NEW SPLINE METHODS FOR SOLVING FIRST AND SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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DOCTOR OF PHILOSOPHY
UNIVERSITI UTARA MALAYSIA
[2016]

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Abstrak

Banyak permasalahan yang timbul daripada pelbagai aplikasi kehidupan nyata boleh menjurus kepada model matematik yang dapat diungkapkan sebagai masalah nilai awal (MNA) dan masalah nilai sempadan (MNS) untuk persamaan pembeza biasa (PPB) peringkat pertama dan kedua. Masalah ini mungkin tidak mempunyai penyelesaian analitik, dengan itu kaedah berangka diperlukan bagi menganggarkan penyelesaian. Apabila sesuatu persamaan pembeza diselesaikan secara berangka, selang pengamiran dibahagikan kepada subselang. Akibatnya, penyelesaian berangka pada titik grid dapat ditentukan melalui pengiraan berangka, manakala penyelesaian antara titik grid masih tidak diketahui. Bagi mencari penyelesaian hampir antara dua titik grid, kaedah splin diperkenalkan. Walau bagaimanapun, kebanyakan keadah splin yang sedia ada digunakan untuk menganggar penyelesaian bagi MNA dan MNS yang tertentu sahaja. Oleh itu, kajian ini membangunkan beberapa kaedah splin baharu yang berasaskan fungsi splin polynomial dan bukan polynomial bagi menyelesaikan MNA dan MNS umum yang berperingkat pertama dan kedua. Analisis penumpuan bagi setiap kaedah splin baharu turut dibincangkan. Dari segi pelaksanaan, kaedah Runge-Kutta tersurat bertahap empat dan berperingkat keempat digunakan bagi mendapat penyelesaian pada titik grid, manakala kaedah splin baharu digunakan untuk memperoleh penyelesaian antara titik grid. Prestasi kaedah splin yang baharu kemudiannya dibandingkan dengan beberapa kaedah splin yang sedia ada dalam menyelesaikan 12 masalah ujian. Secara umumnya, keputusan berangka menunjukkan bahawa kaedah splin baharu memberikan kejituan yang lebih baik daripada kaedah splin yang sedia ada. Oleh itu, kaedah splin baharu adalah alternatif yang berdaya saing dalam menyelesaikan MNA dan MNS berperingkat pertama dan kedua.

Kata kunci: Interpolasi, Keadah splin, Masalah nilai awal, Masalah nilai sempadan, Persamaan pembeza biasa.

Abstract

Many problems arise from various real life applications may lead to mathematical models which can be expressed as initial value problems (IVPs) and boundary value problems (BVPs) of first and second ordinary differential equations (ODEs).These problems might not have analytical solutions, thus numerical methods are needed inapproximating the solutions. When a differential equation is solved numerically, the interval of integration is divided into subintervals.Consequently, numerical solutions at the grid pointscan be determined through numerical computations, whereas the solutions between the grid points still remain unknown. In order to find the approximate solutions between any two grid points, spline methods are introduced. However, most of the existing spline methods are used to approximate the solutions of specific cases of IVPs and BVPs. Therefore, this study develops new spline methods based on polynomial and non-polynomial spline functions for solving general cases of first and second order IVPs and BVPs. The convergence analysis for each new spline method is also discussed. In terms of implementation, the 4-stage fourth order explicit Runge-Kutta method is employed to obtain the solutions at the grid points, while the new spline methods are used to obtain the solutions between the grid points. The performance of the new spline methods are then compared with the existing spline methods in solving12 test problems. Generally, the numerical results indicate that the new spline methods provide better accuracy than their counterparts. Hence, the new spline methods are viable alternatives for solving first and second order IVPs and BVPs.

Keywords: Interpolation, Spline method, Initial value problem, Boundary value problem, Ordinary differential equation.

Acknowledgement

I am grateful to the Almighty Allah for giving me the opportunity to complete my PhD thesis. May peace and blessing of Allah be upon His beloved Prophet Muhammad (SAW), his family, and his companions.

I wish to express my deepest gratitude to my main supervisor, Dr. Teh Yuan Ying and co-supervisor, Assoc. Prof. Dr. Azizan Saaban; who have been very patient in guiding and supporting me from the very beginning of my first arrival here in Malaysia and throughout the completion of this thesis. They assisted me immensely in developing a correct focus for my study and have given me their valuable ideas, insights, comments and suggestions.

I further wish to dedicate this work to the spirit of my father and to my beloved mother, my brothers, my lovely wife, my daughter, and to all who helped me. I wish to express my sincere gratitude to all my dear friends for the friendship rendered and assistance provided during this journey. Finally, I wish to express appreciation to all academic and administrative staff in the College of Arts of Sciences.

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CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Spline functions have been rapidly developed as a result of their applications usefulness. Spline functions with their various categories have many high quality approximation powers as well as structural properties such as zero properties, sign change properties and determinental properties (Dold & Eckmann, 1976). There are many applications of spline functions in applied mathematics and engineering. Some of these applications are data fitting, approximating functions, optimal control problems, integro-differential equation and Computer-Aided Geometric Design (CAGD). It is important to note that programmes based on spline functions have been embedded in various computer applications.

A common consensus is that, Schoenberg (1946) made the first mathematical reference to spline in his interesting article, and this probably was the first time that 'spline' was used in connection with smooth piecewise polynomial approximation. However, it is important to note that the ideas of developing splines were originated from shipbuilding and aircraft industries earlier than computer modeling was available (Dermoune & Preda, 2014). Then, naval architects faced the necessity to draw a smooth curve through a set of points. The answer to this challenge was to put metal weights (called ***knots***) at the points of control so that a thin metal or wooden beam (called a ***spline***) would be bent through the weights (see Figure 1.1). Bending splines from physicist's point of view was important as the weight has some greatest influence at the contact point but further smoothly along the splines. The draftsman added some more weights in order to exercise more control on specific region of the splines. The scheme poses some enormous challenges especially with the exchange of data, and hence, arise the need to describe the shape of the curve mathematically. Mathematically, the equivalence of draftsman's wooden beam is actually cubic polynomial splines. Since then, splines have achieved more importance especially with the evolution of computer as they were used first to replace polynomials in the interpolation and then as an instrument to build flexible and smooth shapes in computer graphics.

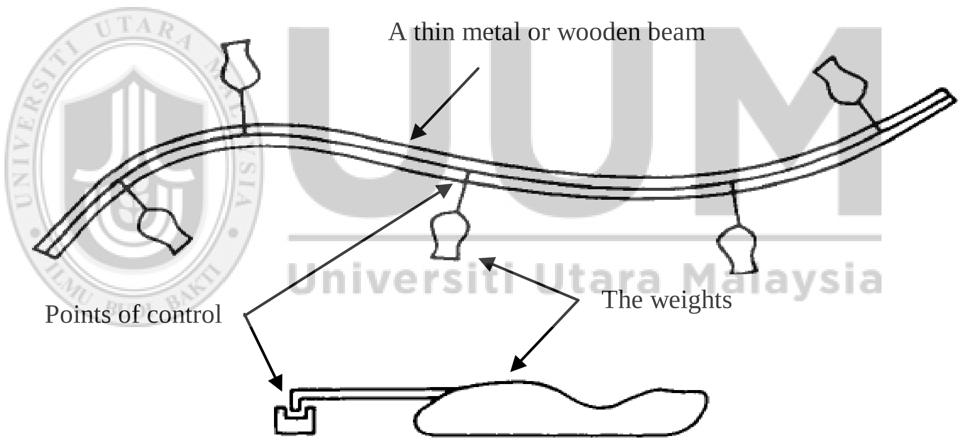

Figure 1.1. A thin metal or wooden beam bent through the weights

During the 1960s and 1970s, there were articles which made substantial contributions to the splines developments such as Schoenberg (1958), Birkhoff and Garabedian (1960), Ahlberg and Nilson (1963), Loscalzo and Talbot (1967), Rubin and Khosla (1976), and Sastry (1976). Though in the 1960s, univariate splines were intensely studied, but the in-depth understanding of it came to light in the 1970s, which gave rise to its treatments in various books. Furthermore, these are some books which discussed splines completely, including Ahlberg, Nilson and Walsh (1967), Prenter (1975), Schumaker (1981), Shikin and Plis (1995), Spath (1995), and De Boor (2001). Notably, some authors in their earliest articles used spline functions to obtain smooth approximate solutions of ordinary differential equations (ODEs), for examples, Loscalzo and Talbot (1967), Bickely (1968), Albasiny and Hoskins (1969), Crank and Gupta (1972), Usmani and Warsi (1980), Sallam and Karablli (1996), Al-Said (1998), and Sallam and Anwar (1999, 2000). All of these articles demonstrate that splines of various degrees can be used to approximate the solutions of first order and second order initial value problems (IVPs) as well as second order boundary value problems (BVPs). Nowadays, many researchers are still publishing their works on this subject, which make this topic remains an active area of research.

Lately, non-polynomial spline methods become a useful tool which efficiently compute accurate solutions of ODEs. Many articles proposed non-polynomial spline methods to find the numerical solutions of second order BVPs, such as Hossam, Sakr and Zahra (2003), Khan (2004), Ramadan, Lashien and Zahra (2007), Rashidinia, Mohammadi and Jalilian (2008), Hamid, Majid and Ismail (2010), Jalilian (2010), and Zarebnia and Sarvari (2012, 2013).

1.2 Existence and Uniqueness of Solutions to Initial Value Problems for First Order Ordinary Differential Equations

The first order initial value problems of ODE is generally represented in the following form

$$u' = f(x,u), \quad u(a) = \eta. \tag{1.1}$$

The most important theorem here is the existence and uniqueness theorem which states the sufficient conditions for a unique solution of (1.1) to exist. This theorem is given as below (Lambert, 1991).

Theorem 1.1 (Existence of unique solution of a first order IVP). Let $f(x,u),$ where $f:\Re\times\Re\rightarrow\Re,$ be defined and continuous for all (x,u) in the region D defined by $a\leq x\leq b, -\infty<u<\infty,$ a and b are finite, and let there exist a constant L such that

$$\left|f(x,u)-f(x,u^*)\right|\leq L\left|u-u^*\right|, \tag{1.2}$$

holds for every $(x,u),(x,u^*)\in D.$ Then for any $\eta\in\Re,$ there exist a unique solution $u(x)$ for the problem (1.1) where $u(x)$ is continuous and differentiable for all $(x,u)\in D.$

The requirement (1.2) is known as Lipschitz condition and the constant L as a Lipschitz constant. If $f(x,u)$ is differentiable with respect to $u,$ then from the mean value theorem

$$f(x,u)-f(x,u^*)=\frac{\partial f(x,\bar{u})}{\partial u}(u-u^*), \tag{1.3}$$

where $\bar{u}$ is a point in the interior of the interval whose end-points are u and u^*, and $(x,u), (x,u^*)$ are both in the region D (Lambert, 1973). Therefore, if we choose

$$L = \sup_{(x,u)\in D} \left| \frac{\partial f(x,u)}{\partial u} \right|, \tag{1.4}$$

then condition (1.2) of Theorem 1.1 is satisfied.

If there are more than one first order ODEs that need to be solved at one time, then we deal with a system of m simultaneous first order ODEs in m dependent variables $u_1, u_2, \ldots, u_m$. If each of these variables satisfies the initial conditions that are prescribed at the same point, then we have an IVP for a first order system (Lambert, 1991)

$$\begin{aligned} u_1' &= f_1(x,u_1,u_2,\cdots,u_m), \quad u_1(a) = \eta_1, \\ u_2' &= f_2(x,u_1,u_2,\cdots,u_m), \quad u_2(a) = \eta_2, \\ &\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ u_m' &= f_m(x,u_1,u_2,\cdots,u_m), \quad u_m(a) = \eta_m. \end{aligned} \tag{1.5}$$

For simplicity, system (1.5) can also be expressed in the following vector form

$$\mathbf{u}' = \mathbf{f}(x,\mathbf{u}), \quad \mathbf{u}(a) = \boldsymbol{\eta}, \tag{1.6}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix},$$

$$\mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_m' \end{pmatrix},$$

$$\mathbf{f}(x,\mathbf{u}) = \begin{pmatrix} f_1(x,u_1,u_2,\cdots,u_m) \\ f_2(x,u_1,u_2,\cdots,u_m) \\ \vdots \\ f_m(x,u_1,u_2,\cdots,u_m) \end{pmatrix},$$

and

$$\mathbf{u}(a) = \begin{pmatrix} u_1(a) \\ u_2(a) \\ \vdots \\ u_m(a) \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{pmatrix} = \boldsymbol{\eta}.$$

Theorem 1.1 readily generalizes to give necessary conditions for the existence of a unique solution to the system (1.6); where the region D now is defined by $a \le x \le b, -\infty < u_i < \infty$ for $i = 1,2,\ldots,m,$ and conditions (1.2) is replaced by the condition

$$\|\mathbf{f}(x,\mathbf{u}) - \mathbf{f}(x,\mathbf{u}^*)\| \le L\,\|\mathbf{u} - \mathbf{u}^*\|, \qquad (1.7)$$

where $(x,\mathbf{u})$ and $(x,\mathbf{u}^*)$ are in $D,$ and $\|.\|$ denotes a vector norm (Lambert, 1973). If $\mathbf{f}(x,\mathbf{u})$ is differentiable with respect to $\mathbf{u},$ then from the mean value theorem

$$\mathbf{f}(x,\mathbf{u}) - \mathbf{f}(x,\mathbf{u}^*) = \frac{\partial \mathbf{f}(x,\bar{\mathbf{u}})}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}^*), \qquad (1.8)$$

where the notation implies that each row of the Jacobian $\partial \mathbf{f}(x,\bar{\mathbf{u}})/\partial \mathbf{u}$ is evaluated at different mean values which are internal points of the line segment from

$(x, \mathbf{u})$ to $(x, \mathbf{u}^*)$, all of which are points in the region D (Lambert, 1973). Therefore, if we choose

$$L = \sup_{(x,\mathbf{u}) \in D} \left| \frac{\partial \mathbf{f}(x, \mathbf{u})}{\partial \mathbf{u}} \right|,$$

then condition (1.7) is satisfied (Lambert, 1991).

Some of the solutions of (1.1) and (1.6) can be obtained analytically. When an IVP can be solved analytically, then this particular problem has one exact solution for (1.1) and *m* exact solutions for (1.6). Numerical integration formulae for (1.1) and (1.6) are used when exact solution(s) cannot be obtained. Numerical integration formulae will give approximate solutions for the exact solutions. There are three popular integration methods for solving (1.1) and (1.6). We can use either linear multistep method, predictor-corrector method or Runge-Kutta method to obtain the approximations for IVPs (1.1) and (1.6). These numerical methods are classical numerical methods and can be found in some well known text books on numerical solution of ODE, see Henrici (1962), Milne (1970), Gear (1971), Stetter (1973), Lambert (1973), Jain (1984), Butcher (1987), Fatunla (1988), Lambert (1991), Hairer and Wanner (1991), Hairer, Norsett and Wanner (1993), Iserles (1996), and Butcher (2003).

1.3 Existence and Uniqueness of Solutions to Boundary Value Problems for First and Second Ordinary Differential Equations

Consider the general form of the first order BVPs given by

$$\begin{aligned} \mathbf{u}' &= \mathbf{f}(x, \mathbf{u}), \\ \mathbf{A}\mathbf{u}(a) &= \boldsymbol{\alpha}, \mathbf{B}\mathbf{u}(b) = \boldsymbol{\beta}, \end{aligned} \tag{1.9}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $m \times m$ matrices. According to Holsapple, Venkataraman and Doman (2004), it is difficult to set up an existence and uniqueness theorem for first order BVPs. Therefore, in this work, we assume that the existence and uniqueness of the solution for equation (1.9) is known over the interval $[a,b]$. Moreover, we also assume that the boundary conditions for equation (1.9) are sufficient for the solution to exist.

Now, consider the general form of the second order ODE given by

$$u'' = f(x, u, u'), \tag{1.10}$$

subject to the mixed boundary conditions of the form

$$\begin{aligned} a_1 u(a) + a_2 u'(a) = A, |a_1| + |a_2| \neq 0, \\ b_1 u(b) + b_2 u'(b) = B, |b_1| + |b_2| \neq 0, \end{aligned} \tag{1.11}$$

where a_1, a_2, b_1, b_2, A and B are given constants. We note that (1.11) becomes the Dirichlet boundary conditions when $a_2 = b_2 = 0.$ On the other hand, (1.11) becomes the Neumann boundary conditions when $a_1 = b_1 = 0.$

The theorem which states the sufficient conditions for existence and uniqueness of the solution of the second order BVP (1.10) - (1.11) is given below (Keller, 1966).

Theorem 1.2 (Existence of unique solution of a second order BVP). Let $f(x,u,u')$ in (1.10) have continuous derivatives which satisfy

i. $\dfrac{\partial f(x,u,u')}{\partial u} > 0$, and

ii. $\dfrac{\partial f(x,u,u')}{\partial u'} \le M$,

for some $M \ge 0, a \le x \le b$ and all continuously differential functions $u(x)$. Let the constants a_i, b_i satisfy

$$a_i \ge 0, b_i \ge 0, i = 1,2; \text{ and } a_1 + b_1 > 0.$$

Then a unique solution of the second order BVP given by (1.10) and (1.11) exists for each A and B.

Consider the following corollary which is a special case of (1.10) and (1.11), where $f(x,u,u') = p(x)u' + q(x)u + r(x)$ with the boundary conditions $u(a) = A$ and $u(b) = B.$ This corollary is given as below (Farago, 2014).

Corollary 1.1 Assume that $f(x,u,u')$ is a linear non-homogeneous function. Consider the second order linear BVP of the form

$$u'' = f(x,u,u') = p(x)u' + q(x)u + r(x), \quad a \le x \le b,$$

subject to the condition

$$u(a) = A; u(b) = B,$$

where $p(x), q(x)$ and $r(x)$ are given continuous functions on $[a,b]$. If $q(x) > 0$ on $[a,b]$, then the given second order linear BVP has a unique solution.

In general, the exact solution of the second order BVPs given by (1.10) subject to the boundary conditions (1.11) does not exist or it is very difficult to obtain. Therefore, numerical integration formulae are needed to solve problem (1.10) with the boundary conditions (1.11). There are three popular integration methods for problem (1.10) with the boundary conditions (1.11). For examples, we can either use finite difference method, finite element method or shooting method to obtain the approximations for problem (1.10) - (1.11). These numerical methods can be found in some well known text books on numerical solutions of second order BVPs, see Zienkiewicz and Morice (1971), Reddy (1993), Gupta (1995), Rao (2001), Stoer and Bulirsch (2002), Press (2007) and Chapra and Canale (2010).

1.4 Preliminaries

In this section, we introduced some preliminary definitions, theorems and notations that will be used throughout this work.

1.4.1 Some Properties of Vector and Matrix

In order to analyze and measure the size of the errors in vector and matrix forms, we introduce the following properties.

Definition 1.1 (Ghufran, 2010). The norm of a vector $\boldsymbol{\upsilon}$ is a nonnegative function $\|.\| : \Re^n \to \Re$ with the following properties:

i. $\|\boldsymbol{\upsilon}\| > 0$ when $\boldsymbol{\upsilon} \neq 0$ and $\|\boldsymbol{\upsilon}\| = 0$ iff $\boldsymbol{\upsilon} = 0$ in $\Re^n$,

ii. $\|\alpha\, \boldsymbol{\upsilon}\| = |\alpha| \|\boldsymbol{\upsilon}\|$ for all $\alpha \in \Re,$ and

iii. $\|\mathbf{u} + \boldsymbol{\upsilon}\| \leq \|\mathbf{u}\| + \|\boldsymbol{\upsilon}\|$ for all $\mathbf{u}, \boldsymbol{\upsilon} \in \Re^n$.

Definition 1.2 (Ghufran, 2010). A matrix norm is a nonnegative real valued function $\|.\| : \Re^{m \times n} \to \Re,$ if for all $\mathbf{A}, \mathbf{B} \in \Re^{m \times n},$ it satisfies the following three axioms:

i. $\|\mathbf{A}\| > 0$ when $\mathbf{A} \neq 0$ and $\|\mathbf{A}\| = 0$ iff $\mathbf{A} = 0,$

ii. $\|\alpha\, \mathbf{A}\| = |\alpha| \|\mathbf{A}\|$ for all $\alpha \in \Re,$

iii. $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$ for all $\mathbf{A}, \mathbf{B} \in \Re^{m \times n}$.

One of the most common vector norms used in numerical linear algebra is infinity norm (∞- norm), and it is defined in Definition 1.3.

Definition 1.3 (Cowlishaw & Fillmore, 2010). The ∞- norm of a vector $\boldsymbol{\upsilon}$ ($\|\boldsymbol{\upsilon}\|_\infty$) is defined by

$$\|\boldsymbol{\upsilon}\|_\infty = \max_{i=1}^{n} |\upsilon_i|.$$

Similarly, we defined the infinity norm (∞- norm) for the matrices in Definition 1.4.

Definition 1.4 (Cowlishaw & Fillmore, 2010). The ∞- norm of the matrix $\mathbf{A}$ ($\|\mathbf{A}\|_\infty$) is defined by

$$\|\mathbf{A}\|_\infty = \max_i \sum_j |a_{ij}|.$$

Definition 1.5 (Engeln-Müllges & Uhlig, 2013). A square matrix $\mathbf{A}$ is said to be diagonally dominant matrix if $|a_{ii}| \geq \sum_{i \neq j} |a_{ij}|$ for all i.

Definition 1.6 (Engeln-Müllges & Uhlig, 2013). A square matrix $\mathbf{A}$ is said to be strictly diagonally dominant matrix if $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$ for all i.

Theorem 1.3 (Varah, 1975). Assume that $\mathbf{A}$ is strictly diagonally dominant matrix by rows and let $\alpha = \min_i (|a_{ii}| - \sum_{i \neq j} |a_{ij}|)$, then $\|\mathbf{A}^{-1}\|_\infty < \dfrac{1}{\alpha}$.

Theorem 1.4 (Lui, 2012). Diagonally dominant matrices are invertible.

1.4.2 Peano Kernels

We begin this subsection with the expansion of a function $f(x)$ in Taylor polynomial plus an error term which expressed as an integral. Thus, if $f^{(n+1)}(x)$ exists on the interval $[a,b]$, then

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + r_n(f),$$

for $a \le x \le b$, where $r_n(f) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt$. Now, we defined the truncated power function in Definition 1.7.

Definition 1.7 (Phillips, 2003). For any fixed real number x and any nonnegative integer n, we write $(x-\tau)_+^n$, which is called a truncated power function (TPF), to denote the function of τ defined for $-\infty < \tau < \infty$ as follows:

$$(x-\tau)_+^n = \begin{cases} (x-\tau)^n & \tau \le x, \\ 0 & \tau \ge x. \end{cases}$$

By the introduction of the TPF, the error term $r_n(f)$ can be written as

$$r_n(f) = \frac{1}{n!}\int_a^b f^{(n+1)}(\tau)(x-\tau)_+^n\,d\tau.$$

Thus, by imposing the TPF for $(x-\tau)^n$, the terminals of the integration in $r_n(f)$ are independent of x. Therefore, this leads us to present Theorem 1.5.

Theorem 1.5 (Sarfraz, Hussain & Nisar, 2010). If $f \in C^{n+1}[a,b]$, $f^{(n+1)}(x)$ absolutely continuous, then

$$E(f) = \int_a^b f^{(n+1)}(\tau)R_x\,[(x-\tau)_+^n]\,d\tau,$$

where the expression $R_x\,[(x-\tau)_+^n]$ is called the Peano kernel.

We note that the Peano kernel function can be rewritten on the interval $[a,b]$ as

$$R_x\,[(x-\tau)_+^n] = \begin{cases} r(\tau, x), & a < \tau < x, \\ s(\tau, x), & x < \tau < b. \end{cases}$$

In order to estimate the interpolation error using the Peano kernel theorem in Theorem 1.5, we introduce Theorem 1.6.

Theorem 1.6 (Li, 2007). If $f \in C^{n+1}[a,b]$ and $f^{(n+1)}(x)$ is bounded, then

$$|E(f)| \le \frac{1}{n!} \| f^{(n+1)}(x) \|_\infty \int_a^b \left| R_x\,[(x-\tau)_+^n] \right| d\tau.$$

1.5 Problem Statement and Scope of Study

Many natural phenomena in various fields of sciences and engineering can be described as mathematical models which involve differential equations. The mathematical representations for these models could range from very simple model which consists of a single differential equation to very complex model which involved more than one differential equations. There is a higher chance for a simpler model to have a known exact solution; otherwise the mathematics involved may be so complex that there is little hope in solving the model analytically (Giordano, Fox, Horton & Weir, 2009). When the exact solutions for differential equations are not known, then we need to approximate them numerically.

When solving differential equations numerically, we have to discretize the interval of integration into smaller sub-intervals. The size of the sub-intervals can be equally fixed (in the case of fixed step-size strategy) or they can be varying (in the case of variable step-size strategy). For both cases, sub-intervals are separated by grid points based on the step-size. Therefore, numerical solutions fall on the grid points are known through numerical computations, whereas the solutions fall between any two subsequent grid points are still unknown. Spline methods are introduced to find the approximate solutions fall between any two subsequent grid points. The main advantage of spline methods is, once the splines have been determined, the numerical solutions at any locations over the interval of intergration are available.

From the literature, spline interpolation techniques or spline methods are widely used to approximate the numerical solutions of second order BVPs (Chang, Yang & Zhao, 2011). Moreover, we have observed from the literature that most of the recent works about the spline methods were developed to approximate special cases of second order BVPs. These cases can be sorted out into four categories; that are

i. $u''(x) = q(x)u(x) + r(x),$ as in Ramadan et al.(2007), Rashidinia, Jalilian and Mohammadi (2009), Zahra, Abd El-Salam, El-Sabbagh and Zaki (2010), Srivastava, Kumar and Mohapatra (2011), and Chen and Wong (2012).

ii. $u''(x) = p(x)u'(x) + q(x)u(x) + r(x),$ as in Hamid et al. (2010), Chang et al. (2011), Hamid, Majid and Ismail (2011), Kalyani and Rama Chandra Rao (2013).

iii. $-(p(x)u'(x))' = r(x),$ as in Caglar, Caglar and Elfaituri (2006) and Rashidinia et al. (2008).

iv. $u''(x) = f(x, u(x)),$ as in Al Bayati et al. (2009), Caglar, Caglar, Özer, Valarıstos and Anagnostopoulos (2010), Liu, Liu and Chen (2011), El hajaji, Hilal, Mhamed and Jalila (2013), and Ogundare (2014).

Moreover, while going through the literature, it is noticed that the majority of the spline methods used to approximate second order BVPs subject to Dirichlet boundary conditions and few for second order BVPs subject to Neumann boundary conditions. On the other hand, many authors used different type of spline methods to approximate the solutions of second order IVPs (Al Bayati, Saeed & Hama-Salh, 2009).

However, for the first order IVPs, it is proven in Loscalzo and Talbot (1967) that for spline methods of degree higher than three, those methods are divergent. Moreover, for the second order IVPs, Micula (1973) showed that every spline method of degree higher than four are divergent. Nevertheless, we noticed that the divergence can be avoided if the spline function appeared in the spline method is carefully defined. To the best of our knowledge, we did not find any spline method used to approximate first order BVPs during our investigation of the spline methods in the literature review.

Through our examinations of the current developments of spline methods in the literature, we identify a few gaps which can be fulfilled in this study:

i. The necessity to develop new spline methods that are based on higher degree polynomial spline functions and non-polynomial spline functions,
ii. New spline methods which guarantee convergence even if higher degree spline functions are employed, and
iii. New spline methods that are capable in solving the following problems with improved numerical accuracy:

a. General first order IVPs (mixtures of autonomous, non-autonomous, linear, nonlinear)

$$u'(x) = f(x, u(x)), \quad u(a) = \eta. \tag{1.12}$$

b. First order BVPs of the form

$$\begin{aligned} &u'(x) = A(x)u(x) + B(x), \\ &Au(a) = \alpha,\ Bu(b) = \beta. \end{aligned} \tag{1.13}$$

c. General second order IVPs (mixtures of linear, nonlinear, with/without the presence of $u'(x)$)

$$\begin{aligned} &u''(x) = f(x, u(x), u'(x)), \\ &u(a) = \alpha \text{ and } u'(a) = \beta. \end{aligned} \tag{1.14}$$

d. General second order BVPs (mixtures of linear, nonlinear, with/without the presence of $u'(x)$)

$$u''(x) = f(x, u(x), u'(x)), \tag{1.15}$$

subject to $u(a) = \alpha_1$ and $u(b) = \beta_1$; or $u'(a) = \alpha_2$ and $u'(b) = \beta_2$.

1.6 Objectives of the Study

The main objective of this study is to develop new spline methods for solving first and second order IVPs and BVPs numerically, which can be accomplished by:

i. Developing two new spline methods based on polynomial spline functions and two new spline methods based on non-polynomial spline functions.

ii. Analyzing the convergence properties for each of the proposed spline method.

iii. Comparing the performance of the proposed spline methods in terms of errors with other existing spline methods in solving first and second order IVPs and BVPs.

1.7 Significance of the Study

New convergent spline methods for solving first and second order ODEs have been introduced in this study. This study shows that the newly developed spline methods do provide alternatives to current spline methods found in the literature. We also proved that the same spline methods can be used for both first and second order IVPs and BVPs, rather than having different spline methods to treat different types of problems separately. This will reduce the number of spline methods that need to be developed. Finally, Loscalzo and Talbot (1967) showed that spline methods of degree greater than three could produce divergent numerical solutions for first order IVPs. Moreover, Micula (1973) proved that spline methods of degree higher than four might generate divergent numerical solutions for second order IVPs. The new spline methods developed in this study do not suffer from these drawbacks.

1.8 Thesis Organization

This thesis consist of six chapters. Chapter One covers a general introduction of the study. It presents the background of this study, existence and uniqueness theorems involving first and second order ODEs, preliminaries, problem statements and scope of study, objectives of the study, significance of the study and thesis organization.

In Chapter Two, the literature review focus on the discussions of existing spline methods in solving first and second order IVPs and BVPs.

In Chapter Three, new quartic and quintic spline methods are developed to approximate the solutions of the first and second order IVPs and BVPs. Convergence analyses of the two proposed spline methods are presented.

Chapter Four presents the construction of new cubic and quintic non-polynomial spline methods for the numerical solutions of the first and second order IVPs and BVPs. Moreover, the convergence analyses for each proposed spline methods are established.

Chapter Five covers the implementations of the new proposed spline methods on 12 test problems found in the current literature. These test problems consist of first and second order IVPs and BVPs with known exact solutions. Using the infinity error norms, the accuracy and the applicability of the proposed spline methods are examined through the comparisons with some existing methods.

Finally, Chapter Six contains conclusions and some suggestions for future research.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

The mathematical modeling of many problems in science and engineering leads to ODEs. Depending upon the form of the boundary conditions to be satisfied by the solution, problems involving ODEs can be divided into two main categories, namely IVPs (conditions prescribed at one end of the domain of analysis) and BVPs (conditions prescribed at both ends of the domain of analysis). Analytic solutions for these problems are not generally available and hence numerical methods must be resorted to (Mai-Duy & Tran-Cong, 2001). In this chapter, reviews of existing spline methods for the numerical solutions of the first order IVPs, the first order BVPs, second order IVPs and second order BVPs are presented.

2.2 Numerical Methods Based on Spline Functions

Literatures show that some researchers such as Schoenberg (1958), Walsh, Ahlberg and Nilson (1962), Ahlberg and Nilson (1963) and Ahlberg et al. (1967), started to study the properties of using spline functions to approximate any function $f(x)$. According to Haque (2006), spline function can be defined in general as a piecewise function in which the pieces joined together in a suitably smooth fashion. (See Figure 2.1).

For the remaining of this chapter, we are going to discuss different types of spline methods for the numerical solutions of first and second order IVPs and BVPs. Readers should note that every spline in this discussion is defined on the partition P of the interval $[a,b]$, unless mentioned otherwise. This partition P is defined as follows

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\},$$

where $x_i = a + ih, i = 0,1,2,\ldots,n,$ and $h = \dfrac{b-a}{n}$ is constant step size.

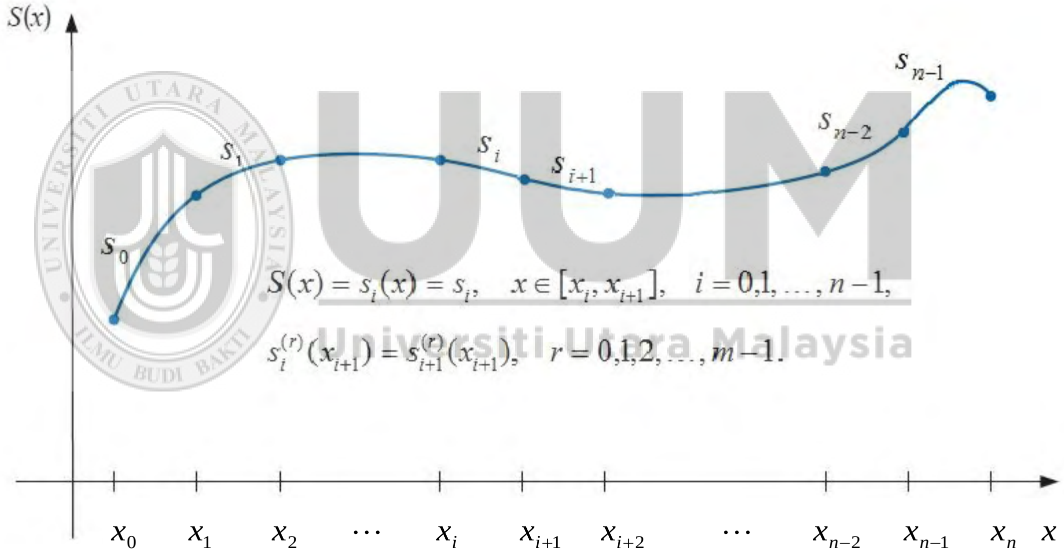

Figure 2.1. A spline function of degree *m* composed of *n* segments joined together at the grid points

2.2.1 Spline Methods for Solving First Order Ordinary Differential Equations

In the literature, Loscalzo and Talbot (1967) constructed a class of spline of degree m $(m \geq 2), S_i(x),$ to find the approximate solution for (1.1) on the interval $[0,b]$. The construction of this spline is started on the subinterval $[0, x_1],$ by defining the first component of $S_i(x)$ as follows

$$S_0(x) = u(0) + u'(0)x + ... + \frac{1}{(m-1)!}u^{(m-1)}(0)x^{m-1} + \frac{1}{m!}a_0 x^m, \tag{2.1}$$

where a_0 is an undetermined coefficient. To determine the value of a_0, $S_0(x)$ should satisfy the IVP at $x = h,$ this yields the equation $S_0'(h) = f(h, S_0(h))$ which solved for a_0. The second component of $S_i(x)$ will be determined on the subinterval $[x_1, x_2]$ as

$$S_1(x) = \sum_{\upsilon=0}^{m-1} \frac{1}{\upsilon!} S^{(\upsilon)}(h)(x-h)^{m-1} + \frac{1}{m!}a_1(x-h)^m. \tag{2.2}$$

In order to find the value of a_1, $S_1(x)$ should satisfy equation $S_1'(2h) = f(2h, S_1(2h))$. By continuing in the same way, a spline function $S_i(x)$, which satisfied the equation $S_i'(\upsilon h) = f(\upsilon h, S_i(\upsilon h)),\ \upsilon = 0,1,\ldots,n,$ is obtained. Moreover, they investigated three cases, i.e. $m = 2, 3$ and $m \geq 4$. It turns out that: when $m = 2$, this quadratic spline is the trapezoidal rule; when $m = 3$, this cubic spline is nothing but another way of writing the Simpson's rule; and the method is divergent if $m \geq 4$, as $h \to 0$.

Patrício (1978) used the cubic spline method presented in Ahlberg et al. (1967), which is given on the subinterval $[x_i, x_{i+1}]$ as

$$S_i(x) = S_i' \frac{(x_{i+1}-x)^2(x-x_i)}{h_i^2} - S_{i+1}' \frac{(x-x_i)^2(x_{i+1}-x)}{h_i^2} + u(x_i)\frac{(x_{i+1}-x)^2[2(x-x_i)+h_i]}{h_i^3} + u(x_{i+1})\frac{(x-x_i)^2[2(x_{i+1}-x)+h_i]}{h_i^3}, \quad (2.3)$$

to approximate the numerical solution of the first order IVP (1.1), where $S_i' = f(x_i, S_i(x_i))$. The order of convergence of the spline method (2.3) is proved to be $O(h^4)$. Moreover, systems of first order IVPs as in (1.6) can be solved using this method.

Sallam and Anwar (1999) created a quadratic spline method for solving the first order IVP (1.1). They wrote the quadratic spline method as

$$s(x) = s_i + hs_i' A(t) + hs_{i+\beta}' B(t), \quad (2.4)$$

where $t = \frac{x-x_i}{h}$, $s_j' = s'(x_j)$, $A(t) = t - \frac{t^2}{2\beta}$, $B(t) = \frac{t^2}{2\beta}$, and $\beta \in (0,1]$. The continuity conditions are used to obtain the following relations

$$s_i = s_{i-1} + h(1-\frac{1}{2\beta})s_{i-1}' + \frac{h}{2\beta} f(x_{i-1+\beta}, s_{i-1+\beta}),$$

and

$$s_i' = (1-\frac{1}{\beta})s_{i-1}' + \frac{1}{\beta} f(x_{i-1+\beta}, s_{i-1+\beta}).$$

They confirmed that the quadratic spline method (2.4) is of order $O(h^2)$, if $\beta \geq 1/2$, and when $\beta < 1/2$ the method is divergent.

In Nikolis (2004), a quadratic trigonometric spline method is proposed to approximate the first order IVP (1.1). This method is defined as a linear combination of the trigonometric basis functions as

$$s(x) = \sum_{i=-2}^{n-1} C_i TB_i^2(x), \tag{2.5}$$

where the trigonometric basis functions, $TB_i^2(x)$ is defined as follows

$$TB_i^2(x) = \frac{1}{\sin(h)\sin(\frac{h}{2})}\begin{cases} \sin^2(\frac{x-x_i}{2}), & x \in [x_i, x_{i+1}), \\ \sin(\frac{x-x_i}{2})\sin(\frac{x_{i+2}-x}{2}) & \\ +\sin(\frac{x_{i+3}-x}{2})\sin(\frac{x-x_{i+1}}{2}), & x \in [x_{i+1}, x_{i+2}), \\ \sin^2(\frac{x_{i+3}-x}{2}), & x \in [x_{i+2}, x_{i+3}], \\ 0 \quad , & \text{otherwise}. \end{cases}$$

Additionally, the order of convergence of this numerical method (2.5) is demonstrated to be $O(h^2)$.

Defez, Soler, Hervas and Santamaria (2005) extended the method in Loscalzo and Talbot (1967) to approximate the solution of a system of linear first order IVPs of the form

$$\mathbf{u}'(x) = \mathbf{A}(x)\mathbf{u}(x) + \mathbf{B}(x). \tag{2.6}$$

They expressed the cubic spline method on the first interval $[x_0, x_1]$, in the form of

$$\mathbf{s}_0(x)=\mathbf{u}(x_0)+\mathbf{u}'(x_0)(x-x_0)+\frac{1}{2!}\mathbf{u}''(x_0)(x-x_0)^2+\frac{1}{3!}\boldsymbol{\alpha}_0(x-x_0)^3. \qquad (2.7)$$

To determine the value of $\boldsymbol{\alpha}_0$, $\mathbf{u}''(x_0)$ is needed, which can be found by differentiating $\mathbf{u}'(x)$ once in order to obtain

$$\mathbf{u}''(x)=(\mathbf{A}'(x)+\mathbf{A}^2(x))\mathbf{u}(x)+\mathbf{A}(x)\mathbf{B}(x)+\mathbf{B}'(x).$$

The first continuity property is imposed at the point x_1, to get

$$\begin{aligned}(\mathbf{I}-\frac{h}{3}\mathbf{A}(x_1))\boldsymbol{\alpha}_0&=\frac{2}{h^2}(\mathbf{A}(x_1)(\mathbf{u}(x_0)+\mathbf{u}'(x_0)h+\frac{1}{2!}\mathbf{u}''(x_0)h^2)\\&\quad+\mathbf{B}(x_1)-\mathbf{u}'(x_0)-\mathbf{u}''(x_0)h).\end{aligned} \qquad (2.8)$$

Equation (2.8) is solved for $\boldsymbol{\alpha}_0$, and therefore, the spline method (2.7) is totally determined on the first subinterval. Similarly, they defined the cubic spline method over the subinterval $[x_i, x_{i+1}]$, $i=1, 2, \ldots, n-1$, as

$$\mathbf{s}_i(x)=\mathbf{s}_{i-1}(x_i)+\mathbf{s}'_{i-1}(x_i)(x-x_i)+\frac{1}{2!}\mathbf{s}''_{i-1}(x_i)(x-x_i)^2+\frac{1}{3!}\boldsymbol{\alpha}_i(x-x_i)^3. \qquad (2.9)$$

Thus, the spline method (2.9) will be completely defined by determining the values of $\boldsymbol{\alpha}_i$ on its corresponding subintervals. Moreover, they revealed that the spline method (2.9) is of order $O(h^4)$.

