) = \phi$. Since U is α-γ-open set containing x, then $x \in \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(U))$ and τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(U))$ is γ-open set. So by Theorem 2.3.15 (1), $x \notin \tau_\gamma$-$Cl(A)$. Hence τ_γ-$Cl(A) \subseteq \tau_{\alpha-\gamma}$-$Cl(A)$. But $\tau_{\alpha-\gamma}$-$Cl(A) \subseteq \tau_\gamma$-$Cl(A)$. Thus, τ_γ-$Cl(A) = \tau_{\alpha-\gamma}$-$Cl(A)$. □

Since every γ-P_S-open set is γ-β-open, then by using Theorem 4.4.58, we have the following corollary.

Corollary 4.4.59. For each $A \subseteq X$. If A is γ-P_S-open, then τ_γ-$Cl(A) = \tau_{\alpha-\gamma}$-$Cl(A)$. $\square$

Theorem 4.4.60. For each $A \subseteq X$. If A is γ-preopen, then τ_γ-$P_SCl(A) \subseteq \tau_\gamma$-$Cl(A)$.

Proof. Suppose that A is γ-preopen, then $A \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(A))$ and hence τ_γ-$P_SCl(A) \subseteq \tau_\gamma$-$P_SCl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(A)))$. Since τ_γ-$Int(\tau_\gamma$-$Cl(A))$ is γ-open set in X and hence it is γ-semiopen. So by Corollary 4.4.56, τ_γ-$P_SCl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(A))) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(A)))$. Since A is γ-preopen set, then A is γ-β-open. So by Theorem 3.2.10, we get τ_γ-$P_SCl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(A))) = \tau_\gamma$-$Cl(A)$. Thus, τ_γ-$P_SCl(A) \subseteq \tau_\gamma$-$Cl(A)$. $\square$

The proof of the following corollary follows directly from Theorem 4.4.58 and Theorem 4.4.60.

Corollary 4.4.61. For each $A \subseteq X$. If A is γ-preopen, then τ_γ-$P_SCl(A) \subseteq \tau_{\alpha-\gamma}$-$Cl(A)$.

$\square$

Since every γ-P_S-open set is γ-preopen, then by using Theorem 4.4.60 and Corollary 4.4.61, we have the following corollary.

Corollary 4.4.62. If A is γ-P_S-open subset of X, then τ_γ-$P_SCl(A) \subseteq \tau_\gamma$-$Cl(A)$ and τ_γ-$P_SCl(A) \subseteq \tau_{\alpha-\gamma}$-$Cl(A)$. $\square$

Theorem 4.4.63. For any subset A of a space X. Then A is γ-semiopen if and only if τ_γ-$P_SCl(A) = \tau_\gamma$-$P_SCl(\tau_\gamma$-$Int(A))$.

Proof. Let A be any γ-semiopen subset of X, then by Theorem 3.2.11, τ_γ-$Cl(A) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(A))$. Since τ_γ-$Int(A) \in \tau_\gamma$-$SO(X)$. Then by Corollary 4.4.56, we have τ_γ-$P_SCl(A) = \tau_\gamma$-$P_SCl(\tau_\gamma$-$Int(A))$.

Conversely, let τ_γ-$P_SCl(A) = \tau_\gamma$-$P_SCl(\tau_\gamma$-$Int(A))$. Since $A \subseteq \tau_\gamma$-$P_SCl(A)$, then $A \subseteq \tau_\gamma$-$P_SCl(\tau_\gamma$-$Int(A))$. Since τ_γ-$Int(A)$ is γ-semiopen set. Then by Corollary 4.4.56, we have τ_γ-$P_SCl(\tau_\gamma$-$Int(A)) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(A))$. Hence $A \subseteq \tau_\gamma$-$Cl(\tau_\gamma$-$Int(A))$. Therefore, A is γ-semiopen set in X. □

4.5 γ-P_S-Generalized Closed Sets

In this section, a new class of sets called γ-P_S-generalized closed using γ-P_S-open set and τ_γ-P_S-closure of a set will be defined. Then, some of its basic properties will be studied.

Definition 4.5.1. Let A be any subset of a topological space (X, τ) with an operation γ on τ is called γ-P_S-generalized closed (γ-P_S-g-closed) if τ_γ-$P_SCl(A) \subseteq G$ whenever $A \subseteq G$ and G is a γ-P_S-open set in X.

The class of all γ-P_S-g-closed sets of X is denoted by τ_γ-$P_SGC(X)$.

Theorem 4.5.2. Every γ-P_S-closed set is γ-P_S-g-closed.

Proof. Let A be any γ-P_S-closed set in a space X and $A \subseteq G$ where G is a γ-P_S-open set in X. Then τ_γ-$P_SCl(A) \subseteq G$ since A is γ-P_S-closed set. Therefore, A is γ-P_S-g-closed set. $\square$

The following example shows that the converse of the Theorem 4.5.2 is not true.

Example 4.5.3. Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Define an operation $\gamma\colon \tau \to P(X)$ by $\gamma(A) = A$ for all $A \in \tau$. Then τ_γ-$P_SC(X) = \{\phi, \{a\}, \{b, c\}, X\}$ and τ_γ-$P_SGC(X)$ = all subsets of X. So $\{b\}$ is γ-P_S-g-closed, but it is not γ-P_S-closed.

Remark 4.5.4. The union of any two γ-P_S-g-closed sets may not be γ-P_S-g-closed.

Remark 4.5.5. The intersection of any two γ-P_S-g-closed sets may not be γ-P_S-g-closed.

Reverse implication of the above lemma does not hold as can be shown in the following example.

Example 4.5.6. Let $X = (0, 1)$ and τ be the usual topology on X. Define an operation γ on τ by $\gamma(U) = U$ for all $U \in \tau$. Let A be the set of rational numbers in X except the singleton set $\{\frac{1}{2}\}$ and B be the set of irrational numbers in X. Then A and B are both γ-P_S-g-closed sets, but $A \cup B$ is not γ-P_S-g-closed.

Theorem 4.5.7. In any topological space (X, τ). The intersection of γ-P_S-g-closed set of X and γ-P_S-closed set of X is γ-P_S-g-closed in X.

Proof. Let A be a γ-P_S-g-closed set in X and let F be a γ-P_S-closed set in X. Suppose that G is γ-P_S-open set such that $A \cap F \subseteq G$. Then $A \subseteq G \cup (X \backslash F)$. Since F is γ-P_S-closed set, then $X \backslash F$ is γ-P_S-open set and hence by Theorem 4.2.4, $G \cup X \backslash F$ is

γ-P_S-open. Since A is γ-P_S-g-closed set, then τ_γ-$P_SCl(A) \subseteq G \cup (X \backslash F)$. Thus, by Theorem 4.4.11 (3),

τ_γ-$P_SCl(A \cap F) \subseteq \tau_\gamma$-$P_SCl(A) \cap \tau_\gamma$-$P_SCl(F) = \tau_\gamma$-$P_SCl(A) \cap F \subseteq (G \cup (X \backslash F)) \cap F = (G \cap F) \cup ((X \backslash F) \cap F) = (G \cap F) \cup \phi \subseteq G$. So τ_γ-$P_SCl(A \cap F) \subseteq G$. Therefore, $A \cap F$ is γ-P_S-g-closed set in X. □

Most of the important results about γ-P_S-g-closed set depend on the following theorem.

Theorem 4.5.8. Let A be a subset of topological space (X, τ) and γ be an operation on τ. Then A is γ-P_S-g-closed if and only if τ_γ-$P_SCl(A) \backslash A$ does not contain any nonempty γ-P_S-closed set.

Proof. Let F be a nonempty γ-P_S-closed set in X such that $F \subseteq \tau_\gamma$-$P_SCl(A) \backslash A$. Then $F \subseteq X \backslash A$ implies $A \subseteq X \backslash F$. Since $X \backslash F$ is γ-P_S-open set and A is γ-P_S-g-closed set, then τ_γ-$P_SCl(A) \subseteq X \backslash F$. That is, $F \subseteq X \backslash \tau_\gamma$-$P_SCl(A)$. Hence $F \subseteq X \backslash \tau_\gamma$-$P_SCl(A) \cap \tau_\gamma$-$P_SCl(A) \backslash A \subseteq X \backslash \tau_\gamma$-$P_SCl(A) \cap \tau_\gamma$-$P_SCl(A) = \phi$. This shows that $F = \phi$. This is contradiction. Therefore, $F \nsubseteq \tau_\gamma$-$P_SCl(A) \backslash A$.

Conversely, let $A \subseteq G$ and G is γ-P_S-open set in X. So $X \backslash G$ is γ-P_S-closed set in X. Suppose that τ_γ-$P_SCl(A) \nsubseteq G$, then τ_γ-$P_SCl(A) \cap X \backslash G$ is a nonempty γ-P_S-closed set such that τ_γ-$P_SCl(A) \cap X \backslash G \subseteq \tau_\gamma$-$P_SCl(A) \backslash A$. Contradiction of hypothesis. Hence τ_γ-$P_SCl(A) \subseteq G$ and so A is γ-P_S-g-closed set. □

Corollary 4.5.9. Let A be a γ-P_S-g-closed subset of topological space (X, τ) with an operation γ on τ. Then A is γ-P_S-closed if and only if τ_γ-$P_SCl(A) \backslash A$ is γ-P_S-closed set.

Proof. Let A be a γ-P_S-closed set. Then τ_γ-$P_SCl(A) = A$ and hence τ_γ-$P_SCl(A)\backslash A = \phi$ which is γ-P_S-closed set.

Conversely, suppose τ_γ-$P_SCl(A)\backslash A$ is γ-P_S-closed and A is γ-P_S-g-closed. Then by Theorem 4.5.8, τ_γ-$P_SCl(A)\backslash A$ does not contain any nonempty γ-P_S-closed set and since τ_γ-$P_SCl(A)\backslash A$ is γ-P_S-closed subset of itself, then τ_γ-$P_SCl(A)\backslash A = \phi$ implies τ_γ-$P_SCl(A) \cap X\backslash A = \phi$. This implies that τ_γ-$P_SCl(A) = A$. Therefore, A is γ-P_S-closed in X. □

Theorem 4.5.10. Let (X, τ) be a topological space and γ be an operation on τ. If a subset A of X is γ-P_S-g-closed and γ-P_S-open, then A is γ-P_S-closed.

Proof. Since A is γ-P_S-g-closed and γ-P_S-open set in X, then τ_γ-$P_SCl(A) \subseteq A$ and so A is γ-P_S-closed. □

Theorem 4.5.11. Let (X, τ) be a topological space and γ be an operation on τ. If a subset A of X is both γ-P_S-g-closed and γ-P_S-open, and F is γ-P_S-closed, then $A \cap F$ is γ-P_S-closed.

Proof. Since A is both γ-P_S-g-closed and γ-P_S-open set. Then by Theorem 4.5.10, A is γ-P_S-closed and since F is γ-P_S-closed, then $A \cap F$ is γ-P_S-closed. □

Corollary 4.5.12. If $A \subseteq X$ is both γ-P_S-g-closed and γ-P_S-open and F is γ-P_S-closed, then $A \cap F$ is γ-P_S-g-closed.

Proof. Follows from Theorem 4.5.11 and the fact that every γ-P_S-closed set is γ-P_S-g-closed. □

Corollary 4.5.13. For any topological space (X, τ). If a subset A of X is γ-P_S-g-closed and γ-P_S-open, then A is γ-pre$g$-closed.

Proof. The proof follows directly from Theorem $4.5.10$ and the fact that every γ-P_S-closed set is γ-preclosed and every γ-preclosed set is γ-pre$g$-closed. $\square$

Theorem 4.5.14. If $A \subseteq (X, \tau)$ is both γ-regular-open and γ-P_S-g-closed, then A is γ-regular-closed and hence it is γ-clopen.

Proof. Let A be both γ-regular-open and γ-P_S-g-closed. Since A is γ-regular-open set. Then A is γ-P_S-open and by Theorem $4.5.10$, A is γ-P_S-closed and so it is γ-preclosed. Again since A is γ-regular-open set, then A is γ-semiopen. Therefore, A is γ-regular-closed in X. Thus A is both γ-open and γ-closed and hence it is γ-clopen. $\square$

Theorem 4.5.15. If a subset A of (X, τ) is both α-γ-open and γ-pre$g$-closed, then A is γ-P_S-g-closed.

Proof. Suppose that A is both α-γ-open and γ-pre$g$-closed set in X. Let $A \subseteq G$ and G be a γ-P_S-open set in X. Since A is α-γ-open. Then A is γ-preopen. Now $A \subseteq A$. By hypothesis, τ_γ-$pCl(A) \subseteq A$. Again since A is α-γ-open, then A is γ-semiopen. By Theorem $4.4.55$, we get τ_γ-$P_SCl(A) \subseteq A \subseteq G$. Thus, A is γ-P_S-g-closed. $\square$

Corollary 4.5.16. If a set A in X is both α-γ-open and α-γ-g-closed, then A is γ-P_S-g-closed.

Proof. The proof is similar to Theorem $4.5.15$ and using Corollary $4.4.56$ to obtain $\tau_{\alpha-\gamma}$-$Cl(A) = \tau_\gamma$-$P_SCl(A)$ for every γ-semiopen set A in X. $\square$

The converse of Theorem $4.5.15$ and Corollary $4.5.16$ are true when A is γ-regular-open as it can be seen from the following corollary.

Corollary 4.5.17. Let A be a γ-regular-open subset of a topological space (X, τ) with an operation γ on τ. Then the following conditions are equivalent:

1. A is γ-P_S-g-closed.

2. A is γ-pre$g$-closed.

3. A is α-γ-g-closed. $\square$

Theorem 4.5.18. In a topological space (X, τ) with an operation γ on τ. Then every subset of X is γ-P_S-g-closed if and only if τ_γ-$P_SO(X) = \tau_\gamma$-$P_SC(X)$.

Proof. Assume that every subset of X is γ-P_S-g-closed. Let $U \in \tau_\gamma$-$P_SO(X)$. Since U is γ-P_S-g-closed. Then by Theorem $4.5.10$, we have U is γ-P_S-closed. Hence τ_γ-$P_SO(X) \subseteq \tau_\gamma$-$P_SC(X)$. If $F \in \tau_\gamma$-$P_SC(X)$, then $X \backslash F \in \tau_\gamma$-$P_SO(X)$ and $X \backslash F$ is γ-P_S-g-closed. Then by Theorem $4.5.10$, $X \backslash F$ is γ-P_S-closed and hence F is γ-P_S-open set. Thus, τ_γ-$P_SC(X) \subseteq \tau_\gamma$-$P_SO(X)$. This means that τ_γ-$P_SO(X) = \tau_\gamma$-$P_SC(X)$.

Conversely, suppose that τ_γ-$P_SO(X) = \tau_\gamma$-$P_SC(X)$ and that $A \subseteq G$ and $G \in \tau_\gamma$-$P_SO(X)$. Then τ_γ-$P_SCl(A) \subseteq \tau_\gamma$-$P_SCl(G) = G$. So A is γ-P_S-g-closed. $\square$

Theorem 4.5.19. Let A, B be subsets of a topological space (X, τ) and γ be an operation on τ. If A is γ-P_S-g-closed and $A \subseteq B \subseteq \tau_\gamma$-$P_SCl(A)$, then B is γ-P_S-g-closed set.

Proof. Let A be any γ-P_S-g-closed set in (X, τ) and $B \subseteq G$ where G is γ-P_S-open. Since $A \subseteq B$, then $A \subseteq G$ and hence τ_γ-$P_SCl(A) \subseteq G$. Since $B \subseteq \tau_\gamma$-$P_SCl(A)$ implies τ_γ-$P_SCl(B) \subseteq \tau_\gamma$-$P_SCl(A)$. Thus τ_γ-$P_SCl(B) \subseteq G$ and this shows that B is γ-P_S-g-closed set. □

From Theorems 4.5.8 and 4.5.19, we obtain the following corollary.

Corollary 4.5.20. Let A, B be subsets of a topological space (X, τ) and γ be an operation on τ. If A is γ-P_S-g-closed and $A \subseteq B \subseteq \tau_\gamma$-$P_SCl(A)$, then τ_γ-$P_SCl(B)\backslash B$ contains no nonempty γ-P_S-closed set. □

Theorem 4.5.21. Let (X, τ) be γ-semiT_1 space and γ be an operation on τ. A set A is γ-P_S-g-closed set in X if and only if A is γ-pre$g$-closed.

Proof. Follows from Theorem 4.2.24 and Corollary 4.4.54 (1). □

Theorem 4.5.22. Let (X, τ) be γ-locally indiscrete space and γ be an operation on τ. Then every γ-P_S-g-closed set in X is γ-g-closed.

Proof. Let A be any γ-P_S-g-closed set in γ-locally indiscrete space (X, τ) and let G be any γ-open set containing A. Then by Theorem 4.2.19, G is γ-P_S-open set. So τ_γ-$P_SCl(A) \subseteq G$ and hence by Corollary 4.4.52 (1), we obtain τ_γ-$Cl(A) \subseteq G$. Since by Remark 2.3.5 $Cl_\gamma(A) \subseteq \tau_\gamma$-$Cl(A)$. Therefore, $Cl_\gamma(A) \subseteq G$. Consequintly, A is γ-g-closed set in X. □

Corollary 4.5.23. Let (X, τ) be any γ-locally indiscrete space and γ be an operation on τ. Then the following conditions are equivalent:

1. A is γ-P_S-g-closed.

2. A is α-γ-g-closed.

3. A is γ-semi$g$-closed.

Proof. Follows directly from Corollary 4.2.21 and using τ_γ-$P_SCl(A) = \tau_{\alpha-\gamma}$-$Cl(A) = \tau_\gamma$-$sCl(A)$. □

The following theorem shows that if a space X is γ-locally indiscrete, then τ_γ-$P_SGC(X)$ is discrete topology.

Theorem 4.5.24. If a topological space (X, τ) is γ-locally indiscrete, then every subset of X is γ-P_S-g-closed.

Proof. Suppose that (X, τ) is γ-locally indiscrete space and $A \subseteq U$ where $U \in \tau_\gamma$-$P_SO(X)$. Then τ_γ-$P_SCl(A) \subseteq \tau_\gamma$-$P_SCl(U)$ and by Corollary 4.3.11, we have τ_γ-$P_SCl(A) \subseteq U$ and so A is γ-P_S-g-closed set in X. □

The following example shows that the converse of Theorem 4.5.24 does not true in general.

Example 4.5.25. Considering the space (X, τ) as defined in Example 4.3.12. Since τ_γ-$P_SO(X, \tau) = \tau_\gamma$-$P_SC(X, \tau)$, then by Theorem 4.5.18, every subset of X is γ-P_S-g-closed, but X is not γ-locally indiscrete space.

Theorem 4.5.26. In a topological space (X, τ) with an operation γ on τ, if τ_γ-$P_SO(X) = \{\phi, X\}$, then every subset of X is a γ-P_S-g-closed.

Proof. Let A be any subset of a topological space (X, τ) and τ_γ-$P_SO(X) = \{\phi, X\}$. Suppose that $A = \phi$, then A is a γ-P_S-g-closed set in X. If $A \neq \phi$, then X is the only γ-P_S-open set containing A and hence τ_γ-$P_SCl(A) \subseteq X$. So A is a γ-P_S-g-closed set in X. □

Corollary 4.5.27. Let γ be a regular operation on τ. If a topological space (X, τ) is γ-hyperconnected, then every subset of X is γ-P_S-g-closed.

Proof. Follows from Theorem 4.2.25 and Theorem 4.5.26. □

Theorem 4.5.28. In any topological space (X, τ) with an operation γ on τ. For an element $x \in X$, the set $X \backslash \{x\}$ is γ-P_S-g-closed or γ-P_S-open.

Proof. Suppose that $X \backslash \{x\}$ is not γ-P_S-open. Then X is the only γ-P_S-open set containing $X \backslash \{x\}$. This implies that τ_γ-$P_SCl(X \backslash \{x\}) \subseteq X$. Thus $X \backslash \{x\}$ is a γ-P_S-g-closed set in X. □

Corollary 4.5.29. In any topological space (X, τ) with an operation γ on τ. For an element $x \in X$, either the set $\{x\}$ is γ-P_S-closed or the set $X \backslash \{x\}$ is γ-P_S-g-closed.

Proof. Suppose $\{x\}$ is not γ-P_S-closed, then $X \backslash \{x\}$ is not γ-P_S-open. Hence by Theorem 4.5.28, $X \backslash \{x\}$ is γ-P_S-g-closed set in X. □

Lemma 4.5.30. Let (X, τ) be a topological space and γ be an operation on τ. A set A in (X, τ) is γ-P_S-g-closed if and only if $A \cap \tau_\gamma$-$P_SCl(\{x\}) \neq \phi$ for every $x \in \tau_\gamma$-$P_SCl(A)$.

Proof. Suppose A is γ-P_S-g-closed set in X and suppose (if possible) that there exists an element $x \in \tau_\gamma$-$P_SCl(A)$ such that $A \cap \tau_\gamma$-$P_SCl(\{x\}) = \phi$. This follows that

$A \subseteq X\backslash\tau_\gamma$-$P_SCl(\{x\})$. Since τ_γ-$P_SCl(\{x\})$ is γ-P_S-closed implies $X\backslash\tau_\gamma$-$P_SCl(\{x\})$ is γ-P_S-open and A is γ-P_S-g-closed set in X. Then τ_γ-$P_SCl(A) \subseteq X\backslash\tau_\gamma$-$P_SCl(\{x\})$. This means that $x \notin \tau_\gamma$-$P_SCl(A)$. This is a contradiction. Hence $A \cap \tau_\gamma$-$P_SCl(\{x\}) \neq \phi$.

Conversely, let G be any γ-P_S-open set in X containing A. To show that τ_γ-$P_SCl(A) \subseteq G$. Let $x \in \tau_\gamma$-$P_SCl(A)$. Then by hypothesis, $A \cap \tau_\gamma$-$P_SCl(\{x\}) \neq \phi$. So there exists an element $y \in A \cap \tau_\gamma$-$P_SCl(\{x\})$. Thus $y \in A \subseteq G$ and $y \in \tau_\gamma$-$P_SCl(\{x\})$. By Theorem 4.4.13, $\{x\} \cap G \neq \phi$. Hence $x \in G$ and so τ_γ-$P_SCl(A) \subseteq G$. Therefore, A is γ-P_S-g-closed set in (X, τ). □

Theorem 4.5.31. For any subset A of a topological space (X, τ). Then $A \cap \tau_\gamma$-$P_SCl(\{x\}) \neq \phi$ for every $x \in \tau_\gamma$-$P_SCl(A)$ if and only if τ_γ-$P_SCl(A)\backslash A$ does not contain any nonempty γ-P_S-closed set.

Proof. The proof is directly from Theorem 4.5.8 and Lemma 4.5.30. □

Corollary 4.5.32. Let A be a subset of a topological space (X, τ) and γ be an operation on τ. Then A is γ-P_S-g-closed if and only if $A = E\backslash F$, where E is γ-P_S-closed set of X and F contains no nonempty γ-P_S-closed set in X.

Proof. Let A be any γ-P_S-g-closed set in (X, τ). Then by Theorem 4.5.8, τ_γ-$P_SCl(A)\backslash A = F$ contains no nonempty γ-P_S-closed set. Let $E = \tau_\gamma$-$P_SCl(A)$ is γ-P_S-closed set such that $A = E\backslash F$.

Conversely, let $A = E\backslash F$, where E is γ-P_S-closed set and F contains no nonempty γ-P_S-closed set. Let $A \subseteq G$ and G is γ-P_S-open set in X. Then $E \cap X\backslash G$ is a

γ-P_S-closed subset of F and hence it is empty. Therefore, τ_γ-$P_SCl(A) \subseteq E \subseteq G$. Thus A is γ-P_S-g-closed set. $\square$

By considering Figure 2.2 and Theorem 4.5.2, the following figure is obtained.

$$\begin{tikzcd}
 & \gamma\text{-regular-closed} \arrow[d] \arrow[dr] & & \\
\gamma\text{-}P_S\text{-}g\text{-closed} & \gamma\text{-}P_S\text{-closed} \arrow[l] \arrow[d] & \gamma\text{-closed} \arrow[r] \arrow[d] & \gamma\text{-}g\text{-closed} \\
\gamma\text{-pre}g\text{-closed} & \gamma\text{-preclosed} \arrow[l] \arrow[d] & \alpha\text{-}\gamma\text{-closed} \arrow[l] \arrow[r] \arrow[d] & \alpha\text{-}\gamma\text{-}g\text{-closed} \\
\gamma\text{-}b\text{-}g\text{-closed} & \gamma\text{-}b\text{-closed} \arrow[l] \arrow[d] & \gamma\text{-semiclosed} \arrow[l] \arrow[r] & \gamma\text{-semi}g\text{-closed} \\
\gamma\text{-}\beta g\text{-closed} & \gamma\text{-}\beta\text{-closed} \arrow[l] & &
\end{tikzcd}$$

Remark: $A \longrightarrow B$ represents A implies B

Figure 4.2. The relations between γ-P_S-g-closed set and various types of γ-g-closed sets

In the sequel, none of the implications that concerning γ-P_S-g-closed set in the Figure 4.2 is reversible. It is notice that γ-P_S-closed set lies strictly between the classes of γ-regular-closed set and γ-P_S-g-closed set. In addition, the relation between γ-P_S-g-closed set and γ-pre$g$-closed set are independent. Similarly the relation between γ-P_S-g-closed set and γ-g-closed set are independent, the relation between γ-P_S-g-closed set and α-γ-g-closed set are independent, the relation between γ-P_S-g-closed set and γ-semi$g$-closed set are independent, the relation between γ-P_S-g-closed set and γ-b-g-closed set are independent and the relation between γ-P_S-g-closed set and γ-βg-closed set are independent.

4.6 γ-P_S-Generalized Open Sets

In this section, the complement of γ-P_S-g-closed set which is γ-P_S-g-open set in a topological space (X, τ) will be studied.

Definition 4.6.1. A subset B of a topological space (X, τ) with an opeation γ on τ is called γ-P_S-generalized open (γ-P_S-g-open) if $X \backslash B$ is γ-P_S-g-closed. The class of all γ-P_S-g-open subsets of a topological space (X, τ) is denoted by τ_γ-$P_SGO(X)$ or τ_γ-$P_SGO(X, \tau)$.

Theorem 4.6.2. A set A is γ-P_S-g-open if $F \subseteq \tau_\gamma$-$P_SInt(A)$ whenever $F \subseteq A$ and F is a γ-P_S-closed set in X.

Proof. Let A be γ-P_S-g-open subset of a space (X, τ) and let F be any γ-P_S-closed set in X such that $F \subseteq A$. Then $X \backslash A \subseteq X \backslash F$ where $X \backslash F$ is γ-P_S-open. Since A is γ-P_S-g-open set, then $X \backslash A$ is γ-P_S-g-closed. Hence τ_γ-$P_SCl(X \backslash A) \subseteq X \backslash F$. By using Theorem 4.4.33 (2), we get $X \backslash \tau_\gamma$-$P_SInt(A) \subseteq X \backslash F$. That is, $F \subseteq \tau_\gamma$-$P_SInt(A)$.

Conversely, let $X \backslash A \subseteq G$ where G is γ-P_S-open set in (X, τ). Then $X \backslash G \subseteq A$ where $X \backslash G$ is γ-P_S-closed. By hypothesis, $X \backslash G \subseteq \tau_\gamma$-$P_SInt(A)$. That is, $X \backslash \tau_\gamma$-$P_SInt(A) \subseteq G$. Thus, by Theorem 4.4.33 (2), τ_γ-$P_SCl(X \backslash A) \subseteq G$. Therefore, $X \backslash A$ is γ-P_S-g-closed and hence A is γ-P_S-g-open. $\square$

It is obvious from Theorem 4.6.2 that every γ-P_S-open set is γ-P_S-g-open, but the converse does not true in general as shown from the following example.

Example 4.6.3. Let (X, τ) be a topological space and γ be an operation on τ as in thref4Ex7. Then the $\{b\}$ is γ-P_S-g-open, but it is not a γ-P_S-open.

The following remark follows directly from Remark 4.5.4 and Remark 4.5.5, and using complements.

Remark 4.6.4.
1. The union of any two γ-P_S-g-open sets may not be γ-P_S-g-open.
2. The intersection of any two γ-P_S-g-open sets may not be γ-P_S-g-open.

Proof. Follows from Remark 4.5.4 and Remark 4.5.5, and using complements. □

Corollary 4.6.5. In any topological space (X, τ). The union of γ-P_S-g-open set of X and γ-P_S-open set of X is γ-P_S-g-open in X.

Proof. The proof follows from Theorem 4.5.7, and using complements. □

Theorem 4.6.6. Let (X, τ) be a topological space and γ be an operation on τ. If a subset A of X is γ-P_S-g-open and γ-P_S-closed, then A is γ-P_S-open.

Proof. Since A is γ-P_S-g-open and γ-P_S-closed set in X. Then by Theorem 4.6.2, $A \subseteq \tau_\gamma$-$P_S Int(A)$. Hence A is γ-P_S-open. □

Corollary 4.6.7. For any topological space (X, τ). If a subset A of X is γ-P_S-g-open and γ-P_S-closed, then A is γ-pre$g$-open.

