# DEFUZZIFICATION OF GROUPS OF FUZZY NUMBERS USING DATA ENVELOPMENT ANALYSIS

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# Abstrak

Penyahkaburan merupakan satu proses kritikal dalam pelaksanaan sistem kabur yang menukar nombor kabur kepada pewakilan rangup. Sebilangan kecil penyelidik telah memberikan tumpuan pada kes yang mana data rangup asal atau output rangup mesti memenuhi suatu set hubungan yang ditentukan dalam data rangup asal. Fenomena ini menunjukkan bahawa data rangup ini secara matematiknya saling bersandar antara satu sama lain. Tambahan pula, nombor kabur ini boleh wujud sebagai satu kumpulan nombor kabur. Oleh itu, tujuan utama tesis ini adalah untuk membangunkan satu kaedah yang menyahkabur kumpulan nombor kabur berasaskan model Charnes, Cooper, and Rhodes (CCR) – Analisis Penyampulan Data (DEA) dengan mengubah suai kaedah pusat graviti (COG) sebagai fungsi objektif. Kekangan mewakili hubungan pada output rangup dan beberapa sekatan pada output rangup yang dibenarkan bagi memenuhi sifat kebergantungan pada output rangup. Berbanding dengan kaedah asas pemprograman linear (LP), kaedah yang dinyatakan lebih cekap, dan mampu menyahkabur nombor kabur tak linear dengan penyelesaian lebih jitu. Kaedah penyahkabur asas CCR-DEA yang dicadangkan juga mampu untuk menyelesaikan nombor kabur tak linear dan memperoleh penyelesaian yang tepat. Selain itu, output rangup yang diperoleh melalui kaedah yang dicadangkan adalah titik terdekat bagi kes output rangup tidak bersandar, dan titik terdekat terbaik bagi titik terdekat bagi kes output rangup bersandar. Kesimpulannya, kaedah penyahkabur CCR-DEA boleh mencipta sama ada output rangup bersandar dengan mengekalkan hubungan atau output rangup tak bersandar tanpa hubungan. Selain itu, kaedah yang dibangunkan merupakan kaedah umum untuk menyahkabur kumpulan nombor kabur atau nombor kabur individu dengan andaian kecembungan bagi fungsi atau keahlian linear atau tak linear.



**Kata kunci:** Penyahkabur, Analisis penyampulan data, Kumpulan nombor kabur, Output rangup bersandar, Output rangup tak bersandar.

# Abstract

Defuzzification is a critical process in the implementation of fuzzy systems that converts fuzzy numbers to crisp representations. Few researchers have focused on cases where the crisp outputs must satisfy a set of relationships dictated in the original crisp data. This phenomenon indicates that these crisp outputs are mathematically dependent on one another. Furthermore, these fuzzy numbers may exist as a group of fuzzy numbers. Therefore, the primary aim of this thesis is to develop a method to defuzzify groups of fuzzy numbers based on Charnes, Cooper, and Rhodes (CCR)-Data Envelopment Analysis (DEA) model by modifying the Center of Gravity (COG) method as the objective function. The constraints represent the relationships and some additional restrictions on the allowable crisp outputs with their dependency property. This leads to the creation of crisp values with preserved relationships and/or properties as in the original crisp data. Comparing with Linear Programming (LP) based model, the proposed CCR-DEA model is more efficient, and also able to defuzzify non-linear fuzzy numbers with accurate solutions. Moreover, the crisp outputs obtained by the proposed method are the nearest points to the fuzzy numbers in case of crisp independent outputs, and best nearest points to the fuzzy numbers in case of dependent crisp outputs. As a conclusion, the proposed CCR-DEA defuzzification method can create either dependent crisp outputs with preserved relationship or independent crisp outputs without any relationship. Besides, the proposed method is a general method to defuzzify groups or individuals fuzzy numbers under the assumption of convexity with linear and non-linear membership functions or relationships.

**Keywords**: Defuzzification, Data envelopment analysis, Groups of fuzzy numbers, Dependent crisp outputs, Independent crisp outputs.

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# List of Abbreviations

| | |
|---|---|
| A&H | Abbasbandy & Hajjari's (2009) method |
| A&Z | Asady & Zendamman's (2007) method |
| AHP | Analytic Hierarchy Process |
| AP | Andersen and Petersen Model |
| BCC | Banker, Charnes and Cooper Model |
| CCR | Charnes, Cooper and Rhodes |
| CCR | Charnes, Cooper and Rhodes Model |
| COA | Center of Area method |
| COG | Center of Gravity method |
| CRS | Constant Returns-to-Scale |
| CV index | Coefficient of Variation method |
| DEA | Data Envelopment Analysis |
| DM | Decision-Maker |
| DMU | Decision Making Unit |
| DRS | Decreasing Returns to Scale |
| FCCR | Fuzzy Charnes, Cooper and Rhodes Model |
| FDH | Free Disposal Hull |
| FGP | Fuzzy Goal Programming |
| FIME | Fuzzy Input-Mix Efficiency |
| FIW | Fuzzy Interval Weight |
| FLP | Fuzzy Linear Programming |
| FSBM | Fuzzy Slacks-Based Measure |
| Fuzzy DEA | Fuzzy Data Envelopment Analysis |
| FWGP | Fuzzy Weighted Goal Programming |
| GP | Goal Programming |
| HM | Height Method |
| HTF | Hospital Tuanku Fauziah |
| IME | Input-Mix Efficiency |
| IRS | Increasing Returns to Scale |

| | |
|---|---|
| IW | Interval Weight |
| IWGP | Interval Weight Goal Programming |
| LGP | Lexicographical Goal Programming |
| LP | Linear Programming |
| MADM | Multiple Attribute Decision Making |
| MAJ | Mehrabian, Alirezaee, and Jahanshahloo |
| Max | Maximum |
| MCDA | Multi Criteria Decision Analysis |
| MCDM | Multi Criteria Decision Making |
| Min | Minimum |
| MODM | Multi-Objective Decision Making |
| MOH | Ministry of Health |
| MOLP | Multi-Objective Linear Programming |
| MOMP | Multi-Objective Mathematical Programming |
| MOP | Multi-Objective Problem |
| MPI | Malmquist Productivity Index |
| NDRS | Non-Decreasing Returns to Scale |
| NIRS | Non-Increasing Returns to Scale |
| NLP | Non-linear Programming |
| OR | Operation Research |
| PPS | Production Possibility Set |
| SBM | Slacks-based Measure (SBM) |
| TpFN | Trapezoidal Fuzzy Number |
| TrFN | Triangular Fuzzy Number |
| VRS | Variable Returns to Scale |
| WAM | Weighted Average Method |
| WEI | Without Explicit Inputs |
| WEO | Without Explicit Inputs |
| WGP | Weighted Goal Programming |

# CHAPTER ONE

# INTRODUCTION

In the theory of classical set, two alternatives are allowed for an element i.e. it either should strictly be a member of a set or should not. The fuzzy set concept developed by Zadeh in 1965, along with its techniques, is an interesting and promising approach to address complex, real-world issues with a new pattern for modelling human logic to improve a simplification model. As a result, more robust and versatile models have been developed (Lai & Hwang, 1992).

## 1.1 Fuzzy System Structure

The term 'system' is defined as an ordered structure containing interdependent and interrelated elements (factors, entities, components, etc.). These elements frequently affect each other (in a direct or indirect way) for the system to exist, maintaining their activity and achieving the system goal. All systems consist of outputs, inputs, mechanisms, and boundaries which are usually identified by the system observer (Huang & Shi, 2002). However, the precise mathematics or crisp representation for modelling a complex system is insufficient due to the imperfect information and knowledge (Sladoje, Lindblad, & Nyström, 2011). Hence, the 'fuzzy system' term could be used in labeling any classification that has a structure and mechanism based on the fuzzy theory (Starczewski, 2013).

In general, a fuzzy representation provides more information about a set than a crisp representation. To replace a crisp representation of sets with a fuzzy representation in fuzzy system applications, the process of fuzzification is applied. As the objective determination of the fuzzy structures of problematic systems is difficult, the crisp representation becomes necessary because it simplifies the conception and clarification. A crisp representation is typically easy to interpret and understand although it displays less information (Clark, Larson, Mordeson, Potter, & Wierman, 2008; Runkler, 2013; Yager & Zadeh, 1992). In order to replace a fuzzy representation of sets into a crisp representation in fuzzy system applications, the process of defuzzification is applied (Siddique, 2014). Figure 1.1 illustrates the structure of a fuzzy system in general.

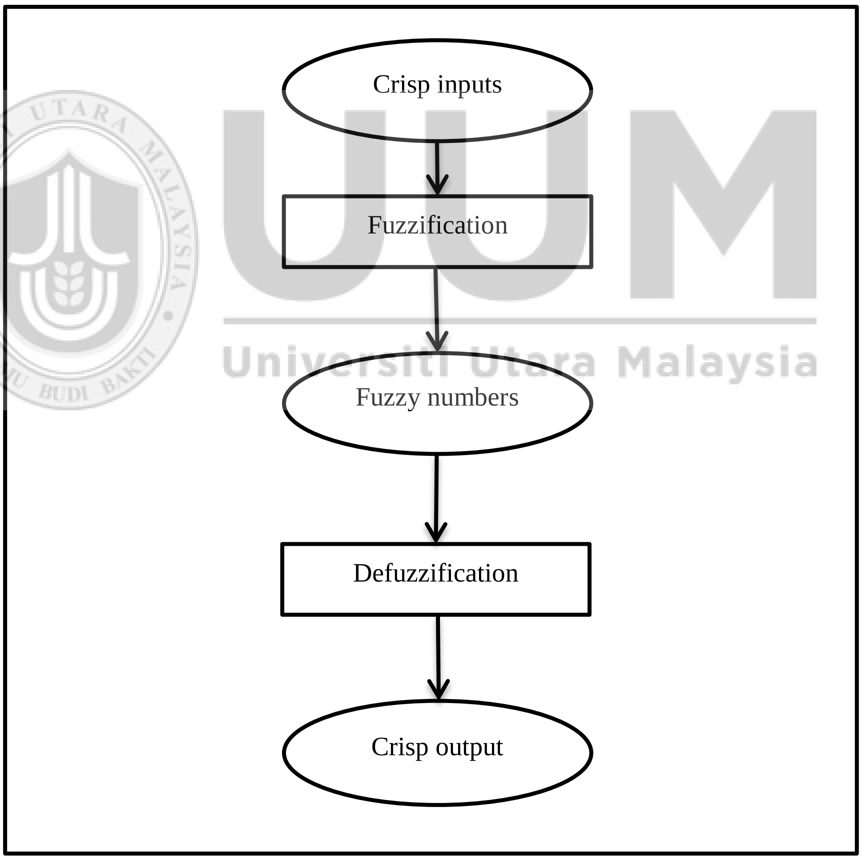



*Figure 1.1.* The structure of fuzzy system

### 1.1.1 Fuzzification

In fuzzy system applications, the original crisp data are commonly characterized as crisp. At this stage, crisp inputs are transformed to fuzzy numbers through a process of mapping and the original crisp is characterised by an element of membership function (Clark et al., 2008). Such functions may be classified as Triangle-shaped and Trapezoidal, discussed in Chapter Two.

### 1.1.2 Defuzzification

Defuzzification is the reverse process of fuzzification. Mathematically, defuzzification of a fuzzy set is the process of conversion of a fuzzy quantity into a crisp value. This process is necessary when a crisp value is to be provided by a fuzzy system to the user. Defuzzification is known as a critical fuzzy system stage that replaces fuzzy numbers with representative of crisp numbers. This definition allows each defuzzification method operation to reduce the fuzzy number to a single numerical value "crisp" which carries the best information and makes a kind of composition of this fuzzy number (Leekwijck & Kerre, 1999; Runkler, 2013). Defuzzification represents the crucial stage in the application of fuzzy systems as it processes the fuzzy numbers to be converted into the real line (Esogbue, Song, & Hearnes, 2000). In other words, most fuzzy systems use the defuzzification process in the last stage of their application.

To date, in most studies in fuzzy set theory applications, defuzzification is viewed as an inevitable phase that involves the reduction of the fuzzy set to single real (crisp) value. This phase should accurately describe the fuzzy set influence on the final value of the particular system output whose result has the best information and should make a kind of assembly about this fuzzy number (Liu, 2007). For this purpose, many defuzzification methods have been suggested, but none of them provides a precise sufficient defuzzified output since a different result is given by each method (Leekwijck & Kerre, 1999). More precisely, defuzzification can also be defined as the process of reversing the core idea of fuzzy sets (Rondeau, Ruelas, Levrat, & Lamotte, 1997). Some researchers introduced another description of defuzzification as the link connecting a fuzzy model with its application environment (Esogbue et al., 2000). Defuzzification is evidently important, if not the most significant step in the determination of the fuzzy set applications success (Bede, 2013; Yager, 1996). Some researchers suggested the defuzzification approach under the classification of ranking fuzzy number methods (Chang & Lee, 1994; Lee, 2000; Wang & Kerre, 2001). This classification is due to the conversion of fuzzy numbers to real tangible numbers (crisp points). These real tangible numbers (crisp points) are corresponded to an easy ranking of fuzzy numbers after defuzzification (Hajjari & Abbasbandy, 2011; Hajjari, 2011; Rouhparvar & Panahi, 2015).

Although many defuzzification methods are available and accessible in the literature, they failed to give an accurate, efficient defuzzified output for a number of applications. Therefore, the selection of an appropriate defuzzification method is significant for certain applications (Runkler, 1997; Siddique, 2014). Unfortunately, no standard rule has been provided for the selection of a particular defuzzification method for certain applications with some conditions or properties.

The option for the most appropriate method depends on the decision maker's knowledge and application applied (Liu, 2007; Runkler, 1997; Siddique, 2014).

The present defuzzification methods are categorized, which are mathematically accepted and broadly utilized by the fuzzy systems. This research used one of the standard methods under the area category, namely Centre of Gravity (COG) developed by Sugeno (1985) and the method suggested by Asady and Zendehnam (2007) known as A&Z method under the heading of distance minimization concept.

#### 1.1.2.1 Method of Center of Gravity

The method of COG was developed as the most commonly used defuzzification method. This method calculates the position at which the left and the right areas are equal. COG refers to the centroid of an area, and the defuzzification method could be expressed as following:



$$x_{COG} = \frac{\int \mu_{\tilde{F}}(x).xdx}{\int \mu_{\tilde{F}}(x).dx} \tag{1.1}$$

where $x_{COG}$ represents the crisp value to the fuzzy number $\tilde{F}$, $\mu_{\tilde{F}}$ is the membership function of $\tilde{F}$.

#### 1.1.2.2 The Method of Asady and Zendehnam

On the basis of the nearest point of a fuzzy number, Asady and Zendehnam (2007) presented a defuzzification method.

The nearest crisp point to a triangular fuzzy number $\mathcal{T} = (x_0, \delta, \beta)$ to be:

$$x_{A\&Z} = x_0 + \frac{\beta - \delta}{4} \tag{1.2}$$

where $\delta$ and $\beta$ are the left and the right fuzzy values respectively, while $x_0$ is the middle value of the fuzzy number $\mathcal{T}$ and $x_{A\&Z}$ is the crisp value to $\mathcal{T}$. There is a number of defuzzification methods reported in the literature that is based on the optimization techniques such as linear programming (LP), non-linear programming (NLP) and Data Envelopment Analysis (DEA). The following sub-section discusses about the optimization techniques and their relation to defuzzification.

## 1.2 Optimization Techniques

These techniques are described as mathematical programming methods used to determine the min and max functions under limitations (Kuester, Mize, & Griffin, 1974). They comprise many techniques, namely LP, NLP, and DEA. Most importantly, these methods have all been utilized in the defuzzification process.

### 1.2.1 Linear Programming

Linear programming (LP) technique is an optimization concept where both the objective function to be optimized and all the constraints are linear regarding the decision variables. The LP problems are defined as a convex problem because the linear functions are convex and the feasible region, which is described as the intersection of the convex constraint functions, which is also defined as a convex region. This assumption of a convex feasible region and convex objective results in only one optimal solution which is universally optimal (Luptacik, 2010). The convex function is defined as follows.

**Definition 1.2.1.1:** A function $g(x)$ is defined on a convex set $\mathbb{G}$ in $\mathbb{R}^n$ called convex on $\mathbb{G}$ if

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y) \quad (1.3)$$

for any $x, y \in \mathbb{G}$ and any $\lambda$ between 0 and 1.

LP's strengths are in the modelling simplicity and the efficient algorithms for its solution (Dowsland, 2014). In practical application, various real-world problems are solved by LP techniques with more complicated objectives or goals, which require optimization rather than one objective. As a result, of the LP limitations, one objective was selected, whereas other objectives were given to be constraints. By introducing multi-objective linear programming (MOLP), it is possible to model these problems more realistically using their techniques such as goal programming (GP) that are commonly utilized for an MOLP problems (Eiselt & Sandblom, 2007; Luptacik, 2010).

The theory of fuzzy set was firstly recognized by Bellman and Zadeh (1970) aiming to solve issues of decision-making problems. This concept was adopted to problems of LP and MOLP by Zimmermann (1978, 1996). The formulation of fuzzy linear programming (FLP) was firstly introduced where the goal has maximized the values that have the smallest membership grade. One of the studies that focus on using FLP and defuzzification is Kikuchi (2000), where the details of this method is presented in Chapter Three. Recently, Verstraete, (2015) proposed a defuzzification method based on FLP model with simulation. Then in the case of non-linear objective and or constraints, the NLP is also considered as a tool under defuzzification.

### 1.2.2 Non-Linear Programming

NLP is parallel to the LP in that it has one objective function, general constraints, and variable bounds. The variation is that the NLP involves at least one non-linear form that may be the objective and / or some or whole of the constraints. Besides, the objective of NLP problems can be defined as a convex function (if minimizing) and concave (if maximizing) and the constraints are defined as a convex set (Luenberger & Ye, 2008). Additionally, some complicated issues are designed to non-convex NLP problems where the objective or any of the constraints is non-convex. Such problems could have multiple feasible regions and locally optimal solution in each region. It may take time, depending upon the number of constraints and the variables, to get to the objective unbounded, that a non-convex problem is infeasible, or that an optimal solution is the "global optimum" across all feasible regions (Luenberger & Ye, 2008; Luptacik, 2010).

Furthermore, the NLP concept covers two parts of problems the constrained and unconstrained (Luenberger & Ye, 2008). Many real systems are inherently expressed as constrained problems because most cases involve complex issue, for instance, the planning, and designing of complex systems. Most frequently, the problems of unconstrained optimization occur in various contexts when the formulation of problem is simple, but explicit functional constraints are involved in formulas that are more complicated. Nevertheless, many constraints problems are often transformed to unconstrained using the constraints to create relations among variables through decreasing the sufficient number of variables such as the problem of control, approximation, and selection problem.

So to avoid the problem of constrained optimization from becoming unconstrained and having a global solution in using the NLP concept in defuzzification, Yager and Filev, (1995) assumed that the crisp outputs have to satisfy a constraint that is forcing the defuzzified value in order to be in the allowable area. It is done by using an expected value of the probability distributions of the fuzzy sets as an optimization problem to find the crisp output. Therefore, there is a need of a guarantee to finding an optimal solution for all such problem when the function and region shapes are convex (LP and some of NLP) or non-convex (NLP). This led us to define one of the powerful optimization techniques that are DEA.



### 1.2.3 Data Envelopment Analysis

DEA is a non-parametric technique, which has been verified to be beneficial for an efficient analysis service organization. DEA is a powerful optimization technique to assess and ascertain the efficient performance of a group of similar units, which is called decision making units (DMUs). It measures the relative efficiency of DMUs and identifies the best practice frontier. It also indicates targets for inefficient units to improve (Cook & Zhu, 2005). DEA was initially recommended to operations research (OR) field, by Charnes, Cooper, and Rhodes (CCR) in the year 1978. The original DEA model the CCR is under the concept of constant returns to scale (CRS). Then an extension of the CCR model to accommodate technologies that exhibit variable returns to scale (VRS) presented by Banker, Charnes, and Cooper (BCC) in (1984). In literature, many models are presented as modifications of traditional DEA models (e.g., Andersen & Petersen, 1993; Jahanshahloo, Pourkarimi, & Zarepisheh, 2006; Lotfi, Jahanshahloo, Mozaffari, & Gerami, 2011). Therefore, all the modification models of DEA are extensions of CCR model, which is expressed as follows:

(CCR model),

$$
\begin{aligned}
& \min \theta \\
& \text{subject to} \\
& \theta x_{p0} - \sum_{j=1}^{n} \lambda_j \, x_{pj} \geq 0 \\
& \sum_{j=1}^{n} \lambda_j \, y_{kj} \geq y_{ko} \\
& \lambda_j \geq 0 \\
& \theta \; free, \;\; j = 1,2,..,n \,, k = 1,2,3,..,s, p = 1,2,3,..,r
\end{aligned}
\tag{1.4}
$$

where $j = 1,2,..,n$ are the numbers of DMUs, $k = 1,2,3,..,s$ are the outputs, $p = 1,2,3,..,r$ are inputs and $x_{pj}$ is $p^{th}$ input of $j^{th}$ DMU, $y_{kj}$ is $k^{th}$ output of $j^{th}$ DMU.

DEA can employ multiple outputs and inputs measured in various units without assuming any particular functional of the boundary and disregarded error of measurements. As an alternative, the best practical technology is the boundary of a reconstructed production possibility subset based on directly enclosing an observations set (Cooper, Seiford, & Tone, 2006; Kumar & Gulati, 2014; Luptacik, 2010). The production possibility set (PPS) is formed which is the collection of all feasible DMUs that are capable of producing output $y = (y_1, y_2, \ldots, y_r)$ by consuming input $x = (x_1, x_2 \ldots, x_s)$. PPS demarcated as $\mathbb{P} = \{(x, y) \in \Re^{r+s} | x \text{ produce } y\}$, where, in numerous applications the PPS is unknown. This implies the way to the utilizing DEA as a vital tool, lies in the capacity to give an appraisal $\mathbb{P}$ from the set of original crisp DMUs under the convex concept (Ali Emrouznejad & Amin, 2009; Fukuyama & Sekitani, 2012). Then the PPS is described as smallest convex set that comprises the data points, as following;

$$\mathbb{P} = \left\{ (X, Y) \middle| \sum_{j=1}^{n} \lambda_j Y_j \geq Y, \sum_{j=1}^{n} \lambda_j X_j \leq X, \ \lambda_j \geq 0, j = 1,2,..,n \right\} \quad (1.5)$$

In 1984, Deprins, Simar, and Tulkens proposed generalized version of the DEA model and then Tulkens (1993) extended it (as cited in Briec & Kerstens, 2006; Kumar & Gulati, 2014), and it is called the Free Disposal Hull (FDH). The FDH model depends only on the operating free disposability assumption with a dull monotone hull as an estimator of technology $\mathbb{P}$. In other words, both DEA and FDH are consistent estimators if the PPS is convex. However, FDH reveals a lower convergence rate (as a result of requiring the less assumption) for DEA. In contrast, supposedly the actual production set not at convex position, the DEA is not dependable of the estimator production set, whereas FDH is at dependable position.

In general, DEA is defined on original crisp units. Finding the distance of every DMU estimated PPS is where the convexity is the underlying conventions of the PPS. This means that PPS under DEA concept can generate all appropriate activity, since in case of non-convex, another extension of DEA, which is FDH can be used. Accordingly, in this research, an approach based on the integration of defuzzification and DEA is introduced to solve convex problem.

### 1.3 Relationship and Dependency

In Section 1.1, we have described in detail the term 'system' which contains a number of components that has inputs and outputs with relationships. Since a primary task of system analysis is the determination and evaluation of the relationship between the inputs and outputs, knowledge and understanding of this relationship is valuable for several reasons (Huang & Shi, 2002). First, an accurate knowledge of the relationship between inputs and outputs for a given system leads to some understanding of the behaviour and inner operations or internal mechanics of that system, and is a vital step to fully understand the operation and nature of the system. Second, where the input and output relationship is adequately known, it is possible to explain past performance of the system and to predict the output or response of the system for any permissible future input. Third, with the output response of the system known for any input, it is then possible to develop means for controlling or influencing the system in some desirable or optimal manner (Sakawa, Nishizaki, & Katagiri, 2011).

In any general system, the relationship between or among components, inputs and outputs can be described by linear and non-linear functions, or else the system is said to have no relationship. In this research, the focus is on the relationships or properties already exist in the system and our aim is to have crisp outputs that satisfy the relationship. Here, the concept of dependency is not about the outputs depending on the inputs, which is naturally the case, but the output also depends on the relationship between or among components of system.

Naturally, a question arises why we need fuzzy systems when we have a crisp relationship, crisp input and also a crisp output for a classical system. The reason for this fact lies in epistemic uncertainty (Bede, 2013). Here, fuzzy systems can fill in the gaps and approximate any desired output with arbitrary precision. In fuzzy systems, the inputs and outputs are all fuzzy. Since the crisp representation is still on demand, this idea leads to the need of the development of defuzzification method that has the ability to deal with a situation where their crisp output depends on a relationship that may exist in the crisp inputs. In other words, a specific defuzzification has to be developed that undertake the restriction or dependency of the output on certain relationship.



Some trails in the literature are addressed in this research that deal with the dependency of the crisp output on relationships or properties in the crisp input. Such trails are presented in the next section with three main issues in defuzzification techniques.

## 1.4 Issues in Defuzzification Techniques

As mentioned formerly, to execute the last phase of any fuzzy system, defuzzification stage is needed. In this section, we consider some issues referred to in the literature.

### 1.4.1 Issue of Dependency in Defuzzification

The inputs in any system usually have some relationships or properties that need to be satisfied in their crisp outputs (Kikuchi, 2000).

Hence, the traditional defuzzification methods such as classical COG or A&Z only deal with independent crisp outputs. Thus, these models are not correctly representing the actual or nature of systems structure in order to mathematically deal with dependency terms as embedded in defuzzification method. Furthermore, these methods can generate similar results even though the given data are with various relationships (Kikuchi, 2000; Sladoje et al., 2011; Verstraete, 2015).

In practice, the standard methods lay no restrictions to reach a crisp representative that intuitively suits to the fuzzy numbers. Figure 1.2 classifies the original data in system with relationships and without relationships and presents some common methods used in each case.

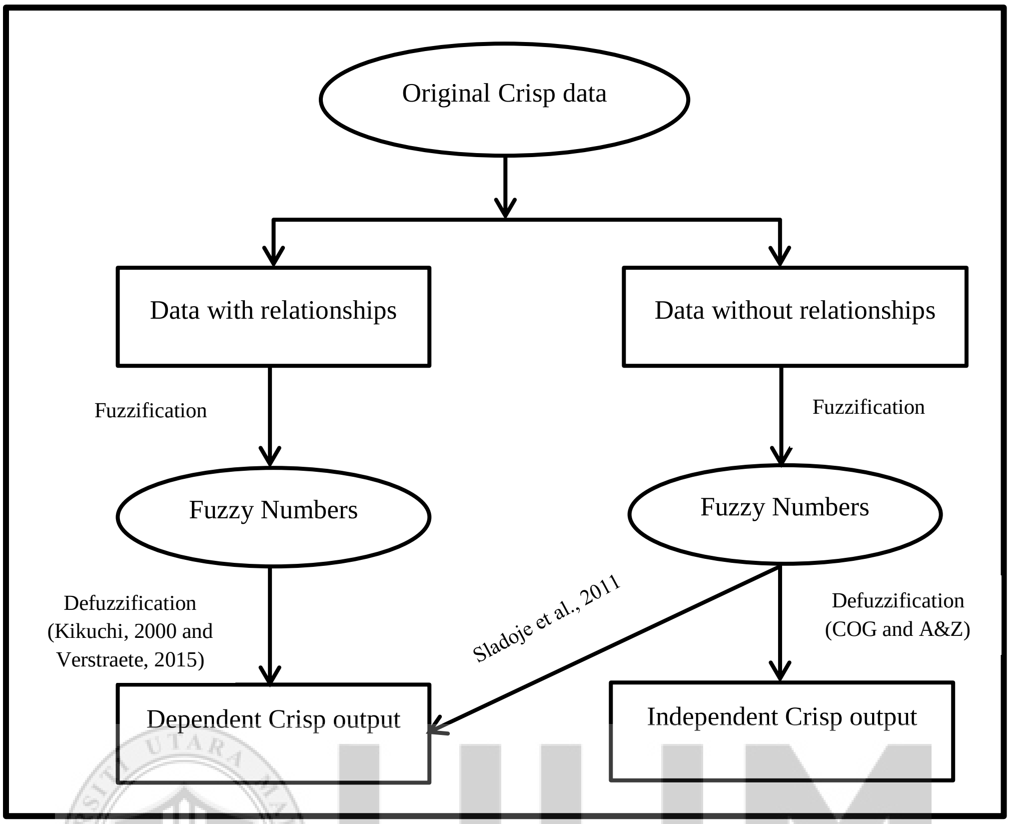



*Figure 1.2.* Classification of original data in system

Based on Figure 1.2 the dependent crisp output can be obtained by using methods such as of Kikuchi (2000) and Verstraete ( 2015), but these methods are based on the FLP model presented by Zimmermann (1978) which are known as max-min approach. It is true that the solution base in the max-min approach may neither be unique nor efficient (Li, Zhang, & Li, 2006; Peidro & Vasant, 2011). While Sladoje et al., (2011) presented the dependent crisp output in case of there are some properties of fuzzy numbers in fuzzification step, that need to be preserved in defuzzified output. While in the case of there are no relationships or properties in crisp data the standard methods such as COG or other method such as A&Z can be used to get the independent crisp output. In this research, we focus on the first case when there are some relationships or properties in original data that need to be satisfied in the crisp output.

### 1.4.2 Issue of Defuzzification Groups of Fuzzy Numbers

In most existing methods of defuzzification, fuzzy numbers are being addressed as individuals but not as group of fuzzy numbers. This means that, instead of considering the fuzzy numbers as individual, this study focuses on a collection of fuzzy numbers. Generally, in practical application, implementing the defuzzification techniques, such as classical COG or A&Z, leads to finding the defuzzified value for each fuzzy number individually (Bede, 2013). Since the computation of defuzzified value involves several mathematical operations, these methods require more calculations but also take more time to produce the output. It becomes more severe when the number of inputs and number of rules increase. Therefore, it is very important to reduce the computational time and requirements by reducing the mathematical operations involved (Siddique, 2014).



Moreover, most practical applications contain a large number of components and a large number of inputs with relationships within the components. These inputs demands the decision makers to employ the fuzzy numbers as groups by applying the method in one time (Hou, 2016).

### 1.4.3 Issue of Defuzzification in Prohibited Zones

Unfortunately, analyzing a defuzzification problem under optimization models requires extra constraints from decision makers in case of including the nonlinearity as memberships or relationships. In the defuzzification usual approach, any point in the output space is assumed to be the permissible values for the defuzzified output. The complexity of defuzzification problem lies in the possibility of having restrictions when the defuzzified value could have only some values from the whole discourse universe (Yager & Filev, 1995). A constraint on this problem must be included as well. This constraint is required to assure the location of defuzzified value or the crisp output is in the permissible area; then the defuzzification problem restricted by legal region could be represented. The problem of constrained optimization under some values becomes unconstrained and has a global solution (Roychowdhury & Pedrycz, 2001; Siddique, 2014; Yager & Filev, 1995).

## 1.5 Problem Statements

The identified gap or problem of this research can be recapitulated as follows. Although the classical defuzzification techniques such as Center of Gravity (COG) and Asady and Zendehnam (2007) (A&Z) emerge as the broadly applied defuzzification techniques under the concept of area and minimization distance, these techniques are being disparaged due to two major shortcomings. Firstly, they disregard the interaction between original crisp data and crisp outputs. Secondly, these methods are ineffectual in dealing with the aspect of defuzzification of fuzzy numbers as groups generated from complex systems.

The first issue can be elucidated as follows: Normally, in the crisp output of defuzzification techniques, some relationships or properties from original data or suggested from decision makers are needed to be met in the crisp output.

In reality, most existing methods of defuzzification attempted to make the fuzzy set estimation (crisp outputs) objectively (Liu, 2007). Nevertheless, being able to represent the decision maker's subjective knowledge is considered to be a significant aspect of the fuzzy set application, thus different perceptions for the defuzzification outputs may be revealed by various decision makers (Liu, 2007). Also, when the estimation of a fuzzy set (crisp outputs) do not satisfy the relationships, each value is adjusted until they meet the relationships in original crisp data indicating their mathematical dependence on one another (Kikuchi, 2000; Sladoje et al., 2011; Verstraete, 2015).

Nonetheless, the classical defuzzification such as COG and A&Z methods are based on independent crisp outputs terms. As a result, these techniques do not exactly reflect the actual physical principle of system or thinking pattern of decision makers (DM). However, the application of optimization techniques such as LP in defuzzification has been found to be helpful in modelling the usual relationships or properties that existed in the original crisp data or offered by DMs (Kikuchi, 2000; Sladoje et al., 2011). Unfortunately, the process of estimating crisp output by applying LP model can turn into unconstrained or global solution in case of non-linear objective or relationships (Yager & Filev, 1995). These complications could limit the decision makers in utilizing the advantageous of LP or NLP techniques as tools to be applicable in real problems.

Meanwhile, the second issue can be highlighted as follows: while implementing defuzzification techniques such as classical COG or A&Z, decision makers tend to employ the fuzzy numbers as groups by applying the method in one time (Hou, 2016). In reality, most real applications contain large number of component with inputs, relationships between these components or their inputs as described in the first issue.

However, it is proven that LP is capable of modelling the relationships among components or inputs and keeping them in the crisp outputs in case of linearity in objective or relationships (Kikuchi, 2000; Verstraete, 2015). While in the case of non-linear objective or relationships, the NLP model attempts to defuzzify fuzzy numbers. Unfortunately, the process of finding the crisp outputs that satisfy the relationships in the original crisp data by applying NLP can turn into a complicated process (Roychowdhury & Pedrycz, 2001; Yager & Filev, 1995). With regards to above-mentioned issues, it can be simplified as follows: Firstly, using standard defuzzification methods to solve the issue in the case of having groups of fuzzy numbers demands a higher computational effort by decision makers as it would require more computational steps when dealing with complex systems that have a large number of components or inputs. Secondly, using optimization techniques, such as LP and NLP in a defuzzification problem, demands more effort by decision makers, as it would require a restriction on outputs to be in the allowable region.



As a result, this research discovers an opportunity or need to offer an optimization procedure, which can provide a nearest crisp outputs to the fuzzy numbers, minimize the number of computational steps in case of groups of fuzzy number. It would be done by applying the multi-objective concept and meet the relationships or properties in original crisp data or that DMs offered to be in outputs. In other words, there is a necessity for the DMs to have a simple and straightforward defuzzification procedure, which concurrently captures the aspect of dependent and independent in crisp outputs and gives the optimal crisp outputs that are the nearest point to the fuzzy numbers. Figure 1.3 simplifies the gap identified through this research.

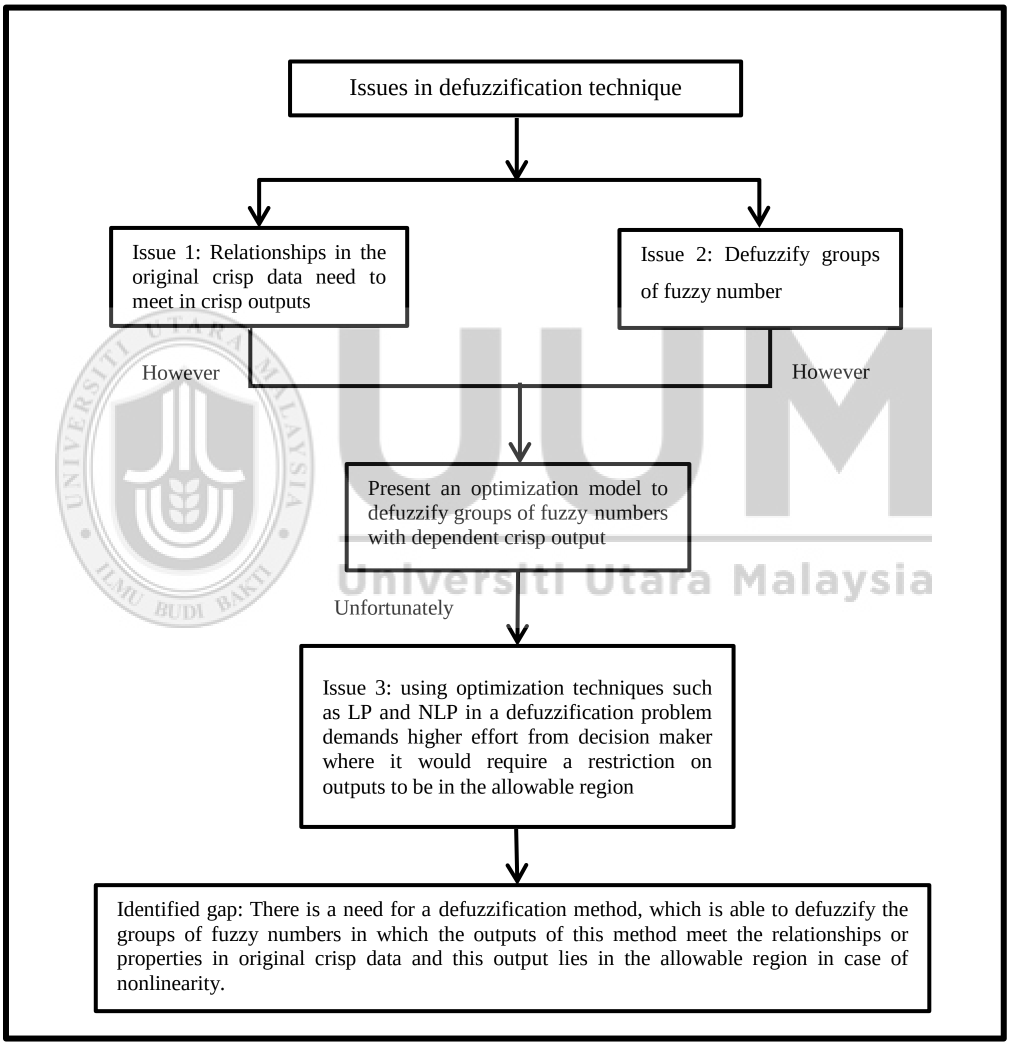



*Figure 1.3.* Problem statement of the research

In resolving the identified gap, several research questions have been formulated as follows.

## 1.6 Research Questions

The following questions are addressed in this research:

1. How to defuzzify groups of fuzzy numbers that satisfy some relationships in the crisp outputs?
2. How to implement this new method in a real problem?
3. How to evaluate the proposed method?
4. How to apply the proposed method in dealing with different types of fuzzy numbers and solving other problems?

## 1.7 Research Objectives

The current research primarily aims at developing a new method of defuzzification to defuzzify groups of fuzzy numbers based on a DEA model that can lead to the creation of crisp output values in which this method is able to satisfy the relationships or properties in the original crisp data and keep them in the solution. The following are the specific objectives that need to be accomplished to meet the desired results:

1. To develop a new defuzzification method by modifying the center of gravity (COG) method.
2. To modify the DEA model with a new objective and extra constraints.
3. To implement the new method to solve real problems.
4. To compare the outcome obtained by the suggested method with other corresponding methods.