In Ogundare and Okecha (2008), two spline methods are developed to solve the first order IVP (1.1). The first method is a quadratic spline method and the second method is a cubic spline method. The iterative numerical schemes are derived as

$$u_{i+1}=u_{i-1}+2hf_i, \qquad (2.10a)$$

$$u_{i+1} = u_i + \frac{h}{2}(3f_i - f_{i-1}), \tag{2.10b}$$

$$u_i = u_{i-1} + \frac{h}{2}(f_i + f_{i-1}), \tag{2.10c}$$

and

$$u_{i+1} = 2u_i - u_{i-1} + \frac{h}{2}(f_{i+1} - f_{i-1}), \tag{2.10d}$$

where $u_i = u(x_i)$ and $f_i = f(x_i, u(x_i))$. The methods (2.10a) – (2.10c) result from the quadratic spline method and they are convergent with order $O(h^2)$. One the other hand, the scheme (2.10d) results from the cubic spline method and it gives convergent solution of order $O(h^3)$.

Tung (2013) generalized the cubic spline method proposed in Defez et al. (2005) to approximate the nonlinear system of first order IVPs (1.6). The cubic spline method is written on the first interval $[x_0, x_1]$, as in equation (2.7). Consequently, in order to find the value of $\boldsymbol{\alpha}_0$, $\mathbf{u}''(x_0)$ is required. Differentiating $\mathbf{u}'(x)$ once to get

$$\mathbf{u}''(x) = \frac{\partial \mathbf{f}(x, \mathbf{u}(x))}{\partial x} + ((vec\mathbf{f}(x, \mathbf{u}(x))^t \otimes \mathbf{I}_r) \frac{\partial \mathbf{f}(x, \mathbf{u}(x))}{\partial vec\mathbf{u}(x)},$$

where $\otimes$ represents the Kronecker product and $\mathbf{I}_r$ is the identity matrix of size r. For a matrix $\mathbf{A} \in C^{m \times n}$, we have

$$vec\mathbf{A} = \begin{pmatrix} \mathbf{A}_{\bullet 1} \\ \vdots \\ \mathbf{A}_{\bullet n} \end{pmatrix}, \text{ where } \mathbf{A}_{\bullet \mathbf{k}} = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{pmatrix}.$$

The continuity of the first derivative is used at the point x_1 to get the value of $\boldsymbol{\alpha}_0$ as

$$\boldsymbol{\alpha}_0 = \frac{2}{h^2}(\mathbf{f}(x_1, \mathbf{u}(x_0) + \mathbf{u}'(x_0)h + \frac{1}{2!}\mathbf{u}''(x_0)h^2 + \frac{1}{3!}\boldsymbol{\alpha}_0 h^3) - \mathbf{u}'(x_0) - \mathbf{u}''(x_0)h). \tag{2.11}$$

Hence, the spline method is fully defined on the first subinterval by solving the above equation. In the same way, the cubic spline method on any arbitrary subinterval $[x_i, x_{i+1}],\ i = 1, 2, \ldots, n-1,$ is stated as in equation (2.9). Thus, the cubic spline method will be identified by determining the values of $\boldsymbol{\alpha}_i$ on its corresponding subinterval. The author concluded that the cubic spline method is of order $O(h^4)$.

However, according to the best of our knowledge, there is no previous spline methods used to approximate first order BVPs. Hence, this is our first attempt to handle this problem.

The highlights of this section are presented in Table 2.1.

Table 2.1

Highlights of Literature Review on Spline Methods for Solving First Order IVPs

Authors	Methods	Advantages	Disadvantages
Loscalzo and Talbot (1967)	Presented a class of spline methods of degree m $(m \geq 2)$	These methods are used to approximate the solution of (1.1). Moreover, for $m = 2$ and $m = 3$, the resultant methods are convergent.	This method is divergent if $m \geq 4$, as $h \to 0$.
Patrício (1978)	Implemented the cubic spline method presented in Ahlberg et al. (1967)	This method approximates the numerical solution of the first order IVP (1.1). The order of convergence of the spline method is $O(h^4)$. Moreover, systems of first order IVPs (1.6) can be solved using this method.	The accuracy of the method can beimproved.
Sallam and Anwar (1999)	Developed a quadratic spline method depending on a parameter β	This method solves the first order IVP (1.1). Furthermore, the quadratic spline method is of order $O(h^2)$ if $\beta \geq 1/2$.	This method is divergent when $\beta < 1/2$.

Table 2.1

Continued

Nikolis (2004)	Proposed a quadratic trigonometric spline method	This method solves the first order IVP (1.1). In addition, the order of convergence of this numerical method is $O(h^2)$.	The method is of lower accuracy.
Defez et al. (2005)	Extended the cubic spline method in Loscalzo and Talbot (1967)	This spline method is of order $O(h^4)$.	This method is used to approximate the solution of linear system of first order IVP.
Ogundare and Okecha (2008)	Developed a class of spline schemes based on quadratic and cubic spline methods	The method solve the general first order IVP (1.1). Moreover, the order of convergent of the schemes based on quadratic spline method is two, while for cubic spline method is three.	The order of accuracy for these methods is low. Moreover, they only consider linear first order IVP.

Table 2.1

Continued

Tung (2013)	Generalized the cubic spline method proposed in Defez et al. (2005)	Approximate the nonlinear system of first order IVPs (1.6). The cubic spline method is of order $O(h^4)$.	For large *m*, *m* is the number of subintervals, the method becomes very complicated due to the Kronecker product.

2.2.2 Spline Methods for Solving Second Order Initial Value Problems

Consider the following second order ODE of the form

$$u''(x) = f(x, u(x)), \tag{2.12}$$

subject to the initial conditions $u(a) = \alpha$ and $u'(a) = \beta$.

Micula in 1973 followed the approach in Loscalzo and Talbot (1967) in constructing a class of spline method of degree m, where m is an integer greater than two. The constructed spline method on the subinterval $[x_i, x_{i+1}]$ is given by

$$S_i(x) = \sum_{\upsilon=0}^{m-1} \frac{1}{\upsilon!} S^{(\upsilon)}(x_i)(x - x_i)^{m-1} + \frac{1}{m!} a_m (x - x_i)^m. \tag{2.13}$$

The proposed spline method (2.13) is used for solution of second order IVPs (2.12). He discovered that the spline method (2.13) diverged when m is greater than four. Consequently, a spline method of degree three and another spline method of degree four are constructed. Moreover, it is proved that these methods are of order $O(h^3)$ and $O(h^5)$, respectively.

Sallam and Karaballi (1996) introduced a quartic spline method for solving second order IVP (2.12). The quartic spline method over the subinterval $[x_i, x_{i+1}]$ is given in the following form

$$s(x) = s_i + hs_i' B(t) + h^2 s_i'' C(t) + h^2 s_{i+1}'' D(t) + h^3 s_i''' E(t), \tag{2.14}$$

where $t = \dfrac{x - x_i}{h}$, $s_j^{(m)} = s^{(m)}(x_j)$, $m = 0, 1, 2$ and 3,

$$B(t) = t,$$

$$C(t) = \frac{t^2(6 - t^2)}{12},$$

$$D(t) = \frac{t^4}{12},$$

and

$$E(t) = \frac{t^3(2 - t)}{12}.$$

From the continuity conditions, the following relations are derived

$$s_i = s_{i-1} + hs_{i-1}' + \frac{h^2}{12}(5f(x_{i-1}, s_{i-1}) + f(x_i, s_i)) + \frac{h^3}{12} s_{i-1}''',$$

$$s_i' = s_{i-1}' + hs_{i-1}' + \frac{h}{3}(2f(x_{i-1}, s_{i-1}) + f(x_i, s_i)) + \frac{h^2}{6} s_{i-1}''',$$

$$s_i'' = f(x_i, s_i),$$

and

$$s_i''' = -s_{i-1}''' + \frac{2}{h}(f(x_i, s_i) - f(x_{i-1}, s_{i-1})).$$

On the first subinterval $[0, x_1]$, the starting values for the scheme are chosen to be

$$s(0) = u(0),$$

$$s'(0) = u'(0),$$

and

$$s'''(0) = u'''(0) = f_x(0, u(0)) + f_u(0, u(0))u'(0).$$

They verified that the quartic spline method (2.14) is of order $O(h^4)$.

Sallam and Anwar (2000) established a quintic spline method for obtaining the solution of second order IVP (2.12). The quintic spline method is considered as

$$s(x) = s_i + hs_i' A(t) + h^2 s_i'' B(t) + h^2 s_{i+1/3}'' C(t) + h^2 s_{i-2/3}'' D(t) + h^2 s_{i+1}'' E(t), \quad (2.15)$$

where $t = \frac{x - x_i}{h}$, $s_j^{(m)} = s^{(m)}(x_j)$, $m = 0, 1$ and 2,

$$A(t) = t,$$

$$B(t) = \frac{t^2}{2} - \frac{11t^3}{12} + \frac{3t^4}{4} - \frac{9t^5}{40},$$

$$C(t) = \frac{3t^3}{2} - \frac{15t^4}{8} + \frac{27t^5}{40},$$

$$D(t) = -\frac{3t^3}{4} + \frac{3t^4}{2} - \frac{27t^5}{40},$$

and

$$F(t) = \frac{t^3}{6} - \frac{3t^4}{8} + \frac{9t^5}{40}.$$

Moreover, the following relations are obtained from the continuity conditions as

$$s_{i-2/3} = s_{i-1} + \frac{1}{3}hs'_{i-1} + \frac{97h^2}{3240} f(x_{i-1}, s_{i-1}) + \frac{19h^2}{540} f(x_{i-2/3}, s_{i-2/3}) - \frac{13h^2}{1080} f(x_{i-1/3}, s_{i-1/3}) + \frac{h^2}{405} f(x_i, s_i),$$

$$s_{i-1/3} = s_{i-1} + \frac{2}{3}hs'_{i-1} + \frac{28h^2}{405} f(x_{i-1}, s_{i-1}) + \frac{22h^2}{135} f(x_{i-2/3}, s_{i-2/3}) - \frac{2h^2}{135} f(x_{i-1/3}, s_{i-1/3}) + \frac{2h^2}{405} f(x_i, s_i),$$

$$s_i = s_{i-1} + hs'_{i-1} + \frac{13h^2}{120} f(x_{i-1}, s_{i-1}) + \frac{3h^2}{10} f(x_{i-2/3}, s_{i-2/3}) + \frac{3h^2}{40} f(x_{i-1/3}, s_{i-1/3}) + \frac{h^2}{60} f(x_i, s_i),$$

$$s'_i = s'_{i-1} + \frac{h}{8}(f(x_{i-1}, s_{i-1}) + 3f(x_{i-2/3}, s_{i-2/3}) + 3f(x_{i-1/3}, s_{i-1/3}) + f(x_i, s_i)),$$

and

$$s''_\alpha = f(x_\alpha, s_\alpha),$$

where $\alpha = \{i-1, i-\frac{2}{3}, i-\frac{1}{3}, i\}$. In order to get the solution for the above system, they selected the starting values at the first subinterval $[0, x_1]$, as follows:

$$s(0) = u(0),\ s'(0) = u'(0).$$

The fourth order of convergence for the quintic spline method (2.15) is verified.

Najafi, Kordrostami and Esmaeilzadeh (2005) introduced a quartic spline method and a quintic spline method to solve second order IVP (2.12). The quartic spline method is expressed on the subinterval $[x_{i-1}, x_i]$ as

$$s(x) = s_{i-1} + hs'_{i-1}\,B(t) + h^2 s''_{i-1}\,C(t) + h^2 s''_i\,D(t) + h^3 s'''_{i-1}\,E(t), \qquad (2.16)$$

where $t = \dfrac{x - x_i}{h}$, $s_j^{(m)} = s^{(m)}(x_j)$, $m = 0, 1, 2$ and 3,

$$B(t) = t,$$

$$C(t) = \frac{t^2(6 - t^2)}{12},$$

$$D(t) = \frac{t^3(2 - t)}{12},$$

and

$$E(t) = \frac{t^4}{12}.$$

Furthermore, the following relations are generated from the continuity conditions

$$s_i = s_{i-1} + hs'_{i-1} + \frac{h^2}{12}(5f(x_{i-1}, s_{i-1}) + f(x_i, s_i)) + \frac{h^3}{12}s'''_{i-1},$$

$$s'_i = s'_{i-1} + hs'_{i-1} + \frac{h}{3}(2f(x_{i-1}, s_{i-1}) + f(x_i, s_i)) + \frac{h^2}{6}s'''_{i-1},$$

$$s''_i = f(x_i, s_i), \qquad (2.17)$$

and

$$s'''_i = -s'''_{i-1} + \frac{2}{h}(f(x_i, s_i) - f(x_{i-1}, s_{i-1})).$$

On the subinterval $[0, x_1]$, they took the initial values for the system (2.17) as follows

$$s(0) = u(0),$$

$$s'(0) = u'(0),$$

and

$$s'''(0) = u'''(0) = f_x(0, u(0)) + f_u(0, u(0))u'(0).$$

In addition, to solve the system (2.17), the values of s_i and s_i' on the right hand side of system (2.17) are calculated by

$$s_i = s_{i-1} + hs'_{i-1} + \frac{h^2}{2!} s''_{i-1} + \frac{h^3}{3!} s'''_{i-1},$$

and

$$s_i' = \frac{s_i - s_{i-1}}{x_i - x_{i-1}},$$

respectively. On the other hand, the quintic spline method is expressed in the form

$$s(x) = s_{i-1} + hs'_{i-1} B(t) + h^2 s''_{i-1} C(t) + h^2 s''_{i-2/3} D(t) + h^2 s''_{i-1/3} E(t) + h^2 s''_i F(t), \quad (2.18)$$

where $t = \frac{x - x_i}{h}$, $s_j^{(m)} = s^{(m)}(x_j)$, $m = 0, 1$ and 2,

$$B(t) = t,$$

$$C(t) = \frac{t^2}{2} - \frac{11t^3}{12} + \frac{3t^4}{4} - \frac{9t^5}{40},$$

$$D(t) = \frac{3t^3}{2} - \frac{15t^4}{8} + \frac{27t^5}{40},$$

$$E(t) = -\frac{3t^3}{4} + \frac{3t^4}{2} - \frac{27t^5}{40},$$

and

$$F(t) = \frac{t^3}{6} - \frac{3t^4}{8} + \frac{9t^5}{40}.$$

By the continuity conditions, the following schemes are derived

$$\begin{aligned}
s_{i-2/3} &= s_{i-1} + \frac{1}{3}hs'_{i-1} + \frac{97h^2}{3240} f(x_{i-1}, s_{i-1}) + \frac{19h^2}{540} f(x_{i-2/3}, s_{i-2/3}) \\
&\quad - \frac{13h^2}{1080} f(x_{i-1/3}, s_{i-1/3}) + \frac{h^2}{405} f(x_i, s_i), \\
s_{i-1/3} &= s_{i-1} + \frac{2}{3}hs'_{i-1} + \frac{28h^2}{405} f(x_{i-1}, s_{i-1}) + \frac{22h^2}{135} f(x_{i-2/3}, s_{i-2/3}) \\
&\quad - \frac{2h^2}{135} f(x_{i-1/3}, s_{i-1/3}) + \frac{2h^2}{405} f(x_i, s_i), \\
s_i &= s_{i-1} + hs'_{i-1} + \frac{13h^2}{120} f(x_{i-1}, s_{i-1}) + \frac{3h^2}{10} f(x_{i-2/3}, s_{i-2/3}) \\
&\quad + \frac{3h^2}{40} f(x_{i-1/3}, s_{i-1/3}) + \frac{h^2}{60} f(x_i, s_i), \\
s'_i &= s'_{i-1} + \frac{h}{8}(f(x_{i-1}, s_{i-1}) + 3f(x_{i-2/3}, s_{i-2/3}) + 3f(x_{i-1/3}, s_{i-1/3}) \\
&\quad + f(x_i, s_i)),
\end{aligned} \tag{2.19}$$

and

$$s''_\alpha = f(x_\alpha, s_\alpha), \alpha = \{i, i - \frac{1}{3}, i - \frac{2}{3}\}.$$

They selected the initial values for the system (2.19) as

$$s(0) = u(0), s'(0) = u'(0) \text{ and } s''_0 = f(x_0, s_0).$$

In order to find the solution for the system (2.19), the values of s_i on the right hand side of system (2.19) are evaluated from

$$s_{\alpha} = \frac{4s_{\alpha-2/3} - s_{\alpha-1/3}}{2}, \; \alpha = \{i, i-\frac{1}{3}, i-\frac{2}{3}\}.$$

In Al Bayati et al. (2009), they presented a quintic spline method to approximate the solution of the second order IVP (2.12). The quintic spline method is written as

$$\begin{aligned} s(x) = A(x)\,s_i + B'(x)\,s_{i+1} + h^2 C(x)\,s_i'' + h^2 D(x)\,s_{i+1}'' \\ + h^3 E(x)\,s_i''' E(t) + h^3 F(x)\,s_{i+1}''', \end{aligned} \qquad (2.20)$$

where

$$A(x) = 1,$$

$$B(x) = x,$$

$$C(x) = \frac{1}{20}(-10x + 10x^2 - 5x^4 + 2x^5),$$

$$D(x) = \frac{1}{20}(-10x + 5x^4 - 2x^5),$$

$$E(x) = \frac{1}{60}(-5x + 10x^3 - 10x^4 + 3x^5),$$

and

$$F(x) = \frac{1}{60}(5x - 5x^4 + 3x^5).$$

They showed that the following relations are obtained from the continuity conditions, i.e.

$$s_i = s_{i-1} + hs_i' - \frac{h^2}{20}(3f(x_{i-1}, s_{i-1}) + 7f(x_i, s_i)) + \frac{h^3}{60}s_i''', \qquad (2.21a)$$

$$s_i' = s_{i-1}' + \frac{h}{2}(f(x_{i-1}, s_{i-1}) + f(x_i, s_i)), \qquad (2.21b)$$

$$s_{i+1}''' = -5s_{i-1}''' + \frac{3}{h}(f(x_{i+1}, s_{i+1}) - f(x_{i-1}, s_{i-1})), \qquad (2.21c)$$

and

$$s_i'' = f(x_i, s_i). \qquad (2.21d)$$

Additionally, the following starting values for the above system are used

$$s(0) = u(0),$$

$$s'(0) = u'(0),$$

and

$$s'''(0) = u'''(0) = f_x(0, u(0)) + f_u(0, u(0))u'(0).$$

In order to find the solution of the system (2.21), the values of s_i and s_i' on the right hand side of system (2.21) are computed by the help of

$$s_i = s_{i-1} + hs_{i-1}' + \frac{h^2}{2!}s_{i-1}'' + \frac{h^3}{3!}s_{i-1}''' + \frac{h^4}{4!}s_{i-1}^{(4)} + \frac{h^5}{5!}s_{i-1}^{(5)},$$

and

$$s_i' = s_{i-1}' + hs_{i-1}'' + \frac{h^2}{2!}s_{i-1}''' + \frac{h^3}{3!}s_{i-1}^{(4)} + \frac{h^4}{4!}s_{i-1}^{(5)},$$

respectively. They illustrated that the spline method (2.20) is of order $O(h^5)$.

Ogundare (2014) derived class of schemes based on cubic and quartic spline methods to approximate the second order IVP (2.12). These schemes are stated as follows

$$u_{i+1} = 2u_i - u_{i-1} + h^2 f_i, \tag{2.22a}$$

$$u_{i+2} = 3u_i - 2u_{i-1} + h^2 (4 f_i - f_{i-1}), \tag{2.22b}$$

$$u_{i+1} = 2u_i - u_{i-1} + \frac{h^2}{24}(f_i + 23 f_{i-2}), \tag{2.22c}$$

$$u_{i+2} = u_{i+1} + u_i - u_{i-1} + \frac{h^2}{3}(9 f_i + f_{i-1} + 2 f_{i-2}), \tag{2.22d}$$

$$u_{i+2} = u_{i+1} + u_i - u_{i-1} + \frac{h^2}{6}(-3 f_{i+1} + 22 f_i + 3 f_{i-1} + 2 f_{i-2}), \tag{2.22e}$$

and

$$u_{i+1} = 2u_i - u_{i-1} + \frac{h^2}{48}(- f_{i+1} + 47 f_i + f_{i-1} + f_{i-2}). \tag{2.22f}$$

It is noted that scheme (2.22a) and scheme (2.22b) are derived from cubic spline, while the other schemes (2.22c) - (2.22f) are formulated from quartic spline. Additionally, it is illustrated that only (2.22a), (2.22c) and (2.22f) are convergent and provide accuracies of order 3, 2 and 2, respectively.

We summarize the highlights of this section in Table 2.2.

Table 2.2

Highlights of Literature Review on Spline Methods for Solving Second Order IVPs

Authors	**Methods**	**Advantages**	**Disadvantages**
Micula (1973)	Constructed a class of spline method of degree m $(m \geq 2)$	Convergence analysis proved that for $m = 3$ and $m = 4,$ the resultant methods provide third and fifth order of accuracy, respectively.	The methods are used to approximate nonlinear second order IVPs without the presence of $u'(x).$ Moreover, The method is divergent if $m > 4,$ as $h \rightarrow 0.$
Sallam and Karaballi (1996)	Developed a quartic spline method	This method is of order $O(h^4).$	This method finds the solutions of nonlinear second order IVPs without the presence of $u'(x).$
Sallam and Anwar (2000)	Established a quintic spline method	This method provide accuracy of order $O(h^4).$	This method solves nonlinear second order IVPs without the presence of $u'(x).$

Table 2.2

Continued

Najafi et al. (2005)	Introduced a quartic and a quintic spline methods	The results generated by these methods are accurate.	These methods are developed to solve nonlinear second order IVPs without the presence of $u'(x)$. Furthermore, the authors do not provide the details of the convergence analysis.
Al Bayati et al. (2009)	Presented a quintic spline method	The spline method produce accuracy of order $O(h^5)$.	This method used to approximate the solutions of nonlinear second order IVPs without the presence of $u'(x)$.
Ogundare (2014)	Derived a class of six schemes based on cubic and quartic spline methods	Schemes (2.22a), (2.22c) and (2.22f) are convergent.	Approximate the solutions of nonlinear second order IVPs without the presence of $u'(x)$. Three out of six developed schemes are divergent. Moreover, the order of accuracy for the converge methods is low.

2.2.3 Spline Methods for Solving Second Order Boundary Value Problems

Consider the general form of the second order ODE given by

$$u''(x) = f(x, u(x), u'(x)), a \le x \le b, \tag{2.23}$$

subject to Dirichlet boundary conditions i.e. $u(a) = \alpha_1$ and $u(b) = \beta_1$; or subject to Neumann boundary conditions i.e. $u'(a) = \alpha_2$ and $u'(b) = \beta_2$.

In the literature, Bickley (1968) proposed a cubic spline method in the form

$$S(x) = a + b(x - x_0) + c(x - x_0)^2 + d_0(x - x_0)^3, \tag{2.24}$$

to approximate the solution of the following second order BVPs:

$$u''(x) = p(x)u'(x) + q(x)u(x) + r(x), \tag{2.25}$$

subject to mix boundary conditions (1.11). This method is started at x_0 in the interval $[x_0, x_1]$. In the next interval $[x_1, x_2]$, the term $d_1(x - x_1)^3$ is also added to the approximate function $S(x)$ in (2.24). On advancing into the next interval $[x_2, x_3]$, the term $d_2(x - x_2)^3$ is appended to the last term. This process is continued until the last interval $[x_{n-1}, x_n]$ is reached. So, this spline can be represented on the interval $[x_i, x_{i+1}]$, in the following form

$$S(x) = a + b(x - x_0) + c(x - x_0)^2 + \sum_{j=0}^{i} d_j (x - x_j)^3. \tag{2.26}$$

Albasiny and Hoskins (1969) used the cubic spline method suggested by Ahlberg et al. (1967), which is given in the form

$$S_i(x) = \frac{M_i}{6h_i}(x_{i+1} - x)^3 + \frac{M_{i+1}}{6h_i}(x - x_i)^3 + (\frac{u(x_{i+1})}{h_i} - \frac{h_i M_{i+1}}{6})(x - x_i)$$
$$+ (\frac{u(x_i)}{h_i} - \frac{h_i M_i}{6})(x_{i+1} - x), \qquad (2.27)$$

to obtain the approximate solution for two specific cases of the second order BVPs associated with Dirichlet boundary conditions. We observe that $M_i = S_i''(x_i), i = 0,1,\ldots,n-1.$ The first case is when the first derivative is absent from (2.25), i.e.

$$u''(x) = q(x)u(x) + r(x), \qquad (2.28)$$

and the second case is when the first derivative is present, i.e. the second order BVPs (2.25). Moreover, they found that the cubic spline method (2.27) is of order $O(h^4)$ for both cases.

Fyfe (1969) used the cubic spline method (2.26) to obtain the approximate solution for the second order BVP (2.25). It is shown that the cubic spline method (2.26) provides accuracy of order $O(h^4)$.

According to Usmani and Warsi (1980), a quintic spline method in the following form

$$S_i(x) = a_i(x - x_i)^5 + b_i(x - x_i)^4 + c_i(x - x_i)^3$$
$$+ d_i(x - x_i)^2 + e_i(x - x_i) + f_i, \qquad (2.29)$$

is derived to find the approximate solution for the second order BVP (2.28) subject to Dirichlet boundary conditions. The coefficients in equation (2.29) are given by

$$a_i = \frac{s_{i+1} - s_i}{120h},$$

$$b_i = \frac{s_i}{24},$$

$$c_i = \frac{M_{i+1} - M_i}{6h} - \frac{h(s_{i+1} + s_i)}{36},$$

$$d_i = \frac{M_i}{2},$$

$$e_i = \frac{u_{i+1} - u_i}{h} - \frac{h(M_{i+1} - 2M_i)}{6} + \frac{h^3(s_{i+1} - s_i)}{360},$$

and

$$f_i = u_i, \quad i = 0,1,\ldots,n,$$

where M_i and s_i are approximations to $u''(x_i)$ and $u^{(4)}(x_i)$, respectively. They proved that the order of convergence for the proposed method (2.29) is $O(h^4)$.

Chawla and Subramanian (1988) constructed a scheme based on the cubic spline method (2.27), given by

$$\begin{aligned} S(x) = \frac{(x_{i+1} - x)^3}{6h} M_i + \frac{(x - x_i)^3}{6h} M_{i+1} + (\tilde{u}_i - \frac{h^2 M_i}{6})(\frac{x_{i+1} - x}{h}) \\ + (\tilde{u}_{i+1} - \frac{h^2 M_{i+1}}{6})(\frac{x - x_i}{h}), \end{aligned} \quad (2.30)$$

for solving second order nonlinear BVPs of the form (2.23) subject to mix boundary conditions (1.11). We note that $M_i = u''(x_i)$ and $\tilde{u}_i$, $i = 0,1,\ldots,n,$ represents the approximate solution obtained using the finite difference method. The spline method (2.30) is shown to be fourth order of accuracy.

Al-Said (1998) formulated a cubic spline method in the form

$$S_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i, \qquad (2.31)$$

to find the approximate solution for the second order BVPs (2.28) subject to Dirichlet boundary conditions. The following equations are established for the coefficients of cubic spline method (2.31)

$$a_i = \frac{1}{12}(T_{i+1} + T_i),$$

$$b_i = \frac{1}{2}F_{i+1/2} - \frac{h}{8}(T_{i+1} + T_i),$$

$$c_i = D_i,$$

and

$$d_i = s_{i+1/2} - \frac{h}{2}D_i - \frac{h^2}{8}F_{i+1/2} + \frac{h^3}{48}(T_{i+1} + T_i),$$

where $s_{i+1/2} = S_i(x_{i+1/2})$, $D_i = S_i'(x_i)$, $F_{i+1/2} = S_i''(x_{i+1/2})$, and $\frac{1}{2}(T_{i+1} + T_i) = S_i'''(x_{i+1/2})$.

In addition, it turns out that the cubic spline method (2.31) is of order $O(h^2)$.

Hossam et al. (2003) derived a quadratic spline method for solving the second order BVP (2.28) subject to Dirichlet boundary conditions. Their quadratic spline method is given by

$$S_i(x) = a_i(x-x_i)^2 + b_i(x-x_i) + c_i. \qquad (2.32)$$

The values of the coefficients are computed as

$$a_i = \frac{1}{2} F_{i+1/2},$$

$$b_i = D_i,$$

and

$$c_i = s_{i+1/2} - \frac{h^2}{8} F_{i+1/2} - \frac{h}{2} D_i,$$

where $s_{i+1/2} = S_i(x_{i+1/2})$, $D_i = S_i'(x_i)$, and $F_{i+1/2} = S_i''(x_{i+1/2})$. They also proved that spline method (2.32) is of order $O(h^2)$.

Khan (2004) developed a cubic non-polynomial spline method with a parameter τ to approximate the second order BVP (2.28) subject to Dirichlet boundary conditions. The cubic non-polynomial spline method is defined as follows

$$\begin{aligned} S(x) = {} & \frac{h^2}{\omega^2 \sin(\omega)} [S''(x_{i+1}) \sin(\frac{x - x_i}{h}) \\ & + S''(x_i) \sin(\frac{x_{i+1} - x}{h})] + \frac{h^2}{\omega^2} [\frac{(x - x_i)}{h} (S''(x_{i+1}) \\ & + \frac{\omega^2}{h^2} S(x_{i+1})) + \frac{(x_{i+1} - x)}{h} (S''(x_i) + \frac{\omega^2}{h^2} S(x_i))], \end{aligned} \tag{2.33}$$

where $\omega = h\tau^{\frac{1}{2}}$. The main relation in this article is given by

$$h^2 (\alpha S''(x_{i-1}) + 2\beta S''(x_i) + \alpha S''(x_{i+1})) = u(x_{i+1}) - 2u(x_i) + u(x_{i-1}),$$

where $\alpha = \frac{\omega \csc(\omega) - 1}{\omega^2}$, $\beta = \frac{1 - \omega \cot(\omega)}{\omega^2}$ and $i = 1, 2, \ldots, n-1$. He verified that this method (2.33) is of order $O(h^2)$ for any values of α and β such that $\alpha + \beta = \frac{1}{2}$, while it is of order $O(h^4)$ for $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$.

Caglar et al. (2006) applied a cubic B-spline method to second order BVP of the form

$$-(p(x)u'(x))' = r(x), \qquad (2.34)$$

subject to Dirichlet boundary conditions. The cubic B-spline method is written as

$$S_i(x) = \sum_{i=-1}^{m+1} b_i B_i(x), \qquad (2.35)$$

where the basis function $B_i(x)$ is given by

$$B_i(x) = \frac{1}{6h^3}\begin{cases} (x-x_i)^3, & x \in [x_i, x_{i+1}], \\ h^3 + 3h^2(x-x_{i+1}) + 3h(x-x_{i+1})^2 & \\ -3(x-x_{i+1})^3, & x \in [x_{i+1}, x_{i+2}], \\ h^3 + 3h^2(x_{i+3}-x) + 3h(x_{i+3}-x)^2 & \\ -3(x_{i+3}-x)^3, & x \in [x_{i+2}, x_{i+3}], \\ (x_{i+4}-x)^3, & x \in [x_{i+3}, x_{i+4}], \\ 0 \quad , & \text{otherwise.} \end{cases}$$

In Ramadan et al. (2007), a quadratic spline method and a cubic spline method are constructed to approximate the solution of the second order BVP (2.28) subject to Neumann boundary conditions. This quadratic spline method is defined in the form

$$S_i(x) = a_i(x-x_i)^2 + b_i(x-x_i) + c_i. \qquad (2.36)$$

Besides, the coefficients are determined as

$$a_i = \frac{1}{2}F_{i+1/2},$$

$$b_i = D_i,$$

and

$$c_i = s_{i+1/2} - \frac{h}{2} D_i - \frac{h^2}{8} F_{i+1/2},$$

where $s_{i+1/2} = S_i(x_{i+1/2})$, $D_i = S_i'(x_i)$, and $F_{i+1/2} = S_i''(x_{i+1/2})$. On the other hand, the cubic spline method is found to be identical to the cubic spline method (2.31). They proved that both quadratic and cubic spline methods are of order $O(h^2)$.

Rashidinia et al. (2008) developed a cubic non-polynomial spline method for the solution of the second order BVP (2.34) subject to Dirichlet boundary conditions. They expressed the cubic non-polynomial spline as

$$S_i(x) = a_i + b_i(x - x_i) + c_i \sin\tau(x - x_i) + d_i \cos\tau(x - x_i). \qquad (2.37)$$

Moreover, the coefficients of the non-polynomial spline (2.37) are computed to be

$$a_i = u_i + \frac{\mu_i}{\tau^2},$$

$$b_i = \frac{u_{i+1} - u_i}{h} + \frac{\mu_{i+1} - \mu_i}{\tau\theta},$$

$$c_i = \frac{\mu_i \cos(\theta) - \mu_{i+1}}{\tau^2 \sin(\theta)},$$

and

$$d_i = -\frac{\mu_i}{\tau^2},$$

where $u_i = u(x_i)$, $\mu_i = S_i''(x_i)$, $\mu_{i+1} = S_i''(x_{i+1})$, and $\theta = h\tau$. The main relation is given by

$$\alpha\,\mu_{i-1} + 2\beta\,\mu_i + \alpha\mu_{i+1} = \frac{1}{h^2}(u(x_{i+1}) - 2u(x_i) + u(x_{i-1})),$$

where $\alpha = \frac{\theta\csc(\theta)-1}{h^2}$, $\beta = \frac{1-\theta\cot(\theta)}{h^2}$ and $i = 1, 2, \ldots, n-1$. They showed that the spline method (2.37) is of order $O(h^2)$ for any values of α and β such that $\alpha + \beta = \frac{1}{2}$, whereas it is optimal $O(h^2)$ for $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$.

Rashidinia et al. (2009) proposed a quintic non-polynomial spline method to approximate the solution of the second order BVP (2.28) subject to Dirichlet boundary conditions. They considered the quintic non-polynomial spline method in the following form

$$\begin{aligned} S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 \\ + e_i \sin\tau(x-x_i) + f_i \cos\tau(x-x_i). \end{aligned} \tag{2.38}$$

The coefficients above are calculated as

$$a_i = u_i - \frac{F_i}{\tau^4},$$

$$b_i = \frac{u_{i+1}-u_i}{h} + \frac{F_i - F_{i+1}}{\tau^3\theta} - \frac{h(\mu_{i+1}+2\mu_i)}{6} - \frac{h(2F_i + F_{i+1})}{6\tau^2},$$

$$c_i = \frac{1}{2}(\mu_i + \frac{F_i}{\tau^2}),$$

$$d_i = \frac{\mu_{i+1}-\mu_i}{6h} + \frac{F_{i+1}-F_i}{6\theta\tau},$$

$$e_i = \frac{F_{i+1} - F_i\cos(\theta)}{\tau^4\sin(\theta)},$$

and

$$f_i = \frac{F_i}{\tau^4},$$

where $\theta = \tau h$, $S_i(x_i) = u_i$, $S_i(x_{i+1}) = u_{i+1}$, $S_i''(x_i) = \mu_i$, $S_i''(x_{i+1}) = \mu_{i+1}$, $S_i^{(4)}(x_i) = F_i$,

and $S_i^{(4)}(x_{i+1}) = F_{i+1}$. It turns out that the main relation of their work is

$$p\mu_{i-2} + r\,\mu_{i-1} + s\mu_i + r\mu_{i+1} + p\mu_{i+2} =$$
$$\frac{1}{h^2}(\alpha(u_{i+2} + u_{i-2}) + 2(\beta - \alpha)(u_{i+1} + u_{i-1}) + (2\alpha - 4\beta)u_i),$$

where

$$p = \alpha_1 + \frac{\alpha}{6},$$

$$r = 2(\frac{1}{6}(2\alpha + \beta) - (\alpha_1 - \beta_1)),$$

$$s = 2(\frac{1}{6}(\alpha + 4\beta) + (\alpha_1 - 2\beta_1)),$$

$$\alpha = \frac{\theta\csc(\theta) - 1}{\theta^2},$$

$$\beta = \frac{1 - \theta\cot(\theta)}{\theta^2},$$

$$\alpha_1 = \frac{1}{\theta^2}(\frac{1}{6} - \alpha),$$

and

$$\beta_1 = \frac{1}{\theta^2}(\frac{1}{3} - \beta).$$

They verified that the spline method (2.38) is of order $O(h^2)$ for $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{4}$, while it is of order $O(h^4)$ when $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{3}$. Moreover, this method is of order $O(h^6)$ if $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$.

Raslan, Al-Jeaid and Aboualnaja (2010) considered the solution of the second order BVP of the form

$$k(x)u''(x) + p(x)u'(x) + q(x)u(x) = f(x, u(x)), \qquad (2.39)$$

subject to Dirichlet boundary conditions, using quartic and quintic B-spline methods. Details are given only for the calculation in the quartic spline case. Therefore, the quartic B-spline method is expressed as follows

$$S_i(x) = \sum_{i=-2}^{m+1} c_i B_i(x), \qquad (2.40)$$

where the basis function $B_i(x)$ is defined as

$$B_i(x) = \frac{1}{24h^4} \begin{cases} (x - x_{i-2})^4, & x \in [x_i, x_{i+1}], \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4, & x \in [x_{i+1}, x_{i+2}], \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4 & \\ +10(x - x_i)^4, & x \in [x_{i+2}, x_{i+3}], \\ (x - x_{i+3})^4 - 5(x - x_{i+2})^4, & x \in [x_{i+3}, x_{i+4}], \\ (x - x_{i+3})^4, & x \in [x_{i+3}, x_{i+4}], \\ 0 \quad , & \text{otherwise.} \end{cases}$$

They demonstrated that the presented spline methods are of order $O(h^4)$.

According to Hamid et al. (2010), the second order BVPs (2.25) subject to Dirichlet boundary conditions, are solved using a cubic non-polynomial B-spline method. The cubic non-polynomial B-spline method is defined as

$$s(x)=\sum_{i=-2}^{n-1} C_i TB_i^3(x), \tag{2.41}$$

where the trigonometric basis functions $TB_i^3(x)$ is defined as follows

$$TB_i^3(x)=\frac{1}{\theta}\begin{cases}\sigma^3(x_i), & x\in[x_i,x_{i+1}],\\ \sigma(x_i)(\sigma(x_i)\varsigma(x_{i+2})+\varsigma(x_{i+3})\sigma(x_{i+1})) & \\ +\varsigma(x_{i+4})\sigma^2(x_{i+1}), & x\in[x_{i+1},x_{i+2}],\\ \sigma(x_i)\varsigma^2(x_{i+3}) & \\ +\varsigma(x_{i+4})(\sigma(x_{i+1})\varsigma(x_{i+3})+\varsigma(x_{i+4})\sigma(x_{i+2})), & x\in[x_{i+2},x_{i+3}],\\ \varsigma^3(x_{i+4}), & x\in[x_{i+3},x_{i+4}].\end{cases}$$

where $\sigma(x_i)=\sin(\frac{x-x_i}{2})$, $\varsigma(x_i)=\sin(\frac{x_i-x}{2})$, and $\theta=\sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})$.

Jalilian (2010) used the quintic non-polynomial spline method (2.38) to approximate the solution of a second order Bratu type problem in one dimension given by

$$\begin{aligned} u''(x)+\lambda e^{u(x)}&=0,\\ u(0)=u(1)&=0. \end{aligned} \tag{2.42}$$

The spline method (2.38) is proved to be of order $O(h^6)$ for $\alpha=\frac{1}{12}$ and $\beta=\frac{5}{12}$.