Proof. The proof follows directly from Theorem 4.6.6 and the fact that every γ-P_S-open set is γ-preopen and every γ-preopen set is γ-pre$g$-open. □

Theorem 4.6.8. If a subset A of a space (X, τ) is both γ-regular-closed and γ-P_S-g-open, then A is γ-regular-open and hence it is γ-clopen.

Proof. The proof is similar to Theorem $4.5.14$ taking $A = X \backslash B$. $\square$

Theorem 4.6.9. If a subset A of (X, τ) is both α-γ-closed and γ-pre$g$-open, then A is γ-P_S-g-open.

Proof. The proof is similar to Theorem $4.5.15$. $\square$

Theorem 4.6.10. If a set A in X is both α-γ-closed and α-γ-g-open, then A is γ-P_S-g-open.

Proof. The proof is similar to Corollary $4.5.16$. $\square$

The converse of Theorem $4.6.9$ and Theorem $4.6.10$ are true when A is γ-regular-closed as it can be seen from the following corollary.

Corollary 4.6.11. Let A be a γ-regular-closed subset of a topological space (X, τ) with an operation γ on τ. Then the following conditions are equivalent:

1. A is γ-P_S-g-open.

2. A is α-γ-g-open.

3. A is γ-pre$g$-open. $\square$

Theorem 4.6.12. In any topological space (X, τ) with an operation γ on τ. For an element $x \in X$, the set $\{x\}$ is either γ-P_S-g-open or γ-P_S-closed.

Proof. Let $\{x\}$ is not γ-P_S-g-open. Then $X\backslash\{x\}$ is not γ-P_S-g-closed. So by Theorem 4.5.28, $X\backslash\{x\}$ is γ-P_S-open and hence $\{x\}$ is γ-P_S-closed set in X. $\square$

Corollary 4.6.13. In any topological space (X, τ) with an operation γ on τ. For an element $x \in X$, either the set $\{x\}$ is γ-P_S-g-open or the set $X\backslash\{x\}$ is γ-P_S-open set in X.

Proof. Suppose $X\backslash\{x\}$ is not γ-P_S-open, then $\{x\}$ is not γ-P_S-closed. Hence by Theorem 4.6.12, $\{x\}$ is γ-P_S-g-open. $\square$

Theorem 4.6.14. In any topological space (X, τ), a set $A \subseteq (X, \tau)$ is γ-P_S-g-closed if and only if τ_γ-$P_SCl(A)\backslash A$ is γ-P_S-g-open set.

Proof. Let F be a γ-P_S-closed set in X such that $F \subseteq \tau_\gamma$-$P_SCl(A)\backslash A$. Since A is γ-P_S-g-closed. Then by Theorem 4.5.8, $F = \phi$. Hence $F \subseteq \tau_\gamma$-$P_SInt(\tau_\gamma$-$P_SCl(A)\backslash A)$. This shows that τ_γ-$P_SCl(A)\backslash A$ is γ-P_S-g-open set.

Conversely, suppose that $A \subseteq G$, where G is a γ-P_S-open set in X. So τ_γ-$P_SCl(A)$ $\cap\ X\backslash G \subseteq \tau_\gamma$-$P_SCl(A) \cap X\backslash A = \tau_\gamma$-$P_SCl(A)\backslash A$. Since $X\backslash G$ is a γ-P_S-closed and hence τ_γ-$P_SCl(A) \cap X\backslash G$ is γ-P_S-closed set in X and τ_γ-$P_SCl(A)\backslash A$ is γ-P_S-g-open set. Then by Theorem 4.6.2, τ_γ-$P_SCl(A) \cap X\backslash G \subseteq \tau_\gamma$-$P_SInt(\tau_\gamma$-$P_SCl(A)\backslash A) = \phi$. By Theorem 4.4.44 (1), τ_γ-$P_SInt(\tau_\gamma$-$P_SCl(A)\backslash A)$ implies that τ_γ-$P_SCl(A) \cap X\backslash G = \phi$ and hence τ_γ-$P_SCl(A) \subseteq G$. This means that A is γ-P_S-g-closed. $\square$

Theorem 4.6.15. Let A and B be subsets of (X, τ). If A is γ-P_S-g-open and τ_γ-$P_SInt(A) \subseteq B \subseteq A$, then B is γ-P_S-g-open set.

Proof. Since τ_γ-$P_SInt(A) \subseteq B \subseteq A$ implies that $X \backslash A \subseteq X \backslash B \subseteq X \backslash \tau_\gamma$-$P_SInt(A)$. By Theorem 4.4.33 (4), we get $X \backslash A \subseteq X \backslash B \subseteq \tau_\gamma$-$P_SCl(X \backslash A)$. Since A is γ-P_S-g-open and then $X \backslash A$ is γ-P_S-g-closed. So by Theorem 4.5.19, $X \backslash B$ is γ-P_S-g-closed and hence B is γ-P_S-g-open. □

Theorem 4.6.16. A subset A in (X, τ) is γ-P_S-g-open if and only if $G = X$ whenever G is γ-P_S-open set in X and τ_γ-$P_SInt(A) \cup X \backslash A \subseteq G$.

Proof. Let G be a γ-P_S-open set in X and τ_γ-$P_SInt(A) \cup X \backslash A \subseteq G$. This implies $X \backslash G \subseteq \tau_\gamma$-$P_SCl(X \backslash A) \cap A = \tau_\gamma$-$P_SCl(X \backslash A) \backslash (X \backslash A)$. Since G is γ-P_S-open and A is γ-P_S-g-open, then $X \backslash G$ is γ-P_S-closed and $X \backslash A$ is γ-P_S-g-closed. So by Theorem 4.5.8, $X \backslash G = \phi$ implies $G = X$.

Conversely, suppose F is a γ-P_S-closed set in X and $F \subseteq A$. Then $X \backslash A \subseteq X \backslash F$ and hence τ_γ-$P_SInt(A) \cup X \backslash A \subseteq \tau_\gamma$-$P_SInt(A) \cup X \backslash F$. Since τ_γ-$P_SInt(A) \cup X \backslash F$ is γ-P_S-open set in X, then by hypothesis τ_γ-$P_SInt(A) \cup X \backslash F = X$. It follows that $F \subseteq \tau_\gamma$-$P_SInt(A)$. Therefore, A is γ-P_S-g-open set in X. □

Theorem 4.6.17. Let (X, τ) be γ-semiT_1 space and γ be an operation on τ. A set A is γ-P_S-g-open set in X if and only if A is γ-pre$g$-open.

Proof. Follows from Corollary 4.3.13, Corollary 4.4.54 (2), Theorem 4.6.2 and Definition 2.3.21 (1). □

Theorem 4.6.18. Let (X, τ) be any γ-locally indiscrete space and γ be an operation on τ. Then the following conditions are equivalent:

1. A is γ-P_S-g-open.

2. A is α-γ-g-open.

3. A is γ-semi$g$-open.

Proof. Follows directly from Theorem 4.6.2, Definition 2.3.21 (2) and (3) and using Corollary 4.2.21 to obtain τ_γ-$P_S Int(A) = \tau_{\alpha-\gamma}$-$Int(A) = \tau_\gamma$-$sInt(A)$. □

4.7 Conclusion

This chapter has constructed γ-P_S-open set, γ-P_S-closed set, γ-P_S-operations, γ-P_S-g-closed set and γ-P_S-g-open set. The related relations and properties have also been discussed.

In the next two chapters, we will develop some types of γ-P_S- functions and γ-P_S- separation axioms by using this γ-P_S-open set.

CHAPTER FIVE

γ-P_S- FUNCTIONS

5.1 Introduction

This chapter defines some new classes of γ-P_S- functions called γ-P_S-continuous, (γ, β)-P_S-continuous and (γ, β)-P_S-irresolute functions by using γ-P_S-open sets. Also, the relationships between them and then between γ-P_S- functions and the other classes of functions are obtained. Some properties and characterizations on these γ-P_S- functions are investigated. These γ-P_S- functions are compared with other types of functions as stated in Section 2.4. Moreover, some other classes of γ-P_S- functions are introduced such as β-P_S-open, β-P_S-closed, (γ, β)-P_S-open, (γ, β)-P_S-closed, $(\gamma, \beta P_S)$-open and $(\gamma, \beta P_S)$-closed functions. Furthermore, we define another kinds of γ-P_S- functions called γ-P_S-g-continuous and some new functions are employed. For each of new constructed γ-P_S- functions, we present some properties and theorems. Also, we apply γ-regular-open set to obtain a new type of functions called completely γ-continuous. Finally, composition of these γ-P_S- functions are given.

5.2 Definitions and Relations of γ-P_S- Functions

In this section, we introduce some types of γ- functions called γ-P_S-continuous, (γ, β)-P_S-continuous and (γ, β)-P_S-irresolute by using γ-P_S-open set. The relations between these functions and other types of γ- functions are investigated.

Definition 5.2.1. Let (X,τ) and (Y,σ) be two topological spaces and γ be an operation on τ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called γ-P_S-continuous at a point $x \in X$ if for each open set V of Y containing $f(x)$, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. If f is γ-P_S-continuous at every point x in X, then f is said to be γ-P_S-continuous.

Definition 5.2.2. Let (X,τ) and (Y,σ) be two topological spaces and γ and β be operations on τ and σ respectively. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called (γ,β)-P_S-continuous at a point $x \in X$ if for each β-open set V of Y containing $f(x)$, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. If f is (γ,β)-P_S-continuous at every point x in X, then f is said to be (γ,β)-P_S-continuous.

Definition 5.2.3. Let (X,τ) and (Y,σ) be two topological spaces and γ and β be operations on τ and σ respectively. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called (γ,β)-P_S-irresolute at a point $x \in X$ if for each β-P_S-open set V of Y containing $f(x)$, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. If f is (γ,β)-P_S-irresolute at every point x in X, then f is said to be (γ,β)-P_S-irresolute.

Remark 5.2.4. It is clear from the Definitions 5.2.1 and 5.2.2 that every γ-P_S-continuous function is (γ,β)-P_S-continuous since every β-open set is open, where β is an operation on σ. However, the converse is not true in general as it can be seen from the following example.

Example 5.2.5. Consider (X,τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Suppose that $Y = \{1,2,3\}$ and $\sigma = \{\phi, Y, \{2\}, \{3\}, \{2,3\}\}$ be a topology on Y. Define an operation $\beta\colon \sigma \to P(Y)$ as follows: for every $B \in \beta$

$$\beta(B) = \begin{cases} B & \text{if } B = \{2\} \\ Cl(B) & \text{if } B \neq \{2\} \end{cases}$$

Then $\sigma_\beta = \{\phi, \{2\}, Y\}$.

Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function defined as follows:

$$f(x) = \begin{cases} 2 & \text{if } x = a \\ 3 & \text{if } x = b \\ 1 & \text{if } x = c \end{cases}$$

Then f is (γ, β)-P_S-continuous, but it is not γ-P_S-continuous since $\{3\}$ is an open set in (Y, σ) containing $f(b) = 3$, but there exist no γ-P_S-open set U in (X, τ) containing b such that $f(U) \subseteq \{3\}$.

Remark 5.2.6. The relation between (γ, β)-P_S-irresolute function and (γ, β)-P_S-continuous function are independent. Similarly the relation between (γ, β)-P_S-irresolute function and γ-P_S-continuous function are independent, as shown from the following examples.

Example 5.2.7. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Suppose that $Y = \{1, 2, 3\}$ and $\sigma = \{\phi, Y, \{2\}, \{2, 3\}\}$ be a topology on Y. Define an operation β on σ such that $\beta\colon \sigma \to P(Y)$ by $\beta(B) = B$ for all $B \in \sigma$. Then σ_β-$P_SO(Y) = \{\phi, Y\}$.

Let $f\colon (X,\tau) \to (Y,\sigma)$ be a function defined as follows:

$$f(x) = \begin{cases} 3 & \text{if } x = a \\ 2 & \text{if } x = b \\ 1 & \text{if } x = c \end{cases}$$

Then the function f is (γ,β)-P_S-irresolute, but f is not (γ,β)-P_S-continuous since $\{2\}$ is a β-open set in (Y,σ) containing $f(b) = 2$, but there exist no γ-P_S-open set U in (X,τ) containing b such that $f(U) \subseteq \{2\}$. By Remark 5.2.4, f is not γ-P_S-continuous.

Example 5.2.8. Consider the space $X = \{a,b,c\}$ with the topologies $\tau = \{\phi, X, \{a\}, \{c\}, \{a,c\}\}$ and $\sigma = \{\phi, X, \{b\}, \{c\}, \{a,b\}, \{b,c\}\}$. Define the operations γ and β on τ and σ respectively as follows:

For every $A \in \tau$, $\gamma(A) = A$ and for every $B \in \sigma$

$$\beta(B) = \begin{cases} B & \text{if } c \in B \\ Cl(B) & \text{if } c \notin B \end{cases}$$

Obviously, $\tau_\gamma = \tau = \tau_\gamma$-$P_SO(X)$, $\sigma_\beta = \{\phi, X, \{c\}, \{a,b\}, \{b,c\}\}$ and σ_β-$P_SO(X) = \{\phi, X, \{c\}, \{a,b\}, \{a,c\}\}$.

Define a function $f\colon (X,\tau) \to (X,\sigma)$ as follows:

$$f(x) = \begin{cases} b & \text{if } x \in \{a,c\} \\ a & \text{if } x = b \end{cases}$$

Clearly, the function f is both (γ, β)-P_S-continuous and γ-P_S-continuous, but f is not (γ, β)-P_S-irresolute since $\{a, c\}$ is a β-P_S-open set in (X, σ) containing $f(b) = a$, there exist no γ-P_S-open set U in (X, τ) containing c such that $f(U) \subseteq \{a, c\}$.

Remark $4.2.9$ (1) lead to the following remark.

Remark 5.2.9. Every γ-P_S-continuous function is γ-precontinuous.

The converse of the Remark $5.2.9$ is not true in general as can be seen from the following example.

Example 5.2.10. Let (X, τ) be a topological space and γ be an operation on τ as in Example $4.2.3$. Let $\sigma = \{\phi, \{a\}, \{a, b\}, X\}$ be another topology on X.
Define a function $f\colon (X, \tau) \to (X, \sigma)$ as follows:

$$f(x) = \begin{cases} c & \text{if } x = a \\ a & \text{if } x = b \\ b & \text{if } x = c \end{cases}$$

Then f is γ-precontinuous, but it is not γ-P_S-continuous since $\{a\}$ is an open set in (X, σ) containing $f(b) = a$, but there exist no γ-P_S-open set U in (X, τ) containing b such that $f(U) \subseteq \{a\}$.

The following examples show that the relation between γ-P_S-continuous function and γ-continuous function are independent in general.

Example 5.2.11. In Example 5.2.5. The function f is γ-continuous, but it is not γ-P_S-continuous.

Example 5.2.12. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Suppose that $Y = \{1, 2, 3\}$ and $\sigma = \{\phi, Y, \{1, 3\}\}$ be a topology on Y. Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = a \\ 2 & \text{if } x = b \\ 3 & \text{if } x = c \end{cases}$$

Obviously, f is γ-P_S-continuous, but it is not γ-continuous since $\{1, 3\} \in \sigma$, but $f^{-1}(\{1, 3\}) = \{a, c\} \notin \tau_\gamma$.

The relation between γ-P_S-continuous function and P_S-continuous function are independent in general as the following examples shown.

Example 5.2.13. In Example 5.2.12. The function f is γ-P_S-continuous, but it is not P_S-continuous since $\{1, 3\} \in \sigma$, but $f^{-1}(\{1, 3\}) = \{a, c\} \notin P_SO(X)$.

Example 5.2.14. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.17. Let $f\colon (X, \tau) \to (X, \tau)$ be a function defined as follows: $f(x) = x$ for all $x \in X$. Then f is P_S-continuous, but f is not γ-P_S-continuous since $\{c\}$ is an open set in (X, τ) containing $f(c) = c$, but there exist no γ-P_S-open set U in (X, τ) containing c such that $f(U) \subseteq \{c\}$.

Now, we introduce a new type of function called completely γ-continuous by using γ-regular-open set in Definition 3.2.1. Some relations between this function with γ-P_S-continuous and γ-continuous functions are examined.

Definition 5.2.15. Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is called completely γ-continuous at a point $x \in X$ if for each open set V of Y containing $f(x)$, there exists a γ-regular-open set U of X containing x such that $f(U) \subseteq V$. If f is completely γ-continuous at every point x in X, then f is said to be completely γ-continuous.

Remark 4.2.9 (2) lead to the following remark.

Remark 5.2.16. Every completely γ-continuous function is γ-P_S-continuous.

The converse of the Remark 5.2.16 is not true in general as it is shown in the following example.

Example 5.2.17. In Example 5.2.8. The function f is γ-P_S-continuous, but f is not completely γ-continuous since $\{b, c\}$ is an open set in (Y, σ) containing $f(a) = b$, but there exist no γ-regular-open set U in (X, τ) containing a such that $f(U) \subseteq \{b, c\}$.

Since every γ-regular-open set is γ-open, then we have the following remark.

Remark 5.2.18. Every completely γ-continuous function is γ-continuous.

The converse of the Remark 5.2.18 is not true in general as it is shown in the following example.

Example 5.2.19. In Example 5.2.8. The function f is γ-continuous, but f is not completely γ-continuous since $\{b, c\}$ is an open set in (Y, σ) containing $f(a) = b$, but there exist no γ-regular-open set U in (X, τ) containing a such that $f(U) \subseteq \{b, c\}$.

The relation between completely continuous function and completely γ-continuous function are independent in general as shown by the following examples.

Example 5.2.20. In Example 5.2.14. The function f is completely continuous, but it is not completely γ-continuous since $\{c\} \in \sigma$, but $f^{-1}(\{c\}) = \{c\}$ is not γ-regular-open set in X.

Example 5.2.21. Consider the space $X = \{a, b, c\}$ with the topologies $\tau = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{\phi, X, \{b\}, \{b, c\}\}$. Define the operations γ on τ as follows: For every $A \in \tau$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{b\} \text{ or } \{c\} \text{ or } \{\text{b, c}\} \\ X & \text{otherwise} \end{cases}$$

Then $\tau_\gamma = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$ and hence τ_γ-$RO(X) = \{\phi, X, \{b\}, \{c\}\}$.

Let $f\colon (X, \tau) \to (X, \sigma)$ be a function defined as follows:

$$f(x) = \begin{cases} a & \text{if } x \in \{a, c\} \\ b & \text{if } x = b \end{cases}$$

Then the function f is completely γ-continuous, but it is not completely continuous since $\{b\} \in \sigma$, but $f^{-1}(\{b\}) = \{b\}$ is not regular-open set in X.

From Remark 5.2.4, Remark 5.2.16, Remark 5.2.18, Remark 5.2.9 and Figure 4.1, we have the following figure.

$$\begin{tikzcd}
(\gamma,\beta)\text{-}P_S\text{-continuous} & \text{completely } \gamma\text{-continuous} \arrow[dl] \arrow[r] & \gamma\text{-continuous} \arrow[d] \\
\gamma\text{-}P_S\text{-continuous} \arrow[u] \arrow[r] & \gamma\text{-precontinuous} \arrow[d] & \alpha\text{-}\gamma\text{-continuous} \arrow[l] \\
 & \gamma\text{-}\beta\text{-continuous} &
\end{tikzcd}$$

Remark: $A \longrightarrow B$ represents A implies B

Figure 5.1. The relations between γ-P_S- functions and various types of γ- functions

5.3 Characterizations and Properties of γ-P_S- Functions

In this section, we give some characterizations and properties of the functions that have been defined in Section 5.2. Then, some composition of γ-P_S- functions will be given.

We start with the most important characterizations of γ-P_S-continuous functions.

Theorem 5.3.1. For a function $f\colon (X,\tau) \to (Y,\sigma)$ and γ be an operation on τ, the following statements are equivalent:

1. f is γ-P_S-continuous.

2. $f^{-1}(V)$ is γ-P_S-open set in X, for every open set V in Y.

3. $f^{-1}(F)$ is γ-P_S-closed set in X, for every closed set F in Y.

4. $f(\tau_\gamma\text{-}P_SCl(A)) \subseteq Cl(f(A))$, for every subset A of X.

5. $\tau_\gamma\text{-}P_SCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$, for every subset B of Y.

6. $f^{-1}(Int(B)) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(B))$, for every subset B of Y.

7. $Int(f(A)) \subseteq f(\tau_\gamma\text{-}P_SInt(A))$, for every subset A of X.

Proof. (1) $\Rightarrow$ (2). Let V be any open set in Y. We have to show that $f^{-1}(V)$ is γ-P_S-open set in X. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By (1), there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. Which implies that $x \in U \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is γ-P_S-open set in X.

(2) $\Rightarrow$ (3). Let F be any closed set of Y. Then $Y \backslash F$ is an open set of Y. By (2), $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F)$ is γ-P_S-open set in X and hence $f^{-1}(F)$ is γ-P_S-closed set in X.

(3) $\Rightarrow$ (4). Let A be any subset of X. Then $f(A) \subseteq Cl(f(A))$ and hence $A \subseteq f^{-1}(Cl(f(A)))$. Since $Cl(f(A))$ is closed set in Y. Then by (3), we have $f^{-1}(Cl(f(A)))$ is γ-P_S-closed set in X. Therefore, $\tau_\gamma\text{-}P_SCl(A) \subseteq f^{-1}(Cl(f(A)))$. Hence $f(\tau_\gamma\text{-}P_SCl(A))$ $\subseteq Cl(f(A))$.

(4) $\Rightarrow$ (5). Let B be any subset of Y. Then $f^{-1}(B)$ is a subset of X. By (4), we have $f(\tau_\gamma\text{-}P_SCl(f^{-1}(B))) \subseteq Cl(f(f^{-1}(B))) = Cl(B)$. Hence $\tau_\gamma\text{-}P_SCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.

(5) $\Leftrightarrow$ (6). Let B be any subset of Y. Then apply (5) to $Y \backslash B$ we obtain $\tau_\gamma\text{-}P_SCl(f^{-1}(Y \backslash B)) \subseteq f^{-1}(Cl(Y \backslash B)) \Leftrightarrow \tau_\gamma\text{-}P_SCl(X \backslash f^{-1}(B)) \subseteq f^{-1}(Y \backslash Int(B)) \Leftrightarrow$

$X\backslash\tau_\gamma$-$P_SInt(f^{-1}(B)) \subseteq X\backslash f^{-1}(Int(B)) \Leftrightarrow f^{-1}(Int(B)) \subseteq \tau_\gamma$-$P_SInt(f^{-1}(B))$. Therefore, $f^{-1}(Int(B)) \subseteq \tau_\gamma$-$P_SInt(f^{-1}(B))$.

(6) $\Rightarrow$ (7). Let A be any subset of X. Then $f(A)$ is a subset of Y. By (6), we have $f^{-1}(Int(f(A))) \subseteq \tau_\gamma$-$P_SInt(f^{-1}(f(A))) = \tau_\gamma$-$P_SInt(A)$. Therefore, $Int(f(A)) \subseteq f(\tau_\gamma$-$P_SInt(A))$.

(7) $\Rightarrow$ (1). Let $x \in X$ and let V be any open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X. By (7), we have $Int(f(f^{-1}(V))) \subseteq f(\tau_\gamma$-$P_SInt(f^{-1}(V)))$. Then $Int(V) \subseteq f(\tau_\gamma$-$P_SInt(f^{-1}(V)))$. Since V is an open set. Then $V \subseteq f(\tau_\gamma$-$P_SInt(f^{-1}(V)))$ implies that $f^{-1}(V) \subseteq \tau_\gamma$-$P_SInt(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is γ-P_S-open set in X which contains x and clearly $f(f^{-1}(V)) \subseteq V$. Hence f is γ-P_S-continuous function. $\square$

Some other properties of γ-P_S-continuous functions are mentioned in the following Theorem 5.3.2.

Theorem 5.3.2. Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function and γ be an operation on τ. Then f is γ-P_S-continuous if and only if τ_γ-$P_SBd(f^{-1}(B)) \subseteq f^{-1}(Bd(B))$, for each subset B of Y.

Proof. Let B be any subset of Y and f be a γ-P_S-continuous function. Then by using Theorem 5.3.1 (2) and (5), we have

$$\begin{aligned} f^{-1}(Bd(B)) &= f^{-1}(Cl(B)\backslash Int(B)) \\ &= f^{-1}(Cl(B))\backslash f^{-1}(Int(B)) \\ &\supseteq \tau_\gamma\text{-}P_SCl(f^{-1}(B))\backslash f^{-1}(Int(B)) \end{aligned}$$

$= \tau_\gamma\text{-}P_S Cl(f^{-1}(B)) \backslash \tau_\gamma\text{-}P_S Int(f^{-1}(Int(B)))$

$\supseteq \tau_\gamma\text{-}P_S Cl(f^{-1}(B)) \backslash \tau_\gamma\text{-}P_S Int(f^{-1}(B))$

$= \tau_\gamma\text{-}P_S Bd(f^{-1}(B))$

Hence $\tau_\gamma\text{-}P_S Bd(f^{-1}(B)) \subseteq f^{-1}(Bd(B))$.

Conversely, let G be any open set in Y. Then $Y\backslash G$ is closed in Y. So by hypothesis, we have $\tau_\gamma\text{-}P_S Bd(f^{-1}(Y\backslash G)) \subseteq f^{-1}(Bd(Y\backslash G)) \subseteq f^{-1}(Cl(Y\backslash G)) = f^{-1}(Y\backslash G)$. By Theorem 4.4.42 (4), $\tau_\gamma\text{-}P_S Cl(f^{-1}(Y\backslash G)) = \tau_\gamma\text{-}P_S Int(f^{-1}(Y\backslash G)) \cup \tau_\gamma\text{-}P_S Bd(f^{-1}(Y\backslash G))$ $\subseteq f^{-1}(Y\backslash G)$. Then $f^{-1}(Y\backslash G)$ is γ-P_S-closed set in X and hence $f^{-1}(G)$ is γ-P_S-open set in X. By Theorem 5.3.1, f is γ-P_S-continuous function. □

Corollary 5.3.3. Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function and γ be an operation on τ. Then f is γ-P_S-continuous if and only if $f(\tau_\gamma\text{-}P_S Bd(A)) \subseteq Bd(f(A))$, for each subset A of X.

Proof. The proof follows directly from Theorem 5.3.2. □

Theorem 5.3.4. Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function and γ be an operation on τ. Then f is γ-P_S-continuous if and only if $f(\tau_\gamma\text{-}P_S D(A)) \subseteq Cl(f(A))$, for each subset A of X.

Proof. Let f be a γ-P_S-continuous function and A be any subset of X. Then by Theorem 5.3.1 (4), we have $f(\tau_\gamma\text{-}P_S Cl(A)) \subseteq Cl(f(A))$ and by Corollary 4.4.15, $f(\tau_\gamma\text{-}P_S D(A)) \subseteq f(\tau_\gamma\text{-}P_S Cl(A))$ which implies that $f(\tau_\gamma\text{-}P_S D(A)) \subseteq Cl(f(A))$.

Conversely, let F be any closed set in Y. Then $f^{-1}(F)$ is subset of X. By hypothesis, we have $f(\tau_\gamma\text{-}P_S D(f^{-1}(F))) \subseteq Cl(f(f^{-1}(F))) = Cl(F) = F$ and hence

τ_γ-$P_SD(f^{-1}(F)) \subseteq f^{-1}(F)$. Then by Theorem 4.4.16, we get $f^{-1}(F)$ is γ-P_S-closed set in X. Therefore, by Theorem 5.3.1, f is γ-P_S-continuous function. □

Corollary 5.3.5. Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function and γ be an operation on τ. Then f is γ-P_S-continuous if and only if τ_γ-$P_SD(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$, for each subset B of Y. □

Lemma 5.3.6. Let γ be an operation on (X,τ). A function $f\colon (X,\tau) \to (Y,\sigma)$ is γ-P_S-continuous if and only if for every $x \in X$ and for neighbourhood O of Y such that $f(x) \in O$, there exists a γ-P_S-neighbourhood P of X such that $x \in P$ and $f(P) \subseteq O$.

Proof. It is clear and hence it is omitted. □

Theorem 5.3.7. Let γ be an operation on (X,τ). A function $f\colon (X,\tau) \to (Y,\sigma)$ is γ-P_S-continuous if and only if the inverse image of every neighbourhood of $f(x)$ is γ-P_S-neighbourhood of $x \in X$.

Proof. The proof follows from Lemma 5.3.6. □

Theorem 5.3.8. The following properties are equivalent for any function $f\colon (X,\tau) \to (Y,\sigma)$, where γ and β are operations on τ and σ respectively.

1. f is (γ,β)-P_S-irresolute.

2. For every $x \in X$ and for every β-P_S-neighbourhood N of Y such that $f(x) \in N$, there exists a γ-P_S-neighbourhood M of X such that $x \in M$ and $f(M) \subseteq N$.