5. To compare the suggested DEA-based model with LP–based model.
6. To apply the proposed method to the numerical problems in the literature with different types of fuzzy numbers.

## 1.8 Scope of the Research

This research develops defuzzification methods using one of the optimization techniques that is DEA of CCR model. Two types of fuzzy numbers are used which are triangular and trapezoidal. The proposed method is compared with the original COG and A&Z methods in the case of independent crisp output and Kikuchi (2000) method in the case of dependent crisp output.

This research focuses on the development of a general defuzzification method based on the assumption of the convex PPS of DEA. A special case of the proposed method is presented in case of linear in relationship, membership function that is LP based model, and the results under DEA and LP are compared.



The application of the proposed method is firstly considered in the allocation problem in a healthcare sector in Malaysia. In addition, the proposed method is applied to finding the optimal weight in GP and solving the issues of ranking of fuzzy numbers, in case of the fuzzy numbers are triangular or trapezoidal and fuzzy number with non-linear membership function.

## 1.9 Significance of Findings

The findings of this research will have the following contributions:

i. New defuzzification methods are constructed based on the DEA model that can create crisp values in case, of original crisp data have relationships need to satisfy in their crisp outputs (dependent crisp outputs) or do not have such relationships (independent crisp outputs)

ii. Theoretically, this research focuses on developing defuzzification methods to address groups of fuzzy number.

iii. Practically, the developed methods can be applied to solve a real problem that is the allocation of an optimal number of beds in hospitals in Malaysia.

iv. The proposed methods can be used in finding the crisp outputs in the following fields:

  - In GP: To determine the ideal weight in the interval weight (IW) approach.
  - In ranking fuzzy numbers with triangular, trapezoidal and non-linear fuzzy numbers.

## 1.10 Overview of the Thesis

This thesis consists of six chapters. Chapter One introduces the present study and fuzzy system structures including fuzzification and defuzzification. The LP, NLP, and DEA under the optimization techniques are included. Discussion on the relationships and dependency besides of issues in defuzzification techniques are also presented. The chapter also includes the problem statement, followed by research questions, outlining the research objectives, research scope and the significance of the study.

Chapter Two presents an overview of fuzzy theory including fuzzy set, fuzzy number, fuzzification and defuzzification processes along with their concepts and methods. The chapter also includes the ranking of fuzzy numbers, the expected value of fuzzy numbers, and the nearest point of fuzzy numbers.

Then, Chapter Three deals with these conceptions, including LP and multi-criteria decision-making (MCDM). Next, it is followed by DEA basic models and their modifications. Then, fuzzy DEA, the applications of each DEA and fuzzy DEA in different sectors are investigated. Multi-objectives decision-making (MODM) concept and one of their techniques, that is GP and its approaches with the interval weight (IW) method to find an optimal weight, are also included. Finally, the DEA model with the defuzzification concept is discussed.

Chapter Four describes research methodology, research design, and research activities. It also illustrates the data collection procedure by discussing the technical approach and developing the methodology and empirical models.

Chapter Five shows the results of the application of the proposed methods in the Malaysian healthcare sector in case where the original crisp data have relationships that need to be satisfied in crisp outputs and in case where there is no relationship. The estimated number of beds is determined by running the proposed model and comparing it with the three existing methods. Furthermore, the application of the proposed method to solve some common issues in the literature, which includes finding the optimal weights in the GP model and the ranking of fuzzy numbers.

The discussion on the results, conclusions, limitations of the study, and suggested future work related to the area of this research are covered in Chapter Six.

# CHAPTER TWO
# AN OVERVIEW OF FUZZY THEORY

There are two major issues hooked with the conventional defuzzification methods as clarified in Chapter One. Firstly, the conventional defuzzification methods ignore the aspect of interdependencies among crisp inputs and crisp outputs during defuzzification process. Secondly, they are incapable of coping with fuzzy numbers as groups in data offered by systems. This chapter is devoted to compiling some significant information about the former issue by reviewing the literature about the concept and methods that would be helpful in constructing the proposed procedure of the research. This chapter begins with the discussion of the basic definitions and concepts of fuzzy set, fuzzy numbers, fuzzification, and defuzzification. Furthermore, it highlights the notion of fuzzy numbers, fuzzification, and defuzzification with their standard types and techniques.



## 2.1 Fuzzy Set

According to the theory of classical set, two alternatives are allowable for an element, i.e. it either should strictly be a member of a set or should not. A fuzzy set is basically an addition from the classical set theory by defining the set elements that do not need to belong to the set but instead require a level of membership (Dubois & Prade, 2000). Nevertheless, a function of membership is defined as the fuzzy set, which is described in the discourse universe as follows.

A universal set is symbolized by $\mathbb{X}$; thus the definition of a fuzzy set $\tilde{F}$ is achieved through its function of membership $\mu_{\tilde{F}} : x \rightarrow [0,1]$. Each element $x \in \mathbb{X}$ is assigned a real number $\mu_{\tilde{F}}(x)$ in the interval [0,1] by the membership function $\mu_{\tilde{F}}(x)$ where the value of $\mu_{\tilde{F}}(x)$ denotes the $x$ membership in $\tilde{F}$. A pair comprising an element $(x)$ denotes a fuzzy set $\tilde{F}$ and its grade $\mu_{\tilde{F}}(x)$, and is therefore shown as.

$$\tilde{F} = \{(x, \mu_{\tilde{F}}(x)), x \in \mathbb{X}\} \tag{2.1}$$

whereas, function on membership completely determines the fuzzy set. For a given ordinary set, the characteristic function as,

$$Ch_F(x) = \begin{cases} 1 & if\ x \in F \\ 0 & if\ x \notin F \end{cases} \tag{2.2}$$

which defines set $F$ as $F = \{x \in \mathbb{X} | Ch_F(x) = 1\}$.

From the above definitions, it is natural to state that a fuzzy set $\tilde{F}$ naturally extends from an ordinal set $F$. It can be noted that crisp sets represent a special case of fuzzy sets since the function range is constrained to the values 0 and 1. In order to produce the function of membership defining a fuzzy set, it is initially important to deal with the empirical outcome. It is practical for a function shape to be determined for each area it examines. The $\mu_F$ is a compilation set of membership values constrained by the value of zero (0) and one (1), representing notch membership in $F$ for every unit, $x \in \mathbb{X}$. Therefore, the fuzzy set contains an object set and its values of its membership. In the same way, a crisp set is defined, but the values of membership for whole elements of the set would be 1. The grade of membership and characteristic function are alternative expressions for the function of membership.

Fuzzy and traditional sets are mainly distinguished by the image of their membership functions. The membership grades of a traditional set take values either 0 or 1in the set $\{0, 1\}$, whereas those of a fuzzy set are realized in the unit interval $[0, 1]$. Elements partially involved in the set can be included in a fuzzy set (Xexéo, 2002). Whenever necessary, a crisp set can be used to define a standard set to be distinguished from a fuzzy set. A crisp set is continuously distinct by a generic set of $\mathbb{X}$. (Zimmermann, 2010) Further explanation on an important concept, in that the fuzzy set is a fuzzy number concept and it is imperative to serve a transfer relation between a fuzzy set and an ordinary set. More details on fuzzy numbers are supplied in next sections.

## 2.2 Types of Fuzzy Number

Fuzzy numbers are real numbers of generalization, and fuzzy numbers are an important part of the fuzzy set that plays essential roles in linguistically-expressions such as "approximately" or "close to". That means a fuzzy number $\tau$ represents the real kernel in the application of a theory of fuzzy set. Where $\tau$ is described as a convex, regular fuzzy set indicated on the real numbers. In literature, the use of fuzzy numbers has been employed in many fields, such as decision sciences (Dubois, 2011), operations research (Kaufmann, 1986) and medicine (Barro & Marin, 2002). The basic definitions in this section are introduced by several scholars (Sakawa et al., 2011; Zimmermann, 2001). Formally, the definition is as follows:

**Definition 2.3.1**: $\mathcal{T}$ expresses as a fuzzy sub-section of a real line $\mathbb{R}$, which is described as $\mathcal{T}: \mathbb{R} \to [0, 1]$. If the following characteristics are satisfied, then $\mathcal{T}$ shall be called as a fuzzy number:

i. $\mathcal{T}$ is normal, where. $\exists\, q \in \mathbb{R}\, and\, \mathcal{T}\,(q_0) = 1$.

ii. $\mathcal{T}$ is fuzzy convex;($\forall \xi \in [0, 1], \exists\, q_1, q_2 \in \mathbb{R}$ then $\mathcal{T}(\xi q_1 + (1 - \xi) q_2) \geq min\{\mathcal{T}(q_1), \mathcal{T}(q_2)\}$).

iii. $\mathcal{T}$ is upper semi-continuous on $\mathbb{R}$ *i.e.* ($\forall \rho > 0\ \exists \gamma > 0\ s.t\ \mathcal{T}(q) - \mathcal{T}(q_0) < \rho, |q - q_0| < \gamma$).

iv. $\mathcal{T}$ is compactly supported i.e. $cl\{q \in \mathbb{R};\ \mathcal{T}(q) > 0\}$, which $cl(\mathbb{A})$ stands for the set $\mathbb{A}$ closure.

A fuzzy numbers division depends on the type of their membership function which describes them (Bede, 2013; Dubois & Prade, 1980; Zimmermann, 1996, 2001). Many types of fuzzy number are addressed in the literature such as Triangular fuzzy number ($TrFN$), Trapezoidal fuzzy number ($TpFN$) and S–Shaped fuzzy number. These varieties of fuzzy numbers are explained as follows.

### 2.2.1 Triangular Fuzzy Numbers

The triangular shape of fuzzy number ($TrFN$) has two line segments represent its functions of membership, one is rising from $\langle a, 0\rangle$ to $\langle m, 1\rangle$ and other is sliding from $\langle m, 1\rangle$ to $\langle b, 0\rangle$. The interval $[a, b]$ refers to its domain. The specification of this number could be by the systematic triplex $\langle a, m, b\rangle$, with $a \leq m \leq b$; therefore, its function of membership is:

$$TrFN[a,m,b](x) = \begin{cases} \dfrac{x-a}{m-a} & a \le x \le m \\ \dfrac{x-b}{m-b} & m < x \le b \end{cases} \quad (2.3)$$

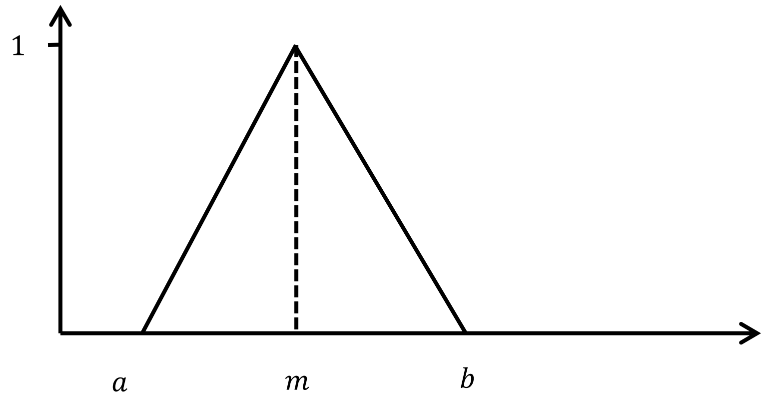



*Figure 2.1*. Triangular shape of fuzzy number $\mathcal{T} = TrFN[a,m,b]$

### 2.2.2 Trapezoidal Fuzzy Number

Identification of $TpFN$, is by a systematic quadrilateral $\langle a,l,r,b\rangle$ with $a \le l \le r \le b$ and its function of membership includes three line pieces. One line is rising from $\langle a,0\rangle$ to $\langle l,1\rangle$; the other line is horizontal having one as a fixed value ranging between $\langle l,1\rangle$ and $\langle r,1\rangle$, while the last line falls from $\langle r,1\rangle$ to $\langle b,0\rangle$. Its function of membership is presented by;

$$TpFN[a,l,r,b](x) = \begin{cases} \dfrac{x-a}{l-a} & a \le x \le l \\ 1 & l < x < r \\ \dfrac{x-b}{r-b} & r \le x \le b \end{cases} \quad (2.4)$$

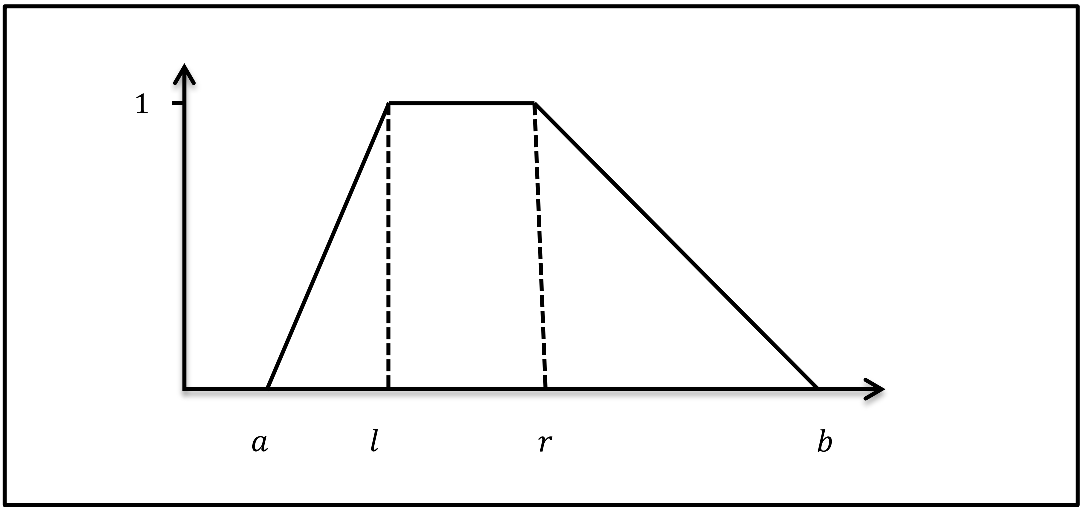



*Figure 2.2.* Trapezoidal shape of fuzzy number $\mathcal{T} = TpFN[a, l, r, b]$.

If $l = r$ then, the $TpFN$ be a $TrFN$, since $TrFN$ is a special situation of the trapezoidal fuzzy number.

## 2.3 Fuzzy Numbers Representation

As mentioned before fuzzy numbers could be divided on the basis of the kind of their membership function which describes them. The functions of linear memberships (trapezoidal and triangular) are appropriate for most cases and offers quick computation, whereas other curves, such as Gaussian and Sigmoid, provide smooth results (Duch & Jankowski, 1997; Duch, 2005).

According to Xexéo (2002), a reasonable strategy examines the development methods or models in fuzzy set theory by starting with simple functions of Triangular and Trapezoidal membership in developing methods; then the models are refined later using more complex functions such as S-shaped.

Among the main assumptions when determining solutions for fuzzy mathematical programming problems is the employment of linear membership functions for the entire fuzzy sets primarily in a decision-making process. In this case, a linear approximation is widely used owing to its simple functions, and its definition of the upper and lower levels of acceptability. If the theory of fuzzy set is deemed to be a formal one, then such an assumption can be accepted (Zimmermann, 1983, 2001).

In contrast, if the theory of fuzzy set is utilized to model actual decision-making processes, then an assumption is made that the resulting models are actual models of real-life situations. Hence, justification is needed. Suitable membership functions were applied to the pioneering study of Zimmermann (1978) to determine a solution to a problem of linear programming (LP) with many objective functions. It demonstrated the results from fuzzy linear programming (FLP) were efficient. This efficiency was also emphasized by Leberling (1981), who made use of hyperbolic membership function to resolve a multi objective linear programming (MOLP) problem and concluded that solutions are provided by the FLP with non-linear memberships function. Meanwhile, Dhingra and Moskowitz (1991) described various non-linear membership function types including exponential, quadratic and logarithmic that resolve a problem.

Similarly, Verma, Biswal and Biswas (1997) also reached the same conclusion when employing the fuzzy programming approach on non-linear (hyperbolic and exponential) membership functions in determining a solution to a multi-objective transportation problem.

Vasant (2006) proposed FLP as per modified S-shaped membership function. Moreover, in a related study, Gupta and Dangar (2010) developed a pair of fuzzy primal-dual quadratic programming problems wherein ambiguous aspiration levels are reflected by an exponential membership function. Also, De and Yadav (2011) utilized a non-linear (exponential) membership function to determine a solution to a multi-objective assignment problem (MOAP) via interactive fuzzy goal programming (FGP) method.

To this end, other non-linear shapes for membership functions such as exponential, hyperbolic and sigmoid memberships function are deemed as non-linear and fuzzy mathematical programming having non-linear membership function that leads to NLP. While a linear membership function uses steer clear of non-linearity, some issues were encountered during determining a solution to problem written in linear membership. In this regard, its non-linear membership counterpart has more flexibility in describing the ambiguity found in the fuzzy parameters for actual real life (Tiwari, Tiwari, Samuel, & Pandey, 2013; Watada, 2001).

Based on the above studies, a fuzzy set can be represented graphically using functions of membership. The discourse universe is represented by an $x$-axis, whereas $y$-axis denotes of the membership degrees in the interval of $[0,1]$. In this research, membership functions are built by the use of simple functions. In defining fuzzy concepts, the use of more complex functions leads to the non-convexity term, which is not covered in this research. Next section presents the techniques and process to create fuzzy data.

## 2.4 Fuzzification Techniques

Fuzzification refers to the first process in every fuzzy system to convert or create a fuzzy number from data. The linguistic terminology are always used to describe the problem (Sivanandam, Sumathi, & Deepa, 2007). This process is performed with experience, intuition, and analysis of the rules and conditions set associated with the variables of input data. This leads to the lack of fixed procedures set for the fuzzification (Pant & Holbert, 2004). Through several studies, some fuzzification approaches are identified. Fuzzification techniques have three different categories: adaptive, manual, and automatic. Adaptive techniques seek an optimal design of the system, whereas manual techniques are most fitting with the obtained evidence from human responses. In the third category, data sets are processed using automatic techniques in order to decide the proper representation of fuzzy set (Clark et al., 2008).



### 2.4.1 Adaptive Techniques

Adaptive techniques seek an optimal design of system used for creating the membership functions to represent data. The most common methods for adaptive techniques are neural networks and genetic algorithms. The neural network represents computer systems that simulate the human brain's operations (Zhou, Purvis, & Kasabov, 1997). Neural networks have numerous advantages, including: (i) No assumption is made on the fitting shape with an input-output data; (ii) empirical well-functioning irrespective of the source and nature of data set; (iii)

Could be utilized on-line or off-line for a continuous refinement of the membership functions facing a new conduct of the system that is being controlled.

Other means used to construct fuzzy data represented by genetic algorithms are referred to techniques that are inspired ‘biologically’ for evolving better fuzzy sets. Accordingly, the computer requires a goal in order to achieve this evolution. The fuzzy set theatrical goals are to present improved fuzzy logic controllers (Treesatayapun, Kantapanit, & Dumronggittigule, 2003).

### 2.4.2 Manual Techniques

Several statistical methods are often used to determine functions of membership. To conclude the fuzzy set shape, various approaches and questions should be investigated. Practically, triangular and trapezoidal fuzzy sets have been proven effective in several applications for the elicitation of critical values. For instance, Watanabe (1993) asserts that the use of frequencies and direct estimation are the two broad categories under which these statistical techniques fall (as cited in Szczepaniak, Lisboa, & Kacprzyk, 2000). The frequency methodology attains the function of membership by measuring the population’s percentage in a test group. The direct estimation methodology derives its values from a sliding scale and elicits responses from experts that grade the compatibility of the object and the set. All the manual techniques suffer from the deficiency that they rely on a very subjective interpretation of words (Szczepaniak et al., 2000).

### 2.4.3 Automatic Techniques

Automatic technique of data transmissions, which primarily process data sets to decide proper representation of a fuzzy set. Automated techniques cover various approaches. The automatic generation differs from the manual methods in that it involves a complete removal of expert(s) from the process (Clark et al., 2008; Szczepaniak et al., 2000). Curve fitting and histograms are typical procedures.

#### 2.4.3.1 Curve Fitting

When there is information set which could be utilized as a premise for building the membership functions of a fuzzy set, a large technique of fuzzification is accessible. The basic methods of curve fitting, such as triangular and trapezoidal curves have generic formula equations (2.3) and (2.4) sequentially. For example, if the equation (2.4) of $TpFN$ is considered by assuming the presence of $n$ data points $\langle c_i, y_i \rangle$, minimization of error, $Er$,



$$Er = \sum_{i=1}^{n} \left(y_i - TpFN(c_i)\right)^2 \qquad (2.5)$$

could be effectively prepared on a computer through the use of standard numerical technique. Minimum squared error estimation on $TpFN$ will be the solution. Practically, a methodology of curve fitting such as methodologies of regression and Lagrangian interpolation in the mathematical principle could be adjusted to find the fuzzy set's membership values.

#### 2.4.3.2 Histograms

Sometimes in some cases of fitting the curve to the data set, the use of histogram is more logical; for instance when there are no input-output pairs, but there is a broad cross-section of non-correlated input values with a controlled output, or when there is no satisfactory control whether by human or machine.

When a data set contains elements of $p$ for sets $\langle c_t, d_t \rangle$ , $t = 1,2,..,p$, then the leading steps are finding $c^{min}$ and $c^{max}$, which represent the minimum nor maximum values is achieved by the leading component in the orderly pair. The following step will divide the $c^{min}$ and $c^{max}$ interlude into $m$ sub-intervals. For each interval $[c^{min}, c^{max}]$ of the element $c_t$ is divided into $m$ sub-intervals

$$\{[c^{min} = c_0, c_1], [c_1, c_2] .., [c_{(m-1)}, c_m = c^{max}]\} \qquad (2.6)$$

where each sub-interval has the width $\Delta c = \frac{c^{max} - c^{min}}{m}$, then each element is labeled as;

$$c_k = c^{min} + k * \Delta c \qquad k = 0,1,2,..,m \qquad (2.7)$$

According to preceding description, the points set can be used in sub-intervals for curve fitting.

### 2.4.4 Overview of Fuzzification Techniques

This section illustrates the most common practices or techniques of fuzzification used. For instance, some studies used averaging measurement to transform crisp (non-fuzzy) numbers into fuzzy numbers. The number is produced as follows:

$$\tilde{c} = \left(c^{min}\ ,\ c^{mid}\ , c^{max}\right) \tag{2.8}$$

where,

$$\left.\begin{array}{l} c^{min} = min\{\ c_1\ ,\ c_2, \dots ,\ c_n\} \\ c^{max} = max\{\ c_1\ , c_2, \dots , c_n\} \end{array}\right\} \tag{2.9}$$

and

$$c^{mid} = \text{the average value of}\ \{\ c_1\ , c_2, \dots ,\ c_n\} \tag{2.10}$$

The $c^{min}$ represents the lowest value of element $c_i$, $c^{max}$ represents the highest value of element $c_i$, while $c^{mid}$ is the average value of element $c_i$.



The mean value is usually at the middle. However, there is some averaging measurement used in such cases as mentioned by Dubois and Prade (2000). For example, the median or mean value, refers to a non-resistant measure of the center because of its sensitivity to unusually small or large data (outliers) (Bodjanova, 2006), is described as:

$$c^{arth} = \frac{1}{n}\sum_{i=1}^{n} c_i \tag{2.11}$$

The geometric mean which is the $n^{\text{th}}$ root of the product of the $n$ non-negative numerical values is described as;

$$c^{geo} = \left(\prod_{i=1}^{n} c_i\right)^{1/n} \tag{2.12}$$

The importance of the geometric mean lies in its strength to lessen the (so low or high) values effect since such very low or extremely high values could bias the arithmetic mean. In other words, the extreme values affect the arithmetic mean more than the geometric mean (Angiz & Sajedi, 2012; Chang, Yeh, & Chang, 2013; Yeh & Chang, 2009), ranging from the lowest and highest value, so they follow the probabilistic path to produce fuzzy sets. The best estimate, in this case, is deemed to be a geometric mean. Measurements of central tendency that suited the geometric mean is represented as an intermediate value of the fuzzy number (Aref & Javadian, 2009).



Some other studies followed another path to create a fuzzy number (Au, Chan, & Wong, 2006), which determined the fuzzy sets' membership functions from the data directly. The class interdependence attribute is maximized to improve the results of classification. Domańska and Wojtylak (2010) introduced a procedure aimed at converting real numbers into fuzzy number. The study generated a particular fuzzy number to be used in a model for estimating the concentrations of pollution.

Nasibov and Peker (2011) investigated the formulas of the parameter for an exponential membership function based on the problem of minimization. The method used in their study aimed at obtaining an exponential membership function, which assumed that the shape of data is a histogram.

Grzegorzewski and Mrówka (2005) suggested a new method of the approximations of a trapezoidal fuzzy number utilizing other fuzzy numbers as input. Voxman (2001) used input represented by a discrete fuzzy number. In their method, Ralescu and Visa (2007) achieved a discrete fuzzy number that was not efficiently computable because of the necessity of using several elements to denote a discrete fuzzy number. Moreno-Garcia, Jimenez Linares, Rodriguez-Benitez, and del Castillo (2013) proposed a new technique to calculate a trapezoidal fuzzy number from discrete raw data. They utilized a linear regression for obtaining the fuzzy number's membership function. Trapezoidal membership functions were used to simplify the use of fuzzy numbers. Their suggested method has a complexity of 0 $(n\ log\ n)$ and is unsupervised.



As mentioned by Starczewski (2013, p. 139), that in the case where is unavailable knowledge about the uncertainty of the given input data, the simplest form called a singleton fuzzification is chosen. In fact, it is the most commonly used method of blurring premises. The singleton fuzzification is a way of representation of a crisp value in the fuzzy set form. Additional knowledge about the uncertainty of inputs allows us to assume a type of a fuzzification function, e.g., in the form of a triangular fuzzy number or Gaussian fuzzy number. The following section discusses the step to convert these fuzzy representatives into the crisp representatives i.e. the defuzzification process.

## 2.5 Overview of Defuzzification

Defuzzification is considered important in the field of research since the early 1990s. Defuzzification transforms a fuzzy set to single crisp value (Hatzichristos & Potamias, 2004; Runkler, 2013). This process occurs by reducing all fuzzy numbers to a single number which often loses some information and is, therefore, incapable of encapsulating the element of uncertainty (Lee, 2000; Leekwijck & Kerre, 1999; Sivanandam et al., 2007). Unfortunately, there is no systematic procedure to be used in the selection of a good strategy of defuzzification; therefore, when considering the application case properties, a systematic procedure should be selected (Lee, 2006).

Some methods of defuzzification have a tendency to create an aggregate output by seeing all fundamentals of the fuzzy set that has identical weights. Another method considers only the elements, which correspond to the optimum of the resulting membership functions. The fundamental methods of defuzzification having a practical importance are Center of Sums (COS), Center of Area (COA), Middle of Maxima (MOM), Height Method (HM), Center of Largest Area (COLA), and First of Maxima (FM).



### 2.5.1 Fundamental Methods

This section presents a number of the widely used defuzzification methods. Some researchers proposed a classification of defuzzification methods (Leekwijck & Kerre, 1999; Runkler, 1997) to evaluate whether the properties are important as the following groups:

a. Area methods: The defuzzification value divides the area under the membership function in two, more, or less equal parts.

b. Distribution methods and derivatives: Conversion of the membership functions into probability distributions, and computation of the expected value. The main advantage of these approaches is the continuity property.

c. Maxima methods and derivatives: Selection of an element from the core of a fuzzy set is considered as the defuzzified value. The main advantage of these approaches is simplicity.

Three primary defuzzification methods which are widely used (Driankov, Hellendoorn, & Reinfrank, 1996; Lee, 2006) are explained in the following three sub-sections.

#### 2.5.1.1 Centroid Method

The basic and the most frequently used method for defuzzification is the centroid method which is developed by Sugeno (1985) under the classification of area method. It is known as the center of gravity (COG) or center of area (COA), which calculates the center of area of membership grades $\mu_{\tilde{F}}(x)$ using the Riemann integral as

$$x_{COG} = \frac{\int \mu_{\tilde{F}}\,(x).\,x\;dx}{\int \mu_{\tilde{F}}\,(x)\;dx} \tag{2.13}$$

where, $x_{COG}$ is the defuzzified value for the interval fuzzy data. There are two ways to calculate the centroid in fuzzy logic systems namely, numerical integration with the use of approximation, and restriction of $\mu_{\tilde{F}}(x)$ to the specific shapes or known as integrals. For the discrete case in which $\mu_{\tilde{F}}$ is defined on a finite universal set, $\{x_1, x_2, x_3, .., x_n\}$, the centroid can be calculated by the discrete centroid given by

$$x_{COG} = \frac{\sum_{i=1}^{n} \mu_{\tilde{F}}(x_i).x_i}{\sum_{i=1}^{n} \mu_{\tilde{F}}(x_i)} \quad (2.14).$$

The origin of the discrete centroid method comes from the height type defuzzification (also known as the center average defuzzification) that disregards the exact shapes of membership functions of fuzzy rule conclusions and takes into account only their height. This approach is equivalent to reducing fuzzy rule conclusions to singleton membership grades. The drawback of the height type defuzzification is that it cannot be applied directly to fuzzy logic systems with logical reasoning, in which the aggregation operation has to be performed previously (Starczewski, 2013). Figure 2.3 represents this method graphically.



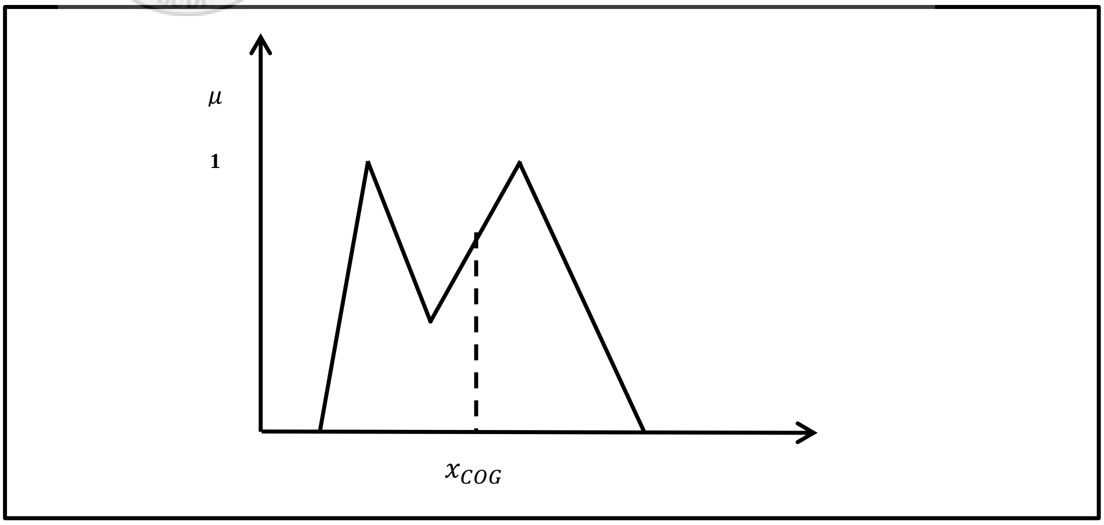



*Figure 2.3.* Centroid method

#### 2.5.1.2 Weighted Average Method

Weighted average method (WAM) is one of the most frequent defuzzification methods used in the fuzzy application. WAM is computationally faster, easier, gives accurate result, and can be expressed as:

$$x_{WAM} = \frac{\sum_i \mu_{\tilde{F}_i}(x) w_i}{\sum_i \mu_{\tilde{F}_i}(x)} \quad (2.15)$$

Where $x_{WAM}$ is the defuzzified output, $\mu_{\tilde{F}_i}(x)$ is the membership of each fuzzy number $\tilde{F}$, $w_i$ is the weight associated with each membership of each fuzzy number.

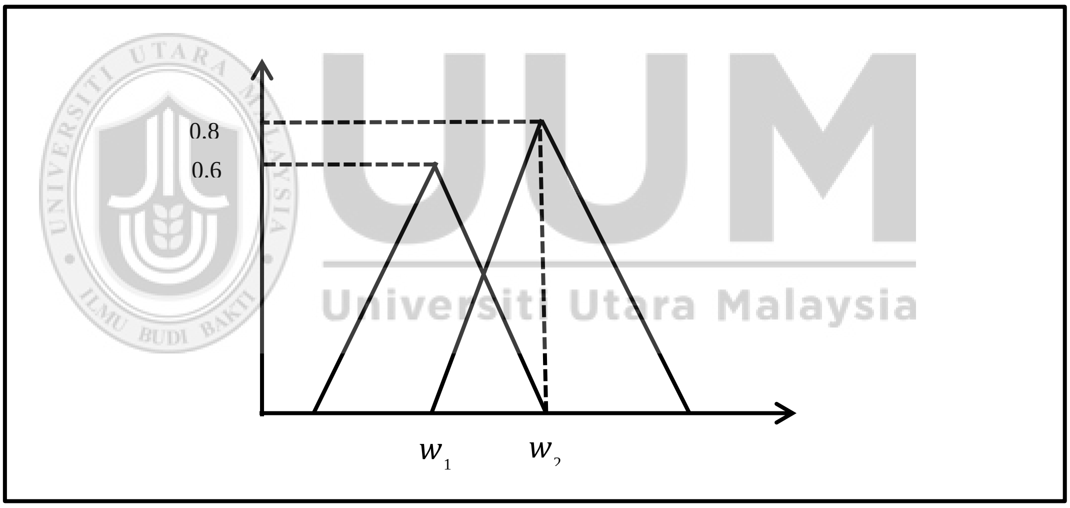



*Figure 2.4.* Weighted average method

From Figure 2.4,

$$x_{WAM} = \frac{w_1(0.6) + w_2(0.8)}{(0.6 + 0.8)}.$$

#### 2.5.1.3 Height Method

In height method (HM) the defuzzified point is the point that has the highest membership function It is also known as the max-membership method which is very fast but is limited to peaked output functions as shown in Figure 2.5. The expression of HM method is given as follows:

$$\mu_{\tilde{F}}(x_{HM}) \geq \mu_{\tilde{F}}(x) \quad , \forall x \in X \tag{2.16}$$

where $x_{HM}$ is the defuzzified output, $\mu_{\tilde{F}}(x)$ is the membership of fuzzy number $\tilde{F}$.

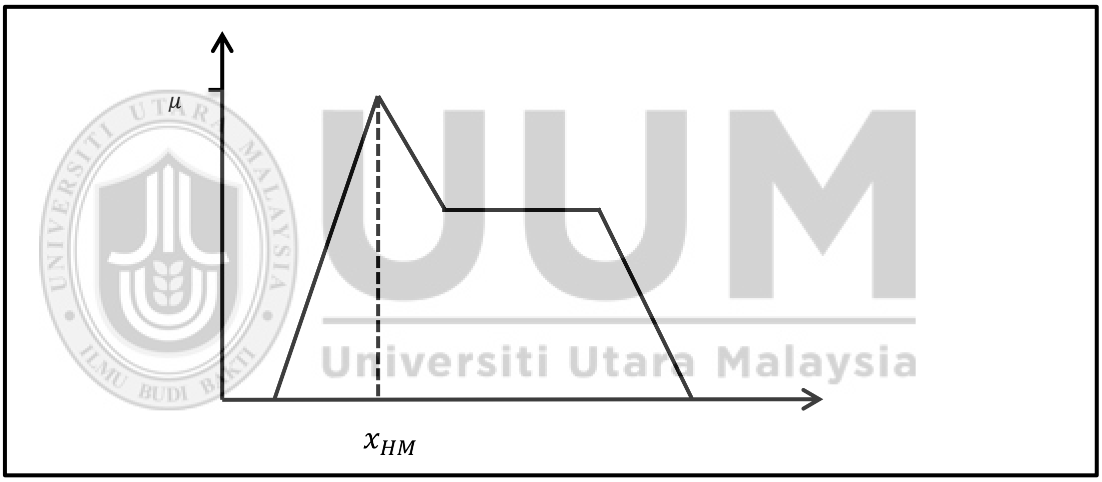



*Figure 2.5.* Height method

### 2.5.2 Modification of Defuzzification Techniques

There is no systematic way for selecting a defuzzification technique (Lee, 1990; Siddique, 2014). For this reason, different strategies proposed in the literature are explained in this section. To begin with, Mabuchi (1993) introduced a defuzzification method to defuzzify fuzzy subsets and interval values using sensitivity analysis along with a type of mini-max principle.

Similarly, Ma et al. (2000) suggested a novel method to defuzzify fuzzy sets through metric distance between two symmetric $TrFN$. They defuzzified a fuzzy number while simultaneously acquiring the fuzziness of original quantity with the help of a close symmetric fuzzy number. Shi and Sen (2000) introduced the Height Weighted Second Maxima (HWSM) and assessed performance using the simulation results. , They Also compared different defuzzification methods as COA, COS, HM and MOM, Center of Largest Area (COLA) and First of Maxima (FM), to evaluate the HWSM of every output. As a result, the defuzzification using their method HWSM provides further enhancements.

Leekwijck and Kerre (2001) brought forward a continuous maxima defuzzification approach by considering the output fuzzy set as the defuzzifed value. The computational time of their method is very efficient. Ginart and Sanchez (2002) introduced a new fast defuzzification method based on the estimates of the centroid position by accommodating the fuzzy output structure in one triangle. This method tested the second order plants control and eventually became a model for Continuous Stirred Tank Reactor (CSTR). Lancaster and Wierman (2003) proposed two new methods in defuzzifying the plateau average (PA) and weighted plateau average (WPA). The concept of PA method is the result of a defuzzification output value and is calculated as the midpoints of all fuzzy outputs average. Evidently, PA is a superior version of the Mean of Maxima (MOM) with the only difference being that PA does not overlook any provided rule output. Meanwhile, WPA is an improved version of PA, which requires higher computation and acquires higher precision than

PA. The final defuzzified output value in WPA is calculated whereas mid-points of the entire output platform is the weighted average.

Banaiyan, Fakhraie, and Mahdiani, (2005) presented three defuzzification method namely trapezoid median average (TMA) and the family of weighted trapezoid median average (WTMA) method. The WTMA has two methods, which are the Trapezoidal WTMA (TWTMA) and the Rectangular WTMA (RWTMA). In TMA method, the defuzzified output is obtained by averaging the mid-points of the entire output trapezoid medians as opposed to the mid-points of the entire output trapezoid platforms by PA method presented by Lancaster and Wierman (2003). TMA is advantageous over PA as the latter attempts to estimate trapezoid center of gravity whereas TMA considers the top and bottom corner of the slopes as well. The outputs of both methods are identical symmetric output membership functions, but TMA offers the superior estimation of the COG in asymmetric membership functions. On the other hand, the WTMA and TWTMA seem to be improved models of TMA. In such methods, the last defuzzified output is calculated through midpoints on entire trapezoid medians weighted average. The two WTMA forms only different in their weights; for example, TWTMA method weights the same areas of the trapezoids membership function.

Banaiyan, Mahdiani, and Fakhraie (2006) compared their three new defuzzification methods introduced in their work in 2005 that were suitable for efficient software and hardware implementations. Their comparative results showed that their new methods give the highest accuracy levels while giving faster and lower software complexities.