Zahra et al. (2010) presented a cubic non-polynomial spline for the numerical solution of the second order BVPs (2.28) subject to Neumann boundary conditions. They wrote the cubic non-polynomial spline method in the form of

$$S_i(x) = a_i \sin\tau(x - x_i) + b_i \cos\tau(x - x_i) + c_i(x - x_i) + d_i. \qquad (2.43)$$

The coefficients of the above method are found to be

$$a_i = -\frac{1}{2\tau^3}(T_{i+1} + T_i),$$

$$b_i = \frac{\tan(\frac{\theta}{2})}{2\tau^3}(T_{i+1} + T_i) - \frac{\sec(\frac{\theta}{2})}{\tau^2}\mu_{i+1/2},$$

$$c_i = D_i + \frac{1}{2\tau^2}(T_{i+1} + T_i),$$

and

$$d_i = S_{i+1/2} + \frac{1}{\tau^2}\mu_{i+1/2} - \frac{h}{2}D_i - \frac{h}{4\tau^2}(T_{i+1} + T_i),$$

where $S_{i+1/2} = S_i(x_{i+1/2})$, $D_i = S_i'(x_i)$, $\mu_{i+1/2} = S_i''(x_{i+1/2})$, $\frac{T_{i+1} + T_i}{2} = S_i'''(x_i)$, and $\theta = h\tau$. Additionally, they derived the main relation as

$$S_{i-3/2} - 2S_{i-1/2} + S_{i+1/2} = h^2(\alpha\,\mu_{i-3/2} + \beta\,\mu_{i-1/2} + \alpha\mu_{i+1/2}), \qquad (2.44)$$

where $\alpha = \frac{\theta - 2\sin(\frac{\theta}{2})}{2\theta^2\sin(\frac{\theta}{2})}$, $\beta = \frac{2\theta\sin^2(\frac{\theta}{2}) + 4\sin(\frac{\theta}{2}) - \theta(1 + \cos(\theta))}{2\theta^2\sin(\frac{\theta}{2})}$ and $i = 2, 3, \ldots, n-1$. Since the previous relation (2.44) gives $(n-2)$ equations in n unknowns, two equations are added, one at each end of the interval of integration using Taylor series and the undetermined coefficients method. These equations are

$$-hS_0' - S_{1/2} + S_{3/2} = h^2(w_0\,\mu_{1/2} + w_1\,\mu_{3/2} + w_2\,\mu_{5/2} + w_3\,\mu_{7/2}) \text{ at } i = 1,$$

and

$$S_{n-3/2} - S_{n-1/2} + hS'_n = h^2(w_0\,\mu_{n-1/2} + w_1\,\mu_{n-3/2} + w_2\,\mu_{n-5/2} + w_3\,\mu_{n-7/2}) \text{ at } i = n.$$

They showed that for $\alpha = \frac{1}{12}$, $\beta = \frac{10}{12}$ and $(w_0, w_1, w_2, w_3) = (1, -\frac{1}{24}, \frac{1}{24}, 0)$, the spline method (2.43) is of order $O(h^3)$. On the other hand, it is of order $O(h^4)$ if $(w_0, w_1, w_2, w_3) = (\frac{6007}{5760}, -\frac{981}{5760}, \frac{981}{5760}, -\frac{247}{5760})$.

Caglar et al. (2010) used the cubic B-spline method (2.35) to find the approximate solution of the second order Bratu type problem (2.42).

In Srivastava et al. (2011), the quintic non-polynomial spline method (2.38) is extended to compute the approximate solution of the second order BVP (2.28) subject to Neumann boundary conditions. They also proved that the spline method is of order $O(h^2)$ for $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{4}$, whilst it is of order $O(h^4)$ for $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{3}$.

Chang et al. (2011) used the cubic B-spline method (2.35) to approximate the solution of the second order BVP (2.25) subject to Dirichlet boundary conditions.

Hamid et al. (2011) solved the second order BVP (2.25) subject to Dirichlet boundary conditions via extended cubic B-spline method. The extended cubic B-spline of order four is an expansion of the cubic B-spline with one shape parameter, namely λ. The extended cubic B-spline method is given by

$$s(x) = \sum_{i=-2}^{n-1} C_i TB_i^4(x), \tag{2.45}$$

where

$$TB_i^4(x) = \frac{1}{24h^4} \begin{cases} -4h(\lambda-1)(x-x_i)^3 + 3\lambda(x-x_i)^4, & x \in [x_i, x_{i+1}], \\ (4-\lambda)h^4 + 12h^3(x-x_{i+1}) \\ +6h^2(2+\lambda)(x-x_{i+1})^2 \\ -12h(x-x_{i+1})^3 - 3\lambda(x-x_{i+1})^4, & x \in [x_{i+1}, x_{i+2}], \\ (16+2\lambda)h^4 - 12h^2(2+\lambda)(x-x_{i+2})^2 \\ +12h(1+\lambda)(x-x_{i+2})^3 \\ -3\lambda(x-x_{i+2})^4, & x \in [x_{i+2}, x_{i+3}], \\ -(h+x_{i+3}-x)^3((\lambda-4) \\ +3\lambda(x-x_{i+3})), & x \in [x_{i+3}, x_{i+4}], \\ 0, & \text{otherwise.} \end{cases}$$

The value of λ can be optimized such that the generated approximate solution has minimal global error.

Liu et al. (2011) suggested an algorithm based on quartic spline method for solving the second order BVP of the form

$$u''(x) = f(x, u(x)), \tag{2.46}$$

subject to Neumann boundary conditions. They introduced the quartic spline method in the form of

$$s(x) = s_{i-1} + hs'_{i-1}\, B(t) + h^2 s''_{i-1}\, C(t) + h^2 s''_i\, D(t) + h^3 s'''_{i-1}\, E(t), \tag{2.47}$$

where $t = \dfrac{x - x_i}{h}$, $s_j^{(m)} = s^{(m)}(x_j)$, $m = 0, 1, 2$ and 3,

$$B(t) = t,$$

$$C(t) = \frac{t^2(6 - t^2)}{12},$$

$$D(t) = \frac{t^4}{12},$$

and

$$E(t) = \frac{t^3(2 - t)}{12}.$$

The following relations are developed from the continuity conditions:

$$s_i = s_{i-1} + hs'_{i-1} + \frac{5h^2}{12} s''_{i-1} + \frac{h^2}{12} s''_i + \frac{h^3}{12} s'''_{i-1},$$

$$s'_i = s'_{i-1} + \frac{2h}{3} s''_{i-1} + \frac{2h}{3} s''_i + \frac{h^2}{6} s'''_{i-1},$$

and

$$s'''_i = -s'''_{i-1} + \frac{2}{h}(s''_i - s''_{i-1}).$$

The main recurrence relation declared in this article is

$$\frac{1}{12} s''_{i+1} + \frac{5}{6} s''_i + \frac{1}{12} s''_{i-1} = \frac{1}{h^2}(s_{i+1} - 2s_i + s_{i-1}).$$

They discovered that scheme (2.47) can attain sixth order of accuracy at the internal grid points and fourth order of accuracy on the boundary or close to the boundary for solving (2.46) in linear form. Moreover, the spline method (2.47) is of order $O(h^4)$.

Al-Said, Noor, Almualim, Kokkinis and Coletsos (2011) used a quartic spline method to derive a numerical algorithm for approximating the solution for the second order BVP (2.28) subject to Dirichlet boundary conditions. The quartic spline method is written in the following form:

$$S_i(x) = a_i(x-x_i)^4 + b_i(x-x_i)^3 + c_i(x-x_i)^3 + d_i(x-x_i) + e_i. \tag{2.48}$$

The coefficients of this method are calculated as

$$a_i = \frac{1}{48}(F_i + F_{i+1}),$$

$$b_i = \frac{1}{6h}(D_{i+1} - D_i) - \frac{h}{24}(F_i + F_{i+1}),$$

$$c_i = \frac{1}{2}D_i,$$

$$d_i = \frac{1}{h}(s_{i+1} - s_i) - \frac{h}{6}(D_{i+1} + 2D_i) + \frac{h^3}{48}(F_i + F_{i+1}),$$

and

$$e_i = s_i,$$

where $s_i = S_i(x_i)$, $s_{i+1} = S_i(x_{i+1})$, $D_i = S_i''(x_i)$, $D_{i+1} = S_i''(x_{i+1})$, and $\frac{1}{2}(F_i + F_{i+1}) = S_i^{(4)}(x_i).$ They found the main relation as

$$s_{i+1} - 2s_i + s_{i-1} = \frac{h^2}{12}(D_{i+1} + 10D_i + D_{i-1}).$$

They also showed that the spline method (2.48) is of order $O(h^4)$.

According to Chen and Wong (2012), a cubic spline method was presented to approximate the second order BVP (2.28) subject to Dirichlet boundary conditions. They expressed the cubic spline method in the form of

$$S_i(x) = a_{i-1}\frac{(x_i - x)^3}{6h} + a_i\frac{(x - x_{i-1})^3}{6h} + b_i\frac{(x_i - x)}{h} + c_i\frac{(x - x_{i-1})}{h}, \qquad (2.49)$$

where

$$a_i = S_i''(x_i),$$

$$b_i = S_{i-1} - \frac{h^2 - p^2}{6}S_{i-1}''(x_{i-1}),$$

$$c_i = S_i - \frac{h^2 - p^2}{6}S_i''(x_i),$$

and

$p \in (0, h]$ is a given constant.

The recurrence relation is stated as

$$(\frac{h^2 - p^2}{6})c_{i+1} + 2(\frac{2h^2 + p^2}{6})c_i + (\frac{h^2 - p^2}{6})c_{i-1} = s_{i+1} - 2s_i + s_{i-1}.$$

In addition, it is proved that the spline method (2.49) is of order $O(h^4)$ if $p = \frac{h}{\sqrt{2}}$, and of order $O(h^2)$ for other values of p.

Shafie and Majid (2012) discussed the implementations of three numerical methods to approximate the solution of the second order BVP (2.25). These methods are the cubic B-spline method (2.35), the shooting method and finite difference method.

Zarebnia and Sarvari (2012) applied the quintic non-polynoimal spline method (2.38) to the Bratu type problem (2.42). They also confirmed that the spline method is of order $O(h^6)$ for $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$.

El hajaji et al. (2013) developed a numerical algorithm for solving the Bratu type problem (2.42) using the cubic B-spline method (2.35) and the generalized Newton method. They deduced that the cubic B-spline method (2.35) is of order $O(h^2)$.

Kalyani and Rama Chandra Rao (2013) constructed a scheme based on the cubic non-polynomial spline method (2.37) to approximate the solution of the second order BVP (2.25) subject to Dirichlet boundary conditions. The following approximation formulae are derived from the finite difference approach for the first derivative:

$$u_i' = \frac{u_{i+1} - u_{i-1}}{2h},$$

$$u_{i+1}' = \frac{3u_{i+1} - 4u_i + u_{i-1}}{2h}, \tag{2.50}$$

and

$$u_{i-1}' = \frac{-3u_{i-1} + 4u_i - u_{i+1}}{2h}.$$

They substituted $\mu_i = u_i''$ in the second order BVP (2.25) to get

$$\mu_i + p_i u_i' + q_i u_i = r_i. \tag{2.51}$$

The formulae (2.50) and (2.51) are imposed into BVP (2.25) to obtain the following main relation:

$$(\frac{3h}{2}\alpha\, p_{i-1} + h\beta p_i - \frac{h}{2}\alpha\, p_{i+1} - h^2\alpha\, q_{i-1} - 1)u_{i-1} + (-2h\alpha\, p_{i-1}$$
$$+ 2h\alpha\, p_{i+1} - 2h^2\beta\, q_{i-1} + 2)u_i + (\frac{h}{2}\alpha\, p_{i-1} - h\beta p_i - \frac{3h}{2}\alpha\, p_{i+1}$$
$$- h^2\alpha\, q_{i-1} - 1)u_{i+1} = -h^2(\alpha\, r_{i-1} + 2\beta\, r_i + \alpha\, r_{i+1}).$$

Zarebnia and Sarvari (2013) used the cubic non-polynomial spline method (2.33) to solve the Bratu type problem (2.42). They illustrated that the presented cubic non-polynomial spline method is of order $O(h^2)$ for $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{3}$, whereas it is of order $O(h^4)$ for $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$.

Dabounou, Khayyari and Lamnii (2014) proposed a cubic non-polynomial B-spline method for the numerical solution of the second order BVP (2.25) subject to Dirichlet and Neumann boundary conditions. They considered the partition

$$P = \{a = x_{-2} = x_{-1} = x_0 < x_1 < \cdots < x_{n-1} < x_n = x_{n+1} = x_{n+2} = b\}.$$

The proposed cubic non-polynomial B-spline method is given by

$$S_i(x) = \sum_{i=-2}^{n+2} c_i HB_i^3(x), \tag{2.52}$$

where

$$HB_i^3(x) = c\begin{cases} -1+\cos(\lambda(x-x_i)), & x\in[x_i, x_{i+1}), \\ 2(\cos(\lambda h)-\cos(\lambda\frac{h}{2})\cos(\lambda(x_{i+1}+\frac{h}{2}-x)), & x\in[x_{i+1}, x_{i+2}), \\ -1+\cos(\lambda(x-x_{i+3})), & x\in[x_{i+2}, x_{i+3}], \\ 0\quad , & \text{otherwise.} \end{cases}$$

Additionally, the particular cubic non-polynomial B-spline functions around the neighborhood of the boundaries are

$$HB_{-2}^{3}(x)=c\begin{cases}2(-1+\cos(\lambda(h-x+a)), & x\in[x_0,x_1),\\ 0\quad, & \text{otherwise,}\end{cases}$$

$$HB_{-1}^{3}(x)=c\begin{cases}1+2\cos(\lambda h)-\cos\lambda(-a+x) & \\ -2\cos\lambda(a+h-x), & x\in[x_0,x_1),\\ -1+\cos\lambda(a+2h-x), & x\in[x_1,x_2),\\ 0\quad, & \text{otherwise,}\end{cases}$$

$$HB_{n+1}^{3}(x)=c\begin{cases}-1+\cos\lambda(b-2h-x), & x\in[x_n,x_{n+1}),\\ 1+2\cos(\lambda h)-\cos\lambda(b-x) & \\ -2\cos\lambda(b-h-x), & x\in[x_{n+1},x_{n+2}),\\ 0\quad, & \text{otherwise,}\end{cases}$$

and

$$HB_{n+2}^{3}(x)=c\begin{cases}2(-1+\cos(\lambda(b-h-x)), & x\in[x_n,x_{n+1}),\\ 0\quad, & \text{otherwise,}\end{cases}$$

where $c=\dfrac{1}{2(-1+\cos(\lambda h))}$. They showed that the spline method (2.52) is of order $O(h^4)$.

According to El Khayyari and Lamnii (2014), a quartic hyperbolic B-spline method is proposed for the numerical solution of the second order BVP (2.25) subject to Dirichlet and Neumann boundary conditions. The following partition is considered

$$P=\{a=x_{-3}=x_{-2}=x_{-1}=x_0<x_1<\cdots<x_{n-1}<x_n=x_{n+1}=x_{n+2}=x_{n+3}=b\}.$$

The quartic hyperbolic B-spline method is given by

$$S_i(x) = \sum_{i=-3}^{n+3} c_i HB_i^4(x), \tag{2.53}$$

where

$$HB_i^4(x) = c \begin{cases} -x + x_i + \sinh(x - x_i), & x \in [x_i, x_{i+1}), \\ x - x_{i+2} + 2(x - x_{i+1})\cosh(h) & \\ + 2\sinh(x_{i+1} - x) + \sinh(x_{i+2} - x), & x \in [x_{i+1}, x_{i+2}), \\ -x + x_{i+2} + 2(x_{i+3} - x)\cosh(h) & \\ -\sinh(x_{i+2} - x) - 2\sinh(x_{i+3} - x), & x \in [x_{i+2}, x_{i+3}), \\ x - x_{i+4} + \sinh(x_{i+4} - x), & x \in [x_{i+3}, x_{i+4}], \\ 0 \quad , & \text{otherwise}, \end{cases}$$

for $c = \dfrac{1}{4h\sin^2(\frac{h}{2})}$ and $i = 0,1,\ldots,n-4.$ Moreover, they defined the quartic hyperbolic B-spline functions around the neighborhood of the boundaries as

$$HB_{-3}^4(x) = \begin{cases} \dfrac{-h + x - a + \sinh(h - x - a)}{-h + \sinh(h)}, & x \in [x_0, x_1), \\ 0, & \text{otherwise}, \end{cases}$$

$$HB_{-2}^{4}(x)=\begin{cases}\dfrac{x-a-\sinh(h)+\sinh(h-x+a)}{h-\sinh(h)}, & x\in[x_0,x_1),\\[2ex] \dfrac{a-x-2x\sinh(h)+2a\sinh(h)+2\sinh(h)}{2h\cosh(h)-2\sinh(h)} \\ +\dfrac{-2\sinh(h-x+a)+\sinh(x-a)}{2h\cosh(h)-2\sinh(h)}, & x\in[x_1,x_2),\\[2ex] \dfrac{x-2h-a+\sinh(2h-x+a)}{2h\cosh(h)-2\sinh(h)}, & x\in[x_2,x_3),\\[2ex] 0, & \text{otherwise,}\end{cases}$$

$$HB_{-1}^{4}(x)=\begin{cases}\dfrac{x-a-\sinh(h)}{-2h+2h\cosh(h)}+\dfrac{x-a+2(x-a)\cosh(h)-2\sinh(h)}{2h\cosh(h)-2\sinh(h)} \\ +\dfrac{2\sinh(h-x+a)-\sinh(x-a)}{2h\cosh(h)-2\sinh(h)}, & x\in[x_0,x_1),\\[2ex] 1+\dfrac{h-\sinh(h)}{-2h+2h\cosh(h)} \\ +\dfrac{2h-x+a-\sinh(2h-x+a)}{2h\cosh(h)-2\sinh(h)} \\ +\dfrac{2(h-x+a)\cosh(h)-\sinh(h)}{4h\sinh^2(\frac{h}{2})} \\ +\dfrac{\sinh(h-x+a)+\sinh(2h-x+a)}{4h\sinh^2(\frac{h}{2})}, & x\in[x_1,x_2),\\[2ex] \dfrac{-3h+x-a+\sinh(3h-x+a)}{4h\sinh^2(\frac{h}{2})}, & x\in[x_2,x_3),\\[2ex] 0, & \text{otherwise,}\end{cases}$$

$$
HB_{n+1}^{4}(x)=\begin{cases}
\dfrac{b-x-3h+\sinh(3h-b+x)}{4h\sinh^{2}(\frac{h}{2})}, & x\in[x_n,x_{n+1}),\\[2ex]
\dfrac{h-\sinh(h)}{2h-2h\cosh(h)}-\dfrac{b-2h-x+\sinh(2h-x+a)}{2h\cosh(h)-2\sinh(h)} & \\
+\dfrac{2(b-2h-x)\cosh(h)+\sinh(h)}{4h\sinh^{2}(\frac{h}{2})} & \\
+\dfrac{\sinh(-b+h+x)+\sinh(2h+x-b)}{4h\sinh^{2}(\frac{h}{2})}, & x\in[x_{n+1},x_{n+2}),\\[2ex]
\dfrac{h(b-x)\cosh(2h)+2h\cos(\frac{3h}{2}-b+x)\sinh(\frac{h}{2})}{2h(-1+\cosh(h))(h\cosh(h)-\sinh(h))} & \\
+\dfrac{2h\sinh(h)-b\sinh(h)+x\sinh(h)-h\sinh(h)}{2h(-1+\cosh(h))(h\cosh(h)-\sinh(h))} & \\
\dfrac{\sinh(h)\sinh(b-x)-h\sinh(h+b-x)}{2h(-1+\cosh(h))(h\cosh(h)-\sinh(h))}, & x\in[x_{n+2},x_{n+3}),\\[2ex]
0, & \text{otherwise,}
\end{cases}
$$

$$
HB_{n+2}^{4}(x)=\begin{cases}
\dfrac{b-x-2h+\sinh(2h-b+x)}{2h\cosh(h)-2\sinh(h)}, & x\in[x_n,x_{n+1}),\\[2ex]
\dfrac{-\sinh(h)(b-x-2h+2(h+x)\cosh(h))}{2(h+\sinh(h))(h\cosh(h)-\sinh(h))} & \\
+\dfrac{-\sinh(h)\sinh(-x+b)+h(b+x)+b\sinh(h)}{2(h+\sinh(h))(h\cosh(h)-\sinh(h))} & \\
+\dfrac{-h(2\sinh(-b+x+h)+\sinh(2h+x-b)}{2(h+\sinh(h))(h\cosh(h)-\sinh(h))}, & x\in[x_{n+1},x_{n+2}),\\[2ex]
0, & \text{otherwise,}
\end{cases}
$$

and

$$HB_{n+3}^4(x) = \begin{cases} \dfrac{h + x - b + \sinh(-h - x + b)}{h - \sinh(h)}, & x \in [x_n, x_{n+1}), \\ 0, & \text{otherwise.} \end{cases}$$

It is showed that the spline method (2.53) is of order $O(h^4)$.

Al-Towaiq and Ala'yed (2014) proposed an algorithm based on cubic spline method on the finite difference method to solve the Bratu type problem (2.42) numerically. The proposed cubic spline method is expressed as

$$\begin{aligned} S_i(x) = \mu_i \frac{(x_{i+1} - x)^3}{6h} + \mu_{i+1} \frac{(x - x_i)^3}{6h} \\ + (w_i - \frac{h^2}{6}\mu_i)\frac{(x_{i+1} - x)}{h} + (w_{i+1} - \frac{h^2}{6}\mu_{i+1})\frac{(x - x_i)}{h}, \end{aligned} \tag{2.54}$$

where $\mu_i = S_i''(x_i)$ and w_i is the approximate solution of $u(x)$ at the point x_i. They found that the main relation is

$$\mu_{i-1} + 4\mu_i + \mu_{i+1} = \frac{6}{h^2}(w_{i-1} - 2w_i + w_{i+1}).$$

El hajaji, Hilal, Jalila and Mhamed (2014) used the cubic B-spline method (2.35) and the generalized Newton method for solving the Troesch's problem given by

$$\begin{aligned} u''(x) = \lambda \sinh(\lambda u(x)), \\ u(0) = 0 \text{ and } u(1) = 1. \end{aligned} \tag{2.55}$$

They showed that the cubic B-spline method is of order $O(h^2)$.

Zarebnia and Hoshyar (2014) presented a cubic non-polynomial spline for the approximate solution of the Bratu type problem (2.42). They derived the cubic non-polynomial spline in the form

$$S_i(x) = a_{i+1/2}\sin\tau(x - x_{i+1/2}) + b_{i+1/2}\cos\tau(x - x_{i+1/2}) + c_{i+1/2}(x - x_{i+1/2}) + d_{i+1/2}. \quad (2.56)$$

The coefficients in formula (2.56) are found to be

$$a_{i+1/2} = -\frac{1}{2\tau^2 \sin(\theta)}(\mu_{i+1/2}\cos(\theta) - \mu_{i+3/2}),$$

$$b_{i+1/2} = -h^2 \frac{\mu_{i+1/2}}{\theta^2},$$

$$c_{i+1/2} = \frac{D_{i+1} + D_{i+1/2}}{2} - \frac{(\mu_{i+1/2}\cos(\theta) - \mu_{i+1/2})(1+\cos(\theta))}{2\tau\sin(\theta)},$$

and

$$d_i = S_{i+1/2} + \frac{1}{\tau^2}\mu_{i+1/2},$$

where $S_{i+1/2} = S_i(x_{i+1/2}) = u_{i+1/2}$, $D_{i+1/2} = S_i'(x_{i+1/2})$, $D_{i+3/2} = S_i'(x_{i+3/2})$, $\mu_{i+1/2} = S_i''(x_{i+1/2})$, $\mu_{i+3/2} = S_i''(x_{i+3/2})$, and $\theta = h\tau$. They obtained the main relation as

$$u_{i-3/2} - 2u_{i-1/2} + u_{i+1/2} = h^2(\alpha\,\mu_{i-3/2} + 2\beta\,\mu_{i-1/2} + \alpha\mu_{i+1/2}), \quad (2.57)$$

where $\alpha = \frac{1}{\theta\sin(\theta)} - \frac{1}{\theta^2}$, $\beta = \frac{1}{\theta^2} - \frac{\cos(\theta)}{\theta\sin(\theta)}$ and $i = 2, 3, \ldots, n-1$. As the formula (2.57) provide $(n-2)$ equations with n unknowns, a couple of equations are added, one at each end of the interval. These extra equations are obtained using Taylor series and undetermined coefficients method as

$$2u_0 - 3u_{1/2} + u_{3/2} = h^2\left(\frac{-1}{120}\mu_0 + \frac{5}{8}\mu_{1/2} + \frac{7}{48}\mu_{3/2} - \frac{1}{80}\mu_{5/2}\right) \text{ at } i = 1,$$

and

$$2u_n - 3u_{n-1/2} + u_{n-3/2} = h^2\left(\frac{-1}{120}\mu_n + \frac{5}{8}\mu_{n-1/2} + \frac{7}{48}\mu_{n-3/2} - \frac{1}{80}\mu_{n-5/2}\right) \text{ at } i = n,$$

where $\mu_j = -\lambda e^{u(x_j)}$. They demonstrated that the spline method (2.56) is of order $O(h^4)$ for $\alpha = \frac{1}{12}$ and $\beta = \frac{10}{12}$.

Rashidinia and Sharifi (2015) applied the quartic B-spline method (2.40) to the second order BVP (2.25) subject to Dirichlet boundary conditions. The resulting system consists of $(n+3)$ equations in $(n+4)$ unknowns. In order to solve this system, one more equation at the point $x_{1/2}$ is added. They proved that this spline method is of order $O(h^5)$.

Finally, the highlights of this section are concluded in Table 2.3.

Table 2.3

Highlights of Literature Review on Spline Methods for Solving Second Order BVPs

Authors	**Methods**	**Advantages**	**Disadvantages**	**Types of boundary conditions**
Bickley (1968)	Proposed a cubic spline method	The method is easy to implement.	The method solves only linear second order BVPs. Besides, he does not provide the error analysis.	Mix boundary conditions
Albasiny and Hoskins (1969)	Employed the cubic spline method suggested by Ahlberg et al. (1967)	The convergence analysis is considered, and it is proved that this method is of order $O(h^4)$.	The method is used to approximate solution for two specific cases of the second order BVPs i.e. linear with/without the presence of $u'(x)$.	Dirichlet boundary conditions

Table 2.3

Continued

Fyfe (1969)	Applied the cubic spline method reported in Bickley (1968)	The convergence analysis is investigated. The method provides accuracy of order $O(h^4)$.	The method approximates only linear second order BVPs.	Dirichlet boundary conditions
Usmani and Warsi (1980)	Developed a quintic spline method	They investigate the convergence analysis. Moreover, the method is illustrated to be of $O(h^4)$.	The method is employed only to linear second order BVPs in which $u'(x)$ does not appear.	Dirichlet boundary conditions
Chawla and Subramanian (1988)	Constructed a scheme based on the cubic spline method proposed by Ahlberg et al. (1967)	The method is shown to be fourth order of convergence. This method also solves linear and nonlinear second order BVPs.	The method can be used only after some other fourth order method such as finite difference method is employed	Mix boundary conditions

Table 2.3

Continued

Al-Said (1998)	Proposed a cubic spline method	The convergence analysis of the method is established in details.	The method can only be applied to linear second order BVPs without the presence of $u'(x)$. The accuracy of the method is low.	Dirichlet boundary conditions
Hossam et al. (2003)	Derived a quadratic spline method	The details of the convergence analysis of this method is provided.	The method only approximates the solution for the linear second order BVPs in which $u'(x)$ does not appear. The accuracy of the method in terms of error is not encouraging.	Dirichlet boundary conditions
Khan (2004)	Developed a cubic non-polynomial spline method	The method can attain fourth order of accuracy.	The method is constructed only for the linear second order BVPs without the presence of $u'(x)$.	Dirichlet boundary conditions

Table 2.3

Continued

Caglar et al. (2006)	Employed a cubic B-spline method	The details of the implementation of this method are presented in step by step fashion.	The method is solely applied to special linear second order BVPs when the term $u(x)$ is absence.	Dirichlet boundary conditions
Ramadan et al. (2007)	Proposed a quadratic and a cubic spline method	The error analysis of each method is presented and it is shown to be of order two for both cases.	The method can only solve the linear second order BVPs without the existence of $u'(x)$. Moreover, the accuracy of these methods are not encouraging.	Neumann boundary conditions
Rashidinia et al. (2008)	Developed a cubic non-polynomial spline method	They provide the full details of the convergence analysis. Furthermore, the method is of order two.	The method is solely employed to linear second order BVPs with the term $u(x)$ is absence. The method is of lower accuracy.	Dirichlet boundary conditions

Table 2.3

Continued

Rashidinia et al. (2009)	Proposed a quintic non-polynomial spline method	The method can achieve second, fourth or sixth order of accuracy depending on the values of α and β in the recurrence relation.	The method is employed only to special case of the linear second order BVPs i.e. when the term $u'(x)$ is absence.	Dirichlet boundary conditions
Raslan et al. (2010)	Derived quartic and quintic B-spline methods	Details are given only for the calculation in the quartic spline case. Both methods can achieve fourth order of accuracy.	Details of derivations are not provided for the quintic spline. The methods are constructed only to approximate a special case of the nonlinear second order BVPs.	Dirichlet boundary conditions

Table 2.3

Continued

Hamid et al. (2010)	Employed a cubic non-polynomial B-spline method	The method is slightly more accurately than the cubic B-spline method if the problems being solved involving trigonometric expressions.	The method can just be applied to the linear second order BVPs. The convergence analysis is not presented for this method.	Dirichlet boundary conditions
Jalilian (2010)	Implemented the quintic non-polynomial spline method reported in Rashidinia et al. (2009)	The method is confirmed to reach sixth order of accuracy for special values of α and β in the recurrence relation.	The method is just valid for special case of nonlinear second order BVP. This problem is known as Bratu type problem (2.42).	Dirichlet boundary conditions
Zahra et al. (2010)	Presented a cubic non-polynomial spline	The method can attain third or fourth order of accuracy depending on the values of α and β in the recurrence relation.	The method constructs only for special case of the linear second order BVPs i.e. when the term $u'(x)$ is not present.	Neumann boundary conditions

Table 2.3

Continued

Caglar et al. (2010)	Used the cubic B-spline method proposed in Caglar et al. (2006)	The implementation of this method is simple and clear.	The method is just applicable for Bratu type problem. Furthermore, they do not provide the error analysis.	Dirichlet boundary conditions
Srivastava et al. (2011)	Applied the quintic non-polynomial spline method reported in Rashidinia et al. (2009)	The method can achieve second or fourth order of accuracy depending on the values of α and β in the recurrence relation.	The method can just be employed to special case of the linear second order BVPs, i.e. BVPs with out the term $u'(x)$	Neumann boundary conditions
Chang et al. (2011)	Employed the cubic B-spline method stated in Caglar et al. (2006)	The implementation of this method is provided in details.	The method is solely applied to the linear second order BVPs. Moreover, the convergence analysis is not given for this method.	Dirichlet boundary conditions

Table 2.3

Continued

Hamid et al. (2011)	Introduced an extended cubic B-spline method of order four based on one parameter λ	The method can achieve more accurate solution by optimizing λ.	The method is developed only for the linear second order BVPs. Moreover, the convergence analysis for this method is not provided.	Dirichlet boundary conditions
Liu et al. (2011)	Proposed an algorithm based on quartic spline method	The spline method is of order four.	The method is just derived for the nonlinear second order BVPs when the term $u'(x)$ is not present.	Neumann boundary conditions
Al-Said et al. (2011)	Developed a quartic spline method	The spline method is of order $O(h^4)$.	The method can just be employed to special case of the linear second order BVPs, i.e. BVPs with out the term $u'(x)$.	Dirichlet boundary conditions

Table 2.3

Continued

Chen and Wong (2012)	Constructed a cubic spline method based on a parameter $p,\ p \in (0,h]$	The convergence analysis is provided and it is proved that this method is of order $O(h^4)$ if $p = \frac{h}{\sqrt{2}}$, and of order $O(h^2)$ for other values of p.	The method can only be implemented to special case of the linear second order BVPs with vanishing $u'(x)$.	Dirichlet boundary conditions
Zarebnia and Sarvari (2012)	Applied the quintic non-polynoimal spline method suggested by Rashidinia et al. (2009)	The method is confirmed to be of order $O(h^6)$ for special values of α and β in the recurrence relation.	The method is just employed to the Bratu type problem.	Dirichlet boundary conditions

Table 2.3

Continued

El hajaji et al. (2013)	Applied the cubic B-spline method stated in Caglar et al. (2006)	The method is of order two.	The method is just employed to the Bratu type problem.	Dirichlet boundary conditions
Kalyani and Rama Chandra Rao (2013)	Used the cubic non-polynomial spline method developed by Khan (2004)	The method can achieve second or fourth order of accuracy depending on the values of α and β in the recurrence relation.	The method is developed solely for the linear second order BVPs.	Dirichlet boundary conditions
Dabounou et al. (2014)	Proposed a cubic non-polynomial B-spline method	The method is of order $O(h^4)$.	The method is just developed for the linear second order BVPs.	Dirichlet and Neumann boundary conditions
El Khayyari and Lamnii (2014)	Presented a quartic hyperbolic B-spline method	The method is of order $O(h^4)$.	The method is developed only for the linear second order BVPs.	Dirichlet and Neumann boundary conditions

Table 2.3

Continued

Al-Towaiq and Ala'yed (2014)	Proposed an algorithm based on cubic spline method on the finite difference method	The method is easy to implement. Moreover, the numerical results confirmed that the method is accurate.	The method can be improved by choosing better extra conditions.	Dirichlet boundary conditions
El hajaji et al. (2014)	Implemented the cubic B-spline method stated in Caglar et al. (2006)	The cubic B-spline method is of order $O(h^2)$.	The method is just valid for Troesch's problem.	Dirichlet boundary conditions
Zarebnia and Hoshyar (2014)	Presented a cubic non-polynomial spline	The method can achieve fourth order of accuracy depending on the values of α and β in the recurrence relation.	The method is just proposed to approximate the Bratu type problem.	Dirichlet boundary conditions

Table 2.3

Continued

Rashidinia and Sharifi (2015)	Applied the quartic B-spline method	This spline method is of order $O(h^5)$.	The method is developed only for the linear second order BVPs.	Dirichlet boundary conditions

CHAPTER THREE
NEW QUARTIC AND QUINTIC SPLINE METHODS

3.1 Introduction

In this chapter, we derive a new quartic spline method and a new quintic spline method to approximate first order IVPs (1.12), first order BVPs (1.13), second order IVPs (1.14) and second order BVPs (1.15). We present the process of derivations, as well as the convergent analysis for each method derived.

3.2 Quartic Spline Method

In this section, we present the construction process of the new quartic spline method as well as the convergence analysis for it.

3.2.1 Construction of Quartic Spline Method

Let P be the partition for the interval $[a,b]$ such that

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\},$$

where $x_i = a + ih,$ and $h = \dfrac{b-a}{n}.$ We let $u(x)$ be the exact solution of problem (1.12), (1.13), (1.14) or (1.15), and s_i be the approximate solution to $u_i = u(x_i)$ obtained by the quartic spline $s_i(x)$ on the interval $[x_i, x_{i+1}]$. Every quartic spline function $S(x)$ has to satisfy the following conditions:

- $S(x) = s_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0,1,\ldots,n-1,$

- $S(a) = u(a), S(b) = u(b),$ and
- $s_i^{(r)}(x_{i+1}) = s_{i+1}^{(r)}(x_{i+1}), \quad r = 0,1,2,3.$

Since quartic spline is of degree four, the third derivative is a linear polynomial, which can be written as follows

$$s_i'''(x) = Z_{i+1}\frac{(x - x_i)}{h} + Z_i\frac{(x_{i+1} - x)}{h}, \tag{3.1}$$

where $Z_i = s_i'''(x_i)$ and $x \in [x_i, x_{i+1}]$. On integrating equation (3.1) three times, we first obtain

$$s_i''(x) = Z_{i+1}\frac{(x - x_i)^2}{2h} - Z_i\frac{(x_{i+1} - x)^2}{2h} + A_i, \tag{3.2a}$$

then followed by

$$s_i'(x) = Z_{i+1}\frac{(x - x_i)^3}{6h} + Z_i\frac{(x_{i+1} - x)^3}{6h} + A_i(x - x_i) + B_i, \tag{3.2b}$$

and finally

$$s_i(x) = Z_{i+1}\frac{(x - x_i)^4}{24h} - Z_i\frac{(x_{i+1} - x)^4}{24h} + A_i(x - x_i)^2 + B_i(x_{i+1} - x) + C_i(x - x_i), \tag{3.2c}$$

where $A_i, B_i,$ and $C_i, i = 0,1,...,n-1,$ are coefficients which need to be determined in terms of u_i, u_{i+1}, μ_i, and Z_i, where $\mu_i = s_i''(x_i)$. In order to derive explicit expressions for the three coefficients of equation (3.2c) i.e. $A_i, B_i,$ and C_i, we define the following relations

$$u_i = s_i(x_i), \tag{3.3}$$

$$u_{i+1} = s_i(x_{i+1}), \tag{3.4}$$

and

$$\mu_i = s_i''(x_i). \tag{3.5}$$

From equations (3.3), (3.4) and (3.5), and by using straightforward calculation, we obtain the following expressions

$$A_i = \frac{\mu_i}{2} + \frac{h}{4} Z_i, \tag{3.6}$$

$$B_i = \frac{u_i}{h} + \frac{h^2}{24} Z_i, \tag{3.7}$$

and

$$C_i = \frac{u_{i+1}}{h} - \frac{h^2}{4} Z_i - \frac{h^2}{24} Z_{i+1} - \frac{h}{2} \mu_i. \tag{3.8}$$

Now, we impose the first and second continuity conditions of quartic spline $s_i(x)$ at the point x_{i+1}, i.e. $s_i^{(r)}(x_{i+1}) = s_{i+1}^{(r)}(x_{i+1}), r = 1,2,$ and the following relations are obtained

$$\frac{5h^2}{24} Z_i + \frac{6h^2}{24} Z_{i+1} + \frac{h^2}{24} Z_{i+2} + \frac{h}{2} \mu_i + \frac{h}{2} \mu_{i+1} = \frac{u_i - 2u_{i+1} + u_{i+2}}{h}, \tag{3.9}$$

and

$$\frac{h}{2} Z_i + \frac{h}{2} Z_{i+1} + \mu_i - \mu_{i+1} = 0, \tag{3.10}$$

respectively. Then, we eliminate μ_i from equations (3.9) and (3.10) to obtain

$$\mu_{i+1} = \frac{u_i - 2u_{i+1} + u_{i+2}}{h^2} + \frac{h}{24} Z_i - \frac{h}{24} Z_{i+2}. \tag{3.11}$$

On substituting equation (3.11) into equation (3.9), we obtain the following main recurrence relation given by

$$\begin{aligned} &Z_{i-1} + 11Z_i + 11Z_{i+1} + Z_{i+2} \\ &= \frac{24}{h^3}(-u_{i-1} + 3u_i - 3u_{i+1} + u_{i+2}), i = 1, \ldots, n-2. \end{aligned} \tag{3.12}$$

Equation (3.12) forms a system of $n-2$ equations with $n+1$ unknowns, which are the Z_i, where $i = 0,\ldots,n$. To solve this system uniquely, we have to add three more conditions at the end points x_0 and x_n. Here, we choose the extra conditions as follows: $Z_0 = u'''(x_0)$, $Z_n = u'''(x_n)$ and $\mu_0 = u''(x_0)$. To obtain the last equation, we eliminate μ_{i+1} from equations (3.9) and (3.10), and substitute $i = 0$, we get

$$12Z_1 + Z_2 = \frac{24}{h^3}(u_0 - 2u_1 + u_2) - 11Z_0 - \frac{24}{h}\mu_0. \tag{3.13}$$

Equations (3.12) and (3.13) form a system of $n-1$ equations with $n-1$ unknowns i.e. Z_i, where $i = 1,2,\ldots,n$-1. These unknowns can be solved using the *MATHEMATICA* software.

To construct an algorithm for the proposed quartic spline method, we can use the following steps:

Step 1: Divide the interval $[a,b]$ into $n-1$ subinterval by taking $x_i = a + ih$, where $h = (b-a)/n$ and $i = 0, 1, \ldots, n.$

Step 2: To obtain the approximate solution u_i at the grid points, we

- apply the explicit 4-stage fourth order Runge-Kutta method to the first order IVP (1.12), or
- apply the approach in Paláncz and Popper (2000), which is based on the explicit 4-stage fourth order Runge-Kutta method, to the first order BVP (1.13), or
- apply the explicit 4-stage fourth order Runge-Kutta method to the second order IVP (1.14), or
- apply shooting method with the explicit 4-stage fourth order Runge-Kutta method to the second order BVP (1.15).

Step 3: Use equations (3.12) and (3.13) to form a system of linear equations, and then solve for the values of $Z_i, i = 1, 2, \dots, n-1$.

Step 4: Use the values of u_i and Z_i obtained from Step 2 and Step 3 to determine the values of A_i, B_i, and C_i. Therefore, the quartic spline method $s_i(x)$ in equation (3.2) is totally defined.

We note that the explicit 4-stage fourth order Runge-Kutta method is used in Step 2 because we want the calculated numerical solutions at the grid points to achieve an accuracy at least as high as the accuracy of the proposed spline methods. Moreover, the explicit 4-stage fourth order Runge-Kutta method represents a suitable compromise between the competing requirements of a low truncation error per step and a low computational cost per step (Azimi & Mozaffari, 2015).