3. The inverse image of every β-P_S-neighbourhood of $f(x)$ is γ-P_S-neighbourhood of $x \in X$.

Proof. (1) $\Leftrightarrow$ (2). The proof is similar to Lemma 5.3.6.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Leftrightarrow$ (1). The proof is similar to Theorem 5.3.7. □

Theorem 5.3.9. Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function and γ and β be operations on τ and σ respectively. Then the following statements are equivalent:

1. f is (γ,β)-P_S-continuous.
2. $f^{-1}(V)$ is γ-P_S-open set in X, for every β-open set V in Y.
3. $f^{-1}(F)$ is γ-P_S-closed set in X, for every β-closed set F in Y.
4. $f(\tau_\gamma\text{-}P_SCl(A)) \subseteq \sigma_\beta\text{-}Cl(f(A))$, for every subset A of X.
5. $\sigma_\beta\text{-}Int(f(A)) \subseteq f(\tau_\gamma\text{-}P_SInt(A))$, for every subset A of X.
6. $f^{-1}(\sigma_\beta\text{-}Int(B)) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(B))$, for every subset B of Y.
7. $\tau_\gamma\text{-}P_SCl(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta\text{-}Cl(B))$, for every subset B of Y.

Proof. Similar to Theorem 5.3.1 and hence it is omitted. □

Theorem 5.3.10. The following properties are equivalent for any function $f\colon (X,\tau) \to (Y,\sigma)$, where γ and β are operations on τ and σ respectively.

1. f is (γ,β)-P_S-irresolute.

2. The inverse image of every β-P_S-open set of Y is γ-P_S-open set in X.

3. The inverse image of every β-P_S-closed set of Y is γ-P_S-closed set in X.

4. $f(\tau_\gamma\text{-}P_SCl(A)) \subseteq \sigma_\beta\text{-}P_SCl(f(A))$, for every subset A of X.

5. $\tau_\gamma\text{-}P_SCl(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta\text{-}P_SCl(B))$, for every subset B of Y.

6. $f^{-1}(\sigma_\beta\text{-}P_SInt(B)) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(B))$, for every subset B of Y.

7. $\sigma_\beta\text{-}P_SInt(f(A)) \subseteq f(\tau_\gamma\text{-}P_SInt(A))$, for every subset A of X.

Proof. Similar to Theorem 5.3.1. □

Definition 5.3.11. Let (X, τ) and (Y, σ) be two topological spaces and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is called β-open if for every open set V of X, $f(V)$ is β-open set in Y.

From the above definition, we have the following theorem.

Theorem 5.3.12. For a function $f\colon (X, \tau) \to (Y, \sigma)$ and β be an operation on σ, the following statements are equivalent:

1. f is β-open.

2. $Int(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta\text{-}Int(B))$, for every $B \subseteq Y$.

3. $f^{-1}(\sigma_\beta\text{-}Cl(B)) \subseteq Cl(f^{-1}(B))$, for every $B \subseteq Y$.

Proof. (1) $\Rightarrow$ (2). Let B be any subset of Y. Then $Int(f^{-1}(B)) \subseteq f^{-1}(B)$ and hence $f(Int(f^{-1}(B))) \subseteq B$. Since $Int(f^{-1}(B))$ is open set in X. Then by hypothesis, we have

$f(Int(f^{-1}(B)))$ is β-open set in Y. Hence $f(Int(f^{-1}(B))) \subseteq \sigma_\beta\text{-}Int(B)$. Therefore, $Int(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta\text{-}Int(B))$.

(2) $\Leftrightarrow$ (3). Let B be any subset of Y. Then apply (2) to $Y\backslash B$ we obtain $Int(f^{-1}(Y\backslash B)) \subseteq f^{-1}(\sigma_\beta\text{-}Int(Y\backslash B)) \Leftrightarrow Int(X\backslash f^{-1}(B)) \subseteq f^{-1}(Y\backslash \sigma_\beta\text{-}Cl(B)) \Leftrightarrow X\backslash Cl(f^{-1}(B)) \subseteq X\backslash f^{-1}(\sigma_\beta\text{-}Cl(B)) \Leftrightarrow f^{-1}(\sigma_\beta\text{-}Cl(B)) \subseteq Cl(f^{-1}(B))$. Thus, $f^{-1}(\sigma_\beta\text{-}Cl(B)) \subseteq Cl(f^{-1}(B))$.

(3) $\Rightarrow$ (1). Let V be any open set of X. Then $f(X\backslash V)$ is a subset of Y. By (3), we have $f^{-1}(\sigma_\beta\text{-}Cl(f(X\backslash V))) \subseteq Cl(f^{-1}(f(X\backslash V)))$ implies that $f^{-1}(\sigma_\beta\text{-}Cl(Y\backslash f(V)))$ $\subseteq Cl(X\backslash V))$ and hence $f^{-1}(Y\backslash \sigma_\beta\text{-}Int(f(V))) \subseteq X\backslash Int(V)$. This implies that $X\backslash f^{-1}(\sigma_\beta\text{-}Int(f(V))) \subseteq X\backslash V$ and hence $V \subseteq f^{-1}(\sigma_\beta\text{-}Int(f(V)))$. So $f(V) \subseteq \sigma_\beta\text{-}Int(f(V))$. This means that $f(V)$ is β-open set in Y. Hence f is β-open. $\square$

By using Theorem 5.3.12, we have Theorem 5.3.13 and Theorem 5.3.14.

Theorem 5.3.13. Let γ and β be operations on τ and σ respectively. If $f\colon (X,\tau) \to (Y,\sigma)$ is a γ-continuous and β-open function and V is β-preopen set of Y, then $f^{-1}(V)$ is γ-preopen set in X.

Proof. Let V be a β-preopen set of Y, then by Theorem 3.2.22, there exists a β-open set U in Y such that $V \subseteq U \subseteq \sigma_\beta\text{-}Cl(V)$. Then $f^{-1}(V) \subseteq f^{-1}(U) \subseteq f^{-1}(\sigma_\beta\text{-}Cl(V))$. Since U is β-open set, then U is open in (Y,σ). Since f is γ-continuous function, then by Theorem 2.4.12, $f^{-1}(U)$ is γ-open set in X. By Theorem 5.3.12, $f^{-1}(\sigma_\beta\text{-}Cl(V)) \subseteq Cl(f^{-1}(V))$ since f is β-open function. But by Remark 2.3.5, $Cl(f^{-1}(V)) \subseteq \tau_\gamma\text{-}Cl(f^{-1}(V))$. Hence we obtain that $f^{-1}(V) \subseteq f^{-1}(U) \subseteq \tau_\gamma\text{-}Cl(f^{-1}(V))$.

Therefore, by Theorem 3.2.22, $f^{-1}(V)$ is γ-preopen set in X. $\square$

Theorem 5.3.14. Let γ and β be operations on τ and σ respectively. If $f\colon (X,\tau) \to (Y,\sigma)$ is a γ-continuous and β-open function and F is β-semiclosed set of Y, then $f^{-1}(F)$ is γ-semiclosed set in X.

Proof. Let F be a β-semiclosed set of Y, then by Corollary 3.2.24, there exists a β-closed set E in Y such that $\sigma_\beta\text{-}Int(E) \subseteq F \subseteq E$. Then $f^{-1}(\sigma_\beta\text{-}Int(E)) \subseteq f^{-1}(F) \subseteq f^{-1}(E)$. Since E is β-closed set, then E is closed in (Y,σ). Since f is γ-continuous function, so by Theorem 2.4.12, $f^{-1}(E)$ is γ-closed set in X, and since f is β-open function, thus by Theorem 5.3.12, $Int(f^{-1}(E)) \subseteq f^{-1}(\sigma_\beta\text{-}Int(E))$. But by Remark 2.3.5, $\tau_\gamma\text{-}Int(f^{-1}(E)) \subseteq Int(f^{-1}(E))$. Hence we obtain that $\tau_\gamma\text{-}Int(f^{-1}(E)) \subseteq f^{-1}(F) \subseteq f^{-1}(E)$. Therefore, by Corollary 3.2.24, $f^{-1}(F)$ is γ-semiclosed set in X. $\square$

Theorem 5.3.15. Let γ and β be operations on τ and σ respectively. If $f\colon (X,\tau) \to (Y,\sigma)$ is a γ-continuous and β-open function and V is β-P_S-open set of Y, then $f^{-1}(V)$ is γ-P_S-open set in X.

Proof. Let V be a β-P_S-open set of Y, then V is a β-preopen set of Y and $V = \bigcup_{i\in I} F_i$ where F_i is β-semiclosed set in Y for each i. Then $f^{-1}(V) = f^{-1}(\bigcup_{i\in I} F_i) = \bigcup_{i\in I} f^{-1}(F_i)$ where F_i is β-semiclosed set in Y for each i. Since f is a γ-continuous and β-open function. Then by Theorem 5.3.13, $f^{-1}(V)$ is γ-preopen set of X and by Theorem 5.3.14, $f^{-1}(F_i)$ is γ-semiclosed set of X for each i. Hence by Theorem 4.2.2, $f^{-1}(V)$ is γ-P_S-open set in X. $\square$

Corollary 5.3.16. Let γ and β be operations on τ and σ respectively. If $f\colon (X,\tau) \to (Y,\sigma)$ is a γ-continuous and β-open function and F is β-P_S-closed set of Y, then $f^{-1}(F)$ is γ-P_S-closed set in X. $\square$

Theorem 5.3.17. Let γ and β be operations on τ and σ respectively. If a function $f\colon (X,\tau) \to (Y,\sigma)$ is both γ-continuous and β-open, then f is (γ,β)-P_S-irresolute.

Proof. The proof follows directly from Theorem 5.3.15 and Theorem 5.3.10. $\square$

Theorem 5.3.18. Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function and γ and β be operations on τ and σ respectively. Then the following properties are equivalent:

1. f is (γ,β)-P_S-irresolute.
2. τ_γ-$P_SBd(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SBd(B))$, for each subset B of Y.
3. $f(\tau_\gamma$-$P_SBd(A)) \subseteq \sigma_\beta$-$P_SBd(f(A))$, for each subset A of X.

Proof. (1) $\Rightarrow$ (2). Let f be a (γ,β)-P_S-irresolute function and B be any subset of (Y,σ). Then by Theorem 5.3.10 (2) and (5), we have

$$\begin{aligned}
\tau_\gamma\text{-}P_SBd(f^{-1}(B)) &= \tau_\gamma\text{-}P_SCl(f^{-1}(B)) \backslash \tau_\gamma\text{-}P_SInt(f^{-1}(B)) \\
&\subseteq f^{-1}(\sigma_\beta\text{-}P_SCl(B)) \backslash \tau_\gamma\text{-}P_SInt(f^{-1}(B)) \\
&\subseteq f^{-1}(\sigma_\beta\text{-}P_SCl(B)) \backslash \tau_\gamma\text{-}P_SInt(f^{-1}(\sigma_\beta\text{-}P_SInt(B))) \\
&= f^{-1}(\sigma_\beta\text{-}P_SCl(B)) \backslash f^{-1}(\sigma_\beta\text{-}P_SInt(B)) \\
&= f^{-1}(\sigma_\beta\text{-}P_SCl(B) \backslash \sigma_\beta\text{-}P_SInt(B)) = f^{-1}(\sigma_\beta\text{-}P_SBd(B)).
\end{aligned}$$

Therefore, τ_γ-$P_SBd(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SBd(B))$.

(2) $\Rightarrow$ (3). Let A be any subset of X. Then $f(A)$ is a subset of Y. Then by (2), we have τ_γ-$P_SBd(f^{-1}(f(A))) \subseteq f^{-1}(\sigma_\beta$-$P_SBd(f(A)))$ implies that τ_γ-$P_SBd(A) \subseteq f^{-1}(\sigma_\beta$-$P_SBd(f(A)))$ and hence $f(\tau_\gamma$-$P_SBd(A)) \subseteq \sigma_\beta$-$P_SBd(f(A))$. This completes the proof.

(3) $\Rightarrow$ (1). Let E be any β-P_S-closed set in Y. Then $f^{-1}(E)$ is a subset of X. So by using part (3), we have $f(\tau_\gamma$-$P_SBd(f^{-1}(E))) \subseteq \sigma_\beta$-$P_SBd(f(f^{-1}(E))) = \sigma_\beta$-$P_SBd(E)$ implies that $f(\tau_\gamma$-$P_SBd(f^{-1}(E))) \subseteq \sigma_\beta$-$P_SBd(E) \subseteq \sigma_\beta$-$P_SCl(E) = E$ and hence $f(\tau_\gamma$-$P_SBd(f^{-1}(E))) \subseteq E$. This implies that τ_γ-$P_SBd(f^{-1}(E)) \subseteq f^{-1}(E)$. Thus, by Theorem 4.4.43 (3), $f^{-1}(E)$ is γ-P_S-closed set in X. Consequently by Theorem 5.3.10, f is (γ, β)-P_S-irresolute function. □

Theorem 5.3.19. Let $f\colon (X, \tau) \to (Y, \sigma)$ be any function and γ and β be operations on τ and σ respectively. Then the following properties are equivalent:

1. f is (γ, β)-P_S-continuous.

2. For each subset A in X, $f(\tau_\gamma$-$P_SBd(A)) \subseteq \sigma_\beta$-$Bd(f(A))$.

3. For each subset B in Y, τ_γ-$P_SBd(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$Bd(B))$.

Proof. The proof is similar to Theorem 5.3.18, and hence it is omitted. □

Theorem 5.3.20. Let $f\colon (X, \tau) \to (Y, \sigma)$ be any function and γ and β be operations on τ and σ respectively. Then the following properties are equivalent:

1. f is (γ, β)-P_S-irresolute.

2. $f(\tau_\gamma$-$P_SD(A)) \subseteq \sigma_\beta$-$P_SCl(f(A))$, for each subset A of X.

3. τ_γ-$P_SD(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SCl(B))$, for each subset B of Y.

Proof. (1) $\Rightarrow$ (2). Let A be any subset of X and f be a (γ, β)-P_S-continuous function. Then by Theorem 5.3.10 (4), we have $f(\tau_\gamma$-$P_SCl(A)) \subseteq \sigma_\beta$-$P_SCl(f(A))$. Then by Corollary 4.4.15, we obtain $f(\tau_\gamma$-$P_SD(A)) \subseteq f(\tau_\gamma$-$P_SCl(A))$ which implies that $f(\tau_\gamma$-$P_SD(A)) \subseteq \sigma_\beta$-$P_SCl(f(A))$.

(2) $\Rightarrow$ (3). Let B be any subset of Y. Then $f^{-1}(B)$ is a subset of X. Then by hypothesis, we get $f(\tau_\gamma$-$P_SD(f^{-1}(B))) \subseteq \sigma_\beta$-$P_SCl(f(f^{-1}(B))) = \sigma_\beta$-$P_SCl(B)$ and hence τ_γ-$P_SD(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SCl(B))$. This completes the proof.

(3) $\Rightarrow$ (1). Let F be any β-P_S-closed set in Y. Then by (3), we have τ_γ-$P_SD(f^{-1}(F))$ $\subseteq f^{-1}(\sigma_\beta$-$P_SCl(F)) = f^{-1}(F)$ and hence τ_γ-$P_SD(f^{-1}(F)) \subseteq f^{-1}(F)$. So by Theorem 4.4.16, we get $f^{-1}(F)$ is γ-P_S-closed set in X. Therefore, by Theorem 5.3.10, f is (γ, β)-P_S-irresolute function. □

Theorem 5.3.21. Let $f\colon (X, \tau) \to (Y, \sigma)$ be any function and γ and β be operations on τ and σ respectively. Then the following properties are equivalent:

1. f is (γ, β)-P_S-continuous.

2. $f(\tau_\gamma$-$P_SD(A)) \subseteq \sigma_\beta$-$Cl(f(A))$, for each subset A of X.

3. τ_γ-$P_SD(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$Cl(B))$, for each subset B of Y.

Proof. The proof is similar to Theorem 5.3.20, and hence it is omitted. □

Theorem 5.3.22. Let $f\colon (X, \tau) \to (Y, \sigma)$ be a surjection function and γ be an operation on τ, then the following statements are equivalent:

1. f is γ-P_S-continuous.

2. $f^{-1}(Int(B)) \subseteq Int(Cl(f^{-1}(B)))$ and $f^{-1}(Int(B)) = \bigcup_{i\in I} F_i$ where F_i is γ-semiclosed set in X, for every subset B in Y.

3. $Cl(Int(f^{-1}(B))) \subseteq f^{-1}(Cl(B))$ and $f^{-1}(Cl(B)) = \bigcap_{i\in I} G_i$ where G_i is γ-semiopen set in X, for every subset B in Y.

4. $f(Cl(Int(A))) \subseteq Cl(f(A))$ and $f^{-1}(Cl(f(A))) = \bigcap_{i\in I} G_i$ where G_i is γ-semiopen set in X, for every subset A in X.

5. $Int(f(A)) \subseteq f(Int(Cl(A)))$ and $f^{-1}(Int(f(A))) = \bigcup_{i\in I} F_i$ where F_i is γ-semiclosed set in X, for every subset A in X.

Proof. It is enough to proof (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1) since the others are obvious.

(1) $\Rightarrow$ (2). Let B be any subset in Y. Then $Int(B)$ is open set in Y. Since f is γ-P_S-continuous, then by Theorem 5.3.1, $f^{-1}(Int(B))$ is γ-P_S-open set in X. By Theorem 4.2.2, we obtain $f^{-1}(Int(B))$ is γ-preopen set in X and $f^{-1}(Int(B)) = \bigcup_{i\in I} F_i$ where F_i is γ-semiclosed set in X, for every subset B in Y. Therefore, $f^{-1}(Int(B)) \subseteq Int(Cl(f^{-1}(Int(B))))$ and $f^{-1}(Int(B)) = \bigcup_{i\in I} F_i$ where F_i is γ-semiclosed set in X. Thus $f^{-1}(Int(B)) \subseteq Int(Cl(f^{-1}(B)))$ and $f^{-1}(Int(B)) = \bigcup_{i\in I} F_i$ where F_i is γ-semiclosed set in X.

(5) $\Rightarrow$ (1). Let V be any open set in Y. Then $f^{-1}(V)$ is a subset of X. By (5), we get $Int(f(f^{-1}(V))) \subseteq f(Int(Cl(f^{-1}(V))))$ and $f^{-1}(Int(f(f^{-1}(V)))) = \bigcup_{i\in I} F_i$ where F_i is γ-semiclosed set in X. Hence $Int(V) \subseteq f(Int(Cl(f^{-1}(V))))$ and $f^{-1}(Int(V)) =$

$\bigcup_{i \in I} F_i$ where F_i is γ-semiclosed set in X. This implies that $V \subseteq f(Int(Cl(f^{-1}(V))))$ and $f^{-1}(V) = \bigcup_{i \in I} F_i$ where F_i is γ-semiclosed set in X and hence $f^{-1}(V) \subseteq Int(Cl(f^{-1}(V)))$ and $f^{-1}(V) = \bigcup_{i \in I} F_i$ where F_i is γ-semiclosed set in X. So $f^{-1}(V)$ is γ-preopen set in X and $f^{-1}(V) = \bigcup_{i \in I} F_i$ where F_i is γ-semiclosed set in X. Therefore, by Theorem 4.2.2, $f^{-1}(V)$ is γ-P_S-open set in X and hence by Theorem 5.3.1, f is γ-P_S-continuous. □

Now, the proofs for the next two theorems are similar to Theorem 5.3.22 and are thus omitted.

Theorem 5.3.23. Let $f\colon (X, \tau) \to (Y, \sigma)$ be a surjection function and γ and β be operations on τ and σ respectively. Then the following statements are equivalent:

1. f is (γ, β)-P_S-continuous.

2. $f^{-1}(\sigma_\beta\text{-}Int(B)) \subseteq Int(Cl(f^{-1}(B)))$ and $f^{-1}(\sigma_\beta\text{-}Int(B)) = \bigcup_{i \in I} F_i$ where F_i is γ-semiclosed set in X, for every subset B in Y.

3. $\sigma_\beta\text{-}Int(f(A)) \subseteq f(Int(Cl(A)))$ and $f^{-1}(\sigma_\beta\text{-}Int(f(A))) = \bigcup_{i \in I} F_i$ where F_i is γ-semiclosed set in X, for every subset A in X.

4. $Cl(Int(f^{-1}(B))) \subseteq f^{-1}(\sigma_\beta\text{-}Cl(B))$ and $f^{-1}(\sigma_\beta\text{-}Cl(B)) = \bigcap_{i \in I} G_i$ where G_i is γ-semiopen set in X, for every subset B in Y.

5. $f(Cl(Int(A))) \subseteq \sigma_\beta\text{-}Cl(f(A))$ and $f^{-1}(\sigma_\beta\text{-}Cl(f(A))) = \bigcap_{i \in I} G_i$ where G_i is γ-semiopen set in X, for every subset A in X. □

Theorem 5.3.24. Let $f\colon (X,\tau) \to (Y,\sigma)$ be a surjection function and γ and β be operations on τ and σ respectively. Then the following properties are equivalent:

1. f is (γ,β)-P_S-irresolute.

2. $f^{-1}(\sigma_\beta\text{-}P_S Int(B)) \subseteq Int(Cl(f^{-1}(B)))$ and $f^{-1}(\sigma_\beta\text{-}P_S Int(B)) = \bigcup_{i\in I} F_i$ where F_i is γ-semiclosed set in X, for every subset B in Y.

3. $Cl(Int(f^{-1}(B))) \subseteq f^{-1}(\sigma_\beta\text{-}P_S Cl(B))$ and $f^{-1}(\sigma_\beta\text{-}P_S Cl(B)) = \bigcap_{i\in I} G_i$ where G_i is γ-semiopen set in X, for every subset B in Y.

4. $f(Cl(Int(A))) \subseteq \sigma_\beta\text{-}P_S Cl(f(A))$ and $f^{-1}(\sigma_\beta\text{-}P_S Cl(f(A))) = \bigcap_{i\in I} G_i$ where G_i is γ-semiopen set in X, for every subset A in X.

5. $\sigma_\beta\text{-}P_S Int(f(A)) \subseteq f(Int(Cl(A)))$ and $f^{-1}(\sigma_\beta\text{-}P_S Int(f(A))) = \bigcup_{i\in I} F_i$ where F_i is γ-semiclosed set in X, for every subset A in X. □

Theorem 5.3.25. A function $f\colon (X,\tau) \to (Y,\sigma)$ with an operation γ on τ is γ-P_S-continuous if and only if f is γ-precontinuous and for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a γ-semiclosed set F in X containing x such that $f(F) \subseteq V$.

Proof. Let $x \in X$ and let V be any open set of Y containing $f(x)$. Since f is γ-P_S-continuous, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. Since U is γ-P_S-open set. Then for each $x \in U$, there exists a γ-semiclosed set F of X such that $x \in F \subseteq U$. Therefore, we get $f(F) \subseteq V$. And also since f is γ-P_S-continuous. Then f is γ-precontinuous.

Conversely, let V be any open set of Y. We have to show that $f^{-1}(V)$ is γ-P_S-open set in X. Since f is γ-precontinuous, then by Theorem 2.4.11, $f^{-1}(V)$ is γ-preopen set in X. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By hypothesis, there exists a γ-semiclosed set F of X containing x such that $f(F) \subseteq V$. Which implies that $x \in F \subseteq f^{-1}(V)$. Therefore, by Definition 4.2.1, $f^{-1}(V)$ is γ-P_S-open set in X. Hence by Theorem 5.3.1, f is γ-P_S-continuous. □

Theorem 5.3.26. If a function $f\colon (X, \tau) \rightarrow (Y, \sigma)$ with an operation γ on τ is γ-P_S-continuous, then for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a γ-semiclosed set F in X such that $x \in F$ and $f(F) \subseteq V$.

Proof. Suppose f be a γ-P_S-continuous function and let V be any open set of Y such that $f(x) \in V$, for each $x \in X$. Then there exists a γ-P_S-open set U of X such that $x \in U$ and $f(U) \subseteq V$. Since U is γ-P_S-open set. Then for each $x \in U$, there exists a γ-semiclosed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$. This completes the proof. □

Theorem 5.3.27. If a function $f\colon (X, \tau) \rightarrow (Y, \sigma)$ is (γ, β)-P_S-irresolute, then for each $x \in X$ and each β-P_S-open set V of Y containing $f(x)$, there exists a γ-semiclosed set F in X such that $x \in F$ and $f(F) \subseteq V$. Furthermore, if f is (γ, β)-precontinuous, then the converse also holds.

Proof. Suppose f be a (γ, β)-P_S-irresolute function and let V be any β-P_S-open set of Y such that $f(x) \in V$, for each $x \in X$. Then there exists a γ-P_S-open set U of X such that $x \in U$ and $f(U) \subseteq V$. Since U is γ-P_S-open set. Then for each $x \in U$, there exists a γ-semiclosed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$.

Now suppose that f is (γ, β)-precontinuous function. Let V be any β-P_S-open set of Y, then $f^{-1}(V)$ is γ-preopen set in X since every β-P_S-open set of Y is β-preopen in Y. We have to show that $f^{-1}(V)$ is γ-P_S-open set in X. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By hypothesis, there exists a γ-semiclosed set F of X containing x such that $f(F) \subseteq V$. Which implies that $x \in F \subseteq f^{-1}(V)$. Therefore, by Definition 4.2.1, $f^{-1}(V)$ is γ-P_S-open set in X. Hence by Theorem 5.3.10, f is (γ, β)-P_S-irresolute. This completes the proof. □

Theorem 5.3.28. Let γ and β be operations on τ and σ respectively. If a function $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-continuous (respectively (γ, β)-P_S-irresolute), then the following properties are true:

1. for each $x \in X$ and each β-open (respectively β-P_S-open) set V of Y such that $f(x) \in V$, there exists a γ-preopen set U of X such that $x \in U$ and $f(U) \subseteq V$.

2. for each $x \in X$ and each β-regular-open set V of Y containing $f(x)$, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$.

Proof. 1) Since every γ-P_S-open set of X is γ-preopen, then it is straightforward by using Definition 5.2.2 (respectively Definition 5.2.3).

2) Since every β-regular-open set of Y is both β-open and β-P_S-open, then the proof follows directly from Definition 5.2.2 and Definition 5.2.3. □

Theorem 5.3.29. Let γ and β be operations on τ and σ respectively. A function $f\colon (X,\tau) \to (Y,\sigma)$ is (γ,β)-P_S-continuous (respectively, (γ,β)-P_S-irresolute), if the following properties are true:

1. for each $x \in X$ and each β-preopen set V of Y such that $f(x) \in V$, there exists a γ-P_S-open set U of X such that $x \in U$ and $f(U) \subseteq V$.

2. for each $x \in X$ and each β-open (respectively, β-P_S-open) set V of Y containing $f(x)$, there exists a γ-regular-open set U of X containing x such that $f(U) \subseteq V$.

Proof. 1) The proof is clear since every β-open (respectively, β-P_S-open) set of Y is β-preopen.

2) Obvious since every γ-regular-open set of X is both γ-open and γ-P_S-open and hence it is omitted. □

Theorem 5.3.30. For any operation γ on τ and $f\colon (X,\tau) \to (Y,\sigma)$ be any function, $X\backslash\tau_\gamma$-$P_SC(f) = \cup\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is an open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$, where τ_γ-$P_SC(f)$ denotes the set of points at which f is γ-P_S-continuous function.

Proof. Let $x \in \tau_\gamma$-$P_SC(f)$. Then there exists open set V in (Y,σ) containing $f(x)$ such that $f(U) \not\subseteq V$ for every γ-P_S-open set U of (X,τ) containing x. Hence $U \cap X\backslash f^{-1}(V) \neq \phi$ for every γ-P_S-open set U of (X,τ) containing x. Therefore, by Theorem 4.4.13, $x \in \tau_\gamma$-$P_SCl(X\backslash f^{-1}(V))$. Then $x \in f^{-1}(V) \cap \tau_\gamma$-$P_SCl(X\backslash f^{-1}(V)) \subseteq \tau_\gamma$-$P_SCl(f^{-1}(V)) \cap \tau_\gamma$-$P_SCl(X\backslash f^{-1}(V)) = \tau_\gamma$-$P_SBd(f^{-1}(V))$. Then $X\backslash\tau_\gamma$-$P_SC(f) \subseteq \cup\,\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$.

Conversely, let $x \notin X\backslash\tau_\gamma$-$P_SC(f)$. Then for each open set V in (Y,σ) containing $f(x)$, $f^{-1}(V)$ is γ-P_S-open set of (X,τ) containing x. Hence $x \in \tau_\gamma$-$P_SInt(f^{-1}(V))$ and hence $x \notin \tau_\gamma$-$P_SBd(f^{-1}(V))$ for every open set V in (Y,σ) containing $f(x)$. Therefore, $X\backslash\tau_\gamma$-$P_SC(f) \supseteq \cup\,\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$. □

Now, the proofs for the next two theorems are similar to Theorem 5.3.30 and are thus omitted.

Theorem 5.3.31. For any operations γ and β on τ and σ respectively and $f\colon (X,\tau) \to (Y,\sigma)$ be any function, $X\backslash\tau_\gamma$-$P_SC(f) = \cup\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is a β-open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$, where τ_γ-$P_SC(f)$ denotes the set of points at which f is (γ,β)-P_S-continuous function. □

Theorem 5.3.32. For any operations γ and β on τ and σ respectively and $f\colon (X,\tau) \to (Y,\sigma)$ be any function, $X\backslash\tau_\gamma$-$P_SC(f) = \cup\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is a β-P_S-open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$, where τ_γ-$P_SC(f)$ denotes the set of points at which f is (γ,β)-P_S-irresolute function. □

Some properties of completely γ-continuous functions are mentioned in the following two results.

Theorem 5.3.33. For a function $f\colon (X,\tau) \to (Y,\sigma)$ and γ be an operation on τ, the following statements are equivalent:

1. f is completely γ-continuous.