In their work, Mahdiani, Banaiyan, Javadi, Fakhraie and Lucas (2013) utilized Banaiyan et al.'s (2005) approaches, which are also referred to as classic standard methods, and applied them in the software field. These methods offer passable accuracy levels as compared to the existing methods as they require less costs of implementation with regards to area, delay, and power consumption in hardware realization while, at the same time, less execution time and instruction count in the software implementation. The defuzzification methods of the software models are created under three platforms, namely, Intel's Pentium IV, IBM's PowerPC, and TI's C62 DSP to reveal that novel methods require less execution time and instruction count of the most common methods used. The hardware models are created to synthesize the present superiority of the novel methods, with regard to the area, delay and power consumption.



Based on the above discussion, it can be concluded that the past defuzzification methods were developed without considering the characteristic of the original crisp data. In this case, the characteristic is referred to any relationship or some properties that must be satisfied in their crisp output.

## 2.6 Overview of Methods in Ranking Fuzzy Numbers

Ranking fuzzy numbers is an important component of the decision process, data analysis, artificial intelligence and socioeconomic practices. It is due to the essence of the measurements. Ranking of fuzzy numbers was suggested by Jain in 1976 and 1977 (as mentioned in Abbasbandy & Hajjari, 2009; Hajjari, 2011) on the base of the maximizing a set perception to rank the fuzzy numbers. There are several methods for ranking fuzzy numbers, which have strengths and weaknesses because of the common characteristics of fuzzy numbers. It is unusual to find and modify new methods of ranking fuzzy numbers because most of the methods give different orders of ranking for the same fuzzy numbers set.

According to Yong and Qi (2005), Yager (1980) and Murakami and Maeda (1984) proposed pioneer methods for ranking fuzzy numbers according to centroid index using a procedure for Technique of Order Preference by Similarity to an Ideal Solution (TOPSIS). Abbasbandy & Asady (2006) introduced a sign distance method of ranking fuzzy numbers to overwhelm the deficiencies in previous methods, such as the coefficient of variation (CV) index, the distance between fuzzy sets, centroid point and original point, and weighted mean. Deng, Zhenfu and Qi (2006) suggested ranking fuzzy numbers method as per radius of gyration. Abbasbandy and Hajjari (2009) presented magnitude method of ranking $TpFN[a, l, r, b](x)$ according to two spreads (left and right) towards some $\alpha-$cut to the trapezoidal fuzzy numbers as:

$$Mag(\hat{X}) = \int_{0}^{1} [x_l(\alpha) + x_r(\alpha) + a + b] f(\alpha) \, d\alpha \tag{2.17}$$

where, $f(\alpha) = \alpha$, $[a, b]$ is the most possible values and $x_l(\alpha), x_r(\alpha)$ are the left and right points of $X_\alpha$, respectively. In this method, larger fuzzy number $\hat{X}$ has a larger $Mag(\hat{X})$.

Among all the existing methods, Wang, Luo and Liang's (2009) method was one of the extensive methods used to solve the fuzziness of coefficients in decision-making problem using the DEA approach. Nejad and Mashinchi (2011) proposed a fuzzy number ranking method as per areas of two spreads (left and right) of a fuzzy number, which can rank several fuzzy numbers and their images (non-normal, normal, trapezoidal and triangular). Then, Xu and Zhai (2012) improved the method for ranking fuzzy numbers by distance minimization based on the two spreads areas (right and left) of the fuzzy number. Their method overcame the drawback of the methods by Asady and Zendehnam (2007)(A&Z) and Abbasbandy and Hajjari (2009) (A&H) when two fuzzy numbers have the same nearest point.



As a conclusion, many ranking methods are developed in ranking fuzzy numbers based on distance minimization (Asady & Zendehnam, 2007; Xu & Zhai, 2012), the areas method (Nejad & Mashinchi, 2011; Wang & Luo, 2009), radius of gyration (Deng et al., 2006), sign distance (Abbasbandy & Asady, 2006; Chen & Hwang, 1992), and the area in between centroid point and original point (Chen, 1985; Chu & Tsao, 2002; Yong & Qi, 2005). Other methods used the left and right deviation degree of fuzzy numbers (Wang, Liu, Fan, & Feng, 2009). Nevertheless, not all method could frequently rank fuzzy numbers and the rankings are acceptable in all cases and situations (Modarres & Sadi-Nezhad, 2001; Xu & Zhai, 2012).

### 2.6.1 Ranking of Fuzzy Numbers Based on the Defuzzification Methods

Defuzzification methods are widely used as effective approach to compare and rank fuzzy numbers (Chang et al., 2013; Chen & Hwang, 1992; Lee, 2000). Many studies ranked fuzzy numbers under the defuzzification concepts. For instance, Chu and Tsao (2002) suggested that fuzzy numbers can be ranked by considering an area in between centroid point and original point mean. Saneifard and Saneifard (2011) suggested a novel method to tackle the issue of crisp element selection by using the information provided by a fuzzy set. This issue arises in fuzzy logic controllers under the defuzzification stage. The suggested methods defuzzify fuzzy numbers into a crisp approximation of its origin through the probability density function and the Mellin transformation of fuzzy numbers. Then, the approximations in crisp values are utilized to organize the fuzzy numbers into certain ranking. Likewise, through defuzzification, the present study attempts to propose a new method that ranks fuzzy numbers. Along with its ranking features, the method eradicates the inaccurate results and handles the shortcomings characterized in the prior ranking.

Rouhparvar and Panahi (2015) presented a method to defuzzify generalized fuzzy numbers according to geometric aspects of the membership function in ranking several fuzzy numbers and images. Obviously, several defuzzification methods proposed in the literature are common methods for evaluating fuzzy numbers in decision-making problems and ranking fuzzy numbers (Asady & Zendehnam, 2007; Fortemps & Roubens, 1996; Rouhparvar & Panahi, 2015; Yager, 1981; Yoshida & Kerre, 2002). There are also researchers who have examined defuzzification methods in various applications (De Campos Ibáñez & Muñoz, 1989; Goetschel & Voxman, 1986), and these methods are used to reduce fuzzy numbers to a single, crisp, numerical value. The result leads to the best information and makes a kind of synthesis of the fuzzy number.

It can be noted that the above review of literature does not indicate that the relationships among the inputs and/ or outputs are addressed whether in the modified or principal defuzzification methods. One of the few related studies that deals with this matter is the work of Kikuchi (2000). He proposed a method to defuzzify fuzzy numbers using a FLP model, where the defuzzified results need to satisfy some relation of the original crisp data and when finding the set of values such as smallest membership grade among them is maximized. Since the Kikuchi's method is used in as comparative method with the proposed method of the study, more details of method and formulation of model is available in the next chapter. Finally, the next section discusses the measurement of how to find the approximation of fuzzy number is done.



## 2.7 Expected Value of Fuzzy Numbers

Initially, the expectation for fuzzy numbers in intervals was defined for the first time by Dubois and Prade (1987) as follows;

$$\mathbb{EI}(\mathcal{T}) = [\int_0^1 \underline{\mathcal{T}}(t) \, , \int_0^1 \overline{\mathcal{T}}(t)] \tag{2.18}$$

whereas, $\underline{\mathcal{T}}(t)$ and $\overline{\mathcal{T}}(t)$ are parametric form of the fuzzy number $\mathcal{T}$. The conception of the expected values of fuzzy numbers were introduced by Heilpern (1992). Later, Bede (2013) defined the expected value of a fuzzy number as the midpoint of the expected interval. Some studies have dealt with the expected value with different aims (Liu & Liu, 2002; Moreno-Garcia et al., 2013; Xue, Tang, & Zhao, 2008).

The expected value (Heilpern, 1992) is employed to prove that the fuzzy number representative in comparison with the distribution of possibilities. In this case, if their expected value is close, then both representatives have the same concept. The qualitative investigation of the fit is then achieved by examining the expected values. In this situation, the concept of regression is applied to the (right and left) lines which minimize the expected error, leading to fit data.

### 2.7.1 Nearest Point of Fuzzy Number

Some trails in the literature focuses on the nearest point to the fuzzy number. First, Grzegorzewski (2002) proved that the interval $\mathbb{EI}(\mathcal{T})$ introduced in Dubois and Prade (1987) is the nearest interval to the fuzzy number $\mathcal{T} = (x_0, \alpha, \beta)$ where $x_0$ known as the mean value of $\mathcal{T}$, is a real number and based on the equation (2.3).

$$\left.\begin{aligned} \alpha &= x_0 - x^l \\ \beta &= x^u - x_0 \end{aligned}\right\} \tag{2.19}$$

where, $\alpha$ and $\beta$ are known as the left and right spreads, respectively. $x^l$ and $x^u$ are the lower value and the upper value of the interval of the fuzzy number. Symbolically, $\mathcal{T}$ is denoted by $(x_0, \alpha, \beta)$ or $(x^l, x_0, x^u)$.

Then, its parametric form respectively is;

$$\underline{\mathcal{T}}(t) = x_0 - \alpha(1 - t) \quad (2.20)$$

$$\overline{\mathcal{T}}(t) \ = x_0 + \beta(1 - t) \quad (2.21)$$

Later, Asady and Zendehnam (2007) presented a defuzzification method based on the nearest point of a fuzzy number by utilizing the above interval. They define the middle point of interval $\mathbb{EI}(\mathcal{T})$ as follows;

$$Mid(\mathcal{T}) = \frac{1}{2}\int_0^1 (\underline{\mathcal{T}}(t) + \overline{\mathcal{T}}(t))dt \quad (2.22)$$

They considered a crisp point as a fuzzy number instead of the nearest interval. Asady and Zendehnam (2007) found the nearest point to the fuzzy number $\mathcal{T}$ which is known as the nearest crisp point $Cr(\mathcal{T}) = Mid(\mathcal{T})$. In case of $\mathcal{T}$ is triangular fuzzy number the nearest point is calculated as follows;



$$Cr(\mathcal{T}) = x_0 + \frac{\beta - \alpha}{4} \quad (2.23)$$

They proved that the distance between the fuzzy number $\mathcal{T}$ and his crisp point $Cr(\mathcal{T})$ is the smallest distance which is calculated as follows;

$$D(\mathcal{T}, Cr(\mathcal{T})) = \left[\int_0^1 (\underline{\mathcal{T}}(t) - Cr(\mathcal{T}))^2 dt + \int_0^1 (\overline{\mathcal{T}}(t) - Cr(\mathcal{T}))^2 dt\right]^{1/2} \quad (2.24)$$

where, the function $D(\mathcal{T}, Cr(\mathcal{T}))$ is the minimum value of distance between a fuzzy number $\mathcal{T}$ and the crisp point $Cr(\mathcal{T})$ and assumed that if $Cr(\mathcal{T}) = Mid(\mathcal{T})$, then $Mid(\mathcal{T})$ is unique.

In 2012, Xu and Zhai introduced a method of ranking fuzzy numbers. They defined an index based on distance in fuzzy number space for characterizing the uncertain degree. Their method is applied and compared with the proposed method of this research is available in Section 5.5.2. While other researchers (e.g., Nasibov & Peker, 2008) focused on finding the nearest fuzzy numbers approximated to the given fuzzy numbers. They found that the most adjacent parametric membership function according to decision maker's willingness by minimization of a distance (Hamming, Euclidean, etc.) between fuzzy number and nearest parametric estimate.

Grzegorzewski and Mrówka (2005) recommended the nearest approximated trapezoidal fuzzy number of a given fuzzy number by minimizing the distance criterion proposed by Ma et al.(2000), and uses the concept of the symmetric triangular fuzzy number to introduce a new approach to defuzzify a general fuzzy quantity. The basic idea of the new method is to obtain the nearest symmetric triangular fuzzy number which fuzzy quantity are related to.



Then Grzegorzewski and Mrówka (2007) presented a correction expression for the approximation operator in their work in 2005, which is called nearest trapezoidal estimate operator preserving expected intermission. Later, Ban (2008) proved by giving examples of trapezoidal approximation of fuzzy number provided by Grzegorzewski and Mrówka (2007) does not necessarily $TpFN$. Therefore, he managed to resolve the problem by finding the nearest $TpFN$ towards an agreed fuzzy number on well-known metric that conserves the predictable interval of the initial fuzzy number. Moreover, Ban and Coroianu (2012) found the nearest real interval, nearest triangular (symmetric) fuzzy number and nearest trapezoidal (symmetric) fuzzy number of a fuzzy number, with regards to the regular Euclidean distance, and preserving the ambiguity.

## 2.8 Summary and Discussion

This chapter commences by presenting a brief survey of the field of fuzzy numbers. Practical applications indicated that the current use of fuzzy numbers is triangular fuzzy numbers, which are principally attributed by their easiness at conceptual and computation. Ideally, merits of utilizing triangular fuzzy numbers in fuzzy modelling is greatly acceptable (Pedrycz, 1994). The easiest form of membership function, triangular fuzzy numbers create an instant answer to the optimization difficulties in fuzzy modelling. Moreover, this chapter provides the ranking fuzzy numbers methods and their relation to defuzzification concepts. Then, the necessary steps in the fuzzy system including the fuzzification and defuzzification concepts with their techniques are also discussed. This chapter shows that a geometric mean is the best-represented midpoint in triangular fuzzy numbers (Aref & Javadian, 2009).

The present research discovers the problem of defuzzification of fuzzy numbers generated from the systems with some relationships or properties. It reveals that the fundamental and modification methods under defuzzification are unable to give a crisp output or to keep the relationships or properties on the original crisp data. As a result, the present research has set its primary goal to develop a defuzzification method that retains the properties or relationships in the original crisp data or in the crisp output. Besides, this estimated crisp output is expected to be an optimal output of the systems under the problem. This lead us to combine the optimization techniques and one of the fundamental defuzzification methods to develop a new defuzzification method that has the ability to perverse the properties in the original crisp data as well as in the crisp output. Toward this purpose, next chapter focuses on multi-criteria decision- making, particularly the LP, DEA, and their relations to defuzzification. By accomplishing the objectives, the present research could offer significant contributions to the field of fuzzy theory.

# CHAPTER THREE
# MULTI-CRITERIA DECISION-MAKING AND DATA ENVELOPMENT ANALYSIS CONCEPTS

Based on previous discussion, specifically in sections 1.4 and 1.5 of Chapter 1, an investigation of a defuzzification formulation is crucial in case we have a relationship or properties that need to satisfy in the crisp output. This chapter focuses on linear programming (LP), and multi-criteria decision-making (MCDM) concepts in relation to Data Envelopment Analysis (DEA). Then, the background, assumptions, models and applications of DEA and fuzzy DEA are presented. It is followed by the discussions on the multi-objective concept and goal programming techniques, interval weight approach concept and formulation, and the relation of DEA and defuzzification.



## 3.1 Overview of Linear Programming

Linear programming (LP) is a subset of mathematical programming and a powerful mathematical modelling technique in operations research (OR). It is designed for maximizing or minimizing a linear function to linear inequality constraints (Dowsland, 2014). The following is the general formulation of LP.

$$\text{Minimize or Maximize} \sum_{j=1}^{n} c_j x_j \tag{3.1a}$$

subject to

$$\sum_{j=1}^{n} a_{ij}\, x_j \geq b_i \qquad i = 1,2, \ldots m \tag{3.1b}$$

$$x_j \geq 0 \qquad j = 1,2, \ldots n \tag{3.1c}$$

We seek values of $x_j$ that optimizes (minimizes or maximizes) the objective characteristic equation (3.1a). The constant $c_j, a_{ij},$ and $b_i$ are known constants. They may be positive or negative subject to the situation. The values of $x_j$ needs to fulfill the constraint equations i.e. equation (3.1b) and be non-negative i.e. equation (3.1c). In other words, the constraints equation (3.1b) defines the decision space, the goal by the objective function, and the type of decision under certainty (Eiselt & Sandblom, 2007).

LP is commonly used in engineering schemes, business organization, oil industry, as well as in numerous applications. Previously, several researches have revealed that variation of practical problems resolved by LP methods have further complications. Regularly, these problems are of several objectives, which are to be optimized. Due to the restrictions of LP, only one objective is designated, while the others are constraints. By the initiation of multi objective linear programming (MOLP), such complications could be demonstrated more accurately (Luptacik, 2010). MOLP is known as a branch of MCDM that includes the techniques under the LP concept (Ishizaka & Nemery, 2013).

In addition, the classical LP models suffer from making decisions in an uncertain environment. So, the fuzzy set concept was adopted to the problems of LP and MOLP by Zimmermann (1978, 1996), with the aim of solving decision-making problems which includes finding the range of values that maximizes the least membership grade between them. This idea of fuzzy LP used by Kikuchi (2000) to defuzzify fuzzy numbers when the original crisp data have some relationships that need to be satisfied with their output is described as follows:

$$max\ h \qquad (3.2a)$$

subject to

$$\left(\frac{\bar{x}_i - x_i^l}{x_i^m - x_i^L}\right) = \mu_{x_i^-} \geq h$$
$$\left(\frac{\bar{x}_i - x_i^u}{x_i^m - x_i^u}\right) = \mu_{x_i^+} \geq h \qquad (3.2b)$$
$$R(\bar{x}_i) = C$$
$$\bar{x}_i, h \geq 0 \qquad i = 1,2..,n.$$

where, $\mu_{x_i}^-(x_i)$ and $\mu_{x_i}^+(x_i)$ represent the left and right-hand side of the membership of fuzzy number and $h$ is the minimum degree of membership where one of the values of $\bar{x}_1, \bar{x}_2, .., \bar{x}_n$ takes while the $R(\bar{x}_i) = C$ is the relationship in the original crisp data. The three values $(x_i^l, x_i^m, x_i^u)$ are the (low, mid and high) values of the support and designated for each $X_i$.

In LP problem, the value of $h$ is maximized. As such, the solution to each variable lies on the support of the membership function. Amongst all the combinations of the values for $x_i$ that satisfy $R(\bar{x}_i) = C$, the one that maximizes the minimum membership grade $\mu_{x_i}(x_i)$ ; $maxmin\ \{\mu_{x_i}^-(x_i), \mu_{x_i}^+(x_i)\}$, is chosen.

In the next section, we first consider the MCDM concepts and their relation with LP. Then, we discuss DEA, which is commonly known as LP application or a non-parametric LP approach (Eiselt & Sandblom, 2007; Ishizaka & Nemery, 2013).

### 3.2 Multi-Criteria Decision-Making

MCDM is a division of a broad class of OR models which are appropriate to address complicated problems involving high uncertainty, contradictory objectives, various forms of information and data, multi interests and perspectives (Mateo, 2012; Ramesh & Zionts, 2013). Practical problems are frequently categorized by numerous incommensurable and inconsistent (conflicting) criteria, and there may be no solution satisfying all criteria concurrently. Thus, by using MCDM, a compromise solution to the problem of conflicting criteria could be determined, and, it will provide a solution to solve the problems of attaining the final result among decision-makers (Gwo-Hshiung & Huang, 2011). In general terms, MCDM is divided into two categories of methods (Kahraman, 2008; Mateo, 2012; Zimmermann, 2001).

a. Multi-attribute decision-making (MADM). Its concept is based on the selection of an alternative from a menu or catalog primarily referred on the prioritized of the alternatives.

b. Multi-objective decision-making (MODM). Its concept is based on the synthesis of an alternative or alternatives by prioritized objectives.

The MODM and MADM are differentiated by the evaluation criteria, i.e. MODM represents objectives and MADM represents attributes. Besides, MADM focuses on problems with distinct decision possibilities. In such problems, the set of decision alternatives are predetermined.

The decision maker (DM) chooses, prioritizes or ranks a fixed amount of sequences of action. Some of MADM techniques are the analytical hierarchy process (AHP), TOPSIS method, the ELECI'RE methods and the DEA (Chen & Hwang, 1992; Hajiagha, Mahdiraji, & Sadat, 2013; Lu, Zhang, Ruan, & Wu, 2007; Mateo, 2012; Sen & Yang, 1998; Velasquez & Hester, 2013).

On the other hand, MODM relates decision problems where a decision aspect is uninterrupted although MODM is not related to problems where the alternatives are decided. The DM's key attention is to produce a "most" promising alternative through confined sources. A usual instance is mathematical programming issues with several objective functions. Some of MODM techniques commonly used are goal programming (GP) (Lu et al., 2007; Mateo, 2012; Sen & Yang, 1998). More discussions on MODM are presented in Section 3.5. Some basic concepts used in MCDM are also discussed in the next sub-sections.



### 3.2.1 Discrimination of Goals and Constraints

The real difference between goals and constraints is rather vague. In fact, being inequalities, both objectives and limitations are of an identical mathematical configuration and, thus, are identical. The dissimilarity between both is in the denotation that is attached to the right-hand parameter of the inequality. It is a goal aspired through the DM, which may or may not be achieved and when it represents an inflexible constraint, it has to be satisfied; otherwise, the solution will be infeasible (Romero & Rehman, 2003; Spronk, 1981). Generally, a goal can be expressed as: $Attribute + Deviational = Target$

Or mathematically as;

$$f(x) + d^{-} - d^{+} = t \quad (3.3)$$

The variables $d^{-}$ and $d^{+}$ represent deviations from a goal achievement from its target $t$. So the amount of violation of a goal in the sense of an under-achievement is denoted by the negative deviational variable $d^{-}$. The positive deviational variable on the other hand is in reverse, in which it specifies the amount by which a goal has exceeded its target. So the total violation of a goal in the sense of an over-achievement is signified by the positive deviational variable $d^{+}$.

The deviational variable is a very useful device for two different reasons. First, it is a simple and interesting way to impart flexibility to constraints, that is, to convert rigid constraints into goals or soft constraints. Second, it is the first step to building a GP, which is the commonly used approach within the general MCDM framework to solve MOLP, as would be explained in sub-section 3.5.1.



### 3.2.2 Pareto Optimality

The concept of Pareto optimality performs a critical function in traditional economic theory and is, likewise, an essential idea in the MCDM paradigm, as all the methods inside this paradigm look for efficient or Pareto optimal solutions.

Pareto optimal solutions, also known as the efficient solutions are the feasible solutions which mean that no other feasible solution can acquire the same or better performance for all the criteria under consideration and strictly better for at least one criterion (Ramesh & Zionts, 2013). In other words, a Pareto optimal solution is a feasible solution that an increase in the value of one criterion can only be reached by degrading the value of at least one additional criterion. All the MCDM techniques aim at acquiring solutions that are efficient within the Paretian sense as defined above. The next section focuses on one of optimization technique associated with the idea of Pareto optimality to find efficiency i.e. DEA (Ray, 2004).

### 3.3 An Overview of DEA

The pioneering idea for the DEA approach was introduced by Farrell (1957) to extend the common concept of efficiency measurement as a ratio of output to input.

$$Efficiency = \frac{Output}{Input} \tag{3.4}$$

The extension was found to measure the technical efficiency in managing problems of several inputs and several outputs as follows:



$$Efficiency = \frac{Sum\ of\ Outputs}{Sum\ of\ Inputs} \tag{3.5}$$

Charnes, Cooper, and Rhodes (1978) revealed the first model of DEA based on the concept of measurement efficiency introduced by Farrell (1957). The DEA concept stems from LP model for the evaluation of relative efficiencies of decision-making units with many inputs and outputs. In DEA, the under study organizations is termed as decision making units (DMUs). DEA is known as a non-parametric mathematical programming approach to frontier estimation, which is widely employed in many fields as a method to evaluate performance and efficiency of DMUs.

The MCDM methods and DEA have been used together in many situations for performance measurement. Though DEA was originally developed as a tool for performance measurement, the linkages between the fields of the DEA and MCDM have been explored. Hence, DEA is now widely accepted as a tool for MCDM (Bouyssou, 1999; Ramanathan, 2003, 2006).

Literature indicates the use of the DEA method as an MCDM tool. The first method integrating both concepts was put forth by Golany (1988) where he used an interactive multi LP model. This model helps DEA to choose the effective DMU rather than just the efficient DMU, the difference being the former will be able to achieve its objective more closely. The aim of the DEA is to define the productivity of a system or DMU by relating it to the ability of DMUs to convert inputs to outputs. Stewart (1996) distinguished and compared traditional goals of DEA and MCDM. Combined usage of MCDM and DEA was also available in many studies (Belton & Vickers, 1993; Doyle & Green, 1993; Ehrgott & Gandibleux, 2003; Keshavarz & Toloo, 2015; Sarkis, 2000; Yougbaré & Teghem, 2007). Moreover, using a multi-objective model with the DEA was introduced in many studies. Lotfi, Jahanshahloo, Soltanifar, Ebrahimnejad and Mansourzadeh (2010) established a similarity between model of DEA and MOLP and presented methods of DEA in solving problem interactively by transforming it into MOLP formulation. The MCDM problems mentioned in the studies above are taken into consideration as a DEA problem without inputs or as a problem in which every alternative has the same amount of every input. Hence, the DEA technique can be applied to identify no dominating alternatives (Chiang Kao, 2010).

The underlying concept of DEA is based on Pareto optimality (Charnes, Cooper, Golany, Seiford, & Stutz, 1985; Sengupta, 2012). A DMU is comparatively effective without the presence of other DMU or combined DMUs which produce almost the similar quantity of all outputs with fewer input and not more of any other input (Luptácik, 2010). It calculates the relative proportion of outputs to inputs for individual unit, with the score figured as 0 –1 or 0 –100%.

The strengths of DEA are as follows;

i. Objectivity i.e. DEA delivers an efficient score of DMUs based on statistical data, and not by means of subjective opinions of individuals (Ramanathan, 2003).
ii. It can be employed by several inputs and outputs and measured in various units (Cooper et al., 2006).
iii. It is non-parametric which means that the requirement of an assumption of a functional form in relation of inputs to outputs is not necessary (Luptácik, 2010).
iv. It differentiates efficient and inefficient DMUs (Ramanathan, 2003).
v. It has the possibility of identifying the required improvement (in inputs, outputs) and the reference set of non-efficient DMUs (Cooper, Seiford, & Tone, 2007).

However, DEA also has few weaknesses, which are:

i. It is powerful to identify the inefficient DMUs as it lists the efficiency score from best to worst. However, it is prosaic in discriminating among the efficient DMUs with the same efficiency score (Angulo-Meza & Lins, 2002; Ramanathan, 2003).

ii. Conventional DEA models have the problem of weak discriminating power which identifies too many DMUs as being efficient, and this arises when the DMUs being evaluated have several inputs and outputs.

iii. Conventional models are characterized with issues attributed to unrealistic weight distribution where some DMUs are rated efficiently only because of their significant weights in one output and/or significantly light weights in one input. In reality, these extreme weights are detrimental and unreasonable. As a result, some inputs and outputs that are assumed significant via decision maker are overlooked from the analysis, resulting in an inaccurate outcome (Allen, Athanassopoulos, Dyson, & Thanassoulis, 1997; Peaw & Mustafa, 2006).

iv. The relation between the number of DMUs and the number of input and output is sometimes specified by some rules of thumb (Avkiran, 2001), such as the rule of thumb suggested by Golany (1988), who proposed that the number of DMUs are ought to be at least twice the number of inputs and outputs aggregated. On the other hand, Banker, Charnes, Cooper, Swarts, and Thomas (1989) as cited in Cook, Tone, and Zhu (2014) suggested that the number of DMUs should be at minimum three times the aggregate of inputs and outputs.

Several basic DEA concepts are discussed in the following sub-sections.

### 3.3.1 Production Possibility Set

In productivity analysis or efficiency measurement in general, the production possibility set (PPS) is the collection of all feasible DMUs that are capable of producing output $y = (y_1, y_2, \dots, y_s)$ and consuming input $x = (x_1, x_2 \dots, x_r)$. The PPS is defined as the set: $\mathbb{P} = \{(x, y) \in \Re^{r+s} | x\ produce\ y\}$, has the properties as follows:

i. Each observation $(X_j, Y_j) \in \mathbb{P}$, where $j = 1,2, .., n$.
ii. If $(X, Y) \in \mathbb{P}$, and $\rho > 0$ then $(\rho X, \rho Y) \in \mathbb{P}, \forall \rho$.
iii. If $(X, Y) \in \mathbb{P}$, and $\exists \bar{X} \geq X\ and\ \bar{Y} \leq Y$ then $(\bar{X}, \bar{Y}) \in \mathbb{P}$.
iv. If $(X, Y) \in \mathbb{P}$, and $(\grave{X}, \grave{Y}) \in \mathbb{P}$ then $(\lambda X + (1-\lambda)\grave{X}, \lambda Y + (1-\lambda)\grave{Y}) \in \mathbb{P}$, for $\lambda \in [0, 1]$.



Now, the set $\mathbb{P}$ defined as the smallest convex set that includes all observation activity based on the above properties is:

$$\mathbb{P} = \left\{ (X, Y) \middle| \sum_{j=1}^{n} \lambda_j Y_j \geq Y, \sum_{j=1}^{n} \lambda_j X_j \leq X, \lambda_j \geq 0, j = 1,2, .., n \right\} \quad (3.6)$$

This research focuses on this meaning of the smallest convex set.

### 3.3.2 Types of Orientation

The DEA models are classified into three directions to reach efficiency. In input orientation models, a comparative reduction in the input variables is utilized as a way to attain competence while in the output models, a comparative upsurge in the output is considered. The third option is signified by the additive and slacks-based measure (SBM) of efficiency models that deal with the input dissipations and output deficits concurrently in a method that jointly maximize both. If the attainment of efficiency or inability to do such is the solitary subject of interest, then these different models will all produce the same outcome in which technical and combined inefficiency will be a concern. It is important to note that the PPS based on the DEA models does not depend on the types of orientation (Charnes, Cooper, Lewin, & Seiford, 1994; Lewin & Seiford, 1997; Zhu, 2009).

### 3.3.3 Types of Return to Scale



Returns to scale is an economic notion that describes the production frontier in a DEA model, which can be either constant or variable. If the change in inputs causes a proportional change in outputs, then this is termed as constant returns to scale (CRS); otherwise, it is called variable returns to scale (VRS). The VRS is further classified as (cumulative) increasing returns to scale (IRS) if the output levels increase at the different rate than that of inputs and declining returns to scale (DRS) when the output levels decrease at a rate different than that of the input levels (Cooper, Seiford, & Tone, 2006, 2007).

The input orientation models and the output orientation models give the same technical efficiency score under the assumption of CRS while in the case of VRS, both models give different efficiency scores (Zhu, 2009).

### 3.3.4 Radial and Non-Radial Models

Radial and non-radial models generally DMUs' measures of efficiency in DEA (Zhu, 2009). Radial models assume that inputs/outputs go through proportional changes while the remaining slacks are overlooked in the efficiency scores. The DEA envelopment models are radial efficiency measures since the models enhance all inputs or outputs of a DMU at a definite proportion. The CCR and BCC models epitomize these models. One of the first Non-radial models introduced by Färe and Lovell (1978), as cited in Thanassoulis, Portela, & Despi (2008) addresses slacks of every input/output in an individual and independent level and includes them in an efficiency degree stated as SBM. The envelopment models and the non-radial DEA models result into same frontier, but may also result into numerous effective targets (even when the envelopment models do not have non-zero slacks) (Zhu, 2009).



## 3.4 DEA Models

Charnes et al. (1978) first introduced DEA, and several DEA models were later developed and enhanced. All DEA models depend on the maximization of DMUs efficiency. The difference between the models stems from the frontier structure and the approach utilized in the projection of inefficient DMUs to the frontier (Cooper et al., 2007). The following terminologies are repeatedly used in the discussion of DEA models.

$j = 1,2,3,..,n$ is the number of DMUs.

$k = 1,2,3,..,s$ is the number of outputs, and $p = 1,2,3,..,r$ is the number of inputs.

$DMU_j$ is the $j^{th}$ $DMU$ and $DMU_0$ is the target DMU under evaluation.

$X_j$ is the input vector of the $j^{th}$ $DMU$, and $Y_j$ is the output vector of the $j^{th}$ $DMU$.

$x_{pj}$ is the $p^{th}$ input of the $j^{th}$ $DMU$.

$y_{kj}$ is the $k^{th}$ output of the $j^{th}$ $DMU$.

$\lambda \in \mathbb{R}^{n\times 1}$ is the column vector of a linear combination of $n$ $DMU$

$\gamma_k$ is the column vector of output weights.

$\delta_p$ is the column vector of input weights.

$\Theta$ is the objective value efficiency.



### 3.4.1 Basic DEA Models

This section introduces two basic DEA models as provided in the literature.

#### 3.4.1.1 The Charnes, Cooper and Rhodes Model

Charnes et al. introduced the primary DEA model, the CCR model in 1978, which aims at measuring efficiency of each DMU under study. Let $DMU_0$ be the unit to be assessed where 0 ranges over $1,2,..,n$. By finding the solution of the fractional programming problem, the values for the input “weights” $(\delta_p)$ $p = 1,2,\ldots,r$, and the output “weights” $(\gamma_k)$ $k = 1,2,\ldots,s$ are the variables that can be obtained.

(CCR –Fractional Programming model),

$$max\ \Theta_0 = \frac{\sum_{k=1}^{s} \gamma_k y_{k0}}{\sum_{p=1}^{r} \delta_p\, x_{p0}} \tag{3.7}$$

subject to

$$\frac{\sum_{k=1}^{s} \gamma_k y_{kj}}{\sum_{p=1}^{r} \delta_p\, x_{pj}} \leq 1 \qquad j = 1,2,\ldots n$$

$$\gamma_k \geq 0 \qquad k = 1,2,\ldots s$$

$$\delta_p \geq 0 \qquad p = 1,2,\ldots r$$

In the above constraints, the “virtual output” vs. “virtual input” ratio must not be higher than 1 in each DMU. The aim is to acquire weights and to maximize the proportion of assessed DMUs. According to the constraints, the ideal value to be achieved is at most 1.

Now, the conversion of the above fraction program to the LP, which is also known as the multiplier approach is presented as follows.

(CCR – LP model),

$$max\ \Theta_0 = \sum_{k=1}^{s} \gamma_k y_{k0} \tag{3.8}$$

subject to

$$\sum_{k=1}^{s} \gamma_k y_{kj} - \sum_{p=1}^{r} \delta_p\, x_{pj} \leq 0 \qquad j = 1,2,\ldots n$$

$$\sum_{p=1}^{r} \delta_p\, x_{p0} = 1$$

$$\gamma_k \geq 0 \qquad k = 1,2,\ldots s$$

$$\delta_p \geq 0 \qquad p = 1,2,\ldots r$$

In the CCR model, the $DMU_0$ is considered efficient if it satisfies the following condition; $\Theta^* = 1$ with at least one optimal $(\gamma^*, \delta^*)$ with $\gamma^* \geq 0$ and $\delta^* \geq 0$. If not, then $DMU_0$ is considered inefficient. Therefore, CCR-inefficiency reflects that either (i) $\Theta^* < 1$ or (ii) $\Theta^* = 1$ with at least a single element $(\gamma^*, \delta^*)$ is zero for each optimal solution of $(LP_0)$. For $DMU_0$ with $\Theta^* < 1$ (CCR-inefficient), it is considered inefficient and, hence, there should be at least a single constraint whose weight $(\gamma^*, \delta^*)$ equalizes between the left and right hand sides or else, $\Theta^*$ may perhaps be enlarged.

Let the set be such that, $j \in \{\{1,2,\dots \text{n}\}$ and $\mathfrak{R}_0^{..} = \{j: \sum_{k=1}^{s} \gamma_k^* y_{kj} = \sum_{p=1}^{r} \delta_p^* x_{pj}\}$. The subset $\mathfrak{R}_0$ of $\mathfrak{R}_0^{..}$ containing CCR-efficient DMUs is referred to as the reference set or the peer group to $DMU_0$. The presence of this set of efficient DMUs causes the $DMU_0$ to be inefficient. Moreover, if the set is stretched by $\mathfrak{R}_0$, it is referred to as the $DMU_0$ efficient frontier.



#### 3.4.1.2 The Banker, Charnes and Cooper Model

Banker, Charnes, and Cooper (BCC) (1984) stretched the CCR model to overcome the shortcomings of the CCR model by comparing DMUs according to general efficiency with constant returns to scale (CRS). It disregards the point that dissimilar DMUs possibly will operate at different scales wherein variable returns to scale (VRS) is considered and compared based on technical efficiency.

The new model BCC incorporates a novel constraint represented by $\sum_{j=1}^{n} \lambda_j = 1$ in order to determine whether or not the operations are carried out in increasing (cumulative), constant or decreasing returns to scale.

If we add $\sum_{j=1}^{n} \lambda_j = 1$ then we obtain VRS models. If we substitute $\sum_{j=1}^{n} \lambda_j = 1$ to $\sum_{j=1}^{n} \lambda_j \leq 1$ then we acquire non-increasing return to scale (NIRS). If we substitute $\sum_{j=1}^{n} \lambda_j = 1$ to $\sum_{j=1}^{n} \lambda_j \geq 1$ then we acquire non-decreasing return to scale (NDRS).

### 3.4.2 Modified DEA Models

After the establishment of the CCR and BCC models, many modifications to DEA model appeared in the literature, such as the Free Disposal Hull approach (FDH) recommended by Deprins, Simar and Tulkens (1984) and further explored by Tulkens in 1993 as generalized form of DEA adaptable returns-to-scale model whereas it depend on disposability assumption of production set and does not confine itself to convex technologies. The FDH model is expressed by the extra limitation $\lambda_j \in \{0, 1\}$ in which. $\lambda_j$ to be binary in the BCC model, so as to relax an assumption of convexity. The FDH formulation is described as;

$$
\begin{aligned}
& min\ \Theta && (3.9) \\
& \text{subject to} && \\
& \Theta x_{p0} - \sum_{j=1}^{n} \lambda_j x_{pj} \geq 0 && p = 1,2, \dots r \\
& \sum_{j=1}^{n} \lambda_j y_{kj} \geq y_{ko} && k = 1,2, \dots s \\
& \sum_{j=1}^{n} \lambda_j = 1 && \\
& \lambda_j \in \{0,1\}, \qquad \Theta\ free && j = 1,2, \dots n
\end{aligned}
$$

A FDH frontier has the stairway character that envelops on the data firmly rather than DEA frontier does. The FDH frontier is congruent with or without interior to the DEA frontier. FDH shall naturally create a greater estimation than DEA. The DEA and FDH do not accept functional form and disregard measurement error. The best exercise for the frontier is to reconstruct PPS grounded upon which enveloping a set of observations. However, in FDH, the PPS composed only the DEA vertices and FDH points interior to vertices.

The CCR model is based on the CRS practical production mixtures which could be scaled up nor down proportionately, and BCC model accepts VRS and is characterized by piecewise linear convex frontier. Thus, FDH, CCR, and BCC models describe diverse PPS and efficiency scores.

In 1985, the additive DEA model suggested by Charnes et al. which differentiate it from the classical DEA model as it does not differentiate between input or output oriented models but this additive model combines both into a single objective function. The model is invaluable as it measures slack and surplus, which linked with every input and output. Sexton, Silkman, and Hogan (1986) suggested the cross-efficiency method to rank DMUs fully. The slack-adjusted DEA model (SADEA) was proposed by Sueyoshi (1999) while the most significant element of ranking DEA models is represented by the super-efficiency DEA models like the Andersen and Petersen(AP) (1993) model. Here Andersen and Petersen developed the AP model as a modified version of DEA to overcome the drawbacks of the conventional DEA models (CCR and BCC) concerning ranking of efficient DMUs.