3.2.2 Convergence Analysis of Quartic Spline Method

Let $s_i(x)$ given by equation (3.2), denotes the quartic spline using the exact values u_i, μ_i and Z_i. Also, let $\tilde{s}_i(x)$ denotes the quartic spline constructed using the values $\tilde{u}_i$, $\tilde{\mu}_i$ and $\tilde{Z}_i$, where $\tilde{u}_i$ is the approximate solution of problem (1.12), (1.13), (1.14) or (1.15), at the grid points which obtained using the explicit 4-stage fourth order Runge-Kutta method, $\tilde{\mu}_i$ and $\tilde{Z}_i$ are the second and the third derivatives of the function $\tilde{s}_i(x)$ at the point $(x_i, \tilde{u}_i)$, respectively. Then, $\tilde{s}_i(x)$ is given by

$$\tilde{s}_i(x) = \tilde{Z}_{i+1}\frac{(x-x_i)^4}{24h} - \tilde{Z}_i\frac{(x_{i+1}-x)^4}{24h} + \tilde{A}_i(x-x_i)^2 + \tilde{B}_i(x_{i+1}-x) + \tilde{C}_i(x-x_i), \quad (3.14)$$

where $x \in [x_i, x_{i+1}]$, and

$$\tilde{A}_i = \frac{\tilde{\mu}_i}{2} + \frac{h}{4}\tilde{Z}_i,$$

$$\tilde{B}_i = \frac{\tilde{u}_i}{h} + \frac{h^2}{24}\tilde{Z}_i,$$

and

$$\tilde{C}_i = \frac{\tilde{u}_{i+1}}{h} - \frac{h^2}{4}\tilde{Z}_i - \frac{h^2}{24}\tilde{Z}_{i+1} - \frac{h}{2}\tilde{\mu}_i.$$

Assume that $e(x)$ defines the error between the exact solution $u(x)$ and the approximate solution generated by spline function $\tilde{s}_i(x)$ for problem (1.12), (1.13), (1.14) or (1.15), which is

$$e(x) = u(x) - \tilde{S}(x), \; x \in [a,b]. \quad (3.15)$$

It is easy to verify that we can rewrite the error function $e(x)$ in equation (3.15) as follows

$$\begin{aligned} e(x) &= [u(x) - S(x)] + [S(x) - \tilde{S}(x)] \\ &= e_I(x) + e_D(x), \end{aligned} \tag{3.16}$$

where $e_I(x)$ is the error caused by spline interpolation and $e_D(x)$ is the error caused by discretization of problem (1.12), (1.13), (1.14) or (1.15). Now, to estimate $e(x)$ we have to estimate $e_I(x),$ and $e_D(x)$ separately. In the rest of the discussion in this section, we have to assume that $u(x) \in C^5[a,b]$ as needed in equations (3.17) - (3.19).

Since our spline is a polynomial of degree four, then we can write $e_I(x)$ over the subinterval (x_i, x_{i+1}) as

$$u(x) - s_i(x) = \frac{u^{(5)}(\zeta_i)}{5!}(x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+1})(x - x_{i+2}), \tag{3.17}$$

for some $\zeta_i \in (x_i, x_{i+1})$. Recall that every subinterval has length of h, and if we let $t = x - x_i$, then equation (3.17) can be rewritten as

$$u(x) - s_i(x) = \frac{u^{(5)}(\zeta_i)}{5!}(2h + t)(h + t)(t)(h - t)(2h - t). \tag{3.18}$$

By using the first derivative test on the expression $(2h+t)(h+t)(t)(h-t)(2h-t)$ in equation (3.18), i.e. $\frac{d}{dt}((2h+t)(h+t)(t)(h-t)(2h-t)) = 0,$ we found that it has maximum value at $t = -\sqrt{\frac{15+\sqrt{145}}{10}}h.$ This maximum value is equal to $3.632h^5$. Then, $\|u(x) - s_i(x)\|_\infty$ is bounded by

$$\left\| u(x) - s_i(x) \right\|_\infty \le 0.0303 h^5 \left\| u^{(5)}(\zeta_i) \right\|_\infty . \tag{3.19}$$

Let $W^5 = \max_{x \in [a,b]} \left\| u^{(5)}(x) \right\|_\infty$. Therefore, it is easy to conclude that

$$\left\| e_I(x) \right\|_\infty \le 0.0303 W^5 h^5. \tag{3.20}$$

In order to estimate the error function $e_D(x)$, we can subtract equation (3.14) from equation (3.2), to obtain

$$\begin{aligned} s_i(x) - \tilde{s}_i(x) = {} & (Z_{i+1} - \tilde{Z}_{i+1}) \frac{(x - x_i)^4}{24h} - (Z_i - \tilde{Z}_i) \frac{(x_{i+1} - x)^4}{24h} \\ & + (A_i - \tilde{A}_i)(x - x_i)^2 + (B_i - \tilde{B}_i)(x_{i+1} - x) \\ & + (C_i - \tilde{C}_i)(x - x_i), \end{aligned} \tag{3.21}$$

where $x \in [x_i, x_{i+1}]$. Let $U = (u_1, \ldots, u_{n-1})^t$, $\tilde{U} = (\tilde{u}_1, \ldots, \tilde{u}_{n-1})^t$, $\mu = (\mu_1, \ldots, \mu_{n-1})^t$, $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_{n-1})^t$, $Z = (Z_1, \ldots, Z_{n-1})^t$, and $\tilde{Z} = (\tilde{Z}_1, \ldots, \tilde{Z}_{n-1})^t$. Therefore, by using the first derivative test on equation (3.21), i.e. $\frac{d}{dx}(s_i(x) - \tilde{s}_i(x)) = 0$ together with the definition of infinity norm, we can obtain

$$\left\| e_D(x) \right\|_\infty \le \left\| U - \tilde{U} \right\|_\infty + \frac{h^2}{8} \left\| \mu - \tilde{\mu} \right\|_\infty + \frac{7h^3}{100} \left\| Z - \tilde{Z} \right\|_\infty . \tag{3.22}$$

Next, we will estimate $\left\| \mu - \tilde{\mu} \right\|_\infty$. We use equation (3.11) to obtain

$$\begin{aligned} \mu_i - \tilde{\mu}_i = {} & \frac{u_{i-1} - \tilde{u}_{i-1}}{h^2} - 2\frac{u_i - \tilde{u}_i}{h^2} + \frac{u_{i+1} + \tilde{u}_{i+1}}{h^2} + \frac{h}{24}(Z_{i-1} - \tilde{Z}_{i-1}) \\ & - \frac{h}{24}(Z_{i+1} - \tilde{Z}_{i+1}). \end{aligned} \tag{3.23}$$

On taking the infinity norm of equation (3.23) gives

$$\|\mu-\tilde{\mu}\|_\infty \le \frac{4}{h^2}\|U-\tilde{U}\|_\infty + \frac{h}{12}\|Z-\tilde{Z}\|_\infty. \quad (3.24)$$

On substituting inequality (3.24) into inequality (3.22), we have

$$\|e_D(x)\|_\infty \le \frac{3}{2}\|U-\tilde{U}\|_\infty + \frac{193}{2400}h^3\|Z-\tilde{Z}\|_\infty. \quad (3.25)$$

To estimate $\|Z-\tilde{Z}\|_\infty$ in inequality (3.21), we let $Q=(q_{i,j})$ denotes a matrix with

$$q_{i,j} = \begin{cases} 12, & i=j=1, \\ 1, & j=i+1,\ i=1,2,\ldots,n-2, \\ 11, & i=j,\ i=2,3,\ldots,n-1, \\ 11, & j=i-1,\ i=2,3,\ldots,n-1, \\ 1, & j=i-2,\ i=3,4,\ldots,n-1, \\ 0, & \text{otherwise.} \end{cases}$$

We also let $J=(j_{m,l})$ to denote a matrix with

$$j_{m,l} = \begin{cases} -2, & m=l=1, \\ 1, & l=m+1,\ m=1,2,\ldots,n-2, \\ -3, & m=l,\ m=2,3,\ldots,n-1, \\ 3, & l=m-1,\ m=2,3,\ldots,n-1, \\ -1, & l=m-2,\ m=3,4,\ldots,n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi=(\frac{24}{h^3}u_0-11Z_0-\frac{24}{h}\mu_0,-\frac{24}{h^3}u_0-Z_0,0,\ldots,0,\frac{24}{h^3}u_n-Z_n)^t$, then the system which contains equations (3.12) and (3.13) can be rewritten in a matrix form as

$$QZ=\frac{24}{h^3}JU+\psi. \quad (3.26)$$

From equation (3.26), one can obtain

$$Q(Z-\tilde{Z}) = \frac{24}{h^3} J(U-\tilde{U}) + \tau(h), \qquad (3.27)$$

where $\tau(h) = (\tau_0(h), \tau_1(h), \ldots, \tau_{n-1}(h))^t$ is the error due to the discretization. By using the mean value theorem on each of the component $\tau_i(h)$, the following expressions can be obtained:

$$\tau_0(h) < c_1 h,\ \tau_1(h) < c_2 h,\ \tau_{n-1}(h) < c_3 h, \qquad (3.28)$$

and

$$\tau_i(h) = -\frac{1}{2} h u^{(4)}(\zeta_i),\ \zeta_i \in (x_i, x_{i+1}),\ i = 2, 3, \ldots, n-2, \qquad (3.29)$$

where c_1, c_2 and c_3 are constants. From inequality (3.28) and equation (3.29), it follows that

$$\|\tau(h)\|_\infty \le c_5 h, \qquad (3.30)$$

where $c_5 = \max\{c_1, c_2, c_3, \frac{1}{2} c_4\}$, and $c_4 = \max_{a \le x \le b} \|u^{(4)}(x)\|_\infty$.

Since Q is invertible perturbed matrix, then $\|Q^{-1}\|_\infty \le 1$, and $\|J\|_\infty = 8.$ Together with equation (3.27) and inequality (3.30), we obtain

$$\|Z-\tilde{Z}\|_\infty \le \frac{192}{h^3} \|U-\tilde{U}\|_\infty + c_5 h. \qquad (3.31)$$

From inequalities (3.25) and (3.31), we obtain

$$\|e_D(x)\|_\infty \le \frac{386}{25} \|U-\tilde{U}\|_\infty + \frac{193}{2400} c_5 h^4. \qquad (3.32)$$

In order to derive a bound for the error function $\|e_D(x)\|_\infty$, we introduce Theorem 3.1.

Theorem 3.1. (Chawla and Subramanian, 1988) Assume that $u(x)$ is the exact solution and $\tilde{u}(x)$ is the approximate solution. If $u(x)$ is sufficiently smooth, then there exists a constant c independent of h such that

$$\|u(x)-\tilde{u}(x)\|_\infty \le c\,h^4.$$

Therefore, from inequality (3.31) together with Theorem 3.1, we have

$$\|e_D(x)\|_\infty \le c_6 h^4, \tag{3.33}$$

where $c_6 = \frac{386}{25}c + \frac{193}{2400}c_5$. Finally, from equation (3.16) and inequalities (3.20) and (3.33), we can get

$$\|e(x)\|_\infty \le \|e_I(x)\|_\infty + \|e_D(x)\|_\infty \le c_7 h^4, \tag{3.34}$$

where $c_7 = \frac{0.0303(b-a)W^5}{3} + c_6$. We summarize the above convergent analysis in the following remark.

Remark: With the assumptions of Theorem 3.1, if $\tilde{S}(x)$ is the quartic spline method (3.14) that used to approximate the solution of problem (1.12), (1.13), (1.14) or (1.15), i.e $u(x)$, then

$$\|u(x)-\tilde{S}(x)\|_\infty \le c_7 h^4, \tag{3.35}$$

where $c_7 = \dfrac{0.0303(b-a)W^5}{3} + c_6$.

3.3 Quintic Spline Method

In this section, the derivation and convergent analysis of the new quintic spline method are established.

3.3.1 Construction of Quintic Spline Method

Consider the following partition for the interval $[a,b]$, i.e.

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\},$$

where $x_i = a + ih$, and $h = \dfrac{b-a}{n}$. We have $u(x)$ denotes the exact solution of problem (1.12), (1.13), (1.14) or (1.15), and s_i represent the approximate solution to $u_i = u(x_i)$ acquired by the quintic spline $s_i(x)$ on the interval $[x_i, x_{i+1}]$. The following conditions have to be satisfied by every quintic spline function $S(x)$:

- $S(x) = s_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0,1,\ldots,n-1,$
- $S(a) = u(a), S(b) = u(b),$ and
- $s_i^{(r)}(x_{i+1}) = s_{i+1}^{(r)}(x_{i+1}), \quad r = 0,1,2,3,4.$

The fourth derivative of quintic spline is a linear polynomial; therefore, it can be expressed as

$$s_i^{(4)}(x) = Z_{i+1}\frac{(x-x_i)}{h} + Z_i\frac{(x_{i+1}-x)}{h}, \tag{3.36}$$

where $Z_i = s_i^{(4)}(x_i)$ and $x \in [x_i, x_{i+1}]$.

We integrated equation (3.36) four times to produce the following equations

$$s_i'''(x) = Z_{i+1}\frac{(x-x_i)^2}{2h} - Z_i\frac{(x_{i+1}-x)^2}{2h} + A_i, \tag{3.37a}$$

$$s_i''(x) = Z_{i+1}\frac{(x-x_i)^3}{6h} + Z_i\frac{(x_{i+1}-x)^3}{6h} + A_i(x-x_i) + B_i, \tag{3.37b}$$

$$\begin{aligned} s_i'(x) &= Z_{i+1}\frac{(x-x_i)^4}{24h} - Z_i\frac{(x_{i+1}-x)^4}{24h} + A_i(x-x_i)^2 \\ &+ B_i(x_{i+1}-x) + C_i, \end{aligned} \tag{3.37c}$$

$$\begin{aligned} s_i(x) &= Z_{i+1}\frac{(x-x_i)^5}{120h} + Z_i\frac{(x_{i+1}-x)^5}{120h} + A_i(x-x_i)^3 + B_i(x_{i+1}-x)^2 \\ &+ C_i(x-x_i) + D_i(x_{i+1}-x), \end{aligned} \tag{3.37d}$$

where A_i, B_i, C_i, and D_i, $i = 0,1,\ldots,n-1$, are coefficients that can be determined using u_i, u_{i+1}, μ_i, η_i, and Z_i, where $\mu_i = s_i''(x_i)$ and $\eta_i = s_i'''(x_i)$. To determine the four coefficients in (3.37d) i.e. A_i, B_i, C_i, and D_i, we first define:

$$u_i = s_i(x_i), \tag{3.38}$$

$$u_{i+1} = s_i(x_{i+1}), \tag{3.39}$$

$$\mu_i = s_i''(x_i), \tag{3.40}$$

and

$$\eta_i = s_i'''(x_i). \tag{3.41}$$

By substituting x_i and x_{i+1} into $s_i(x)$ and equations (3.38)-(3.41), and by using straightforward calculation, the following equations are obtained

$$A_i = \frac{h}{12} Z_i + \frac{\eta_i}{6}, \tag{3.42}$$

$$B_i = -\frac{h^2}{12} Z_i + \frac{\mu_i}{2}, \tag{3.43}$$

$$C_i = \frac{u_{i+1}}{h} - \frac{h^3}{12} Z_i - \frac{h^3}{120} Z_{i+1} - \frac{h^2}{6} \eta_i, \tag{3.44}$$

and

$$D_i = \frac{u_i}{h} + \frac{9h^3}{120} Z_i - \frac{h}{2} \mu_i. \tag{3.45}$$

On using the first, second and third continuity conditions of quintic spline $s_i(x)$ at the point x_{i+1}, i.e. $s_i^{(r)}(x_{i+1}) = s_{i+1}^{(r)}(x_{i+1}),\ r = 1,2,3,$ we get the following equations

$$\frac{11h^3}{120} Z_i + \frac{8h^3}{120} Z_{i+1} + \frac{h^3}{120} Z_{i+2} + \frac{2h^2}{6} \eta_i + \frac{h^2}{6} \eta_{i+1} + \frac{h}{2} \mu_i + \frac{h}{2} \mu_{i+1} = \frac{u_i - 2u_{i+1} + u_{i+2}}{h}, \tag{3.46}$$

$$\frac{4h^2}{12} Z_i + \frac{2h^2}{12} Z_{i+1} + h\eta_i + \mu_i - \mu_{i+1} = 0, \tag{3.47}$$

and

$$\frac{h}{2} Z_i + \frac{h}{2} Z_{i+1} + \eta_i - \eta_{i+1} = 0, \tag{3.48}$$

respectively. From equation (3.47), it follows that

$$\eta_i = -\frac{4h}{12} Z_i - \frac{2h}{12} Z_{i+1} - \frac{\mu_i}{h} + \frac{\mu_{i+1}}{h}. \tag{3.49}$$

On substituting equation (3.49) into equation (3.48) and equation (3.46), we obtain

$$\frac{h}{6}Z_i + \frac{4h}{6}Z_{i+1} + \frac{h}{6}Z_{i+2} - \frac{\mu_i}{h} + \frac{2\mu_{i+1}}{h} - \frac{\mu_{i+2}}{h} = 0, \tag{3.50}$$

and

$$\begin{aligned} &-\frac{7h^3}{360}Z_i - \frac{16h^3}{360}Z_{i+1} - \frac{7h^3}{360}Z_{i+2} + \frac{h}{6}\mu_i + \frac{4h}{6}\mu_{i+1} + \frac{h}{6}\mu_{i+2} \\ &= \frac{u_i - 2u_{i+1} + u_{i+2}}{h}, \end{aligned} \tag{3.51}$$

respectively. On eliminating μ_i and μ_{i+2} from equation (3.50) and equation (3.51), we get

$$\mu_{i+1} = \frac{u_i - 2u_{i+1} + u_{i+2}}{h^2} - \frac{3h^2}{360}Z_i - \frac{24h^2}{360}Z_{i+1} - \frac{3h^2}{360}Z_{i+2}. \tag{3.52}$$

On substituting equation (3.52) into equation (3.50), the main recurrence relation is found to be

$$Z_{i-1} + 26Z_i + 66Z_{i+1} + 26Z_{i+2} + Z_{i+3} = \frac{120}{h^4}(u_{i-1} - 4u_i + 6u_{i+1} - 4u_{i+2} + u_{i+3}), \tag{3.53}$$

where $i = 1, \ldots, n-3.$

Equation (3.53) gives $n-3$ equations in $n+1$ unknowns, which are the Z_i, where $i = 0, \ldots, n.$ In order to get a unique solution for this system, we need additional four equations at the end points x_0 and x_n. Therefore, we choose these extra equations as follows: $Z_0 = u^{(4)}(x_0)$, $Z_n = u^{(4)}(x_n)$, $\mu_0 = u''(x_0)$ and $\mu_n = u''(x_n)$.

On replacing μ_{i+1} and μ_{i+2} in equation (3.51) by their values obtained from equation (3.52), and then substituting $i = 0$ in equation (3.51) which yields

$$\mu_0 = \frac{18h^2}{120} Z_0 + \frac{65h^2}{120} Z_1 + \frac{26h^2}{120} Z_2 + \frac{h^2}{120} Z_3 - \frac{-2u_0 + 5u_1 - 4u_2 + u_3}{h^2}. \quad (3.54)$$

Lastly, On substituting the values of μ_0 and Z_0 from the extra conditions into equation (3.54) leads to the second to last equation as

$$65Z_1 + 26Z_2 + Z_3 = \frac{120}{h^4}(-2u_0 + 5u_1 - 4u_2 + u_3) + \frac{120}{h^2}\mu_0 - 18Z_0. \quad (3.55)$$

Likewise, substituting μ_i and μ_{i+1} in equation (3.51) by their values from equation (3.52), and replacing i by $n-2$ in equation (3.51) form the last equation which is given by

$$Z_{n-3} + 26Z_{n-2} + 65Z_{n-1} = \frac{120}{h^4}(-u_n + 3u_{n-1} - 3u_{n-2} + u_{n-3}) + \frac{120}{h^2}\mu_n - 18Z_n. \quad (3.56)$$

Equations (3.53), (3.55) and (3.56) leads to a $n-1$ by $n-1$ system which can be solved using the *MATHEMATICA* software.

To compute the approximate solution of problem (1.12), (1.13), (1.14) or (1.15), using the proposed quintic spline method on the interval $[a,b]$, we design an algorithm which consists of four steps as follows:

Step 1: The interval $[a,b]$ is divided into n subinterval by the means of points

$x_i = a + ih,$ where $h = (b-a)/n$ and $i = 0,1,\ldots,n.$

Step 2: In order to generate the numerical solution u_i at the point x_i, we

- apply the explicit 4-stage fourth order Runge-Kutta method to the first order IVP (1.12), or
- employ the approach in Paláncz and Popper (2000), which is based on the explicit 4-stage fourth order Runge-Kutta method, to the first order BVP (1.13), or
- use the explicit 4-stage fourth order Runge-Kutta method to the second order IVP (1.14), or
- exercise the shooting method with the explicit 4-stage fourth order Runge-Kutta method to the second order BVP (1.15).

Step 3: Solve the constructed system which contains equations (3.53), (3.55) and (3.56), for $Z_i, i = 1,2,\ldots,n-1$.

Step 4: Determine the values of the coefficients A_i, B_i, C_i, and D_i by using the values of u_i and Z_i from Step 2 and Step 3. Thus, the quintic spline method $s_i(x)$ in equation (3.37d) is completely determined.

3.3.2 Convergence Analysis of Quintic Spline Method

In order to investigate the convergence analysis for the proposed quintic spline method, we start by introducing $\tilde{s}_i(x)$ to represent the quintic spline method (3.37) constructed by means of the values $\tilde{u}_i, \tilde{\mu}_i, \tilde{\eta}_i$ and $\tilde{Z}_i$, where $\tilde{u}_i$ denotes the approximate solution of problem (1.12), (1.13), (1.14) or (1.15) at the points x_i which generated by the explicit 4-stage fourth order Runge-Kutta method, whereas

$\tilde{\mu}_i$, $\tilde{\eta}_i$ and $\tilde{Z}_i$ are the second, third and fourth derivatives of the function $\tilde{s}_i(x)$ at the point $(x_i, \tilde{u}_i)$, respectively. Then, $\tilde{s}_i(x)$ is defined as

$$\begin{aligned}\tilde{s}_i(x) = \tilde{Z}_{i+1}\frac{(x-x_i)^5}{120h} + \tilde{Z}_i\frac{(x_{i+1}-x)^5}{120h} + \tilde{A}_i(x-x_i)^3 + \tilde{B}_i(x_{i+1}-x)^2 \\ + \tilde{C}_i(x-x_i) + \tilde{D}_i(x_{i+1}-x),\end{aligned} \tag{3.57}$$

where $x \in [x_i, x_{i+1}]$, and

$$\tilde{A}_i = \frac{h}{12}\tilde{Z}_i + \frac{\tilde{\eta}_i}{6},$$

$$\tilde{B}_i = -\frac{h^2}{12}\tilde{Z}_i + \frac{\tilde{\mu}_i}{2},$$

$$\tilde{C}_i = \frac{\tilde{u}_{i+1}}{h} - \frac{h^3}{12}\tilde{Z}_i - \frac{h^3}{120}\tilde{Z}_{i+1} - \frac{h^2}{6}\tilde{\eta}_i,$$

and

$$\tilde{D}_i = \frac{\tilde{u}_i}{h} + \frac{9h^3}{120}\tilde{Z}_i - \frac{h}{2}\tilde{\mu}_i.$$

Here, we defined the error function $e(x)$ as the difference between the exact solution $u(x)$ and the approximate solution obtained by spline method $\tilde{s}_i(x)$ in equation (3.57) for problem (1.12), (1.13), (1.14) or (1.15). Therefore, $e(x)$ is expressed by the following equation

$$e(x) = u(x) - \tilde{S}(x), \; x \in [a,b]. \tag{3.58}$$

We note that $e(x)$ can be revised as

$$e(x) = [u(x) - S(x)] + [S(x) - \tilde{S}(x)] \\ = e_I(x) + e_D(x), \tag{3.59}$$

where $e_I(x)$ and $e_D(x)$ denote the errors caused by spline interpolation and discretization of problem (1.12), (1.13), (1.14) or (1.15), respectively. As a result, to estimate $e(x)$, we have to estimate $e_I(x)$ and $e_D(x)$ separately. Throughout this discussion in this section, we are required to assume that $u(x) \in C^6[a,b]$, as we are employing this assumption in the upcoming equations.

As the degree of the quintic spline method is five, then $e_I(x)$ can be rewritten over the subinterval (x_i, x_{i+1}) as

$$u(x) - s_i(x) = \frac{u^{(6)}(\zeta_i)}{6!}(x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+1})(x - x_{i+2})(x - x_{i+3}), \tag{3.60}$$

for some $\zeta_i \in (x_i, x_{i+1})$. On substituting $t = x - x_i$ in equation (3.60), this yields

$$u(x) - s_i(x) = \frac{u^{(6)}(\zeta_i)}{6!}(2h + t)(h + t)(t)(h - t)(2h - t)(3h - t), \tag{3.61}$$

where h denotes the length of each subinterval. The expression $(2h+t)(h+t)(t)(h-t)(2h-t)(3h-t)$ in equation (3.61), has a maximum value at $t = \frac{h(3 - \sqrt{3}\sqrt{35 + 8\sqrt{7}})}{6}$ via the first derivative test. Moreover, the maximum value is equal to $16.901h^6$. Then, $\|u(x) - s_i(x)\|_\infty$ is bounded by

$$\|u(x) - s_i(x)\|_\infty \leq 0.0234h^6 \|u^{(6)}(\zeta_i)\|_\infty . \tag{3.62}$$

Let $W^6 = \max_{x \in [a,b]} \|u^{(6)}(x)\|_\infty$. Thus, we can conclude that

$$\|e_I(x)\|_\infty \le 0.0234W^6h^6. \tag{3.63}$$

To estimate the error function $e_D(x)$, equation (3.57) can be subtracted from equation (3.37d), to give

$$\begin{aligned} s_i(x) - \tilde{s}_i(x) &= (Z_{i+1} - \tilde{Z}_{i+1})\frac{(x-x_i)^5}{120h} + (Z_i - \tilde{Z}_i)\frac{(x_{i+1}-x)^5}{120h} \\ &+ (A_i - \tilde{A}_i)(x-x_i)^3 + (B_i - \tilde{B}_i)(x_{i+1}-x)^2 \\ &+ (C_i - \tilde{C}_i)(x-x_i) + (D_i - \tilde{D}_i)(x_{i+1}-x), \end{aligned} \tag{3.64}$$

where $x \in [x_i, x_{i+1}]$. Let $U = (u_1, \ldots, u_{n-1})^t$, $\tilde{U} = (\tilde{u}_1, \ldots, \tilde{u}_{n-1})^t$, $\mu = (\mu_1, \ldots, \mu_{n-1})^t$, $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_{n-1})^t$, $\eta = (\eta_1, \ldots, \eta_{n-1})^t$, $\tilde{\eta} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_{n-1})^t$, $Z = (Z_1, \ldots, Z_{n-1})^t$, and $\tilde{Z} = (\tilde{Z}_0, \ldots, \tilde{Z}_n)^t$. By applying the first derivative test and infinity norm on equation (3.64), we get

$$\|e_D(x)\|_\infty \le \|U - \tilde{U}\|_\infty + \frac{h^2}{8}\|\mu - \tilde{\mu}\|_\infty + \frac{7h^3}{100}\|\eta - \tilde{\eta}\|_\infty + \frac{3h^4}{100}\|Z - \tilde{Z}\|_\infty. \tag{3.65}$$

Now, we will estimate $\|\eta - \tilde{\eta}\|_\infty$ using equation (3.49) as

$$\eta_i - \tilde{\eta}_i = -\frac{4h}{12}(Z_i - \tilde{Z}_i) - \frac{2h}{12}(Z_{i+1} - \tilde{Z}_{i+1}) - \frac{1}{h}(\mu_i - \tilde{\mu}_i) + \frac{1}{h}(\mu_{i+1} - \tilde{\mu}_{i+1}). \tag{3.66}$$

Then, taking the infinity norm on equation (3.66) yields

$$\|\eta - \tilde{\eta}\|_\infty \le \frac{2}{h}\|\mu - \tilde{\mu}\|_\infty + \frac{h}{2}\|Z - \tilde{Z}\|_\infty. \tag{3.67}$$

After that, we substitute inequality (3.67) into inequality (3.65), and as a result, we have

$$\|e_D(x)\|_\infty \le \|U-\tilde{U}\|_\infty + \frac{212h^2}{800}\|\mu-\tilde{\mu}\|_\infty + \frac{13h^4}{200}\|Z-\tilde{Z}\|_\infty. \qquad (3.68)$$

On using equation (3.52), we can estimate $\|\mu-\tilde{\mu}\|_\infty$ as follows

$$\begin{aligned}\mu_i-\tilde{\mu}_i &= \frac{1}{h^2}(u_{i-1}-\tilde{u}_{i-1}) - \frac{2}{h^2}(u_i-\tilde{u}_i) + \frac{1}{h^2}(u_{i+1}-\tilde{u}_{i+1}) \\ &\quad - \frac{h^2}{120}(Z_{i-1}-\tilde{Z}_{i-1}) - \frac{8h^2}{120}(Z_i-\tilde{Z}_i) - \frac{h^2}{120}(Z_{i+1}-\tilde{Z}_{i+1}).\end{aligned} \qquad (3.69)$$

Therefore, from equation (3.69), we obtain

$$\|\mu-\tilde{\mu}\|_\infty \le \frac{4}{h^2}\|U-\tilde{U}\|_\infty + \frac{10h^2}{120}\|Z-\tilde{Z}\|_\infty. \qquad (3.70)$$

On substituting inequality (3.70) into inequality (3.68), we get

$$\|e_D(x)\|_\infty \le \frac{1648}{800}\|U-\tilde{U}\|_\infty + \frac{212h^4}{9600}\|Z-\tilde{Z}\|_\infty. \qquad (3.71)$$

To estimate $\|Z-\tilde{Z}\|_\infty$, we let $Q=(q_{i,j})$ represents a matrix with

$$q_{i,j} = \begin{cases} 65, & i=j=1, \text{ and } i=j=n-1, \\ 1, & j=i+2,\ i=1,2,\ldots,n-2, \\ 26, & j=i+1,\ i=1,2,\ldots,n-2, \\ 66, & i=j,\ i=2,3,\ldots,n-1, \\ 26, & j=i-1,\ i=2,3,\ldots,n-1, \\ 1, & j=i-2,\ i=3,4,\ldots,n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, we also let $J=(j_{m,l})$ to designate a matrix with

$$j_{m,l} = \begin{cases} 5, & m = l = 1, \text{ and } m = l = n-1 \\ 1, & l = m+2,\ m = 1,2,\ldots,n-2, \\ -4, & l = m+1,\ m = 1,2,\ldots,n-2, \\ 6, & m = l,\ m = 2,3,\ldots,n-1, \\ -4, & l = m-1,\ m = 2,3,\ldots,n-1, \\ 1, & l = m-2,\ m = 3,4,\ldots,n-1, \\ 0, & \text{otherwise.} \end{cases}$$

In order to write equations (3.53), (3.55) and (3.56) in terms of matrix form, we let

$$\psi = (-\frac{120}{h^4}u_0 - Z_0 + \frac{120}{h^2}\mu_0, \frac{120}{h^4}u_0 - Z_0, 0, \ldots, 0, \frac{120}{h^4}u_n - Z_n,$$
$$-\frac{120}{h^4}u_n - Z_n + \frac{120}{h^2}\mu_n)^t.$$

As a result, this system can be written as

$$QZ = \frac{120}{h^4}JU + \psi. \qquad (3.72)$$

From equation (3.72), we obtain

$$Q(Z - \tilde{Z}) = \frac{120}{h^4}J(U - \tilde{U}) + \tau(h), \qquad (3.73)$$

where $\tau(h) = (\tau_0(h), \tau_1(h), \ldots, \tau_{n-1}(h))^t$ is the error due to the discretization. By using the mean value theorem on each of the component $\tau_i(h)$, we have

$$\tau_0(h) < c_1 h, \tau_1(h) < c_2 h, \tau_{n-2}(h) < c_3 h, \tau_{n-1}(h) < c_4 h, \qquad (3.74)$$

and

$$\tau_i(h) = -hu^{(5)}(\zeta_i),\ \zeta_i \in (x_i, x_{i+1}),\ i = 2, 3, \ldots, n-3, \qquad (3.75)$$

where c_1, c_2, c_3 and c_4 are constants. From inequality (3.74) and equation (3.75), it follows that

$$\|\tau(h)\|_\infty \le c_6 h, \tag{3.76}$$

where $c_6 = \max\{c_1, c_2, c_3, c_4, c_5\}$, and $c_5 = \max_{a \le x \le b} \|u^{(5)}(x)\|_\infty$.

We note that the matrix Q is invertible because it is strictly diagonally dominant matrix, which implies $\|Q^{-1}\|_\infty \le \frac{1}{12}$. Moreover, the infinity norm for the matrix J is computed to be 16 i.e. $\|J^{-1}\|_\infty = 16$. On using this information together with equation (3.73) and inequality (3.76), we acuire

$$\|Z - \tilde{Z}\|_\infty \le \frac{160}{h^4}\|U - \tilde{U}\|_\infty + \frac{1}{12} c_6 h. \tag{3.77}$$

From inequalities (3.71) and (3.77), we obtain

$$\|e_D(x)\|_\infty \le \frac{839}{150}\|U - \tilde{U}\|_\infty + \frac{53}{28800} c_6 h^5. \tag{3.78}$$

From inequality (3.78) together with Theorem 3.1, we have

$$\|e_D(x)\|_\infty \le c_7 h^4, \tag{3.79}$$

where $c_7 = \frac{839}{150}c + \frac{53(b-a)c_6}{28800}$. Finally, from equation (3.59), inequalities (3.63) and (3.79), we get

$$\|e(x)\|_\infty \le \|e_I(x)\|_\infty + \|e_D(x)\|_\infty \le c_8 h^4, \tag{3.80}$$

where $c_8 = \dfrac{0.02347(b-a)^2 W^6}{25} + c_7$. Last but not least, we summarize the above convergence analysis in the following remark.

Remark: With the assumptions of Theorem 3.1, if $\tilde{S}(x)$ is the quintic spline method (3.57) that used to approximate the solution of problem (1.12), (1.13), (1.14) or (1.15), i.e. $u(x)$, then

$$\left\| u(x) - \tilde{S}(x) \right\|_\infty \le c_8 h^4, \tag{3.81}$$

where $c_8 = \dfrac{0.02347(b-a)^2 W^6}{25} + c_7$.

CHAPTER FOUR
NEW CUBIC AND QUINTIC NON-POLYNOMIAL SPLINE METHODS

4.1 Introduction

This chapter considers the derivations of two new non-polynomial spline methods for the approximate solution of first order IVPs (1.12), first order BVPs (1.13), second order IVPs (1.14) and second order BVPs (1.15). Moreover, the convergent analysis for each derived method is considered as well.

4.2 Cubic Non-polynomial Spline Method

The construction process and the convergent analysis of the new cubic non-polynomial spline method are described in this section.

4.2.1 Construction of Cubic Non-polynomial Spline Method

During our investigations through the literature, we note that there exists some non-polynomial splines that are based on the trigonometric functions sine and cosine together with polynomial. Likwise, there also exists non-polynomial splines which depend on sine hyperbolic, cosine hyperbolic functions and polynomial. Therefore, in this section, we examine the derivation of new spline method which contains the functions sine, cosine, and their hyperbolic counterparts, in solving first and second order ODEs. Hence, let the interval $[a,b]$ being divided into $n-1$ equal subintervals

$[x_i, x_{i+1}]$, each of length h, where $x_i = a + ih$, and $h = \dfrac{b-a}{n}$. For each i-th segment, the cubic non-polynomial spline $s_i(x)$ on the interval $[x_i, x_{i+1}]$ has the form

$$s_i(x) = a_i \sin k(x - x_i) + b_i \cos k(x - x_i) + c_i \sinh k(x - x_i) + d_i \cosh k(x - x_i)\ , \quad (4.1)$$

where a_i, b_i, c_i, and d_i, $i = 0,1,\ldots,n-1$, are finite constants and k is a free parameter.

The following conditions have to be satisfied by each cubic non-polynomial spline function $S(x)$:

- $S(x) = s_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0,1,\ldots,n-1,$
- $S(a) = u(a), S(b) = u(b)$, and
- $s_i^{(r)}(x_{i+1}) = s_{i+1}^{(r)}(x_{i+1}), \quad r = 0,1,2,$

where $u(x_i)$ is the exact solution of problem (1.12), (1.13), (1.14) or (1.15), and s_i be the approximate solution to $u_i = u(x_i)$ generated by the cubic non-polynomial spline $s_i(x)$ on the subinterval $[x_i, x_{i+1}]$.

To obtain expressions for the four coefficients of equation (4.1) in terms of u_i, u_{i+1}, and μ_i, the following relations are defined for each value of i:

$$u_i = s_i(x_i), \quad (4.2)$$

$$u_{i+1} = s_i(x_{i+1}), \quad (4.3)$$

$$\mu_i = s_i''(x_i), \quad (4.4)$$

and

$$\mu_{i+1} = s_i''(x_{i+1}). \tag{4.5}$$

On applying equations (4.2), (4.3), (4.4) and (4.5) to the cubic non-polynomial spline $s_i(x)$, we acquired the following equations

$$a_i = -\frac{u_i \cot(kh)}{2} + \frac{u_{i+1}}{2\sin(kh)} + \frac{\mu_i \cot(kh)}{2k^2} - \frac{\mu_{i+1}}{2k^2 \sin(kh)}, \tag{4.6}$$

$$b_i = \frac{u_i}{2} - \frac{\mu_i}{2k^2}, \tag{4.7}$$

$$c_i = -\frac{u_i \coth(kh)}{2} + \frac{u_{i+1}}{2\sinh(kh)} - \frac{\mu_i \coth(kh)}{2k^2} + \frac{\mu_{i+1}}{2k^2 \sinh(kh)}, \tag{4.8}$$

and

$$d_i = \frac{u_i}{2} + \frac{\mu_i}{2k^2}. \tag{4.9}$$

The continuity condition of the first derivative of the cubic non-polynomial spline $s_i(x)$ is exploited at the point x_{i+1}, i.e. $s_i'(x_{i+1}) = s_{i+1}'(x_{i+1})$, to get

$$a_i\, k\cos(kh) - b_i\, k\sin(kh) + c_i\, k\cosh(kh) + d_i\, k\sinh(kh) = a_{i+1}\, k + c_{i+1}\, k, \tag{4.10}$$

where $i = 0,1,\ldots,n-2.$ On substituting the results in equations (4.6) - (4.9) into equation (4.10), this gives

$$\begin{aligned}
&\left(\frac{\cot(kh)\ \cos(kh)}{2} + \frac{\sin(kh)}{2} - \frac{\coth(kh)\ \cosh(kh)}{2} + \frac{\sinh(kh)}{2}\right)\mu_i \\
&+(-\cot(kh) + \coth(kh))\mu_{i+1} + \left(\frac{1}{2\ \sin(kh)} - \frac{1}{2\ \sinh(kh)}\right)\mu_{i+2} \\
&= k^2\Bigg(\left(\frac{\cot(kh)\ \cos(kh)}{2} + \frac{\sin(kh)}{2} + \frac{\coth(kh)\ \cosh(kh)}{2} - \frac{\sinh(kh)}{2}\right)u_i \\
&+ (-\cot(kh) - \coth(kh))u_{i+1} + \left(\frac{1}{2\ \sin(kh)} + \frac{1}{2\ \sinh(kh)}\right)u_{i+2}\Bigg).
\end{aligned} \tag{4.11}$$

On simplifying equation (4.11), we acquire the following main recurrence relation

$$\alpha_1\mu_i + \alpha_2\mu_{i+1} + \alpha_1\mu_{i+2} = k^2(\beta_1 u_i + \beta_2 u_{i+1} + \beta_1 u_{i+2}), \tag{4.12}$$

where $i = 0,1,\ldots,n-1,$

$$\alpha_1 = \frac{\sinh(kh)}{2} - \frac{\sin(kh)}{2},$$

$$\alpha_2 = -\cos(kh)\sinh(kh) + \cosh(kh)\sin(kh),$$

$$\beta_1 = \frac{\sinh(kh)}{2} + \frac{\sin(kh)}{2},$$

and

$$\beta_2 = -\cos(kh)\sinh(kh) - \cosh(kh)\sin(kh).$$

Equation (4.12) gives a system of $n-1$ equations in $n+1$ unknowns, which are the μ_i, $i = 0,1,\ldots,n-1,n.$ This system can be solved uniquely by adding two extra conditions at the points x_0 and x_n. For this reason, we choose $\mu_0 = u''(x_0),$ and $\mu_n = u''(x_n).$ By substituting the values of μ_0 and μ_n in equation (4.12), the last two equations are

$$\alpha_2\mu_1 + \alpha_1\mu_2 = k^2(\beta_1 u_0 + \beta_2 u_1 + \beta_1 u_2) - \alpha_1\mu_0, \tag{4.13}$$

and

$$\alpha_1\mu_{n-2} + \alpha_2\mu_{n-1} = k^2(\beta_1 u_{n-2} + \beta_2 u_{n-1} + \beta_1 u_n) - \alpha_1\mu_n. \tag{4.14}$$

This system can be solved for μ_i, $i = 0,1,\ldots,n-1,n,$ using the *MATHEMATICA* software.