2. $f^{-1}(V)$ is γ-regular-open set in X, for every open set V in Y.

3. $f^{-1}(F)$ is γ-regular-closed set in X, for every closed set F in Y.

Proof. The proof is similar to Theorem 5.3.1 and hence it is omitted. □

Corollary 5.3.34. Let γ and β be operations on τ and σ respectively. If a function $f\colon (X,\tau) \to (Y,\sigma)$ with an operation γ on τ is completely γ-continuous, then the following properties are true:

1. for each $x \in X$ and each regular-open set V of Y containing $f(x)$, there exists a γ-regular-open set U of X containing x such that $f(U) \subseteq V$.

2. for each $x \in X$ and each β-open set V of Y such that $f(x) \in V$, there exists a γ-regular-open set U of X such that $x \in U$ and $f(U) \subseteq V$.

Proof. 1) Since every regular-open set of Y is open, then it is obvious using Definition 5.2.15 to obtain the proof.

2) Obvious since every β-open set of Y is open. □

By using the sets in Definition 4.4.37, we have the following new functions.

Definition 5.3.35. Let (X,τ) and (Y,σ) be two topological spaces and γ be an operation on τ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called τ-γ-P_S-continuous (respectively, γ-γ-P_S-continuous) if for each open set V of Y, $f^{-1}(V)$ is τ-γ-P_S-open (respectively, γ-γ-P_S-open) sets in X.

The relation between the functions in Definition 5.3.35, γ-continuous function and γ-P_S-continuous function are shown in the Theorem 5.3.36 and Theorem 5.3.37.

Theorem 5.3.36. For a function $f\colon (X,\tau) \to (Y,\sigma)$ and γ be an operation on τ, the following statements are equivalent:

1. f is γ-γ-P_S-continuous and γ-P_S-continuous.

2. f is γ-γ-P_S-continuous and γ-continuous.

3. f is γ-P_S-continuous and γ-continuous.

Proof. The proof follows directly from Theorem 4.4.38. □

Theorem 5.3.37. For a function $f\colon (X,\tau) \to (Y,\sigma)$ and γ be an operation on τ, the following statements are equivalent:

1. f is τ-γ-P_S-continuous and γ-P_S-continuous.

2. f is τ-γ-P_S-continuous and continuous.

3. f is γ-P_S-continuous and continuous.

Proof. Follows directly from Theorem 4.4.39. □

Theorem 5.3.38. Let (X,τ) be γ-regular space and γ be an operation on τ. Then a function $f\colon (X,\tau) \to (Y,\sigma)$ is γ-γ-P_S-continuous if and only if f is τ-γ-P_S-continuous.

Proof. This is an immediate consequence of Theorem 4.4.40. □

Corollary 5.3.39. Let (X, τ) be γ-regular space and γ be an operation on τ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is γ-P_S-continuous if and only if f is P_S-continuous.

Proof. This is an immediate consequence of Theorem 4.2.18. $\square$

Corollary 5.3.40. Let (X, τ) be γ-regular space and γ be an operation on τ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is completely γ-continuous if and only if f is completely continuous.

Proof. This is an immediate consequence of Remark 3.2.7. $\square$

Corollary 5.3.41. Let (Y, σ) be β-regular space and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is γ-P_S-continuous if and only if f is (γ, β)-P_S-continuous.

Proof. Follows directly from Remark 2.3.31. $\square$

Corollary 5.3.42. Let (X, τ) be γ-semiT_1 space and γ be an operation on τ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is γ-P_S-continuous if and only if f is γ-precontinuous.

Proof. This is an immediate consequence of Theorem 4.2.24. $\square$

Corollary 5.3.43. Let (X, τ) be γ-semiT_1 space and (Y, σ) be β-semiT_1 space, where γ and β be operations on τ and σ respectively. A function $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-irresolute if and only if f is (γ, β)-precontinuous.

Proof. This is an immediate consequence of Theorem 4.2.24. $\square$

Corollary 5.3.44. Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function and let (X, τ) be γ-locally indiscrete topological spaces and γ be an operation on τ. Then f is γ-P_S-continuous if and only if f is γ-continuous.

Proof. This is an immediate consequence of Theorem 4.2.19. □

Corollary 5.3.45. Let (Y, σ) be β-locally indiscrete space and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-irresolute if and only if f is (γ, β)-P_S-continuous.

Proof. Follows directly from Theorem 4.2.19. □

Corollary 5.3.46. Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function and β be a regular operation on σ. If (Y, σ) is β-hyperconnected space, then f is (γ, β)-P_S-irresolute.

Proof. This is an immediate consequence of Theorem 4.2.25. □

Now, more γ-P_S- functions are defined by using γ-P_S-open set.

Definition 5.3.47. Let (X, τ) and (Y, σ) be two topological spaces, and γ and β be operations on τ and σ respectively. A function $f\colon (X, \tau) \to (Y, \sigma)$ is called β-P_S-open (repectively, $(\gamma, \beta P_S)$-open and (γ, β)-P_S-open) if for every open (repectively, γ-open and γ-P_S-open) set V of X, $f(V)$ is β-P_S-open set in Y.

Definition 5.3.48. Let (X, τ) and (Y, σ) be two topological spaces, and γ and β be operations on τ and σ respectively. A function $f\colon (X, \tau) \to (Y, \sigma)$ is called β-P_S-closed (repectively, $(\gamma, \beta P_S)$-closed and (γ, β)-P_S-closed) if for every closed (repectively, γ-closed and γ-P_S-closed) set F of X, $f(F)$ is β-P_S-closed set in Y.

Remark 5.3.49. Every β-P_S-open (respectively, β-P_S-closed) function is $(\gamma, \beta P_S)$-open (respectively, $(\gamma, \beta P_S)$-closed), but the converse is not true as it is shown in the following example.

Example 5.3.50. Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{c\}, \{b, c\}, X\}$ and $Y = \{1, 2, 3\}$ with the topology $\sigma = \{\phi, Y, \{2\}, \{1, 3\}\}$. Define operations β on σ by $\beta(B) = B$ for all $B \in \sigma$ and γ on τ as follows:

For every $A \in \tau$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{c\} \\ X & \text{otherwise} \end{cases}$$

Then $\tau_\gamma = \{\phi, X, \{c\}\}$ and hence τ_γ-$P_SO(X) = \{\phi, X\}$.

Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function defined as follows:

$$f(x) = \begin{cases} 3 & \text{if } x = a \\ 1 & \text{if } x = b \\ 2 & \text{if } x = c \end{cases}$$

So f is $(\gamma, \beta P_S)$-open (respectively, $(\gamma, \beta P_S)$-closed) function, but f is not β-P_S-open (respectively, β-P_S-closed) since $\{c\} \in \tau$, but $f(\{b, c\}) = \{1, 2\}$ is not β-P_S-open set in (Y, σ). Again since $\{a\}$ is closed set in (X, τ), but $f(\{a\}) = \{3\}$ is not β-P_S-closed set in (Y, σ).

Remark 5.3.51. Let β be an operation on (Y, σ). A function $f\colon (X, \tau) \to (Y, \sigma)$ is β-P_S-open if and only if for every $x \in X$ and for every neighbourhood N of x, there exists a β-P_S-neighbourhood M of Y such that $f(x) \in M$ and $M \subseteq f(N)$.

Remark 5.3.52. Let γ and β be operations on τ and σ respectively. A function $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-open if and only if for every $x \in X$ and for every

γ-P_S-neighbourhood N of x, there exists a β-P_S-neighbourhood M of Y such that $f(x) \in M$ and $M \subseteq f(N)$.

Theorem 5.3.53. The following statements are equivalent for a function $f\colon (X,\tau) \to (Y,\sigma)$ with an operation β on σ:

1. f is β-P_S-open.

2. $f(Int(A)) \subseteq \sigma_\beta$-$P_SInt(f(A))$, for every $A \subseteq X$.

3. $Int(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SInt(B))$, for every $B \subseteq Y$.

Proof. The proof is similar to Theorem 5.3.1. □

Theorem 5.3.54. The following properties of f are equivalent for a function $f\colon (X,\tau) \to (Y,\sigma)$ with an operation β on σ:

1. f is β-P_S-closed.

2. $f^{-1}(\sigma_\beta$-$P_SCl(B)) \subseteq Cl(f^{-1}(B))$, for every $B \subseteq Y$.

3. σ_β-$P_SCl(f(A)) \subseteq f(Cl(A))$, for every $A \subseteq X$.

4. σ_β-$P_SD(f(A)) \subseteq f(Cl(A))$, for every $A \subseteq X$.

Proof. The proof is similar to Theorem 5.3.1. □

Theorem 5.3.55. Let γ and β be operations on τ and σ respectively. For any function $f\colon (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:

1. f is (γ,β)-P_S-open.

2. $f(\tau_\gamma\text{-}P_SInt(A)) \subseteq \sigma_\beta\text{-}P_SInt(f(A))$, for every $A \subseteq X$.

3. $\tau_\gamma\text{-}P_SInt(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta\text{-}P_SInt(B))$, for every $B \subseteq Y$.

Proof. Similar to Theorem 5.3.10. □

Theorem 5.3.56. Let γ and β be operations on τ and σ respectively. For any function $f\colon (X,\tau) \to (Y,\sigma)$, the following conditions are equivalent:

1. f is (γ,β)-P_S-closed.

2. $f^{-1}(\sigma_\beta\text{-}P_SCl(B)) \subseteq \tau_\gamma\text{-}P_SCl(f^{-1}(B))$, for every $B \subseteq Y$.

3. $\sigma_\beta\text{-}P_SCl(f(A)) \subseteq f(\tau_\gamma\text{-}P_SCl(A))$, for every $A \subseteq X$.

4. $\sigma_\beta\text{-}P_SD(f(A)) \subseteq f(\tau_\gamma\text{-}P_SCl(A))$, for every $A \subseteq X$.

Proof. The proof is similar to Theorem 5.3.10. □

Definition 5.3.57. Let $id\colon \tau \to P(X)$ be the identity operation. If f is (id,β)-P_S-closed, then for every γ-P_S-closed set F of X, $f(F)$ is β-P_S-closed set in Y.

Theorem 5.3.58. If a function f is bijective and $f^{-1}\colon (Y,\sigma) \to (X,\tau)$ is (id,β)-P_S-irresolute, then f is (id,β)-P_S-closed.

Proof. Follows from Definition 5.2.3 and Definition 5.3.57. □

Theorem 5.3.59. Let γ and β be operations on τ and σ respectively. Suppose that a function $f\colon (X,\tau) \to (Y,\sigma)$ is both (γ,β)-P_S-irresolute and (γ,β)-P_S-closed, then:

1. For every γ-P_S-g-closed set A of (X,τ), the image $f(A)$ is β-P_S-g-closed in (Y,σ).

2. For every β-P_S-g-closed set B of (Y, σ) the inverse set $f^{-1}(B)$ is γ-P_S-g-closed in (X, τ).

Proof. (1) Let G be any β-P_S-open set in (Y, σ) such that $f(A) \subseteq G$. Since f is (γ, β)-P_S-irresolute function, then by Theorem 5.3.10 (2), $f^{-1}(G)$ is γ-P_S-open set in (X, τ). Since A is γ-P_S-g-closed and $A \subseteq f^{-1}(G)$, we have τ_γ-$P_SCl(A) \subseteq f^{-1}(G)$, and hence $f(\tau_\gamma$-$P_SCl(A)) \subseteq G$. Since τ_γ-$P_SCl(A)$ is γ-P_S-closed set and f is (γ, β)-P_S-closed, then $f(\tau_\gamma$-$P_SCl(A))$ is β-P_S-closed set in Y. Therefore, σ_β-$P_SCl(f(A)) \subseteq \sigma_\beta$-$P_SCl(f(\tau_\gamma$-$P_SCl(A))) = f(\tau_\gamma$-$P_SCl(A)) \subseteq G$. This implies that $f(A)$ is β-P_S-g-closed in (Y, σ).

(2) Let H be any γ-P_S-open set of (X, τ) such that $f^{-1}(B) \subseteq H$. Let $C = \tau_\gamma$-$P_SCl(f^{-1}(B)) \cap (X \backslash H)$, then C is γ-P_S-closed set in (X, τ). Since f is (γ, β)-P_S-closed function. Then $f(C)$ is β-P_S-closed in (Y, σ). Since f is (γ, β)-P_S-irresolute function, then by using Theorem 5.3.10 (4), we have $f(C) = f(\tau_\gamma$-$P_SCl(f^{-1}(B))) \cap f(X \backslash H) \subseteq \sigma_\beta$-$P_SCl(B) \cap f(X \backslash H) \subseteq \sigma_\beta$-$P_SCl(B) \cap (Y \backslash B)$. This implies that $f(C) = \phi$, and hence $C = \phi$. So τ_γ-$P_SCl(f^{-1}(B)) \subseteq H$. Therefore, $f^{-1}(B)$ is γ-P_S-g-closed in (X, τ). □

Remark 5.3.60. Every β-P_S-open (respectively, β-P_S-closed) function is β-preopen (respectively, β-preclosed), but the converse is not true as it is shown in the following example.

Example 5.3.61. Let $X = \{a, b, c\}$ with the topologies $\tau = \{\phi, \{c\}, \{b, c\}, X\}$ and $\sigma = \{\phi, X, \{b\}, \{a, c\}\}$. Define an operation β on σ by $\beta(A) = A$ for all $A \in \sigma$. Let

$f\colon (X,\tau) \to (X,\sigma)$ be a function defined as follows:

$$f(x) = \begin{cases} a & \text{if } x = a \\ c & \text{if } x = b \\ b & \text{if } x = c \end{cases}$$

Then f is both β-preopen and β-preclosed, but f is not β-P_S-open and β-P_S-closed function since $\{b,c\} \in \tau$ and $\{a\}$ is closed set in (X,τ), but $f(\{b,c\}) = \{b,c\}$ is not β-P_S-open set in (X,σ) and $f(\{a\}) = \{a\}$ is not β-P_S-closed set in (X,σ), respectively.

Corollary 5.3.62. Let (Y,σ) be β-semiT_1 space and β be an operation on σ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is β-P_S-open if and only if f is β-preopen.

Proof. Follows from Theorem 4.2.24. □

Corollary 5.3.63. Let (X,τ) be γ-semiT_1 space and (Y,σ) be β-semiT_1 space. A function $f\colon (X,\tau) \to (Y,\sigma)$ is (γ,β)-P_S-closed if and only if f is (γ,β)-preclosed.

Proof. This is an immediate consequence of Theorem 4.2.24. □

Corollary 5.3.64. Let (Y,σ) be β-locally indiscrete topological spaces and β be an operation on σ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is β-P_S-open if and only if f is β-open.

Proof. Follows from Theorem 4.2.19. □

Theorem 5.3.65. Let (X,τ) be γ-locally indiscrete space and $f\colon (X,\tau) \to (Y,\sigma)$ be a function, then the following properties of f are equivalent:

1. (γ,β)-P_S-open.

2. (γ, β)-P_S-closed.

3. $(\gamma, \beta P_S)$-closed.

4. $(\gamma, \beta P_S)$-open.

Proof. Follows directly from Theorem 4.2.19, Corollary 4.3.9, Theorem 4.3.10 and Corollary 4.3.11. □

Theorem 5.3.66. Let (X, τ) be γ-regular space. A function $f\colon (X, \tau) \to (Y, \sigma)$ is $(\gamma, \beta P_S)$-open (respectively, $(\gamma, \beta P_S)$-closed) if and only if f is β-P_S-open (respectively, β-P_S-closed).

Proof. This is an immediate consequnce of Remark 2.3.31. □

Corollary 5.3.67. Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function and γ be a regular operation on τ. If (X, τ) is γ-hyperconnected space, then f is (γ, β)-P_S-open.

Proof. This is an immediate consequence of Theorem 4.2.25. □

Theorem 5.3.68. Let (Y, σ) be a topological space and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is β-P_S-closed if and only if for each subset S of Y and each open set O in X containing $f^{-1}(S)$, there exists a β-P_S-open set R in Y containing S such that $f^{-1}(R) \subseteq O$.

Proof. Suppose that f is β-P_S-closed function and let O be an open set in X containing $f^{-1}(S)$, where S is any subset in Y. Then $f(X \backslash O)$ is β-P_S-open set in Y. If we put $R = Y \backslash f(X \backslash O)$. Then R is β-P_S-closed set in Y such that $S \subseteq R$ and $f^{-1}(R) \subseteq O$.

Conversely, let F be closed set in X. Let $S = Y\backslash f(F) \subseteq Y$. Then $f^{-1}(S) \subseteq X\backslash F$ and $X\backslash F$ is open set in X. By hypothesis, there exists a β-P_S-open set R in Y such that $S = Y\backslash f(F) \subseteq R$ and $f^{-1}(R) \subseteq X\backslash F$. For $f^{-1}(R) \subseteq X\backslash F$ implies $R \subseteq f(X\backslash F) \subseteq Y\backslash f(F)$. Hence $R = Y\backslash f(F)$. Since R is β-P_S-open set in Y. Then $f(F)$ is β-P_S-closed set in Y. Therefore, f is β-P_S-closed function. □

Theorem 5.3.69. Let (Y, σ) be a topological space and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-closed if and only if for each subset S of Y and each γ-P_S-open set O in X containing $f^{-1}(S)$, there exists a β-P_S-open set R in Y containing S such that $f^{-1}(R) \subseteq O$.

Proof. The proof is similar to Theorem 5.3.68. □

Theorem 5.3.70. A function $f\colon (X, \tau) \to (Y, \sigma)$ is $(\gamma, \beta P_S)$-closed if and only if for each subset S of Y and each γ-open set O in X containing $f^{-1}(S)$, there exists a β-P_S-open set R in Y containing S such that $f^{-1}(R) \subseteq O$.

Proof. The proof is similar to Theorem 5.3.68. □

Definition 5.3.71. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be γ-P_S-homeomorphism, if f is bijective, γ-P_S-continuous and f^{-1} is γ-P_S-continuous.

Definition 5.3.72. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be (γ, β)-P_S-homeomorphism, if f is bijective, (γ, β)-P_S-irresolute and f^{-1} is (γ, β)-P_S-irresolute.

Theorem 5.3.73. The following statements are equivalent for a bijective function $f\colon (X, \tau) \to (Y, \sigma)$ with an operation β on σ.

1. f is β-P_S-closed.

2. f is β-P_S-open.

3. f^{-1} is β-P_S-continuous.

Proof. It is clear. □

Theorem 5.3.74. For a bijective function $f\colon (X,\tau) \to (Y,\sigma)$ with an operation γ on τ. The following properties of f are equivalent:

1. f^{-1} is (γ,β)-P_S-irresolute.

2. f is (γ,β)-P_S-open.

3. f is (γ,β)-P_S-closed.

Proof. Obvious. □

Theorem 5.3.75. The following conditions of f are equivalent for a bijective function $f\colon (X,\tau) \to (Y,\sigma)$ and for operations γ on τ and β on σ.

1. f is (γ,β)-P_S-homeomorphism.

2. f is (γ,β)-P_S-irresolute and (γ,β)-P_S-open.

3. f is (γ,β)-P_S-irresolute and (γ,β)-P_S-closed.

4. $f(\tau_\gamma\text{-}P_SCl(A)) = \sigma_\beta\text{-}P_SCl(f(A))$ for each subset A of X.

Proof. Straightforward. □

Definition 5.3.76. Let (X, τ) and (Y, σ) be two topological spaces and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is called σ-β-P_S-open (repectively β-β-P_S-open) if for every open set V of X, $f(V)$ is σ-β-P_S-open (repectively β-β-P_S-open) sets in Y.

Theorem 5.3.77. For a function $f\colon (X, \tau) \to (Y, \sigma)$ and β be an operation on σ, the following statements are equivalent:

1. f is β-β-P_S-open (repectively, β-β-P_S-closed) and β-P_S-open (repectively, β-P_S-closed).

2. f is β-β-P_S-open (repectively, β-β-P_S-closed) and β-open (repectively, β-closed).

3. f is β-P_S-open (repectively, β-P_S-closed) and β-open (repectively, β-closed).

Proof. Follows from Theorem 4.4.38. □

Theorem 5.3.78. For a function $f\colon (X, \tau) \to (Y, \sigma)$ and β be an operation on σ, the following statements are equivalent:

1. f is σ-β-P_S-open (repectively, σ-β-P_S-closed) and β-P_S-open (repectively, β-P_S-closed).

2. f is σ-β-P_S-open (repectively, σ-β-P_S-closed) and open (repectively, closed).

3. f is β-P_S-open (repectively, β-P_S-closed) and open (repectively, closed).

Proof. Follows from Theorem 4.4.39. □

Theorem 5.3.79. Let (Y, σ) be β-regular space and β be an operation on σ. Then a function $f\colon (X, \tau) \to (Y, \sigma)$ is β-β-P_S-open (respectively, β-β-P_S-closed) if and only if f is σ-β-P_S-open (respectively, σ-β-P_S-closed).

Proof. This is an immediate consequence of Theorem 4.4.40. □

In the end of this section, conditions for composition of two functions are established in order to obtain γ-P_S-continuous, (γ,β)-P_S-continuous and (γ,β)-P_S-irresolute functions.

Theorem 5.3.80. Let γ and β be operations on the topological spaces (X,τ) and (Y,σ) respectively. Let $f\colon (X,\tau)\to(Y,\sigma)$ and $g\colon(Y,\sigma)\to(Z,\rho)$ be functions. Then the composition function $g\circ f\colon (X,\tau)\to(Z,\rho)$ is γ-P_S-continuous if f and g satisfy one of the following conditions:

1. f is γ-P_S-continuous and g is continuous.

2. f is (γ,β)-P_S-continuous and g is β-continuous.

3. f is (γ,β)-P_S-irresolute and g is β-P_S-continuous.

Proof. (1) Let V be an open subset of (Z,ρ). Since g is continuous, then $g^{-1}(V)$ is open in (Y,σ). Since f is γ-P_S-continuouse, then by Theorem 5.3.1, $(g\circ f)^{-1}(V)=f^{-1}(g^{-1}(V))$ is γ-P_S-open in (X,τ). Therefore, $g\circ f$ is γ-P_S-continuous.

The proofs of (2) and (3) are similars to (1). □

Corollary 5.3.81. Let γ and β be operations on the topological spaces (X,τ) and (Y,σ) respectively. Let $f\colon (X,\tau)\to(Y,\sigma)$ and $g\colon(Y,\sigma)\to(Z,\rho)$ be functions. Then the composition function $g\circ f\colon (X,\tau)\to(Z,\rho)$ is γ-P_S-continuous if f and g satisfy one of the following conditions:

1. f is completely γ-continuous and g is continuous.

2. f is γ-P_S-continuous and g is β-continuous.

3. f is completely γ-continuous and g is β-continuous.

4. f is (γ, β)-P_S-continuous and g is completely β-continuous.

5. f is (γ, β)-P_S-irresolute and g is completely β-continuous.

Proof. Follows from Theorem 5.3.80, Remark 5.2.4, Remark 5.2.16 and Remark 5.2.18. □

Theorem 5.3.82. Let γ, β and α be operations on the topological spaces (X, τ), (Y, σ) and (Z, ρ) respectively. Let $f\colon (X, \tau) \to (Y, \sigma)$ and $g\colon (Y, \sigma) \to (Z, \rho)$ be functions. Then the composition function $g \circ f\colon (X, \tau) \to (Z, \rho)$ is (γ, α)-P_S-continuous if f and g satisfy one of the following conditions:

1. f is (γ, β)-P_S-irresolute and g is (β, α)-P_S-continuous.

2. f is (γ, β)-P_S-irresolute and g is β-P_S-continuous.

Proof. (1) The proof is similar to Theorem 5.3.80.

(2) The proof follows directly from the part (1) since every β-P_S-continuous is (β, α)-P_S-continuous. □

Theorem 5.3.83. Let γ, β and α be operations on the topological spaces (X, τ), (Y, σ) and (Z, ρ) respectively. If the functions $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-irresolute and $g\colon (Y, \sigma) \to (Z, \rho)$ is (β, α)-P_S-irresolute. Then the composition function $g \circ f\colon (X, \tau) \to (Z, \rho)$ is (γ, α)-P_S-irresolute.

Proof. It is clear. □

Corollary 5.3.84. The composition of two completely γ-continuous functions is completely γ-continuous function.

Proof. Obvious. □

Corollary 5.3.85. Let α be an operation on the topological space (Z, ρ). If the function $f\colon (X, \tau) \to (Y, \sigma)$ is open (repectively, closed) and $g\colon (Y, \sigma) \to (Z, \rho)$ is α-P_S-open (repectively, α-P_S-closed). Then $g \circ f\colon (X, \tau) \to (Z, \rho)$ is α-P_S-open (repectively, α-P_S-closed).

Proof. Obvious. □

Corollary 5.3.86. Let γ, β and α be operations on the topological spaces (X, τ), (Y, σ) and (Z, ρ) respectively. Let $f\colon (X, \tau) \to (Y, \sigma)$ and $g\colon (Y, \sigma) \to (Z, \rho)$ be any functions. Then the following are holds:

1. If f is β-open (repectively, β-closed) and g is (β, α)-P_S-open (repectively, (β, α)-P_S-closed), then $g \circ f\colon (X, \tau) \to (Z, \rho)$ is α-P_S-open (repectively, α-P_S-closed).

2. If f is (γ, β)-P_S-open (repectively, (γ, β)-P_S-closed) and g is (β, α)-P_S-open (repectively, (β, α)-P_S-closed), then $g \circ f\colon (X, \tau) \to (Z, \rho)$ is (γ, α)-P_S-open (repectively, (γ, α)-P_S-closed).

3. If f is $(\gamma, \beta P_S)$-open (repectively, $(\gamma, \beta P_S)$-closed) and g is (β, α)-P_S-open

(repectively, (β, α)-P_S-closed), then $g \circ f\colon (X, \tau) \rightarrow (Z, \rho)$ is $(\gamma, \alpha P_S)$-open (repectively, $(\gamma, \alpha P_S)$-closed).

Proof. It is clear. □

Theorem 5.3.87. Let γ be an operation on the topological space (X, τ). If $f\colon (X, \tau) \rightarrow (Y, \sigma)$ is a function, $g\colon (Y, \sigma) \rightarrow (Z, \rho)$ is open and injective, and $g \circ f\colon (X, \tau) \rightarrow (Z, \rho)$ is γ-P_S-continuous. Then f is γ-P_S-continuous.

Proof. Let V be an open subset of Y. Since g is open, $g(V)$ is open subset of Z. Since $g \circ f$ is γ-P_S-continuous and g is injective, then $f^{-1}(V) = f^{-1}(g^{-1}(g(V)) = (g \circ f)(g(V))$ is γ-P_S-open in X, which proves that f is γ-P_S-continuous. □

Theorem 5.3.88. Let γ and β be operations on the topological spaces (X, τ) and (Y, σ) respectively. Let $f\colon (X, \tau) \rightarrow (Y, \sigma)$ and $g\colon (Y, \sigma) \rightarrow (Z, \rho)$ be any functions. Then the following are holds:

1. If f is β-P_S-open and surjective, and $g \circ f\colon (X, \tau) \rightarrow (Z, \rho)$ is continuous. Then g is γ-P_S-continuous.

2. If f is continuous and surjective, and $g \circ f\colon (X, \tau) \rightarrow (Z, \rho)$ is β-P_S-open. Then g is β-P_S-open.

Proof. Similar to Theorem 5.3.87. □

Theorem 5.3.89. If $f\colon (X, \tau) \rightarrow (Y, \sigma)$ is a function, $g\colon (Y, \sigma) \rightarrow (Z, \rho)$ is (β, α)-P_S-open and injective, and $g \circ f\colon (X, \tau) \rightarrow (Z, \rho)$ is (γ, α)-P_S-irresolute. Then f is (γ, β)-P_S-irresolute.

Proof. Let V be an β-P_S-open subset of Y. Since g is (β, α)-P_S-open, then $g(V)$ is α-P_S-open subset of Z. Since $g \circ f$ is (γ, α)-P_S-irresolute and g is injective, then $f^{-1}(V) = f^{-1}(g^{-1}(g(V)) = (g \circ f)(g(V))$ is γ-P_S-open in X, which proves that f is (γ, β)-P_S-irresolute. □

Theorem 5.3.90. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$ be any functions. Then the following are holds:

1. If f is (γ, β)-P_S-open and surjective, and $g \circ f : (X, \tau) \to (Z, \rho)$ is (γ, α)-P_S-irresolute, then g is (γ, β)-P_S-irresolute.

2. If f is (γ, β)-P_S-irresolute and surjective, and $g \circ f : (X, \tau) \to (Z, \rho)$ is (γ, α)-P_S-open, then g is (β, α)-P_S-open.

Proof. The proof is similar to Theorem 5.3.89. □

5.4 γ-P_S-g-Continuous Functions

In this section, a new class of γ-P_S- functions called γ-P_S-g-continuous by using γ-P_S-g-closed set as in Section 4.5 will be defined. Some theorems and properties for this function are studied.

Definition 5.4.1. Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ. A function $f : (X, \tau) \to (Y, \sigma)$ is called γ-P_S-g-continuous if the inverse image of every closed set in Y is γ-P_S-g-closed set in X.

The following theorem is the most important characterizations of γ-P_S-g-continuous functions.

Theorem 5.4.2. For a function $f\colon (X,\tau) \to (Y,\sigma)$ with an operation γ on τ, the following statements are equivalent:

1. f is γ-P_S-g-continuous.