The AP model is consistent with the BCC model with an exception that DMUs are omitted in the reference set and, therefore, efficient DMUs may obtain a value of more than one. The super-efficiency models approach has its basis in the measurement of the distance of the unit under evaluation, $DMU_0$, from a novel efficient frontier provided through the removal of the unit from the set of units. The AP model provides a rating of the efficient units similar to the rating of the inefficient unit.

Efficient units are where $\Theta^* \geq 1$ while inefficient ones are where $0 < \Theta^* \leq 1$. Two difficulties are noted in the model. First, for special data in input-oriented case, the AP model may not be feasible, and, second, for some DMUs with input /output near to zero, the AP model is not suitable. As a final note, it is important to state that these difficulties are only in input-oriented models but not in the output-oriented ones.



Mehrabian, Alirezaee and Jahanshahloo (MAJ) (1999) modified the AP model to rank efficient units and proposed the MAJ model. The MAJ model eliminates the difficulties ascending from the AP model. However, MAJ model may not be feasible in certain cases as extreme sensitivity exists in small variations of data because some DMUs associate small values for some inputs. So, to solve this issue, Saati, Zerafat Angiz, Memariani and Jahanshahloo (2001) suggested some models which remove the difficulties of infeasibility in the AP and MAJ models. Moreover, they verified that the improved version is always possible and project the DMUs in both input and output orientation.

Tone (2001) proposed slacks-based measure of efficiency (SBM), a non-radial approach which pacts with input/output slacks straight. The SBM returns efficient measures in between 0 and 1 and gives unity if the concerned DMU is on the frontiers of the production prospect set with no input/output slacks. SBM varies from the outdated radial measures of efficiency that does not take into account the existence of slacks. Then in 2002, Tone used his proposed SBM model to measure super-efficiency and rank efficient DMUs. As Tone's model is known as Super SBM, the SBM must progress first to classify efficient and inefficient DMUs, and then Super SBM must progress only for the efficient DMUs. Super SBM may not be efficient, but it is useful when the number of DMUs is small compared with the number of criteria for evaluation.

Jahanshahloo et al. (2006) modified the MAJ model to address the problem that might occur in the MAJ model where the ranking of efficient DMUs might change while some inputs of inefficient DMUs changes, minus inflicting variation on PPS. On the other hand, the ranking in the MAJ model is done for each excluded efficient DMUs from the set of the original observed DMUs with regards to the new found PPS.

For the MAJ model, if the efficient DMUs and the new PPSs remain unchanged, it is expected that the ranking also remains unchanged. Emrouznejad and Amin (2009) showed that using the standard DEA models for the observations containing ratio data as input and/or output may result in incorrect efficiency scores. A set of modification DEA models takes into consideration the correct convexity of DMUs when a ratio variable is included in the assessment model, and these modification models can only be used if the nominator and denominator of the ratio variables are known.

Du, Chen, Chen, Cook and Zhu (2012), proposed and developed additive integer-valued efficiency and super-efficiency models that deal with the case of when some inputs and/or outputs can only take integer values. Therefore, they directly dealt with slacks to calculate efficiency and super-efficiency scores. In the same vein, Chen, Du, Huo and Zhu (2012) proposed an integer DEA model developed based on the additive DEA model and an integer super-efficiency with undesirable inputs and outputs. Firstly, the proposed model based on the additive DEA model was used in which the input and output slacks were used to compute the efficiency scores. Then, an integer super-efficiency model was used to discriminate the performance of efficient firms.



Fang, Lee, Hwang and Chung (2013) proposed an alternative two-stage approach so that the projection identified would be strongly Pareto efficient and the efficiency score is the same as with Tone's approach. They showed that the proposed approach provides the same super-efficiency score as that given by the Super SBM model when the evaluated DMU is efficient and the same efficiency score as that obtained by the SBM model when the evaluated DMU is inefficient. Toloo (2013) and Toloo and Kresta (2014) developed a DEA model without explicit inputs (called DEA-WEI) and without explicit outputs (called DEA–WEO), respectively, to find the most efficient unit when inputs or outputs are not directly considered. In addition, Toloo (2014) proposed a new integrated mixed integer programming and DEA (MIP-DEA) model to find the most efficient suppliers in the presence of imprecise data.

Tone and Tsutsui (2015) discussed the case when there exist non-convex frontiers, which cannot be identified by the traditional DEA models. They developed a scale and cluster-adjusted DEA model that assumes scale efficiency and clustering of DMUs. The scale and cluster-adjusted score reflects the inefficiency of the DMUs after removing the inefficiencies caused by scale-demerits and accounting for in-cluster inefficiency. This model can identify non-convex (S-shaped) frontiers reasonably. Furthermore, they proposed a new scheme for the evaluation of scale elasticity.

Huguenin (2015) provided an AHP-based approach to select the most suitable DEA model. In order to avoid a biased model selection and potential opportunistic behavior from decision makers, Huguenin argued that such criteria should not be oriented towards the results of the alternative DEA models. Also, Khalili-Damghani, Tavana and Haji-Saami (2015) proposed a DEA method for measuring the performance of combined cycle power plants in the presence of pollution production and data uncertainty. The proposed model contributes to the uncertainties in the input and output data, by using interval data when considering undesirable outputs. The efficiency scores of the DMUs are determined as interval values, developing a group of indices to distinguish between the efficient and inefficient DMUs.

In doing so, the most economic scale size for the efficient DMUs is determined, beside determining practical benchmarks for the inefficient DMUs.

It is clear that various types of DEA models have been proposed as a modification or an extension of traditional DEA models (CCR and BCC) where some of them are under the economic concepts of return to scale (CRS and VRS). Other measures of efficiency in the DEA models are radial and non-radial models. Moreover, these models assume that all data are known exactly without any variation. In real decision-making and evaluation problems, ordinal preference information and/or fuzzy data are often encountered. When all or parts of input and output data are fuzzy data, several approaches have been proposed to deal with them in the framework of DEA under fuzzy DEA.

### 3.4.3 Overview of Fuzzy DEA



In standard DEA models, the contribution, which is in the form of input and output statistical information, has a precise value on a proportionate measure. At present, DEA models have been transformed in order to highlight undefined information, where most of the input and output data have been found to be ambiguous. Inaccurate information is often stated through fuzzy numbers, rank ordering or restricted data interims.

Bellman and Zadeh (1970) proposed the fuzzy set theory with the aim of solving problems in decision-making. Since 1992, DEA researchers began to use fuzzy concepts to measure the exact value and output of DMUs. According to Hatami-

Marbini, Emrouznejad and Tavana (2011), some of the existing approaches related to solving uncertain or fuzzy data for DEA include the tolerance approaches, the fuzzy ranking approaches, the $\alpha$-level based approaches, the defuzzification approaches and the possibility approach. In addition, Emrouznejad, Tavana, and Hatami-Marbini (2014) expanded this classification and added two new groups; the fuzzy arithmetic and the fuzzy random type-2 fuzzy set.

Sengupta (1992), a pioneering scholar and author, proposed a mathematical programming method wherein fuzziness is included in the DEA model, and the tolerance levels in the objective function and constraint abuses are exclusively defined. Meanwhile, following Sengupta's (1992) method, in 1998, Triantis and Girod proposed yet another mathematical programming approach which transforms fuzziness into a DEA model using the membership function values. In 2000, Kao and Liu further improved the fuzzy data input and output into measurable interims by using the $\alpha$-level sets.



Fundamentally, Guo and Tanaka (2001) were recognized as pioneers in using the developed fuzzy DEA models, grounding their applications on related possibilities and necessity measures. In 2002, the α-level sets approach proposed by Kao and Liu (2000), was further extended by Saati, Memariani and Jahanshahloo (2002), who defined the fuzzy DEA model as a possibilistic programming problem or the possibility approach, which was later transformed it into an interval programming.

Subsequently, Lertworasirikul, Fang, Joines and Nuttle (2003a) suggested three similar fuzzy DEA models. These models considered the uncertainties in fuzzy objectives and fuzzy constraints using the possibility approach. Following this, Lertworasirikul, Fang, Joines, and Nuttle (2003b) later proposed a fuzzy DEA model which uses the credibility approach. In the credibility approach, the fuzzy variables were replaced with predictable tributes, identified according to the credibility measures. At the same time, Lertworasirikul, Fang, Nuttle and Joines (2003) developed a fuzzy BCC model where the possibility and credibility approaches provide the relationship between the primal and dual models of fuzzy BCC. They used the credibility approach to reveal the way to obtain the efficiency value for each DMU as a representative of its potential range.

Furthermore, Jahanshahloo, Soleimani-damaneh, and Nasrabadi (2004) proposed a membership function of two triangular fuzzy numbers to solve the slack-based measure (SBM) model with fuzzy inputs and outputs. Their membership function is defined based on the Carlsson and Fullér's (2001) determination of the relation between two triangular fuzzy numbers by using a possibilistic mean value. Molavi, Aryanezhad, and Alizadeh (2005) introduced two more modified models of fuzzy DEA, in which the objective function and the ambiguous constraints of the fuzzy CCR model were converted into breakable conditions, by integrating LR-fuzzy numbers and the ranking method. Meanwhile, Liu (2008) developed an fuzzy DEA technique, in order to find the effective measures that are rooted in the concept of the assurance region (AR), especially when some observations were deemed to be of fuzzy numbers. Noteworthy, in order to develop and accomplish the new technique, Liu (2008) applied an $\alpha$-level approach and integrated it with Zadeh's (1978) extension principle, in order to transform the fuzzy DEA-AR model into a pair of parametric mathematical programs.

Significantly, Liu (2008) controlled the lower and upper bounds of the efficiency scores of the DMUs and determined the membership function of the efficiency by using different possibility levels.

On the other hand, Liu and Chuang (2009) presented fuzzy SBM to evaluate the efficiency of DEA models based on $\alpha$-cuts when the objective and constraints are fuzzy. The presented method overcame the shortcoming of the model of Jahanshahloo et al. (2004). Their method, fuzzy SBM, assumed that the solution is in the suitable variables interval where their alternatives make the model non-linear. Therefore, by maximizing the suitable alternatives, the model is made linear and by solving a linear programming problem in $\alpha$-cuts, it becomes possible to produce a reliable and robust solution for possibilistic mathematical programming problems in general and fuzzy SBM model in particular. Wen, You, and Kang (2010) proposed a new fuzzy DEA model, fuzzy CCR, based on credibility measure presented in Liu and Liu (2002) as well as a ranking method. Due to the character of the fuzzy programming, they designed a hybrid algorithm combined with fuzzy simulation and genetic algorithm to compute the fuzzy DEA model.

The advantages and disadvantages of approaches including the fuzzy ranking approach, the defuzzification approach, the tolerance approach, and the α-level based approach were addressed by Zerafat, Emrouznejad and Mustafa (2010). They recommended the α-level approach to maintain the model fuzziness through the maximization of the inputs and outputs membership functions. Hatami-Marbini, Saati and Tavana (2011) proposed an interactive evaluation model to measure the relative efficiencies of a set of DMUs in fuzzy DEA by considering the DMs' preferences. Meanwhile Wang and Chin (2011) proposed a fuzzy expected value approach for DEA in which the fuzzy inputs and fuzzy outputs were first weighted and their expected values were then used to measure the optimistic and pessimistic efficiencies of DMUs in fuzzy environments. The two efficiencies were finally geometrically averaged for the purposes of ranking and identifying the best performing DMU.

In 2012, Zhou, Zhao, Lui and Ma successfully developed a comprehensive fuzzy DEA model which could be generalized over related conditions, within the assurance regions, based on the DEA model. They used the α-cut approach to calculate the upper and lower bounds of the efficiency score, which has been selectively given a value of $\alpha$.

Zerafat, Emrouznejad and Mustafa (2012) developed an alternative approach that could provide measures of fuzzy efficiency for DMUs when there are fuzzy observations. The main idea behind this method is the transformation of the fuzzy CCR model into a crisp LP problem through the application of an alternative α- cut approach that considers the solution to lie in the interval and describes the appropriate variables for it. Their model was presented as a multi-objective programming with three objectives, two being of minimization (the distance function of inputs and outputs) and the third one of the basic DEA model. This is described as follows;

$$min\, z_{pj} = \sum_{f} \mu_{\tilde{x}_{pj}}(x_{pjf})\, |\delta_p \bar{x}_{pj} - \delta_p\, x_{pjf}| \qquad \forall p, j \qquad (3.10)$$

$$min\, z_{kj} = \sum_{g} \mu_{\tilde{y}_{kj}}(y_{kjg})\, |\gamma_k \bar{y}_{kj} - \gamma_k y_{kjg}| \qquad \forall k, j$$

$$max\, z_0 = \sum_{k=1}^{s} \gamma_k y_{k0}$$

subject to

$$\sum_{p=1}^{r} \delta_p \bar{x}_{p0} = 1$$

$$\sum_{k=1}^{s} \gamma_k \bar{y}_{kj} - \sum_{p=1}^{r} \delta_p \bar{x}_{pj} \leq 0 \qquad \forall j$$

$$x^{l}{}_{pj} \leq \bar{x}_{pj} \leq x^{u}{}_{pj} \qquad \forall p, j$$

$$y^{l}{}_{kj} \leq \bar{y}_{kj} \leq y^{u}{}_{kj} \qquad \forall k, j$$

$$\gamma_k, \delta_p \geq \varepsilon \qquad \forall k, p$$

where, $x_{pjf}$ and $y_{kjg}$ represent the length of input value and output value of $x_{pj}$ and $y_{kj}$ respectively, located in the intersection of $\alpha_f$ and sides of the corresponding $TrFN$.

After including $\left|\grave{x}_{pjf}\right| = x_{pjf}^{+} + x_{pjf}^{-}$ and $\left|\grave{y}_{kjg}\right| = y_{kjg}^{+} + y_{kjg}^{-}$ assumptions model (3.10) is defined as,

$$
\begin{aligned}
& min\, z_{pj} = \sum_{f} \mu_{\check{x}_{pj}}\left(x_{pjf}\right)\left(x_{pjf}^{+} + x_{pjf}^{-}\right) && \forall p, j \qquad (3.11)\\
& min\, z_{kj} = \sum_{g} \mu_{\check{y}_{kj}}\left(y_{kjg}\right)\left(y_{kjg}^{+} + y_{kjg}^{-}\right) && \forall k, j\\
& max\, z_{0} = \sum_{k=1}^{s} \hat{y}_{k0} &&\\
& \quad subject\ to &&\\
& \quad \sum_{p=1}^{r} \hat{x}_{p0} = 1 &&\\
& \quad \sum_{k=1}^{s} \hat{y}_{kj} - \sum_{p=1}^{r} \hat{x}_{pj} \leq 0 && \forall j\\
& \quad \hat{x}_{pj} - \delta_{p} x_{pjf} - \left(x_{pjf}^{+} - x_{pjf}^{-}\right) = 0 && \forall p, j, f\\
& \quad \hat{y}_{kj} - \gamma_{k} y_{kjg} - \left(y_{kjg}^{+} - y_{kjg}^{-}\right) = 0 && \forall k, j, g\\
& \quad \delta_{p} x_{pj}^{l} \leq \hat{x}_{pj} \leq \delta_{p} x_{pj}^{u} && \forall p, j\\
& \quad \gamma_{k} y_{kj}^{l} \leq \hat{y}_{kj} \leq \gamma_{k} y_{kj}^{u} && \forall k, j\\
& \quad \delta_{p}\, , \gamma_{k} \geq \epsilon && \forall p, k
\end{aligned}
$$



The model (3.11) is a MOLP, thus it can be solved using one of the multi-objective techniques. Puri and Yadav (2013) proposed some fuzzy approaches based on input-oriented models of CCR and SBM by using $\alpha$-cut approach. Next, the results of these models were applied to compute input mix-efficiency (IME). Moreover, the new correlation method was proposed to calculate the fuzzy correlation coefficients between fuzzy inputs and fuzzy outputs by using expected value approach. Then, a new ranking method based on fuzzy (FIME) to rank DMUs was also presented.

Similarity, Puri and Yadav (2014b) presented the concept of FIME and fuzzy output mix-efficiency (FOME), defined by proposing the output-oriented FCCR model and output-oriented fuzzy SBM (FSBM) model with fuzzy input and fuzzy output data.

Puri and Yadav (2014a) proposed a fuzzy DEA model to handle unwanted ambiguous outputs, which can be answered using the crisp LP $\alpha$-cut approach. Furthermore, the cross-efficiency technique was used in order to increase the discrimination control of the suggested models and to rank the efficient DMUs at every $\alpha$ in (0, 1].

More recently, Dotoli, Epicoco, Falagario and Sciancalepore ( 2015) presented a cross-efficiency model to deal with uncertainty in inputs and outputs by estimating them using a $TrFN$. Each DMU fuzzy efficiency was obtained by using only a set of weights, derived as a compromise between the set targets. It is done by maximizing the modal value, minimizing the difference between the pessimistic value and the modal value, and maximizing the difference between the optimistic value and the modal value. In order to discern the efficient DMUs under uncertainty, coefficients resulting from maximizing the efficiency of each DMU, were then used to evaluate the efficiency of all the others. The cross efficiency of each DMU is the mean value of its efficiency measures while varying the weights i.e. stepping from a self-evaluation to a comparative one. The results were finally defuzzified by the means of the center of the area of their distributions. Bray, Caggiani, and Ottomanelli (2015) proposed a fuzzy DEA model to evaluate the efficiency of transport services system including a set of international container ports. They used the concept of fuzzy set theory with a DEA model, to offer a more objective evaluation in vague environment.

From the above discussion of the literature, all fuzzy DEA approaches presented are powerful, but the shortcomings may appear in the way of the treatment of fuzzy data in DEA model. For example, with the defuzzification approach the fuzziness in inputs and outputs is effectively ignored. The tolerance approach treats fuzzy inequality and equality instead of fuzzy inputs and fuzzy outputs. The ranking approach of Guo and Tanaka (2001) uses only one number at a given level to compare fuzzy efficiencies. With the possibility approach, the numerical computation is more complicated in the case of fuzzy data with non-linear membership functions (Tlig & Rebai, 2009).

### 3.4.4 Application of DEA and Fuzzy DEA



The DEA and fuzzy DEA models are widely applied to real world applications, such as banking, education, health care, and hospital efficiency.

#### 3.4.4.1 Application of DEA

Various applications of DEA have been reported in the literature, such as those to banking sector. According to Rangan, Grabowski, Aly, and Pasurka (1988), the pioneering DEA work was applied to investigate bank efficiency by Sherman and Gold in 1985. Sherman and Gold utilized the CCR model to make a comparison between the operating efficiencies of 14 bank branches. Similarly, Brockett, Chames, Cooper, Huang and Sun (1997) suggested cone ratio DEA models to monitor the early warning systems that could be used by bank regulatory agencies. Their illustrative examples were developed from 1984 and 1985 performances data of the 16 largest banks in Texas. Meanwhile, Porembski, Breitenstein, and Alpar (2005) included German bank branches in determining the outliers and inefficient branches by using DEA and Sammon's mapping. This model showed how to conceive the relations between the reference set of inefficient and efficient DMUs. This conception is useful to obtain the first insight into the situation, particularly when there are many DMUs.

Liu (2010) employed the Malmquist productivity index approach to investigate the technical efficiency and productivity change of commercial banks in Taiwan over the period 1997–2001. In 2011, Paradi, Rouatt, and Zhu presented a two-stage DEA analysis approach applied to a Canadian bank's national branch network. Both CCR and BCC models were used to evaluate the branch performance in three different dimensions: production, profitability, and intermediation. Then, the second stage was accomplished by using a modified output-oriented SBM model that incorporates the efficiency scores of the three first-stage models as outputs with unity as input.

Wanke and Barros (2014) evaluated the efficiency of Brazilian banks by using a two-stage DEA. In the first stage, called cost efficiency, the number of branches and employees were used to attain a certain level of administrative and personnel expenses per year. In the second stage, called productive efficiency, these expenses allowed the consecution of two important net outputs: equity and permanent assets. The network-DEA centralized efficiency model was adopted here to optimize both stages simultaneously. The results indicate that Brazilian banks are heterogeneous, with some focusing on cost efficiency and others on productive efficiency.

The other key applications of DEA are in healthcare. For instance, Liu, Lu, Lu, and Lin (2013a) examined the performance of hospitals with the inclusion of nursing homes, primary care, and care programs. Nunamaker and Sherman conducted two independent, empirical tests of DEA on hospitals in 1983 and 1984. Morey, Fine, and Loree (1990) compared the allocative efficiencies of 60 hospitals in the USA while Al-Share (1998) evaluated the robustness of the DEA-CCR and BCC models and compared the efficiency of Jordanian hospitals by using three methods of DEA, Ratio Analysis, and Cobb-Douglas.

Giokas (2001) estimated the relative efficiency of public, general, and teaching hospitals in Greece. The efficient cost of hospitals was estimated and compared with the original crisp cost. In addition, they were concerned with the use of two different estimation techniques (average and frontier estimates) as a means of ascertaining specific estimates of the marginal costs of hospital services for public, general, and teaching hospitals. Garavaglia, Lettieri, Agasisti and Lopez (2011) applied DEA to find the efficiency of nursing homes in Italy. Asandului, Roman, and Fatulescu (2014) used DEA to evaluate the efficiency of public healthcare systems in 30 European states. Furthermore, they used CCR model comprising three output variables, including life expectancy at birth, health-adjusted life expectancy, and infant mortality rate, in addition to three input variables, which were the number of doctors, the number of hospital beds, and the percentage of public health expenditures.

In general, the above studies not only demonstrated that DEA is an effective technique for evaluating the efficiency of healthcare organizations, but they also reflect the variety of problems in healthcare management which can be handled by DEA (Liu, Lu, Lu, & Lin, 2013b). Moreover, DEA has been applied to evaluate the relative efficiencies of universities, university departments, or colleges. Earlier studies concerning the applications of DEA in the higher education context include Avkiran's (2001) study which examined the relative efficiency of Australian universities by developing three models of performance, namely, overall performance, performance on delivery of educational services, and performance on fee-paying enrollments. The findings based on the 1995 data show that the university sector was performing well on technical and scale efficiency.



Johnes (2006) discussed the issue of the measurement of the technical efficiency of English universities. From an output-oriented perspective, efficiency is defined as the ratio of a university original crisp output to the maximum output, which could be achieved given its input levels. Also, at the same time, Köksal and Nalçaci (2006) used a dual CCR-AR model and MCDEA-CCR-AR model to measure the efficiency of an engineering college in Turkey. The results of the dual CCR-AR model is found to be more appropriate than the MCDEA-CCR-AR model because it is easier to apply, and it provides not only an efficiency score but also suggests how an inefficient department can become efficient via target setting.

Katharaki and Katharakis (2010) estimated the efficiency of 20 public universities in Greece through quantitative analysis (including performance indicators, DEA, and econometric procedures). The findings show inefficiency in terms of human resources management while also identifying a clear opportunity to increase research activity and, hence, research income. Kuah and Wong (2011) provided a DEA model containing 16 inputs and outputs to measure the efficiency of 30 universities based on their teaching and research activities. Altamirano-Corro and Peniche-Vera (2014) established an approach to measure institutional efficiency by combining AHP and DEA. The majority of the results obtained using AHP correlated with those of DEA also reflected a widespread perception about how the performance of a university might be evaluated. The modelling of AHP and DEA combined offers DMs an opportunity to learn more about the educational systems in order to define policies that permit academic authorities to make better decisions in the short and long term.



Many other studies also used DEA to evaluate and measure the efficiency of organizations in different sectors, for instance in agriculture (Khoshroo, Mulwa, Emrouznejad, & Arabi, 2013), hotels (Ashrafi, Seow, Lee, & Lee, 2013; Assaf, Barros, & Josiassen, 2012), and industry and production (Du, Liang, Chen, & Bi, 2010), to name a few.

#### 3.4.4.2 Application of Fuzzy DEA

In a fuzzy environment, the fuzzy DEA has many applications in different fields as shown by previous studies. For instance, Triantis (1997) and Triantis and Girod (1998) presented an application of fuzzy DEA to evaluate the efficiency performance of a newspaper preprint insertion production line. Kao and Liu (2003) applied DEA to rank 24 libraries in Taiwan university with fuzzy observations while Wu, Yang, and Liang (2006) applied input-oriented fuzzy DEA to measure efficiency in banking in Canada, where a total of 808 bank branches were involved in their study. Of all the branches, 600 branches were from Ontario, 82 branches from Quebec, and 126 branches from Alberta.

Liu and Chuang (2009) developed a fuzzy DEA-AR method to measure the efficiency of the university libraries in Taiwan with fuzzy observations. They found that the proposed method was able to calculate the fuzzy efficiency score when the input and output data were represented as convex fuzzy numbers. Wang, Luo, and Liang (2009) proposed two new fuzzy DEA models constructed from the perspective of fuzzy arithmetic to deal with fuzziness in input and output data in DEA. The new fuzzy DEA models were formulated as LP models and can be solved to determine fuzzy efficiencies of a group of DMUs. An analytical fuzzy ranking approach was developed to compare and rank the fuzzy efficiencies of the DMUs. The proposed fuzzy DEA models and the ranking approach were applied to evaluate the performances of eight manufacturing enterprises in China. Hsiao, Chern, Chiu, and Chiu (2011) proposed the use of a fuzzy super-efficiency SBM (Fuzzy Super SBM) DEA to analyze the operational performance of 24 commercial banks facing problems on loan and investment parameters with vague characteristics. Then, they found that the Fuzzy SBM fuzzy super-efficiency slack-based measure of efficiency (Fuzzy Super SBM) can effectively characterize uncertainty, and has a higher capability to evaluate bank efficiency than the conventional fuzzy DEA approach.

A new approach to computing the Malmquist productivity index (MPI) under the VRS and scale efficiency change for every DMU in the fuzzy environment was proposed by Hatami-Marbini, Tavana and Emrouznejad (2012). The MPI transformation involved the presentation of fuzzy data as trapezoidal fuzzy number through the application of the α-level based approach. This study presented an application of the proposed approach in the study in healthcare to demonstrate the simplicity and efficacy of the procedures and algorithms in hospital efficiency. The previous studies have shown that DEA and fuzzy DEA are powerful tools to help identify the reference sets for inefficient institutions and objectively determine productivity improvements. The DEA models are primarily used in the case of crisp data and or fuzzy data in evaluating and finding the efficiency of DMUs.

## 3.5 Multi-Objective Decision-Making Methods

Multi-objective decision-making MODM (also known as multi-objective mathematical programming, (MOMP), or Pareto optimization) is an extension to the mathematical programming theory that involves decisions, which depend on the maximization or minimization of multiple objective functions, that need to be optimized subject to a set of constraints (Deb, 2014; Lu et al., 2007). The general formulation of an MOMP problem is as follows:

$$max\ or\ min\ \{f_1(x), f_2(x), \dots, f_n(x)\} \tag{3.12}$$
$$\text{subject to}$$
$$x \in S$$

where, $x$ is the vector of the decision variables, $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ are the objective functions (linear or non-linear) to be optimized, and $S$ is the set of all feasible solution.

MOMP methods such as MOLP are techniques used to solve such MCDM problems. Many decision-making problems can be formulated as MOLP (also known as multi-objective optimization problems (MOP)). Normally, there hardly exists a feasible solution that optimizes all objective functions in MOP, to be considered candidates for a final decision-making solution. Then, the concept of Pareto optimal solution (or known as an efficient solution, vector minimum, or non-dominated solution) is introduced (Konak, Coit, & Smith, 2006; Lu et al., 2007). It is an issue how decision makers decide the final solution from the set of Pareto optimal solutions (Murty, 2010).



In contrast to single-objective optimization, a solution to a multi-objective problem is more of a concept than a definition. Typically, there is no single global solution, and it is often necessary to determine a set of points that all fit a predetermined definition for an optimum (Diwekar, 2008). The predominant concept of defining the optimal point is that of Pareto optimality which is defined in Section 3.2.4. Hence, some other techniques for MOP methods have been developed to this end including goal programming (GP), the weighted sum (Scalarization) method, the ε-constraint method and Multi-level programming. In this research, we focus on the GP as a technique to solve MOLP problem as described next.

### 3.5.1 Goal Programming

Goal programming (GP) was developed as an extension of the LP model in the 1950s. GP was originally proposed by Charnes, Cooper, and Ferguson (1955) and Charnes and Cooper (1961), which was later improved by Ignizio (1976 and 1983), and (Charnes & Cooper, 1977, as cited in Jones & Tamiz, 2010).

The GP approach could be the most prevalent method used to handle multi-objective problems in practice. This approach has the added conveniences in that different units can be used to measure different objective functions, as well as all the objective functions are not necessary to be in the same form (either maximization or minimization) neither in the LP model (Murty, 2010). It can be thought of as an extension of LP to handle multiple, normally conflicting objective measures. Each of these measures is given a goal or target value to be achieved. Unwanted deviations from this set of target values are then minimized in an achievement function. This function can be a vector or a weighted sum dependent on the GP variant used. As the target is deemed to satisfy the decision maker(s), an underlying satisfying philosophy is assumed (Jones & Tamiz, 2010). In other words, the main idea behind the GP method is to find solutions that are close to the predefined targets. Therefore, in the GP method, the decision maker should fix the targets for each objective function. He or she then solves a single objective program aiming at minimizing the sum of deviations to the targets.

The main advantage of GP is its computational efficiency, provided that the target values are known, and if the goals are in a feasible region (Jones & Tamiz, 2010).

GP will generally produce a dominated solution if the target point is chosen in a feasible region (Caramia & Dell'Olmo, 2008; Romero, 2014). However, if the targets are wrong, then a feasible region is difficult to approach, in which case GP could be very inefficient. Nevertheless, GP may prove useful in situations where a linear or piecewise-linear approximation of the objective functions can be made, because of the availability of excellent computer programs for such approximations, along with the possibility of eliminating dominated goal points easily (Jones & Tamiz, 2010).

Cooper (2005) examined how the two types of models i.e. DEA and GP relate to each other where the "additive model" of DEA is shown to have the same structure as a GP model in which only "one-sided deviations" are permitted. However, the objectives are differently oriented because GP is directed to planning future performances, whereas DEA is directed to evaluating past performances, as part of the control function of management. In other words, Cooper (2005) defined GP as directed to the problems of management "planning" while DEA is directed to problems in the "control" and evaluation of activities.



GP does not pose the question of maximizing multiple objectives, but rather it attempts to find specific goal values of these objectives. There are a number of different GP techniques developed since GP was introduced in 1961. The major approaches are Lexicographic GP, and Min-Max (Chebyshev) GP and Weighted GP. There are also other GP variants that have been used including non-linear GP, fuzzy

GP, and fractional GP but the formulation of these are not distinctly different from single objective forms. These methods are described next.

#### 3.5.1.1 Lexicographic Goal Programming

In the lexicographic goal programming (LGP) model, decision makers prioritize their goals into different priority levels such as 1, 2, 3 etc. Each of this priority level may contain one or more goals. If a priority level contains two or more goals, these goals should be weighted as the same. The main idea behind this model is that a lower priority level goal must not be achieved at the expense of higher priority goals. This means that if the minimum total weighted deviation of priority 1 goals has a value of $N_1$, then it must be ensured that this value remains the same while looking to minimize the total weighted deviations of priority 2 goals. Similarly, if the weighted deviations of the priority 2 goals are valued as $N_2$, then this value must remain the same while seeking to minimize the total weighted deviation of priority 3 goals (Jones & Tamiz, 2010; Romero, 2014).

#### 3.5.1.2 Min-Max Goal Programming

The notion of min-max GP method is that the solution sought is the one that minimizes the maximum deviation from any single goal. Consider a simple problem, where there are 3 goals, all on a single priority level. Assume that the detrimental deviations from the goals are listed as, $D_1$, $D_2$ and $D_3$ respectively. Weights are assigned as $W_1$, $W_2$ and $W_3$, respectively. A dummy variable $\psi$ is used to measure the maximum deviation from any of the goal. Based on the concept of min-max GP, the problem is descripted as follows:

$$min\, \psi \tag{3.13}$$

subject to

$$W_i D_i - \psi \leq 0 \qquad i = 1,2,3$$

The min-max method provides the most balanced solution; it minimizes the maximum deviation from the goals, which represents maximum equity (Romero, 2014).

#### 3.5.1.3 Weighted Goal Programming

Charnes and Cooper first presented weighted goal programming (WGP) in 1961. This is similar to the weighting method of multi-objective optimization. Instead of assigning weights to different objective functions directly, weights are assigned to different goals in this method. The WGP considers all goals simultaneously as they are embodied in a composite objective function. This composite function tries to minimize the sum of all the deviations between the goals and their aspirational levels. The deviations are weighted according to the relative importance for the DM of each goal. The description of this method involves identifying objectives, setting goal (target value for each objective), assigning weights to each goal and then developing a normalized single objective function. Each goal, $i$ has its achievement value $f_i$ which is equal to the target $t_i$.

Satisficing philosophy allows under-achievement or over-achievement of each of the goals; deviational variables $d_i^-$ (for under-achievement) and $d_i^+$ (for over-achievement) are introduced as: $f_i + d_i^- - d_i^+ = t_i$.

If under-achievement is desirable then $d_i^+$ is minimized, while $d_i^-$ can take any positive value. Where over-achievement is desirable, $d_i^-$ is minimized while $d_i^+$ can have any positive value. The WGP objective function is then:

$$min \sum_{i=1}^{n} \left( w_{in} d_i^- + w_{ip} d_i^+ \right) \quad (3.14)$$

Here, $n$ is the total number of objectives, $w_{in}$ is the weights assigned to under-achievement deviational variables $d_i^-$ and $w_{ip}$ is the weights given to over-achievement deviational variables $d_i^+$. These weights can be calculated by using some techniques such as Analytic Hierarchy Process (AHP) proposed by Saaty, (1980 as cited in Saaty, 2008). Some other ways to get these weights are presented in the next section. On the other hand, it can be concluded that WGP technique is the suitable and important method to solve multi-objective problem of this research when DMs do not have any pre-emptive ordering of the objective functions. Instead of prioritizing the objective functions in different levels, they assign different weights for deviation variable of each objective function in the single priority level.

### 3.6 An Overview of Interval Weights

In multi-objective optimization, the relative importance of one objective over another is defined as the weight of the first objective. Weights are significant in determining the solution to a MOLP problem according to the requirements of DMs. In GP, weights associated with unwanted deviational variables measure the relative importance of the respective objective. Different methodologies that derive weights or priorities have been previously studied by (Chen & Tsai, 2001; Pekelman & Sen, 1974; Srinivasan, 1976; Zhang, Li, & Li, 2011).

However, the goal values for different objectives cannot be defined precisely in most cases. Thus, to address such imprecision, the fuzzy programming (FP) approach in the multi-objective linear optimization area based on fuzzy set theory of (Zadeh, 1965) was introduced by Zimmermann (1978). A membership function in FP is defined based on aspiration levels and tolerance limits. A max-min approach is then used to achieve the desired solution. In some cases, however, tolerance limits cannot be defined in highly sensitive decision situations. In order to resolve this problem, the GP approach in a fuzzy environment (FGP) was introduced by Narasimhan, (1980). Thereafter, FGP has been extensively studied (e.g. Hannan, 1981; Pal et al., 2003; Tiwari, Dharmar, & Rao, 1987) and applied to different real-life problems (e.g., Biswas & Pal, 2005; Pal & Sen, 2008). In such cases, fuzzy weights were employed to solve multi-objective fuzzy fractional programming problems (Pal et al., 2003). For previously explored methodologies in GP or FGP, weights of relative importance are defined as crisp values.

Additionally, the interval programming approach is a popular tool for solving MOP that involves interval uncertainty. Interval programming, which is based on interval arithmetic, was introduced by Moore (1979, as cited in Moore, Kearfortt, & Cloud, 2009), but the interval programming approach was introduced in the GP area by Inuiguchi and Kume (1991). Meanwhile, Saaty and Vargas (1987) introduced the concept of uncertainty in weight structure. The priorities determined from a pairwise interval comparison matrix were also suggested by Sugihara, Ishii, and Tanaka (2004). Wang and Elhag (2007) determined interval weights from the interval comparison matrix.

In these methodologies, the target achievement function is presented as the weighted summation of unwanted deviational variables. Weights (in interval form) are regulated by using a pairwise interval judgment matrix via the GP methodology (Pal & Sen, 2008). The problem is in the form of an interval programming problem at this point. Interval goals are modified into standard goals by using the IGP approach (Wang & Elhag, 2007). The sum of unwanted deviations associated with their respective goals is considered to achieve the goal values within the specified range and construct the regret function of the final executable model. Thus, the problem is resolved through standard GP methodology.

Similarly, Sen and Pal (2013) proposed an alternative method that employs interval weights to provide a solution to GP problem. On the other hand, their method focused on the interval as (min, max) which includes just two extreme values from all responses. In the next sub-sections, we describe the interval concepts and how to find the weights by using IGP as proposed by Sen and Pal (2013).

### 3.6.1 Arithmetic Definition of Interval

An interval $\mathbb{A}$ can be defined as an ordered pair. A closed interval $\mathbb{A}$ (called an interval number) is defined by $\mathbb{A} = [a^L, a^U] = \{a: a^L \leq a \leq a^U, a \in \Re\}$ where $a^L$ and $a^U$ are the left and right boundaries, respectively, of the interval $\mathbb{A}$ on the real line $\Re$. For a precise situation, $\mathbb{A} = [a, a]$ signifies the real number which is an $a$. Now, for intervals $\mathbb{A}_1 = [a_1^L, a_1^U]$ and $\mathbb{A}_2 = [a_2^L, a_2^U]$ the various interval arithmetic operations as presented in Sen and Pal (2013), are defined as follows:

- The binary operation addition between two interval numbers $\mathbb{A}_1$ and $\mathbb{A}_2$ is defined as : $\mathbb{A}_1 + \mathbb{A}_2 = [a_1^L + a_2^L,\ a_1^U + a_2^U]$
- The multiplication of two interval numbers, $\mathbb{A}_1$ and $\mathbb{A}_2$, is defined as:

$$\mathbb{A}_1 * \mathbb{A}_2 = [min(a_1^L a_2^L, a_1^L a_2^U, a_1^U a_2^L, a_1^U a_2^U), max(a_1^L a_2^L, a_1^L a_2^U, a_1^U a_2^L, a_1^U a_2^U)].$$

- The division of two interval numbers $\mathbb{A}_1$ and $\mathbb{A}_2$, is defined as:

$$\frac{\mathbb{A}_1}{\mathbb{A}_2} = \left[min\left(\frac{a_1^L}{a_2^L}, \frac{a_1^L}{a_2^U}, \frac{a_1^U}{a_2^L}, \frac{a_1^U}{a_2^U}\right), max\left(\frac{a_1^L}{a_2^L}, \frac{a_1^L}{a_2^U}, \frac{a_1^U}{a_2^L}, \frac{a_1^U}{a_2^U}\right)\right]$$

$$a_2^L, a_2^U \neq 0$$

For a particular case, when $(a_1^L, a_1^U, a_2^L, a_2^U) > 0$ then, $\frac{\mathbb{A}_1}{\mathbb{A}_2} = [\frac{a_1^L}{a_2^U}, \frac{a_1^U}{a_2^L}]$.