In order to obtain numerical solution using the proposed cubic non-polynomial spline method, we construct the following algorithm:

Step 1: The interval $[a,b]$ is partitioned into n subintervals by inserting $n-1$ equally spaced points, i.e. $x_i = a + ih,$ where $h = (b-a)/n$ and $i = 1, 2,\ldots, n-1.$

Step 2: To find the approximate solution u_i at the grid points, we

- employ the explicit 4-stage fourth order Runge-Kutta method to first order IVP (1.12), or
- exercise the approach in Paláncz and Popper (2000), which is based on the explicit 4-stage fourth order Runge-Kutta method, to first order BVP (1.13), or
- apply the explicit 4-stage fourth order Runge-Kutta method to second order IVP (1.14), or
- use the shooting method together with the explicit 4-stage fourth order Runge-Kutta method to second order BVP (1.15).

Step 3: The linear system which consists of equations (4.12), (4.13) and (4.14) is solved for the values of μ_i, $i = 1,2,3,\ldots,n-1.$

Step 4: Use the values of u_i and μ_i generated from Step 2 and Step 3 to obtain the values of a_i, b_i, c_i, and d_i. Therefore, the cubic non-polynomial spline method $s_i(x)$ in equation (4.1) is wholly defined.

4.2.2 Convergence Analysis of Cubic Non-polynomial Spline Method

Assume that the cubic non-polynomial spline $s_i(x)$ given by equation (4.1) is developed using the exact values u_i and μ_i. Let the cubic non-polynomial spline constructed using the values $\tilde{u}_i$ and $\tilde{\mu}_i$ being denoted by $\tilde{s}_i(x)$, where $\tilde{u}_i$ is obtained using the explicit 4-stage fourth order Runge-Kutta method when solving problem (1.12), (1.13), (1.14) or (1.15), and $\tilde{\mu}_i$ represents the second derivative of $\tilde{s}_i(x)$ at the point $(x_i, \tilde{u}_i)$. Thus, $\tilde{s}_i(x)$ can be given in the form

$$\tilde{s}_i(x) = \tilde{a}_i \sin k(x - x_i) + \tilde{b}_i \cos k(x - x_i) + \tilde{c}_i \sinh k(x - x_i) + \tilde{d}_i \cosh k(x - x_i), \qquad (4.15)$$

where $x \in [x_i, x_{i+1}]$, and

$$\tilde{a}_i = -\frac{\tilde{u}_i \cot(kh)}{2} + \frac{\tilde{u}_{i+1}}{2\sin(kh)} + \frac{\tilde{\mu}_i \cot(kh)}{2k^2} - \frac{\tilde{\mu}_{i+1}}{2k^2 \sin(kh)}, \qquad (4.16)$$

$$\tilde{b}_i = \frac{\tilde{u}_i}{2} - \frac{\tilde{\mu}_i}{2k^2}, \qquad (4.17)$$

$$\tilde{c}_i = -\frac{\tilde{u}_i \coth(kh)}{2} + \frac{\tilde{u}_{i+1}}{2\sinh(kh)} - \frac{\tilde{\mu}_i \coth(kh)}{2k^2} + \frac{\tilde{\mu}_{i+1}}{2k^2 \sinh(kh)}, \qquad (4.18)$$

and

$$\tilde{d}_i = \frac{\tilde{u}_i}{2} + \frac{\tilde{\mu}_i}{2k^2}. \qquad (4.19)$$

Assume that $e(x)$ denotes the error between the exact solution $u(x)$ and the approximate solution $\tilde{s}_i(x)$ for problem (1.12), (1.13), (1.14) or (1.15). Therefore, $e(x)$ can be defined as follows

$$e(x) = u(x) - \tilde{S}(x),\ x \in [a,b]. \tag{4.20}$$

It is simple to verify that we can write $e(x)$ in equation (4.20) as

$$\begin{aligned} e(x) &= [u(x) - S(x)] + [S(x) - \tilde{S}(x)] \\ &= e_I(x) + e_D(x), \end{aligned} \tag{4.21}$$

where $e_I(x)$ and $e_D(x)$ are the errors due to the cubic non-polynomial spline interpolation and discretization of problem (1.12), (1.13), (1.14) or (1.15), respectively. The estimation of $e(x)$ in equation (4.21) requires the estimations of both $e_I(x)$ and $e_D(x)$. In the remainder of the discussion in this section, we suppose that $u(x) \in C^4[a,b]$, since up to the fourth derivative of the solution $u(x)$ is needed.

On using Theorem 2.5, $e_I(x)$ can be defined on the interval $[x_i, x_{i+1}]$ as follows

$$u(x) - s_i(x) = \frac{1}{6}\int_{x_i}^{x_{i+1}} u^{(4)}(\tau)\, R_x\, [(x-\tau)_+^3]\, d\tau, \tag{4.22}$$

where $R_x\, [(x-\tau)_+^3]$ is the Peano kernel. According to Definition 2.7, we can revise the Peano kernel function on the subinterval $[x_i, x_{i+1}]$ as

$$R_x\, [(x-\tau)_+^3] = \begin{cases} r(\tau, x), & x_i < \tau < x, \\ s(\tau, x), & x < \tau < x_{i+1}. \end{cases}$$

By taking the absolute value of equation (4.22) on the interval $[x_i, x_{i+1}]$ and with the help of Theorem 2.6, we have

$$\left|u(x)-s_i(x)\right| \le \frac{1}{6}\left\|u^{(4)}(\tau)\right\|_\infty \int_{x_i}^{x_{i+1}} \left|R_x\ [(x-\tau)_+^3]\right| d\tau. \tag{4.23}$$

Therefore, we can rewrite the integral $\int_{x_i}^{x_{i+1}} \left|R_x\ [(x-\tau)_+^3]\right| d\tau$ in inequality (4.23) as

$$\int_{x_i}^{x_{i+1}} \left|R_x\ [(x-\tau)_+^3]\right| d\tau = \int_{x_i}^{x} \left|r(\tau,x)\right| d\tau + \int_{x}^{x_{i+1}} \left|s(\tau,x)\right| d\tau. \tag{4.24}$$

To evaluate the Peano kernel $R_x\ [(x-\tau)_+^3],$ it can be written as

$$\begin{aligned} R_x\ [(x-\tau)_+^3] = (x-\tau)_+^3 & - [\frac{(x_{i+1}-\tau)_+^3}{2\sin\theta} - \frac{6h^2(x_{i+1}-\tau)_+}{2\theta^2\sin\theta} \\ & - \frac{(x_i-\tau)_+^3\cot\theta}{2} + \frac{6h^2(x_i-\tau)_+\cot\theta}{2\theta^2}]\sin k(x-x_i) \\ & - [\frac{(x_i-\tau)_+^3}{2} - \frac{6h^2(x_i-\tau)_+}{2\theta^2}]\cos k(x-x_i) \\ & - [\frac{(x_{i+1}-\tau)_+^3}{2\sinh\theta} + \frac{6h^2(x_{i+1}-\tau)_+}{2\theta^2\sinh\theta} - \frac{(x_i-\tau)_+^3\coth\theta}{2} \\ & - \frac{6h^2(x_i-\tau)_+\coth\theta}{2\theta^2}]\sinh k(x-x_i) - [\frac{(x_i-\tau)_+^3}{2} \\ & - \frac{6h^2(x_i-\tau)_+}{2\theta^2}]\cosh k(x-x_i), \end{aligned} \tag{4.25}$$

where $\theta = kh.$ Thus, the Peano kernel $R_x\ [(x-\tau)_+^3]$ can be rewritten depending on the value of $\tau,$ i.e. if $x_i < \tau < x,$ equation (4.25) becomes

$$r(\tau,x)=(x-\tau)^3-[\frac{(x_{i+1}-\tau)^3}{2\sin\theta}-\frac{3h^2(x_{i+1}-\tau)}{\theta^2\sin\theta}]\sin k(x-x_i)$$
$$-[\frac{(x_{i+1}-\tau)^3}{2\sinh\theta}+\frac{3h^2(x_{i+1}-\tau)}{\theta^2\sinh\theta}]\sinh k(x-x_i). \quad (4.26)$$

Otherwise, if $x<\tau<x_{i+1}$, then equation (4.25) becomes

$$s(\tau,x)=-[\frac{(x_{i+1}-\tau)^3}{2\sin\theta}-\frac{3h^2(x_{i+1}-\tau)}{\theta^2\sin\theta}]\sin k(x-x_i)$$
$$-[\frac{(x_{i+1}-\tau)^3}{2\sinh\theta}+\frac{3h^2(x_{i+1}-\tau)}{\theta^2\sinh\theta}]\sinh k(x-x_i). \quad (4.27)$$

We observed that $r(x,x)=0$ if and only if $x=x_i$ or $x=x_{i+1}$.

In order to find the values of τ such that $r(\tau,x)=0$, we assume that $\rho=x-\tau$ in equation (4.26). As a result, we get

$$r(\tau,x)=\rho^3-[\frac{(x_{i+1}-x)^3+3(x_{i+1}-x)^2\rho+3(x_{i+1}-x)\rho^2+\rho^3}{2\sin\theta}$$
$$-\frac{3h^2((x_{i+1}-x)+\rho)}{\theta^2\sin\theta}]\sin k(x-x_i)-[\frac{(x_{i+1}-x)^3+3(x_{i+1}-x)^2\rho}{2\sinh\theta} \quad (4.28)$$
$$+\frac{3(x_{i+1}-x)\rho^2+\rho^3}{2\sinh\theta}+\frac{3h^2((x_{i+1}-x)+\rho)}{\theta^2\sinh\theta}]\sinh k(x-x_i)=0.$$

Equation (4.28) can be solved by using *MATHEMATICA* software to conclude that $r(\tau,x)\neq 0$ for all $\tau\in(x_i,x)$. Furthermore, for all $\tau\in(x_i,x)$, $r(\tau,x)>0$, i.e. $|r(\tau,x)|=r(\tau,x)$.

Likewise, to determine the values of τ such that $s(\tau,x)=0$, we also use the assumption $\rho=x-\tau$ in equation (4.27), and we acquire

$$s(\tau,x) = -[\frac{(x_{i+1}-x)^3+3(x_{i+1}-x)^2\rho+3(x_{i+1}-x)\rho^2+\rho^3}{2\sin\theta}$$

$$-\frac{3h^2((x_{i+1}-x)+\rho)}{\theta^2\sin\theta}]\sin k(x-x_i)-[\frac{(x_{i+1}-x)^3+3(x_{i+1}-x)^2\rho}{2\sinh\theta} \tag{4.29}$$

$$+\frac{3(x_{i+1}-x)\rho^2+\rho^3}{2\sinh\theta}+\frac{3h^2((x_{i+1}-x)+\rho)}{\theta^2\sinh\theta}]\sinh k(x-x_i)=0.$$

By using *MATHEMATICA* software to solve equation (4.29), we notice that the only value for τ such that $s(\tau,x)=0,$ is $\tau = x_{i+1},$ which does not belong to the interval $(x,x_{i+1}).$ Thus, $s(\tau,x)\neq 0$ for all $\tau\in(x,x_{i+1}).$ Moreover, for all $\tau\in(x,x_{i+1}),$ $s(\tau,x)>0,$ i.e. $|s(\tau,x)| = s(\tau,x).$ In view of the fact that the functions $s(\tau,x)$ and $r(\tau,x)$ are positives on their respective intervals, and hence, equation (4.24) becomes

$$\int_{x_i}^{x_{i+1}} \left| R_x\,[(x-\tau)_+^3] \right| d\tau = \int_{x_i}^{x} r(\tau,x)\,d\tau + \int_{x}^{x_{i+1}} s(\tau,x)\,d\tau. \tag{4.30}$$

The integrations of equation (4.28) and equation (4.29) on their respective intervals by means of the *MATHEMATICA* software produce

$$\int_{x_i}^{x} r(\tau,x)\,d\tau = -\frac{1}{4}h^4(\sigma)^4 + (\frac{h^4(1-\sigma)^4}{8\sin(\theta)}-\frac{h^4}{8\sin(\theta)}-\frac{3h^4(1-\sigma)^2}{8\theta^2\sin(\theta)}$$

$$+\frac{h^4}{8\theta^2\sin(\theta)})\sin(\theta\sigma)+(\frac{h^4(1-\sigma)^4}{8\sinh(\theta)}-\frac{h^4}{8\sinh(\theta)} \tag{4.31}$$

$$+\frac{3h^4(1-\sigma)^2}{8\theta^2\sinh(\theta)}-\frac{h^4}{8\theta^2\sinh(\theta)})\sinh(\theta\sigma),$$

and

$$\int_{x}^{x_{i+1}} s(\tau, x)\, d\tau = \left(-\frac{h^4(1-\sigma)^4}{8\sin(\theta)} + \frac{3h^4(1-\sigma)^2}{8\theta^2 \sin(\theta)}\right)\sin(\theta\sigma) - \left(\frac{h^4(1-\sigma)^4}{8\sinh(\theta)} + \frac{3h^4(1-\sigma)^2}{8\theta^2 \sinh(\theta)}\right)\sinh(\theta\sigma), \tag{4.32}$$

where $\sigma = \dfrac{x - x_i}{h}$. On substituting equation (4.31) and equation (4.32) into equation (4.30), we have

$$\int_{x_i}^{x_{i+1}} \left| R_x \, [(x-\tau)_+^3] \right| d\tau = f(x), \tag{4.33}$$

where

$$f(x) = -\frac{1}{4} h^4 (\sigma)^4 + \left(-\frac{h^4}{8\sin(\theta)} + \frac{h^4}{8\theta^2 \sin(\theta)}\right)\sin(\theta\sigma) + \left(-\frac{h^4}{8\sinh(\theta)} - \frac{h^4}{8\theta^2 \sinh(\theta)}\right)\sinh(\theta\sigma). \tag{4.34}$$

Thus, inequality (4.23) can be written as

$$\left| u(x) - s_i(x) \right| \leq \frac{1}{6} \left\| u^{(4)}(\tau) \right\|_\infty f(x). \tag{4.35}$$

On maximizing both sides of inequality (4.35), we get

$$\max_{[x_i, x_{i+1}]} \left| u(x) - s_i(x) \right| \leq \frac{1}{6} \left\| u^{(4)}(\tau) \right\|_\infty \max_{[x_i, x_{i+1}]} f(x). \tag{4.36}$$

From equation (4.34), it is confirmed that

$$\max_{[x_i, x_{i+1}]} \left| f(x) \right| \leq c_1 h^4, \tag{4.37}$$

where c_1 is a constant. On substituting inequality (4.37) into inequality (4.36), this yields

$$\|u(x)-s_i(x)\|_\infty \le \|u^{(4)}(\tau)\|_\infty c_1 h^4. \tag{4.38}$$

Hence, the error function $\|e_I(x)\|_\infty$ can be estimated on the interval $[a,b]$ as shown below:

$$\|e_I(x)\|_\infty \le c_3 h^4, \tag{4.39}$$

where $c_2 = \max_{x\in[a,b]} \|u^{(4)}(x)\|_\infty$, and $c_3 = c_2 \times c_1$.

In order to complete the estimation of $e(x)$, we have to estimate $e_D(x)$. Therefore, we subtract equation (4.15) from equation (4.1), to gain

$$\begin{aligned} s_i(x)-\tilde{s}_i(x) =& (a_i-\tilde{a}_i)\sin k(x-x_i)+(b_i-\tilde{b}_i)\cos k(x-x_i) \\ &+(c_i-\tilde{c}_i)\sinh k(x-x_i)+(d_i-\tilde{d}_i)\cosh k(x-x_i), \end{aligned} \tag{4.40}$$

where

$$a_i-\tilde{a}_i = -\frac{(u_i-\tilde{u}_i)\cot(\theta)}{2}+\frac{(u_{i+1}-\tilde{u}_{i+1})}{2\sin(\theta)}+\frac{h^2(\mu_i-\tilde{\mu}_i)\cot(\theta)}{2\theta^2}-\frac{h^2(\mu_{i+1}-\tilde{\mu}_{i+1})}{2\theta^2\sin(\theta)}, \tag{4.41}$$

$$b_i-\tilde{b}_i = \frac{(u_i-\tilde{u}_i)}{2}-\frac{h^2(\mu_i-\tilde{\mu}_i)}{2\theta^2}, \tag{4.42}$$

$$c_i-\tilde{c}_i = -\frac{(u_i-\tilde{u}_i)\coth(\theta)}{2}+\frac{(u_{i+1}-\tilde{u}_{i+1})}{2\sinh(\theta)}-\frac{h^2(\mu_i-\tilde{\mu}_i)\coth(\theta)}{2\theta^2}+\frac{h^2(\mu_{i+1}-\tilde{\mu}_{i+1})}{2\theta^2\sinh(\theta)}, \tag{4.43}$$

and

$$d_i-\tilde{d}_i = \frac{(u_i-\tilde{u}_i)}{2}+\frac{h^2(\mu_i-\tilde{\mu}_i)}{2\theta^2}, \tag{4.44}$$

where $\theta = kh$.

Let $U=(u_1,\ldots,u_{n-1})^t$, $\widetilde{U}=(\widetilde{u}_1,\ldots,\widetilde{u}_{n-1})^t$, $\mu=(\mu_1,\ldots,\mu_{n-1})^t$, and $\widetilde{\mu}=(\widetilde{\mu}_1,\ldots,\widetilde{\mu}_{n-1})^t$. Thus, by applying the first derivative test and infinity norm to equation (4.40), we obtain

$$\|e_D(x)\|_\infty \le \left\|U-\widetilde{U}\right\|_\infty + \frac{h^2}{8}\|\mu-\widetilde{\mu}\|_\infty. \tag{4.45}$$

In order to estimate $\|e_D(x)\|_\infty$, we first estimate $\|\mu-\widetilde{\mu}\|_\infty$. Let $B=(b_{ij})$ designates a matrix defined as

$$B=\frac{1}{\theta^2}\begin{bmatrix} \alpha_2 & \alpha_1 & 0 & & \cdots & & 0 \\ \alpha_1 & \alpha_2 & \alpha_1 & 0 & & & \\ 0 & \alpha_1 & \alpha_2 & \alpha_1 & 0 & & \vdots \\ & & \ddots & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \alpha_1 & \alpha_2 & \alpha_1 & 0 \\ & & & 0 & \alpha_1 & \alpha_2 & \alpha_1 \\ 0 & & \cdots & & 0 & \alpha_1 & \alpha_2 \end{bmatrix}.$$

Besides, let $V=(v_{ij})$ denotes a tridiagonal matrix defined as

$$V=\begin{bmatrix} \beta_2 & \beta_1 & 0 & & \cdots & & 0 \\ \beta_1 & \beta_2 & \beta_1 & 0 & & & \\ 0 & \beta_1 & \beta_2 & \beta_1 & 0 & & \vdots \\ & & \ddots & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \beta_1 & \beta_2 & \beta_1 & 0 \\ & & & 0 & \beta_1 & \beta_2 & \beta_1 \\ 0 & & \cdots & & 0 & \beta_1 & \beta_2 \end{bmatrix}.$$

Therefore, our system, which consists of equations (4.12)-(4.14), can be represented in matrix form as

$$B\mu=\frac{1}{h^2}VU+C, \tag{4.46}$$

where $C = (\frac{\beta_1}{h^2} u_0 - \frac{\alpha_1}{\theta^2} \mu_0, 0, \ldots, 0, \frac{\beta_1}{h^2} u_n - \frac{\alpha_1}{\theta^2} \mu_n)^t$.

From equation (4.46), we get

$$B(\mu - \tilde{\mu}) = \frac{1}{h^2} V(U - \tilde{U}) + \tau(h), \qquad (4.47)$$

where $\tau(h) = (\tau_0(h), \tau_1(h), \ldots, \tau_{n-1}(h))^t$. By using the mean value theorem on each of the component $\tau_i(h)$, we obtain

$$\tau_0(h) < c_4 h^2, \tau_{n-1}(h) < c_5 h^2,$$

and

$$\|\tau_i(h)\|_\infty \le c_6 h^2, \ i = 1, 2, \ldots, n-2,$$

where c_4, c_5 and c_6 are constants. Thus, form the above inequalities, we can get

$$\|\tau(h)\|_\infty \le c_7 h^2, \qquad (4.48)$$

where $c_7 = \max\{c_4, c_5, c_6\}$.

As B is strictly diagonally dominant matrix over the interval $[0.1,1]$, then B^{-1} exists and $\|B^{-1}\|_\infty \le 30.$ Moreover, we find that $\|V\|_\infty \le 3.96.$ On substituting inequality (4.48) into equation (4.47), we have

$$\|\mu - \tilde{\mu}\|_\infty \le \frac{119}{h^2} \|U - \tilde{U}\|_\infty + 30 c_7 h^2. \qquad (4.49)$$

So, the substitution of inequality (4.49) into inequality (4.45) gives

$$\|e_D(x)\|_\infty \le \|U - \tilde{U}\|_\infty + 14.85 \|U - \tilde{U}\|_\infty + 3.75 c_7 h^4, \qquad (4.50)$$

$$\|e_D(x)\|_\infty \le 15.85\|U-\tilde{U}\|_\infty + 3.75c_7h^4. \tag{4.51}$$

By using Theorem 3.1, we may conclude that

$$\|e_D(x)\|_\infty \le c_8h^4 \tag{4.52}$$

where $c_8 = 15.85c + 3.75c_7$. Hence, on substituting inequalities (4.52) and (4.39) into equation (4.21), we obtain

$$\|e(x)\|_\infty \le c_9h^4, \tag{4.53}$$

where $c_9 = c_3 + c_8$. We summarize the above convergence analysis in the following remark.

Remark: With the assumptions of Theorem 3.1, if $\tilde{S}(x)$ is the cubic non-polynomial spline method (4.15) that used to approximate the solution of problem (1.12), (1.13), (1.14) or (1.15), i.e. $u(x)$, then

$$\|u(x)-\tilde{S}(x)\|_\infty \le c_9h^4, \tag{4.54}$$

where $c_9 = c_3 + c_8$.

4.3 Quintic Non-polynomial Spline Method

The development process and convergent analysis of the new quintic non-polynomial spline method are examined in this section.

4.3.1 Construction of Quintic Non-polynomial Spline Method

In the previous section, we have examined a cubic non-polynomial spline which solely consists of four transcendental functions, i.e. sine, cosine, sine hyperbolic and cosine hyperbolic. In this section, we extend the cubic non-polynomial spline to a new quintic non-polynomial spline, by adding a linear polynomial to the cubic non-polynnomial spline. Thus, let us consider $h=(b-a)/n,$ n being a positive integer, so that the partition of the interval $[a,b]$ is given by

$$P=\{a=x_0<x_1<\ldots<x_n=b\},$$

where $x_i=a+ih,\ i=0,1,\ldots,n-1.$ On each interval $[x_i,x_{i+1}],$ we let u_i and s_i represent the exact solution and approximate solution of problem (1.12), (1.13), (1.14) or (1.15) at the point x_i, respectively. Each quintic non-polynomial spline function $S(x)$ has to satisfy the following conditions:

- $S(x)=s_i(x),\quad x\in[x_i,x_{i+1}],\quad i=0,1,\ldots,n-1,$
- $S(a)=u(a), S(b)=u(b),$ and
- $s_i^{(r)}(x_{i+1})=s_{i+1}^{(r)}(x_{i+1}),\quad r=0,1,2,3,4,5.$

For every interval $[x_i,x_{i+1}],$ the quintic non-polynomial spline can be written in the following form

$$\begin{aligned} s_i(x) &= a_i\sin k(x-x_i)+b_i\cos k(x-x_i)+c_i\sinh k(x-x_i) \\ &\quad + d_i\cosh k(x-x_i)+f_i(x-x_i)+g_i, \end{aligned} \tag{4.55}$$

where a_i, b_i, c_i, d_i, f_i and g_i, $i = 0,1,\ldots,n-1,$ are coefficients to be specified and k is a free parameter. Therefore, to determine the coefficients of the quintic non-polynomial spline (4.55) in terms of u_i, u_{i+1}, μ_i, μ_{i+1}, Z_i and Z_{i+1}, we define

$$u_i = s_i(x_i), \tag{4.56}$$

$$u_{i+1} = s_i(x_{i+1}), \tag{4.57}$$

$$\mu_i = s_i''(x_i), \tag{4.58}$$

$$\mu_{i+1} = s_i''(x_{i+1}), \tag{4.59}$$

$$Z_i = s_i^{(4)}(x_i), \tag{4.60}$$

and

$$Z_{i+1} = s_i^{(4)}(x_{i+1}). \tag{4.61}$$

On substituting $s_i(x)$, its second and fourth derivatives in equations (4.56) - (4.61), this leads to a system of six equations with six unknowns. By solving this system, we are able to obtain the values of the coefficients as given below:

$$a_i = \frac{\mu_i \cot(kh)}{2k^2} - \frac{\mu_{i+1}}{2k^2 \sin(kh)} - \frac{Z_i \cot(kh)}{2k^4} + \frac{Z_{i+1}}{2k^4 \sin(kh)}, \tag{4.62}$$

$$b_i = -\frac{\mu_i}{2k^2} + \frac{Z_i}{2k^4}, \tag{4.63}$$

$$c_i = -\frac{\mu_i \coth(kh)}{2k^2} + \frac{\mu_{i+1}}{2k^2 \sinh(kh)} - \frac{Z_i \coth(kh)}{2k^4} + \frac{Z_{i+1}}{2k^4 \sinh(kh)}, \tag{4.64}$$

$$d_i = \frac{\mu_i}{2k^2} + \frac{Z_i}{2k^4}, \tag{4.65}$$

$$f_i = \frac{u_{i+1} - u_i}{h} + \frac{Z_i - Z_{i+1}}{hk^4}, \tag{4.66}$$

and

$$g_i = u_i - \frac{Z_i}{k^4}. \tag{4.67}$$

After that, the first, third and fifth continuity conditions of the quintic non-polynomial spline $s_i(x)$ are imposed at the knot x_{i+1}, i.e. $s_i^{(r)}(x_{i+1}) = s_{i+1}^{(r)}(x_{i+1}),\ r = 1,3,$ and $5.$ This leads to

$$\begin{aligned} &a_i k\cos(kh) - b_i k\sin(kh) + c_i k\cosh(kh) + d_i k\sinh(kh) + f_i \\ &= a_{i+1}k + c_{i+1}k + f_{i+1}, \end{aligned} \tag{4.68}$$

$$\begin{aligned} &-a_i k^3\cos(kh) + b_i k^3\sin(kh) + c_i k^3\cosh(kh) + d_i k^3\sinh(kh) \\ &= -a_{i+1}k^3 + c_{i+1}k^3, \text{ and} \end{aligned} \tag{4.69}$$

$$\begin{aligned} &a_i k^5\cos(kh) - b_i k^5\sin(kh) + c_i k^5\cosh(kh) + d_i k^5\sinh(kh) \\ &= a_{i+1}k^5 + c_{i+1}k^5, \end{aligned} \tag{4.70}$$

respectively. By substituting the coefficients a_i, b_i, c_i, d_i, f_i and g_i defined in equations (4.62) - (4.67) into equations (4.68), (4.69) and (4.70), we acquire

$$
\begin{aligned}
&(-\frac{\cot(kh)\cos(kh)}{2k^3}-\frac{\sin(kh)}{2k^3}-\frac{\coth(kh)\cosh(kh)}{2k^3}+\frac{\sinh(kh)}{2k^3}+\frac{1}{hk^4})Z_i \\
&+(\frac{\cot(kh)}{k^3}+\frac{\coth(kh)}{k^3}-\frac{2}{hk^4})Z_{i+1}+(-\frac{1}{2k^3\sin(kh)}-\frac{1}{2k^3\sinh(kh)}+\frac{1}{hk^4})Z_{i+2} \\
&+(\frac{\cot(kh)\cos(kh)}{2k}+\frac{\sin(kh)}{2k}-\frac{\coth(kh)\cosh(kh)}{2k}+\frac{\sinh(kh)}{2k})\mu_i \\
&+(-\frac{\cot(kh)}{k}+\frac{\coth(kh)}{k})\mu_{i+1}+(\frac{1}{2k\sin(kh)}-\frac{1}{2k\sinh(kh)})\mu_{i+2} \\
&=\frac{u_i-2u_{i+1}+u_{i+2}}{h},
\end{aligned}
\tag{4.71}
$$

$$
\begin{aligned}
&(\frac{\cot(kh)\cos(kh)}{2k}+\frac{\sin(kh)}{2k}-\frac{\coth(kh)\cosh(kh)}{2k}+\frac{\sinh(kh)}{2k})Z_i \\
&+(-\frac{\cot(kh)}{k}+\frac{\coth(kh)}{k})Z_{i+1}+(\frac{1}{2k\sin(kh)}-\frac{1}{2k\sinh(kh)})Z_{i+2} \\
&+(-\frac{k\cot(kh)\cos(kh)}{2}-\frac{k\sin(kh)}{2}-\frac{k\coth(kh)\cosh(kh)}{2}+\frac{k\sinh(kh)}{2})\mu_i \\
&+(k\cot(kh)+k\coth(kh))\mu_{i+1}+(-\frac{k}{2\sin(kh)}-\frac{k}{2\sinh(kh)})\mu_{i+2}=0,
\end{aligned}
\tag{4.72}
$$

and

$$
\begin{aligned}
&(-\frac{k\cot(kh)\cos(kh)}{2}-\frac{k\sin(kh)}{2}-\frac{k\coth(kh)\cosh(kh)}{2}+\frac{k\sinh(kh)}{2})Z_i \\
&+(k\cot(kh)+k\coth(kh))Z_{i+1}+(-\frac{k}{2\sin(kh)}-\frac{k}{2\sinh(kh)})Z_{i+2} \\
&+k^3(\frac{\cot(kh)\cos(kh)}{2}+\frac{\sin(kh)}{2}-\frac{\coth(kh)\cosh(kh)}{2}+\frac{\sinh(kh)}{2})\mu_i \\
&+(-k^3\cot(kh)+k^3\coth(kh))\mu_{i+1}+(\frac{k^3}{2\sin(kh)}-\frac{k^3}{2\sinh(kh)})\mu_{i+2}=0,
\end{aligned}
\tag{4.73}
$$

respectively. The eliminations of μ_i and μ_{i+2} from equations (4.72) and (4.73) yield

$$-\csc(kh)\operatorname{csch}(kh)\, Z_i + \cot(kh)\operatorname{csch}(kh)\, Z_{i+1} + \coth(kh)\csc(kh)\, Z_{i+1}$$
$$-\csc(kh)\operatorname{csch}(kh) Z_{i+2} + k^2(\coth(kh)\csc(kh) \tag{4.74}$$
$$-\cot(kh)\operatorname{csch}(kh))\mu_{i+1} = 0.$$

As a result, from equation (4.74), we have

$$\mu_{i+1} = -\frac{Z_i - (\cos(kh) + \cosh(kh))Z_{i+1} + Z_{i+2}}{k^2(\cos(kh) - \cosh(kh))}. \tag{4.75}$$

The following recurrence relation is obtained by eliminating μ_i, μ_{i+1} and μ_{i+2} from equations (4.71) and (4.73):

$$Z_i - 2Z_{i+1} + Z_{i+2} = k^4(u_i - 2u_{i+1} + u_{i+2}), \tag{4.76}$$

where $i = 0,1,\ldots,n-3,n-2.$

Equation (4.76) provides a system of $n-1$ equations in $n+1$ unknowns, which are the Z_i, $i = 0,1,\ldots,n-1,n.$ As a result of choosing $Z_0 = u^{(4)}(x_0)$ and $Z_n = u^{(4)}(x_n)$ as extra conditions, this system can be solved uniquely.

To obtain the last two equations, we substitute the values of Z_0 and Z_n in equation (4.76), to get

$$-2Z_1 + Z_2 = k^4(u_0 - 2u_1 + u_2) - Z_0, \tag{4.77}$$

and

$$-2Z_{n-2} + Z_{n-1} = k^4(u_{n-2} - 2u_{n-1} + u_n) - Z_n. \tag{4.78}$$

As a result, a system of $n-1$ equations with $n-1$ unknowns has been formed. These unknowns can be solved using the *MATHEMATICA* software. Finally, to fully determine the proposed quintic non-polynomial spline, we choose $\mu_0 = u''(x_0)$ and $\mu_n = u''(x_n)$. Thus, our quintic non-polynomial spline $s_i(x)$ in equation (4.55) is completely defined.

To construct an algorithm for the proposed quintic non- polynomial spline method, we use the following steps:

Step 1: Divide the interval $[a,b]$ into n subintervals by the points $x_i = a + ih$, where $h = (b-a)/n$ and $i = 0, 1, \ldots, n$.

Step 2: To attain the approximate solution u_i at the grid points, we

- implement the explicit 4-stage fourth order Runge-Kutta method to first order IVP (1.12), or
- apply the approach in Paláncz and Popper (2000), which is based on the explicit 4-stage fourth order Runge-Kutta method, to first order BVP (1.13), or
- employ the explicit 4-stage fourth order Runge-Kutta method to second order IVP (1.14), or
- exercise the shooting method with the explicit 4-stage fourth order Runge-Kutta method to second order BVP (1.15).

Step 3: Equations (4.76), (4.77) and (4.78) forms a linear system which can be solved for the values of Z_i, for $i = 1, 2, \ldots, n-1$.

Step 4: The values of u_i and Z_i produced from Step 2 and Step 3 are used to establish the values of a_i, b_i, c_i, d_i, f_i and g_i. Hence, the quintic non-polynomial spline method $s_i(x)$ in equation (4.55) is fully defined.

4.3.2 Convergence Analysis of Quintic Non-polynomial Method

We consider $s_i(x)$ and $\tilde{s}_i(x)$ to represent the quintic non-polynomial spline (4.55) using the exact values u_i, μ_i and Z_i. and the approximate values $\tilde{u}_i, \tilde{\mu}_i$ and $\tilde{Z}_i$, respectively. The approximate solution $\tilde{u}_i$ is obtained with the help of explicit 4-stage fourth order Runge-Kutta method of the problems mentioned in (1.12), (1.13), (1.14) or (1.15), and $\tilde{\mu}_i$ and $\tilde{Z}_i$ are the second and fourth derivatives of $\tilde{s}_i(x)$ at the point $(x_i, \tilde{u}_i)$, respectively. Then, $\tilde{s}_i(x)$ takes the following form

$$\begin{aligned}\tilde{s}_i(x) = {}& \tilde{a}_i \sin k(x-x_i) + \tilde{b}_i \cos k(x-x_i) + \tilde{c}_i \sinh k(x-x_i) \\ &+ \tilde{d}_i \cosh k(x-x_i) + \tilde{f}_i (x-x_i) + \tilde{g}_i,\end{aligned} \tag{4.79}$$

where

$$\tilde{a}_i = \frac{\tilde{\mu}_i \cot(kh)}{2k^2} - \frac{\tilde{\mu}_{i+1}}{2k^2 \sin(kh)} - \frac{\tilde{Z}_i \cot(kh)}{2k^4} + \frac{\tilde{Z}_{i+1}}{2k^4 \sin(kh)}, \tag{4.80}$$

$$\tilde{b}_i = -\frac{\tilde{\mu}_i}{2k^2} + \frac{\tilde{Z}_i}{2k^4}, \tag{4.81}$$

$$\tilde{c}_i = -\frac{\tilde{\mu}_i \coth(kh)}{2k^2} + \frac{\tilde{\mu}_{i+1}}{2k^2 \sinh(kh)} - \frac{\tilde{Z}_i \coth(kh)}{2k^4} + \frac{\tilde{Z}_{i+1}}{2k^4 \sinh(kh)}, \tag{4.82}$$

$$\tilde{d}_i = \frac{\tilde{\mu}_i}{2k^2} + \frac{\tilde{Z}_i}{2k^4}, \tag{4.83}$$

$$\tilde{f}_i = \frac{\tilde{u}_{i+1} - \tilde{u}_i}{h} + \frac{\tilde{Z}_i - \tilde{Z}_{i+1}}{hk^4}, \tag{4.84}$$

and

$$\tilde{g}_i = \tilde{u}_i - \frac{\tilde{Z}_i}{k^4}. \tag{4.85}$$

The convergence of the proposed method can be proved by defining $e(x)$ to be the difference between the exact solution $u(x)$ and the approximate solution by the quintic non-polynomial spline function $\tilde{s}_i(x)$ for problem (1.12), (1.13), (1.14) or (1.15). As a result, we get

$$e(x) = u(x) - \tilde{S}(x),\ x \in [a,b]. \tag{4.86}$$

It is easy to show that $e(x)$ in equation (4.86) could be rewritten as follows

$$\begin{aligned} e(x) &= [u(x) - S(x)] + [S(x) - \tilde{S}(x)] \\ &= e_I(x) + e_D(x), \end{aligned} \tag{4.87}$$

where $e_I(x)$ is the error caused by the quintic non-polynomial spline interpolation and $e_D(x)$ is the error on account of discretization of problem (1.12), (1.13), (1.14) or (1.15). So, to estimate $e(x)$, we need to estimate $e_I(x)$ and $e_D(x)$ separately. It will be assumed throughout this discussion that $u(x) \in C^5[a,b]$.

The error function $e_I(x)$ can be estimate via Theorem 2.5. Therefore, we defined $e_I(x)$ on the interval $[x_i, x_{i+1}]$ as

$$u(x) - s_i(x) = \frac{1}{24}\int_{x_i}^{x_{i+1}} u^{(5)}(\tau)\, R_x\,[(x-\tau)_+^4]\, d\tau, \tag{4.88}$$

where $R_x\,[(x-\tau)_+^4]$ denotes the Peano kernel. Therefore, by using Definition 2.7, the Peano kernel function on the subinterval $[x_i, x_{i+1}]$ can be expressed as

$$R_x\,[(x-\tau)_+^4] = \begin{cases} r(\tau, x), & x_i < \tau < x, \\ s(\tau, x), & x < \tau < x_{i+1}. \end{cases}$$

Taking the absolute value for equation (4.88) on the interval $[x_i, x_{i+1}]$ and by using Theorem 2.6, this gives

$$\left|u(x) - s_i(x)\right| \le \frac{1}{24}\left\|u^{(5)}(x)\right\|_\infty \int_{x_i}^{x_{i+1}} \left|R_x\,[(x-\tau)_+^4]\right| d\tau. \tag{4.89}$$

The integral $\int_{x_i}^{x_{i+1}} \left|R_x\,[(x-\tau)_+^4]\right| d\tau$ on the right hand side of inequality (4.89) can be written as

$$\int_{x_i}^{x_{i+1}} \left|R_x\,[(x-\tau)_+^4]\right| d\tau = \int_{x_i}^{x} \left|r(\tau, x)\right| d\tau + \int_{x}^{x_{i+1}} \left|s(\tau, x)\right| d\tau. \tag{4.90}$$

In order to evaluate the Peano kernel $R_x\,[(x-\tau)_+^4]$, it can be expressed as

$$
\begin{aligned}
R_x\,[(x-\tau)_+^4] = (x-\tau)_+^4 &- [\frac{24h^4}{2\theta^4 \sin\theta} - \frac{12h^2 (x_{i+1}-\tau)_+^2}{2\theta^2 \sin\theta} - \frac{24h^4 \cot\theta}{2\theta^4} \\
&+ \frac{12h^2 (x_i-\tau)_+^2 \cot\theta}{2\theta^2}]\sin k(x-x_i) - [\frac{24h^4}{2\theta^4} \\
&- \frac{12h^2 (x_i-\tau)_+^2}{2\theta^2}]\cos k(x-x_i) - [\frac{24h^4}{2\theta^4 \sinh\theta} + \frac{12h^2 (x_{i+1}-\tau)_+^2}{2\theta^2 \sinh\theta} \\
&- \frac{24h^4 \coth\theta}{2\theta^4} - \frac{12h^2 (x_i-\tau)_+^2 \coth\theta}{2\theta^2}]\sinh k(x-x_i) \\
&- [\frac{24h^4}{2\theta^4} + \frac{12h^2 (x_i-\tau)_+^2}{2\theta^2}]\cosh k(x-x_i) - \frac{(x_{i+1}-\tau)_+^4}{h}(x-x_i) \\
&- [(x_i-\tau)_+^4 - \frac{24h^4}{2\theta^4}],
\end{aligned}
\qquad (4.91)
$$

where $\theta = kh$. As a consequence, from equation (4.91), we note that the Peano kernel could be reformulated as follows

$$
\begin{aligned}
r(\tau,x) = (x-\tau)^4 &- [\frac{24h^4}{2\theta^4 \sin\theta} - \frac{12h^2 (x_{i+1}-\tau)^2}{2\theta^2 \sin\theta} - \frac{24h^4 \cot\theta}{2\theta^4}]\sin k(x-x_i) \\
&- \frac{24h^4}{2\theta^4}\cos k(x-x_i) - [\frac{24h^4}{2\theta^4 \sinh\theta} + \frac{12h^2 (x_{i+1}-\tau)^2}{2\theta^2 \sinh\theta} \\
&- \frac{24h^4 \coth\theta}{2\theta^4}]\sinh k(x-x_i) - \frac{24h^4}{2\theta^4}\cosh k(x-x_i) \\
&- \frac{(x_{i+1}-\tau)^4}{h}(x-x_i) + \frac{24h^4}{2\theta^4},
\end{aligned}
\qquad (4.92)
$$

if $x_i < \tau < x$; otherwise

$$s(\tau,x) = -[\frac{24h^4}{2\theta^4 \sin\theta} - \frac{12h^2(x_{i+1}-\tau)^2}{2\theta^2 \sin\theta} - \frac{24h^4 \cot\theta}{2\theta^4}]\sin k(x-x_i) - \frac{24h^4}{2\theta^4}\cos k(x-x_i) - [\frac{24h^4}{2\theta^4 \sinh\theta} + \frac{12h^2(x_{i+1}-\tau)^2}{2\theta^2 \sinh\theta} - \frac{24h^4 \coth\theta}{2\theta^4}]\sinh k(x-x_i) - \frac{24h^4}{2\theta^4}\cosh k(x-x_i) - \frac{(x_{i+1}-\tau)^4}{h}(x-x_i) + \frac{24h^4}{2\theta^4}, \tag{4.93}$$

if $x < \tau < x_{i+1}$. We note that $r(x,x) \neq 0$ for all $x \in (x_i, x_{i+1})$.