2. The inverse image of every open set in Y is γ-P_S-g-open set in X.

3. For each point $x \in X$ and each open set V of Y containing $f(x)$, there exists a γ-P_S-g-open set U of X containing x such that $f(U) \subseteq V$.

Proof. Straightforward. □

Corollary 5.4.3. Every γ-P_S-continuous function is γ-P_S-g-continuous.

Proof. Obvious since every γ-P_S-closed set is γ-P_S-g-closed. □

The converse of the above remark does not true as seen from the following example.

Example 5.4.4. Let (X,τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Suppose that $Y = \{1,2,3\}$ and $\sigma = \{\phi, Y, \{1\}, \{1,3\}\}$ be a topology on Y.

Let $f\colon (X,\tau) \to (Y,\sigma)$ be a function defined as follows:

$$f(x) = \begin{cases} 3 & \text{if } x = a \\ 2 & \text{if } x = b \\ 1 & \text{if } x = c \end{cases}$$

Then f is γ-P_S-g-continuous, but f is not γ-P_S-continuous since $\{2,3\}$ is closed in (Y,σ), but $f^{-1}(\{2,3\}) = \{a,b\}$ is not γ-P_S-closed set in (X,τ).

Corollary 5.4.5. Let $f\colon (X,\tau) \to (Y,\sigma)$ be a function and γ be an operation on τ. If (X,τ) is γ-locally indiscrete space, then f is γ-P_S-g-continuous.

Proof. This is an immediate consequence of Theorem 4.5.24. □

Theorem 5.4.6. Let γ be an operation on the topological space (X,τ). If the functions $f\colon (X,\tau) \to (Y,\sigma)$ is γ-P_S-g-continuous and $g\colon (Y,\sigma) \to (Z,\rho)$ is continuous. Then the composition function $g \circ f\colon (X,\tau) \to (Z,\rho)$ is γ-P_S-g-continuous.

Proof. It is clear. □

Theorem 5.4.7. Let γ be an operation on the topological space (X,τ). If $f\colon (X,\tau) \to (Y,\sigma)$ is a function, $g\colon (Y,\sigma) \to (Z,\rho)$ is closed and injective, and $g\circ f\colon (X,\tau) \to (Z,\rho)$ is γ-P_S-g-continuous. Then f is γ-P_S-g-continuous.

Proof. Let F be a closed subset of Y. Since g is closed, $g(F)$ is closed subset of Z. Since $g \circ f$ is γ-P_S-g-continuous and g is injective, then $f^{-1}(F) = f^{-1}(g^{-1}(g(F)) = (g \circ f)(g(F))$ is γ-P_S-g-closed in X, which proves that f is γ-P_S-g-continuous. □

Definition 5.4.8. Let (X,τ) and (Y,σ) be two topological spaces and β be an operation on σ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called β-P_S-g-closed if for every closed set F of X, $f(F)$ is β-P_S-g-closed set in Y.

Remark 5.4.9. Every β-P_S-closed function is β-P_S-g-closed.

The converse of the above remark does not true as seen from the following example.

Example 5.4.10. In Example 5.3.61, The function f is β-P_S-g-closed, but f is not β-P_S-closed since $\{a\}$ is closed set in (X, τ), but $f(\{a\}) = \{a\}$ is not β-P_S-closed set in (X, σ).

Theorem 5.4.11. Let (Y, σ) be a topological space and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is β-P_S-g-closed if and only if for each subset S of Y and each open set O in X containing $f^{-1}(S)$, there exists a β-P_S-g-open set R in Y such that $S \subseteq R$ and $f^{-1}(R) \subseteq O$.

Proof. Suppose that f is β-P_S-g-closed function and let O be an open set in X containing $f^{-1}(S)$, where S is any subset in Y. Then $f(X \backslash O)$ is β-P_S-g-open set in Y. If we put $R = Y \backslash f(X \backslash O)$. Then R is β-P_S-g-closed set in Y containing S such that $f^{-1}(R) \subseteq O$.

Conversely, let F be closed set in X. Let $S = Y \backslash f(F) \subseteq Y$. Then $f^{-1}(S) \subseteq X \backslash F$ and $X \backslash F$ is open set in X. By hypothesis, there exists a β-P_S-g-open set R in Y such that $S = Y \backslash f(F) \subseteq R$ and $f^{-1}(R) \subseteq X \backslash F$. For $f^{-1}(R) \subseteq X \backslash F$ implies $R \subseteq f(X \backslash F) \subseteq Y \backslash f(F)$. Hence $R = Y \backslash f(F)$. Since R is β-P_S-g-open set in Y. Then $f(F)$ is β-P_S-g-closed set in Y. Therefore, f is β-P_S-g-closed function. $\square$

Corollary 5.4.12. Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function and β be an operation on σ. If (Y, σ) is β-locally indiscrete space, then f is β-P_S-g-closed.

Proof. This is an immediate consequence of Theorem 4.5.24. $\square$

Definition 5.4.13. Let (X, τ) and (Y, σ) be two topological spaces and β be an operation on σ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called β-P_S-g-open if for every open set V of X, $f(V)$ is β-P_S-g-open set in Y.

Definition 5.4.14. A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be γ-P_S-g-homeomorphism, if f is bijective, γ-P_S-g-continuous and f^{-1} is γ-P_S-g-continuous.

Theorem 5.4.15. The following statements are equivalent for a bijective function $f\colon (X,\tau) \to (Y,\sigma)$ with an operation β on σ.

1. f is β-P_S-g-closed.

2. f is β-P_S-g-open.

3. f^{-1} is β-P_S-g-continuous.

Proof. It is clear. □

Corollary 5.4.16. Let α be an operation on the topological space (Z,ρ). If the function $f\colon (X,\tau) \to (Y,\sigma)$ is closed (repectively, open) and $g\colon (Y,\sigma) \to (Z,\rho)$ is α-P_S-g-closed (repectively, α-P_S-g-open). Then the composition function $g \circ f\colon (X,\tau) \to (Z,\rho)$ is α-P_S-g-closed (repectively α-P_S-g-open).

Proof. Obvious. □

Corollary 5.4.17. Let β be an operation on the topological space (Y,σ). If $g\colon (Y,\sigma) \to (Z,\rho)$ is a function, $f\colon (X,\tau) \to (Y,\sigma)$ is β-P_S-g-open and surjective, and $g \circ f\colon (X,\tau) \to (Z,\rho)$ is continuous. Then g is γ-P_S-g-continuous.

Proof. Similar to Theorem 5.4.7. □

Corollary 5.4.18. Let β be an operation on the topological space (Y, σ). If $g\colon (Y, \sigma) \to (Z, \rho)$ is a function, $f\colon (X, \tau) \to (Y, \sigma)$ is continuous and surjective, and $g \circ f\colon (X, \tau) \to (Z, \rho)$ is β-P_S-g-closed. Then g is β-P_S-g-closed.

Proof. Similar to Theorem $5.4.7$. □

5.5 Conclusion

This chapter defined some types of γ-P_S- functions such as γ-P_S-continuous, (γ, β)-P_S-continuous, (γ, β)-P_S-irresolute, β-P_S-open, β-P_S-closed, (γ, β)-P_S-open, (γ, β)-P_S-closed, $(\gamma, \beta P_S)$-open, $(\gamma, \beta P_S)$-closed and γ-P_S-g-continuous in terms of γ-P_S-open, γ-P_S-closed and γ-P_S-g-closed sets. The function completely γ-continuous by using γ-regular-open set has been studied. The relations, composition, properties and characterizations of these functions have also been investigated.

CHAPTER SIX

γ-P_S- SEPARATION AXIOMS

6.1 Introduction

This chapter introduces some new types of γ-P_S- separation axioms in topological space (X, τ). Firstly, we define γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$ using γ-P_S-open sets. Then some new types of γ-P_S- separation axioms called γ-P_S-regular, γ-P_S-normal, γ-P_S-R_0 and γ-P_S-R_1 spaces are established. The relations between these γ-P_S- separation axioms and other known types of γ- separation axioms are also obtained. Some basic characterizations and properties of these spaces are studied. Finally, some other conditions which related to γ-P_S- functions are discussed.

6.2 γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$

In this section, we introduce some types of γ-P_S- separation axioms called γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$ using γ-P_S-open set. The relation between these γ-P_S- spaces and other types of γ- spaces will be investigated.

Definition 6.2.1. A topological space (X, τ) with an operation γ on τ is said to be:

1. γ-P_S-T_0 if for each pair of distinct points x, y in X, there exists a γ-P_S-open sets G such that either $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$.

2. γ-P_S-T_1 if for each pair of distinct points x, y in X, there exist two γ-P_S-open sets

G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.

3. γ-P_S-T_2 if for each pair of distinct points x, y in X, there exist two γ-P_S-open sets G and H containing x and y respectively such that $G \cap H = \phi$.

4. γ-P_S-$T_{\frac{1}{2}}$ if every γ-P_S-g-closed set in X is γ-P_S-closed.

The following are some basic properties of γ-P_S-T_i spaces for $i = 0, \frac{1}{2}, 1, 2$.

Theorem 6.2.2. Let (X, τ) be a topological space and γ be an operation on τ. Then X is γ-P_S-T_0 if and only if τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$, for every pair of distinct points x, y of X.

Proof. Let (X, τ) be a γ-P_S-T_0 and x, y be any two distinct points of X. Then there exists a γ-P_S-open set G containing x or y (say x, but not y). Then $X \backslash G$ is a γ-P_S-losed set, which does not contain x, but contains y. Since τ_γ-$P_SCl(\{y\})$ is the smallest γ-P_S-closed set containing y, τ_γ-$P_SCl(\{y\}) \subseteq X \backslash G$, and so $x \notin \tau_\gamma$-$P_SCl(\{y\})$. Consequently τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$.

Conversely, suppose for any $x, y \in X$ with $x \neq y$, τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. Now, let $z \in X$ such that $z \in \tau_\gamma$-$P_SCl(\{x\})$, but $z \notin \tau_\gamma$-$P_SCl(\{y\})$. Now, we claim that $x \in \tau_\gamma$-$P_SCl(\{y\})$. For, if $x \in \tau_\gamma$-$P_SCl(\{y\})$, then $\{x\} \subseteq \tau_\gamma$-$P_SCl(\{y\})$, which implies that τ_γ-$P_SCl(\{x\}) \subseteq \tau_\gamma$-$P_SCl(\{y\})$. This is contradiction to the fact that $z \notin \tau_\gamma$-$P_SCl(\{y\})$. Consequently x belongs to the γ-P_S-open set $X \backslash \tau_\gamma$-$P_SCl(\{y\})$ to which y does not belong. It gives that X is γ-P_S-T_0 space. $\square$

Theorem 6.2.3. Let (X, τ) be a topological space and γ be an operation on τ. Then X is γ-P_S-T_1 if and only if every singleton set in X is γ-P_S-closed.

Proof. Suppose (X, τ) be γ-P_S-T_1. Let $x \in X$. Then for any point $y \in X$ such that $x \neq y$, there exists a γ-P_S-open set G such that $y \in G$ but $x \notin G$. Thus, $y \in G \subseteq X \backslash \{x\}$. This implies that $X \backslash \{x\} = \cup\{G : y \in X \backslash \{x\}\}$. So, by Theorem 4.2.4, $X \backslash \{x\}$ is γ-P_S-open set in X. Hence $\{x\}$ is γ-P_S-closed set in X.

Conversely, suppose every singleton set in X is γ-P_S-closed. Let $x, y \in X$ such that $x \neq y$. This implies that $x \in X \backslash \{y\}$. By hypothesis, we get $X \backslash \{y\}$ is a γ-P_S-open set contains x but not y. Similarly $X \backslash \{x\}$ is a γ-P_S-open set contains y but not x. Therefore, a space X is γ-P_S-T_1. □

Theorem 6.2.4. For a topological space (X, τ) with an operation γ on τ. The following conditions are equivalent.

1. X is γ-P_S-T_2.
2. If $x \in X$, then there exists a γ-P_S-open set G containing x such that $y \notin \tau_\gamma$-$P_SCl(G)$ for each $y \in X$.
3. For each $x \in X$, $\cap\{\tau_\gamma$-$P_SCl(G)$: G is a γ-P_S-open set containing $x\} = \{x\}$.

Proof. (1) ⇒ (2) Let X be any γ-P_S-T_2 space. For each $x, y \in X$ with $x \neq y$, then there exist two γ-P_S-open sets G and H containing x and y respectively such that $G \cap H = \phi$. This implies that $G \subseteq X \backslash H$ and hence τ_γ-$P_SCl(\{G\}) \subseteq X \backslash H$ since $X \backslash H$ is γ-P_S-closed set in X. Therefore, $y \notin \tau_\gamma$-$P_SCl(G)$.

(2) ⇒ (3) Straightforward.

(3) ⇒ (1) Let $x, y \in X$ with $x \neq y$, then by hypothesis there exists a γ-P_S-open set G containing x such that $y \notin G$ and hence $y \notin \tau_\gamma$-$P_SCl(G)$. Then $y \in X \backslash \tau_\gamma$-$P_SCl(G)$ and

$X\backslash\tau_\gamma$-$P_SCl(G)$ is γ-P_S-open set. So $G \cap X\backslash\tau_\gamma$-$P_SCl(G) = \phi$. Therefore, X is γ-P_S-T_2 space. □

The following remark follows directly from Definition 6.2.1 (3).

Remark 6.2.5. If for each pair of distinct points x, y in a topological space (X, τ), there exist two γ-P_S-open sets G and H containing x and y respectively such that τ_γ-$P_SCl(G)$ $\cap\, \tau_\gamma$-$P_SCl(H) = \phi$. Then X is γ-P_S-T_2.

Theorem 6.2.6. For any topological space (X, τ) with an operation γ on τ. Then X is γ-P_S-$T_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is γ-P_S-closed or γ-P_S-open.

Proof. Let X be a γ-P_S-$T_{\frac{1}{2}}$ space and let $\{x\}$ is not γ-P_S-closed set in X. By Corollary 4.5.29, $X\backslash\{x\}$ is γ-P_S-g-closed. Since X is γ-P_S-$T_{\frac{1}{2}}$, then $X\backslash\{x\}$ is γ-P_S-closed set which means that $\{x\}$ is γ-P_S-open set in X.

Conversely, let F be any γ-P_S-g-closed set in the space (X, τ). We have to show that F is γ-P_S-closed (that is τ_γ-$P_SCl(F) = F$). Let $x \in \tau_\gamma$-$P_SCl(F)$. By hypothesis $\{x\}$ is γ-P_S-closed or γ-P_S-open for each $x \in X$. So we have two cases:

Case (1): If $\{x\}$ is γ-P_S-closed set. Suppose $x \notin F$, then $x \in \tau_\gamma$-$P_SCl(F)\backslash F$ contains a nonempty γ-P_S-closed set $\{x\}$. A contradiction since F is γ-P_S-g-closed set and according to the Theorem 4.5.8. Hence $x \in F$. This follows that τ_γ-$P_SCl(F) \subseteq F$ and so τ_γ-$P_SCl(F) = F$. This means that F is γ-P_S-closed set in X. Thus a space X is γ-P_S-$T_{\frac{1}{2}}$.

Case (2): If $\{x\}$ is γ-P_S-open set. Then by Theorem 4.4.13, $F \cap \{x\} \neq \phi$ which implies that $x \in F$. So τ_γ-$P_SCl(F) \subseteq F$. Thus F is γ-P_S-closed. Therefore, X is

γ-P_S-$T_{\frac{1}{2}}$ space. $\square$

Corollary 6.2.7. If a space (X,τ) is γ-P_S-$T_{\frac{1}{2}}$, then the set $\{x\}$ is γ-P_S-closed or γ-regular-open for each $x \in X$.

Proof. The proof is directly from Theorem 6.2.6 and Remark 4.2.33. $\square$

The relations between the γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$ are given as follows:

Theorem 6.2.8. Let (X,τ) be a topological space and γ be an operation on τ. Then the following statements are holds:

1. If X is γ-P_S-T_2, then it is γ-P_S-T_1.
2. If X is γ-P_S-T_1, then it is γ-P_S-$T_{\frac{1}{2}}$.
3. If X is γ-P_S-$T_{\frac{1}{2}}$, then it is γ-P_S-T_0.

Proof. (1) Directly from Definition 6.2.1 (2) and (3).

(2) Directly from Theorem 6.2.3 and Theorem 6.2.6.

(3) Let $x, y \in X$ such that $x \neq y$. Since X is γ-P_S-$T_{\frac{1}{2}}$ space. Then by Theorem 6.2.6, the set $\{x\}$ is either γ-P_S-closed or γ-P_S-open. If $\{x\}$ is γ-P_S-closed, then $X \backslash \{x\}$ is γ-P_S-open. Hence $y \in X \backslash \{x\}$ and $x \notin X \backslash \{x\}$. So X is γ-P_S-T_0. Or, if $\{x\}$ is γ-P_S-open. Then $x \in \{x\}$ and $y \notin \{x\}$ and hence X is γ-P_S-T_0 space $\square$

The converse of the above remark does not always true as shown from the following example.

Example 6.2.9. Let (X, τ) be any infinite set with the cofinite topology and γ be an operation on τ. Define an operation γ on τ by $\gamma(A) = A$ for all $A \in \tau$. Simplify the space X is γ-T_1 and hence it is γ-preT_1 and γ-semiT_1. Then by Corollary 6.2.15, X is γ-P_S-T_1, but not γ-P_S-T_2, since for x and y in X, there is no a pair of disjoint γ-P_S-open sets, one containing x and the other containing y.

Example 6.2.10. In Example 4.2.20, the space (X, τ) is γ-P_S-$T_{\frac{1}{2}}$, but it is not γ-P_S-T_1 since for the points a and d in X, there is no γ-P_S-open set containing d but not a.

Example 6.2.11. Consider (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Then the space (X, τ) is γ-P_S-T_0, but it is not γ-P_S-$T_{\frac{1}{2}}$ since the set $\{a, b\}$ is γ-P_S-g-closed, but it is not γ-P_S-closed.

More relations between the γ-P_S-T_i with other types of γ-preT_i for $i = 0, \frac{1}{2}, 1, 2$ are provided as follows:

Theorem 6.2.12. Let (X, τ) be a topological space and γ be an operation on τ. If X is γ-P_S-T_i, then X is γ-preT_i for $i = 0, \frac{1}{2}, 1, 2$.

Proof. The proof is obvious for the cases $i = 0, 1, 2$ since every γ-P_S-open set is γ-preopen. Now, when $i = \frac{1}{2}$, then let X be a γ-P_S-$T_{\frac{1}{2}}$ space. Then by Theorem 6.2.6, every singleton set is γ-P_S-closed or γ-P_S-open. This implies that every singleton set is γ-preclosed or γ-preopen. Therefore, by Theorem 2.5.6 (1), X is γ-pre$T_{\frac{1}{2}}$ space. $\square$

The converse of the Theorem 6.2.12 may not be true as seen in the following example.

Example 6.2.13. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.7. Then the space (X, τ) is γ-preT_i, but it is not γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$.

Corollary 6.2.14. Let (X, τ) be a topological space and γ be an operation on τ. If X is γ-P_S-T_i, then X is γ-βT_i for $i = 0, \frac{1}{2}, 1, 2$.

Proof. The proof follows directly from Theorem 6.2.12 and the fact that every γ-preopen set is γ-β-open. □

Notice that the converse of Theorem 6.2.12 is true when a space X is γ-semiT_1 as shown in the following.

Corollary 6.2.15. Let (X, τ) be a γ-semiT_1 space. Then (X, τ) is γ-P_S-T_i if and only if (X, τ) is γ-preT_i for $i = 0, \frac{1}{2}, 1, 2$.

Proof. The proof follows from Theorem 4.2.24 and Theorem 4.5.21. □

The following theorem is a relation between the γ-P_S-T_i with other types of γ-semiT_i for $i = 0, 1$.

Theorem 6.2.16. Let (X, τ) be a topological space and γ be an operation on τ. If X is γ-P_S-T_i, then X is γ-semiT_i for $i = 0, 1$.

Proof. (1) For $i = 0$, let X be a γ-P_S-T_0 space and x and y be any two distinct points of X. Then there exists a γ-P_S-open set G containing x or y (say, x but not y). Then by Definition 4.2.1, there exists a γ-semiclosed set F such that $x \in F \subseteq G$. So $X \backslash F$ is a γ-semiopen set containing y, and it is obvious that $x \notin X \backslash F$. Therefore, X is γ-semiT_0 space.

(2) For $i = 1$, then the proof is similar to the part (1). $\square$

Definitions 4.2.1 and 6.2.1 lead us to the following remark.

Remark 6.2.17. Let (X, τ) be a topological space and γ be an operation on τ, then the following are true:

1. If X is γ-P_S-T_0, then for each pair of distinct points x, y in X, there exists a γ-semiclosed set containing one, but not the other.

2. If X is γ-P_S-T_1, then for each pair of distinct points x, y in X, there exist two γ-semiclosed sets one containing x but not y, and the other containing y but not x.

3. If X is γ-P_S-T_2, then for each pair of distinct points x, y in X, there exist two disjoint γ-semiclosed sets one containing x and the other containing y.

From Theorem 6.2.8, Theorem 6.2.12, Corollary 6.2.14, Theorem 6.2.16 and Theorem 2.5.7. The Figure 6.1 is illustrated.

$$
\begin{tikzcd}
\gamma\text{-}P_S\text{-}T_2 \arrow[r] \arrow[d] & \gamma\text{-}P_S\text{-}T_1 \arrow[r] \arrow[d] & \gamma\text{-}P_S\text{-}T_{\frac{1}{2}} \arrow[r] \arrow[d] & \gamma\text{-}P_S\text{-}T_0 \arrow[d] \\
\gamma\text{-pre}T_2 \arrow[r] \arrow[d] & \gamma\text{-pre}T_1 \arrow[r] \arrow[d] & \gamma\text{-pre}T_{\frac{1}{2}} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \gamma\text{-pre}T_0 \arrow[d] \\
\gamma\text{-}\beta T_2 \arrow[r] & \gamma\text{-}\beta T_1 \arrow[r] & \gamma\text{-}\beta T_{\frac{1}{2}} \arrow[r, leftrightarrow] & \gamma\text{-}\beta T_0 \\
\gamma\text{-semi}T_2 \arrow[r] \arrow[u] & \gamma\text{-semi}T_1 \arrow[r] \arrow[u] & \gamma\text{-semi}T_{\frac{1}{2}} \arrow[r] \arrow[u] & \gamma\text{-semi}T_0 \arrow[u] \\
\gamma\text{-}P_S\text{-}T_2 \arrow[r] & \gamma\text{-}P_S\text{-}T_1 \arrow[r] \arrow[u] & \gamma\text{-}P_S\text{-}T_{\frac{1}{2}} \arrow[r] & \gamma\text{-}P_S\text{-}T_0 \arrow[u]
\end{tikzcd}
$$

Remark: $A \longrightarrow B$ represents A implies B

Figure 6.1. The relations between γ-P_S- separation axioms and other types of γ- separation axioms

It is notice from Figure 6.1, that if a toplogical space (X,τ) is γ-P_S-T_0, then X is γ-pre$T_{\frac{1}{2}}$ and hence it is γ-$\beta T_{\frac{1}{2}}$. Also, if X is γ-semiT_0, then X is both γ-pre$T_{\frac{1}{2}}$ and γ-$\beta T_{\frac{1}{2}}$. Moreover, there is no relation between γ-P_S-$T_{\frac{1}{2}}$ space and γ-semi$T_{\frac{1}{2}}$ space. Also, the spaces γ-P_S-T_2 and γ-semiT_2 are independent. Finally, the spaces γ-preT_i and γ-semiT_i for $i = 1, 2$ are independent.

Theorem 6.2.18. A topological space (X,τ) is γ-P_S-$T_{\frac{1}{2}}$ if and only if τ_γ-$P_SGC(X) = \tau_\gamma$-$P_SC(X)$.

Proof. Follows from Definition 6.2.1 (4) and Theorem 4.5.2. $\square$

The next corollary follows directly from Theorem 6.2.18.

Corollary 6.2.19. A topological space (X, τ) is γ-P_S-$T_{\frac{1}{2}}$ if and only if τ_γ-$P_SGO(X) = \tau_\gamma$-$P_SO(X)$. □

Corollary 6.2.20. Let (X, τ) be γ-P_S-$T_{\frac{1}{2}}$ space and γ be an operation on τ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is γ-P_S-continuous if and only if f is γ-P_S-g-continuous.

Proof. Follows from Theorem 6.2.18, Theorem 5.3.1 and Theorem 5.4.2. □

Corollary 6.2.21. Let (Y, σ) be β-P_S-$T_{\frac{1}{2}}$ space and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is β-P_S-g-closed if and only if f is β-P_S-closed.

Proof. Follows from Theorem 6.2.18. □

Theorem 6.2.22. Let γ and β be operations on τ and σ respectively. Let $f\colon (X, \tau) \to (Y, \sigma)$ be an injective, (γ, β)-P_S-irresolute and (γ, β)-P_S-closed function. If (Y, σ) is β-P_S-$T_{\frac{1}{2}}$, then (X, τ) is γ-P_S-$T_{\frac{1}{2}}$.

Proof. Let G be any γ-P_S-g-closed set of (X, τ). Since f is (γ, β)-P_S-irresolute and (γ, β)-P_S-closed function. Then by Theorem 5.3.59 (1), $f(G)$ is β-P_S-g-closed in (Y, σ). Since (Y, σ) is β-P_S-$T_{\frac{1}{2}}$, then $f(G)$ is β-P_S-closed in Y. Again, since f is (γ, β)-P_S-irresolute, then by Theorem 5.3.10, $f^{-1}(f(G))$ is γ-P_S-closed in X. Hence G is γ-P_S-closed in X since f is injective. Therefore, a space(X, τ) is γ-P_S-$T_{\frac{1}{2}}$. □

Theorem 6.2.23. Let γ and β be operations on τ and σ respectively. Let a function $f\colon (X, \tau) \to (Y, \sigma)$ be a surjective, (γ, β)-P_S-irresolute and (γ, β)-P_S-closed. If (X, τ) is γ-P_S-$T_{\frac{1}{2}}$, then (Y, σ) is β-P_S-$T_{\frac{1}{2}}$.

Proof. Let H be a β-P_S-g-closed set of (Y, σ). Since a function f is (γ, β)-P_S-irresolute and (γ, β)-P_S-closed. Then by Theorem 5.3.59 (2), $f^{-1}(H)$ is γ-P_S-g-closed in (X, τ). Since (X, τ) is γ-P_S-$T_{\frac{1}{2}}$, then we have, $f^{-1}(H)$ is γ-P_S-closed set in X. Again, since f is (γ, β)-P_S-closed function, then $f(f^{-1}(H))$ is β-P_S-closed in Y. Therefore, H is β-P_S-closed in Y since f is surjective. Hence (Y, σ) is β-P_S-$T_{\frac{1}{2}}$ space. □

Theorem 6.2.24. Assume that $f\colon (X, \tau) \to (Y, \sigma)$ is a (γ, β)-P_S-homeomorphism function. If (X, τ) is γ-P_S-$T_{\frac{1}{2}}$, then (Y, σ) is β-P_S-$T_{\frac{1}{2}}$.

Proof. Let $\{y\}$ be any singleton set of (Y, σ). Then there exists an element x of X such that $y = f(x)$. So by hypothesis and Theorem 6.2.6, we have $\{x\}$ is γ-P_S-closed or γ-P_S-open set in X. By using Theorem 5.3.10, $\{y\}$ is β-P_S-closed or β-P_S-open set. Then by Theorem 6.2.6, (Y, σ) is β-P_S-$T_{\frac{1}{2}}$ space. □

Theorem 6.2.25. Let γ and β be operations on τ and σ respectively. Let $f\colon (X, \tau) \to (Y, \sigma)$ be an injective (γ, β)-P_S-irresolute function. If (Y, σ) is β-P_S-T_2, then (X, τ) is γ-P_S-T_2.

Proof. Let x_1 and x_2 be any distinct points of a space (X, τ). Since f is an injective function and (Y, σ) is β-P_S-T_2. Then there exist two β-P_S-open sets G_1 and G_2 in Y such that $f(x_1) \in G_1$, $f(x_2) \in G_2$ and $G_1 \cap G_2 = \phi$. Since f is (γ, β)-P_S-irresolute, then by Theorem 5.3.10, $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are γ-P_S-open sets in (X, τ) containing x_1 and x_2 respectively. Hence $f^{-1}(G_1) \cap f^{-1}(G_2) = \phi$. Therefore, (X, τ) is γ-P_S-T_2. □

Theorem 6.2.26. Let $f\colon (X, \tau) \to (Y, \sigma)$ be an injective (γ, β)-P_S-irresolute function. If (Y, σ) is β-P_S-T_i, then (X, τ) is γ-P_S-T_i for $i = 0, 1$.

Proof. The proof is similar to Theorem 6.2.25. □

Corollary 6.2.27. Let $f\colon (X,\tau) \rightarrow (Y,\sigma)$ be an injective γ-continuous and β-open function. If (Y,σ) is β-P_S-T_i, then (X,τ) is γ-P_S-T_i for $i = 0, 1, 2$.