### 3.6.2 Definition of Interval Weights

Interval weights (IW) are a way to compute the weights in the presence of uncertainty in decision-making techniques. The IWs are derived from pairwise interval judgment matrix. Most real-world assessment issues include numerous criteria that are often in conflict and it is occasionally required to conduct a trade-off analysis in MCDM. As such, the estimation of the relative weights of criteria plays an important role in the MCDM process. Among many frameworks developed for weight estimation, pair wise comparison matrices provide a natural framework to elicit preferences from decision makers and have been used in several weight generation methods (Wang, Yang, & Xu, 2005a, 2005b).

### 3.6.3 Determination of Interval Weights

Weights of the importance of unwanted deviational variable are used to represent the relative importance of the respective criteria. It is more realistic to measure the relative importance in interval form rather than the deterministic values. If $[W_i^L, W_i^U]$ where $W_i^L, W_i^U > 0$ be IW of an importance of the objective $Z_i$ and also the pairwise judgments are precise, then interval comparison matrix $\mathbb{A}$ can be presented as follows:



$$\mathbb{A} = \begin{pmatrix} 1 & \frac{[W_1^L, W_1^U]}{[W_2^L, W_2^U]} & \cdots & \frac{[W_1^L, W_1^U]}{[W_n^L, W_n^U]} \\ \frac{[W_2^L, W_2^U]}{[W_1^L, W_1^U]} & 1 & \cdots & \frac{[W_2^L, W_2^U]}{[W_n^L, W_n^U]} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{[W_n^L, W_n^U]}{[W_1^L, W_1^U]} & \frac{[W_n^L, W_n^U]}{[W_2^L, W_2^U]} & \cdots & 1 \end{pmatrix} \tag{3.15}$$

where $\frac{[W_i^L\ ,\ W_i^U]}{[W_j^L\ ,\ W_j^U]}$ represents the relative importance of objective $i$ over $j$ using arithmetic of interval as defined in Section 3.6.1. Matrix $\mathbb{A}$ in 3.15 can be simplified as follows:

$$\mathbb{A} = \begin{pmatrix} 1 & \left[\frac{W_1^L}{W_2^U}, \frac{W_1^U}{W_2^L}\right] & \dots & \left[\frac{W_1^L}{W_n^U}, \frac{W_1^U}{W_n^L}\right] \\ \left[\frac{W_2^L}{W_1^U}, \frac{W_2^U}{W_1^L}\right] & 1 & \dots & \left[\frac{W_2^L}{W_n^U}, \frac{W_2^U}{W_n^L}\right] \\ \dots & \dots & \dots & \dots \\ \left[\frac{W_n^L}{W_1^U}, \frac{W_n^U}{W_1^L}\right] & \left[\frac{W_n^L}{W_2^U}, \frac{W_n^U}{W_2^L}\right] & \dots & 1 \end{pmatrix} \quad (3.16)$$

If $(i\ ,j)$th element of the matrix defined in (3.16), is designated by $[L_{ij}\ ,U_{ij}]$ then $L_{ij} = \frac{W_i^L}{W_j^U}$ , $U_{ij} = \frac{W_i^U}{W_j^L}$ $[L_{ij}\ ,U_{ij}]$ and clearly for

$$L_{ij} \times U_{ij} = 1 \quad , \quad i\ ,j = 1,2,\dots,n \quad (3.17)$$

still, the two relations

$$\mathbb{A}^L\, W^U = W^U + (n-1)W^L \quad (3.18)$$

and

$$\mathbb{A}^U W^L = W^L + (n-1)W^U \quad (3.19)$$

satisfied where

$$\mathbb{A}^L = \begin{pmatrix} 1 & \frac{W_1^L}{W_2^U} & \dots & \frac{W_1^L}{W_n^U} \\ \frac{W_2^L}{W_1^U} & 1 & \dots & \frac{W_2^L}{W_n^U} \\ \dots & \dots & \dots & \dots \\ \frac{W_n^L}{W_1^U} & \frac{W_n^L}{W_2^U} & \dots & 1 \end{pmatrix}$$

and,

$$\mathbb{A}^U = \begin{pmatrix} 1 & \frac{W_1^U}{W_2^L} & \dots & \frac{W_1^U}{W_n^L} \\ \frac{W_2^U}{W_1^L} & 1 & \dots & \frac{W_1^L}{W_n^L} \\ \dots & \dots & \dots & \dots \\ \frac{W_n^U}{W_1^L} & \frac{W_n^L}{W_2^L} & \dots & 1 \end{pmatrix} \qquad (3.20)$$

$W^L$ and $W^U$ are represent the lower and upper weight vector defined as the first interval $W^L = [W_1^L, W_2^L, \dots, W_n^L]^T$ and $W^U = [W_1^U, W_2^U, \dots, W_n^U]^T$.



### 3.6.4 Determination of Errors

In practical cases, pairwise comparison judgment is not hundred percent correct and, obviously the relation (3.17) is not satisfied. $\mathbb{E}_1$ and $\mathbb{E}_2$ are of the several errors that occur in satisfying relations (3.18) and (3.19) and these errors can be expressed as follows:

$$\left.\begin{aligned} \mathbb{E}_1 &= (\mathbb{A}^L - I)\, W^U - (n-1)\, W^L \\ \mathbb{E}_2 &= (\mathbb{A}^U - I)\, W^L - (n-1)\, W^U \end{aligned}\right\} \qquad (3.21)$$

The goal is to achieve the weights $W^L$ and $W^U$ in such a way that the error is zero. Then considering the target values as zero the goal expression can be written as:

$$\left.\begin{array}{l}(\mathbb{A}^L - I)\, W^U - (n-1)\, W^L + d_1^- - d_1^+ = 0 \\ (\mathbb{A}^U - I)\, W^L - (n-1)\, W^U + d_2^- - d_2^+ = 0\end{array}\right\} \quad (3.22)$$

where $d_i^-, d_i^+ (i = 1,2)$ represents the vector of the deviational variables of the dimension that is the same as $[W^L, W^U]$ and $i = 1,2,..,n$ . is the number of goals. Given that our target is to obtain the exact value zero, we have to minimize the sum of both under- and over-deviations associated with their respective goals. The executable GP model can be expressed according to the proposal by several authors (e.g., Makui et al., 2010; Sen & Pal, 2013; Wang & Elhag, 2007) as:

$$Min\mathcal{G} = \sum_{e=1}^{2}\sum_{i=1}^{n} ( d_{ei}^- + d_{ei}^+ ) \quad (3.23)$$

Thus, to satisfy the goal equations in (3.21) and (3.22) the following goals should be satisfied:

$$W_j^L + \sum_{\substack{i=1 \\ i\neq j}}^{n} W_i^U \geq 1 \ , \quad (3.24)$$

$$W_j^U + \sum_{\substack{i=1 \\ i\neq j}}^{n} W_i^L \leq 1$$

$$W^U - W^L \geq 0$$

$$W^L, W^U \geq 0$$

$$e = 1,2.,\ \ j = 1,2,..,n$$

By solving the GP model in equations (3.23) and (3.24), the determined weights (in an interval form) are presented as follows:

$$\{[w_1^L, w_1^U], [w_2^L, w_2^U], \dots\ [w_n^L, w_n^U]\}$$

### 3.7 DEA and Defuzzification

The idea of using the DEA model and defuzzification concept in the literature first appeared in Lertworasirikul (2002) and Lertworasirikul, Fang, Joines, et al. (2003a). The idea is called the 'defuzzification approach' used to solve fuzzy DEA. This approach was developed to defuzzify fuzzy inputs and fuzzy outputs into crisp. Then, the fuzzy CCR model converts to crisp CCR and solved. According to Hatami-Marbini, Emrouznejad, et al. (2011), the defuzzification approach suffered from a lack of attention may be because, in this approach, the fuzziness in the inputs and outputs is effectively ignored. Along the same line, Lee, Shen, and Chyr (2005) also proposed fuzzy DEA models for CCR and BCC by defuzzifying fuzzy inputs and outputs into crisp values and using them in conventional DEA models. Furthermore, Juan (2009) proposed a two-stage decision support model by using a hybrid DEA and case-based reasoning model. In this approach, the center of gravity method was used to transform the fuzzy data into crisp data and build a conventional CCR model.



Hatami-Marbini, Saati, and Makui (2009) presented a defuzzification approach to solving a fuzzy CCR model with fuzzy inputs and outputs in the form of trapezoidal fuzzy numbers. This approach transforms the fuzzy model into crisp LP model by ranking fuzzy numbers method proposed by Asady and Zendehnam (2007). Also, the obtained efficiencies from the proposed model reflect the inherent fuzziness in assessment problems. As can be seen, most of the previous studies used the defuzzification methods with DEA to convert fuzzy inputs and/or fuzzy outputs to crisp data before using the DEA models. In this research a new insight of using DEA and defuzzification concept is considered, where the DEA is used to develop a new defuzzification method, which is the first time addressed in the literature.

## 3.8 Summary and Discussion

Some crucial findings from this chapter, which would be helpful in achieving the primary goal of this research, can be recapped as follows:

The concentration is on the CCR model because it was the original DEA model. All other models are extensions of the CCR model obtained by either modifying the PPS of the CCR model or adding slack variables in the objective function. Fuzzy sets are used with DEA to handle the problems where some data inputs or/and outputs are ambiguous. One of the existing approaches to solving fuzzy DEA is the defuzzification approach, where some methods under defuzzification are used to convert fuzzy inputs and /or outputs to crisp inputs and outputs. Then, the DEA model is used with the new crisp data in the second stage.



Hence, the relation between DEA and MCDM was discussed, followed by the MOLP concept and some of their techniques.

GP was introduced as one of the flexible techniques used to solve MOP problems. On the other hand, the interval weight concept was introduced as one of the common methods to find the optimal weights in the GP.

The next chapter discusses the research methodology and model development. The present research develops a new method that uses some modifications on the COG with DEA as a tool to solve a defuzzification problem in the case where some relationships or properties in the original crisp data need to be satisfied with the crisp result. Thus, a problem is deciphered by using a standard DEA model, and the concept of PPS is explained in the next chapter.

# CHAPTER FOUR
# RESEARCH METHODOLOGY

The main problem of defuzzification methods, especially concerning original crisp data with relationships or with some properties that need to be kept in crisp output, was identified in the previous chapters, particularly in Section 1.5. We also identified the promising technique that can be utilized to solve the problem. It should be noted that the proposed method is a general one and with slight modifications that can be easily applied to all types of relationships that need to be satisfied in crisp output. Consequently, this chapter describes the methodology to achieve the objectives, that is, to develop a defuzzification technique that has the ability to preserve the relationships of the original crisp data. The details of how to accomplish the objectives are discussed in this chapter, particularly in terms of 4.1 research design, 4.2 research activities, 4.3 research framework, 4.4 types of data and source, 4.5 the fuzzification steps, 4.6 development of defuzzification method , 4.7 algorithm of the proposed defuzzification method, and 4.8 evaluation and comparison of the proposed method. This chapter ends with a summary and discussion in 4.9.



## 4.1 Research Design

The main goal of this research is to develop a defuzzification method that is able to defuzzify groups of fuzzy numbers created from original crisp data that have some relationships or properties that need to be satisfied in crisp outputs. A DEA optimization technique is chosen as a basis model because of its good features as discussed in Chapter Three.

The research focuses on how this defuzzification method is developed by modifying the Center of Gravity (COG) method as the objective function. This modification is based on the minimization of the distinct of the crisp outputs with all fuzzy numbers points in each interval of fuzzy numbers, to find the best approximation of each fuzzy numbers that satisfy the properties of the original crisp data. In addition, a modification of a DEA model is presented by ignoring the primary objective of the CCR model since our focus is not on finding efficiency but in using DEA as a tool by considering the PPS of CCR model as the region that has all possible activities. Moreover, new constraints are included in the model to represent the relationships. Then, the modification of the COG method as the objective is presented as the main objective of our proposed model. Furthermore, the suggested method does not only depend on the endpoints or midpoints of intervals corresponding to the fuzzy numbers, but it is looking at the best approximation of fuzzy number that satisfies some properties among these three values.



On the other hand, two sets of secondary data are collected from the healthcare sector in Malaysia. These hospital admission data are used as a basis to implement the proposed method to estimate the suitable number of beds for the hospitals. The first set of the hospital admission data is presented as having some relationships that need to be satisfied in crisp outputs. The other set is considered data that do not have the relationships to be preserved in crisp outputs. Experimentation and comparison were carried out to validate the two sets of data to fulfill the main and specific objectives of this research.

Therefore, the results of the proposed method with different cases are compared with the results from Kikuchi's method, the COG, and Assady and Zendemann (A&Z) methods to quantify the performance of the proposed method. Since both COG and A&Z methods deal with an independent crisp output, which differs from the proposed method, the results are reported by ignoring the relationships among the data.

## 4.2 Research Activities

The research activities are summarized in Figure 4.1 and 4.2. This section discusses how to define the problem, define the variables and the relationships, collect data and formulate the mathematical model, and develop a method based on the DEA model.

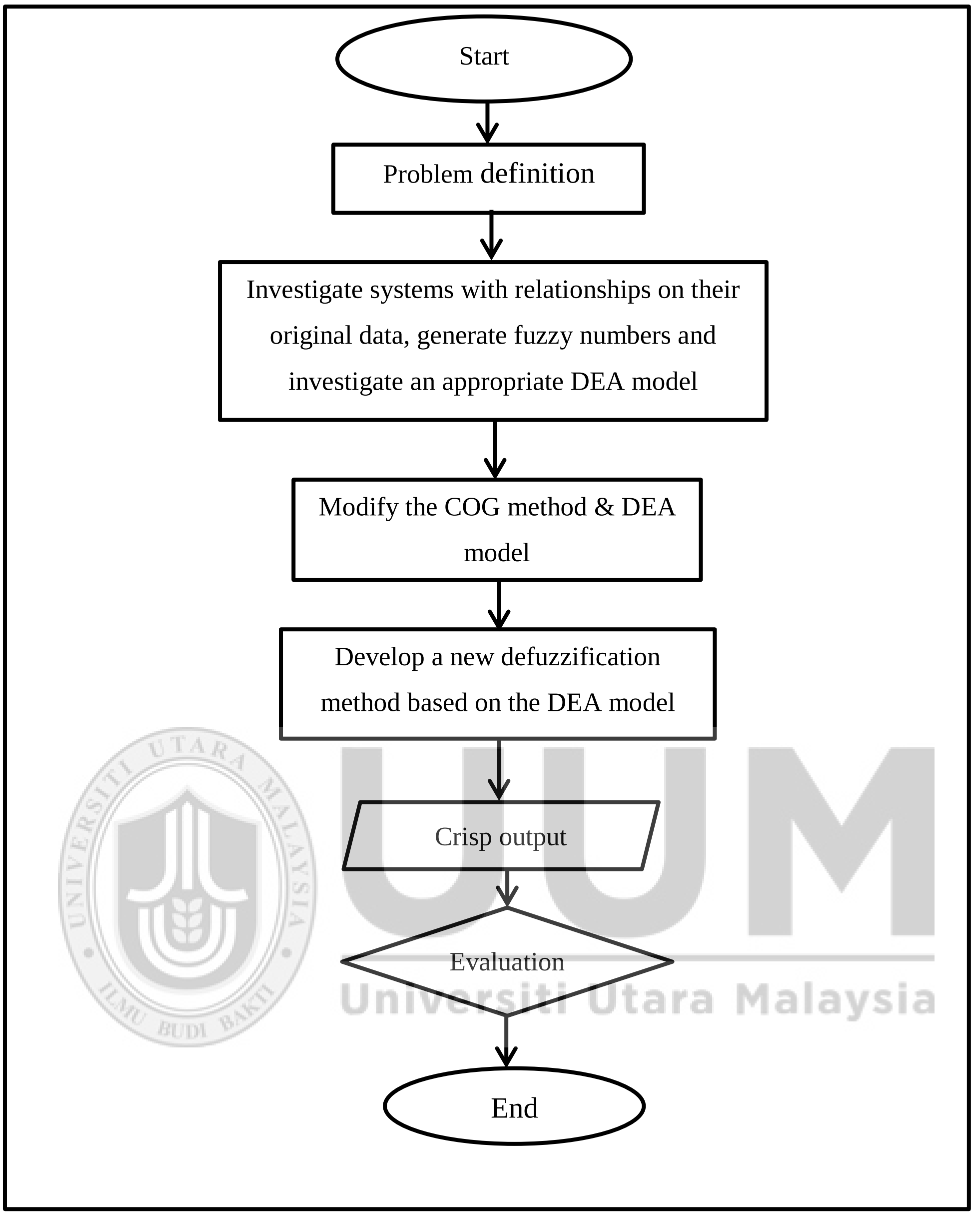
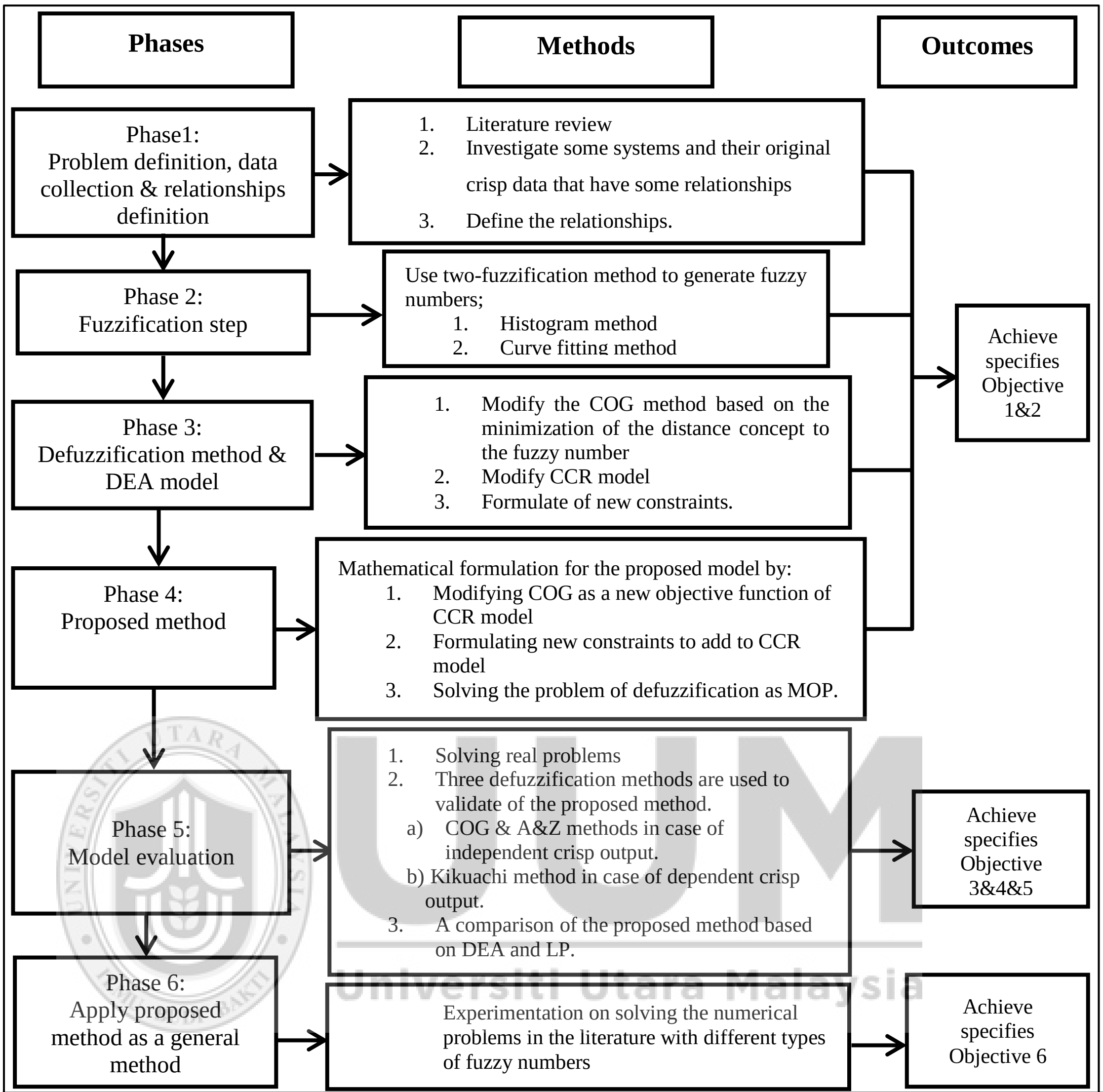



*Figure 4.1.* Structure chart of the research activities




**OBJECTIVES**

The current research primarily aims at developing a new method of defuzzification to defuzzify groups of fuzzy numbers based on a DEA model that can lead to the creation of crisp output values in which this method is able to satisfy the relationships or properties in the original crisp data and keep them in the solution. The following are the specific objectives that need to be accomplished to meet the desired results:

1. To develop a new defuzzification method by modifying the center of gravity (COG) method.
2. To modify the DEA model with new objective and extra constraints.
3. To implement the new method to solve real problems.
4. To compare the outcome obtained by the suggested method with other corresponding methods.
5. To compare the suggested DEA-based model with LP–based model.
6. To apply the proposed method to the numerical problems in the literature with different types of fuzzy numbers

*Figure 4.2.* Phase of research and its activities

## 4.3 Research Framework

This research aims at developing methods to defuzzify groups of fuzzy numbers that can satisfy some relationships or properties in the crisp output. In order to achieve the objective, this research is conducted through six phases of research activities: problem definition, data collection and fuzzy numbers generation, model development, model evaluation, and generalization of the model. In the model evaluation, model validation and a comparison of the proposed model based on DEA and LP are involved (refer to Figure 4.1 and 4.2).

Data are collected by investigating some systems with real problems and generating fuzzy numbers using fuzzification. Then, an appropriate DEA model is chosen based on the assumption of the production possibility set (PPS). After identifying all objectives and constraints, the defuzzification formulation is written in a mathematical form. Two variants of the method are then developed using the DEA model as a tool. The proposed method is then evaluated through validation and comparison. Furthermore, the proposed method can be considered as a general method since it has the ability to solve different problems with different relationships and fuzzy numbers. Figure 4.3 displays the research framework of the proposed defuzzification method, which starts with defining variables from a predefined system with some relationship. The next five sub-sections explain the research framework in detail.

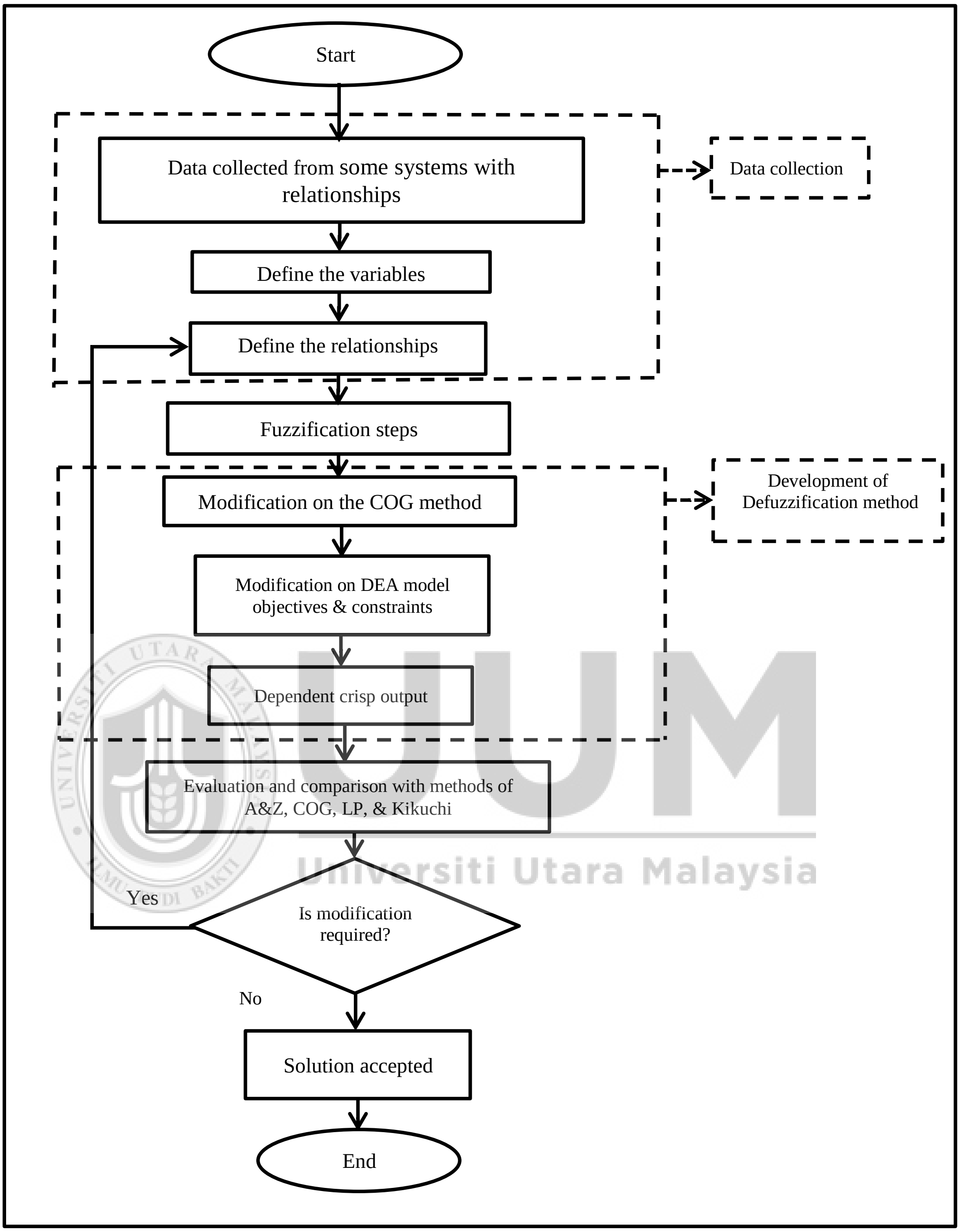



*Figure 4.3.* Research framework of the defuzzification method

### 4.3.1 Problem Definition and Data Collection

The foremost concern of this research is to develop a defuzzification method that can defuzzify groups of fuzzy numbers. These groups are generated from systems with some relationships or properties in the original crisp data that need to be kept in a crisp output (or defuzzified value). In other words, the proposed method would deal with the original crisp data that have some relationships which lead to the dependent output. That is, there exist some additional restrictions to the allowable crisp outputs i.e. crisp outputs are dependent. Besides that, the proposed method is also implemented to solve problems in the case where the systems do not have relationships or properties in the original crisp data need to be kept in the crisp output (or defuzzified value), which means that the crisp outputs are independent. Data collection is one of the most important components of research since it is necessary to verify whether the proposed method can be adopted or not. In this research, two sets of secondary data are collected from a Malaysian hospital. More details on the data types and source can be found in Section 4.4.

### 4.3.2 Generating Fuzzy Numbers

Generating fuzzy numbers are done by converting the original crisp data to the fuzzy number; this process is known as fuzzification. The details of the fuzzification method and the generation of fuzzy numbers are presented in Section 4.5.

### 4.3.3 Method Development

Defuzzification methods were developed to defuzzify groups of fuzzy numbers based on systems with relationships in their original crisp data or not. In other words, there exist some ancillary restrictions to the allowable crisp outputs i.e. crisp outputs are dependent or independent. In this phase, we explain the process to convert fuzzy numbers into crisp, which is known as defuzzification. First, a COG method is used with some modifications included as an objective in our proposed models. We use the modified COG method as the objective, which is included in the DEA model with enhancement of some of the constraints. Besides, developing the defuzzification method based on the DEA model, we also construct a different defuzzification method but based on an LP model.

On the other hand, each of the proposed methods based on DEA and LP model considers both cases of systems with or without relationships in their original crisp data. The formulation of each method is then developed and solved using Mathematica 9 software. The core of defuzzification, nearest points to the fuzzy numbers, and the best nearest point to each fuzzy number are explained. Also, the appropriate DEA model is included, and a detailed explanation of the method development can be found in Section 4.6.

### 4.3.4 Method Validation

Three defuzzification methods are used to validate the proposed method. Two are used in the case where there are no relationships, and one is in the event where there are relationships in the original crisp data. The implementation details of this comparison are shown in Chapter Five.

### 4.3.5 Application of the Proposed Method as a General Method

Application of the proposed methods is performed to handle different problems and different types of fuzzy numbers in the case whether the original crisp data have relationships or not. This phase is important to highlight the ability of the proposed defuzzification method in solving other problems in previous studies. The related numerical examples of that selected problems are also included in Chapter Five.

## 4.4 Types of Data and Data Source

Two types of data are used in this research, real data, and numerical examples. The real data with and without relationships are included and collected from the healthcare sector in Malaysia. The source of our real data with relationships is the Hospital Tuanku Fauziah (HTF) in Perlis while the data without relationships are derived from the database of the Ministry of Health (MOH) in Malaysia. The data from HTF cover the number of patients admitted to the hospital for the period of two years between 2013 and 2014. The data is considered with relationships since we are also given information on the available number of beds in each of the 19 wards in the hospital, the total number of available beds in the hospital (384 beds), and the number of patients based on their ward entrance. The data without relationships cover the number of patients admitted to the seven hospitals in Klang Valley for the period of three years between 2008 and 2010 but with no extra information. The only information available is that the patients are categorized into five groups based on their age (i.e. Toddler, Schoolchildren, Adult, Old, and Elderly). There is no information related to number of beds available that might be considered as the relationship of the crisp data.

The second type of data, related numerical examples with defuzzification, is collected from previous studies to evaluate the proposed method to solve other different problems. Several types of data needed in this research along with their sources are explained in detail in sub-sections 4.4.1 to 4.4.3.

### 4.4.1 Data with Relationships

Nineteen group of patients admitted to the Hospital Tuanku Fauziah (HTF) in Perlis are considered in this research based on their record in such wards as ($Ward_1$, $Ward_2$,..$Ward_{19}$) for the period of two years between 2013 and 2014. We consider these real data as an original crisp data with relationships because the source provides us with the total number of beds in each ward and the hospital in general. The total number of available beds in the hospital is considered as the relationship, which the crisp outputs (total number of beds to be allocated to the selected number of wards) need to satisfy. Other types of relations such as the number of doctors or nurses are not considered since the information is not available.

### 4.4.2 Data without Relationships

Five groups of patients based on their ages (Toddler, Schoolchildren, Adult, Old, and Elderly) are considered in this research based on their record. We consider these data without relationships because the source provides us with the number of recorded patients based on their age without giving any information about the total number of available beds, the number of the wards, or any other information. Therefore, the proposed method handles this problem by assuming that there are no relationships between these five groups. Then, the proposed method estimates the number of beds in each group. This process is done by estimating the optimal number of beds in each group based on the number of patients admitted every day for the period from 2008 to 2010. A list of all patients is depicted in Appendix B.

### 4.4.3 Numerical Examples

Two different applications are used to show how the proposed method can handle various problems. The first case is about using the proposed method to solve the problem of finding the optimal weights in solving a GP model. The second case is using the proposed method in the event of no relationships that need to be satisfied in crisp outputs (independent crisp outputs) to handle the ranking of the fuzzy number problem in finding the correct order.

## 4.5 The Fuzzification Steps

After collecting our real data, i.e. two types of original crisp data with and without relationships, the fuzzification process is needed in order to fuzzify our original crisp data because our data come from an uncertain environment. The fuzzification process is performed to convert or generate fuzzy numbers from the original crisp data by using the methods of curve fitting and histograms based. Automatic techniques are used to determine the appropriate fuzzy set of representation as described in sub-section 2.4.3. Generating $n$ triangular fuzzy numbers that correspond to each group is identified via equations (2.8) by using the minimum and maximum values as the left and right hand side boundaries respectively, via equations (2.9) and the geometric mean as a middle point via equation (2.12). In order to generate the triangular fuzzy number in this step, a curve fitting method to the data is assumed.

## 4.6 Development of Defuzzification Method

The method used to defuzzify groups of fuzzy number relies on the minimization of the distance concept based on their membership function. As mentioned in Section 3.1, one of the fewer studies that considered the concept of dependent crisp outputs is the Kikuchi (2000) method. Therefore, the results by the proposed defuzzification method are possible to be compared with results from the Kikuchi's method since they are developed using the same concept addressed in the following chapters. However, other studies focused on the concept of the nearest interval and nearest point to find the best representative point to the fuzzy numbers as shown in Section 2.7.

### 4.6.1 The Best Nearest Point

The notion of the nearest point was discussed in Section 2.7.1 with some methods under this concept. One of the common methods is the A&Z method, which presents a defuzzification method to defuzzify fuzzy numbers that rely on minimizing the distance between them and the new point (crisp point). The core of their method is the base of our proposed method where some predefined constraints and a modification of COG method is respected, and a new nearest point is considered. In our work, we are looking for a crisp point that gives us the minimum distance between other elements in the interval of the fuzzy number by considering the membership function. Besides that, that crisp point has to satisfy some relationship while the A&Z method does not have any constraint or condition on the crisp output even through there is a relationship in the data.



### 4.6.2 Appropriate DEA Model

Most of the DEA models presented in the literature and some of them were discussed in Section 3.4 are considered an extension of the CCR model under the economic concept of constant returns to scale (CRS) or the BCC model under the variable returns to scale (VRS). The models are generally used to evaluate DMUs and find the efficient one. Model (1.4) is divided such all LP models into the objective function and constraints namely, the left- hand sides and right-hand sides of the constraints. The left-hand side content the $\lambda$'s that generates the PPS corresponding to the CCR model which generate all possible solutions. The right-hand side and the objective function lead DMUs to the frontier.

Thus, the DMUs located on the efficiency frontier have considered the relevant ideal points in the DEA evaluation. However, this research is not about evaluation or finding the efficient DMU. In this research, a new method is introduced using the CCR-DEA model as a tool to defuzzify groups of fuzzy numbers. In other words, we use the left hand side of the model (1.4) with our new objective and extra constraints. In the traditional efficiency analysis by DEA, the ideal point is always on the frontier, but in our case, the ideal may not always be on the frontier, but it can be probed within PPS.

## 4.7 Algorithm of the Proposed Defuzzification Method

This section provides the algorithm of the proposed defuzzification method for group of fuzzy numbers as portrayed in Figure 4.2 and 4.3. This research is now all set to develop a defuzzification method which is able to create a new crisp output which is a best approximated value to the fuzzy number when dealing with 'dependency between original crisp data and the crisp output' and 'fuzzy numbers as group' simultaneously, for a more reliable result.

The proposed method is configured with the convergence of five key components namely the COG method, A&Z method, CCR-DEA model, LP model and WGP model. Figure 4.4 illustrates the algorithm steps of development the proposed defuzzification method.

**Step 1**: Defining problem and fuzzy numbers, and identifying the relationship.

**Step 2**: Partitioning interval of fuzzy numbers.

**Step 3**: Estimating approximated crisp outputs by modification on COG method and CCR-DEA model.

**Step 4**: Formulating a general defuzzification method to determining the best approximated crisp output under the convex assumption of PPS of CCR-DEA model.

**<u>Case1</u>**: Non-linear objective with any relationship using mathematical formulation in Section 4.7.4.1 model (4.9).

**<u>Case 2</u>**: Linear objective with any relationship using mathematical formulation in Section 4.7.4.2 model (4.10).

**<u>Case 3</u>**: Linear objective with a linear relationship using mathematical formulation in Section 4.7.4.3 model (4.11).

**Step 5**: Solving a MOLP problem using mathematical formulation in Section 4.7.5 model (4.12) by using the WGP technique.

*Figure 4.4.* Algorithm of the proposed defuzzification method

The details and steps of this algorithm are presented in next sub-sections.

### 4.7.1 Step 1: Defining the Problem and Identifying the Relationship

Based on step 1 of the algorithm as summarized in Figure 4.4, firstly, we define the problem of defuzzification of groups of fuzzy numbers that depend on some relationships in the original crisp data need to be fulfilled in the crisp outputs.

For this purpose, we assume that $x_{ip}$ , is the original crisp data where $i = 1,2,..n$, and $n$ is the number of groups which is collected from the system with relationships. While $p = 1, 2, .., P$ is the number of observations in each group. Three variables for each group are defined as;

$$\left.\begin{aligned} x_i^l &= \min\{x_{i1}, x_{i2}, \dots, x_{ip}\} \\ x_i^u &= \max\{x_{i1}, x_{i2}, \dots, x_{ip}\} \\ x_i^m &= \text{the average value of}\{x_{i1}, x_{i2}, \dots, x_{ip}\} \end{aligned}\right\} \quad (4.1)$$

Based on equation (4.1), $x_i^l$, $x_i^m$, and $x_i^u$, represent lower, middle, and upper value of each group.



However, if the fuzzy numbers are not defined yet, we need to generate the fuzzy numbers using certain methods. In this research, we utilize the method in phase two, as presented in Figure 4.2, and explained in Section 2.4.4. The fuzzification process is performed where each of $n$ triangular fuzzy number ($TrFN$) correspond to each of $n$ groups is identified via equation (4.1) as follows;

$$TrFN\ \tilde{x}_i = \left(x_i^l, x_i^m, x_i^u\right) \quad (4.2)$$

Based on equation (4.1), $x_i^l$, $x_i^m$, and $x_i^u$, represent lower, middle, and upper value of a $TrFN\ \tilde{x}_i$. Then the representative interval of each group is presented as $\left[x_i^l, x_i^u\right]$. Based on the equations (2.6) and (2.7) each interval of fuzzy number $\left[x_i^l, x_i^u\right]$ is divided into $m$ sub-intervals as; $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots [x_{m-1}, x_m]$, where $x^{min} = x_0 < x_1 < \cdots < x_{m-1} < x_m = x^{max}$. This step is known as a partition of the interval of fuzzy number, where the number of partition $m$ is equal to the number of produced sub-intervals. At the same time, we suggest using the arithmetic mean via equation (2.11), as the middle point of the triangular fuzzy number to show no real differences appear in the results as described in Chapter 5 when there are no outliers.

Second, the relationship(s) among these groups are expressed as constraint(s).

$$R(\bar{x}_i) = \mathrm{C} \qquad\qquad i = 1,2,..n \qquad (4.3)$$

where, $R$ represents different types of relationships such as ( $\sum_{i=1}^{n} \bar{x}_i$, $\prod_{i=1}^{n} \bar{x}_i$), $\bar{x}_i$ is the optimal solution of each group (the crisp output) and $\mathrm{C}$ is a constant.



### 4.7.2 Step 2: Partitioning Interval of Fuzzy Numbers

Then, the interval $[x_i^l, x_i^u]$, of the fuzzy number $\tilde{x}_i$, is divided into $m$ sub-intervals as described in Section 2.4.3.2 as

$$\left\{\left[x_i^l = x_{i_0}, x_{i_1}\right], \left[x_{i_1}, x_{i_2}\right].., \left[x_{i_{(m-1)}}, x_{i_m} = x_i^u\right]\right\} \qquad (4.4)$$

Then, each element in this process (called partition) is labeled as

$$x_k = x^l + k * \Delta x \qquad k = 0,1,2,..,m \qquad (4.5)$$

where, $\Delta x = \frac{x^u - x^l}{m}$ , is the width of each sub-interval, $m$ is the number of sub-intervals and $k$ is the number of elements in these sub-intervals which is equal to the number of strips. Finally building the triangular membership functions for the representation problem as equation (2.3) for each element in the sub-intervals is done.