To determine the values of τ such that $r(\tau,x) = 0$, we let $\rho = x - \tau$. On substituting this assumption into equation (4.92), we obtain

$$r(\tau,x) = \rho^4 - [\frac{12h^4}{\theta^4 \sin\theta} - \frac{6h^2((x_{i+1}-x)^2 + 2\rho(x_{i+1}-x) + \rho^2)}{\theta^2 \sin\theta} - \frac{12h^4 \cot\theta}{\theta^4}]\sin k(x-x_i) - \frac{24h^4}{2\theta^4}\cos k(x-x_i) - [\frac{12h^4}{2\theta^4 \sinh\theta} + \frac{6h^2((x_{i+1}-x)^2 + 2\rho(x_{i+1}-x) + \rho^2)}{\theta^2 \sinh\theta} - \frac{12h^4 \coth\theta}{\theta^4}]\sinh k(x-x_i) - \frac{12h^4}{\theta^4}\cosh k(x-x_i) - \frac{((x_{i+1}-x)^4 + 4\rho(x_{i+1}-x)^3 + 6\rho^2(x_{i+1}-x)^2}{h}(x-x_i) + \frac{4\rho^3(x_{i+1}-x) + \rho^4)}{h}(x-x_i) + \frac{12h^4}{\theta^4} = 0. \tag{4.94}$$

By using *MATHEMATICA* software to solve equation (4.94), we find that there is no real root for $r(\tau,x)$ for all $\tau \in (x_i, x)$. Moreover, for all $\tau \in (x_i, x)$, $r(\tau,x) < 0$, i.e. $|r(\tau,x)| = -r(\tau,x)$.

Similarly, in order to find out the values of τ such that $s(\tau, x) = 0,$ the same assumption $\rho = x - \tau$ is substituted in equation (4.93) and this gives

$$
\begin{aligned}
s(\tau, x) = -[&\frac{12h^4}{\theta^4 \sin\theta} - \frac{6h^2((x_{i+1} - x)^2 + 2\rho(x_{i+1} - x) + \rho^2)}{\theta^2 \sin\theta} \\
&- \frac{12h^4 \cot\theta}{\theta^4}]\sin k(x - x_i) - \frac{24h^4}{2\theta^4}\cos k(x - x_i) \\
&- [\frac{12h^4}{2\theta^4 \sinh\theta} + \frac{6h^2((x_{i+1} - x)^2 + 2\rho(x_{i+1} - x) + \rho^2)}{\theta^2 \sinh\theta} \\
&- \frac{12h^4 \coth\theta}{\theta^4}]\sinh k(x - x_i) - \frac{12h^4}{\theta^4}\cosh k(x - x_i) \\
&- \frac{((x_{i+1} - x)^4 + 4\rho(x_{i+1} - x)^3 + 6\rho^2 (x_{i+1} - x)^2}{h}(x - x_i) \\
&+ \frac{4\rho^3 (x_{i+1} - x) + \rho^4)}{h}(x - x_i) + \frac{12h^4}{\theta^4} = 0.
\end{aligned} \tag{4.95}
$$

By using *MATHEMATICA* software to solve equation (4.95), we note that $s(\tau, x) \neq 0$ for all $\tau \in (x, x_{i+1})$. Furthermore, the function $s(\tau, x)$ is negative for all $\tau \in (x, x_{i+1}),$ i.e. $|s(\tau, x)| = -s(\tau, x).$ Therefore, the integrals in equation (4.90) can be written as

$$\int_{x_i}^{x_{i+1}} \left| R_x\, [(x - \tau)_+^4] \right| d\tau = -\int_{x_i}^{x} r(\tau, x)\, d\tau - \int_{x}^{x_{i+1}} s(\tau, x)\, d\tau. \tag{4.96}$$

On integrating each integral in the right hand side of equation (4.96) on their respective intervals, we get

$$\int_{x_i}^{x} r(\tau,x)\,d\tau = \frac{1}{5}h^5(\sigma)^5 - \Big(\frac{12h^5\sigma}{\theta^4\sin(\theta)} + \frac{2h^5(1-\sigma)^3 - 2h^5}{\theta^2\sin(\theta)}$$

$$- \frac{12h^5\sigma\cot(\theta)}{\theta^4}\Big)\sin(\theta\sigma) - \frac{12h^5\sigma}{\theta^4}\cos(\theta\sigma)$$

$$- \Big(\frac{12h^5\sigma}{\theta^4\sinh(\theta)} - \frac{2h^5(1-\sigma)^3 - 2h^5}{\theta^2\sinh(\theta)} \tag{4.97}$$

$$- \frac{12h^5\sigma\cot(\theta)}{\theta^4}\Big)\sinh(\theta\sigma) - \frac{12h^5\sigma}{\theta^4}\cosh(\theta\sigma)$$

$$+ \frac{h^5}{5}((1-\sigma)^5 - 1)\sigma + \frac{12h^5\sigma}{\theta^4},$$

and

$$\int_{x_i}^{x} s(\tau,x)\,d\tau = -\Big(\frac{12h^5(1-\sigma)}{\theta^4\sin(\theta)} - \frac{2h^5(1-\sigma)^3}{\theta^2\sin(\theta)}$$

$$- \frac{12h^5(1-\sigma)\cot(\theta)}{\theta^4}\Big)\sin(\theta\sigma) - \frac{12h^5(1-\sigma)}{\theta^4}\cos(\theta\sigma)$$

$$- \Big(\frac{12h^5(1-\sigma)}{\theta^4\sinh(\theta)} + \frac{2h^5(1-\sigma)^3}{\theta^2\sinh(\theta)} \tag{4.98}$$

$$- \frac{12h^5(1-\sigma)\cot(\theta)}{\theta^4}\Big)\sinh(\theta\sigma) - \frac{12h^5(1-\sigma)}{\theta^4}\cosh(\theta\sigma)$$

$$- \frac{h^5(1-\sigma)^5}{5}\sigma + \frac{12h^5(1-\sigma)}{\theta^4},$$

where $\sigma = \dfrac{x - x_i}{h}$. On substituting equation (4.97) and equation (4.98) into equation (4.96), we get

$$\int_{x_i}^{x_{i+1}} \left| R_x\,[(x-\tau)_+^4] \right| d\tau = f(x), \tag{4.99}$$

where

$$f(x)=\frac{1}{5}h^5(\sigma)^5+(\frac{-12h^5}{\theta^4\sin(\theta)}+\frac{2h^5}{\theta^2\sin(\theta)}+\frac{12h^5\cot(\theta)}{\theta^4})\sin(\theta\sigma)$$
$$-\frac{12h^5}{\theta^4}\cos(\theta\sigma)+(\frac{h^5}{\theta^4\sinh(\theta)}-\frac{2h^5}{\theta^2\sinh(\theta)}$$
$$-\frac{12h^5\coth(\theta)}{\theta^4})\sinh(\theta\sigma)-\frac{12h^5\sigma}{\theta^4}\cosh(\theta\sigma)-\frac{h^4}{4}\sigma+\frac{12h^5\sigma}{\theta^4}.$$

On substituting equation (4.99) into inequality (4.89), we have

$$|u(x)-s_i(x)|\le\frac{1}{24}\|u^{(5)}(\tau)\|_\infty f(x). \tag{4.100}$$

Taking the maximum on both sides of inequality (4.100), this yields

$$\|u(x)-s_i(x)\|_\infty\le\frac{c_1}{24}\|u^{(5)}(\tau)\|_\infty h^4, \tag{4.101}$$

where $c_1h^4=\max_{x\in(x_i,x_{i+1})} f(x)$. Therefore, we can estimate the error function $\|e_I(x)\|_\infty$ on the interval $[a,b]$ as

$$\|e_I(x)\|_\infty\le c_3h^4 \tag{4.102}$$

where $c_3=\frac{c_1\times c_2}{24}$ and $c_2=\max_{x\in[a,b]}\|u^{(5)}(x)\|_\infty$.

Next, to estimate the error function $e_D(x)$, subtract equation (4.79) from equation (4.55) to obtain

$$\begin{aligned} s_i(x)-\tilde{s}_i(x)=&(a_i-\tilde{a}_i)\sin k(x-x_i)+(b_i-\tilde{b}_i)\cos k(x-x_i)\\ &+(c_i-\tilde{c}_i)\sinh k(x-x_i)+(d_i-\tilde{d}_i)\cosh k(x-x_i)\\ &+(f_i-\tilde{f}_i)(x-x_i)+(g_i-\tilde{g}_i), \end{aligned} \tag{4.103}$$

where

$$a_i - \tilde{a}_i = \frac{h^2(\mu_i - \tilde{\mu}_i)\cot(\theta)}{2\theta^2} - \frac{h^2(\mu_{i+1} - \tilde{\mu}_{i+1})}{2\theta^2 \sin(\theta)} - \frac{h^4(Z_i - \tilde{Z}_i)\cot(\theta)}{2\theta^4} + \frac{h^4(Z_{i+1} - \tilde{Z}_{i+1})}{2\theta^4 \sin(\theta)}, \tag{4.104}$$

$$b_i - \tilde{b}_i = -\frac{h^2(\mu_i - \tilde{\mu}_i)}{2\theta^2} + \frac{h^4(Z_i - \tilde{Z}_i)}{2\theta^4}, \tag{4.105}$$

$$c_i - \tilde{c}_i = -\frac{h^2(\mu_i - \tilde{\mu}_i)\coth(\theta)}{2\theta^2} + \frac{h^2(\mu_{i+1} - \tilde{\mu}_{i+1})}{2\theta^2 \sinh(\theta)} - \frac{h^4(Z_i - \tilde{Z}_i)\coth(\theta)}{2\theta^4} + \frac{h^4(Z_{i+1} - \tilde{Z}_{i+1})}{2\theta^4 \sinh(\theta)}, \tag{4.106}$$

$$d_i - \tilde{d}_i = \frac{h^2(\mu_i - \tilde{\mu}_i)}{2\theta^2} + \frac{h^4(Z_i - \tilde{Z}_i)}{2\theta^4}, \tag{4.107}$$

$$f_i - \tilde{f}_i = \frac{(u_{i+1} - \tilde{u}_{i+1}) - (u_i - \tilde{u}_i)}{h} + \frac{h^3(Z_i - \tilde{Z}_i)}{\theta^4} - \frac{h^3(Z_{i+1} - \tilde{Z}_{i+1})}{\theta^4}, \tag{4.108}$$

and

$$g_i - \tilde{g}_i = (u_i - \tilde{u}_i) - \frac{h^4(Z_i - \tilde{Z}_i)}{\theta^4}, \tag{4.109}$$

where $\theta = kh$.

Let $U = (u_1, \ldots, u_{n-1})^t$, $\tilde{U} = (\tilde{u}_1, \ldots, \tilde{u}_{n-1})^t$, $\mu = (\mu_1, \ldots, \mu_{n-1})^t$, $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_{n-1})^t$, $Z = (Z_1, \ldots, Z_{n-1})^t$, and $\tilde{Z} = (\tilde{Z}_1, \ldots, \tilde{Z}_{n-1})^t$. Therefore, by using the infinity norm together with the first derivative test on equation (4.103), we can see that

$$\|e_D(x)\|_\infty \le \|U - \tilde{U}\|_\infty + \frac{h^2}{8}\|\mu - \tilde{\mu}\|_\infty + \frac{14h^4}{1000}\|Z - \tilde{Z}\|_\infty. \tag{4.110}$$

Next, to estimate $\|e_D(x)\|_\infty$, we have to estimate $\|\mu-\tilde{\mu}\|_\infty$ and $\|Z-\tilde{Z}\|_\infty$. We noted that $\|\mu-\tilde{\mu}\|_\infty$ can be estimated using equation (4.75) as follows

$$\mu_i-\tilde{\mu}_i=-h^2\,\frac{(Z_{i-1}-\tilde{Z}_{i-1})-(\cos\theta+\cosh\theta)(Z_i-\tilde{Z}_i)+(Z_{i+1}-\tilde{Z}_{i+1})}{\theta^2(\cos\theta-\cosh\theta)}. \quad (4.111)$$

Hence, on applying the infinity norm to equation (4.111), we acquire

$$\|\mu-\tilde{\mu}\|_\infty \le \frac{h^2}{12}\|Z-\tilde{Z}\|_\infty. \quad (4.112)$$

On substituting inequality (4.112) into inequality (4.110), we have

$$\|e_D(x)\|_\infty \le \|U-\tilde{U}\|_\infty + 0.025h^4\|Z-\tilde{Z}\|_\infty. \quad (4.113)$$

In order to estimate $\|Z-\tilde{Z}\|_\infty$, we let $B=(b_{i\,j})$ denotes a tridiagonal matrix defined as

$$B=\begin{bmatrix} \frac{-2}{\theta^4} & \frac{1}{\theta^4} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{\theta^4} & \frac{-2}{\theta^4} & \frac{1}{\theta^4} & 0 & 0 & \cdots & \\ 0 & \frac{1}{\theta^4} & \frac{-2}{\theta^4} & \frac{1}{\theta^4} & 0 & \cdots & \vdots \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & & \frac{1}{\theta^4} & \frac{-2}{\theta^4} & \frac{1}{\theta^4} & 0 \\ & & & & \frac{1}{\theta^4} & \frac{-2}{\theta^4} & \frac{1}{\theta^4} \\ 0 & & \cdots & & 0 & \frac{1}{\theta^4} & \frac{-2}{\theta^4} \end{bmatrix}.$$

We also, let $V=(v_{i\,j})$ represents a tridiagonal matrix defined as

$$V = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & \\ 0 & 1 & -2 & 1 & 0 & \cdots & \vdots \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & & 1 & -2 & 1 & 0 \\ & & & & 1 & -2 & 1 \\ 0 & & \cdots & & 0 & 1 & -2 \end{bmatrix}.$$

Then, our system which consist of equations (4.76), (4.77) and (4.78), is able to be expressed in matrix form as

$$BZ = \frac{1}{h^4} VU + C, \tag{4.114}$$

where $C = (\frac{1}{h^4}u_0 - \frac{1}{\theta^4}Z_0, 0, \ldots, 0, \frac{1}{h^4}u_n - \frac{1}{\theta^4}Z_n)^t$. From equation (4.114), we have

$$B(Z - \tilde{Z}) = \frac{1}{h^4} V(U - \tilde{U}) + \tau(h), \tag{4.115}$$

where $\tau(h) = (\tau_0(h), \tau_1(h), \ldots, \tau_{n-1}(h))^t$. By means of the mean value theorem on each of the component $\tau_i(h)$, we can get

$$\tau_0(h) < c_4, \tau_{n-1}(h) < c_5,$$

and

$$\|\tau_i(h)\|_\infty \le c_6, \; i = 1, 2, \ldots, n-2,$$

where c_4, c_5 and c_6 are constants. Therefore, by using these two inequalities, it follows that

$$\|\tau(h)\|_\infty \le c_7, \tag{4.116}$$

where $c_7 = \max\{c_4, c_5, c_6\}$.

Since B is diagonally dominant matrix, then B is invertible, and by using *MATHEMATICA* software, we find $\left\|B^{-1}\right\|_\infty \leq 0.07$ and $\|V\|_\infty = 4.$ On substituting inequality (4.116) into equation (4.115), we obtain

$$\left\|Z-\tilde{Z}\right\|_\infty \leq \frac{28}{100h^4}\left\|U-\tilde{U}\right\|_\infty + 0.07c_7. \tag{4.117}$$

Consequently, we can estimate the error function $\|e_D(x)\|_\infty$ as

$$\|e_D(x)\|_\infty \leq 0.007\left\|U-\tilde{U}\right\|_\infty + 0.002c_7h^4. \tag{4.118}$$

By using Theorem 3.1, we can obtain the following result

$$\|e_D(x)\|_\infty \leq c_8h^4, \tag{4.119}$$

where $c_8 = 0.007c + 0.002c_7$. Thus, on substituting inequalities (4.102) and (4.119) into equation (4.87), we acquire

$$\|e(x)\|_\infty \leq c_9h^4, \tag{4.120}$$

where $c_9 = c_3 + c_8$. We summarize the above convergent analysis in the following remark.

Remark: With the assumptions of Theorem 3.1, if $\tilde{S}(x)$ is the quintic non-polynomial spline method (4.79) that used to approximate the solution of problem (1.12), (1.13), (1.14) or (1.15), i.e. $u(x)$, then

$$\left\|u(x)-\tilde{S}(x)\right\|_\infty \leq c_9h^4, \tag{4.121}$$

$c_9 = c_3 + c_8.$

CHAPTER FIVE
NUMERICAL RESULTS AND DISCUSSIONS

5.1 Introduction

In this chapter, some test problems consist of a variety of first and second order IVPs and BVPs, that are chosen to check the accuracy of the new spline methods. Comparisons with existing spline methods in terms of accuracy are carried out as well. For every test problem, numerical accuracy of all spline methods is investigated through the error $\left|u(x_{i+j}) - \tilde{S}(x_{i+j})\right|$ generated over the integration interval. We note that $u(x_{i+j})$ represents the theoretical solution and $\tilde{S}(x_{i+j})$ represents the numerical solution at the point $x_{i+j} = a + (i + j)h,$ for $i = 0,1,2,\ldots,n-1,$ n is the number of subintervals and $0 \le j < 1.$ Moreover, the value of n and the points of comparison are stated for each test problem.

It is worth to mention that we extend some test problems domain in order to test the efficiency of the developed spline methods on large domain. Moreover, we compare the numerical results of the new spline methods with some existing methods of the same order at different values of j to verify the capability of the developed methods to perform well for every test problem. For the test problems involving first and second order IVPs, the numerical results obtained from our new spline methods are compared with the numerical results obtained from the cubic spline method of order $O(h^4)$ in Tung (2013).

For the test problems involving second order BVPs with Dirichlet boundary conditions, the numerical results generated from our spline methods are compared with those generated by: i) the quartic spline method of order $O(h^4)$ in Al-Said et al. (2011), ii) the cubic spline method of order $O(h^4)$ in Al-Towaiq and Ala'yed (2014) and iii) the quartic B-spline method of order $O(h^5)$ in Rashidinia and Sharifi (2015). For test problems involving second order BVPs with Neumann boundary conditions, we compared the numerical results obtained from our spline methods with the numerical results obtained from the quartic spline method of order $O(h^4)$ in Liu et al. (2011). Last but not least, the graphs of $\log_{10}[|u(x_i)-\tilde{S}(x_i)|]$ vs. x_i are also presented. The logarithm scale is used in the graphs because it allows a large range of error values to be displayed without small values of error being compressed down into bottom of the graphs.

5.2 Numerical Comparisons Involving Initial Value Problems

In this section, we have used 6 test problems involving first and second order IVPs to illustrate the accuracy of the new developed spline methods in terms of error. The findings are discussed at the end of this section.

***Problem* 1** (Ayinde & Ibijola, 2015)

$u'(x)=u(x)+x^2, \quad u(0)=1, n=40, x\in[0,4].$

The theoretical solution is given by $u(x)=-2-2x-x^2+3e^x$. The exact and approximate solutions are evaluated at the point $x_{i+1/4}$, as shown in Table 5.1.

Problem 2 (Rehman, Mumtaz & Iftikhar, 2012)

$u'(x) = u(x)^2 - x^4 \sin^2(x) + x^2 \cos(x) + 2x\sin(x), u(0) = 0, n = 10, x \in [0,1].$

The theoretical solution is given by $u(x) = x^2 \sin(x).$ The exact and numerical solutions are calculated at the points $x_{i+1/4}, x_{i+1/2}$ and $x_{i+3/4},$ as presented in Table 5.2.

***Problem* 3** (Yahaya & Badmus, 2009)

$u''(x) = u'(x),, u(0) = 1, u'(0) = \frac{1}{2}, n = 20, x \in [0,2].$

The theoretical solution is given by $u(x) = 1 - e^x.$ We calculate the exact and approximate solutions at the points $x_{i+1/3}$ and $x_{i+2/3},$ as illustrated in Table 5.3.

***Problem* 4** (Jator & Li, 2009)

$u''(x) = x\ u'(x)^2, u(0) = 1, u'(0) = \frac{1}{2}, n = 10, x \in [0,1].$

The theoretical solution is given by $u(x) = 1 + \frac{1}{2}\ln(\frac{2+x}{2-x}).$ We evaluate the exact and numerical solutions at the points $x_{i+1/4}, x_{i+1/2}$ and $x_{i+3/4},$ as displayed in Table 5.4.

***Problem* 5** (Tung, 2013)

$$u_1'(x) = -1 + e^x - \sin(x) + \sin(u_2(x)),$$

$$u_2'(x) = \frac{1}{4 + u_1(x)^2} - \frac{1}{5 + e^{2x} + 2e^x \cos(x) - \sin^2(x)},$$

$$u_1(0) = 2, u_2(0) = \frac{\pi}{2}, n = 40, x \in [0,4].$$

The theoretical solution is given by $u_1(x) = e^x + \cos(x)$ and $u_2(x) = \frac{\pi}{2}$. The exact and approximate solutions are computed at the point $x_{i+1/3}$, as in Table 5.5.

***Problem* 6** (Tung, 2013)

The following linear matrix differential equations is a Sylvester matrix differential problem of the form

$$U'(x) = A(x)U(x) + U(x)B(x) + D(x), x \in [a,b],$$
$$U(a) = U_a,$$

where $U(x)$, $A(x)$, $B(x)$ and $D(x) \in C^{r \times r}$. According to Tung (2013), the constant coefficients case has been investigated extensively in the literature. However, the numerical treatment in the variable coefficients case has received little attention. For instance, consider the following Sylvester problem with

$$A(x) = \begin{pmatrix} 0 & xe^x \\ x & 0 \end{pmatrix}, B(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, D(x) = \begin{pmatrix} -(1+x^2)e^x & -2xe^x \\ 1 - xe^{-x} & -x^2 \end{pmatrix},$$

$$U(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, n = 40, x \in [0,4].$$

The theoretical solution for this problem is given by $U(x) = \begin{pmatrix} e^{-x} & 0 \\ x & 1 \end{pmatrix}$. We compute the exact and numerical solutions at the point $x_{i+1/2}$, as given in Table 5.6.

***Problem* 6** arises frequently in many important fields such as optimal control, differential games theory, invariant embedding and scattering processes and spectral factorization. For details, one can refer to Freiling, Jank and Sarychev (2000) and references cited therein.

In ***Problem* 5** and ***Problem* 6**, we first evaluate the error between the theoretical and the approximate solutions, and then take the infinity norm of this computed error. The maximum of these errors are tabulated for each subinterval as shown in Table 5.5 and Table 5.6.

Table 5.1

*Errors Obtained by Different Spline Methods in **Problem** 1*

x	Tung (2013)	New Quartic Spline	New Quintic Spline	New Cubic Non-polynomial Spline	New Quintic Non-polynomial Spline
0.025	2.2032(-7)	5.2307(-8)	8.8446(-9)	1.0631(-6)	2.9134(-7)
0.125	3.4926(-6)	7.4403(-9)	6.2046(-8)	2.1387(-7)	8.9034(-7)
0.225	1.1082(-6)	1.8959(-7)	1.4816(-7)	5.8690(-7)	1.4210(-6)
0.325	4.3184(-6)	2.1806(-7)	2.7386(-7)	6.8434(-7)	1.8781(-6)
0.425	2.2577(-6)	4.8601(-7)	4.4615(-7)	9.1659(-7)	2.2544(-6)
0.525	5.4553(-6)	6.1616(-7)	6.7362(-7)	1.1841(-6)	2.5413(-6)
0.625	3.7491(-6)	1.0039(-6)	9.6588(-7)	1.5254(-6)	2.7293(-6)
0.725	6.9932(-6)	1.2746(-6)	1.3341(-6)	1.9440(-6)	2.8071(-6)
0.825	5.6845(-6)	1.8267(-6)	1.7910(-6)	2.4547(-6)	2.7620(-6)
0.925	9.0468(-6)	2.2891(-6)	2.3511(-6)	3.0714(-6)	2.5796(-6)
1.025	8.1944(-6)	3.0642(-6)	3.0312(-6)	3.8109(-6)	2.2430(-6)
1.125	1.1763(-5)	3.7851(-6)	3.8502(-6)	4.6924(-6)	1.7333(-6)
1.225	1.1445(-5)	4.8595(-6)	4.8299(-6)	5.7375(-6)	1.0287(-6)
1.325	1.5327(-5)	5.9261(-6)	5.9949(-6)	6.9708(-6)	1.0459(-7)
1.425	1.5646(-5)	7.3990(-6)	7.3735(-6)	8.4207(-6)	1.0673(-6)
1.525	1.9976(-5)	8.9244(-6)	8.9977(-6)	1.0119(-5)	2.5189(-6)
1.625	2.1066(-5)	1.0924(-5)	1.0904(-5)	1.2102(-5)	4.2867(-6)
1.725	2.6011(-5)	1.3055(-5)	1.3133(-5)	1.4412(-5)	6.4120(-6)
1.825	2.8042(-5)	1.5748(-5)	1.5733(-5)	1.7095(-5)	8.9417(-6)
1.925	3.3813(-5)	1.8671(-5)	1.8756(-5)	2.0204(-5)	1.1929(-5)
2.025	3.7005(-5)	2.2270(-5)	2.2264(-5)	2.3800(-5)	1.5435(-5)
2.125	4.3863(-5)	2.6229(-5)	2.6323(-5)	2.7951(-5)	1.9526(-5)
2.225	4.8496(-5)	3.1009(-5)	3.1011(-5)	3.2734(-5)	2.4282(-5)
2.325	5.6769(-5)	3.6313(-5)	3.6417(-5)	3.8238(-5)	2.9788(-5)
2.425	6.3199(-5)	4.2625(-5)	4.2639(-5)	4.4560(-5)	3.6145(-5)
2.525	7.3301(-5)	4.9671(-5)	4.9788(-5)	5.1813(-5)	4.3464(-5)
2.625	8.1977(-5)	5.7964(-5)	5.7991(-5)	6.0123(-5)	5.1871(-5)
2.725	9.4426(-5)	6.7260(-5)	6.7391(-5)	6.9632(-5)	6.1508(-5)
2.825	1.0592(-4)	7.8105(-5)	7.8150(-5)	8.0502(-5)	7.2538(-5)
2.925	1.2136(-4)	9.0296(-5)	9.0445(-5)	9.2915(-5)	8.5141(-5)
3.025	1.3639(-4)	1.0442(-4)	1.0449(-4)	1.0707(-4)	9.9523(-5)
3.125	1.5565(-4)	1.2033(-4)	1.2050(-4)	1.2321(-4)	1.1591(-4)
3.225	1.7512(-4)	1.3867(-4)	1.3877(-4)	1.4159(-4)	1.3457(-4)
3.325	1.9922(-4)	1.5934(-4)	1.5949(-4)	1.6250(-4)	1.5580(-4)
3.425	2.2427(-4)	1.8305(-4)	1.8327(-4)	1.8625(-4)	1.7991(-4)
3.525	2.5449(-4)	2.0981(-4)	2.0981(-4)	2.1329(-4)	2.0729(-4)
3.625	2.8655(-4)	2.4040(-4)	2.4108(-4)	2.4381(-4)	2.3835(-4)
3.725	3.2451(-4)	2.7491(-4)	2.7393(-4)	2.7907(-4)	2.7356(-4)
3.825	3.6538(-4)	3.1421(-4)	3.1234(-4)	3.1667(-4)	3.1345(-4)
3.925	4.1310(-4)	3.5912(-4)	3.6007(-4)	3.6811(-4)	3.5871(-4)

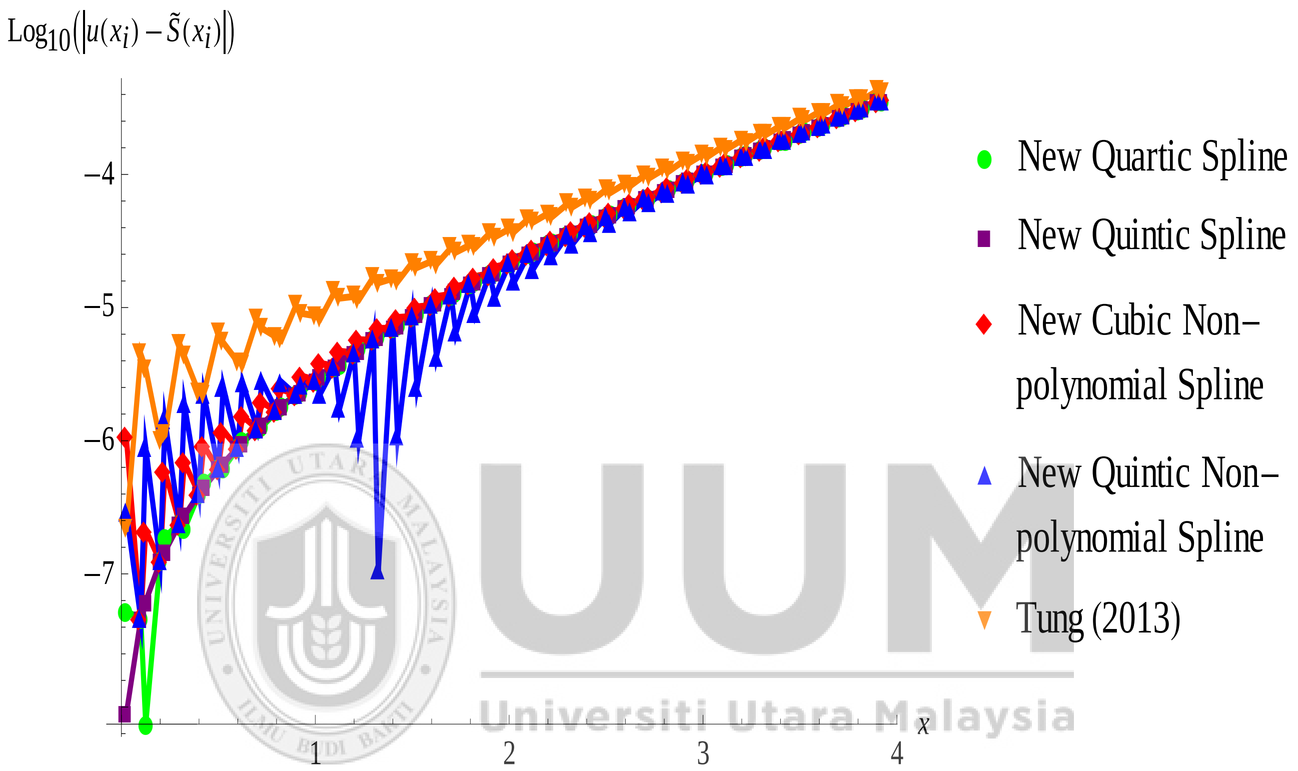

Figure 5.1. Graph of $\log_{10}[|u(x_i)-\tilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, and the spline method from Tung (2013), correspond to ***Problem* 1**

Table 5.2

Errors Obtained by Different Spline Methods in ***Problem* 2**

x	**Tung (2013)**	**New Quartic Spline**	**New Quintic Spline**	**New Cubic Non-polynomial Spline**	**New Quintic Non-polynomial Spline**
0.025	4.1746(-8)	2.4968(-7)	1.8254(-8)	2.1753(-7)	4.0665(-7)
0.05	2.9491(-7)	3.9647(-7)	3.6698(-8)	2.2364(-7)	6.1071(-7)
0.075	7.7569(-7)	2.9428(-7)	5.5136(-8)	2.1526(-9)	4.6404(-7)
0.125	8.5843(-7)	4.3941(-7)	8.8971(-8)	4.3837(-7)	1.2555(-6)
0.15	8.7888(-7)	5.3855(-7)	1.0278(-7)	6.7762(-7)	1.8021(-6)
0.175	1.5776(-6)	3.8330(-7)	1.1312(-7)	2.3934(-7)	1.3201(-6)
0.225	1.6594(-6)	1.4994(-7)	1.1717(-7)	6.8824(-7)	2.0552(-6)
0.25	1.4480(-6)	3.1706(-7)	1.1061(-7)	1.1691(-6)	2.9205(-6)
0.275	2.3517(-6)	2.4360(-7)	9.9202(-8)	5.3637(-7)	2.1485(-6)
0.325	2.4361(-6)	4.0105(-7)	5.5771(-8)	1.0220(-6)	2.8028(-6)
0.35	2.0009(-6)	4.5301(-7)	1.9227(-8)	1.7523(-6)	3.9426(-6)
0.375	3.0949(-6)	2.4117(-7)	3.0828(-8)	9.5192(-7)	2.9493(-6)
0.425	3.1914(-6)	4.4688(-7)	1.8254(-7)	1.4903(-6)	3.5053(-6)
0.45	2.5484(-6)	6.9338(-7)	2.8017(-7)	2.4707(-6)	4.8582(-6)
0.475	3.8173(-6)	7.1093(-7)	3.8583(-7)	1.5391(-6)	3.7389(-6)
0.525	3.9425(-6)	3.0501(-7)	6.2946(-7)	2.2059(-6)	4.1948(-6)
0.55	3.1162(-6)	3.7675(-7)	7.8663(-7)	3.4683(-6)	5.6874(-6)
0.575	4.5451(-6)	7.2788(-7)	9.8569(-7)	2.4522(-6)	4.5663(-6)
0.625	4.7246(-6)	1.7707(-6)	1.5348(-6)	3.1716(-6)	4.9430(-6)
0.65	3.7487(-6)	2.2032(-6)	1.8543(-6)	4.6539(-6)	6.4935(-6)
0.675	5.3244(-6)	2.4269(-6)	2.1659(-6)	3.6200(-6)	5.5233(-6)
0.725	5.5948(-6)	2.5268(-6)	2.7596(-6)	4.9720(-6)	5.8625(-6)
0.75	4.5129(-6)	2.8624(-6)	3.1285(-6)	6.9903(-6)	7.3784(-6)
0.775	6.2262(-6)	3.5001(-6)	3.6398(-6)	5.9857(-6)	6.7334(-6)
0.825	6.6392(-6)	5.2244(-6)	4.7521(-6)	6.1698(-6)	7.0806(-6)
0.85	5.5075(-6)	6.0090(-6)	5.3650(-6)	7.6307(-6)	8.4488(-6)
0.875	7.3568(-6)	6.5974(-6)	6.1417(-6)	6.7953(-6)	8.3084(-6)
0.925	7.9876(-6)	7.4327(-6)	7.9061(-6)	1.2859(-5)	8.6773(-6)
0.95	6.8794(-6)	8.0807(-6)	8.7850(-6)	1.7870(-5)	9.7450(-6)
0.975	8.8775(-6)	9.0846(-6)	9.6426(-6)	1.6940(-5)	1.0272(-5)

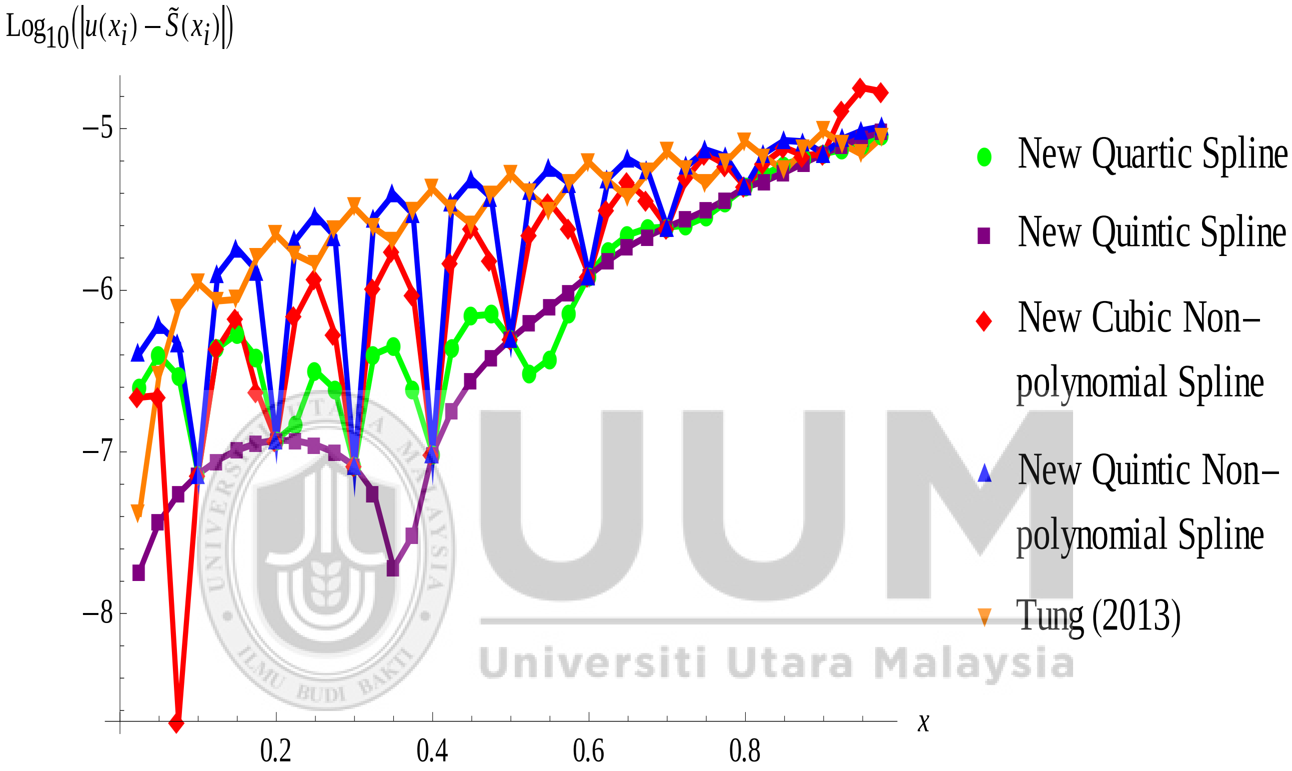

Figure 5.2. Graph of $\log_{10}[|u(x_i)-\widetilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, and the spline method from Tung (2013), correspond to ***Problem* 2**