Proof. Directly follows from Theorem 6.2.25, Theorem 6.2.26 and Theorem 5.3.17 since every γ-continuous and β-open function f is (γ,β)-P_S-irresolute. □

Theorem 6.2.28. Let γ and β be operations on τ and σ respectively. Let $f\colon (X,\tau) \rightarrow (Y,\sigma)$ be an injective γ-P_S-continuous function. If (Y,σ) is T_i, then (X,τ) is γ-P_S-T_i for $i = 0, 1, 2$.

Proof. It is enough to proof for one case of i (say $i = 2$) since the proofs of the other cases are similar.

Let f be an injective γ-P_S-continuous function and $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since (Y,σ) is T_2, then there exist two open sets G_1 and G_2 in (Y,σ) such that $f(x_1) \in G_1$, $f(x_2) \in G_2$ and $G_1 \cap G_2 = \phi$. Since f is γ-P_S-continuous, then by Theorem 5.3.1 (2), $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are γ-P_S-open sets in (X,τ) containing x_1 and x_2 respectively. Hence $f^{-1}(G_1) \cap f^{-1}(G_2) = \phi$. So (X,τ) is γ-P_S-T_2. □

Theorem 6.2.29. Assume that a function $f\colon (X,\tau) \rightarrow (Y,\sigma)$ is a surjective and (γ,β)-P_S-open. If (X,τ) is γ-P_S-T_i, then (Y,σ) is β-P_S-T_i for $i = 0, 1, 2$.

Proof. It is enough to proof for one case of i (say $i = 2$) since the proofs of the other cases are similar.

Let a function f be a surjective and (γ,β)-P_S-open and $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Then there exist distinct points x_1 and x_2 of X such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Since (X, τ) is γ-P_S-T_2 space, there exist γ-P_S-open sets V_1 and V_2 such that $x_1 \in V_1$, $x_2 \in V_2$ and $V_1 \cap V_2 = \phi$. Since f is (γ, β)-P_S-open, then $f(V_1)$ and $f(V_2)$ are β-P_S-open sets in (Y, σ) such that $y_1 = f(x_1) \in f(V_1)$ and $y_2 = f(x_2) \in f(V_2)$. This implies that $f(V_1) \cap f(V_2) = \phi$. Hence (Y, σ) is β-P_S-T_2. $\square$

The proof of the next theorem is similar to Theorem 6.2.29.

Theorem 6.2.30. Assume that a function $f\colon (X, \tau) \to (Y, \sigma)$ is a surjective and β-P_S-open. If (X, τ) is T_i, then (Y, σ) is β-P_S-T_i for $i = 0, 1, 2$.

Proof. Obvious. $\square$

The following new space in terms of τ_γ-P_S-closure will help us to give more relations and properties of γ-P_S-T_i spaces for $i = 0, \frac{1}{2}, 1, 2$.

Definition 6.2.31. A topological space (X, τ) with an operation γ on τ, is said to be γ-P_S-symmetric if for each x, y in X, then $x \in \tau_\gamma$-$P_SCl(\{y\})$ implies that $y \in \tau_\gamma$-$P_SCl(\{x\})$.

Lemma 6.2.32. Let (X, τ) be a topological space and γ be an operation on τ. Then X is γ-P_S-symmetric if and only if the singleton set $\{x\}$ is γ-P_S-g-closed, for each $x \in X$.

Proof. Let $\{x\} \subseteq G$ and G is γ-P_S-open set in X. Suppose that τ_γ-$P_SCl(\{x\}) \nsubseteq G$. Then τ_γ-$P_SCl(\{x\}) \cap X \backslash G \neq \phi$. Let $y \in \tau_\gamma$-$P_SCl(\{x\}) \cap X \backslash G$, then $y \in \tau_\gamma$-$P_SCl(\{x\})$. Since a space X is γ-P_S-symmetric, then $x \in \tau_\gamma$-$P_SCl(\{y\}) \subseteq X \backslash G$ and hence $x \notin G$. This is a contradiction of the assumption. This means that the singleton set $\{x\}$ is γ-P_S-g-closed, for each $x \in X$.

Conversely, suppose that $x \in \tau_\gamma$-$P_SCl(\{y\})$, but $y \notin \tau_\gamma$-$P_SCl(\{x\})$. Then $\{y\} \subseteq X \backslash \tau_\gamma$-$P_SCl(\{x\})$ and $X \backslash \tau_\gamma$-$P_SCl(\{x\})$ is γ-P_S-open set in X. Then by hypothesis, we have τ_γ-$P_SCl(\{y\}) \subseteq X \backslash \tau_\gamma$-$P_SCl(\{x\})$. Therefore $x \in X \backslash \tau_\gamma$-$P_SCl(\{x\})$. This is contradiction. Therefore, $y \in \tau_\gamma$-$P_SCl(\{x\})$ and hence X is γ-P_S-symmetric space. □

Theorem 6.2.33. For any topological space (X, τ) with an operation γ on τ. If τ_γ-$P_SO(X) = \tau_\gamma$-$P_SC(X)$, then (X, τ) is γ-P_S-symmetric.

Proof. Since τ_γ-$P_SO(X) = \tau_\gamma$-$P_SC(X)$, then by Theorem 4.5.18, every subset of (X, τ) is γ-P_S-g-closed. This means that evey singleton sets are γ-P_S-g-closed and hence by Lemma 6.2.32, the space (X, τ) is γ-P_S-symmetric. □

From Theorem 6.2.33 and Theorem 4.3.10, we have the following corollary.

Corollary 6.2.34. If (X, τ) is γ-locally indiscrete space, then (X, τ) is γ-P_S-symmetric.

Proof. Clear. □

Remark 6.2.35. If (X, τ) is γ-locally indiscrete space such that $\tau_\gamma \neq P(X)$, then (X, τ) will be not γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$.

Corollary 6.2.36. If (X, τ) is γ-hyperconnected space and γ is a regular operation on τ, then (X, τ) will be not γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$. This means that if (X, τ) is γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$, then it is not γ-hyperconnected.

Proof. This is an immediate consequence of Theorem 4.2.25. □

Corollary 6.2.37. If (X, τ) is γ-hyperconnected space and γ is a regular operation on τ, then (X, τ) is γ-P_S-symmetric.

Proof. Directly follows from Theorem $6.2.33$ and Theorem $4.2.25$. $\square$

The relation between γ-P_S-symmetric and γ-P_S-T_1 spaces are shown in the following theorem.

Theorem 6.2.38. If (X, τ) is γ-P_S-T_1 space, then it is γ-P_S-symmetric.

Proof. By Theorem $6.2.3$, every singleton sets are γ-P_S-closed in a γ-P_S-T_1 space (X, τ). Since every γ-P_S-closed set is γ-P_S-g-closed. Then by Lemma $6.2.32$, (X, τ) is γ-P_S-symmetric. $\square$

But the converse of Theorem $6.2.38$ may not be true as in the next example shows.

Example 6.2.39. In Example $4.2.17$, the space (X, τ) is γ-P_S-symmetric, but it is not γ-P_S-T_1 since τ_γ-$P_SO(X) = \{\phi, X\}$.

Sicne every γ-P_S-T_2 space is γ-P_S-T_1, then by Theorem $6.2.38$ that every γ-P_S-T_2 space is γ-P_S-symmetric. But there is no relation between γ-P_S-T_0 and γ-P_S-symmetric spaces.

Theorem 6.2.40. Let (X, τ) be a topological space and γ be an operation on τ. Then (X, τ) is γ-P_S-T_1 if and only if (X, τ) is γ-P_S-T_0 and γ-P_S-symmetric.

Proof. If (X, τ) is γ-P_S-T_1, then by Theorem $6.2.8$ and Theorem $6.2.38$, (X, τ) is both γ-P_S-T_0 and γ-P_S-symmetric respectively.

Conversely, let x, y be any two distinct points of γ-P_S-T_0 space (X, τ), then by hypothesis there exists a γ-P_S-open set G containing x or y (say x, but not y). This means that $x \in G \subseteq X \backslash \{y\}$. Then $G \cap \{y\} = \phi$ and hence $x \notin \tau_\gamma$-$P_SCl(\{y\})$. Since (X, τ) is

γ-P_S-symmetric, then $y \notin \tau_\gamma$-$P_SCl(\{x\})$ which implies that there exists a γ-P_S-open set H such that $y \in H \subseteq X \backslash \{x\}$. Therefore, the space (X, τ) is γ-P_S-T_1. □

Theorem 6.2.41. Let (X, τ) be any γ-P_S-symmetric space and γ be an operation on τ. Then the following properties are equivalent:

1. X is γ-P_S-T_1.

2. X is γ-P_S-$T_{\frac{1}{2}}$.

3. X is γ-P_S-T_0.

Proof. The proof of the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are follows directly from Theorem 6.2.8.

(3) $\Rightarrow$ (1) It is clear from Theorem 6.2.40 since (X, τ) is γ-P_S-symmetric space. □

Definition 6.2.42. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then the γ-P_S-kernel of A is denoted by γ-$P_Sker(A)$ and is defined as follows:

$$\gamma\text{-}P_Sker(A) = \cap\{G : A \subseteq G \text{ and } G \in \tau_\gamma\text{-}P_SO(X, \tau)\}$$

In other words, γ-$P_Sker(A)$ is the intersection of all γ-P_S-open sets of (X, τ) containing A.

The following remark follows directly from Definition 6.2.42.

Remark 6.2.43. Let $A \subseteq (X, \tau)$ and γ be an operation on τ. Then the following are true:

1. $A \subseteq \gamma$-$P_Sker(A)$.

2. If A is γ-P_S-open set in X, then γ-$P_Sker(A) = A$.

The following example shows that the converse of the above remark is not true in general.

Example 6.2.44. In Example 4.2.3, the γ-$P_Sker(\{b\}) = \{b, c\}$ and hence γ-$P_Sker(A) \nsubseteq A$. Again, γ-$P_Sker(\{c\}) = \{c\}$, but the set $\{c\}$ is not γ-P_S-open in X.

Theorem 6.2.45. Let A be a subset of a topological space (X, τ) and γ be an operation on τ. Then γ-$P_Sker(A) = \{x \in X$: τ_γ-$P_SCl(\{x\}) \cap A \neq \phi\}$.

Proof. Let $x \in \gamma$-$P_Sker(A)$. Suppose that τ_γ-$P_SCl(\{x\}) \cap A = \phi$. Then $x \notin X \backslash \tau_\gamma$-$P_SCl(\{x\})$ and $X \backslash \tau_\gamma$-$P_SCl(\{x\})$ is γ-P_S-open set in X such that $A \subseteq X \backslash \tau_\gamma$-$P_SCl(\{x\})$. Hence $x \notin \gamma$-$P_Sker(A)$. This is contradiction of the hypothesis. So τ_γ-$P_SCl(\{x\}) \cap A \neq \phi$.

Now, for the other part, let $x \in X$ such that τ_γ-$P_SCl(\{x\}) \cap A \neq \phi$. Suppose that $x \notin \gamma$-$P_Sker(A)$. Then there exists a γ-P_S-open set G containing A and $x \notin G$ and hence $x \notin A$. So, let $y \in \tau_\gamma$-$P_SCl(\{x\}) \cap A$. Then G is a γ-P_S-open set containing y but does not contain x. This means that τ_γ-$P_SCl(\{x\}) \cap A = \phi$ which is contradiction of the assumption. Therefore, $x \in \gamma$-$P_Sker(A)$. □

Theorem 6.2.46. Let (X, τ) be a topological space with an operation γ on τ and let $x, y \in X$. Then $y \in \gamma$-$P_Sker(\{x\})$ if and only if $x \in \tau_\gamma$-$P_SCl(\{y\})$.

Proof. Suppose that $y \notin \gamma$-$P_Sker(\{x\})$. Then there exists a γ-P_S-open set G containing x such that $y \notin G$. That is, $x \notin \tau_\gamma$-$P_SCl(\{y\})$.

Conversely, the proof is similar to the above case. □

Theorem 6.2.47. Let $A \subseteq (X, \tau)$ and γ be an operation on τ. Then A is γ-P_S-g-closed if and only if τ_γ-$P_SCl(A) \subseteq \gamma$-$P_Sker(A)$.

Proof. Suppose that A is γ-P_S-g-closed. Then τ_γ-$P_SCl(A) \subseteq G$, whenever $A \subseteq G$ and G is γ-P_S-open. Let $x \in \tau_\gamma$-$P_SCl(A)$. Suppose that $x \notin \gamma$-$P_Sker(A)$. Then there exists a γ-P_S-open set G containing A and $x \notin G$. This implies that $x \notin \tau_\gamma$-$P_SCl(A)$. Which is contradiction to the hypothesis. Hence $x \in \gamma$-$P_Sker(A)$ and so τ_γ-$P_SCl(A) \subseteq \gamma$-$P_Sker(A)$.

Conversely, let τ_γ-$P_SCl(A) \subseteq \gamma$-$P_Sker(A)$. If G is any γ-P_S-open set containing A. Then γ-$P_Sker(A) \subseteq G$. That is τ_γ-$P_SCl(A) \subseteq \gamma$-$P_Sker(A) \subseteq G$. Therefore, A is γ-P_S-g-closed set in X. □

Corollary 6.2.48. Let (X, τ) be a topological space and γ be an operation on τ. Then X is γ-P_S-symmetric if and only if τ_γ-$P_SCl(\{x\}) \subseteq \gamma$-$P_Sker(\{x\})$ for every $x \in X$.

Proof. The proof is immediate consequence of Theorem 6.2.47 and Lemma 6.2.32. □

Corollary 6.2.49. If (X, τ) is either γ-locally indiscrete or γ-hyperconnected space, then τ_γ-$P_SCl(\{x\}) \subseteq \gamma$-$P_Sker(\{x\})$ for every $x \in X$.

Proof. Straightforward from Corollary 6.2.34, Corollary 6.2.37 and Corollary 6.2.48. □

Theorem 6.2.50. For any points x and y in a topological space (X, τ) with an operation γ on τ. Then τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ if and only if γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$.

Proof. Let τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. This means that there is a point p in X such that $p \in \tau_\gamma$-$P_SCl(\{x\})$ and $p \notin \tau_\gamma$-$P_SCl(\{y\})$. Then there exists a γ-P_S-open set containing p and x, but not containing y. Hence by Definition 6.2.42, $y \notin \gamma$-$P_Sker(\{x\})$ and so γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$.

Conversely, suppose that γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$. That is, there is a point p in X such that $p \in \gamma$-$P_Sker(\{x\})$ and $p \notin \gamma$-$P_Sker(\{y\})$. When the point $p \in \gamma$-$P_Sker(\{x\})$, then by Theorem 6.2.45, τ_γ-$P_SCl(\{p\}) \cap \{x\} \neq \phi$. This means that $x \in \tau_\gamma$-$P_SCl(\{p\})$. Now when the point $p \notin \gamma$-$P_Sker(\{y\})$. Then by Theorem 6.2.45, τ_γ-$P_SCl(\{p\}) \cap \{y\} = \phi$. Since $x \in \tau_\gamma$-$P_SCl(\{p\})$ which implies that τ_γ-$P_SCl(\{x\}) \subseteq \tau_\gamma$-$P_SCl(\{p\})$. So τ_γ-$P_SCl(\{x\}) \cap \{y\} = \phi$. Therefore, τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. This completes the proof. $\square$

6.3 γ-P_S-Regular and γ-P_S-Normal Spaces

In this section, γ-P_S-regular and γ-P_S-normal spaces are which two new types of γ-P_S- separation axioms will be defined. Some characterizations of γ-P_S-regular and γ-P_S-normal spaces are investigated.

Definition 6.3.1. A topological space (X, τ) with an operation γ on τ is said to be γ-P_S-regular if for each γ-P_S-closed set F of X not containing x, there exist disjoint γ-P_S-open sets G and H such that $x \in G$ and $F \subseteq H$.

The following are some significant characterizations of γ-P_S-regular spaces.

Theorem 6.3.2. Let (X, τ) be a topological space and γ be an operation on τ. Then the following conditions are holds:

1. X is γ-P_S-regular.

2. For each point x in X and each γ-P_S-open set U containing x, there exists a γ-P_S-open set V such that $x \in V \subseteq \tau_\gamma\text{-}P_SCl(V) \subseteq U$.

3. For each γ-P_S-closed set F of X, $\cap \{\tau_\gamma\text{-}P_SCl(H) : F \subseteq H$ and $H \in \tau_\gamma\text{-}P_SO(X)\} = F$.

4. For each subset S of X and each γ-P_S-open set G of X such that $S \cap G \neq \phi$, there exists a γ-P_S-open set H of X such that $S \cap H \neq \phi$ and $\tau_\gamma\text{-}P_SCl(H) \subseteq G$.

5. For each nonempty subset S of X and each γ-P_S-closed set F of X such that $S \cap F = \phi$, there exists γ-P_S-open sets M and N of X such that $S \cap M \neq \phi$, $F \subseteq N$ and $M \cap N = \phi$.

6. For each $x \in X$ and each γ-P_S-closed set $F \subseteq X$ such that $x \notin F$, there exists γ-P_S-open set M and γ-P_S-g-open set N such that $x \in M$, $F \subseteq N$ and $M \cap N = \phi$.

7. For each $S \subseteq X$ and each γ-P_S-closed set $F \subseteq X$ such that $S \cap F = \phi$, there exists γ-P_S-open set M and γ-P_S-g-open set M such that $S \cap M \neq \phi$, $F \subseteq N$ and $M \cap N = \phi$.

Proof. (1) $\Rightarrow$ (2) Let x be any point in X and U be any γ-P_S-open set in X containing x. Then $X \backslash U$ is γ-P_S-closed set such that $x \notin X \backslash U$. Since X is γ-P_S-regular space, then there exists γ-P_S-open sets V and H such that $x \in V$ and $X \backslash U \subseteq H$ and $V \cap H = \phi$. This implies that $V \subseteq X \backslash H$ and hence $x \in V \subseteq \tau_\gamma$-$P_S Cl(V) \subseteq U$.

(2) $\Rightarrow$ (3) Let F be any γ-P_S-closed set of X. Suppose that $x \in X \backslash F$, where $X \backslash F$ is γ-P_S-open set and $x \in X$. Then by (2), there exists a γ-P_S-open set V such that $x \in V \subseteq \tau_\gamma$-$P_S Cl(V) \subseteq X \backslash F$. This implies that $F \subseteq X \backslash \tau_\gamma$-$P_S Cl(V)$. Let $H = X \backslash \tau_\gamma$-$P_S Cl(V)$, then H is γ-P_S-open set such that $V \cap H = \phi$. Thus, by Theorem 4.4.13, $x \notin \tau_\gamma$-$P_S Cl(H)$. Therefore, $\cap \{\tau_\gamma$-$P_S Cl(H) : F \subseteq H$ and $H \in \tau_\gamma$-$P_S O(X)\} \subseteq F$.

(3) $\Rightarrow$ (4) Let S be any subset of X and G be any γ-P_S-open set of X such that $S \cap G \neq \phi$. Let $x \in S \cap G$. Then $x \notin X \backslash G$. Since $X \backslash G$ is γ-P_S-closed, then by (3), there exists a γ-P_S-open set O such that $X \backslash G \subseteq O$ and $x \notin \tau_\gamma$-$P_S Cl(O)$. If we take $H = X \backslash \tau_\gamma$-$P_S Cl(O)$, then H is a γ-P_S-open set of X containing x and $S \cap H \neq \phi$. Now $H \subseteq X \backslash O$ and hence τ_γ-$P_S Cl(H) \subseteq X \backslash O \subseteq G$.

(4) $\Rightarrow$ (5) Let S be any nonempty subset of X and F be any γ-P_S-closed set of X such that $S \cap F = \phi$. Then $X \backslash F$ is γ-P_S-open set and $S \cap X \backslash F \neq \phi$. By (4), there exists a γ-P_S-open set M of X such that $S \cap M \neq \phi$ and τ_γ-$P_S Cl(M) \subseteq X \backslash F$. Let $N = X \backslash \tau_\gamma$-$P_S Cl(M)$, then $F \subseteq N$ and $M \cap N = \phi$.

(5) $\Rightarrow$ (6) Let x be any point X and a γ-P_S-closed set $F \subseteq X$ such that $x \notin F$. Then $\{x\}$ is a subset of X and $\{x\} \cap F = \phi$. Hence by using (5), there exists γ-P_S-open sets M and N of X such that $\{x\} \cap M \neq \phi$, $F \subseteq N$ and $M \cap N = \phi$. Since every γ-P_S-open set is γ-P_S-g-open, then N is a γ-P_S-g-open set such that $x \in M$, $F \subseteq N$ and $M \cap N = \phi$.

(6) $\Rightarrow$ (7) For any subset S of X and any γ-P_S-closed set F in X such that $S \cap F = \phi$.

Suppose $x \in S$, then $x \notin F$. By (6), there exists γ-P_S-open set M and γ-P_S-g-open set N such that $x \in M$, $F \subseteq N$ and $M \cap N = \phi$. Hence $S \cap M \neq \phi$.

(7) $\Rightarrow$ (1) For each $x \in X$ and each γ-P_S-closed set F of X such that $x \notin F$. Then $\{x\}$ is a subset of X and $\{x\} \cap F = \phi$. So by using (7), there exists γ-P_S-open set M and γ-P_S-g-open set N such that $\{x\} \cap M \neq \phi$, $F \subseteq N$ and $M \cap N = \phi$. This implies that $x \in M$, $F \subseteq N$ and $M \cap N = \phi$. Now, let $H = \tau_\gamma$-$P_SInt(N)$. Thus, by Theorem 4.6.2, we obtain $F \subseteq H$ and $M \cap H = \phi$. Therefore, a space X is γ-P_S-regular. □

Theorem 6.3.3. The following statements are holds for any topological space (X, τ) and any operation γ on τ.

1. X is γ-P_S-regular.
2. For each γ-P_S-closed set F of X, $\cap\ \{\tau_\gamma$-$P_SCl(H) : F \subseteq H$ and $H \in \tau_\gamma$-$P_SO(X)\} = F$.
3. For each γ-P_S-closed set F of X, $\cap\ \{\tau_\gamma$-$P_SCl(H) : F \subseteq H$ and $H \in \tau_\gamma$-$P_SGO(X)\} = F$.

Proof. (1) $\Rightarrow$ (2) It is clear from Theorem 6.3.2.

(2) $\Rightarrow$ (3) It is evident that every γ-P_S-open set is γ-P_S-g-open.

(3) $\Rightarrow$ (1) Let F be a γ-P_S-closed set in X and let $x \in X$ such that $x \notin F$. Then by (3), there exists a γ-P_S-g-open set H such that $F \subseteq H$ and $x \notin \tau_\gamma$-$P_SCl(H)$. Since F is γ-P_S-closed and H is a γ-P_S-g-open, then by Theorem 4.6.2, we get $F \subseteq \tau_\gamma$-$P_SInt(H)$. Let $U = \tau_\gamma$-$P_SInt(H)$ which is γ-P_S-open and hence $F \subseteq U$. Also, let

$V = X \backslash \tau_\gamma$-$P_SCl(H)$ is a γ-P_S-open set containing x such that $V \cap U = \phi$. Hence X is γ-P_S-regular space. □

Theorem 6.3.4. Let (X, τ) be a γ-locally indiscrete space and γ be a regular operation on τ. Then X is γ-P_S-regular if and only if for each γ-P_S-closed set F of X not containing x in X, there exist γ-P_S-open sets G and H such that $x \in G$, $F \subseteq H$ and τ_γ-$P_SCl(G) \cap \tau_\gamma$-$P_SCl(H) = \phi$.

Proof. Let F be a γ-P_S-closed set in X and $x \in X$ does not belong to F. Since (X, τ) is γ-P_S-regular space, then there exist γ-P_S-open sets G_0 and H such that $x \in G_0$, $F \subseteq H$ and $G_0 \cap H = \phi$ which implies $G_0 \cap \tau_\gamma$-$P_SCl(H) = \phi$. Since τ_γ-$P_SCl(H)$ is γ-P_S-closed such that $x \notin \tau_\gamma$-$P_SCl(H)$ and X is γ-P_S-regular, then there exist γ-P_S-open sets U and V such that $x \in U$, τ_γ-$P_SCl(H) \subseteq V$ and $U \cap V = \phi$. This implies that τ_γ-$P_SCl(U) \cap V = \phi$. If we take $G = G_0 \cap U$. Since (X, τ) is a γ-locally indiscrete space and γ is a regular operation on τ, then by Theorem 4.2.23, G is γ-P_S-open set. So G and H are γ-P_S-open sets such that $x \in G$, $F \subseteq H$ and τ_γ-$P_SCl(G)$ $\cap\, \tau_\gamma$-$P_SCl(H) = \phi$.

The converse is trivial. □

Notice that γ-regular-open, γ-open, α-γ-open and γ-semiopen sets can be substitute to γ-P_S-open set in Theorem 6.3.4 (this is because of Corollary 4.2.21). Again, τ_γ-closure, $\tau_{\alpha-\gamma}$-closure and τ_γ-semi-closure of a set can be replaced by τ_γ-P_S-closure.

Theorem 6.3.5. The following statements are equivalent for for any topological space (X, τ) with an operation γ on τ.

1. X is γ-P_S-regular.

2. For each point x in X and each γ-P_S-open set H containing x, there exists a γ-P_S-open set G containing x such that τ_γ-$P_SCl(G) \subseteq H$.

3. For each point $x \in X$ and for each γ-P_S-closed set F of X not containing x, there exists a γ-P_S-open set G containing x such that τ_γ-$P_SCl(G) \cap F = \phi$.

Proof. (1) $\Rightarrow$ (2) It is similar to Theorem 6.3.2.

(2) $\Rightarrow$ (3) Let $x \in X$ and let F be a γ-P_S-closed set of X such that $x \notin F$. Then $X \backslash F$ is γ-P_S-open set containing x. So by (2), there exists a γ-P_S-open set $H \subseteq X$ containing x such that τ_γ-$P_SCl(G) \subseteq X \backslash F$. This implies that τ_γ-$P_SCl(G) \cap F = \phi$.

(3) $\Rightarrow$ (1) Let F be any γ-P_S-closed set in X such that $x \notin F$. Then by (3), there exists a γ-P_S-open set G containing x such that τ_γ-$P_SCl(G) \cap F = \phi$ which implies that $F \subseteq X \backslash \tau_\gamma$-$P_SCl(G)$. Since $X \backslash \tau_\gamma$-$P_SCl(G)$ is γ-P_S-open such that $X \backslash \tau_\gamma$-$P_SCl(G) \cap G = \phi$. Therefore, X is γ-P_S-regular space. □

The following example shows that the relation between the γ-P_S-regularity and γ-P_S-T_i for $i = 0, 1$ are independent.

Example 6.3.6. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.7. The space (X, τ) is γ-P_S-regular, but it is not γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$.

Again, let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. The space (X, τ) is γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$, but it is not γ-P_S-regular since the set $\{b\}$ is γ-P_S-closed not containing c, then there is no disjoint γ-P_S-open sets G and H such that $c \in G$ and $\{b\} \subseteq H$.

The relation between γ-P_S-regularity and γ-pre-regularity are independent while they are equivalent when a topological space (X, τ) is γ-semiT_1 as can be seen in the following two examples and theorem.

Example 6.3.7. In Example 4.2.3, the space (X, τ) is γ-pre-regular, but it is not γ-P_S-regular since the set $\{b\}$ is γ-P_S-closed not containing c, then there is no disjoint γ-P_S-open sets G and H such that $c \in G$ and $\{b\} \subseteq H$.

Example 6.3.8. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.17. Then the space (X, τ) is γ-P_S-regular, but it is not γ-pre-regular since the set $\{a, b\}$ is γ-preclosed not containing c, then there is no disjoint γ-preopen sets G and H such that $c \in G$ and $\{a, b\} \subseteq H$.

Theorem 6.3.9. Let (X, τ) be a γ-semiT_1 space. Then (X, τ) is γ-P_S-regular if and only if (X, τ) is γ-pre-regular.

Proof. The proof follows from Theorem 4.2.24 and Corollary 4.3.13. □

In the next results, we give some properties of γ-P_S-regular spaces which is related to γ-P_S- functions.

Theorem 6.3.10. Let $f\colon (X, \tau) \to (Y, \sigma)$ be a bijective, (γ, β)-P_S-irresolute and (γ, β)-P_S-closed, where γ and β are operations on τ and σ respectively. If (X, τ) is γ-P_S-regular, then (Y, σ) is also β-P_S-regular.

Proof. Let E be a β-P_S-closed set of (Y, σ) such that $y \notin E$. Since f is (γ, β)-P_S-irresolute function, then by Theorem 5.3.10 (3), $f^{-1}(E)$ is γ-P_S-closed in

(X, τ). Since f is bijective, let $f(x) = y$, then $x \notin f^{-1}(E)$. Since (X, τ) is γ-P_S-regular space, then there exist disjoint γ-P_S-open sets G and H in X such that $x \in G$ and $f^{-1}(E) \subseteq H$. Since f is (γ, β)-P_S-closed, then by Theorem 5.3.69, we have there exists a β-P_S-open set U in Y containing E such that $f^{-1}(U) \subseteq H$. Since f is bijective (γ, β)-P_S-closed function. Then by Theorem 5.3.74, $y \in f(G)$ and $f(G)$ is β-P_S-open set in Y. Thus, $G \cap f^{-1}(U) = \phi$ and hence $f(G) \cap U = \phi$. Therefore, (Y, σ) is β-P_S-regular. □

Theorem 6.3.11. Let $f\colon (X, \tau) \rightarrow (Y, \sigma)$ be a surjective (γ, β)-P_S-irresolute, (γ, β)-P_S-open and (γ, β)-P_S-closed, where γ and β are operations on τ and σ respectively. If (X, τ) is γ-P_S-regular, then (Y, σ) is also β-P_S-regular.