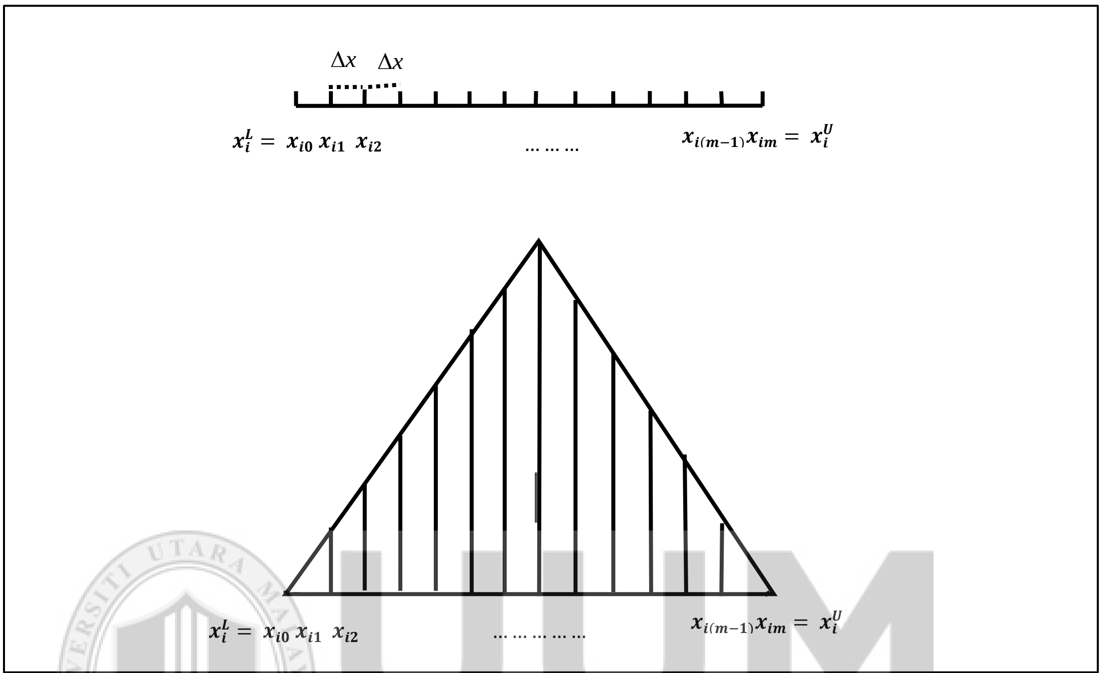



*Figure 4.5.* Building the triangular membership function of sub-intervals elements



Figure 4.5 illustrates the elements of these sub-intervals with their width $\Delta x$. The partition of the region, which is formed by the triangular membership function, divides the region into $m$ sub-intervals with $k$ strips.

### 4.7.3 Step 3: Estimating Approximated Crisp Outputs

Based on the Step 3 of the algorithm in Figure 4.4 a modification on the COG method and CCR-DEA model is presented in next sub-sections.

#### 4.7.3.1 Modification on the COG method

The first part of Step 3 of the algorithm, a description of an objective function as a modification of the COG method described in equation (2.14) of an optimization problem is introduced. In the proposed method we assume that the $x_i$ in equation (2.14) which is a fixed point for each fuzzy number as $x_i = \left|\bar{x}_i - x_{i_k}\right|$ which represents the distance of crisp output $\bar{x}_i$ with all elements in the interval $\left[x_i^l, x_i^u\right]$ by partitioning the interval of each fuzzy number into $m$ sub-intervals using equation (4.4).

The process of partition the interval $\left[x_i^l, x_i^u\right]$ into $m$ sub-intervals improves our estimated crisp output. Where the estimated crisp outputs are chosen from the sub-intervals as an approximation value in that interval for each fuzzy number $i$. These values are known as the estimated crisp outputs $\bar{x}_1, \bar{x}_2, .., \bar{x}_n$, which are changeable, and would be the best approximated to the fuzzy numbers. For each partitioning of fuzzy numbers under our method, the values of fuzzy numbers are optimized to produce best optimal solution which would satisfy the suggested relationships. Then a process of partitioning stopped when the stable results are achieved or when the solutions remain the same even though the number of partitioning is increasing. The new objective is obtained by minimization of distance (Euclidean) between each element in the interval of a fuzzy number and its approximated value as given in the following equation,

$$min \frac{\sum_{k=0}^{m} \mu_{\tilde{x}_i}\left(x_{i_k}\right) \left|\bar{x}_i - x_{i_k}\right|}{\sum_{k=0}^{m} \mu_{\tilde{x}_i}\left(x_{i_k}\right)} \tag{4.6}$$

instead of equation (2.14) presented in Section 2.5.1.1.

The problem is to find $\bar{x}_i$, which is the approximated values to the fuzzy number $i$ that minimizes the objective function in equation (4.6), and satisfies the constraints in crisp outputs.

#### 4.7.3.2 Modification on the CCR-DEA Model

The second part of Step 3 of the algorithm, by considering the CCR model presented in model (1.4), $k$ DMUs are created according to the sub-intervals in equation (4.4) where the PPS of these DMUs generates all of the possible solutions in the fuzzy interval. That means that the DEA constraints assure us that the defuzzified value (crisp outputs) lies in the allowable region.

In other words, the elements $x_1^l = x_{1_0} < x_{1_1} <..< x_{1_{(m-1)}} < x_{1_m} = x_1^u$, for the first fuzzy number $i = 1$ represents the first input of $DMU_k$ $(k = 0,1,2,...\ m)$ that are used to produce the PPS corresponding to the CCR while the single output corresponding to $DMU_k$ is assumed to be one. The inputs and a single output of each DMUs are illustrated in the following Table 4.1.

Table 4.1

*Illustration of Inputs and Outputs of DMUs*

| I&O<br>k | $I_1$ | $I_2$ | . | . | $I_{n-1}$ | $I_n$ | $O_1$ |
|---|---|---|---|---|---|---|---|
| $DMU_0$ | $x_1^L = x_{1_0}$ | $x_2^L = x_{2_0}$ | . | . | $x_{n-1}^L = x_{(n-1)_0}$ | $x_n^L = x_{n_0}$ | 1 |
| $DMU_1$ | $x_{1_1}$ | $x_{2_1}$ | . | . | $x_{(n-1)1}$ | $x_{n_1}$ | 1 |
| $DMU_2$ | $x_{1_2}$ | $x_{2_2}$ | . | . | $x_{(n-1)_2}$ | $x_{n_2}$ | 1 |
| . | . | . | . | . | . | . | . |
| . | . | . | . | . | . | . | . |
| $DMU_{m-1}$ | $x_{1_{(m-1)}}$ | $x_{2_{(m-1)}}$ | . | . | $x_{(n-1)_{(m-1)}}$ | $x_{n_{(m-1)}}$ | 1 |
| $DMU_m$ | $x_1^u = x_{1m}$ | $x_2^u = x_{2m}$ | . | . | $x_{n-1}^u = x_{(n-1)m}$ | $x_n^u = x_{n_m}$ | 1 |

In order to explain Table 4.1 further, Figure 4.6 is referred to which illustrates the inputs and DMUs created by the partition of the first fuzzy number to produce $m$ sub-intervals with $k$ elements (i.e. $k = m + 1$ the number of elements in $m$ sub-intervals).



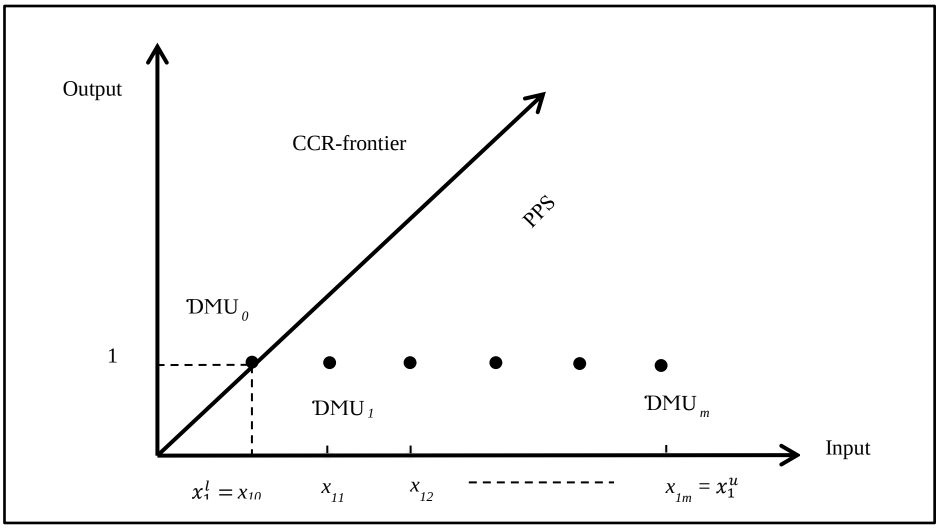



*Figure 4.6.* Inputs and DMUs created by the partition of the first fuzzy number

We replace the main objective of CCR in (1.4) by $n$ objective functions based on the equation (4.6) where these objective functions are approximated to the COG by minimizing the distances from crisp outputs $\bar{x}_i$ with all elements in the fuzzy interval and constraints including λ 's produce the PPS that correspond to the CCR model (1.4) as follows;

$$min \frac{\sum_{k=0}^{m} \mu_{\tilde{x}_i}(x_{i_k}) \left|\bar{x}_i - x_{i_k}\right|}{\sum_{k=0}^{m} \mu_{\tilde{x}_i}(x_{i_k})} \qquad i = 1,2, \dots, n \tag{4.7}$$

subject to

$$\sum_{k=0}^{m} \lambda_k x_{i_k} \leq \bar{x}_i$$

$$\sum_{k=0}^{m} \lambda_k \geq 1$$

The last step of CCR modification is achieved by adding extra constraints to the model (1.4). In the proposed method, the additional restriction to the allowable crisp outputs i.e. dependent crisp outputs are expressed as constraints $R(\bar{x}_i) = C$ where $R(\bar{x}_i)$, $i = 1,2, .. n$ is the relationship and $C$ is a constant. Also, the constraint that includes all the intervals of the crisp outputs $\bar{x}_i$ are added

$$x_i^l \leq \bar{x}_i \leq x_i^u, \quad i = 1,2, \dots, n \tag{4.8}$$

### 4.7.4 Step 4: Mathematical Formulation of the Proposed Method

In this section, the fourth step of the algorithm is achieved by introducing the general formulation for the proposed method based on the three cases presented as sub-sections.

#### 4.7.4.1 Case 1: Non-Linear with Any Relationship

In this case the general formulation of the proposed method is introduced as

$$min \frac{\sum_{k=0}^{m} \mu_{\tilde{x}_i}(x_{i_k}) \left|\bar{x}_i - x_{i_k}\right|}{\sum_{k=0}^{m} \mu_{\tilde{x}_i}(x_{i_k})} \tag{4.9}$$

subject to

$$\sum_{k=0}^{m} \lambda_k x_{i_k} \leq \bar{x}_i$$

$$\sum_{k=0}^{m} \lambda_k \geq 1$$

$$R(\bar{x}_i) = C$$

$$\frac{x_i^l}{x_i^u} \leq \bar{x}_i \leq 1$$

$$\lambda_k \geq 0 \qquad i = 1,2, \dots n, \quad k = 0,1,2, \dots m$$

The above model is a multi-objective non-linear programming (MONLP) problem where its optimal solution takes place at the following intervals;

$$x_i^l \leq \bar{x}_i \leq x_i^u \qquad i = 1,2, \dots n$$

This model is solved only once, unlike DEA models, in which DMU evaluation requires the calculation of many models. Moreover, this model can deal with linear and non-linear relationships.

#### 4.7.4.2 Case 2: Linear Objective with any Relationship

For Case 2, we transfer the non-linear model (4.9) to a linear form by assuming that $z_{i_k} = \bar{x}_i - x_{i_k}$ and $|z_{i_k}| = z_{i_k}^{+} + z_{i_k}^{-}$ $\forall(i,k)$. The MONLP model (4.9) is then proposed as follows:

$$min \frac{\sum_{k=0}^{m} \mu_{\tilde{x}_i}(x_{i_k})\,(z_{i_k}^{+} + z_{i_k}^{-})}{\sum_{k=0}^{m} \mu_{\tilde{x}_i}(x_{i_k})} \tag{4.10}$$

subject to

$$\sum_{k=0}^{m} \lambda_k x_{i_k} \leq \bar{x}_i$$

$$\sum_{k=0}^{m} \lambda_k \geq 1$$

$$R(\bar{x}_i) = C$$

$$x_i^l \leq \bar{x}_i \leq x_i^u$$

$$\bar{x}_i - x_{i_k} - \left(z_{i_k}^{+} - z_{i_k}^{-}\right) = 0$$

$$\lambda_k \geq 0 \qquad i = 1,2,\dots n\,, \qquad k = 0,1,2,\dots m$$

When $R\ (\bar{x}_i)$ the relationship and $\mu_{\tilde{x}_i}(x_{ik})$ the membership function are linear, the above model is a multi-objective linear programming model (MOLP). These axioms lead us to Case 3.

#### 4.7.4.3 Case 3: Linear Objective with a Linear Relationship

A special case of model (4.10) when each objective and relationships are linear is derived using LP concept. The proposed method in model (4.10) is now modified as follows;

$$min \frac{\sum_{k=0}^{m} \mu_{\tilde{x}_i}(x_{i_k})\,(z_{i_k}^{+} + z_{i_k}^{-})}{\sum_{k=0}^{m} \mu_{\tilde{x}_i}(x_{i_k})} \tag{4.11}$$

subject to

$$R(\bar{x}_i) = C$$

$$x_i^l \leq \bar{x}_i \leq x_i^u$$

$$\bar{x}_i - x_{i_k} - (z_{i_k}^{+} - z_{i_k}^{-}) = 0 \qquad i = 1,2, \dots n\,, k = 0,1,2, \dots m$$

The model (4.11) can be used when all objectives and constraints, including the relationship(s) are linear. In such cases, no need to use the proposed CCR-DEA model. Both, models (4.10) and (4.11), can be solved using the weighted goal programming model (WGP).



### 4.7.5 Step 5: Solving a Multi-Objective Problem

The WGP technique is a suitable method to solve multi-objective problem in cases when DMs do not have any pre-emptive ordering of the objective functions. Instead of prioritizing the objective functions in different levels, they assign different weights to deviation variable of each objective function in the single priority level. However, we assume that each objective is equally important and allocate equal weight without losing generality. Then, the MOLP model described as follows;

$$min \sum_{i=1}^{n} w_i d_i \tag{4.12}$$

subject to

$$\frac{\sum_{k=0}^{m} \mu_{\tilde{x}_i}\left(x_{i_k}\right)\left(z_{i_k}^{+} + z_{i_k}^{-}\right))}{\sum_{k=0}^{m} \mu_{\tilde{x}_i}\left(x_{i_k}\right)} - d_i \leq t`_i$$

$$\sum_{k=0}^{m} \lambda_k x_{i_k} \leq \bar{x}_i$$

$$\sum_{k=0}^{m} \lambda_k \geq 1$$

$$R\ (\bar{x}_i) = C$$

$$x_i^l \leq \bar{x}_i \leq\ x_i^u$$

$$\bar{x}_i - x_{i_k} - \left(z_{i_k}^{+} - z_{i_k}^{-}\right) = 0$$

$$\lambda_k \geq 0 \qquad i = 1,2, \dots n, k = 0,1,2, \dots m$$

In model (4.12), $w_i$ ($i = 1,2, \dots n$ ) denotes positive penalty weights. Weights may be assigned to the deviation variables to show preference to a certain group of fuzzy numbers. Decision makers could supply the weights or alternatively, determine them by using MCDM techniques such as AHP developed by Saaty, (1980 as cited in Saaty, 2008).

It should be noted that the problem in this specific study is not about weights. However, we assume that each objective is equally important and allocate equal weight without losing generality.

That is $(w_1 = w_2 = \cdots = w_n = 1/n)$ allocated to each weight for this model $d_i$ $(i = 1,2, \ldots n)$ measures the over-achievement from the target point that is $t`_i$ $(i = 1,2, \ldots n)$ which is obtained by computing the MOLP model as a single objective $n$ times (i.e. by considering each objective individually).

## 4.8 Evaluation and Comparison of the Proposed Method

Figure 4.2 and 4.3 illustrate that the fifth phase is about the evaluation of the performance of the proposed method. We must carefully check the value of defuzzified value obtained by proposed method with others to show how our method has satisfied the relationships in the crisp output by comparing it with two common methods under defuzzification concept. The two methods under defuzzification concept are the COG and the A&Z methods presented in sub-sections 2.5.1.1 and 2.7.1.



These methods are used to evaluate the proposed method in cases where whether there are relationships or no relationships that need to be satisfied in the crisp outputs. The Kikuchi (2000) method presented in model (3.2) is used to evaluate the proposed method in the case where there is a relationship that needs to be satisfied in the crisp outputs. In addition, a comparison of the proposed method based DEA model and LP based model is presented.

## 4.9 Summary and Discussion

In this chapter, we presented how the new methods to defuzzify groups of fuzzy numbers are developed, that is, by extending the defuzzification concept with the presence of restrictions to the crisp outputs research. Two alternative approaches to defuzzification under restriction in crisp outputs are proposed. The first method is based on the modelling of the defuzzification process using the DEA concept. The second method involves the modelling of the defuzzification process as a constrained LP problem.

The presented methods focus on defuzzifying groups of fuzzy numbers based on the concept of minimizing the distance between the crisp output and the other points in the interval of fuzzy numbers. The proposed method in this study is firstly presented and discussed. This is because defuzzification with dependent data is a very specific case when we deal with real applications problem in fuzzy systems. The structure of this method by using DEA model is first presented in the literature. Furthermore, the suggested method is not only dependent on the endpoints or midpoints of intervals corresponding to the fuzzy numbers but it is looking for the best approximation of fuzzy number that satisfies some properties.



Chapter Five concerns with the implementation, validation, and evaluation of the proposed method in solving a real problem and in the case of original crisp data with and without relationships. Besides that, an application of the proposed method in solving different cases with different types of fuzzy numbers is also provided. In doing so, the proposed method works as a general method since it can deal with different types of fuzzy numbers.

# CHAPTER FIVE
# APPLICATION OF THE PROPOSED METHOD

This chapter describes the steps of applying the proposed defuzzification method to solving some problems. First, the beds allocation problems in hospitals in Malaysia are introduced with two sets of secondary data. Section 5.1 briefly describes the beds allocation problem in Malaysia and some related literature. A detailed elaboration of two types of data from hospitals in Malaysia is presented next. The first type is considered the original crisp data that have relationships that need to be kept in the outputs (dependent crisp outputs). The second type is data without any relationships that need to be satisfied in the outputs (independent crisp outputs). These two types of data are described in Section 5.2.

The implementation of the suggested methods based on DEA with the two types of data to get dependent crisp outputs and independent crisp outputs is achieved in Section 5.3. Our third and fourth specific objectives, which are related to the implementation of the proposed method, are discussed in Section 5.3. The implementation of the proposed method based on the LP concept with two types of data to accomplish the fifth objective of the research is presented in Section 5.4.

Furthermore, the application of the proposed method for solving some common issues in the literature including finding the optimal weights in the goal programming (GP) model and the ranking of fuzzy numbers is explained in Section 5.5 to meet the last research objective. Finally, Section 5.6 offers a summary of the discussion of the application of the proposed method.

## 5.1 Background of Beds Allocation Problems

Hospital facilities are striving to minimize resource use by adopting new strategies and tools or by optimizing existing ones. Limited bed availability and the need to include growing health costs have intensified the search for options to distribute the best number of beds for each branch or organization for patient who require them. According to the Malaysian Economic Planning Unit (2013), the three main indicators for the healthcare services subcomponent are the number of hospital beds, doctor to population ratio and hospital waiting time for outpatients. The number of facility hospital admissions in Malaysia has shown to growth, for example by 37.6% to 2.1 million people in 2009 compared to 1.5 million people in 2000 (Ministry of Health Malaysia, 2011b).

This phenomenon ought to receive greater attention, in particular by medical institution directors and the policy makers, because it impacts the structure and organization of the healthcare center and treatment. Also, the affected patient's level of health and satisfaction are jeopardy if their demand no longer fits with the hospital's resources. Overall, the problem of beds allocation, if not addressed well, will have a negative effect on the entire health care systems because it impacts the destiny, economical price sanatorium, and countrywide care budgets (Bottle, Aylin, & Majeed, 2006; Caley & Sidhu, 2011).

The rates of patients admitted to government hospitals are expanded in various states. As an instance in 2003, Negeri Sembilan and Perlis had the highest rates of admissions with 9.9% and 10.5%, respectively, while Kedah and Penang had the lowest rates with 1.3% and 1.4%, respectively (Ministry of Health Malaysia, 2011a). In 2012, 2.6 million admissions were recorded, and about 19 million were visible as outpatients. In the evaluation of approximately 2.1 million admissions registered in 2010, about 17.6 million were as outpatients. Here is also a growth in the admission of oversea patients (110,572 oversea inpatients in 2011 compared to 95,250 in 2008), which contributed to using the MOH hospital beds. Through 2015, it was anticipated a five percentage increase annually in the quantity of medical institution admission. This means additional admissions of about 150,000 to 200,000 patients each year, which is likely to have an effect on the infrastructure capacity of current day centers. Then again, the variety of hospital beds has only accelerated with the growth from 33,211 beds to about 38,978 beds from 2010 to 2012 respectively, an increase of 5,767 beds (Abdullah, 2014).



### 5.1.1 Related Literature of Beds Allocation Problems

The process of determining the optimal number or size of hospital resources is not easy because the number of admissions or arrivals and the length of stay behave in an uncertain manner. Due to the crucial time and limited resources, such as the number of available beds, healthcare professionals need to make wise decisions in identifying the optimal number of beds as beds are categorized into many types depending on the type of treatments or length of stay (Nguyen et al., 2005).

The number of beds has been chosen to be the most crucial resource in many studies conducted to find suitable methods to solve the problems Marcon, Kharraja, Smolski, Luquet, and Viale (2003) used a simulation flow technique, whereas

Kokangul (2008) solved it in a stochastic manner by using non-linear mathematical models with an opportunity distribution functions of data. A simulation version was used to generate data to outline the relationships between the dominance parameters and the number of beds. Also, an integer linear programming model was used to fixed the problem of beds planning in hospitals by Bachouch, Guinet and Hajri-Gabouj (2012).

Very few studies considered a bed allocation problem as a multi-objective problem. Kim, Ira, Young and Buckley (2000) demonstrated with the computer-simulation model that there is no monocular dominant solution to the bed-allocation trouble, consequently indicating the multi-objective nature of the trouble. Meanwhile, Oddoye, Yaghoobi, Tamiz, Jones and Schmidt (2007) considered three main resources of beds, nurses and doctors by using a WGP model in an MODM problem to evaluate the performance of a Medical Assessment Unit (MAU) in the UK and minimize the delay time in beds allocation for patients with using different scenarios. Oddoye, Jones, Tamiz and Schmidt (2009) extended their work in 2007 by using these factors: the length of stay, the number of beds, nurses and doctors in the MAU. Thereafter, a GP model and a simulation model were applied to perform a trade-off analysis.



The hourly allocation schemes of resources were deployed, within the MAU, to help minimize delays and increase the flow of patients. Manaf and Nooi (2009) discussed the quality of employees in public hospitals. There are also studies on optimizing other hospital resources, such as operating room planning and scheduling (Cardoen, Demeulemeester, & Beliën, 2010). Ma and Demeulemeester (2013) considered a multi-level integrated approach of mathematical programming and simulation analysis to hospital case mix and capacity problem.

The present research contributes to the literature by using the defuzzification method as a method to estimate the optimal number of beds under uncertain condition by including the total number of available beds as a constraint.

## 5.2 Information of the Data Collection

Both sets of secondary data are about the number of patients warded in hospitals. The first set of data is collected from Hospital Tuanku Fauziah (HTF) in Perlis and the second set from the database of the Ministry of Health (MOH) of Malaysia. The first set of data is classified as data with relationships because the information of the available number of beds in each of the 19 wards and the number of available beds in the hospital as a total is given. The data covered the recorded number of patients admitted in each day for two years (2013 and 2014).

On the other hand, the second set of data is classified as data without relationships because only the recorded numbers of patients admitted on each day during a period of three years from 2008 to 2010 in seven hospitals are available. There is no other information available, such as the available number of beds in each hospital or the total number of beds. The second set of data classifies patients based on their age (Toddler, Schoolchildren, Adult, Old, and Elderly). The two sets of data along with their sources are explained in detail in the next sub-sections.

### 5.2.1 Data from Hospital Tuanku Fauziah

Eighteen groups of warded patients are considered which based on their record in such wards in Hospital Tuanku Fauziah (HTF) in Perlis for the year 2013 and 2014. We consider these data as real with relationships because the source provided us with the total numbers of beds in each ward and in the hospital in general. The result from the proposed method would offer some recommendations for the management of the hospital on how to distribute the correct number of beds for each ward. This process is done by estimating the optimal number of beds for each ward based on the total number of available beds and recorded number of warded patients every day. A list of all 19 wards and the total number of beds in each ward are depicted in Table 5.1. In this research, we consider only 18 wards and we omit the $Ward_{19}$, which has only one bed. Hence, the total number of available beds is 383 beds instead of 384 beds.

Table 5.1

*List of Wards in HTF and the Number of Available Beds in Years (2013 and 2014)*

| No | Ward | No. of beds |
|---|---|---|
| 1 | $Ward_1$ | 4 |
| 2 | $Ward_2$ | 22 |
| 3 | $Ward_3$ | 2 |
| 4 | $Ward_4$ | 27 |
| 5 | $Ward_5$ | 5 |
| 6 | $Ward_6$ | 39 |
| 7 | $Ward_7$ | 10 |
| 8 | $Ward_8$ | 16 |
| 9 | $Ward_9$ | 18 |
| 10 | $Ward_{10}$ | 20 |
| 11 | $Ward_{11}$ | 39 |
| 12 | $Ward_{12}$ | 28 |
| 13 | $Ward_{13}$ | 28 |
| 14 | $Ward_{14}$ | 28 |
| 15 | $Ward_{15}$ | 24 |
| 16 | $Ward_{16}$ | 28 |
| 17 | $Ward_{17}$ | 20 |
| 18 | $Ward_{18}$ | 25 |
| 19 | $Ward_{19}$ | 1 |
| Total number of available beds | | 384 |



### 5.2.2 Data from Database of the Ministry of Health in Malaysia

The hospital admissions to seven hospitals in Klang Valley for three years from 2008 to 2010 are obtained from the MOH of Malaysia. The data are classified into five groups of patients based on their age (Toddler, Schoolchildren, Adult, Old, and Elderly). We consider these real data as without relationships because the source provides us with the recorded number of patients based on their age without any information of the total number of available beds or the number of wards or any other information. Therefore, the proposed method handles this problem by assuming this data mathematically as being independent, and there is no relationship between the five groups. Then, the proposed method would estimate the correct number of beds for each group. This process is done by estimating the optimal number of beds for each group based on the recorded patients warded every day from 2008 to 2010. A list of the number of all patients is depicted in Appendix B.

## 5.3 Application of the Proposed Method in the Hospital Management

This section discusses the application of the proposed method in real life to solve the problem of allocating the number of beds for the different medical, surgical departments and groups of patients in a hospital. The evidence that the patterns of patient arrivals differ between departments confounds this task. To reiterate, the present case study is an attempt to assist management in determining the optimal number of beds in each department where the hospital has a limited number of beds.



To achieve the third and fourth objectives, five phases of research activities explained in Figure 4.1, and 4.2 are carried out. Phase one is about the problem definition where two types of problem are considered. The first problem is when the original crisp data have some relationships that need to be satisfied in crisp outputs (dependent crisp outputs) (see sub-section 5.3.1) while the second problem is presented in sub-section 5.3.2 when the original crisp data do not have any relationships be satisfied in the crisp outputs (independent crisp outputs).

### 5.3.1 Implementation in Hospital Tuanku Fauziah

We apply the proposed methodology to real-life situations by estimating the required number of hospital beds in the different wards of Hospital Tuanku Fauziah (HTF). In the second phase of the research, we collect data and generate the fuzzy numbers. The following describes the phase.

Data are collected on the number of beds used by patients during the period of 2013 and 2014. The hospital patients are divided into 18 groups based on admissions to medical or surgical departments. In order to avoid observations with zero values, we gather data every five days. Hence, the total observation is 146 blocks days instead of 730 days.

The overall data of the recorded patients in 18 wards at Hospital Tuanku Fauziah (HTF) in Perlis are presented in Appendix A. This case study aims at aiding managers in determining the optimal number of beds to be allocated to each ward because the number of available beds at this hospital is limited. The minimum value, maximum value, geometric mean, and arithmetic mean of each ward are calculated by using equations (2.9), (2.11) and (2.12), respectively. The result is shown in Table 5.2.

Table 5.2

*Statistics of Sample for the Recorded Patients in 18 Wards in HTF in Years (2013 and 2014)*

| Ward | Min | Max | Geometric mean | Arithmetic mean |
|---|---|---|---|---|
| $Ward_1$ | 1 | 10 | 4.134 | 4.5 |
| $Ward_2$ | 11 | 39 | 24.497 | 25.055 |
| $Ward_3$ | 1 | 9 | 2.498 | 2.993 |
| $Ward_4$ | 1 | 14 | 4.43 | 4.829 |
| $Ward_5$ | 4 | 24 | 12.884 | 13.384 |
| $Ward_6$ | 11 | 44 | 24.906 | 25.986 |
| $Ward_7$ | 3 | 90 | 11.647 | 13.822 |
| $Ward_8$ | 1 | 44 | 4.122 | 5.658 |
| $Ward_9$ | 30 | 179 | 57.731 | 59.952 |
| $Ward_{19}$ | 74 | 154 | 108.095 | 109.055 |
| $Ward_{11}$ | 17 | 154 | 37.415 | 40.123 |
| $Ward_{12}$ | 24 | 180 | 43.07 | 45.795 |
| $Ward_{13}$ | 25 | 158 | 44.24 | 46.233 |
| $Ward_{14}$ | 30 | 174 | 51.659 | 54.349 |
| $Ward_{15}$ | 1 | 182 | 35.784 | 42.514 |
| $Ward_{16}$ | 1 | 69 | 7.697 | 10.952 |
| $Ward_{17}$ | 3 | 61 | 20.474 | 22.315 |
| $Ward_{18}$ | 27 | 155 | 44.842 | 47.541 |

Table 5.3 provides the generated fuzzy numbers $TrFN$ as $\tilde{x} = (x^{min}, x^{mid}, x^{max})$ of each ward by considering the geometric mean in Table 5.2 as the middle point of the fuzzy numbers in the second column. Meanwhile, column three in Table 5.3 presents the fuzzy number of each ward by considering the arithmetic mean in Table 5.2 as the middle point. Table 5.3 shows the fuzzy numbers with the geometric mean and arithmetic mean as the middle points of the fuzzy numbers in column 2 and 3 respectively.

Table 5.3

*Generated Fuzzy Number with Geometric Mean and Arithmetic Mean in Each Ward in HTF in Years (2013 and 2014)*

| Ward | Fuzzy numbers with Geometric mean | Fuzzy numbers with Arithmetic mean |
|---|---|---|
| $Ward_1$ | (1,4.134,10) | (1,4.500,10) |
| $Ward_2$ | (11,24.497,39) | (11,25.055,39) |
| $Ward_3$ | (1,2.498,9) | (1,2.993,9) |
| $Ward_4$ | (1,4.430,14) | (1,4.829,14) |
| $Ward_5$ | (4,12.884,24) | (4,13.384,24) |
| $Ward_6$ | (11,24.906,44) | (11,25.986,44) |
| $Ward_7$ | (3,11.647,90) | (3,13.822,90) |
| $Ward_8$ | (1,4.122,44) | (1,5.658,44) |
| $Ward_9$ | (30,57.731,179) | (30,59.952,179) |
| $Ward_{19}$ | (74,108.095,154) | (74,109.055,154) |
| $Ward_{11}$ | (17,37.415,154) | (17,40.123,154) |
| $Ward_{12}$ | (24,43.070,180) | (24,45.795,180) |
| $Ward_{13}$ | (25,44.240,158) | (25,46.233,158) |
| $Ward_{14}$ | (30,51.659,174) | (30,54.349,174) |
| $Ward_{15}$ | (1,35.784,182) | (1,42.514,182) |
| $Ward_{16}$ | (1,7.697,69) | (1,10.952,69) |
| $Ward_{17}$ | (3,20.474,61) | (3,22.315,61) |
| $Ward_{18}$ | (27,44.842,155) | (27,47.541,155) |



The fuzzy number of each group represents the inputs in DEA. Then, the interval of each fuzzy number $i$ is divided into $m = 500$ sub-intervals, where the elements in each interval of fuzzy number are considered as $(x_{1_k}, x_{2_k}, \dots x_{18_k})$ (i.e $input_{1_k}$, $input_{2_k}$,… $input_{18_k}$) of $DMU_k$ in DEA. We still do the partition until we get a stable result that is the solutions are unchanged, and when the results are not stable, we increase the number of partition up to 500 sub-intervals. As mentioned in sub-section 4.7.3.2 each element in first fuzzy number interval is labeled as the first input of $DMU_k$ $k = 0,1,2, \dots m,$ and the elements of the second fuzzy number interval are the second inputs of $DMU_k$ and so on until we reach the last fuzzy number.

Figure 5.1 shows that each fuzzy number and its points in the sub-interval represent the inputs of $k$ DMUs. This figure also illustrates the generating of $k$ DMUs as presented in Table 4.1.

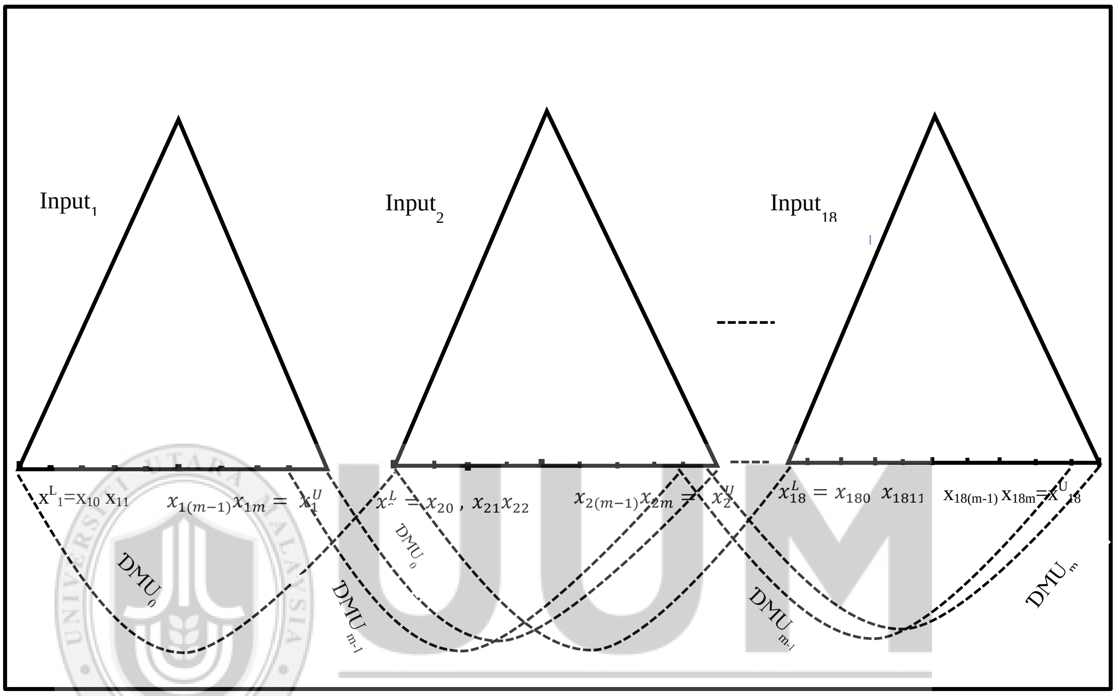





*Figure 5.1.* The fuzzy numbers and their points in sub-interval as inputs of $k$ DMUs

#### 5.3.1.1 Implementation with Geometric Mean

Consider the geometric mean as the middle points for each of the 18 fuzzy numbers. For each fuzzy number, the sub-intervals produced are as follows: when $m = 1$ then $k = 0, 1$, which means one interval and two elements and from equations (4.4 and 4.5). The element of the corresponding interval is $x_0 = x^l$ and $x_1 = x^u$ which lead to the main interval of the fuzzy number $[x^l, x^u]$. Then, when $m = 2$ partition, the interval $[x^l, x^u]$ is divided into two sub-intervals with three elements, where

$k = 0, 1,2$ and so on until we reach $m$ sub-intervals with $k = m + 1$ which is the number of all elements of the corresponding sub-interval.

The formulation indicates that no result could be obtained that satisfy the relationship (the total number of beds is 383) or the objective function and constraints when $m = 1$. However, the results start to appear from $m = 2$ until $m = 500$.

Table 5.4

*Results of the Proposed Method with Geometric Mean for Data from HTF*

| Wards | Fuzzy numbers | Crisp outputs | | | | | | |
|---|---|---|---|---|---|---|---|---|
| i | $(x_i^l, x_i^{gem}, x_i^u)$ | m=2 | m=10 | m=40 | m=50 | m=100 | m=250 | m=500 |
| $Ward_1$ | (1,4.134,10) | 4 | 1 | 2 | 2 | 2 | 2 | 2 |
| $Ward_2$ | (11,24.497,39) | 11 | 11 | 15 | 15 | 15 | 15 | 15 |
| $Ward_3$ | (1,2.498,9) | 5 | 1 | 2 | 2 | 2 | 2 | 2 |
| $Ward_4$ | (1,4.430,14) | 5 | 1 | 2 | 2 | 2 | 2 | 2 |
| $Ward_5$ | (4,12.884,24) | 5 | 4 | 7 | 6 | 6 | 6 | 6 |
| $Ward_6$ | (11,24.906,44) | 27 | 11 | 15 | 15 | 15 | 15 | 15 |
| $Ward_7$ | (3,11.647,90) | 3 | 11 | 9 | 8 | 8 | 8 | 8 |
| $Ward_8$ | (1,4.122,44) | 22 | 5 | 3 | 3 | 3 | 3 | 3 |
| $Ward_9$ | (30,57.731,179) | 30 | 43 | 41 | 42 | 42 | 42 | 42 |
| $Ward_{19}$ | (74,108.095,154) | 114 | 80 | 84 | 84 | 84 | 84 | 84 |
| $Ward_{11}$ | (17,37.415,154) | 17 | 30 | 27 | 28 | 28 | 28 | 28 |
| $Ward_{12}$ | (24,43.070,180) | 24 | 38 | 35 | 34 | 34 | 34 | 34 |
| $Ward_{13}$ | (25,44.240,158) | 25 | 38 | 35 | 35 | 35 | 35 | 35 |
| $Ward_{14}$ | (30,51.659,174) | 30 | 41 | 41 | 41 | 41 | 41 | 41 |
| $Ward_{15}$ | (1,35.784,182) | 1 | 19 | 15 | 16 | 16 | 16 | 16 |
| $Ward_{16}$ | (1,7.697,69) | 1 | 7 | 5 | 5 | 5 | 5 | 5 |
| $Ward_{17}$ | (3,20.474,61) | 32 | 3 | 9 | 9 | 9 | 9 | 9 |
| $Ward_{18}$ | (27,44.842,155) | 27 | 39 | 36 | 36 | 36 | 36 | 36 |
| Sum of the estimated No. of beds | | 383 | 383 | 383 | 383 | 383 | 383 | 383 |

Table 5.4 shows that a stable result starts to emerge in the partition $m = 50$, which means that the optimal solution of the estimated number of beds of each ward appears in partition $m = 50$. In other words, a stable result appears when we have

50 sub-intervals. Furthermore, it can be noted that the sum of all estimated values under each partition (dependent crisp outputs) satisfy the relationship that the total number of all beds is 383 ($R(\bar{x}_i) = C = 383$) in all cases with different number of partitions.