Table 5.3

Errors Obtained by Different Spline Methods in ***Problem* 3**

x	**Tung (2013)**	**New Quartic Spline**	**New Quintic Spline**	**New Cubic Non-polynomial Spline**	**New Quintic Non-polynomial Spline**
$0.0\overline{3}$	1.6107(-7)	4.5412(-8)	2.6678(-8)	6.3707(-7)	2.7367(-8)
$0.0\overline{6}$	8.6871(-7)	7.6964(-8)	5.4743(-8)	5.3460(-7)	5.5395(-8)
$0.1\overline{3}$	9.7576(-7)	9.5114(-8)	1.1671(-7)	2.1409(-7)	1.1681(-7)
$0.1\overline{6}$	4.0098(-7)	1.3327(-7)	1.5088(-7)	2.8327(-7)	1.5097(-7)
$0.2\overline{3}$	4.3817(-7)	2.4408(-7)	2.2589(-7)	4.6076(-7)	2.2593(-7)
$0.2\overline{6}$	1.1198(-6)	2.8950(-7)	2.6692(-7)	4.9241(-7)	2.6696(-7)
$0.3\overline{3}$	1.2487(-6)	3.3469(-7)	3.5663(-7)	5.5468(-7)	3.5670(-7)
$0.3\overline{6}$	7.3323(-7)	3.8852(-7)	4.0561(-7)	6.0617(-7)	4.0567(-7)
$0.4\overline{3}$	8.0111(-7)	5.3022(-7)	5.1242(-7)	7.2034(-7)	5.1248(-7)
$0.4\overline{6}$	1.4641(-6)	5.9373(-7)	5.7055(-7)	7.7780(-7)	5.7061(-7)
$0.5\overline{3}$	1.6227(-6)	6.7462(-7)	6.9701(-7)	9.0230(-7)	6.9709(-7)
$0.5\overline{6}$	1.1691(-6)	7.4923(-7)	7.6569(-7)	9.7115(-7)	7.6576(-7)
$0.6\overline{3}$	1.2762(-6)	9.3210(-7)	9.1477(-7)	1.1208(-6)	9.1485(-7)
$0.6\overline{6}$	1.9290(-6)	1.0195(-6)	9.9556(-7)	1.2015(-6)	9.9564(-7)
$0.7\overline{3}$	2.1273(-6)	1.1477(-6)	1.1706(-6)	1.3764(-6)	1.1707(-6)
$0.7\overline{6}$	1.7398(-6)	1.2497(-6)	1.2653(-6)	1.4712(-6)	1.2654(-6)
$0.8\overline{3}$	1.8971(-6)	1.4869(-6)	1.4702(-6)	1.6761(-6)	1.4703(-6)
$0.8\overline{6}$	2.5495(-6)	1.6056(-6)	1.5808(-6)	1.7867(-6)	1.5809(-6)
$0.9\overline{3}$	2.8003(-6)	1.7962(-6)	1.8198(-6)	2.0257(-6)	1.8199(-6)
$0.9\overline{6}$	2.4850(-6)	1.9340(-6)	1.9487(-6)	2.1546(-6)	1.9488(-6)
$1.0\overline{3}$	2.7066(-6)	2.2427(-6)	2.2267(-6)	2.4326(-6)	2.2268(-6)
$1.0\overline{6}$	3.3705(-6)	2.4022(-6)	2.3764(-6)	2.5823(-6)	2.3765(-6)
$1.1\overline{3}$	3.6899(-6)	2.6746(-6)	2.6990(-6)	2.9049(-6)	2.6991(-6)
$1.1\overline{6}$	3.4554(-6)	2.8590(-6)	2.8726(-6)	3.0785(-6)	2.8727(-6)
$1.2\overline{3}$	3.7594(-6)	3.2612(-6)	3.2461(-6)	3.4520(-6)	3.2462(-6)
$1.2\overline{6}$	4.4488(-6)	3.4739(-6)	3.4468(-6)	3.6527(-6)	3.4470(-6)
$1.3\overline{3}$	4.8576(-6)	3.8530(-6)	3.8784(-6)	4.0844(-6)	3.8785(-6)
$1.3\overline{6}$	4.7155(-6)	4.0979(-6)	4.1101(-6)	4.3162(-6)	4.1102(-6)
$1.4\overline{3}$	5.1249(-6)	4.6218(-6)	4.6077(-6)	4.8133(-6)	4.6079(-6)
$1.4\overline{6}$	5.8571(-6)	4.9034(-6)	4.8747(-6)	5.0801(-6)	4.8749(-6)
$1.5\overline{3}$	6.3817(-6)	5.4211(-6)	5.4476(-6)	5.6550(-6)	5.4478(-6)
$1.5\overline{6}$	6.3471(-6)	5.7444(-6)	5.7548(-6)	5.9628(-6)	5.7549(-6)
$1.6\overline{3}$	6.8914(-6)	6.4260(-6)	6.4132(-6)	6.6139(-6)	6.4134(-6)
$1.6\overline{6}$	7.6874(-6)	6.7965(-6)	6.7659(-6)	6.9641(-6)	6.7662(-6)
$1.7\overline{3}$	8.3612(-6)	7.4940(-6)	7.5218(-6)	7.7476(-6)	7.5219(-6)
$1.7\overline{6}$	8.4540(-6)	7.9185(-6)	7.9263(-6)	8.1616(-6)	7.9264(-6)
$1.8\overline{3}$	9.1708(-6)	8.8007(-6)	8.7920(-6)	8.9241(-6)	8.7926(-6)
$1.8\overline{6}$	1.0057(-5)	9.2829(-6)	9.2553(-6)	9.3519(-6)	9.2559(-6)
$1.9\overline{3}$	1.0922(-5)	1.0245(-5)	1.0248(-5)	1.0731(-5)	1.0253(-5)
$1.9\overline{6}$	1.1168(-5)	1.0828(-5)	1.0779(-5)	1.1393(-5)	1.0784(-5)

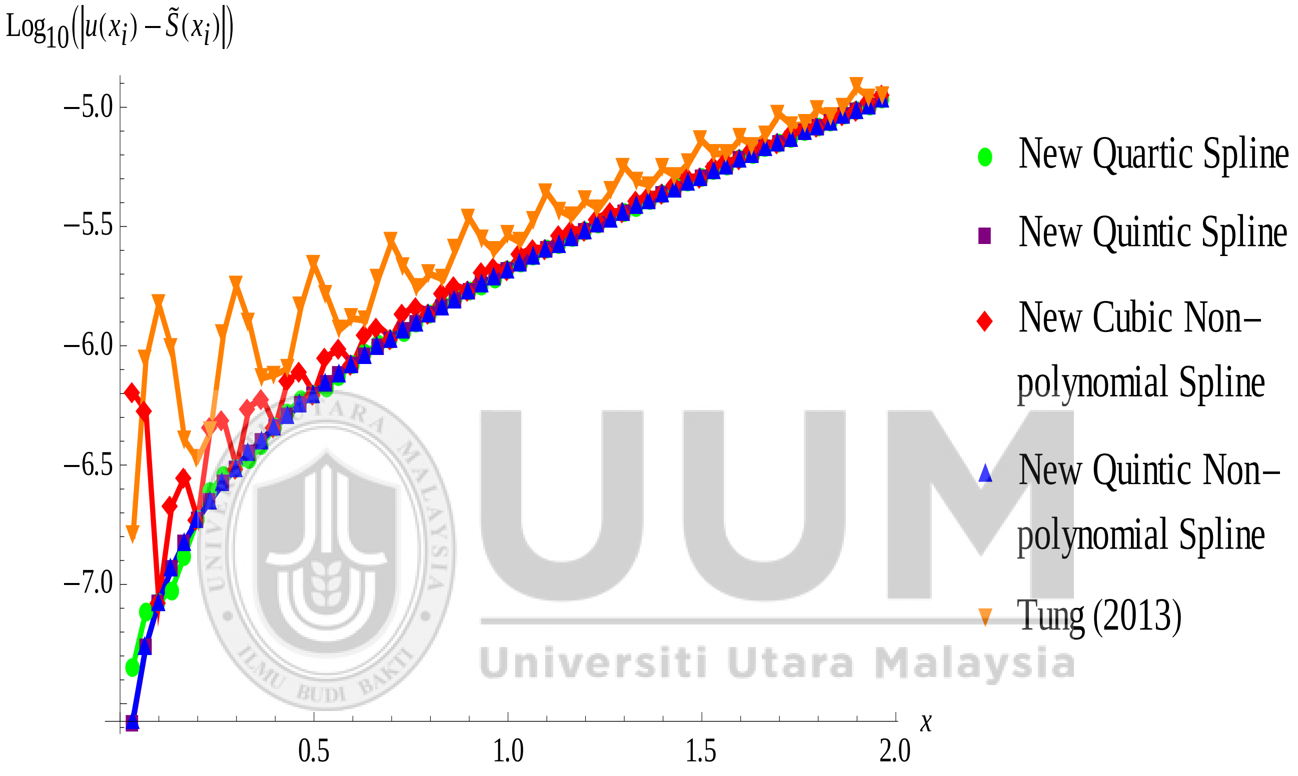

Figure 5.3. Graph of $\log_{10}[|u(x_i)-\tilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, and the spline method from Tung (2013), correspond to ***Problem* 3**

Table 5.4

Errors Obtained by Different Spline Methods in ***Problem* 4**

x	Tung (2013)	New Quartic Spline	New Quintic Spline	New Cubic Non-polynomial Spline	New Quintic Non-polynomial Spline
0.025	2.1191(-9)	1.2701(-8)	2.5862(-9)	5.0953(-7)	1.6330(-7)
0.05	1.5487(-8)	2.1529(-8)	5.2036(-9)	6.3338(-7)	2.4755(-7)
0.075	4.4017(-8)	2.0985(-8)	7.8510(-9)	3.4150(-7)	1.9486(-7)
0.125	9.8874(-8)	1.8183(-10)	1.3142(-8)	2.7612(-8)	5.1703(-7)
0.15	1.3539(-7)	8.0225(-13)	1.5874(-8)	1.3227(-7)	7.358(-7)
0.175	1.9221(-7)	9.0316(-9)	1.8737(-8)	6.9068(-8)	5.4731(-7)
0.225	2.0089(-7)	3.4675(-8)	2.4612(-8)	1.3585(-7)	8.5554(-7)
0.25	1.7776(-7)	4.4489(-8)	2.7695(-8)	2.4102(-7)	1.2026(-6)
0.275	2.0795(-7)	4.4750(-8)	3.0967(-8)	1.2158(-7)	8.8303(-7)
0.325	2.7171(-7)	2.4496(-8)	3.7638(-8)	8.0896(-8)	1.1656(-6)
0.35	3.2102(-7)	2.6090(-8)	4.1221(-8)	1.7927(-7)	1.6299(-6)
0.375	4.3496(-7)	3.6673(-8)	4.5227(-8)	8.2083(-8)	1.1884(-6)
0.425	4.5985(-7)	6.3333(-8)	5.3866(-8)	6.2158(-8)	1.4294(-6)
0.45	4.0490(-7)	7.6035(-8)	5.8561(-8)	1.5306(-7)	1.9930(-6)
0.475	4.5749(-7)	7.8888(-8)	6.3671(-8)	6.0494(-8)	1.4446(-6)
0.525	5.4551(-7)	6.0430(-8)	7.4032(-8)	1.8684(-8)	1.6208(-6)
0.55	6.1144(-7)	6.6607(-8)	7.9875(-8)	9.5400(-8)	2.2558(-6)
0.575	8.1255(-7)	8.1348(-8)	8.6964(-8)	1.7636(-8)	1.6241(-6)
0.625	8.7724(-7)	1.1110(-7)	1.0351(-7)	3.0183(-8)	1.6994(-6)
0.65	7.9732(-7)	1.3103(-7)	1.1275(-7)	3.3575(-8)	2.3626(-6)
0.675	9.1985(-7)	1.4051(-7)	1.2262(-7)	2.4476(-8)	1.6839(-6)
0.725	1.0683(-6)	1.2804(-7)	1.4141(-7)	1.4055(-7)	1.5996(-6)
0.75	1.1602(-6)	1.4648(-7)	1.5276(-7)	1.2904(-7)	2.2225(-6)
0.775	1.5216(-6)	1.7221(-7)	1.6837(-7)	1.5414(-7)	1.5539(-6)
0.825	1.6864(-6)	2.0776(-7)	2.0959(-7)	1.7306(-7)	1.2109(-6)
0.85	1.6023(-6)	2.4252(-7)	2.3288(-7)	1.2291(-7)	1.6819(-6)
0.875	1.9166(-6)	2.6829(-7)	2.5584(-7)	1.0578(-7)	1.1147(-6)
0.925	2.2281(-6)	3.1639(-7)	2.9324(-7)	7.4256(-7)	3.3934(-7)
0.95	2.3845(-6)	4.1609(-7)	3.1957(-7)	1.0837(-6)	4.7105(-7)
0.975	3.1104(-6)	4.9066(-7)	3.6296(-7)	9.6739(-7)	1.5654(-7)

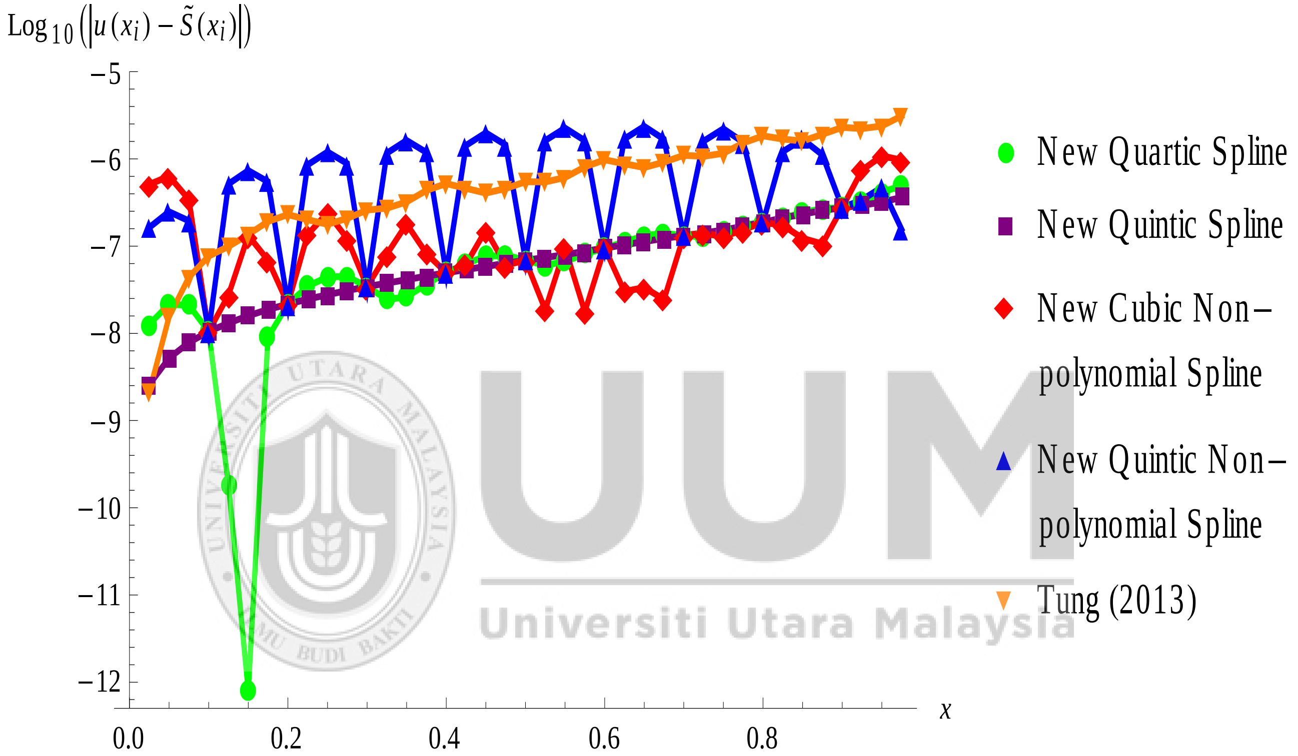

Figure 5.4. Graph of $\log_{10}[|u(x_i) - \widetilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, and the spline method from Tung (2013), correspond to ***Problem* 4**

Table 5.5

Errors Obtained by Different Spline Methods in ***Problem*** **5**

x	**Tung (2013)**	**New Quartic Spline**	**New Quintic Spline**	**New Cubic Non-polynomial Spline**	**New Quintic Non-polynomial Spline**
$0.0\overline{3}$	3.1344(-7)	1.5993(-8)	1.1607(-9)	9.5836(-7)	1.1571(-9)
$0.1\overline{3}$	1.6859(-6)	2.5180(-8)	4.6533(-9)	1.5435(-7)	4.6424(-9)
$0.2\overline{3}$	3.8398(-7)	8.9819(-9)	8.1988(-9)	3.7070(-7)	8.1808(-9)
$0.3\overline{3}$	1.7578(-6)	3.2382(-8)	1.1847(-8)	3.1342(-7)	1.1820(-8)
$0.4\overline{3}$	4.5825(-7)	2.7999(-9)	1.5653(-8)	3.2929(-7)	1.5616(-8)
$0.5\overline{3}$	1.8357(-6)	4.0314(-8)	1.9681(-8)	3.2550(-7)	1.9633(-8)
$0.6\overline{3}$	5.4096(-7)	7.0264(-9)	2.4004(-8)	3.2701(-7)	2.3944(-8)
$0.7\overline{3}$	1.9248(-6)	4.9554(-8)	2.8702(-8)	3.2724(-7)	2.8628(-8)
$0.8\overline{3}$	6.3818(-7)	1.7183(-8)	3.3868(-8)	3.2808(-7)	3.3779(-8)
$0.9\overline{3}$	2.0320(-6)	6.0826(-8)	3.9600(-8)	3.2912(-7)	3.9495(-8)
$1.0\overline{3}$	7.5747(-7)	2.9792(-8)	4.6012(-8)	3.3058(-7)	4.5888(-8)
$1.1\overline{3}$	2.1656(-6)	7.5015(-8)	5.3223(-8)	3.3246(-7)	5.3079(-8)
$1.2\overline{3}$	9.0804(-7)	4.5825(-8)	6.1369(-8)	3.3479(-7)	6.1202(-8)
$1.3\overline{3}$	2.3359(-6)	9.3184(-8)	7.0593(-8)	3.3754(-7)	7.0404(-8)
$1.4\overline{3}$	1.1010(-6)	6.6445(-8)	8.1057(-8)	3.4064(-7)	8.0842(-8)
$1.5\overline{3}$	2.5548(-6)	1.1659(-7)	9.2931(-8)	3.4400(-7)	9.2688(-8)
$1.6\overline{3}$	1.3494(-6)	9.3029(-8)	1.0641(-7)	3.4752(-7)	1.0613(-7)
$1.7\overline{3}$	2.8366(-6)	1.4676(-7)	1.2169(-7)	3.5106(-7)	1.2138(-7)
$1.8\overline{3}$	1.6688(-6)	1.2720(-7)	1.3898(-7)	3.5449(-7)	1.3864(-7)
$1.9\overline{3}$	3.1983(-6)	1.8543(-7)	1.5855(-7)	3.5775(-7)	1.5817(-7)
$2.0\overline{3}$	2.0777(-6)	1.7088(-7)	1.8065(-7)	3.6072(-7)	1.8023(-7)
$2.1\overline{3}$	3.6597(-6)	2.3469(-7)	2.0556(-7)	3.6339(-7)	2.0509(-7)
$2.2\overline{3}$	2.5976(-6)	2.2634(-7)	2.3359(-7)	3.6572(-7)	2.3307(-7)
$2.3\overline{3}$	4.2446(-6)	2.9701(-7)	2.6509(-7)	3.6772(-7)	2.6451(-7)
$2.4\overline{3}$	3.2543(-6)	2.9626(-7)	3.0040(-7)	3.6941(-7)	2.9977(-7)
$2.5\overline{3}$	4.9807(-6)	3.7532(-7)	3.3995(-7)	3.7081(-7)	3.3925(-7)
$2.6\overline{3}$	4.0782(-6)	3.8382(-7)	3.8416(-7)	3.8340(-7)	3.8339(-7)
$2.7\overline{3}$	5.9014(-6)	4.7309(-7)	4.3352(-7)	4.3268(-7)	4.3267(-7)
$2.8\overline{3}$	5.1056(-6)	4.9285(-7)	4.8855(-7)	4.8762(-7)	4.8762(-7)
$2.9\overline{3}$	7.0464(-6)	5.9451(-7)	5.4983(-7)	5.4880(-7)	5.4880(-7)
$3.0\overline{3}$	6.3801(-6)	6.2791(-7)	6.1797(-7)	6.1685(-7)	6.1685(-7)
$3.1\overline{3}$	8.4635(-6)	7.4458(-7)	6.9372(-7)	6.9246(-7)	6.9246(-7)
$3.2\overline{3}$	7.9542(-6)	7.9450(-7)	7.7772(-7)	7.7639(-7)	7.7639(-7)
$3.3\overline{3}$	1.0210(-5)	9.2936(-7)	8.7108(-7)	8.6948(-7)	8.6948(-7)
$3.4\overline{3}$	9.8911(-6)	9.9931(-7)	9.7407(-7)	9.7264(-7)	9.7264(-7)
$3.5\overline{3}$	1.2356(-5)	1.1562(-6)	1.0893(-6)	1.0869(-6)	1.0869(-6)
$3.6\overline{3}$	1.2268(-5)	1.2506(-6)	1.2141(-6)	1.2134(-6)	1.2134(-6)
$3.7\overline{3}$	1.4987(-5)	1.4316(-6)	1.3585(-6)	1.3534(-6)	1.3533(-6)
$3.8\overline{3}$	1.5177(-5)	1.5811(-6)	1.5037(-6)	1.5079(-6)	1.5080(-6)
$3.9\overline{3}$	1.8204(-5)	1.5403(-6)	1.6972(-6)	1.6797(-6)	1.6801(-6)

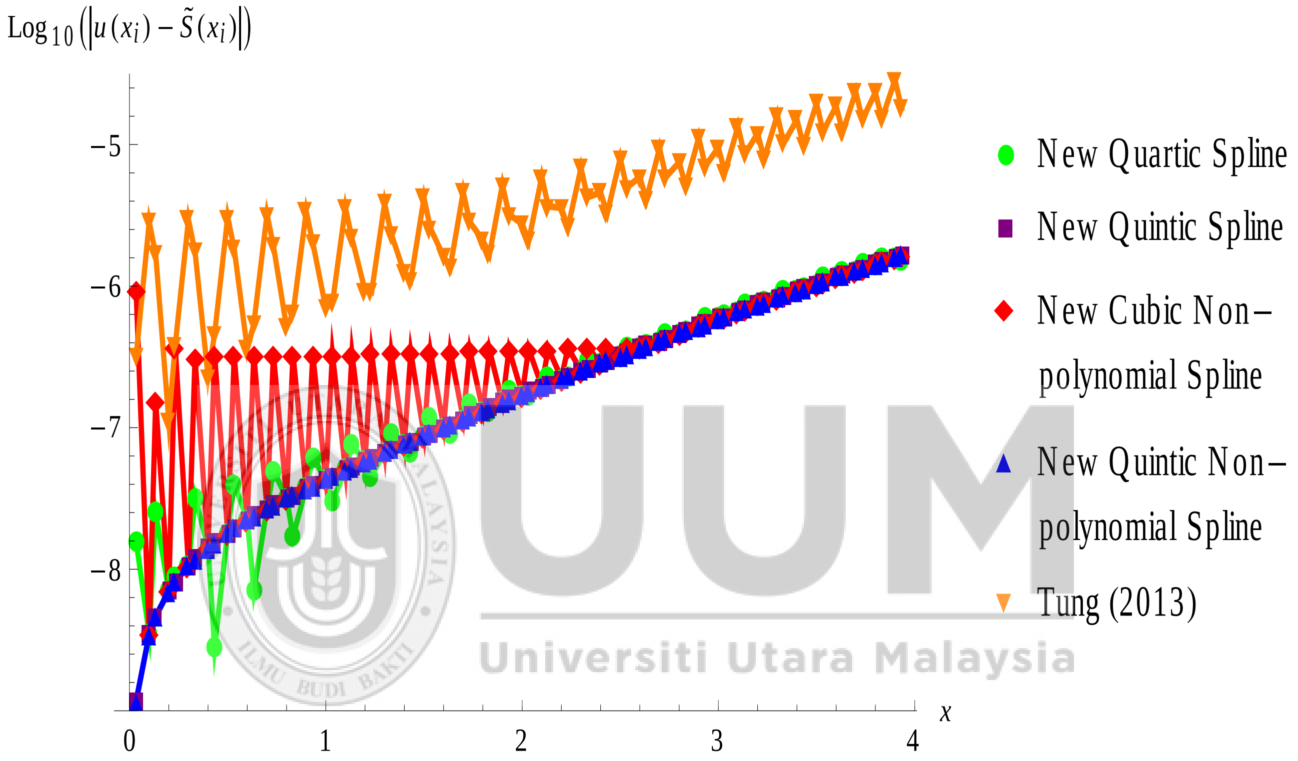

Figure 5.5. Graph of $\log_{10}[|u(x_i)-\tilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, and the spline method from Tung (2013), correspond to ***Problem* 5**

Table 5.6

*Errors Obtained by Different Spline Methods in **Problem** 6*

x	Tung (2013)	New Quartic Spline	New Quintic Spline	New Cubic Non-polynomial Spline	New Quintic Non-polynomial Spline
0.05	4.1959(-7)	2.1271(-8)	1.6520(-8)	6.4524(-7)	1.6401(-8)
0.15	3.9293(-7)	4.6745(-8)	4.6483(-8)	1.7945(-7)	4.6414(-8)
0.25	3.6903(-7)	7.1010(-8)	7.1302(-8)	3.4353(-7)	7.1332(-8)
0.35	3.4762(-7)	1.0106(-7)	1.0223(-7)	3.5534(-7)	1.0227(-7)
0.45	3.2946(-7)	1.5995(-7)	1.5875(-7)	4.2111(-7)	1.5872(-7)
0.55	3.1295(-7)	2.2155(-7)	2.2276(-7)	4.8266(-7)	2.2277(-7)
0.65	3.0096(-7)	2.9361(-7)	2.9239(-7)	5.5295(-7)	2.9239(-7)
0.75	2.8896(-7)	3.6452(-7)	3.6573(-7)	6.2611(-7)	3.6573(-7)
0.85	2.8339(-7)	4.4232(-7)	4.4110(-7)	7.0153(-7)	4.4110(-7)
0.95	2.7540(-7)	5.1570(-7)	5.1692(-7)	7.7733(-7)	5.1692(-7)
1.05	2.7625(-7)	5.9294(-7)	5.9172(-7)	8.5214(-7)	5.9172(-7)
1.15	3.3328(-7)	6.6295(-7)	6.6417(-7)	9.2458(-7)	6.6417(-7)
1.25	3.1315(-7)	7.3425(-7)	7.3303(-7)	9.9345(-7)	7.3303(-7)
1.35	4.4777(-7)	7.9597(-7)	7.9719(-7)	1.0576(-6)	7.9719(-7)
1.45	5.2104(-7)	8.5684(-7)	8.5562(-7)	1.1160(-6)	8.5562(-7)
1.55	5.9795(-7)	9.0617(-7)	9.0739(-7)	1.1678(-6)	9.0739(-7)
1.65	8.5046(-7)	9.5284(-7)	9.5162(-7)	1.2120(-6)	9.5162(-7)
1.75	9.6205(-7)	9.8629(-7)	9.8750(-7)	1.2479(-6)	9.8751(-7)
1.85	1.3250(-6)	1.0154(-6)	1.0142(-6)	1.2746(-6)	1.0142(-6)
1.95	1.4849(-6)	1.0299(-6)	1.0311(-6)	1.2915(-6)	1.0311(-6)
2.05	1.9928(-6)	1.0384(-6)	1.0372(-6)	1.2976(-6)	1.0372(-6)
2.15	2.2197(-6)	1.0306(-6)	1.0318(-6)	1.2922(-6)	1.0318(-6)
2.25	2.9155(-6)	1.0152(-6)	1.0139(-6)	1.2743(-6)	1.0139(-6)
2.35	3.2349(-6)	9.8136(-7)	9.8258(-7)	1.2429(-6)	9.8258(-7)
2.45	4.1715(-6)	9.3784(-7)	9.3662(-7)	1.1970(-6)	9.3663(-7)
2.55	4.6165(-6)	8.7358(-7)	8.7480(-7)	1.1352(-6)	8.7481(-7)
2.65	5.8582(-6)	7.9689(-7)	7.9567(-7)	1.0560(-6)	7.9568(-7)
2.75	6.4714(-6)	6.9640(-7)	6.9762(-7)	9.5803(-7)	6.9762(-7)
2.85	8.0956(-6)	5.8003(-7)	5.7881(-7)	8.3922(-7)	5.7881(-7)
2.95	8.9303(-6)	4.3597(-7)	4.3719(-7)	6.9761(-7)	4.3720(-7)
3.05	1.1029(-5)	3.8379(-7)	3.8346(-7)	5.7511(-7)	3.8346(-7)
3.15	1.2150(-5)	4.2279(-7)	4.2312(-7)	5.7166(-7)	4.2312(-7)
3.25	1.4830(-5)	4.6655(-7)	4.6622(-7)	5.6616(-7)	4.6622(-7)
3.35	1.6318(-5)	5.1267(-7)	5.1300(-7)	5.5963(-7)	5.1300(-7)
3.45	1.9705(-5)	7.0136(-7)	7.0258(-7)	5.6368(-7)	7.0257(-7)
3.55	2.1652(-5)	1.0398(-6)	1.0386(-6)	7.7629(-7)	1.0386(-6)
3.65	2.5887(-5)	1.4182(-6)	1.4194(-6)	1.1663(-6)	1.4194(-6)
3.75	2.8402(-5)	1.8504(-6)	1.8492(-6)	1.5614(-6)	1.8492(-6)
3.85	3.3648(-5)	2.3314(-6)	2.3326(-6)	2.1744(-6)	2.3327(-6)
3.95	3.6857(-5)	2.8765(-6)	2.8750(-6)	2.2334(-6)	2.8751(-6)

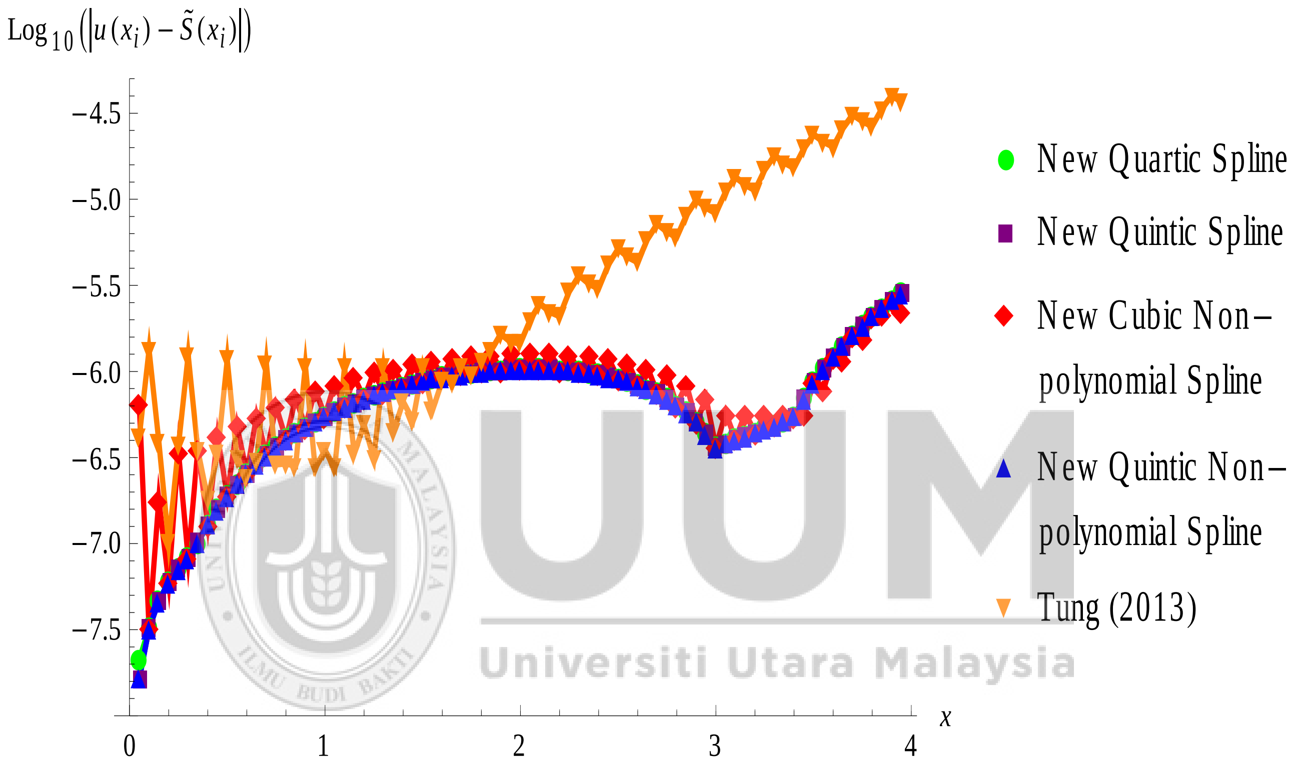

Figure 5.6. Graph of $\log_{10}[|u(x_i)-\tilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, and the spline method from Tung (2013), correspond to ***Problem 6***

In this section, a total of 6 IVPs from the literatures are solved by the new proposed spline methods and the existing spline method of Tung (2013). These 6 test problems can be divided into three groups:

i. ***Problem* 1** and ***Problem* 2** are first order IVPs.

ii. ***Problem* 3** and ***Problem* 4** are second order IVPs.

iii. ***Problem* 5** and ***Problem* 6** are matrix differential equations with initial conditions.

For most of the problems being tested, from Table 5.1 - Table 5.6, it can be observed that the new presented spline methods are more accurate than the cubic spline method in Tung (2013). Specifically, we can see from Table 5.1 - Table 5.4 that, the errors caused by the developed spline methods are smaller than those from the cubic spline method at the beginning of the interval of integration. Besides, the errors results from these methods become closer to each other towards the end of the interval. Over all, we observe from Table 5.5 and Table 5.6 that, the errors due to the developed spline methods are less than those produce from the cubic spline method on the entire interval of integration.

Particularly, in Table 5.1, Table 5.3 and Table 5.5, it can be noticed that the accuracy of the new spline methods for solving the related problems is higher than the cubic spline method in Tung (2013). In addition, it is apparent in Table 5.4 that the results of the new quartic, quintic and cubic non-polynomial spline methods outperform the cubic spline method in Tung (2013) for solving the corresponding problem. Whereas, the results of the cubic spline method in Tung (2013) are found better than those results generated by the new quintic non-polynomial spline method.

From Table 5.2, we can see that the results generated by the new quartic and quintic spline methods are more accurate than the cubic spline method in Tung (2013) in solving ***Problem* 2**. Moreover, the numerical results obtained from cubic and quintic non-polynomial spline methods are comparable with those obtained from the cubic spline method in Tung (2013) in solving the same problem. The results shown in Table 5.6 implies the efficiency of the proposed methods (except for the cubic non-polynomial spline method) in terms of accuracy, over the cubic spline method in Tung (2013) in solving ***Problem* 6**.

The numerical results of the new quintic non-polynomial spline method in Table 5.1 compared favorably with the new quartic spline method and the new quintic spline method, and more accurate than the new cubic non-polynomial spline method for solving ***Problem* 1**. Additionally, the numerical results in Table 5.2 and Table 5.4 seem to indicate that the new quintic spline method is more accurate compared to other new spline methods for the solutions of ***Problem* 2** and ***Problem* 4**, respectively.

It is clear that the accuracy of the new quintic spline method and the new quintic non-polynomial spline method are comparable when solving ***Problem* 3** and ***Problem* 5**. Last but not least, it can be noticed from the numerical results in Table 5.6 that the accuracy of the new quartic, quintic and quintic non-polynomial spline methods are comparable in solving ***Problem* 6**.

5.3 Numerical Comparisons Involving Boundary Value Problems

In this section, 6 test problems involving first and second order BVPs have been used to demonstrate the performance of the new spline methods in terms of error. We observe that ***Problem* 7** and ***Problem* 8** are first order BVPs, ***Problem* 9** and ***Problem* 10** are second order BVPs subject to Dirichlet boundary conditions, whereas ***Problem* 11** and ***Problem* 12** are second order BVPs subject to Neumann boundary conditions. The findings are examined later in this section.

***Problem* 7** (Ascher & Chan, 1991)

$$\begin{pmatrix} u_1'(x) \\ u_2'(x) \end{pmatrix} = \begin{pmatrix} -\cos(2x) & 1+\sin(2x) \\ -1+\sin(2x) & \cos(2x) \end{pmatrix} \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} + e^x \begin{pmatrix} -\cos(2x)+\sin(2x) \\ 2-\cos(2x)-\sin(2x) \end{pmatrix},$$

$u_1(0)=1,\ u_2(4)=e^4,\ n=40,\ x\in[0,4].$

The theoretical solutions are given by $u_1(x)=e^x$ and $u_2(x)=e^x$. We compute the exact and numerical solutions at the point $x_{i+3/4}$, as shown in Table 5.7.

***Problem* 8** (Holsapple et al., 2004)

$$\begin{pmatrix} u_1'(x) \\ u_2'(x) \\ u_3'(x) \\ u_4'(x) \end{pmatrix} = \begin{pmatrix} u_3(x) \\ u_4(x) \\ u_2(x) \\ u_1(x) \end{pmatrix},$$

$u_1(-1)=-2+\dfrac{1}{e}+e,\ u_2(-1)=-2+\dfrac{1}{e}+e,\ u_1(1)=2,\ u_2(1)=2,\ n=20,\ x\in[-1,1].$

The theoretical solutions are given by

$$u_1(x) = \frac{e^{-x}(-2e + e^2 - e^{2x} + 2e^{1+2x})}{-1+e^2}, \;\; u_2(x) = \frac{e^{-x}(-2e + e^2 - e^{2x} + 2e^{1+2x})}{-1+e^2},$$

$$u_3(x) = \frac{e^{-x}(2e - e^2 - e^{2x} + 2e^{1+2x})}{-1+e^2}, \text{ and } u_4(x) = \frac{e^{-x}(2e - e^2 - e^{2x} + 2e^{1+2x})}{-1+e^2}.$$

The exact and numerical solutions are evaluated at the points $x_{i+1/3}$ and $x_{i+2/3}$, as presented in Table 5.8.

***Problem* 9** (Islam & Shirin, 2011)

$$u''(x) = \frac{2}{x^2}u(x) - \frac{1}{x}, u(2) = u(3) = 0, n = 10, x \in [2,3].$$

The theoretical solution is given by $u(x) = \frac{1}{38}(-5x^2 + 19x - \frac{36}{x})$. We calculate the exact and numerical solutions at the points $x_{i+1/4}, x_{i+1/2}$ and $x_{i+3/4}$, as in Table 5.9.

***Problem* 10** (Pandey, 2016)

$$u''(x) = \frac{e^{2u(x)} + (u'(x))^2}{2}, u(0) = 0, u(2) = -\ln(3), n = 20, x \in [0,2].$$

The theoretical solution is given by $u(x) = \ln(\frac{1}{1+x})$. The exact and numerical solutions are computed at the points $x_{i+1/3}$ and $x_{i+2/3}$, as illustrated in Table 5.10.

***Problem* 11** (Liu et al., 2011)

$$u''(x) = -xu(x) + (3 - x - x^2 + x^3)\sin(x) + 4x\cos(x), u'(0) = -1, u'(1) = 2\sin(1), n = 10,$$
$$x \in [0,1].$$

The theoretical solution is given by $u(x)=(x^2-1)\sin(x).$ The exact and numerical solutions are calculated at the points $x_{i+1/4}$, $x_{i+1/2}$ and $x_{i+3/4}$, as displayed in Table 5.11.

***Problem* 12** (Lakestani & Dehghan, 2006)

$$u''(x)=2u(x)^3,\ u'(0)=-1,\ u'(1)=-\frac{1}{4},\ n=10,\ x\in[0,1].$$

The theoretical solution is given by $u(x)=\dfrac{1}{1+x}.$ We evaluate the results of exact and numerical solutions at the points $x_{i+1/4}$, $x_{i+1/2}$ and $x_{i+3/4}$, as given in Table 5.12.

Lastly, for ***Problem* 7** and ***Problem* 8**, the error between the theoretical and approximate solutions are evaluated first, followed by taking the infinity norm of the computed error. Then, we tabulate the maximum of these errors for each subinterval as in Table 5.7 and Table 5.8.