Proof. Similar to Theorem 6.3.10. □

Theorem 6.3.12. Let $f\colon (X, \tau) \rightarrow (Y, \sigma)$ be a surjective β-pre-anti-continuous, β-P_S-open and β-P_S-closed function with the operation β on σ. If (X, τ) is regular, then (Y, σ) is β-P_S-regular.

Proof. Let E be a β-P_S-closed set of (Y, σ) containing y. Then E is β-preclosed. Since a function f is β-pre-anti-continuous, then $f^{-1}(E)$ is closed set in X containing $x \in X$, where $y = f(x)$. Since a space X is regular, there exists an open set V such that $x \in V \subseteq Cl(V) \subseteq f^{-1}(E)$ which implies that $y \in f(V) \subseteq f(Cl(V)) \subseteq E$. Since f is β-P_S-open and β-P_S-closed function, then $f(V)$ and $f(Cl(V))$ are β-P_S-open and β-P_S-closed sets in Y respectively. Then σ_β-$P_SCl(f(Cl(V))) \subseteq E$ and hence $y \in$

$f(V) \subseteq \sigma_\beta\text{-}P_S Cl(V) \subseteq \sigma_\beta\text{-}P_S Cl(f(Cl(V))) \subseteq E$. Consequently, by Theorem 6.3.2, (Y, σ) is β-P_S-regular. $\square$

Theorem 6.3.13. Let γ and β be operations on τ and σ respectively. Let $f\colon (X, \tau) \to (Y, \sigma)$ be an injective, (γ, β)-P_S-irresolute and (γ, β)-P_S-closed. If (Y, σ) is β-P_S-regular, then (X, τ) is also γ-P_S-regular.

Proof. Suppose that a set F is any γ-P_S-closed set of (X, τ) and $x \notin F$. Since f is (γ, β)-P_S-closed function. Then $f(F)$ is β-P_S-closed set in Y and $f(x) \notin f(F)$. Since (Y, σ) is β-P_S-regular. Then there exist disjoint β-P_S-open sets U and V in Y such that $f(x) \in U$ and $f(F) \subseteq V$. Since f is injective, then $x \in f^{-1}(U)$, $F \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Since f is (γ, β)-P_S-irresolute, then by Theorem 5.3.10 (2), $f^{-1}(U)$ and $f^{-1}(V)$ are γ-P_S-open sets in X. So (X, τ) is γ-P_S-regular. $\square$

Theorem 6.3.14. Let γ be an operation on τ. Let $f\colon (X, \tau) \to (Y, \sigma)$ be an injective γ-P_S-continuous and γ-pre-anti-closed. If (Y, σ) is regular, then (X, τ) is γ-P_S-regular.

Proof. Let F be a γ-P_S-closed set of X containing x. Then F is γ-preclosed. Since a function f is γ-pre-anti-closed, then $f(F)$ is closed set in Y containing $y \in Y$, where $y = f(x)$. Since a space (Y, σ) is regular, then there exists an open set V such that $y \in V \subseteq Cl(V) \subseteq f(F)$ which implies that $x \in f^{-1}(V) \subseteq f^{-1}(Cl(V)) \subseteq F$. Since f is γ-P_S-continuous function, then by Theorem $5.3.1$ (2) and (3), $f^{-1}(V)$ and $f^{-1}(Cl(V))$ are γ-P_S-open and γ-P_S-closed sets in X respectively. Then $\tau_\gamma\text{-}P_S Cl(f^{-1}(Cl(V))) \subseteq F$ and hence $x \in f^{-1}(V) \subseteq \tau_\gamma\text{-}P_S Cl(f^{-1}(V)) \subseteq \tau_\gamma\text{-}P_S Cl(f^{-1}(Cl(V))) \subseteq F$. Therefore, by Theorem $6.3.2$, (X, τ) is γ-P_S-regular space. $\square$

Now, γ-P_S-normal space which is another type of γ-P_S- separation axioms is examined.

Definition 6.3.15. A topological space (X, τ) with an operation γ on τ is said to be γ-P_S-normal if for each pair of disjoint γ-P_S-closed sets E, F of X, there exist disjoint γ-P_S-open sets G and H such that $E \subseteq G$ and $F \subseteq H$.

Some important characterizations of γ-P_S-normal spaces are investigated as follows.

Theorem 6.3.16. Let (X, τ) be a γ-locally indiscrete space and γ be a regular operation on τ. Then X is γ-P_S-normal if and only if for each pair of disjoint γ-P_S-closed sets F and E of X, there exist disjoint γ-P_S-open sets G and H such that $E \subseteq G$, $F \subseteq H$ and τ_γ-$P_SCl(G) \cap \tau_\gamma$-$P_SCl(H) = \phi$.

Proof. Let E and F be disjoint γ-P_S-closed sets in (X, τ). Since (X, τ) is γ-P_S-normal space, there exist disjoint γ-P_S-open sets G and H such that $E \subseteq W$, $F \subseteq H$ and $W \cap H = \phi$. This implies that $W \cap \tau_\gamma$-$P_SCl(H) = \phi$. Since τ_γ-$P_SCl(H)$ is γ-P_S-closed set such that $E \cap \tau_\gamma$-$P_SCl(H) = \phi$ and X is γ-P_S-normal, then there exist γ-P_S-open sets U and V such that $E \subseteq U$, τ_γ-$P_SCl(H) \subseteq V$ and $U \cap V = \phi$ which implies that τ_γ-$P_SCl(U) \cap V = \phi$. If we put $G = W \cap U$. Since a space X is γ-locally indiscrete and γ is a regular operation on τ, then by Theorem 4.2.23, G is γ-P_S-open set in X. Hence G and H are γ-P_S-open sets such that $E \subseteq G$, $F \subseteq H$ and τ_γ-$P_SCl(G) \cap \tau_\gamma$-$P_SCl(H) = \phi$.

Conversely, suppose E and F are disjoint γ-P_S-closed sets in X. Then by hypothesis, we have there exist disjoint γ-P_S-open sets G and H such that $E \subseteq G$, $F \subseteq H$ and τ_γ-$P_SCl(G) \cap \tau_\gamma$-$P_SCl(H) = \phi$. This implies that $G \cap H = \phi$ and hence (X, τ) is γ-P_S-normal space. □

Notice that γ-P_S-open set in Theorem $6.3.16$ can be replaced by γ-regular-open, γ-open, α-γ-open and γ-semiopen sets according to Corollary $4.2.21$. Again, τ_γ-P_S-closure can be replaced by τ_γ-closure, $\tau_{\alpha-\gamma}$-closure and τ_γ-semi-closure of a set.

Theorem 6.3.17. The following conditions are equivalent for for any topological space (X, τ) with an operation γ on τ.

1. X is γ-P_S-normal.
2. For each γ-P_S-closed set F in X and each γ-P_S-open set G containing F, there exists a γ-P_S-open set H containing F such that τ_γ-$P_SCl(H) \subseteq G$.
3. For each pair of disjoint γ-P_S-closed sets F and E of X, there exists a γ-P_S-open set G containing E such that τ_γ-$P_SCl(G) \cap F = \phi$.

Proof. (1) $\Rightarrow$ (2) Let F be any γ-P_S-closed set in X and G be any γ-P_S-open set in X such that $F \subseteq G$. Then $F \cap X \backslash G = \phi$ and $X \backslash G$ is γ-P_S-closed. By hypothesis, there exist γ-P_S-open sets H and W such that $F \subseteq H$ and $X \backslash G \subseteq W$ and $H \cap W = \phi$ which implies τ_γ-$P_SCl(H) \cap W = \phi$. Then τ_γ-$P_SCl(H) \subseteq X \backslash W \subseteq G$. So $F \subseteq H$ and τ_γ-$P_SCl(H) \subseteq G$.

(2) $\Rightarrow$ (3) Let E and F be γ-P_S-closed sets of X such that $E \cap F = \phi$. Then $E \subseteq X \backslash F$ and $X \backslash F$ is γ-P_S-open set containing E. By (2), there exists a γ-P_S-open set $G \subseteq X$ containing E such that τ_γ-$P_SCl(G) \subseteq X \backslash F$. This implies that τ_γ-$P_SCl(G) \cap F = \phi$.

(3) $\Rightarrow$ (1) Let E and F be disjoint γ-P_S-closed sets of X . Then by (3), there exists a γ-P_S-open set G containing E such that τ_γ-$P_SCl(G) \cap F = \phi$ which implies that

$F \subseteq X \backslash \tau_\gamma$-$P_S Cl(G)$. Since $X \backslash \tau_\gamma$-$P_S Cl(G)$ is γ-P_S-open such that $X \backslash \tau_\gamma$-$P_S Cl(G) \cap G = \phi$. Therefore, X is γ-P_S-normal. □

Theorem 6.3.18. Let (X, τ) be a topological space and γ be an operation on τ. Then the following statements are holds:

1. X is γ-P_S-normal.

2. For each pair of disjoint γ-P_S-closed sets F and E of X, there exist disjoint γ-P_S-g-open sets G and H such that $E \subseteq G$ and $F \subseteq H$.

3. For each γ-P_S-closed set F in X and each γ-P_S-open set U containing F, there exists a γ-P_S-g-open set V such that $F \subseteq V \subseteq \tau_\gamma$-$P_S Cl(V) \subseteq U$.

4. For each γ-P_S-closed set F in X and each γ-P_S-g-open set U containing F, there exists a γ-P_S-g-open set V such that $F \subseteq V \subseteq \tau_\gamma$-$P_S Cl(V) \subseteq \tau_\gamma$-$P_S Int(U)$.

5. For each γ-P_S-closed set F in X and each γ-P_S-g-open set U containing F, there exists a γ-P_S-open set V such that $F \subseteq V \subseteq \tau_\gamma$-$P_S Cl(V) \subseteq \tau_\gamma$-$P_S Int(U)$.

6. For each γ-P_S-g-closed set F in X and each γ-P_S-open set U containing F, there exists a γ-P_S-open set V such that τ_γ-$P_S Cl(F) \subseteq V \subseteq \tau_\gamma$-$P_S Cl(V) \subseteq U$.

7. For each γ-P_S-g-closed set F in X and each γ-P_S-open set U containing F, there exists a γ-P_S-g-open set V such that τ_γ-$P_S Cl(F) \subseteq V \subseteq \tau_\gamma$-$P_S Cl(V) \subseteq U$.

Proof. (1) ⇒ (2) It is obvious, since every γ-P_S-open set is γ-P_S-g-open.

(2) $\Rightarrow$ (3) Let F be any γ-P_S-closed set in X and let U be any γ-P_S-open set containing F. Then $X\backslash U$ is γ-P_S-closed set in X and hence by using (2), there exist disjoint γ-P_S-g-open sets G and H such that $F \subseteq G$ and $X\backslash U \subseteq H$. Since $X\backslash U$ is γ-P_S-closed and H is γ-P_S-g-open. Then by Theorem 4.6.2, $X\backslash U \subseteq \tau_\gamma$-$P_SInt(H)$ and $G \cap \tau_\gamma$-$P_SInt(H) = \phi$. So by using Corollary 4.4.34, τ_γ-$P_SCl(G) \cap \tau_\gamma$-$P_SInt(H) = \phi$ which implies that τ_γ-$P_SCl(G) \subseteq X\backslash\tau_\gamma$-$P_SInt(H) \subseteq U$. Therefore, $F \subseteq G \subseteq \tau_\gamma$-$P_SCl(G) \subseteq U$.

(3) $\Rightarrow$ (4) Let $F \subseteq X$ be any γ-P_S-closed set and let U be any γ-P_S-open set $F \subseteq U$. Then τ_γ-$P_SInt(U)$ is γ-P_S-open. Then by (3), we have there exists a γ-P_S-g-open set V such that $F \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq \tau_\gamma$-$P_SInt(U)$.

(4) $\Rightarrow$ (5) For each γ-P_S-closed set F in X and each γ-P_S-g-open set U containing F. Then by (4), there exists a γ-P_S-g-open set V such that $F \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq \tau_\gamma$-$P_SInt(U)$. Since $F \subseteq V$ where F is γ-P_S-closed set and U is γ-P_S-g-open set, then by Theorem 4.6.2, $F \subseteq \tau_\gamma$-$P_SInt(V)$. Since τ_γ-$P_SCl(\tau_\gamma$-$P_SInt(V)) \subseteq \tau_\gamma$-$P_SCl(V)$. Therefore, τ_γ-$P_SInt(V)$ is a γ-P_S-open set such that $F \subseteq \tau_\gamma$-$P_SInt(V) \subseteq \tau_\gamma$-$P_SCl(\tau_\gamma$-$P_SInt(V)) \subseteq \tau_\gamma$-$P_SInt(U)$.

(5) $\Rightarrow$ (6) Suppose that $F \subseteq U$, where U is γ-P_S-open set and F is γ-P_S-g-closed set of X. So by Definition 4.5.1, τ_γ-$P_SCl(F) \subseteq U$. Since every γ-P_S-open set is γ-P_S-g-open (that is, U is γ-P_S-g-open) and τ_γ-$P_SCl(F)$ is γ-P_S-closed, then by (5), there exists a γ-P_S-open set V such that τ_γ-$P_SCl(F) \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq \tau_\gamma$-$P_SInt(U) = U$.

(6) $\Rightarrow$ (7) The proof is clear.

(7) $\Rightarrow$ (1) Suppose that E and F are two γ-P_S-closed sets of X such that $E \cap F = \phi$. This implies that $F \subseteq X \backslash E$, F is γ-P_S-g-closed and $X \backslash E$ is γ-P_S-open. Then by (7), there exists a γ-P_S-g-open set V such that τ_γ-$P_SCl(F) \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq X \backslash E$. By Theorem 4.6.2, $F \subseteq \tau_\gamma$-$P_SInt(V)$. Now $E \subseteq X \backslash \tau_\gamma$-$P_SCl(V)$. Let $G = X \backslash \tau_\gamma$-$P_SCl(V)$ and $H = \tau_\gamma$-$P_SInt(V)$ are γ-P_S-open sets of X such that $E \subseteq G$, $F \subseteq H$ and $G \cap H = \phi$. Consequently, X is γ-P_S-normal space. □

The relation between γ-P_S-normality and γ-pre-normality are independent. However, they are equivalent when a topological space (X, τ) is γ-semiT_1 as can be explained in the following two examples and theorem.

Example 6.3.19. Let (X, τ) be a topological space and γ be an operation on τ as in Example 4.2.3. Then the space (X, τ) is γ-pre-regular, but it is not γ-P_S-regular since the set $\{b\}$ is γ-P_S-closed not containing c, then there is no disjoint γ-P_S-open sets G and H such that $c \in G$ and $\{b\} \subseteq H$.

Example 6.3.20. In Example 4.2.17. Then the space (X, τ) is γ-P_S-normal, but it is not γ-pre-normal since $\{a\}$ and $\{b\}$ are disjoint γ-preclosed sets, then there is no disjoint γ-preopen sets G and H such that $\{a\} \in G$ and $\{b\} \subseteq H$.

Theorem 6.3.21. Let (X, τ) be a γ-semiT_1 space. Then (X, τ) is γ-P_S-normal if and only if (X, τ) is γ-pre-normal.

Proof. The proof follows from Theorem 4.2.24 and Corollary 4.3.13. □

Lemma 6.3.22. If (X, τ) is γ-P_S-T_1 space. Then every γ-P_S-normal space X is γ-P_S-regular.

Proof. Let F be a γ-P_S-closed set in X and $x \in X$ does not belong to F. Since X is γ-P_S-T_1. Then by Theorem 6.2.3, $\{x\}$ is γ-P_S-closed. So $\{x\}$ and F are two disjoint γ-P_S-closed sets of X. Since X is γ-P_S-normal, then there exist disjoint γ-P_S-open sets G and H such that $x \in \{x\} \subseteq G$ and $F \subseteq H$. Hence X is γ-P_S-regular. □

The following example shows that the converse of Lemma 6.3.22 is not true in general.

Example 6.3.23. In Example 4.2.7, the space (X, τ) is both γ-P_S-regular and γ-P_S-normal, but (X, τ) is not γ-P_S-T_1 since every γ-P_S-open set containing b contains c also.

As we mentioned in Theorem 6.2.8 (1) that every γ-P_S-T_2 space is γ-P_S-T_1 while the converse is true only when a space X is γ-P_S-regular as shown in the following Lemma 6.3.24.

Lemma 6.3.24. If (X, τ) is γ-P_S-regular and γ-P_S-T_1 space. Then it is γ-P_S-T_2.

Proof. Similar to Lemma 6.3.22. □

From Lemma 6.3.22 and Lemma 6.3.24, we have the following theorem.

Theorem 6.3.25. If (X, τ) is γ-P_S-normal and γ-P_S-T_1 space. Then it is γ-P_S-T_2.

Proof. Obvious. □

In the end of this section, we obtain some properties of γ-P_S-normal spaces which is related to γ-P_S- functions.

Theorem 6.3.26. For any operations γ and β on τ and σ respectively. If $f\colon (X,\tau) \to (Y,\sigma)$ is bijective (γ,β)-P_S-irresolute and (γ,β)-P_S-open and (X,τ) is γ-P_S-normal. Then (Y,σ) is also β-P_S-normal.

Proof. Suppose that E_1 and E_2 are disjoint β-P_S-closed sets of (Y,σ) and f is (γ,β)-P_S-irresolute function, then by Theorem 5.3.10 (3), $f^{-1}(E_1)$ and $f^{-1}(E_2)$ are γ-P_S-closed sets in (X,τ). Since (X,τ) is γ-P_S-normal space, then there exist disjoint γ-P_S-open sets G_1 and G_2 in X such that $f^{-1}(E_1) \subseteq G_1$ and $f^{-1}(E_2) \subseteq G_2$. Since a function f is (γ,β)-P_S-open and bijective, then $f(G_1)$ and $f(G_2)$ are β-P_S-open sets in (Y,σ) such that $E_1 \subseteq f(G_1)$, $E_2 \subseteq f(G_2)$ and $f(G_2) \cap f(G_2) = \phi$. Then (Y,σ) is β-P_S-normal space. $\square$

Corollary 6.3.27. Let $f\colon (X,\tau) \to (Y,\sigma)$ be a surjective (γ,β)-P_S-irresolute, (γ,β)-P_S-open and (γ,β)-P_S-closed, where γ and β are operations on τ and σ respectively. If (X,τ) is γ-P_S-normal, then (Y,σ) is also β-P_S-normal.

Proof. Directly follows from Theorem 6.3.26. $\square$

Theorem 6.3.28. Let $f\colon (X,\tau) \to (Y,\sigma)$ be a bijective β-pre-anti-continuous, β-P_S-open function with the operation β on σ. If (X,τ) is normal, then (Y,σ) is β-P_S-normal.

Proof. Let F_1 and F_2 are disjoint β-P_S-closed sets in (Y,σ). Then F_1 and F_2 are disjoint β-preclosed sets. Since f is β-pre-anti-continuous, then the sets $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are closed sets in (X,τ). Since (X,τ) is normal, then there exist disjoint open sets V_1 and

V_2 such that $f^{-1}(F_1) \subseteq V_1$ and $f^{-1}(F_2) \subseteq V_2$. Since f is bijective β-P_S-open function, then $f(V_1)$ and $f(V_2)$ are β-P_S-open sets in (Y, σ) such that $F_1 \subseteq f(V_1)$, $F_2 \subseteq f(V_2)$ and $f(V_2) \cap f(V_2) = \phi$. Hence (Y, σ) is β-P_S-normal space. $\square$

Theorem 6.3.29. Let γ and β be operations on τ and σ respectively. Let $f \colon (X, \tau) \to (Y, \sigma)$ be an injective, (γ, β)-P_S-irresolute and (γ, β)-P_S-closed. If (Y, σ) is β-P_S-normal, then (X, τ) is also γ-P_S-normal.

Proof. Assume that f is (γ, β)-P_S-closed and, F_1 and F_2 are disjoint γ-P_S-closed sets of (X, τ). Then $f(F_1)$ and $f(F_2)$ are β-P_S-closed sets in (Y, σ). Since (Y, σ) is β-P_S-normal space, then there exist disjoint β-P_S-open sets H_1 and H_2 in X such that $f(F_1) \subseteq H_1$ and $f(F_2) \subseteq H_2$. Since f is injective, then $F_1 \subseteq f^{-1}(H_1)$, $F_2 \subseteq f^{-1}(H_2)$ and $f^{-1}(H_1) \cap f^{-1}(H_2) = \phi$. Since f is (γ, β)-P_S-irresolute, then by Theorem 5.3.10 (2), $f^{-1}(H_1)$ and $f^{-1}(H_2)$ are γ-P_S-open sets in (X, τ). Therefore, (X, τ) is γ-P_S-normal. $\square$

Theorem 6.3.30. For any operation γ on τ. If the function $f \colon (X, \tau) \to (Y, \sigma)$ is injective γ-P_S-continuous and γ-pre-anti-closed, and (Y, σ) is normal. Then (X, τ) is γ-P_S-normal.

Proof. Let F_1 and F_2 are disjoint γ-P_S-closed sets of (X, τ). Then F_1 and F_2 are disjoint γ-preclosed sets. Since f is γ-pre-anti-closed, then $f(F_1)$ and $f(F_2)$ are closed sets in (Y, σ). Since a space (Y, σ) is normal, then there exist open sets V_1 and V_2 in Y such that $f(F_1) \subseteq V_1$, $f(F_2) \subseteq V_2$ and $V_1 \cap V_2 = \phi$. Now since f is injective γ-P_S-continuous, then by Theorem 5.3.1 (2), $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are γ-P_S-open sets in (X, τ) such that

$F_1 \subseteq f^{-1}(V_1)$, $F_2 \subseteq f^{-1}(V_2)$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. So (X, τ) is γ-P_S-normal space. $\square$

6.4 γ-P_S-R_0 and γ-P_S-R_1 Spaces

In this section, another types of γ-P_S- separation axioms which are γ-P_S-R_0 and γ-P_S-R_1 spaces will be presented. Some of their characterizations will be discussed.

The definition of γ-P_S-R_0 space in terms of γ-P_S-open and $\tau_\gamma P_S$-closure operator is defined as follows:

Definition 6.4.1. A topological space (X, τ) with an operation γ on τ is said to be γ-P_S-R_0 if every γ-P_S-open set contains the τ_γ-P_S-closure of each of its singletons.

In other words, a space (X, τ) is said to be γ-P_S-R_0 if G is a γ-P_S-open set and $x \in G$, then τ_γ-$P_S Cl(\{x\}) \subseteq G$.

Some important characterizations of γ-P_S-R_0 spaces and relations between γ-P_S-R_0 space and other types of γ-P_S- separation axioms are established.

Theorem 6.4.2. For any topological space (X, τ) and any operation γ on τ. If τ_γ-$P_S Cl(\{x\}) \subseteq F$ whenever F is γ-semiclosed set in X and $x \in F$. Then (X, τ) is γ-P_S-R_0 space.

Proof. Let G be any γ-P_S-open set in X such that $x \in G$. Then Definition 4.2.1, there exists a γ-semiclosed set F in X such that $x \in F \subseteq G$. So by hypothesis,

τ_γ-$P_SCl(\{x\}) \subseteq F$ and hence τ_γ-$P_SCl(\{x\}) \subseteq G$. This means that (X, τ) is γ-P_S-R_0. □

Theorem 6.4.3. If a topological space (X, τ) is γ-P_S-R_0, then τ_γ-$Cl(\tau_\gamma$-$Int(\{x\})) \subseteq G$ whenever G is γ-P_S-open set in X and $x \in G$.

Proof. Straightforward from Definition 6.4.1 and Remark 4.4.21. □

Theorem 6.4.4. Let (X, τ) be a topological space and γ be an operation on τ. Then the following conditions are equivalent:

1. X is γ-P_S-R_0.
2. If for each γ-P_S-closed set E in X such that $x \notin E$, then there exists a γ-P_S-open set G such that $E \subseteq G$ and $x \notin G$.
3. If for each γ-P_S-closed set E in X such that $x \notin E$, then τ_γ-$P_SCl(\{x\}) \cap E = \phi$.
4. If for each distinct points $x, y \in X$, then either τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \phi$ or τ_γ-$P_SCl(\{x\}) = \tau_\gamma$-$P_SCl(\{y\})$.

Proof. (1) ⇒ (2) Let E be any γ-P_S-closed set in X and $x \notin E$. Since (X, τ) is γ-P_S-R_0, then τ_γ-$P_SCl(\{x\}) \subseteq X \backslash E$. Let $G = X \backslash \tau_\gamma$-$P_SCl(\{x\})$. Then G is γ-P_S-open set in X such that $E \subseteq G$ and $x \notin G$.

(2) ⇒ (3) Let E be any γ-P_S-closed set in X does not containing x. Then by (2), there exists $G \in \tau_\gamma$-$P_SO(X, \tau)$ such that $E \subseteq G$ and G does not containing x. This means that $\{x\} \cap G = \phi$ and hence by Corollary 4.4.34, we get τ_γ-$P_SCl(\{x\}) \cap G = \phi$ since G is γ-P_S-open set. That is, τ_γ-$P_SCl(\{x\}) \cap E = \phi$.

(3) $\Rightarrow$ (4) Let τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ for distinct elements x and y in X. Then there exists an element $z \in \tau_\gamma$-$P_SCl(\{x\})$ such that $z \notin \tau_\gamma$-$P_SCl(\{y\})$ (or $z \in \tau_\gamma$-$P_SCl(\{y\})$ such that $z \notin \tau_\gamma$-$P_SCl(\{x\})$). Then by Theorem 4.4.13, there exists γ-P_S-open set G of X such that $z \in G$ and $z \notin G$. So $\{x\} \cap G \neq \phi$ and $\{y\} \cap G = \phi$. This means that $x \in G$ and $x \notin \tau_\gamma$-$P_SCl(\{y\})$. Therefore, by (3), τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \phi$.

(4) $\Rightarrow$ (1) Let G be any γ-P_S-open set in X such that $x \in G$. For each point $y \in G$ and $x \neq y$. Then $x \notin \tau_\gamma$-$P_SCl(\{y\})$ which implies that τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. So by hypothesis, τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \phi$ for each $y \in X \backslash G$. Hence τ_γ-$P_SCl(\{x\}) \cap (\bigcup_{y \in X\backslash G} \tau_\gamma$-$P_SCl(\{y\})) = \phi$ for each $y \in X \backslash G$. On the other hand, since G is γ-P_S-open set in X and $y \in X \backslash G$ implies τ_γ-$P_SCl(\{y\}) \subseteq X \backslash G$. That is, $X \backslash G = \bigcup_{y \in X\backslash G} \tau_\gamma$-$P_SCl(\{y\})$. Therefore, τ_γ-$P_SCl(\{x\}) \cap X \backslash G = \phi$ and hence τ_γ-$P_SCl(\{x\}) \subseteq G$. Then by Definition 6.4.1, a space X is γ-P_S-R_0. $\square$

Corollary 6.4.5. A space (X, τ) is γ-P_S-R_0 if and only if τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ implies τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \phi$ for every x and y in X.

Proof. Directly follows from Theorem 6.4.4. $\square$

Theorem 6.4.6. A space (X, τ) is γ-P_S-R_0 if and only if γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$ implies γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\}) = \phi$ for every x and y in X.

Proof. Assume that (X, τ) is γ-P_S-R_0 space. By Theorem 6.2.50, if γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$, then τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ for every points x and y in X. Hence by Corollary 6.4.5, τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \phi$ for every points x and y in X. To prove γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\}) = \phi$. Suppose there is a point $p \in X$ such that $p \in \gamma$-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\})$. Then $p \in \gamma$-$P_Sker(\{x\})$ and $p \in \gamma$-$P_Sker(\{y\})$. When $p \in \gamma$-$P_Sker(\{x\})$, then by Theorem 6.2.46, $x \in \tau_\gamma$-$P_SCl(\{p\})$. Since $x \in \tau_\gamma$-$P_SCl(\{x\})$. This means that τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{p\}) \neq \phi$. Thus by Theorem 6.4.4, τ_γ-$P_SCl(\{x\}) = \tau_\gamma$-$P_SCl(\{p\})$. When $p \in \gamma$-$P_Sker(\{y\})$, then by the same way, we can obtain τ_γ-$P_SCl(\{y\}) = \tau_\gamma$-$P_SCl(\{p\})$. This means that τ_γ-$P_SCl(\{x\}) = \tau_\gamma$-$P_SCl(\{y\})$ which is contradiction. Therefore, γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\}) = \phi$ for every x and y in X.