#### 5.3.1.2 Implementation with Arithmetic Mean

Table 5.5 shows the estimated number of the beds, but when the middle point of fuzzy numbers is the arithmetic mean.

Table 5.5

*Results of the Proposed Method with Arithmetic Mean for Data from HTF*

| Ward | Fuzzy numbers | Crisp values | | | | | | |
|---|---|---|---|---|---|---|---|---|
| i | $(x_i^l, x_i^{arth}, x_i^u)$ | m=2 | m=10 | m=40 | m=50 | m=100 | m=250 | m=500 |
| $Ward_1$ | (1,4.500,10) | 5 | 2 | 2 | 2 | 2 | 2 | 2 |
| $Ward_2$ | (11,25.055,39) | 25 | 16 | 15 | 15 | 15 | 15 | 15 |
| $Ward_3$ | (1,2.993,9) | 5 | 2 | 2 | 2 | 2 | 2 | 2 |
| $Ward_4$ | (1,4.829,14) | 5 | 2 | 2 | 2 | 2 | 2 | 2 |
| $Ward_5$ | (4,13.384,24) | 4 | 7 | 7 | 7 | 7 | 7 | 7 |
| $Ward_6$ | (11,25.986,44) | 16 | 21 | 20 | 20 | 20 | 20 | 20 |
| $Ward_7$ | (3,13.822,90) | 10 | 6 | 5 | 5 | 5 | 5 | 5 |
| $Ward_8$ | (1,5.658,44) | 5 | 2 | 2 | 2 | 2 | 2 | 2 |
| $Ward_9$ | (30,59.952,179) | 30 | 37 | 39 | 39 | 39 | 39 | 39 |
| $Ward_{19}$ | (74,109.055,154) | 82 | 90 | 92 | 92 | 92 | 92 | 92 |
| $Ward_{11}$ | (17,40.123,154) | 37 | 25 | 23 | 23 | 23 | 23 | 23 |
| $Ward_{12}$ | (24,45.795,180) | 24 | 28 | 30 | 30 | 30 | 30 | 30 |
| $Ward_{13}$ | (25,46.233,158) | 25 | 31 | 31 | 31 | 31 | 31 | 31 |
| $Ward_{14}$ | (30,54.349,174) | 30 | 38 | 36 | 36 | 36 | 36 | 36 |
| $Ward_{15}$ | (1,42.514,182) | 37 | 28 | 28 | 28 | 28 | 28 | 28 |
| $Ward_{16}$ | (1,10.952,69) | 10 | 3 | 4 | 4 | 4 | 4 | 4 |
| $Ward_{17}$ | (3,22.315,61) | 3 | 11 | 9 | 9 | 9 | 9 | 9 |
| $Ward_{18}$ | (27,47.541,155) | 30 | 34 | 36 | 36 | 36 | 36 | 36 |
| Sum of the estimated No. of beds | | 383 | 383 | 383 | 383 | 383 | 383 | 383 |



Table 5.5 shows that a stable result starts to emerge from $m = 40$, which means that the optimal solution of the number of beds for each ward appeared earlier than

in the case of the geometric means. It can be noted that the sum of all estimated values under each partition also satisfy the relationship that total number of all beds is 383 for all cases with different number of partitions.

#### 5.3.1.3 Estimated Number of Beds, as an Interval

So, from the results obtained in Table 5.4 and 5.5 with geometric mean and arithmetic mean, respectively, with the real number of available beds in each ward, we can introduce an estimated number of beds as an interval for each ward as described in Table 5.6.

Table 5.6
*Available Number of Beds and the Estimated Number of Beds under Geometric and Arithmetic Mean for Data from HTF in Years (2013 and 2014)*

| Ward | Available beds | Estimated No. of beds with Geometric mean | Estimated No. of beds with the Arithmetic mean |
|---|---|---|---|
| $Ward_1$ | 4 | 2 | 2 |
| $Ward_2$ | 22 | 15 | 15 |
| $Ward_3$ | 2 | 2 | 2 |
| $Ward_4$ | 5 | 2 | 2 |
| $Ward_5$ | 27 | 6 | 7 |
| $Ward_6$ | 39 | 15 | 20 |
| $Ward_7$ | 10 | 8 | 5 |
| $Ward_8$ | 16 | 3 | 2 |
| $Ward_9$ | 18 | 42 | 39 |
| $Ward_{19}$ | 20 | 84 | 92 |
| $Ward_{11}$ | 39 | 28 | 23 |
| $Ward_{12}$ | 28 | 34 | 30 |
| $Ward_{13}$ | 28 | 35 | 31 |
| $Ward_{14}$ | 28 | 41 | 36 |
| $Ward_{15}$ | 24 | 16 | 28 |
| $Ward_{16}$ | 25 | 5 | 4 |
| $Ward_{17}$ | 28 | 9 | 9 |
| $Ward_{18}$ | 20 | 36 | 36 |
| Total No. of beds | 383 | 383 | 383 |

The results described in Table 5.6, could help the hospital managers in planning the suitable number of available beds in each ward. It is important to be note that the estimated number of beds based on the proposed method does not consider the size of wards. For instance, in $Ward_{12}$, the minimum number of beds is 28, and the maximum number of beds is 34. This means that the proposed method can give an interval of the estimated value of beds in each ward by using different middle points for each fuzzy number.

#### 5.3.1.4 Results of COG and A&Z Methods

In order to validate the proposed method, we use two defuzzification methods, namely, the COG and the method of A&Z (2007) to defuzzify the groups of fuzzy numbers. This comparison supports our proposed procedure to defuzzify groups of fuzzy numbers and keeps some relationships in the crisp outputs. Table 5.7 shows the results of the COG method using equation (2.13) and the A&Z method using equation (2.23) to the 18 groups of fuzzy numbers with geometric means as the middle point.

The COG and A&Z methods provide us with the estimated number of beds as individuals without fulfilling any requirement to the total number of available beds in the hospital. In such case, the total number of estimated numbers of beds by using the COG and the A&Z method is 848 and 771, respectively, which are very different from the real available number of beds of 383.

Table 5.7

*Estimated Number of Beds Based on the COG and A&Z Methods for Data from HTF*

| Ward | Fuzzy numbers | COG | A&Z |
|---|---|---|---|
| Ward$_{1}$ | (1,4.134,10) | 5 | 5 |
| Ward$_{2}$ | (11,24.497,39) | 25 | 25 |
| Ward$_{3}$ | (1,2.498,9) | 4 | 4 |
| Ward$_{4}$ | (1,4.430,14) | 6 | 6 |
| Ward$_{5}$ | (4,12.884,24) | 14 | 13 |
| Ward$_{6}$ | (11,24.906,44) | 27 | 26 |
| Ward$_{7}$ | (3,11.647,90) | 35 | 29 |
| Ward$_{8}$ | (1,4.122,44) | 16 | 13 |
| Ward$_{9}$ | (30,57.731,179) | 89 | 81 |
| Ward$_{19}$ | (74,108.095,154) | 112 | 111 |
| Ward$_{11}$ | (17,37.415,154) | 69 | 61 |
| Ward$_{12}$ | (24,43.070,180) | 82 | 73 |
| Ward$_{13}$ | (25,44.240,158) | 76 | 68 |
| Ward$_{14}$ | (30,51.659,174) | 85 | 77 |
| Ward$_{15}$ | (1,35.784,182) | 73 | 64 |
| Ward$_{16}$ | (1,7.697,69) | 26 | 21 |
| Ward$_{17}$ | (3,20.474,61) | 28 | 26 |
| Ward$_{18}$ | (27,44.842,155) | 76 | 68 |
| Sum of the estimated No. of beds | | 848 | 771 |

In other words, these methods fail to provide an acceptable solution to the problem in such case. In the next sub-section, we compare the performance of our method using the Kikuchi's method.

#### 5.3.1.5 Results of Kikuchis' Method

The current section introduces the estimated number of available beds in each of the 18 wards in HTF using the Kikuchi's method presented in Section 3.1. By applying Kikuchi's method model (3.2), the estimated number of beds is shown in Table 5.8. Then, the results obtained by the Kikuchi's' method is compared with the result of the proposed method presented in Table 5.4 with a geometric mean as the middle point.

The results of the two methods provide us with the estimated number of beds for each ward, which is nearly approximated and there is no big difference found in the estimated number of beds for each ward. The two methods keep the relationship in the crisp output and give the sum of the estimated number of beds equal to the total number of available beds, which are 383 beds.

Table 5.8

*Estimated Number of Beds Based on Kikuchis' Method and Proposed Method for Data from HTF*

| Ward | Fuzzy numbers | Kikuchis' method | Proposed method |
|---|---|---|---|
| | $h$ | 0.3966823 | 0.225123818 |
| $Ward_1$ | (1,4.134,10) | 3 | 2 |
| $Ward_2$ | (11,24.497,39) | 17 | 15 |
| $Ward_3$ | (1,2.498,9) | 2 | 2 |
| $Ward_4$ | (1,4.430,14) | 3 | 2 |
| $Ward_5$ | (4,12.884,24) | 8 | 6 |
| $Ward_6$ | (11,24.906,44) | 17 | 15 |
| $Ward_7$ | (3,11.647,90) | 7 | 8 |
| $Ward_8$ | (1,4.122,44) | 3 | 3 |
| $Ward_9$ | (30,57.731,179) | 41 | 42 |
| $Ward_{19}$ | (74,108.095,154) | 88 | 84 |
| $Ward_{11}$ | (17,37.415,154) | 26 | 28 |
| $Ward_{12}$ | (24,43.070,180) | 32 | 34 |
| $Ward_{13}$ | (25,44.240,158) | 33 | 35 |
| $Ward_{14}$ | (30,51.659,174) | 39 | 41 |
| $Ward_{15}$ | (1,35.784,182) | 15 | 16 |
| $Ward_{16}$ | (1,7.697,69) | 4 | 5 |
| $Ward_{17}$ | (3,20.474,61) | 10 | 9 |
| $Ward_{18}$ | (27,44.842,155) | 35 | 36 |
| Sum of the estimated No. of beds | | 383 | 383 |

But when we consider the objective of Kikuchis' method that assumes the results are the set of values for $x_i$, such that the smallest membership grade among them is maximized.

In the other words, among all the values for $x_i$ that satisfy the relationship, the one that maximizes the minimum membership grade $\mu_{x_i}$; $maxmin\{\mu_{x_i}^-, \mu_{x_i}^+\}$, is chosen as equation (5.1), which is presented in model (3.2) as equation (3.2b).

$$\begin{aligned} \left(\frac{x_i - x_i^l}{x_i^m - x_i^l}\right) = \mu_{x_i}^- \geq h \\ \left(\frac{x_i - x_i^u}{x_i^m - x_i^u}\right) = \mu_{x_i}^+ \geq h \end{aligned} \tag{5.1}$$

where, $\mu_{x_i}^-(x_i)$ and $\mu_{x_i}^+(x_i)$ represent the left and right hand side of the membership of fuzzy number and $h$ is the minimum degree of membership that one of the values of $x_1, x_2, \ldots\ x_{18}$ takes.

Then, by applying equation (5.1), the value of $h$ for all $(i = 1,2, \ldots 18)$ with the estimated number of beds obtained from our proposed method is illustrated in Table 5.9.

Table 5.9

*Left and Right Membership Function Values with the Estimated Number of Beds for Each Fuzzy Numbers Based on the Proposed Method for Data from HTF*

| Ward | Fuzzy numbers | $\boldsymbol{\mu_i^-}$ | $\boldsymbol{\mu_i^+}$ |
|---|---|---|---|
| $Ward_1$ | (1, 4.134, 10) | 0.3190810 | 1.3637913 |
| $Ward_2$ | (11, 24.497, 39) | 0.2963622 | 1.6548300 |
| $Ward_3$ | (1, 2.498, 9) | 0.6675567 | 1.0765918 |
| $Ward_4$ | (1, 4.430, 14) | 0.2915452 | 1.2539185 |
| $Ward_5$ | (4, 12.884, 24) | 0.2251238 | 1.6192875 |
| $Ward_6$ | (11, 24.906, 44) | 0.2876456 | 1.5188017 |
| $Ward_7$ | (3, 11.647, 90) | 0.5782352 | 1.0465458 |
| $Ward_8$ | (1, 4.122, 44) | 0.6406150 | 1.0281358 |
| $Ward_9$ | (30, 57.731, 179) | 0.4327287 | 1.1297199 |
| $Ward_{19}$ | (74, 108.095, 154) | 0.2932981 | 1.5248884 |
| $Ward_{11}$ | (17, 37.415, 154) | 0.5388195 | 1.0807565 |
| $Ward_{12}$ | (24, 43.070, 180) | 0.5243838 | 1.0662382 |
| $Ward_{13}$ | (25, 44.240, 158) | 0.51975052 | 1.0812236 |
| $Ward_{14}$ | (30, 51.659, 174) | 0.5078720 | 1.0871253 |
| $Ward_{15}$ | (1, 35.784, 182) | 0.4312328 | 1.1353067 |
| $Ward_{16}$ | (1, 7.697, 69) | 0.5972824 | 1.0439946 |
| $Ward_{17}$ | (3, 20.474, 61) | 0.3433673 | 1.2831269 |
| $Ward_{18}$ | (27, 44.842, 155) | 0.5044278 | 1.080267 |
| *maxmin*{ $\mu_i^-$ ,$\mu_i^+$ } | | 0.2251238 | |

Table 5.9 provides the values of the left and right membership function of each fuzzy number $i = 1,2,..,18$ with the estimated number of beds obtained by using proposed method. While, by using equation (5.1), the value of h is obtained, and this value $h$ maximizes the minimum membership grade $\mu_{x_i}$ as $maxmin\{\mu_{x_i}^-, \mu_{x_i}^+\}$ is $(h = 0.225123818)$ by our proposed method. The value $h$ that maximizes the minimum membership grade with the estimated number of beds obtained from Kikuchi's' method is $(h = 0.3966823)$ as displayed in Table 5.8.

It is clear that the estimated number of beds obtained by using the proposed method provides a minimum value of $h$ that maximize the minimum membership grade in comparison to the solution obtained by the Kikuchi's method.

In this research, the results of the proposed method work under the assumption of the nearest point, where the results reach the nearest point of the fuzzy numbers better than the points reached by the A&Z method. Since the Kikuchi's method and the proposed method work under the same concept that is the relationship in the original crisp data need to be satisfied in the crisp outputs, the results from the Kikuchi's method and proposed method are compared using the equation (2.24) by finding the minimum distance between each crisp output and its fuzzy number.

The results obtained by equation (2.24) based on the Kikuchi's method and proposed method are presented in Table 5.10.

Table 5.10

*Minimum Distance between Each Crisp Output and Its Fuzzy Number Based on the Kikuchis' Method and the Proposed Method for Data from HTF*

| Wards | Fuzzy numbers | Distance based on the Kikuchis' method | Distance based on the proposed method |
|---|---|---|---|
| $Ward_1$ | (1, 4.134, 10) | 4.5 | 5.4 |
| $Ward_2$ | (11, 24.497, 39) | 15.8 | 17.9 |
| $Ward_3$ | (1, 2.498, 9) | 4.2 | 4.2 |
| $Ward_4$ | (1, 4.430, 14) | 6.9 | 7.8 |
| $Ward_5$ | (4, 12.884, 24) | 11.2 | 13.3 |
| $Ward_6$ | (11, 24.906, 44) | 18.8 | 20.8 |
| $Ward_7$ | (3, 11.647, 90) | 49.4 | 48.5 |
| $Ward_8$ | (1, 4.122, 44) | 24.0 | 24.0 |
| $Ward_9$ | (30, 57.731, 179) | 85.3 | 84.4 |
| $Ward_{19}$ | (74, 108.095, 154) | 46.2 | 50.4 |
| $Ward_{11}$ | (17, 37.415, 154) | 77.6 | 75.8 |
| $Ward_{12}$ | (24, 43.070, 180) | 89.0 | 87.2 |
| $Ward_{13}$ | (25, 44.240, 158) | 75.8 | 74.0 |
| $Ward_{14}$ | (30, 51.659, 174) | 82.1 | 80.3 |
| $Ward_{15}$ | (1, 35.784, 182) | 103.5 | 102.6 |
| $Ward_{16}$ | (1, 7.697, 69) | 38.7 | 37.8 |
| $Ward_{17}$ | (3, 20.474, 61) | 33.3 | 34.3 |
| $Ward_{18}$ | (27, 44.842, 155) | 72.5 | 71.6 |

Table 5.10 shows the results of calculation of minimum distance between crisp outputs and the fuzzy numbers by using equation (2.24). It can be seen that the proposed method has better approximated crisp outputs to the fuzzy numbers in 9 wards, which are $Ward_{7,}$ $Ward_{9,}$ $Ward_{11,}$ $Ward_{12}$ , $Ward_{13,}$ $Ward_{14}$ , $Ward_{715}$ , $Ward_{16}$ , and $Ward_{18}$ since in all cases the minimum distance is less than Kikuchi's. Kikuchi has only six values with less minimum distance compared to our proposed method, while they have two equal values in $Ward_3$ and $Ward_8$. As a whole, our proposed method performs better than Kikuchi's.

### 5.3.2 Implementation with Data from the Malaysia MOH

Data are collected on the number of beds used by patients for the duration of 1096 days. The data of the hospital patients are provided into five categories based on the age of patients as [Toddler (T), Schoolchildren (S), Adult (A), Old (O), and Elderly (E)], with the number of patients recorded every day in seven Malaysian hospitals. Because the geometric mean is used to find the middle point of each fuzzy number and also the DEA model is used, the data with zero observations need to be adjusted. For these reasons, we replace the days with zero value of the recorded number of patients into one, which include the days (3/9/2008; 19/9/2009; 26/3/2010 and 19/6/2010).

Table 5.11 illustrates the five groups of fuzzy numbers with geometric mean as $\tilde{x}_i = \left(x_i^l, x_i^{gem}, x_i^u\right),\ i = 1,2,3,4,5$ by using equations (4.2).

Table 5.11
*Generated Fuzzy Numbers for Each Group of Patients for Data from Malaysia MOH*

| | Toddler | School-children | Adult | Old | Elderly |
|---|---|---|---|---|---|
| Min | 9 | 1 | 3 | 8 | 7 |
| Geometric mean | 37.083 | 7.359 | 19.120 | 31.956 | 54.848 |
| Max | 95 | 38 | 77 | 67 | 89 |
| Fuzzy Number | (9,37.083,95) | (1,7.359,38) | (3,19.120,77) | (3,19.120,77) | (7,54.848,89) |

After generating fuzzy numbers to each group of patients as shown in Table 5.11, we start partitioning each interval of the fuzzy numbers as follows.

When $m = 1$, it means that the interval is considered one part (i.e. one interval) and from equation (4.4) and (4.5), the element of the corresponding interval is $x_0 = x^l$, $x_1 = x^u$ and the interval is $[x^l, x^u]$. This indicates that no result could be obtained that satisfy the objectives or constraints when $m = 1$. Then, when $m = 2$, the interval is partitioned into two sub-intervals indicating that the results start to appear from this partition $m = 2$ until $m = 500$. Table 5.12 illustrates the crisp outputs of each of the five groups of fuzzy numbers with the number of partitions (sub-intervals).

Table 5.12

*Results of the Proposed Method for Data from Malaysia MOH*

| No | Groups of patient | Fuzzy numbers | Crisp outputs | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| i | | $(x_i^l, x_i^m, x_i^u)$ | m=2 | m=10 | m=30 | m=51 | m=52 | m=100 | m=500 |
| 1 | Toddler | (9,37.083,95) | 52 | 44 | 46 | 45 | 45 | 45 | 45 |
| 2 | Schoolchildren | (1,7.359,38) | 20 | 15 | 14 | 14 | 14 | 14 | 14 |
| 3 | Adult | (3,19.120,77) | 40 | 32 | 30 | 31 | 31 | 31 | 31 |
| 4 | Old | (8,31.956,67) | 37 | 37 | 35 | 35 | 35 | 35 | 35 |
| 5 | Elderly | (7,54.848,89) | 48 | 48 | 51 | 51 | 51 | 51 | 51 |
| Sum of the estimated No. of beds | | | 197 | 176 | 176 | 176 | 176 | 176 | 176 |

As shown in Table 5.12, the proposed method presents different results and different summation of the estimated number of beds under each partition. The total number of partition is chosen to be 500 is done at random, and if the stable result does not appear until that value, the number of partitions has to be increased. However, the results start stabilizing at $m = 51$, where the optimal solution is considered. In this case, stable results mean that estimated number of beds for each group is the same even though the number of partitions is more than 51. Besides that, the sum of the estimated number of beds is always equal to a specific value, 176, even though we do not define any relationship to be satisfied in the crisp output by ignoring all constraints that present the relationships in model (4.10).

### 5.3.2.1 Results of COG and A&Z with Data from Malaysia MOH

In order to validate the proposed method in the case where no relationships to be satisfied in the crisp outputs (independent crisp outputs), we use two defuzzification methods, namely, COG and the method by A&Z.

Table 5.13
*Results of the COG and A&Z Methods for Data from Malaysia MOH*

| No. | Fuzzy numbers $(x_i^l, x_i^{geo}, x_i^u)$ | COG | A&Z |
|---|---|---|---|
| 1 | (9,37.083,95) | 47 | 45 |
| 2 | (1,7.359,38) | 15 | 14 |
| 3 | (3,19.120,77) | 33 | 31 |
| 4 | (8,31.956,67) | 36 | 35 |
| 5 | (7,54.848,89) | 50 | 51 |
| Sum of the estimated No. of beds | | 181 | 176 |

Table 5.13 shows that the results obtained by the A&Z method are the same as those obtained by the proposed method in Table 5.12, which appeared in partition $m = 51$. However, the results obtained using the COG method are different from those obtained by the proposed method  and A&Z method with a different total number of estimated numbers of beds. In terms of the distance between each fuzzy number and the crisp output obtained by the A&Z method and the proposed method, both

methods give the same smallest distance which are calculated using equation (2.24) as presented in Table 5.14.

Table 5.14
*Smallest Distance Based on Crisp Outputs of Proposed Method and A&Z Method for Data from Malaysia MOH*

| No. | Fuzzy numbers $(x_i^l, x_i^{geo}, x_i^u)$ | The smallest distance based the proposed method | The smallest distance based the A&Z method |
|---|---|---|---|
| 1 | (9,37.083,95) | 35.6395 | 35.6395 |
| 2 | (1,7.359,38) | 15.9181 | 15.9181 |
| 3 | (3,19.120,77) | 31.4559 | 31.4559 |
| 4 | (8,31.956,67) | 24.1958 | 24.1958 |
| 5 | (7,54.848,89) | 33.5982 | 33.5982 |

This finding leads us to say that the proposed method gives the nearest point to the fuzzy numbers in case of no relationships. Furthermore, this finding and the results obtained in Table 5.10 support our proposed method with its new suggestion concept that the crisp output is the best nearest point in the case where some relationships in the original crisp data need to be satisfied in the crisp outputs.



## 5.4 Implementation of the Proposed Method Based on LP Model

In order to compare the performance of the proposed method based on CCR-DEA model, a special case of our proposed method based on LP model is developed as discussed in Section 4.7.3.4 and model (4.11). This is probable because the relationship $R(\bar{x}_i)$ and the membership function $\mu_{\tilde{x}_i}(x_{i_k})$ are linear. In this case, we consider the data with and without relationships to estimate the number of beds. In the first case, we consider the problem of HTF, that is, data with relationships.

Table 5.15

*Results Based on LP Model with Geometric Mean for Data from HTF in Years (2013 and 2014)*

| Ward | Fuzzy numbers | Crisp outputs | | | | | | |
|---|---|---|---|---|---|---|---|---|
| i | $(x_i^l, x_i^{gem}, x_i^u)$ | m=2 | m=10 | m=50 | m=101 | m=102 | m=250 | m=500 |
| Ward$_1$ | (1,4.134,10) | 1 | 1 | 2 | 2 | 2 | 2 | 2 |
| Ward$_2$ | (11,24.497,39) | 11 | 13 | 15 | 15 | 15 | 15 | 15 |
| Ward$_3$ | (1,2.498,9) | 1 | 1 | 2 | 2 | 2 | 2 | 2 |
| Ward$_4$ | (1,4.430,14) | 1 | 1 | 2 | 2 | 2 | 2 | 2 |
| Ward$_5$ | (4,12.884,24) | 4 | 6 | 6 | 7 | 7 | 7 | 7 |
| Ward$_6$ | (11,24.906,44) | 11 | 11 | 15 | 15 | 15 | 15 | 15 |
| Ward$_7$ | (3,11.647,90) | 3 | 11 | 8 | 8 | 8 | 8 | 8 |
| Ward$_8$ | (1,4.122,44) | 1 | 5 | 3 | 3 | 3 | 3 | 3 |
| Ward$_9$ | (30,57.731,179) | 30 | 44 | 42 | 42 | 42 | 42 | 42 |
| Ward$_{19}$ | (74,108.095,154) | 74 | 76 | 84 | 84 | 84 | 84 | 84 |
| Ward$_{11}$ | (17,37.415,154) | 17 | 30 | 28 | 27 | 27 | 27 | 27 |
| Ward$_{12}$ | (24,43.070,180) | 24 | 39 | 34 | 35 | 34 | 34 | 34 |
| Ward$_{13}$ | (25,44.240,158) | 91 | 38 | 35 | 35 | 35 | 35 | 35 |
| Ward$_{14}$ | (30,51.659,174) | 30 | 44 | 41 | 40 | 41 | 41 | 41 |
| Ward$_{15}$ | (1,35.784,182) | 1 | 14 | 16 | 16 | 16 | 16 | 16 |
| Ward$_{16}$ | (1,7.697,69) | 24 | 7 | 5 | 5 | 5 | 5 | 5 |
| Ward$_{17}$ | (3,20.474,61) | 32 | 3 | 9 | 9 | 9 | 9 | 9 |
| Ward$_{18}$ | (27,44.842,155) | 27 | 39 | 36 | 36 | 36 | 36 | 36 |
| Sum of the estimated No. of beds | | 383 | 383 | 383 | 383 | 383 | 383 | 383 |



Table 5.15 summarizes the results of the proposed method with LP model for the number of partitions that start from $m = 2$ to $m = 500$ and the estimated number of beds that satisfied the relationships. The stable results under this model appeared in $m = 102$. Meanwhile, the results in the case of using the proposed method with DEA concept as in Table 5.4 show that the stable results appear in $m = 50$. This finding leads us to suggest that the proposed method with the DEA concept is superior to the proposed method using the LP concept since the stable results appeared earlier, that is, at $m = 50$ while with LP the results became stable at $m = 102$.

Table 5.16

*Results of the Proposed Method based LP-Model with Geometric Mean of Data from Malaysia MOH.*

| No | Groups of patient | Fuzzy numbers | Crisp outputs | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| i | | $(x_i^l, x_i^m, x_i^u)$ | m=2 | m=10 | m=30 | m=51 | m=52 | m=100 | m=500 |
| 1 | Toddler | (9,37.083,95) | 52 | 44 | 46 | 45 | 45 | 45 | 45 |
| 2 | Schoolchildren | (1,7.359,38) | 20 | 15 | 14 | 14 | 14 | 14 | 14 |
| 3 | Adult | (3,19.120,77) | 40 | 32 | 30 | 31 | 31 | 31 | 31 |
| 4 | Old | (8,31.956,67) | 37 | 37 | 35 | 35 | 35 | 35 | 35 |
| 5 | Elderly | (7,54.848,89) | 48 | 48 | 51 | 51 | 51 | 51 | 51 |
| | Sum of the estimated No. of beds | | 197 | 176 | 176 | 176 | 176 | 176 | 176 |

In addition, Table 5.16 shows the results of the proposed method based on LP model when no relationships in the original crisp data needs to be satisfied in the crisp outputs using the data from the database of the Ministry of Health. The table shows that the proposed method under DEA and LP concepts give stable results under the same number of partition, i.e. $m = 51$.



## 5.5 Application of the Proposed Method with Numerical Examples

Another research objective to achieve is related to the applicability of the proposed method to solve other problems. It is applied to numerical examples in two different applications. Firstly, a new insight of the interval weight (IW) method in goal programming (GP) is presented to fulfill the optimal weights. Secondly, the application of the proposed method for ranking fuzzy numbers is introduced. In the previous sections, the method is applied to only triangular fuzzy numbers. Here, the ranking of each triangular and trapezoidal fuzzy numbers is presented.

### 5.5.1 Interval Weights

The majority of the real-world problems are characterized by their multiple objectives, which may contradict one another. In order to determine solutions to these problems, GP was proposed by Charnes and Cooper (1961) as an effective method. The relative importance of one objective over another in the context of multi-objectives programming (MOP) is defined as the first objective's weight. In this regard, weights are significant in the determination of a solution to a problem based on diverse DMs subjective requirements. Weights related to unwanted deviational variables in GP gauge the relative significance of the relative objective. Various approaches deriving weights or priorities have been explored in earlier studies as mentioned in Chapter Three.

Research on IW associated with unwanted deviational variables in the weighted goal programming (WGP) or fuzzy weighted goal programming (FWGP) area remains lacking in the literature. To discuss such an uncertain weight structure, weights associated with unwanted deviational variables in the goal achievement function have been considered as a fuzzy interval form in the proposed approach. The interval weight goal programming (IWGP) methodology is an appropriate technique for solving this problem. Owing to the involvement of interval uncertainty in MOP problems, the interval programming approach is an ideal method to be applied. Therefore, in order to shed light on the uncertain weight structure, weights related to unwanted deviational variables in the goal achievement function are deemed as fuzzy numbers form in the proposed method. At this end, the fuzzy interval weight goal programming (FIWGP) method is ideally designed to solve this problem.

Moreover, the target achievement function is presented as the unwanted deviational variables weighted summation, where weights are regulated through the use of a pairwise interval judgment matrix in the GP method. At this point, the issue is in the form of interval programming, where interval goals are transformed into standard ones through IWGP (Wang & Elhag, 2007). The sum of unwanted deviations in relation to respective goals is believed to achieve the desired goal values within a particular range after which the regret function of the final executable model is developed. In other words, the problem is resolved via a standard GP methodology. The primary advantage of our proposed method is that the suitable weights for achieving goals can be apportioned in the approximate decision environment based on their significance.

On the other hand, prior studies focused on the interval as (min, max) which includes just two extreme values from all responses (Sen & Pal, 2013). Since the weights are given by the DMs who have different opinions, background, and experience, these responses should be treated in a fuzzy environment. Therefore, the need arises to find a method covering all weights that come from the DMs. In order to solve such cases, the interval weight is represented as a $TrFN$ with a geometric mean as a middle point for this fuzzy number. Since the fuzzy numbers generalize closed intervals (Bede, 2013), this fuzzy number is defined as a fuzzy interval weight (FIW) under the problem of solving and of finding the optimal weights in GP. Hence, this research contributes by proposing fuzzy IWs to determine solutions for MOLP problems. In doing so, this study specifically applies the proposed method to defuzzify these groups of fuzzy numbers representing IWs based on DEA.

Figure 5.2 illustrates the algorithm of solving a MOLP problem based on the FIW and IW methods using GP and FGP techniques.

| |
|---|
| **Step 1**: Defining the aspiration levels of each objective goals. |
| **Step 2**: Determining the membership goals associated with each objective. |
| **Step 3**: Considering the weights. |
| **Step 4:** Finding the optimal weight. |
| **Step 5**: Solving an MOLP problem using GP and FGP. |

*Figure 5.2.* Algorithm of solving an MOLP using GP and FGP

Two examples of a MOLP problem are addressed in this section to compare the results based on the optimal weight using FIW and IW method as follows:

**Example 5.5.1.1**

Consider a three objective problem presented in Sen and Pal (2013) as follows:

$$\begin{aligned} &max\, Z_1 = 70x_1 - 30x_2 \\ &max\, Z_2 = 3x_1 + 8x_2 \\ &max\, Z_3 = -\, 4x_1 + x_2 \\ &\quad subjected\ to \\ &\qquad 2x_1 + x_2 \geq 8 \\ &\qquad x_1 + x_2 \geq 5 \\ &\qquad x_1 - 2x_2 \geq -6 \\ &\qquad 5x_1 - 2x_2 \leq 18 \\ &\qquad x_1, x_2 \geq 0 \end{aligned} \tag{5.2}$$

The solution details of this MOLP problem are obtained based on the algorithm in Figure 5.2 in the following steps. In terms of the algorithm introduced in Section 4.7, the proposed method (FIW) is used in Step 4 to get the optimal weight, and then these weights are used in solving the MOLP problem by using GP model.

**Step 1**: Consider the calculated individual minimum and maximum values of each the three objectives $Z_1(x)$, $Z_2(x)$ and $Z_3(x)$ as shown in Table 5.17.

Table 5.17

*Individual Minimum and Maximum Values of Each of Objective Functions*

| Objective functions | Minimum $(Z_i(x))=L_i$ | Maximum $(Z_i(x))=U_i$ |
|---|---|---|
| $Z_1(x)$ | 20 | 250 |
| $Z_2(x)$ | 20 | 66 |
| $Z_3(x)$ | -18 | -4 |



**Step 2**: Determine the membership goals associated with each objective that can be expressed as shown in Figure 5.3.

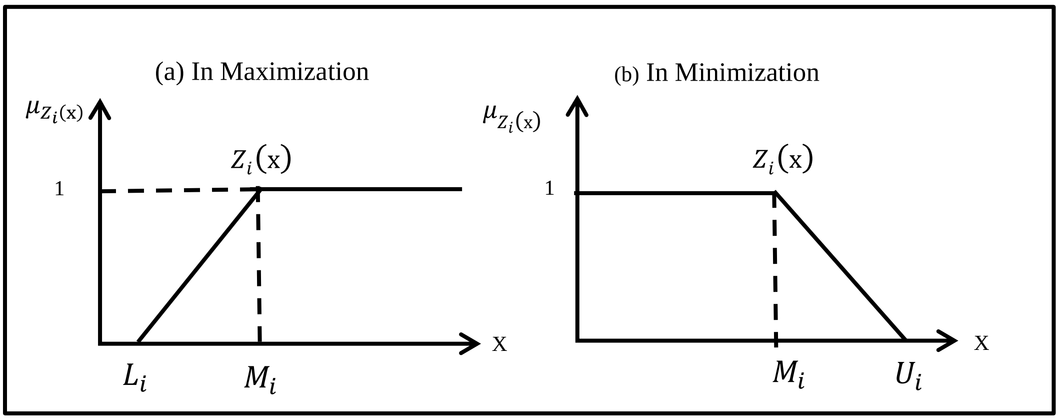



*Figure 5.3.* Membership associated with maximization and minimization objectives

The generic form of the FGP problem can be presented as:

$$Z_i(X) \gtrsim M_i \qquad i = 1,2,\dots,I_1 \tag{5.3}$$

$$Z_i(X) \lesssim M_i \qquad i = (I_1+1),(I_1+2),\dots,I$$

subject to

$$X \in F = \left\{ X \in R^n \middle| \begin{Bmatrix} AX \geq C \\ AX \leq C \end{Bmatrix}, X \geq 0, C \in R^m \right\}$$

Now, description of fuzzy goals is defined as follows:

The fuzzy goals take the form of either $Z_i(X) \gtrsim M_i$ or $Z_i(X) \lesssim M_i$, depending on whether the objectives should be maximized or minimized, where $M_i$ is the imprecise aspiration level of the $i^{th}$ objective, $X$ is the vector of the decision variables and $\gtrsim$ and $\lesssim$ represent the fuzziness of $\geq$ and $\leq$ restrictions, respectively (Ehrgott & Gandibleux, 2003; Zimmermann, 1978). In a decision-making situation, fuzzy goals are characterized by their respective membership functions. Thus, the following membership function, which corresponds to each objective function are introduced as:

The membership function $\mu_i$ for the $i^{th}$ fuzzy goal maximizing $Z_i(x)$ can be described as;

$$\mu_i(x) = \begin{cases} 1 & Z_i(x) \geq M_i \\ \dfrac{Z_i(x) - L_i}{M_i - L_i} & L_i \leq Z_i(x) \leq M_i \\ 0 & Z_i(x) < L_i \end{cases} \tag{5.4}$$

where, $i = 1,2,\dots,I_1$

On the other hand, the membership function $\mu_i$ for the $i^{th}$ fuzzy goal for minimizing $Z_i(x)$ can be defined as.

$$\mu_i(x) = \begin{cases} 1 & Z_i(x) \leq M_i \\ \dfrac{U_i - Z_i(x)}{U_i - M_i} & M_i \leq Z_i(x) \leq U_i \\ 0 & Z_i(x) \geq U_i \end{cases} \tag{5.5}$$

where, $i = (I_1 + 1), (I_1 + 2), \ldots, I,$ $L_i$ is the minimum value for the $i^{th}$ fuzzy goal and $U_i$ is the maximum value for the $i^{th}$ fuzzy goal, $i = 1,2, \ldots, I_1$, is the number of goals under maximization and $i = (I_1 + 1), (I_1 + 2), ., I$ is the number of goals under minimization and $I$ is the total number of goals.

Consequently, the membership goals of the defined membership functions with the highest membership value (unity) are presented as follows:



$$\frac{Z_i(x) - L_i}{M_i - L_i} + \rho_i^- - \rho_i^+ = 1 \qquad i = 1,2, \ldots, I_1 \tag{5.6}$$

$$\frac{U_i - Z_i(x)}{U_i - M_i} + \rho_i^- - \rho_i^+ = 1 \qquad i = (I_1 + 1), (I_1 + 2), \ldots, I \tag{5.7}$$

where, $\rho_i^-, \rho_i^+ \geq 0$ are under and over-deviational variables concerned with achieving the aspired level of the $i^{th}$ membership goal.

Since the goals in this example are all in maximization, so the membership goals associated with each objective can be expressed using equations (5.4) and (5.6) as follows:

$$\begin{aligned}
&(1/230)(70\,x_1 - 30\,x_2 - 20) + \rho_1^- - \rho_1^+ = 1 \\
&(1/46)(3\,x_1 + 8\,x_2 - 20) \quad + \rho_2^- - \rho_2^+ \ = 1 \\
&(1/14)(-4\,x_1 + \ x_2 - 18) + \rho_3^- - \rho_3^+ \ \ = 1
\end{aligned} \tag{5.8}$$

**Step 3:** In order to explain the approach further, weights are used as presented in (Chaloob, Ramli, & Nawawi, 2016) with the geometric mean as the middle point. The summary of weights is given in Table 5.18.

Table 5.18
*Summary of the Weights*

| | Weight 1 | Weight 2 | Weight 3 |
|---|---|---|---|
| Max weight | 0.799 | 0.511 | 0.231 |
| Min weight | 0.42 | 0.077 | 0.059 |
| Geometric mean | 0.603 | 0.216 | 0.085 |

Source: (Chaloob et al., 2016)

The imprecise pairwise comparison matrix could be presented by using the formulations defined in equation (3.20) as follows:

$$\left.\begin{aligned}
\mathbb{A}^L &= \begin{pmatrix} 1 & 0.822 & 1.818 \\ 0.096 & 1 & 0.333 \\ 0.074 & 0.115 & 1 \end{pmatrix} \\
\mathbb{A}^U &= \begin{pmatrix} 1 & 10.377 & 13.542 \\ 1.217 & 1 & 8.661 \\ 0.550 & 3.0 & 1 \end{pmatrix}
\end{aligned}\right\} \tag{5.9}$$

**Step 4**: This step consists of two parts. The first part focuses on how to obtain the weights by using our proposed method FIW, while the second part concerns about how to get the weights by using the IW. This part would solve the problem of finding weights in the example.

**i. Obtaining weights using proposed method FIW**

From Table 5.18, and equation (4.2) the fuzzy number is $\widetilde{W}_i = \left(W_i^L, W_i^{mid}, W_i^U\right)$, which are the representative of the real weights are obtained as follows:

$$\widetilde{W}_1 = (0.42, 0.603, 0.799)$$
$$\widetilde{W}_2 = (0.077, 0.216, 0.511)$$
$$\widetilde{W}_3 = (0.059, 0.085, 0.231)$$

By applying the proposed method using model (4.12), we divided each interval $[W_i^L, W_i^U]$ into $m = 100$, sub-intervals. Until we get a stable weight, we stop the partition process. If not, we increase the number of partitions. The optimal weights are obtained in $m = 47$ partitions, as follows;

$$W_1 = 0.61\,;\; W_2 = 0.27;\; W_3 = 0.12 \tag{5.10}$$

**ii. Obtaining weights using interval weight IW**

By using the pairwise comparison matrix in (5.9) and the GP model for determination of weights (in an interval form) based on equations (3.21, 3.22, 3.23, and 3.24) as in Sen and Pal (2013). The formulation of IW can be presented as:

$$min\,Z = \sum_{j=1}^{2}\sum_{i=1}^{3}\left(d_{ji}^{-} + d_{ji}^{+}\right) \quad (5.11)$$

subjected to

$$(0)w_1^U + (0.822)w_2^U + (1.818)w_3^U - (2)w_1^L + d_{11}^- - d_{11}^+ = 0$$

$$(0.096)w_1^U + (0)w_2^U + (0.333)w_3^U - (2)w_2^L + d_{12}^- - d_{12}^+ = 0$$

$$(0.074)w_1^U + (0.115)w_2^U + (0)w_3^U - (2)w_3^L + d_{13}^- - d_{13}^+ = 0$$

$$(0)w_1^L + (10.377)w_2^L + (13.542)w_3^L - (2)w_1^U + d_{21}^- - d_{21}^+ = 0$$

$$(1.217)w_1^L + (0)w_2^L + (8.661)w_3^L - (2)w_2^U + d_{22}^- - d_{22}^+ = 0$$

$$(0.550)w_1^L + (3.0)w_2^L + (0)w_3^L - (2)w_3^U + d_{23}^- - d_{23}^+ = 0$$

$$w_1^L + w_2^U + w_3^U \geq 1\,,\ w_2^L + w_1^U + w_3^U \geq 1\,,\ w_3^L + w_1^U + w_2^U \geq 1$$

$$w_1^U + w_2^L + w_3^L \leq 1,\ w_2^U + w_1^L + w_3^L \leq 1\,,\ w_3^U + w_1^L + w_2^L \leq 1$$

$$w_1^U \geq w_1^L\,,\quad w_2^U \geq w_2^L\,,\quad w_3^U \geq w_3^L$$



The weights as intervals using LINGO (ver.14) are as follows:

$$\left.\begin{aligned}[w_1^L, w_1^U] &= [0.54, 0.78]\\ [w_2^L, w_2^U] &= [0.10, 0.38]\\ [w_3^L, w_3^U] &= [0.08, 0.12]\end{aligned}\right\} \quad (5.12)$$

At the end of this step, two set of weights are obtained as the first step to solve the examples.

**Step 5:** By using the results of weights obtained from (5.10) and (5.12), the GP formulation can be expressed as follows:

$$min\,G = 0.61\rho_1^- + 0.27\rho_2^- + 0.12\rho_3^- \quad (5.13)$$

$$min\,G = [0.54, 0.78]\rho_1^- + [0.10, 0.38]\rho_2^- + [0.08, 0.12]\rho_3^- \quad (5.14)$$

to satisfy

$$\left.\begin{array}{l}(1/230)(70\,x_1 - 30\,x_2 - 20) + \rho_1^- - \rho_1^+ = 1\\ (1/46)(3\,x_1 + 8\,x_2 - 20) \quad + \rho_2^- - \rho_2^+ \quad = 1\\ (1/14)(-4\,x_1 + \,x_2 - 18) \quad + \rho_3^- - \rho_3^+ \quad = 1\end{array}\right\} \tag{5.15}$$

subject to

$$\left.\begin{array}{r}2x_1 \;+ x_2 \geq 8\\ x_1 \;+\; x_2 \;\geq 5\\ x_1 - 2x_2 \geq -6\\ 5x_1 - 2x_2 \leq 18\\ x_1, x_2 \qquad \geq 0\end{array}\right\} \tag{5.16}$$

First, the objective function in (5.13) based on the weights obtained by using proposed method (fuzzy interval weight). The GP and FGP results are;

$$(x_1, x_2) = (6, 6), \text{ with } (Z_1, Z_2, Z_3) = (250, 66, -18).$$

Consequently, using interval arithmetic, the objective function in (5.14) can be solved to obtain the target interval as follows: $[t_1^L, t_1^U] = [0.1720497,\ 0.2567702]$. Using the procedure defined in Sen and Pal (2013), the goal expression can be written as follows:

$$\left.\begin{array}{l}(0.54)\rho_1^- + (0.10)\rho_2^- + (0.08)\rho_3^- + \gamma_{1L}^- - \gamma_{1L}^+ \;= 0.1720497\\ (0.78)\rho_1^- + (0.38)\rho_2^- + (0.12)\rho_3^- + \gamma_{1U}^- - \gamma_{1U}^+ = 0.2567702\end{array}\right\} \tag{5.17}$$

Then, the executable GP model can be expressed as follows:

$$min\; G = \gamma_{1L}^- + \gamma_{2U}^+$$

Furthermore, the goal relations and a set of constraints in (5.15) and (5.16) are satisfied. The problem is solved using LINGO (Ver. 14.0) and the solution is obtained as follows:

The GP results are $(x_1,\ x_2) = (3, 2)$, with $(Z_1,\ Z_2,\ Z_3) = (150, 25, -10)$.

The FGP the results are $(x_1,\ x_2) = (4, 1)$, with $(Z_1,\ Z_2,\ Z_3) = (250, 20, -15)$.

The preceding results show that the FIW provides optimal weights with the main advantage of giving better results in terms of an optimal solution when the main interval is divided into sub-intervals.

Table 5.19 illustrates the advantages of the proposed method (FIW) compared to the conventional interval weight (IW) in achieving the objective values of standard and fuzzy objectives in GP.



Table 5.19

*Comparison of the Objective Values Obtained Under FIW and IW*

| Methods | $X_1$ | $X_2$ | $Z_1$ | $Z_2$ | $Z_3$ |
|---|---|---|---|---|---|
| FIW by the proposed method under GP&FGP | 6 | 6 | 250 | 66 | -18 |
| IW under GP | 3 | 2 | 150 | 25 | -10 |
| IW under FGP | 4 | 1 | 250 | 20 | -15 |

The results in Table 5.19 show that the FIW method gives optimal solutions for each $X_1 = 6$ and $X_2 = 6$ under standard GP and FGP, where these values give a maximum solution for each objective. The FIW method achieves better objective values than the conventional interval weight (IW).

**Examples 5.5.1.2**

Consider the following MOLP problem with three objective presented in (Winston & Goldberg, 2004) as follows;

$$
\begin{aligned}
\max Z_1 &= 2x_1 + 5x_2 + x_3 \\
\min Z_2 &= 4x_1 - 3x_2 + x_3 \\
\min Z_3 &= x_2 - 2x_3 \\
&\text{subjected to} \\
&4x_1 + x_3 \leq 10 \\
&5x_1 + x_2 \leq 20 \\
&x_1, x_2, x_3 \geq 0
\end{aligned}
\tag{5.18}
$$

By considering the algorithm steps in Figure 5.3, the calculated individual minimum and maximum values of each the three objective $Z_1(x)$, $Z_2(x)$ and $Z_3(x)$, are shown in Table 5.20.

Table 5.20

*Individual Minimum and Maximum Values of Each of Objective Functions in Example 5.5.1.2*

| Objectives | Minimum $(Z_i(x))=L_i$ | Maximum $(Z_i(x))=U_i$ |
|---|---|---|
| $Z_1(x)$ | 50 | 110 |
| $Z_2(x)$ | -60 | -30 |
| $Z_3(x)$ | -20 | 20 |

Since the three goals in this example are one maximization and two minimizations, the membership goals associated with each objective can be expressed using equations (5.4), (5.6) and (5.5), (5.7) as follows:

$$\left.\begin{array}{ll}(1/60)(2x_1 + 5x_2 + x_3 - 50) + \rho_1^- - \rho_1^+ & = 1 \\ (1/30)(-4x_1 + 3x_2 - x_3 - 30) \quad + \rho_2^- - \rho_2^+ & = 1 \\ (1/40)\ (20 - x_2 + 2x_3) + \rho_3^- - \rho_3^+ & = 1\end{array}\right\}$$

By applying the same pairwise comparison matrix of weights used in example 5.5.1.1, the problem is solved, and the solution is obtained as described in Table 5.21. In this example, we use the same values of weights obtained in example 5.5.1.1 since the purpose of this example is to show that the same set of weights can also give optimal solutions for a different problem.

Table 5.21

*Comparison of Objective Values Obtained under FIW and IW*

| Methods | $X_1$ | $X_2$ | $X_3$ | $Z_1$ | $Z_2$ | $Z_3$ |
|---|---|---|---|---|---|---|
| FIW by the proposed method under GP | 0 | 20 | 0 | 100 | -60 | 20 |
| FIW by the proposed method under FGP | 0 | 20 | 10 | 110 | -50 | 0 |
| IW under GP&FGP | 0 | 0 | 0 | 0 | 0 | 0 |

From Table 5.21, even though we use the same matrix weights from the first example, 5.5.1.1 in the second example, 5.5.1.2, we find that the optimal solution of problem is by using weights from the proposed method. This means that the proposed method whether under GP or FGP is better than the IW under GP & FGP method.

### 5.5.2 Ranking of Fuzzy Numbers

This section discusses the problem of ranking fuzzy numbers. A critical issue that appears in fuzzy numbers is how to compare and rank them. As mentioned in Section 2.6, many ranking methods have been developed to rank fuzzy numbers. Each of the ranking methods has its strengths and shortcomings. As mentioned in Xu and Zhai (2012), A&Z method and Abbasbandy and Hajjari (A&H) (2009) method give the same order to two different fuzzy numbers. In this section, another application of the proposed method is addressed to solve the problem of ranking of fuzzy numbers with some numerical examples from previous studies.

In this example, the proposed method is used as a case of independent crisp output. It should be noted that the proposed method could handle one of the shortcomings of the previous methods by ranking the fuzzy numbers as groups with any number of them at the same time.



Up to now, the discussion of the application of the proposed method only deals with triangular fuzzy numbers. So, in this section, the ranking problem with triangular fuzzy numbers, trapezoidal fuzzy numbers, and non-linear fuzzy numbers are constructed. For this purpose, the comparison between the results of the proposed method and those of the methods presented in Nejad and Mashinchi (2011) and Xu and Zhai (2012) is considered in the following examples.

**Example 5.5.2.1:** Consider a five sets of fuzzy numbers presented in Nejad and Mashinchi (2011) as follows:

Set 1: $A_1$ = (1, 1, 3), $A_2$ = (1, 1, 7);
Set 2: $A_1$ = (2, 4, 6), $A_2$ = (1, 5, 6), $A_3$ = (3, 5, 6).
Set 3: $A_1$ = (2, 3, 8), $A_2$ = (2, 3, 7, 8), $A_3$ = (2, 3, 10)
Set 4: $A_1$ = (1, 5, 5), $A_2$ = (2, 3, 5, 5);
Set 5: $A_1$ = (2, 4, 6), $A_2$ = (1, 5, 6).

The ranking orders by numerous methods are described in Table 5.22 (as cited in Nejad and Mashinchi, 2011). The orders are obtained from Wang et al.'s (2009) method for Set 1 and Set 2 and the CV Uniform Distribution by Cheng (1998) is applied to Set 2, which varies from other methods and the ranking order caused from these methods are improper. The outcomes achieved from the suggested method in these parts are alike to the ones acquired from other methods, except for the cases of the issues mentioned above. The proposed method gives the same ranking order to the fuzzy number $A_1$ and $A_2$, in Set 2 and Set 3 with the same representative value of each of them. This means that the proposed method can handle all triangular fuzzy numbers in Set 1 to Set 5 and find the representative number or the crisp representation for each of them at one time.

It is found that for the proposed method, the number of times the process is run depends on how many types of fuzzy numbers to be ranked. This situation is different for other methods because the number of times the process is run depends on how many fuzzy numbers that has to be ranked. Table 5.22 shows the comparison of the results of the proposed method to those of other methods that are cited in Nejad and Mashinchi (2011).

Table 5.22

*Results of Ranking Fuzzy Numbers in Example 5.5.2.1*

| Methods | Fuzzy number | Set 1 | Set 2 | Set 3 | Set 4 | Set 5 |
|---|---|---|---|---|---|---|
| Wang et al.(2009) | $A_1$ | 0 | 0 | 0 | 0 | 0.792 |
| | $A_2$ | 0 | 0 | 0.444 | 1.100 | 0.784 |
| | $A_3$ | | 1.857 | 0.444 | | |
| Ranking results | | $A_1 \sim A_2$ | $A_1 \sim A_2 \prec A_3$ | $A_1 \prec A_2 \sim A_3$ | $A_1 \prec A_2$ | $A_2 \prec A_1$ |
| Sign distance p = 1 | $A_1$ | 3 | 8 | 8 | 8 | 8 |
| | $A_2$ | 5 | 8.5 | 10 | 7.5 | 8.5 |
| | $A_3$ | | 9.5 | 9 | | |
| Ranking results | | $A_1 \prec A_2$ | $A_1 \prec A_2 \prec A_3$ | $A_1 \prec A_3 \prec A_2$ | $A_2 \prec A_1$ | $A_1 \prec A_2$ |
| Sign distance p = 2 | $A_1$ | 2.309 | 5.889 | 6.218 | 5.944 | 5.889 |
| | $A_2$ | 4.472 | 6.377 | 7.916 | 5.598 | 6.377 |
| | $A_3$ | | 6.831 | 7.257 | | |
| Ranking results | | $A_1 \prec A_2$ | $A_1 \prec A_2 \prec A_3$ | $A_1 \prec A_3 \prec A_2$ | $A_2 \prec A_1$ | $A_1 \prec A_2$ |
| Cheng (1998) distance | $A_1$ | 1.725 | 4.031 | 4.358 | 3.707 | 4.031 |
| | $A_2$ | 3.027 | 4.035 | 5.025 | 3.768 | 4.035 |
| | $A_3$ | | 4.694 | 5.020 | | |
| Ranking results | | $A_1 \prec A_2$ | $A_1 \prec A_2 \prec A_3$ | $A_1 \prec A_3 \prec A_2$ | $A_1 \prec A_2$ | $A_1 \prec A_2$ |
| Chu and Tsao (2002) | $A_1$ | 0.741 | 2 | 1.986 | 1.986 | 2 |
| | $A_2$ | 1.2 | 2.118 | 2.500 | 1.908 | 2.118 |
| | $A_3$ | | 2.374 | 2.222 | | |
| Ranking results | | $A_1 \prec A_2$ | $A_1 \prec A_2 \prec A_3$ | $A_1 \prec A_3 \prec A_2$ | $A_2 \prec A_1$ | $A_1 \prec A_2$ |
| Deng et al., (2006) | $A_1$ | 0.707 | 1.667 | 1.850 | 1.546 | 1.667 |
| | $A_2$ | 1.354 | 1.69 | 2.850 | 2.086 | 1.69 |
| | $A_3$ | | 1.922 | 2.167 | | |
| Ranking results | | $A_1 \prec A_2$ | $A_1 \prec A_2 \prec A_3$ | $A_1 \prec A_3 \prec A_2$ | $A_1 \prec A_2$ | $A_1 \prec A_2$ |
| Cheng(1998) CV uniform | $A_1$ | 0.133 | 0.167 | 0.397 | 0.242 | 0.167 |
| | $A_2$ | 0.667 | 0.292 | 0.433 | 0.151 | 0.292 |
| | $A_3$ | | 0.083 | 0.630 | | |
| Ranking results | | $A_2 \prec A_1$ | $A_2 \prec A_1 \prec A_3$ | $A_3 \prec A_2 \prec A_1$ | $A_1 \prec A_2$ | $A_2 \prec A_1$ |
| Nejad and Mashinchi( 2011) | $A_1$ | 0.014 | 1.35 | 0.145 | 1.778 | 1.277 |
| | $A_2$ | 0.067 | 1.5 | 0.259 | 1.622 | 1.35 |
| | $A_3$ | | 2.63 | 0.231 | | |
| Ranking results | | $A_1 \prec A_2$ | $A_1 \prec A_2 \prec A_3$ | $A_1 \prec A_3 \prec A_2$ | $A_2 \prec A_1$ | $A_1 \prec A_2$ |
| proposed method | $A_1$ | 1.59 | 4 | 4.13 | 3.83 | 4 |
| | $A_2$ | 2.76 | 4.16 | 5 | 3.75 | 4.16 |
| | $A_3$ | | 4.73 | 4.71 | | |
| Ranking results | | $A_1 \prec A_2$ | $A_1 \prec A_2 \prec A_3$ | $A_1 \prec A_3 \prec A_2$ | $A_2 \prec A_1$ | $A_1 \prec A_2$ |



The next examples considered are the cases presented in Xu and Zhai (2012), where the A&Z method and A&H method failed to give the correct order to fuzzy numbers in some situations. Three examples are presented to illustrate the efficiency of our proposed method in giving the correct ranking order. On the other hand, Xu and Zhai (2012) presented an improved method to rank the fuzzy numbers using the distance minimization concept. They proved that some existing fuzzy numbers are ranked in the same order using the methods of A&Z and A&H. This is not acceptable, since these fuzzy numbers do not place in an equivalence class. This means that these methods cannot discriminate some types of fuzzy numbers.

**Example 5.5.2.2**

Consider four fuzzy numbers presented in Xu and Zhai, (2012) as follows;

$A_1$ = (-4, 1, 2), $A_2$ = (-1.75, 0.25, 1.256), $A_3$ = (-7, 2, 3), and $A_4$ = (-2, 0, 1, 1).

Based on the Xu and Zhai, (2012) A&Z's distance minimization method gives the same nearest point to each of fuzzy numbers. Table 5.23 illustrates the results of Xu and Zhai's method, A&Z method, and the proposed method.

Table 5.23

*Results of Ranking of Fuzzy Numbers in Example 5.5.2.2*

| Fuzzy number | Asady and Zendehnam.(2007) | Xu and Zhai, (2012) | Proposed method |
|---|---|---|---|
| $A_1$ | 0 | 6.67 | -0.13 |
| $A_2$ | 0 | 1.54 | -0.02 |
| $A_3$ | 0 | 19.33 | -0.29 |
| $A_4$ | 0 | 2.33 | -0.001 |
| Ranking results | $A_1 \sim A_2 \sim A_3 \sim A_4$ | $A_3 \prec A_1 \prec A_4 \prec A_2$ | $A_3 \prec A_1 \prec A_2 \prec A_4$ |

Table 5.23 shows that A&Z method gives the same order as $A_1 \sim A_2 \sim A_3 \sim A_4$ to the four fuzzy numbers.

On the other hand, the ranking order by using the proposed method and Xu and Zhai's method are $A_3 \prec A_1 \prec A_2 \prec A_4$ and $A_3 \prec A_1 \prec A_4 \prec A_2$ respectively. Since there is no method yet that can always give a satisfactory solution to every situation, in this example, the proposed method provides a different ranking order than Xu and Zhai's method. In this case, we have three different rankings by using three different methods.

**Example 5.5.2.3**

Consider the following two $TrFN$ presented in Xu and Zhai (2012), $A_1$ = (-1, 0, 1), and $A_2$ = (-4, 1, 2).

Table 5.24

*Results of Ranking Fuzzy Numbers in Example 5.5.2.3*

| Fuzzy numbers | Asady and Zendehnam.(2007) | Xu and Zhai, (2012) | Proposed method |
|---|---|---|---|
| $A_1$ | 0 | 0.67 | 0 |
| $A_2$ | 0 | 6.67 | -0.13 |
| Ranking results | $A_1 \sim A_2$ | $A_2 \prec A_1$ | $A_2 \prec A_1$ |

Table 5.24 shows that the proposed method solves the drawback of A&Z method in giving the correct order of fuzzy numbers. Lastly, the proposed method is able to give a ranking order that is similar to that in Xu and Zhai's method as $A_2 \prec A_1$.

**Example 5.5.2.4**

Two types of fuzzy numbers presented in Nejad and Mashinchi (2011) are considered, that are $TrFN$ $A_1$=(1, 2, 5), and a fuzzy number with non-linear membership, $A_2$=(1,2,2,4,1) as show in Figure 5.4.

The non-linear membership function of $A_2$ is described as follows;

$$\mu_{A_2}(x) = \begin{cases} [1-(x-2)^2]^{\wedge(0.5)} & 1 \le x \le 2 \\ [1-\frac{1}{4}(x-2)^2]^{\wedge(0.5)} & 2 \le x \le 4 \\ 0 & otherwise \end{cases}$$

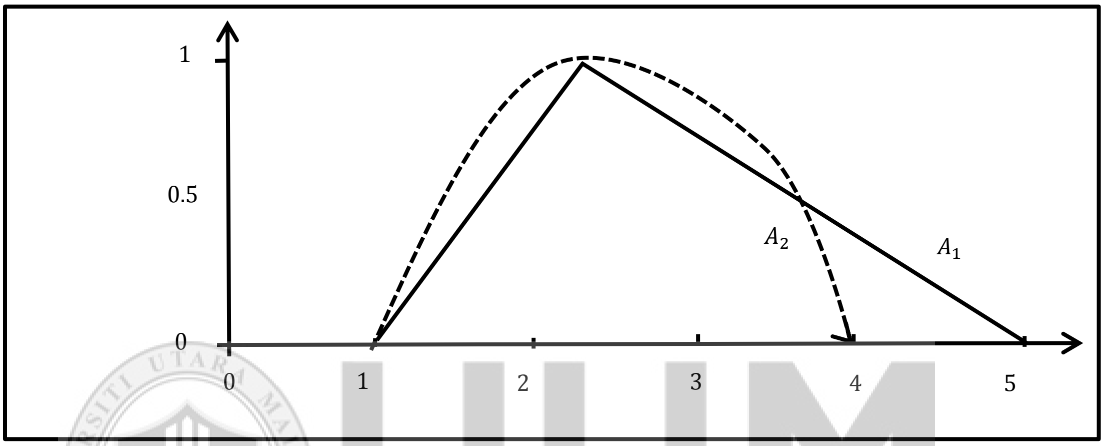



*Figure 5.4.* Fuzzy numbers in example 5.5.2.4

Using the proposed method, we get the crisp output of each fuzzy number $A_1$ = 2.55 and $A_2$ = 2.4 respectively. Therefore, the ranking order is $A_2 \prec A_1$. In comparing the results of the proposed method with those presented in Nejad and Mashinchi (2011), the results of other methods are the same as those of the proposed method except the results of Deng, Zhu, and Liu (2006, as cited in Nejad and Mashinchi, 2011) which gives $A_1 \prec A_2$, which is an unreasonable result.

Table 5.25

*Results of Ranking of Fuzzy Numbers in Example 5.5.2.4*

| Method | $A_1$ | $A_2$ | Ranking results |
|---|---|---|---|
| Nejad and Mashinchi (2011) | 0.274 | 0.190 | $A_2 \prec A_1$ |
| Wang et al., (2009) | 0.2154 | 0 | $A_2 \prec A_1$ |
| Asady and Zendehnam (2007) | 2.5 | 2.3600 | $A_2 \prec A_1$ |
| Chu and Tsao (2012) | 1.245 | 1.182 | $A_2 \prec A_1$ |
| Deng et al. (2006) | 1.143 | 2.045 | $A_1 \prec A_2$ |
| Cheng distance | 2.717 | 2.473 | $A_2 \prec A_1$ |
| Proposed method | 2.55 | 2.4 | $A_2 \prec A_1$ |

Table 5.25 shows that the proposed method is successful in giving the correct ranking order to the fuzzy number in case of the non-linear membership function.

**Example 5.5.2.5**

Consider the following two trapezoidal fuzzy numbers presented in Xu and Zhai (2012) as follows:

$A_1$= (-4, 0, 0.5, 1.5), and $A_2$= (-1.5, -0.5, 0.5, 1.5)

Based on Xu and Zhai (2012), the shortcoming of the magnitude method for ranking trapezoidal fuzzy numbers in case of having the same ranking order is considered. Table 5.26 illustrates the results of Xu and Zhai's method, A&H's method, and the proposed method.

Table 5.26

*Results of Ranking Two Trapezoidal Fuzzy Numbers in Example 5.5.2.5*

| Fuzzy numbers | A&H (2009) | Xu and Zhai (2012) | Proposed method |
|---|---|---|---|
| $A_1$ | 0 | -0.5 | -0.536 |
| $A_2$ | 0 | 0 | 0.001 |
| Ranking results | $A_1 \sim A_2$ | $A_1 \prec A_2$ | $A_1 \prec A_2$ |

Table 5.26 shows that the proposed method is successful in finding the correct order of each fuzzy number as $A_1 \prec A_2$, which is the same order of Xu and Zhai's method. On the other hand, the A&H's method cannot discriminate them.

## 5.6 Summary and Discussion



In this chapter, an evaluation of the application of the proposed method to solve real problems in the healthcare sector in Malaysia and some numerical examples from previous studies is presented. First, two types of data are used to estimate the expected number of beds in hospitals in Malaysia. The data are then used to validate the feasibility of the procedure in solving the real-world problems. We can summarize the finding in this application as follows:

1. The proposed defuzzification method can be used as a method to estimate the optimal number of beds under uncertain matter by including the total number of available beds as a constraint.
2. The proposed methods under DEA concept and LP concept are efficient in handling the problem of finding the crisp outputs in a case study of the

optimal number of beds. Efficient results in each case with dependent and independent crisp outputs are obtained.

3. The results obtained by using the proposed method with DEA concept are superior to those with LP concept because stable results are obtained earlier in the case of dependent outputs.
4. To conduct the validity of the proposed method, we use two defuzzification methods, namely, the COG and A&Z methods to defuzzify groups of fuzzy numbers. This comparison supports our proposed method to defuzzify groups of fuzzy numbers on three accounts:

   a. The proposed method gives crisp outputs, which satisfy the relationships in the original crisp data. However, each method of COG and A&Z fail to provide the crisp outputs that meet the relationships.

   b. The proposed method provides the nearest point to the fuzzy numbers in case of no relationships, and the results obtained are the same as the A&Z results.

   c. The results of the minimum distance between the crisp outputs and their fuzzy numbers based on the proposed method and Kikuchi's method support our proposed method with its new suggestion concept that the crisp output is the best nearest point in the case where some relationships in the original crisp data need to be satisfied in crisp outputs.

5. The implementation of the proposed methods offers recommendations to the hospital management how to distribute the correct number of beds for each ward. This process is done by estimating the optimal number of beds for each ward based on the total number of available beds and recorded patients.

Secondly, this chapter investigated the application of the proposed method in solving some problems that appear in the literature using numerical examples. We can summarize the findings of this application by the following contributions:

1. The proposed method is efficient in handling the problem of finding the optimal weights used in the GP.
2. The proposed method contributes to the ranking of fuzzy numbers field as a new defuzzification method to ranking fuzzy numbers.
3. The proposed method is able to handle the shortcoming of two common methods in the literature by giving the correct ranking order.
4. Ranking trapezoidal fuzzy numbers, which is an extension of the proposed method to deal with other types of fuzzy numbers.
5. One of the advantages of using our proposed method for ranking fuzzy numbers is that it can deal with fuzzy numbers as groups and find the ranking order at the same time.
6. The proposed method can handle the case of ranking fuzzy numbers with a non- linear membership function.

# CHAPTER SIX
# SUMMARY AND CONCLUSIONS

This chapter concludes the presentation of this thesis with the following four sections. The first section summarizes accomplishment of research objectives. The second section describes the significant contribution of this research and includes some concluding comments. The limitations of the research and some recommendation for future research are prescribed in the third and fourth section. The principal goal of this thesis is to develop a new defuzzification method to defuzzify groups of fuzzy numbers based on a DEA model, specifically the DEA-CCR model, which can lead to the creation of crisp values that are able to satisfy some relationships or properties in the original crisp data and keep them in the solution. In order to achieve the primary objective, there are specific objectives that need to be fulfilled.



The main objective is accomplished by modification of the COG method based on the minimization of the distance concept as a new objective function. Then the DEA model is further modified with this new objective function and extra constraints that consider the relationships and or properties to be fulfilled for the defuzzification problem. Then, in order to show the feasibility or ability of the developed method, it is implemented to solve real problems. The evaluation of the performance of the proposed method is done by comparing the obtained results from the proposed method with other existing methods. Also, an application of the proposed method based on the DEA as a general method is achieved by comparing the proposed defuzzification methods based on DEA and LP models and solving other problems with other types of fuzzy numbers.

## 6.1 Accomplishment of Research Objectives

The research has successfully developed a new defuzzification method that is able to create crisp values, which satisfy the relationships that exist in the original crisp data through six specific objectives. In the first and second objectives, the new defuzzification method is developed based on the DEA model by including the modification of COG method based on the minimization of the distance concept as a new objective function with additional constraints addressed in the CCR model. The development of the methodology is discussed from Sections 4.4 to 4.7.

The third objective is the implementation of the developed method in solving real problems is discussed in Section 5.3. Two cases involving the allocations of beds in a hospital are solved where the first one concerns about data with the relationship and the second data set has no relationship. This means that the developed defuzzification method reads as a generalized method since it can be used for data with or without the relationship.

The fourth objective is the comparison of the outcome obtained by the suggested method with other corresponding methods is examined in Section 5.3. Firstly, three existing defuzzification methods are used in case where there is a relationship in the original crisp data to evaluate the proposed method to show how it is successful in providing a crisp output that satisfies the relationships in the original crisp data i.e.

the total number of available beds (383 bed). Two of these methods are the COG and Asady and Zendehnam’s (2007), which provide 848 and 771 number of estimated beds, respectively, which are very different from the real available number of beds, 383. This means that these methods have failed to provide an acceptable solution to the problems in such case. The third method is the Kikuchi's method, which is designed to satisfy the crisp output with some relationships in the original crisp data. The proposed method and the Kikuchi's method provide an estimated number of beds for each ward, which is nearly approximated, and there are no big differences found in the estimated number of beds for each ward. Moreover, the two methods keep the relationship in the crisp output and give the sum of the estimated number of beds equal to the total number of available beds that is 383 beds. However, the proposed method provides a minimum value that maximizes the minimum membership grade than the solution obtained by Kikuchi’s method. The findings are discussed in sub-sections 5.3.1.4 and 5.3.1.5.



Secondly, only the COG and Asady and Zendehnam (A&Z) method are used to evaluate the proposed method in the case where no relationships need to be satisfied in the crisp outputs. In this case, the implementation of the proposed method shows that our proposed method can deal with data when no relationship needs to be satisfied in the crisp output by ignoring all constraints presented in the relationships. Furthermore, the obtained results suggest that the proposed method gives the nearest point (nearest crisp output) to the fuzzy numbers in the case of no relationship. This result is similar to that of A&Z's method. In other words, this leads us to support our proposed method with a new suggestion concept that the crisp output obtained is the best nearest point in the case of some relationships in the original data need to be satisfied in the crisp outputs.

The fifth objective is the comparison of the suggested DEA-based model with LP–based model is discussed in Section 5.4. The implementation shows that the proposed method based on the DEA is superior to the LP based, since the stable results are obtained earlier at the number of partitions, $m = 50$ while the results from the LP based method become stable at $m = 102$ in the case where a relationship in the original crisp data needs to be satisfied in the crisp outputs. Meanwhile, in the case of no relationship in the crisp original data, the two methods give the same results at the same number of partition, $m = 51$.

The sixth objective is applying the proposed method to the numerical problems in the literature with different types of fuzzy numbers is presented and discussed in Section 5.5. Two common issues meet this objective, which are the interval weights approach in the GP model, and the optimal results of GP and FGP implementation described in sub-section 5.5.1. Then the ranking of fuzzy numbers with different methods, some shortcomings in the A&Z's method (2007) and Abbasbandy, and Hajjari's (2009) method are explained in sub-section 5.5.2.

In summary, this research has succeeded in achieving the primary objectives and sub-objectives by testing the model through the experiments shown in Chapter Five.

### 6.2 Contributions of the Research

Our research has contributed modestly towards the understanding of the defuzzification formulation problem. The discussion on the contribution of this research is divided into two aspects: contribution to the knowledge and the practical contribution. The contribution to the knowledge focuses on methodology while the practical contribution looks at the application benefits of the defuzzification formulation in the real applications.

#### 6.2.1 Contribution to the Knowledge

The principal theoretical contribution of this research is to develop a defuzzification method that can lead to the creation of crisp values that is able to meet some relationships or properties in the original data (original crisp data) and keep them in the solution (crisp outputs) since there is no systematic way for selecting a defuzzification technique (Lee, 1990). This research develops methods can be considered an enhancement in cases where the original crisp data have or do not have relationships that need to be met in the solution. For this reason, this study puts its main attention to the modification of the COG method under the minimization of the distance concept. In addition, new insights into the DEA and LP concepts are used to develop two new methods in cases with relationships and without relationships in the original crisp data. The DEA model is used as a tool with a new objective and further constraints. Besides, the LP model uses the same new objective and constraints. Regarding the new insights in the objective and constraints, we propose two new methods under each concept (DEA and LP) to defuzzify fuzzy numbers in two cases with and without relationships.

The proposed method based on the DEA and LP model has successfully provided a solution to defuzzify fuzzy numbers and to find the crisp representative values of fuzzy numbers based on each of them. However, the main difference between these two methods is that the proposed method with DEA can handle the problems with linearity as well as nonlinearity in relationships or membership functions. Hence, the contribution of this research is in the use of the DEA model with its assumption of convexity in PPS that includes some linear and non-linear problems. This leads to the generality of the proposed method based on the DEA model that can handle different cases of linearity and nonlinearity.

The most meaningful contribution of this research is the development of a scientific method to defuzzify groups of fuzzy numbers, which is a new concept. In addition to its ability in handling original data with relationships, the developed defuzzification method can also handle cases when the original data have no relationships or properties that need to be satisfied in the crisp outputs. Thus, the user can expect an optimal solution of any systems, such as healthcare systems or factories systems, with any data, relationships, and fuzzy numbers. The appropriate and optimal solution depends on the needs of the systems.

### 6.2.2 Practical Contribution

In terms of practical benefits, this research is useful to DMs, who are working in a fuzzy environment. Firstly, the proposed method can be used as a first step to solving any problem that needs an optimal solution. In this research, we consider the problem of finding the optimal weights as a first step and then used them to solve an MOLP problem. Moreover, the proposed method shows that the existing interval weight procedures do not always give an optimal solution. However, the proposed method can provide optimal weights under each GP and FGP model when the particular interval is divided into sub-intervals. In this case, our method does not focus on the two values (min, max) of the response, unlike the previous methods. The proposed fuzzy interval weights (FIW) enables fuzzy goals to accomplish their aspired levels based on their relative importance are considered in an uncertain environment of the problem.



Secondly, the other primary benefits of this research are that the proposed method concentrates on using the fuzzy numbers as groups or individuals depending on the systems. Furthermore, this study provides a method for ranking fuzzy numbers by addressing some of the shortcomings of previous methods, such as the A&Z and A&H methods as mentioned in Xu and Zhai (2012)

Finally, to attest the ability of the proposed methods in solving real problems, this study contributes to the research literature in using the defuzzification method as a method to estimate the optimal number of beds in an uncertain environment. This condition allows us to control the resource allocation, such as the number of beds by including the total number of available beds as a constraint. The results appear in Section 5.3 and its sub-sections.

### 6.2.3 Research Contribution and Concluding Comments

In previous studies, defuzzification in DEA has been restricted to convert either the individual fuzzy inputs and/or fuzzy outputs to crisp data before using DEA models. However, this research has filled the gap on how to use CCR-DEA as a tool to defuzzify groups of fuzzy numbers. The DEA model is used to develop a new defuzzification method by keeping some relationships or properties in the original data as well as in the crisp output. The proposed approach ensures that decisions are made by ensuring that the crisp output satisfies the relationships in the original crisp data. In cases where there is no solution obtained by the proposed method, this means that no optimal solution satisfies the relationships.



These cases highlight the role of the decision makers in taking the proper relationships that need to be filled in the crisp outputs, suggesting that the proposed method is a very useful planning tool that gives an optimal solution to the authentic relationships. At the same time, its computation is not complicated as in other methods for complex membership functions. Besides, in the case of non-linear, no extra constraints are needed to find the crisp outputs in the allowable region to be adjusted as the solutions.

## 6.3 Limitations of the Research

This research has some limitations as follows:

1. Due to the DEA assumptions, each DMU has, at least, one positive input and one positive output value. As a result, the method could not deal with inputs or outputs (fuzzy numbers) with negative values.

2. Because the PPS of the CCR model has convexity assumptions, the proposed method with complex non-linear relationships or membership functions may not give a solution.

3. In ranking applications, the proposed method could not deal with two different types of a fuzzy number at the same time.

4. Because of the use of a geometric mean to generate the middle point of each fuzzy number, the observation cannot be zero.



## 6.4 Recommendations for Future Research

The current research offers the following opportunities for future researchers:

1. Concerning DEA, the defuzzification approach could be applied to other DEA models since this research only focuses on the CCR model. Furthermore, the CCR model can be applied to other DEA models to handle cases of non-positive inputs or outputs.

2. The proposed method can be generalized to the FDH model to handle problems with non-convexity assumption in cases of some non-linear relationships or membership functions.

3. The suitability of the pattern of the membership function is an issue of interest. Moreover, the impact of the form of the membership function on the crisp results also warrants attention. The physical interpretation of the membership functions requires investigation. The linear membership function may not be satisfactory in all applications. For further research, one could experiment with other forms of membership functions, such as hyperbolic, logistic, and S-shaped, etc.

4. In future studies, the developed defuzzification approach can be employed in other real problems that occur in different systems with other different relationships.



5. Researchers could also apply other fuzzification techniques to generate fuzzy numbers.

6. An extension of the proposed method can be done to deal with some other properties that DMs like to satisfy in crisp outputs that may not appear in the original crisp data. This can be done by including other objectives to expect the crisp output that meets other priorities.

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