Conversely, suppose that γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$ implies γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\}) = \phi$ for every x and y in a space (X, τ). If τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$, then by Theorem 6.2.50, γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$. So by hypothesis, γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\}) = \phi$. This implies that τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \phi$ since $p \in \tau_\gamma$-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\})$ which implies that $p \in \tau_\gamma$-$P_SCl(\{x\})$ and hence $x \in \gamma$-$P_Sker(\{p\})$., So γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{p\}) \neq \phi$. Then by hypothesis, γ-$P_Sker(\{x\}) = \gamma$-$P_Sker(\{p\})$. Similarly, we have γ-$P_Sker(\{y\}) \cap \gamma$-$P_Sker(\{p\}) \neq \phi$. Then by hypothesis, γ-$P_Sker(\{x\}) = \gamma$-$P_Sker(\{p\}) = \gamma$-$P_Sker(\{y\})$. This is a contradiction. Therefore, τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \phi$. Consequently, by Corollary 6.4.5, (X, τ) is γ-P_S-R_0 space. $\square$

Theorem 6.4.7. The following properties of a topological space (X, τ) with an operation γ on τ are equivalent:

1. X is γ-P_S-R_0.

2. If for every nonempty subset A of X and every γ-P_S-open set U in X such that $A \cap U \neq \phi$, then there exists a γ-P_S-closed set E such that $A \cap E \neq \phi$ and $E \subseteq U$.

3. $U = \cup\{E : E \subseteq U$ and $E \in \tau_\gamma$-$P_SC(X, \tau)\}$ for every $U \in \tau_\gamma$-$P_SO(X, \tau)$.

4. $E = \cap\{U : E \subseteq U$ and $U \in \tau_\gamma$-$P_SO(X, \tau)\}$ for every $E \in \tau_\gamma$-$P_SC(X, \tau)$.

5. τ_γ-$P_SCl(\{x\}) \subseteq \gamma$-$P_Sker(\{x\})$ for every element x in X.

Proof. (1) $\Rightarrow$ (2) Let A be a nonempty subset of X and U is γ-P_S-open set in X such that $A \cap U \neq \phi$. Then there exists a point $x \in A \cap U$ and hence $x \in U$. Since (X, τ) is γ-P_S-R_0, then τ_γ-$P_SCl(\{x\}) \subseteq U$. Let $E = \tau_\gamma$-$P_SCl(\{x\})$. Then E is γ-P_S-closed set in X such that $A \cap E \neq \phi$ and $E \subseteq U$.

(2) $\Rightarrow$ (3) Let $U \in \tau_\gamma$-$P_SO(X, \tau)$. Then $\cup\{E : E \subseteq U$ and $E \in \tau_\gamma$-$P_SC(X, \tau)\} \subseteq U$. Let x be any point in U. Then there exists $E \in \tau_\gamma$-$P_SC(X, \tau)\}$ such that $x \in E$ and $E \subseteq U$. So $x \in E \subseteq \cup\{E : E \subseteq U$ and $E \in \tau_\gamma$-$P_SC(X, \tau)\}$. Therefore, $U = \cup\{E : E \subseteq U$ and $E \in \tau_\gamma$-$P_SC(X, \tau)\}$.

(3) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (5) Let x be any element of X and let $y \in X$ such that $y \notin \gamma$-$P_Sker(\{x\})$. This means that there exists a γ-P_S-open set V in X containing x such that $y \notin V$ and hence τ_γ-$P_SCl(\{y\}) \cap V = \phi$. Then by (4), $\cap\{U : \tau_\gamma$-$P_SCl(\{y\}) \subseteq U$ and $U \in \tau_\gamma$-$P_SO(X, \tau)\} \cap V = \phi$.This means that there exists γ-P_S-open set U in X such that $x \notin U$ and τ_γ-$P_SCl(\{y\}) \subseteq U$. Then $\{x\} \cap U = \phi$ and hence τ_γ-$P_SCl(\{x\}) \cap U = \phi$ and $y \notin \tau_\gamma$-$P_SCl(\{x\})$. Therefore, $y \notin \tau_\gamma$-$P_SCl(\{x\})$. Hence τ_γ-$P_SCl(\{x\}) \subseteq \gamma$-$P_Sker(\{x\})$.

(5) $\Rightarrow$ (1) Let G be any γ-P_S-open set in X containing x. Let $y \in \gamma$-$P_S ker(\{x\})$, then by Theorem 6.2.46, $x \in \tau_\gamma$-$P_S Cl(\{y\})$ and hence $y \in U$. This follows that γ-$P_S ker(\{x\}) \subseteq U$. Thus $x \in \tau_\gamma$-$P_S Cl(\{x\}) \subseteq \gamma$-$P_S ker(\{x\}) \subseteq U$. Then a space (X, τ) is γ-P_S-R_0. □

Theorem 6.4.8. A topological space (X, τ) is γ-P_S-R_0 if and only if τ_γ-$P_S Cl(\{x\}) = \gamma$-$P_S ker(\{x\})$ for all x in X.

Proof. Let (X, τ) be a γ-P_S-R_0 space. Then by Theorem 6.4.7 (5), τ_γ-$P_S Cl(\{x\}) \subseteq \gamma$-$P_S ker(\{x\})$ for every element x in X. Let $y \in \gamma$-$P_S ker(\{x\})$ and by Theorem 6.2.46, $x \in \tau_\gamma$-$P_S Cl(\{y\})$. So by Corollary 6.4.5, τ_γ-$P_S Cl(\{x\}) = \tau_\gamma$-$P_S Cl(\{y\})$. Thus $y \in \tau_\gamma$-$P_S Cl(\{x\})$. This means that γ-$P_S ker(\{x\}) \subseteq \tau_\gamma$-$P_S Cl(\{x\})$. It follows that τ_γ-$P_S Cl(\{x\}) = \gamma$-$P_S ker(\{x\})$ for all x in X.

Conversely, the proof is follows directly from Theorem 6.4.7 and hence it is omitted. □

Theorem 6.4.9. The following properties are equivalent for a topological space (X, τ) with an operation γ on τ:

1. X is γ-P_S-R_0.

2. $x \in \tau_\gamma$-$P_S Cl(\{y\})$ if and only if $y \in \tau_\gamma$-$P_S Cl(\{x\})$.

Proof. (1) $\Rightarrow$ (2) Let X be a γ-P_S-R_0 space and let $x \in \tau_\gamma$-$P_S Cl(\{y\})$. Then τ_γ-$P_S Cl(\{x\}) \cap \tau_\gamma$-$P_S Cl(\{y\}) \neq \phi$. Hence by Corollary 6.4.5, τ_γ-$P_S Cl(\{x\}) = \tau_\gamma$-$P_S Cl(\{y\})$. So $y \in \tau_\gamma$-$P_S Cl(\{x\})$. Similarly, $y \in \tau_\gamma$-$P_S Cl(\{x\})$ implies $x \in \tau_\gamma$-$P_S Cl(\{y\})$.

(2) $\Rightarrow$ (1) Let G be any γ-P_S-open set in X containing x. Let $y \notin G$, then $x \notin \tau_\gamma$-$P_SCl(\{y\})$ and hence $y \notin \tau_\gamma$-$P_SCl(\{x\})$ (by hypothesis). So τ_γ-$P_SCl(\{x\}) \subseteq G$. This shows that X is γ-P_S-R_0 space. $\square$

Corollary 6.4.10. Let (X, τ) be a topological space and γ be an operation on τ. Then (X, τ) is γ-P_S-R_0 if and only if (X, τ) is γ-P_S-symmetric.

Proof. This is an immediate consequence of Theorem 6.4.9 and Definition 6.2.31. $\square$

Corollary 6.4.11 can be constructed by applying Corollary 6.2.48, Corollary 6.4.5 and Corollary 6.4.10.

Corollary 6.4.11. Let (X, τ) be a topological space and γ be an operation on τ. Then X is γ-P_S-symmetric if and only if τ_γ-$P_SCl(\{x\}) = \gamma$-$P_Sker(\{x\})$ for every $x \in X$.

Proof. Straightforward. $\square$

Consequently, some results on γ-P_S-R_0 space are obtained by using Corollary 6.4.10.

Corollary 6.4.12. Let (X, τ) be a topological space and γ be an operation on τ. Then X is γ-P_S-R_0 if and only if every singleton sets in X is γ-P_S-g-closed.

Proof. Directly follows from Corollary 6.4.10 and Lemma 6.2.32. $\square$

From Corollary 6.4.10 and Theorem 6.2.33, we have the following corollary.

Corollary 6.4.13. If τ_γ-$P_SO(X, \tau) = \tau_\gamma$-$P_SC(X, \tau)$, then the topological space (X, τ) is γ-P_S-R_0.

Proof. Obvious. □

The proof of the next corollary follows directly from Corollary 6.4.10, Corollary 6.2.34 and Corollary 6.2.37.

Corollary 6.4.14. If (X, τ) is either γ-locally indiscrete or γ-hyperconnected space, then the topological space (X, τ) is γ-P_S-R_0.

Proof. It is clear. □

The relation between the γ-P_S-R_0 and γ-P_S-T_1 spaces are shown in the following theorem.

Theorem 6.4.15. If (X, τ) is γ-P_S-T_1 space, then it is γ-P_S-R_0.

Proof. The proof is an immediate consequence of Corollary 6.4.10 and Theorem 6.2.38. □

The converse of Theorem 6.4.15 does not true in general as stated in Example 4.2.17 where the space (X, τ) is γ-P_S-R_0, but it is not γ-P_S-T_1. However, the converse of this Theorem 6.4.15 is true when a space (X, τ) is γ-P_S-T_0 as shown in the following Corollary 6.4.16.

Corollary 6.4.16. Let (X, τ) be a topological space and γ be an operation on τ. Then (X, τ) is γ-P_S-T_1 if and only if (X, τ) is γ-P_S-R_0 and γ-P_S-T_0.

Proof. This is an immediate consequence of Corollary 6.4.10 and Theorem 6.2.40. □

Theorem 6.4.17. The following conditions are equivalent for a topological space (X, τ) with an operation γ on τ:

1. X is γ-P_S-R_0.

2. If F is γ-P_S-closed set in X, then $F = \gamma$-$P_S ker(F)$.

3. If F is γ-P_S-closed set in X containing x, then γ-$P_S ker(\{x\}) \subseteq F$.

4. γ-$P_S ker(\{x\}) \subseteq \tau_\gamma$-$P_S Cl(\{x\})$ for every element $x \in X$.

Proof. (1) $\Rightarrow$ (2) Let F be a γ-P_S-closed set in a γ-P_S-R_0 space (X, τ) and $x \notin F$. Then $X \backslash F$ is γ-P_S-open set such that $x \in X \backslash F$. By hypothesis, τ_γ-$P_S Cl(\{x\}) \subseteq X \backslash F$. This implies that τ_γ-$P_S Cl(\{x\}) \cap F = \phi$. Hence by Theorem 6.2.45, $x \notin \gamma$-$P_S ker(F)$. So γ-$P_S ker(F) \subseteq F$. But in general, $F \subseteq \gamma$-$P_S ker(F)$ (by Remark 6.2.43 (1)). This follows that $F = \gamma$-$P_S ker(F)$.

(2) $\Rightarrow$ (3) For any γ-P_S-closed set F in X containing x, $\{x\} \subseteq F$ which implies that γ-$P_S ker(\{x\}) \subseteq \gamma$-$P_S ker(F) = F$.

(3) $\Rightarrow$ (4) Since τ_γ-$P_S Cl(\{x\})$ is γ-P_S-closed containing x. Then by (3), γ-$P_S ker(\{x\}) \subseteq \tau_\gamma$-$P_S Cl(\{x\})$.

(4) $\Rightarrow$ (1) Suppose $x \in \tau_\gamma$-$P_S Cl(\{y\})$, then by Theorem 6.2.46, $y \in \gamma$-$P_S ker(\{x\})$. By using (4), we get $y \in \tau_\gamma$-$P_S Cl(\{x\})$. So $x \in \tau_\gamma$-$P_S Cl(\{y\})$ implies $y \in \tau_\gamma$-$P_S Cl(\{x\})$. Similarly, we can show that $y \in \tau_\gamma$-$P_S Cl(\{x\})$ implies that $x \in \tau_\gamma$-$P_S Cl(\{y\})$. Therefore, by Theorem 6.4.9, a space (X, τ) is γ-P_S-R_0. $\square$

Theorem 6.4.18. Let γ and β be operations on τ and σ respectively. If $f\colon (X,\tau) \to (Y,\sigma)$ is an surjective (γ,β)-P_S-irresolute and (γ,β)-P_S-closed function and (X,τ) is γ-P_S-R_0, then (Y,σ) is also β-P_S-R_0.

Proof. Let U be any β-P_S-open set in (Y,σ) and y be any point in U. Since f is (γ,β)-P_S-irresolute, then by Theorem 5.3.10 (2), $f^{-1}(U)$ is γ-P_S-open set in (X,τ). Since (X,τ) is γ-P_S-R_0 space, for a point $x \in f^{-1}(y)$, τ_γ-$P_SCl(\{x\}) \subseteq f^{-1}(U)$. Since f is (γ,β)-P_S-closed, then σ_β-$P_SCl(\{y\}) = \sigma_\beta$-$P_SCl(\{f(x)\}) \subseteq f(\tau_\gamma$-$P_SCl(\{x\})) \subseteq U$. Therefore, (Y,σ) is β-P_S-R_0 space. □

Corollary 6.4.19. Let γ and β be operations on τ and σ respectively. If $f\colon (X,\tau) \to (Y,\sigma)$ is an surjective (γ,β)-P_S-closed, γ-continuous and β-open function and (X,τ) is γ-P_S-R_0, then (Y,σ) is also β-P_S-R_0.

Proof. Follows directly from Theorem 6.4.18 and Theorem 5.3.17. □

Definition 6.4.20. A topological space (X,τ) with an operation γ on τ is said to be weakly γ-P_S-R_0 if $\bigcap_{x\in X} \tau_\gamma$-$P_SCl(\{x\}) = \phi$.

Theorem 6.4.21. Let (X,τ) be a topological space and γ be an operation on τ. Then (X,τ) is weakly γ-P_S-R_0 if and only if γ-$P_Sker(\{x\}) \neq X$ for every $x \in X$.

Proof. Let (X,τ) be a weakly γ-P_S-R_0 space. Suppose that γ-$P_Sker(\{y\}) = X$ for every $y \in X$. Then $\{y\} \neq G$ where G is any proper γ-P_S-open subset of X. Then $y \in \tau_\gamma$-$P_SCl(\{x\})$ for every $x \in X$ and hence $y \in \bigcap_{x\in X} \tau_\gamma$-$P_SCl(\{x\})$. This is a contradiction.

Conversely, let γ-$P_Sker(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \bigcap_{x \in X} \tau_\gamma$-$P_SCl(\{x\})$. Then every γ-P_S-open set containing y must contain every point of x. Hence X is the unique γ-P_S-open set containing y. So γ-$P_Sker(\{y\}) = X$. This is a contradiction. Therefore, (X, τ) is weakly γ-P_S-R_0 space. □

The following Remark 6.4.22 follows directly from Definitions 6.4.1 and 6.4.20.

Remark 6.4.22. Every γ-P_S-R_0 space is weakly γ-P_S-R_0.

But in general the converse of the above Remark 6.4.22 does not hold as shown by the following Example 6.4.23.

Example 6.4.23. In Example 4.2.3, the space (X, τ) is weakly γ-P_S-R_0, but it is not γ-P_S-R_0 since $\{a, c\}$ is γ-P_S-open set in X containing c such that τ_γ-$P_SCl(\{c\}) = \{b, c\} \nsubseteq \{a, c\}$.

Remark 6.4.24. Let γ and β be operations on τ and σ respectively. If $f\colon (X, \tau) \to (Y, \sigma)$ is an injective (γ, β)-P_S-closed function and (X, τ) is weakly γ-P_S-R_0, then (Y, σ) is also weakly β-P_S-R_0.

Now, γ-P_S-R_1 space in terms of γ-P_S-open and $\tau_\gamma P_S$-closure operator is defined as follows:

Definition 6.4.25. A topological space (X, τ) with an operation γ on τ is said to be γ-P_S-R_1 if for x and y in X with τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$, there exist disjoint γ-P_S-open sets G and H such that τ_γ-$P_SCl(\{x\}) \subseteq G$ and τ_γ-$P_SCl(\{y\}) \subseteq H$.

The relation between the γ-P_S-R_0 and γ-P_S-R_1 spaces are shown in the following theorem.

Theorem 6.4.26. If (X, τ) is γ-P_S-R_1 space, then it is γ-P_S-R_0.

Proof. Let G be any γ-P_S-open set in γ-P_S-R_1 space (X, τ) containing x. Let $y \notin G$. If $y \notin G$, and since $x \notin \tau_\gamma$-$P_SCl(\{y\})$. Then τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. Thus, there exists a γ-P_S-open set H such that τ_γ-$P_SCl(\{y\}) \subseteq H$ and $x \notin H$. This implies that $y \notin \tau_\gamma$-$P_SCl(\{x\})$ and hence τ_γ-$P_SCl(\{x\}) \subseteq G$. Therefore, the space (X, τ) is γ-P_S-R_0. □

But the converse of the above Theorem 6.4.26 does not true in general as seen from the following example.

Example 6.4.27. Let a space $X = \{x, y, z\}$ with the discrete topology τ on X. Define an operation γ on τ by: for every $A \in \tau$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{x, y\} \text{ or } \{x, z\} \text{ or } \{y, z\} \\ X & \text{if } otherwise \end{cases}$$

Clearly, $\tau_\gamma = \{\phi, \{x, y\}, \{x, z\}, \{y, z\}, X\} = \tau_\gamma$-$P_SO(X)$. Then the space (X, τ) is γ-P_S-R_0, but it is not γ-P_S-R_1 since for the points $x, y \in X$ such that τ_γ-$P_SCl(\{x\}) = \{x\} \neq \{y\} = \tau_\gamma$-$P_SCl(\{y\})$, but there is no disjoint γ-P_S-open sets G and H such that τ_γ-$P_SCl(\{x\}) \subseteq G$ and τ_γ-$P_SCl(\{y\}) \subseteq H$.

Theorem 6.4.28. If (X, τ) is γ-P_S-T_2 space, then it is γ-P_S-R_1.

Proof. Let x and y be any elements in γ-P_S-T_2 space (X,τ) such that τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. Then by Theorem 6.2.8 (1), (X,τ) is γ-P_S-T_1 and hence by Theorem 6.2.3, τ_γ-$P_SCl(\{x\}) = \{x\}$ and τ_γ-$P_SCl(\{y\}) = \{y\}$ for every $x, y \in X$. Then $x \neq y$ and hence by hypothesis, there exist disjoint γ-P_S-open sets G and H containing x and y respectively. This follows that τ_γ-$P_SCl(\{x\}) \subseteq G$ and τ_γ-$P_SCl(\{y\}) \subseteq H$. Consequently, (X,τ) is γ-P_S-R_1 space. □

So, the converse of the Theorem 6.4.28 is not true in general as described in Example 4.2.7 where the space (X,τ) is γ-P_S-R_1, but it is not γ-P_S-T_2. The converse of the Theorem 6.4.28 is only true when a space (X,τ) is γ-P_S-T_0 as shown in the following Theorem 6.4.29.

Theorem 6.4.29. Let (X,τ) be a topological space and γ be an operation on τ. Then (X,τ) is γ-P_S-T_2 if and only if (X,τ) is γ-P_S-R_1 and γ-P_S-T_0.

Proof. Let x and y be any distinct points in X and (X,τ) be a γ-P_S-R_1 and γ-P_S-T_0 space. Then by Theorem 6.4.26, (X,τ) is γ-P_S-R_0 and hence by Corollary 6.4.16, (X,τ) is γ-P_S-T_1. Then by Theorem 6.2.3, τ_γ-$P_SCl(\{x\}) = \{x\}$ and τ_γ-$P_SCl(\{y\}) = \{y\}$ for every distinct points $x, y \in X$. This means that τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ and hence by Definition 6.4.25, there exist disjoint γ-P_S-open sets G and H such that τ_γ-$P_SCl(\{x\}) \subseteq G$ and τ_γ-$P_SCl(\{y\}) \subseteq H$. It follows that $x \in G$ and $y \in H$. Therefore, (X,τ) is γ-P_S-T_2.

Conversely, if (X,τ) is γ-P_S-T_2 space, then by Theorem 6.2.8 and Theorem 6.4.28, (X,τ) is γ-P_S-T_0 and γ-P_S-R_1 respectively. This completes the proof. □

Therefore, from Theorem 6.4.28, Theorem 6.4.29 and Theorem 6.2.8, we have the following corollary.

Corollary 6.4.30. Let (X, τ) be any topological space and γ be an operation on τ. Then the following statements are equivalent:

1. (X, τ) is γ-P_S-R_1 and γ-P_S-T_0.

2. (X, τ) is γ-P_S-R_1 and γ-P_S-$T_{\frac{1}{2}}$.

3. (X, τ) is γ-P_S-R_1 and γ-P_S-T_1.

4. (X, τ) is γ-P_S-T_2.

Proof. It is clear and hence it is omitted. □

Corollary 6.4.31. If (X, τ) is γ-P_S-T_1 space. Then every γ-P_S-regular space (X, τ) is γ-P_S-R_1.

Proof. This is an immediate consequence of Lemma 6.3.24 and Theorem 6.4.28. □

The proof of the following corollary is follows directly from Corollary 6.4.31 and Lemma 6.3.22.

Corollary 6.4.32. If (X, τ) is γ-P_S-normal and γ-P_S-T_1 space. Then it is γ-P_S-R_1.

Proof. Obvious. □

The following are some significant characterizations of γ-P_S-R_1 spaces.

Remark 6.4.33. A topological space (X, τ) is γ-P_S-R_1 if and only if for x and y in X with τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$, there exist disjoint γ-P_S-closed sets E and F such that $x \in E$, $y \notin E$, $y \in F$, $x \notin F$ and $E \cup F = X$.

Theorem 6.4.34. A topological space (X, τ) is γ-P_S-R_1 if and only if for x and y in X with γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$, there exist disjoint γ-P_S-open sets G and H such that τ_γ-$P_SCl(\{x\}) \subseteq G$ and τ_γ-$P_SCl(\{y\}) \subseteq H$.

Proof. Follows directly from Theorem 6.2.50. □

Theorem 6.4.35. A topological space (X, τ) is γ-P_S-R_1 if and only if for $x \in X \backslash \tau_\gamma$-$P_SCl(\{y\})$ implies that x and y have disjoint γ-P_S-open neighborhoods.

Proof. Suppose that $x \in X \backslash \tau_\gamma$-$P_SCl(\{y\})$. Then τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ and hence x and y have disjoint γ-P_S-open neighborhoods.

Conversely, first, to show that (X, τ) is γ-P_S-R_0. Let W be a γ-P_S-open set and $x \in W$. Suppose that $y \notin W$. Then τ_γ-$P_SCl(\{y\}) \cap W = \phi$ and $x \notin \tau_\gamma$-$P_SCl(\{y\})$. There exist γ-P_S-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Hence, τ_γ-$P_SCl(\{x\}) \subseteq \tau_\gamma$-$P_SCl(U)$ and τ_γ-$P_SCl(\{x\}) \cap V \subseteq \tau_\gamma$-$P_SCl(U) \cap V = \phi$. Therefore, $y \notin \tau_\gamma$-$P_SCl(\{x\})$. Consequently, τ_γ-$P_SCl(\{x\}) \subseteq G$ and (X, τ) is γ-P_S-R_0. Next, to show that (X, τ) is γ-P_S-R_1. Suppose that τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. Then, we can assume that there exists $p \in \tau_\gamma$-$P_SCl(\{x\})$ such that $p \notin \tau_\gamma$-$P_SCl(\{y\})$. There exist γ-P_S-open sets G and H such that $p \in G$, $y \in H$ and $G \cap H = \phi$. Since $p \in \tau_\gamma$-$P_SCl(\{x\})$, $x \in G$. Since (X, τ) is γ-P_S-R_0, we obtain

τ_γ-$P_SCl(\{x\}) \subseteq G$, τ_γ-$P_SCl(\{y\}) \subseteq H$ and $G \cap H = \phi$. This shows that (X, τ) is γ-P_S-R_1 space. $\square$

Corollary 6.4.36. A topological space (X, τ) is γ-P_S-R_1 if and only if τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ implies that x and y have disjoint γ-P_S-open neighborhoods.

Proof. Assume that $x \in X \backslash \tau_\gamma$-$P_SCl(\{y\})$. Then τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ and hence by hypothesis, x and y have disjoint γ-P_S-open neighborhoods. Thus by Theorem 6.4.35, (X, τ) is γ-P_S-R_1 space. The proof of the converse part is obvious. $\square$

6.5 Conclusion

This chapter defined some types of γ-P_S- separation axioms such as γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$ using γ-P_S-open sets. The spaces γ-P_S-regular, γ-P_S-normal, γ-P_S-R_0 and γ-P_S-R_1 which are types of γ-P_S- separation axioms have been discussed. The relations, properties and characterizations of these spaces have also been investigated.

CHAPTER SEVEN

CONCLUSIONS AND FUTURE RESEARCH

7.1 Introduction

In the second section of this chapter, the conclusions of the research drawn from the present study is presented. Finally, suggestions for future research related to the topics presented in the thesis will be outlined and discussed.

7.2 Conclusions of the Research

In this research, we have introduced a new class of sets called γ-regular-open sets in a topological space (X, τ) together with its complement which was γ-regular-closed. Some results of these sets were obtained. Also, we defined and investigated a new class of functions called completely γ-continuous in terms of γ-regular-open set. Also, we defined some classes of γ- spaces called γ- extremally disconnected, γ- locally indiscrete and γ- hyperconnected spaces. Several characterizations of these γ- spaces were studied and discussed.

Again, we defined a another new class of sets called γ-P_S-open sets in a topological space (X, τ) together with its complements which was γ-P_S-closed. By applying γ-P_S-open and γ-P_S-closed sets, the operations τ_γ-P_S-closure, τ_γ-P_S-interior, τ_γ-P_S-derived set and τ_γ-P_S-boundary of a set have been introduced and investigated as well as studied some of its properties. Some other γ-P_S- sets have been defined.

We extended the class of γ-P_S- sets to introduced γ-P_S-generalized closed sets by using γ-P_S-open set and τ_γ-P_S-closure operator. By applying this set and its complement, we defined some new classes of γ-P_S- functions called γ-P_S-g-continuous, β-P_S-g-closed and β-P_S-g-open

Furthermore, new types of γ-P_S- functions called γ-P_S-continuous, (γ, β)-P_S-continuous and (γ, β)-P_S-irresolute functions in terms of γ-P_S-open sets have been defined. Also, the relationships between these γ-P_S- functions and other known classes of functions were given and discussed. In addition, some other types of γ-P_S- functions called β-P_S-open, β-P_S-closed, (γ, β)-P_S-open, (γ, β)-P_S-closed, $(\gamma, \beta P_S)$-open and $(\gamma, \beta P_S)$-closed functions have been studied. Some basic properties and preservation theorems of these γ-P_S- functions were discussed and investigated. Finally, we given some composition of these γ-P_S- functions.

Finally, some new classes of γ-P_S- separation axioms called γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$ spaces introduced by using γ-P_S-open, γ-P_S-closed and γ-P_S-g-closed sets. Also, the relationships between γ-P_S-T_i spaces and other types of γ- separation axioms were obtained. Next, we defined more γ-P_S- separation axioms namely γ-P_S-regular and γ-P_S-normal spaces via γ-P_S-open and γ-P_S-closed sets. Further, γ-P_S-R_1 and

γ-P_S-R_2 spaces have been introduced. Some properties and characterizations of these γ-P_S- spaces are studied.

7.3 Suggestions for Future Research

Future research is to introduce some other sets by using γ-P_S-open sets in a topological space (X, τ) such as γ-P_S-θ-closed and generalized γ-P_S-closed. Also, some γ-P_S- functions can be done is almost γ-P_S-continuous, almost (γ, β)-P_S-continuous, almost (γ, β)-P_S-irresolute, weakly γ-P_S-continuous, weakly (γ, β)-P_S-continuous, weakly (γ, β)-P_S-irresolute, contra γ-P_S-continuous, contra (γ, β)-P_S-continuous, contra (γ, β)-P_S-irresolute, slightly γ-P_S-continuous, slightly (γ, β)-P_S-continuous, slightly (γ, β)-P_S-irresolute, strongly γ-P_S-continuous, strongly θ-γ-P_S-continuous, strongly θ-(γ, β)-P_S-continuous and strongly θ-(γ, β)-P_S-irresolute functions. Some other γ-P_S- functions can also be studied and investigated.

Moreover, some γ-P_S- spaces such as γ-P_S-compact, γ-P_S-closed and γ-P_S-connected spaces can be defined.

The work carried out in this thesis has revealed many promising areas of further research. We intend to introduce γ-P_S-open sets for bitopological space and ideal topological space. A few of these areas worthy of further investigations can be briefly summarized as follows:

- **Bitopological space**. γ-P_S- operations can be studied in bitopological space to define the notions of γ-P_S- sets, γ-P_S-functions, and γ-P_S- separation axioms in bitopological space. The notions such as γ-P_S-open set, γ-P_S-closed set, γ-P_S- operations and γ-P_S-g-closed set between bitopological spaces can be investigated and studied. In addition, by using these notions we can develop γ-P_S-functions, and γ-P_S- separation axioms between bitopological spaces and investigate some relations between them, and characterizations of their can be obtained.

- **Ideal topological space**. The study of the notions of γ-P_S-open set, γ-P_S-closed set, γ-P_S- operations and γ-P_S-g-closed set can be extended in an ideal topological space to define the notions of γ-P_S-functions, and γ-P_S- separation axioms between ideal topological spaces. In fact, some relations between them, properties and theorems can also be studied.

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