Table 5.7

*Errors Obtained by Different Spline Methods in **Problem** 7*

x	**New Quartic Spline**	**New Quintic Spline**	**New Cubic Non-polynomial Spline**	**New Quintic Non-polynomial Spline**
0.075	8.4312(-6)	8.4144(-6)	8.4145(-6)	8.4138(-6)
0.175	9.1755(-6)	9.1868(-6)	9.1867(-6)	9.1864(-6)
0.275	9.8934(-6)	9.8767(-6)	9.8769(-6)	9.8769(-6)
0.375	1.0455(-5)	1.0465(-5)	1.0465(-5)	1.0465(-5)
0.475	1.0947(-5)	1.0930(-5)	1.0930(-5)	1.0930(-5)
0.575	1.1242(-5)	1.1252(-5)	1.1252(-5)	1.1252(-5)
0.675	1.1430(-5)	1.1412(-5)	1.1412(-5)	1.1412(-5)
0.775	1.1389(-5)	1.1397(-5)	1.1397(-5)	1.1397(-5)
0.875	1.1773(-5)	1.1753(-5)	1.1753(-5)	1.1753(-5)
0.975	1.3322(-5)	1.3331(-5)	1.3331(-5)	1.3331(-5)
1.075	1.4864(-5)	1.4843(-5)	1.4843(-5)	1.4843(-5)
1.175	1.6242(-5)	1.6250(-5)	1.6250(-5)	1.6250(-5)
1.275	1.7535(-5)	1.7513(-5)	1.7513(-5)	1.7513(-5)
1.375	1.8590(-5)	1.8596(-5)	1.8596(-5)	1.8596(-5)
1.475	1.9494(-5)	1.9470(-5)	1.9470(-5)	1.9470(-5)
1.575	2.0109(-5)	2.0112(-5)	2.0113(-5)	2.0112(-5)
1.675	2.0537(-5)	2.0510(-5)	2.0511(-5)	2.0510(-5)
1.775	2.0662(-5)	2.0664(-5)	2.0664(-5)	2.0664(-5)
1.875	2.0615(-5)	2.0586(-5)	2.0586(-5)	2.0586(-5)
1.975	2.0306(-5)	2.0304(-5)	2.0304(-5)	2.0304(-5)
2.075	1.9894(-5)	1.9862(-5)	1.9862(-5)	1.9862(-5)
2.175	1.9319(-5)	1.9314(-5)	1.9315(-5)	1.9315(-5)
2.275	1.8767(-5)	1.873(-5)	1.8731(-5)	1.8731(-5)
2.375	1.8196(-5)	1.8186(-5)	1.8187(-5)	1.8187(-5)
2.475	1.7802(-5)	1.7761(-5)	1.7762(-5)	1.7762(-5)
2.575	1.7546(-5)	1.7531(-5)	1.7532(-5)	1.7532(-5)
2.675	1.7606(-5)	1.7559(-5)	1.7560(-5)	1.7560(-5)
2.775	1.7909(-5)	1.7888(-5)	1.7888(-5)	1.7889(-5)
2.875	1.8580(-5)	1.8526(-5)	1.8526(-5)	1.8527(-5)
2.975	1.9466(-5)	1.9437(-5)	1.9438(-5)	1.9439(-5)
3.075	2.0592(-5)	2.0529(-5)	2.0530(-5)	2.0531(-5)
3.175	2.1678(-5)	2.1640(-5)	2.1641(-5)	2.1642(-5)
3.275	2.2600(-5)	2.2527(-5)	2.2528(-5)	2.2529(-5)
3.375	2.2908(-5)	2.2857(-5)	2.2859(-5)	2.2861(-5)
3.475	2.2293(-5)	2.2209(-5)	2.2208(-5)	2.2210(-5)
3.575	2.0115(-5)	2.0045(-5)	2.0050(-5)	2.0053(-5)
3.675	1.5871(-5)	1.5781(-5)	1.5774(-5)	1.5771(-5)
3.775	8.7505(-6)	8.6352(-6)	8.6490(-6)	8.6657(-6)
3.875	3.4479(-6)	3.3638(-6)	3.3577(-6)	3.3731(-6)
3.975	1.6562(-5)	1.7337(-5)	1.7345(-5)	1.7397(-5)

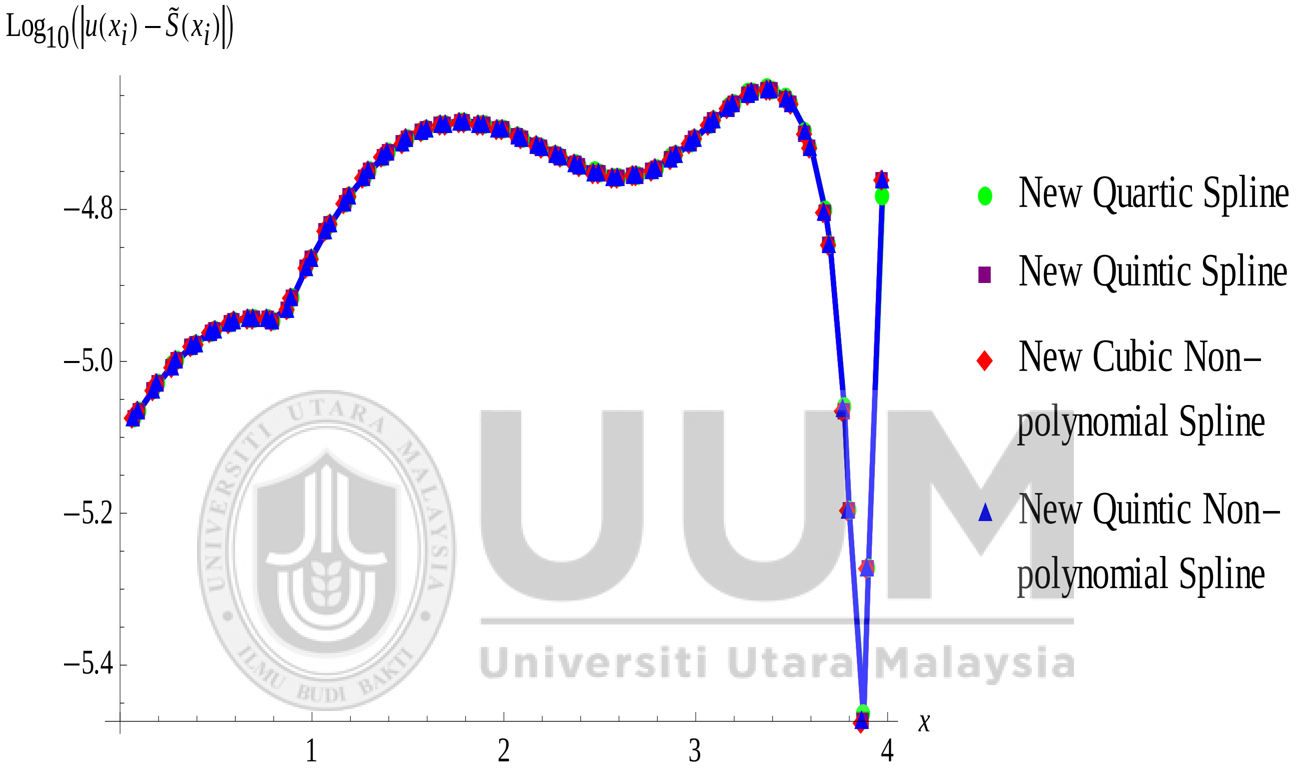

Figure 5.7. Graph of $\log_{10}[|u(x_i) - \tilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, correspond to ***Problem* 7**

Table 5.8

Errors Obtained by Different Spline Methods in ***Problem*** **8**

x	**New Quartic Spline**	**New Quintic Spline**	**New Cubic Non-polynomial Spline**	**New Quintic Non-polynomial Spline**
$-0.9\overline{6}$	6.9389(-7)	7.1162(-7)	7.1191(-7)	7.1205(-7)
$-0.9\overline{3}$	6.6144(-7)	6.8285(-7)	6.8309(-7)	6.8328(-7)
$-0.8\overline{6}$	6.5253(-7)	6.3084(-7)	6.3078(-7)	6.3092(-7)
$-0.8\overline{3}$	6.2573(-7)	6.0750(-7)	6.0747(-7)	6.0757(-7)
$-0.7\overline{6}$	5.4747(-7)	5.6551(-7)	5.6554(-7)	5.6550(-7)
$-0.7\overline{3}$	5.2525(-7)	5.4662(-7)	5.4664(-7)	5.4661(-7)
$-0.6\overline{6}$	5.3420(-7)	5.1270(-7)	5.1271(-7)	5.1272(-7)
$-0.6\overline{3}$	5.1578(-7)	4.9754(-7)	4.9755(-7)	4.9755(-7)
$-0.5\overline{6}$	4.5224(-7)	4.7040(-7)	4.7040(-7)	4.7040(-7)
$-0.5\overline{3}$	4.4294(-7)	4.5826(-7)	4.5826(-7)	4.5826(-7)
$-0.4\overline{6}$	4.6221(-7)	4.7215(-7)	4.7210(-7)	4.7211(-7)
$-0.4\overline{3}$	4.7987(-7)	4.8969(-7)	4.8964(-7)	4.8965(-7)
$-0.3\overline{6}$	5.3065(-7)	5.2128(-7)	5.2123(-7)	5.2124(-7)
$-0.3\overline{3}$	5.4455(-7)	5.3537(-7)	5.3532(-7)	5.3533(-7)
$-0.2\overline{6}$	5.5060(-7)	5.6023(-7)	5.6018(-7)	5.6019(-7)
$-0.2\overline{3}$	5.6088(-7)	5.7102(-7)	5.7098(-7)	5.7098(-7)
$-0.1\overline{6}$	5.9906(-7)	5.8938(-7)	5.8933(-7)	5.8934(-7)
$-0.1\overline{3}$	6.0580(-7)	5.9694(-7)	5.9689(-7)	5.9690(-7)
$-0.0\overline{6}$	5.9952(-7)	6.0884(-7)	6.0879(-7)	6.0880(-7)
$-0.0\overline{3}$	6.0268(-7)	6.1316(-7)	6.1312(-7)	6.1312(-7)
$0.0\overline{3}$	6.2849(-7)	6.1850(-7)	6.1844(-7)	6.1845(-7)
$0.0\overline{6}$	6.2797(-7)	6.1947(-7)	6.1942(-7)	6.1943(-7)
$0.1\overline{3}$	6.0895(-7)	6.1795(-7)	6.1789(-7)	6.1790(-7)
$0.1\overline{6}$	6.0453(-7)	6.1539(-7)	6.1534(-7)	6.1534(-7)
$0.2\overline{3}$	6.1687(-7)	6.0654(-7)	6.0648(-7)	6.0649(-7)
$0.2\overline{6}$	6.0826(-7)	6.0017(-7)	6.0011(-7)	6.0012(-7)
$0.3\overline{3}$	5.7469(-7)	5.8334(-7)	5.8328(-7)	5.8329(-7)
$0.3\overline{6}$	5.6149(-7)	5.7278(-7)	5.7272(-7)	5.7273(-7)
$0.4\overline{3}$	5.5781(-7)	5.4712(-7)	5.4706(-7)	5.4706(-7)
$0.4\overline{6}$	5.3952(-7)	5.3189(-7)	5.3183(-7)	5.3184(-7)
$0.5\overline{3}$	4.8806(-7)	4.9632(-7)	4.9625(-7)	4.9627(-7)
$0.5\overline{6}$	4.6405(-7)	4.7583(-7)	4.7576(-7)	4.7578(-7)
$0.6\overline{3}$	4.4018(-7)	4.2908(-7)	4.2901(-7)	4.2900(-7)
$0.6\overline{6}$	4.0971(-7)	4.0262(-7)	4.0256(-7)	4.0254(-7)
$0.7\overline{3}$	3.3513(-7)	3.4303(-7)	3.4291(-7)	3.4299(-7)
$0.7\overline{6}$	2.9719(-7)	3.0973(-7)	3.0959(-7)	3.0970(-7)
$0.8\overline{3}$	2.4818(-7)	2.3570(-7)	2.3580(-7)	2.3554(-7)
$0.8\overline{6}$	2.0302(-7)	1.9461(-7)	1.9482(-7)	1.9443(-7)
$0.9\overline{3}$	1.9159(-7)	2.1026(-7)	2.1097(-7)	2.1141(-7)
$0.9\overline{6}$	2.4940(-7)	2.5738(-7)	2.5823(-7)	2.5858(-7)

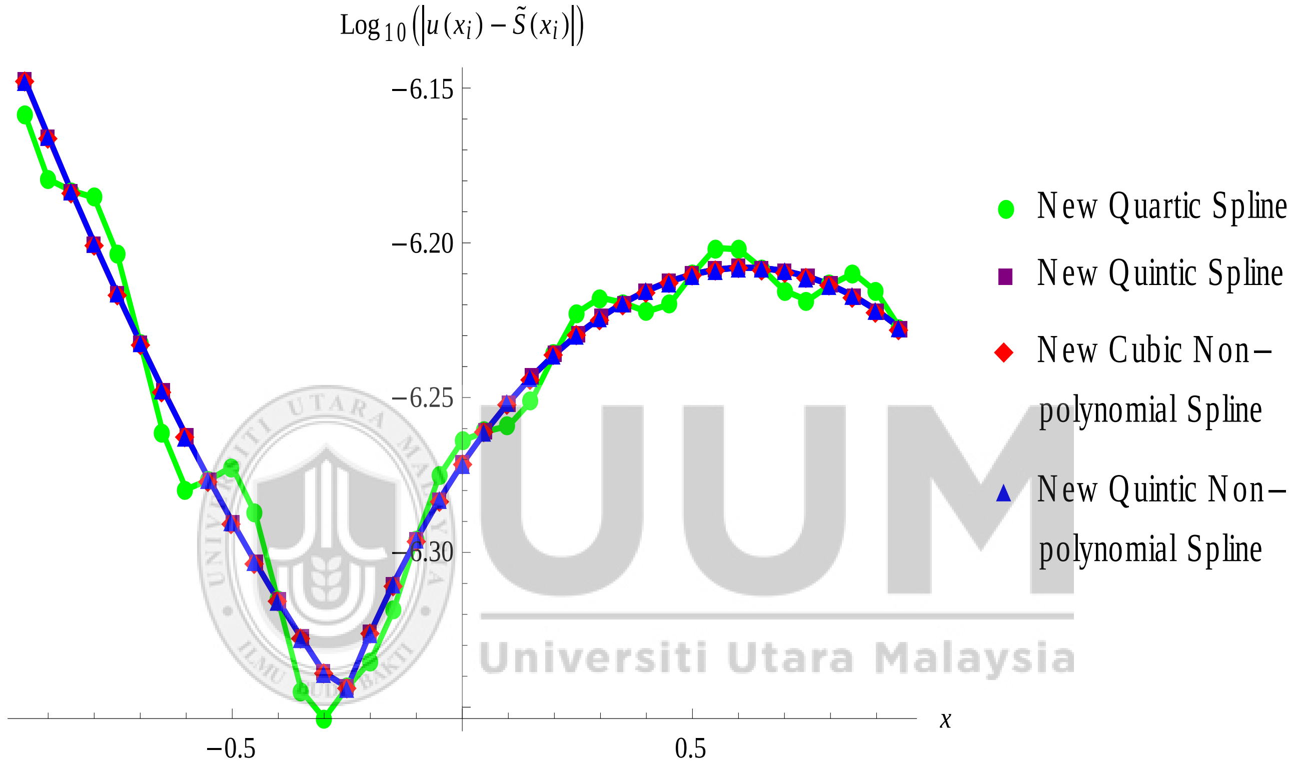

Figure 5.8. Graph of $\log_{10}[|u(x_i) - \widetilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, correspond to ***Problem* 8**

Table 5.9

*Errors Obtained by Different Spline Methods in **Problem** 9*

x	Al-Said et al. (2011)	Al-Towaiq and Ala'yed (2014)	Rashidinia and Sharifi (2015)	New Quartic Spline	New Quintic Spline	New Cubic Non-polynomial Spline	New Quintic Non-polynomial Spline
2.025	5.6189(-7)	2.2076(-4)	1.2573(-8)	2.5718(-8)	5.8742(-9)	3.4252(-7)	6.4028(-8)
2.05	7.8975(-7)	2.2834(-4)	1.8787(-8)	4.3002(-8)	1.0403(-8)	4.2561(-7)	9.8960(-8)
2.075	5.7429(-7)	1.2181(-4)	2.4979(-8)	4.0042(-8)	1.3296(-8)	2.2971(-7)	8.4218(-8)
2.125	5.2868(-7)	5.9219(-5)	4.9539(-8)	1.0741(-8)	1.7609(-8)	1.7802(-8)	1.7632(-7)
2.15	7.2686(-7)	6.1327(-5)	5.4377(-8)	1.6849(-8)	2.0047(-8)	4.0820(-8)	2.4232(-7)
2.175	5.3479(-7)	3.2712(-5)	5.5646(-8)	1.6768(-9)	2.2357(-8)	1.8169(-8)	1.8767(-7)
2.225	3.9756(-7)	1.5820(-5)	6.2101(-8)	4.9622(-8)	2.6241(-8)	5.1736(-8)	2.3388(-7)
2.25	5.3324(-7)	1.6316(-5)	6.2415(-8)	6.2850(-8)	2.7597(-8)	1.0359(-7)	3.1442(-7)
2.275	3.9919(-7)	8.7050(-6)	6.3723(-8)	5.5364(-8)	2.8395(-8)	4.8101(-8)	2.3897(-7)
2.325	3.9132(-7)	4.2315(-6)	7.4815(-8)	1.7353(-9)	2.9365(-8)	1.5202(-8)	2.5325(-7)
2.35	5.1966(-7)	4.3910(-6)	7.5439(-8)	7.2126(-9)	2.9770(-8)	5.8552(-8)	3.3690(-7)
2.375	3.8970(-7)	2.3092(-6)	7.3194(-8)	4.9865(-9)	2.9964(-8)	2.2496(-8)	2.5392(-7)
2.425	2.8016(-7)	1.0211(-6)	6.9873(-8)	5.4482(-8)	2.9921(-8)	1.3587(-8)	2.4611(-7)
2.45	3.6227(-7)	9.3054(-7)	6.7131(-8)	6.5596(-8)	2.9684(-8)	5.0931(-8)	3.2499(-7)
2.475	2.7616(-7)	4.1449(-7)	6.5873(-8)	5.5881(-8)	2.9201(-8)	1.8270(-8)	2.4364(-7)
2.525	2.8887(-7)	1.1361(-7)	6.9766(-8)	8.7363(-10)	2.7899(-8)	7.4923(-9)	2.2067(-7)
2.55	3.7565(-7)	4.4319(-7)	6.8140(-8)	9.6619(-9)	2.7150(-8)	3.9702(-8)	2.8945(-7)
2.575	2.8309(-7)	5.8523(-7)	6.3994(-8)	8.6904(-10)	2.6249(-8)	1.3333(-8)	2.1598(-7)
2.625	1.8901(-7)	1.4826(-6)	5.5297(-8)	4.9323(-8)	2.4187(-8)	5.4891(-9)	1.8281(-7)
2.65	2.3855(-7)	2.8705(-6)	5.0874(-8)	5.9234(-8)	2.3051(-8)	3.2801(-8)	2.3796(-7)
2.675	1.8190(-7)	2.7947(-6)	4.8185(-8)	4.8252(-8)	2.1786(-8)	1.1051(-8)	1.7656(-7)
2.725	2.0635(-7)	5.8176(-6)	4.8006(-8)	7.6271(-9)	1.9071(-8)	6.9355(-9)	1.3678(-7)
2.75	2.6735(-7)	1.0904(-5)	4.5103(-8)	1.9163(-8)	1.7640(-8)	3.1679(-8)	1.7607(-7)
2.775	1.9821(-7)	1.0562(-5)	3.9863(-8)	9.6370(-9)	1.6104(-8)	1.4271(-8)	1.2944(-7)
2.825	1.1313(-7)	2.1793(-5)	2.8037(-8)	3.8747(-8)	1.2838(-8)	1.6642(-9)	8.5678(-8)
2.85	1.4207(-7)	4.0869(-5)	2.2628(-8)	4.8575(-8)	1.1144(-8)	1.6918(-8)	1.0786(-7)
2.875	1.0419(-7)	3.9492(-5)	1.9087(-8)	3.7067(-8)	9.3928(-9)	4.0501(-9)	7.7611(-8)
2.925	1.3559(-7)	8.1334(-5)	1.6468(-8)	2.5042(-8)	5.8367(-9)	2.7937(-8)	3.1766(-8)
2.95	1.7999(-7)	1.5245(-4)	1.2791(-8)	4.3607(-8)	3.9985(-9)	6.0594(-8)	3.6289(-8)
2.975	1.2600(-7)	1.4735(-4)	6.8758(-9)	3.7077(-8)	2.0460(-9)	5.0253(-8)	2.3219(-8)

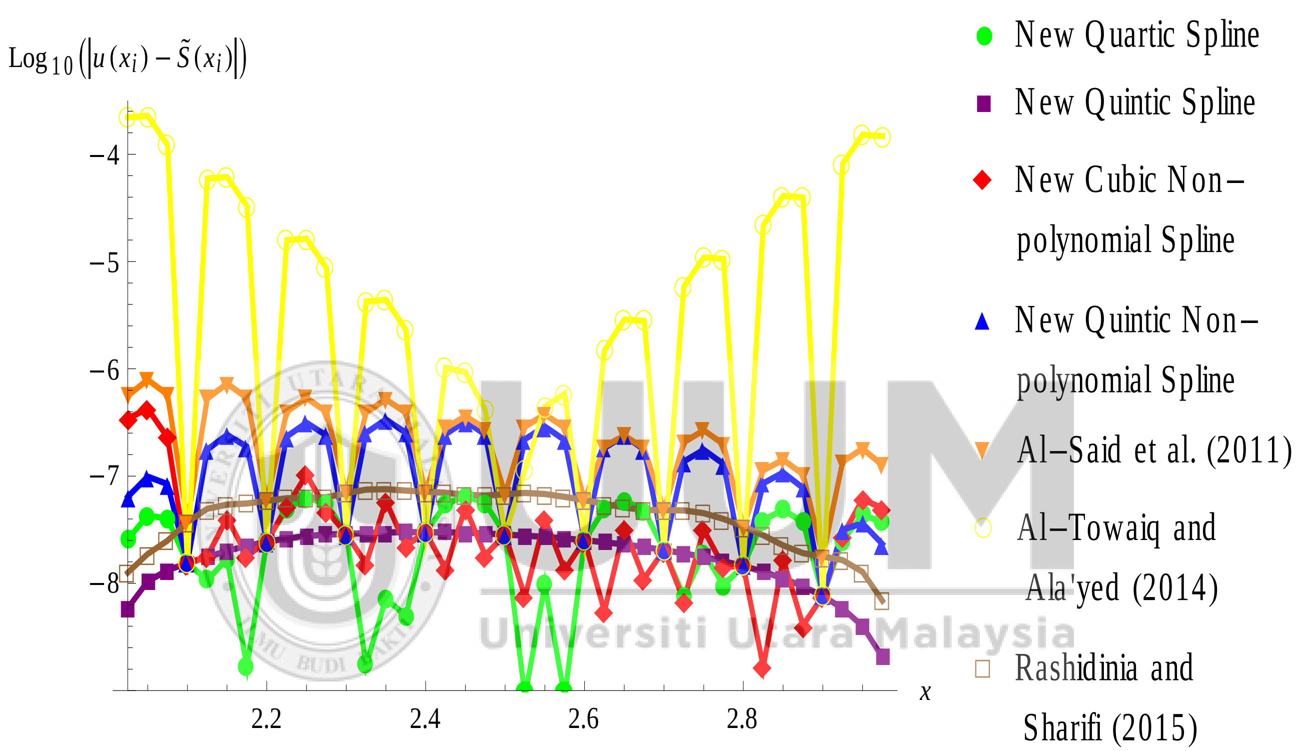

Figure 5.9. Graph of $\log_{10}[|u(x_i) - \tilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, and the spline methods from Al-Said et al. (2011), Al-Towaiq and Ala'yed (2014) and Rashidinia and Sharifi (2015), correspond to ***Problem* 9**

Table 5.10

Errors Obtained by Different Spline Methods in ***Problem*** **10**

x	**Al-Said et al. (2011)**	**Al-Towaiq and Ala'yed (2014)**	**Rashidinia and Sharifi (2015)**	**New Quartic Spline**	**New Quintic Spline**	**New Cubic Non-polynomial Spline**	**New Quintic Non-polynomial Spline**
$0.0\overline{3}$	-	4.8176(-4)	2.8083(-7)	4.5360(-7)	1.5169(-7)	3.2149(-6)	1.7013(-6)
$0.0\overline{6}$	-	3.2598(-4)	4.5543(-7)	6.2362(-7)	2.4676(-7)	2.4583(-6)	2.0142(-6)
$0.1\overline{3}$	-	1.2957(-4)	9.6810(-7)	7.9295(-8)	3.4605(-7)	3.4017(-7)	4.4641(-6)
$0.1\overline{6}$	-	8.7871(-5)	1.0452(-6)	8.8610(-9)	3.9997(-7)	1.0440(-7)	4.6346(-6)
$0.2\overline{3}$	-	3.4662(-5)	1.1039(-6)	8.6302(-7)	4.8508(-7)	2.0604(-7)	5.9797(-6)
$0.2\overline{6}$	-	2.3448(-5)	1.1360(-6)	9.1223(-7)	5.0691(-7)	1.7557(-7)	6.0676(-6)
$0.3\overline{3}$	-	9.0962(-6)	1.3653(-6)	1.0767(-7)	5.2682(-7)	1.6886(-7)	6.7530(-6)
$0.3\overline{6}$	-	6.1291(-6)	1.3634(-6)	1.3306(-7)	5.3526(-7)	1.2963(-7)	6.7914(-6)
$0.4\overline{3}$	-	2.7695(-6)	1.2459(-6)	9.3846(-7)	5.4285(-7)	1.9056(-7)	7.0685(-6)
$0.4\overline{6}$	-	1.9394(-6)	1.2325(-6)	9.4950(-7)	5.4119(-7)	1.7208(-7)	7.0761(-6)
$0.5\overline{3}$	-	3.3565(-7)	1.3684(-6)	1.1672(-7)	5.3039(-7)	2.4156(-7)	7.0923(-6)
$0.5\overline{6}$	-	1.4580(-7)	1.3407(-6)	1.1889(-7)	5.2375(-7)	2.1963(-7)	7.0805(-6)
$0.6\overline{3}$	-	5.2860(-7)	1.1565(-6)	9.0854(-7)	5.0688(-7)	2.4379(-7)	6.9254(-6)
$0.6\overline{6}$	-	4.4749(-7)	1.1283(-6)	9.0516(-7)	4.9690(-7)	2.2477(-7)	6.9011(-6)
$0.7\overline{3}$	-	3.0232(-7)	1.2341(-6)	6.3771(-8)	4.7445(-7)	2.3311(-7)	6.6312(-6)
$0.7\overline{6}$	-	2.9593(-7)	1.1986(-6)	5.6914(-8)	4.6272(-7)	2.1512(-7)	6.5986(-6)
$0.8\overline{3}$	-	3.5078(-7)	9.8683(-7)	8.4218(-7)	4.3801(-7)	2.0931(-7)	6.2503(-6)
$0.8\overline{6}$	-	3.2625(-7)	9.5440(-7)	8.3315(-7)	4.2526(-7)	1.9266(-7)	6.2123(-6)
$0.9\overline{3}$	-	3.1438(-7)	1.0536(-6)	1.0094(-8)	3.9903(-7)	1.7844(-7)	5.8095(-6)
$0.9\overline{6}$	-	2.9644(-7)	1.0167(-6)	2.0452(-8)	3.8581(-7)	1.6287(-7)	5.7679(-6)
$1.0\overline{3}$	-	2.9298(-7)	7.9269(-7)	7.6447(-7)	3.5909(-7)	1.4280(-7)	5.3269(-6)
$1.0\overline{6}$	-	2.7462(-7)	7.6003(-7)	7.5332(-7)	3.4571(-7)	1.2821(-7)	5.2828(-6)
$1.1\overline{3}$	-	2.6429(-7)	8.6164(-7)	8.9375(-8)	3.1890(-7)	1.0434(-7)	4.8148(-6)
$1.1\overline{6}$	-	2.4791(-7)	8.2562(-7)	1.0094(-7)	3.0558(-7)	9.0602(-8)	4.7692(-6)
$1.2\overline{3}$	-	2.2923(-7)	5.9576(-7)	6.8504(-7)	2.7903(-7)	6.4314(-8)	4.2819(-6)
$1.2\overline{6}$	-	2.1146(-7)	5.6428(-7)	6.7329(-7)	2.6587(-7)	5.1354(-8)	4.2353(-6)
$1.3\overline{3}$	-	2.1428(-7)	6.7174(-7)	1.6806(-7)	2.3975(-7)	2.3309(-8)	3.7342(-6)
$1.3\overline{6}$	-	2.055(-7)	6.3741(-7)	1.7984(-7)	2.2683(-7)	1.0950(-8)	3.6869(-6)
$1.4\overline{3}$	-	1.1784(-7)	4.0434(-7)	6.0762(-7)	2.0125(-7)	1.7221(-8)	3.1760(-6)
$1.4\overline{6}$	-	7.9324(-8)	3.7447(-7)	5.9591(-7)	1.8861(-7)	2.8716(-8)	3.1284(-6)
$1.5\overline{3}$	-	3.2342(-7)	4.8893(-7)	2.4392(-7)	1.6362(-7)	6.0281(-8)	2.6105(-6)
$1.5\overline{6}$	-	3.9916(-7)	4.5647(-7)	2.5551(-7)	1.5129(-7)	7.2245(-8)	2.5628(-6)
$1.6\overline{3}$	-	5.9895(-7)	2.2127(-7)	5.336(-7)	1.2691(-7)	9.3039(-8)	2.0400(-6)
$1.6\overline{6}$	-	9.4666(-7)	1.9307(-7)	5.2229(-7)	1.1488(-7)	1.0080(-7)	1.9922(-6)
$1.7\overline{3}$	-	2.6887(-6)	3.1471(-7)	3.1719(-7)	9.1123(-8)	1.6231(-7)	1.4660(-6)
$1.7\overline{6}$	-	3.9240(-6)	2.8403(-7)	3.2941(-7)	7.9400(-8)	1.8358(-7)	1.4183(-6)
$1.8\overline{3}$	-	9.7345(-6)	4.7116(-8)	4.7227(-7)	5.6212(-8)	9.2699(-8)	8.8992(-7)
$1.8\overline{6}$	-	1.4405(-5)	2.0461(-8)	4.7131(-7)	4.4776(-8)	6.1601(-8)	8.4232(-7)
$1.9\overline{3}$	-	3.6478(-5)	1.4907(-7)	4.7731(-7)	2.2249(-8)	5.3840(-7)	3.1252(-7)
$1.9\overline{6}$	-	5.3849(-5)	1.2001(-7)	5.8743(-7)	1.1107(-8)	7.0094(-7)	2.6509(-7)

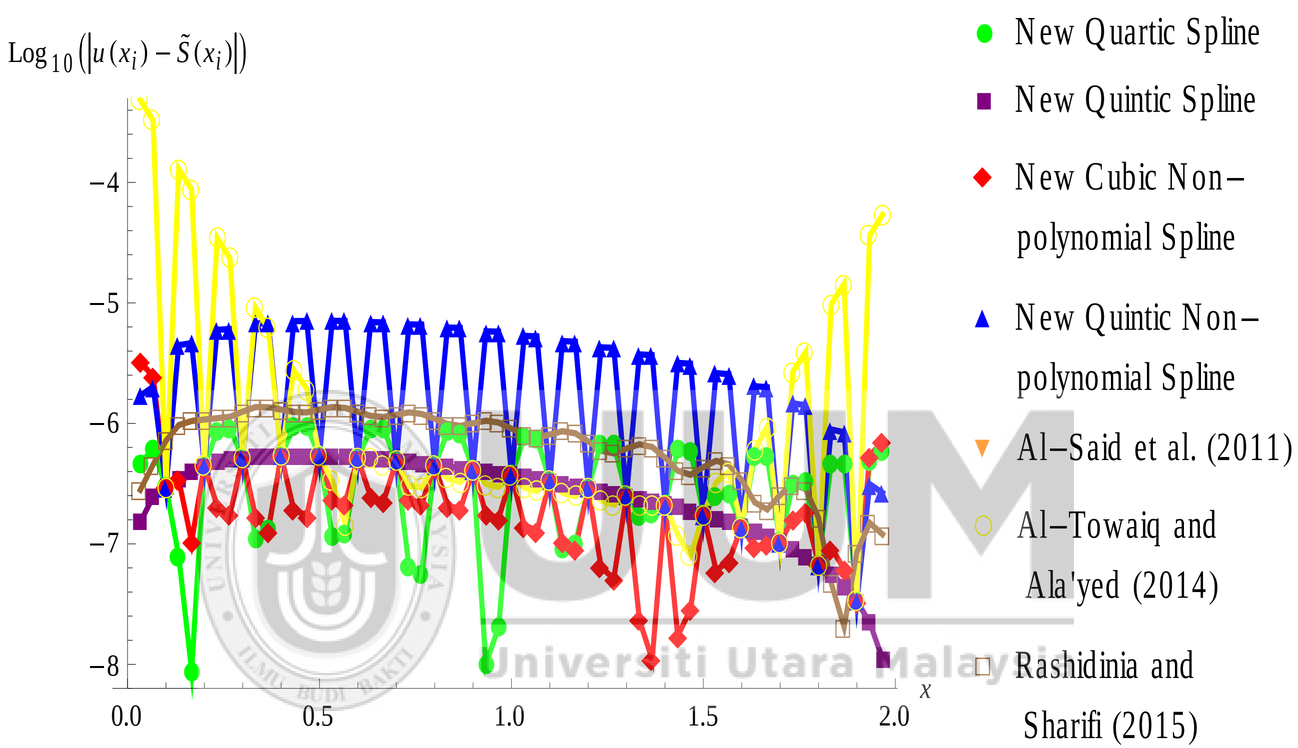

Figure 5.10. Graph of $\log_{10}[|u(x_i) - \tilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, and the spline methods from Al-Said et al. (2011), Al-Towaiq and Ala'yed (2014) and Rashidinia and Sharifi (2015), correspond to ***Problem* 10**

Table 5.11

Errors Obtained by Different Spline Methods in ***Problem*** **11**

X	**Liu et al. (2011)**	**New Quartic Spline**	**New Quintic Spline**	**New Cubic Non-polynomial Spline**	**New Quintic Non-polynomial Spline**
0.025	4.2929(-5)	6.4722(-6)	6.7537(-6)	6.5174(-6)	6.3281(-6)
0.05	4.3512(-5)	6.2194(-6)	6.6747(-6)	6.4134(-6)	6.0261(-6)
0.075	4.4879(-5)	6.2284(-6)	6.5956(-6)	6.5377(-6)	6.0756(-6)
0.125	4.8937(-5)	6.8053(-6)	6.4367(-6)	5.9101(-6)	5.0930(-6)
0.15	5.0294(-5)	6.8159(-6)	6.3573(-6)	5.5780(-6)	4.4538(-6)
0.175	5.0857(-5)	6.5628(-6)	6.2782(-6)	5.9267(-6)	4.8462(-6)
0.225	5.0966(-5)	5.8371(-6)	6.1192(-6)	5.3123(-6)	3.9460(-6)
0.25	5.1507(-5)	5.5878(-6)	6.0398(-6)	4.7575(-6)	3.0069(-6)
0.275	5.2820(-5)	5.5988(-6)	5.9613(-6)	5.3234(-6)	3.7120(-6)
0.325	5.6769(-5)	6.1704(-6)	5.8041(-6)	4.7298(-6)	2.9496(-6)
0.35	5.8047(-5)	6.1879(-6)	5.7260(-6)	3.9605(-6)	1.7712(-6)
0.375	5.8520(-5)	5.9406(-6)	5.6496(-6)	4.7337(-6)	2.7370(-6)
0.425	5.8489(-5)	5.2118(-6)	5.4979(-6)	4.1819(-6)	2.1677(-6)
0.45	5.8912(-5)	4.9742(-6)	5.4232(-6)	3.2184(-6)	8.3215(-7)
0.475	6.0095(-5)	4.9963(-6)	5.3510(-6)	4.1853(-6)	1.9864(-6)
0.525	6.3848(-5)	5.5702(-6)	5.2094(-6)	3.6519(-6)	1.6641(-6)
0.55	6.4965(-5)	5.6057(-6)	5.1409(-6)	2.4920(-6)	2.7455(-7)
0.575	6.5265(-5)	5.3764(-6)	5.0760(-6)	3.6381(-6)	1.5252(-6)
0.625	6.4975(-5)	4.6584(-6)	4.9521(-6)	3.2712(-6)	1.5019(-6)
0.65	6.5193(-5)	4.4465(-6)	4.8936(-6)	2.0174(-6)	1.8074(-7)
0.675	6.6157(-5)	4.4946(-6)	4.8394(-6)	3.3184(-6)	1.4174(-6)
0.725	6.9582(-5)	5.0917(-6)	4.7384(-6)	2.6296(-6)	1.7423(-6)
0.75	7.0448(-5)	5.1624(-6)	4.6929(-6)	1.0128(-6)	6.2916(-7)
0.775	7.0482(-5)	4.9685(-6)	4.6540(-6)	2.4690(-6)	1.7246(-6)
0.825	6.9790(-5)	4.2797(-6)	4.5901(-6)	3.3480(-6)	2.4428(-6)
0.85	6.9709(-5)	4.1034(-6)	4.5644(-6)	2.5026(-6)	1.6925(-6)
0.875	7.0362(-5)	4.1933(-6)	4.5442(-6)	4.0111(-6)	2.5043(-6)
0.925	7.3315(-5)	4.9116(-6)	4.5131(-6)	5.2884(-7)	3.6587(-6)
0.95	7.3835(-5)	5.1140(-6)	4.5068(-6)	4.6913(-6)	3.4405(-6)
0.975	7.3511(-5)	5.0035(-6)	4.5123(-6)	2.8602(-6)	3.8122(-6)

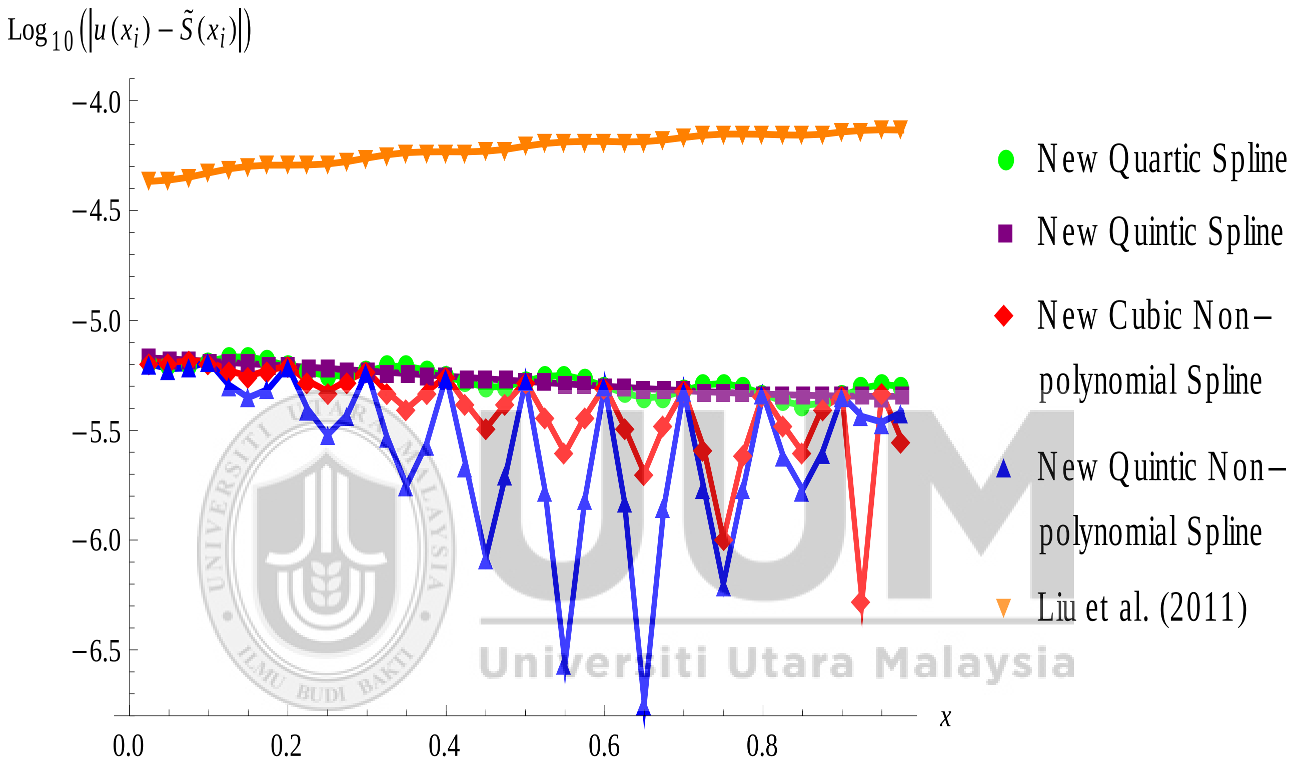

Figure 5.11. Graph of $\log_{10}[|u(x_i)-\tilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, and the spline method from Liu et al. (2011), correspond to ***Problem* 11**

Table 5.12

Errors Obtained by Different Spline Methods in ***Problem*** **12**

x	**Liu et al. (2011)**	**New Quartic Spline**	**New Quintic Spline**	**New Cubic Non-polynomial Spline**	**New Quintic Non-polynomial Spline**
0.025	9.6982(-5)	6.4105(-6)	7.4973(-6)	1.8073(-5)	2.1094(-6)
0.05	9.4611(-5)	4.7190(-6)	6.5411(-6)	1.9825(-5)	1.6576(-6)
0.075	8.8736(-5)	4.3109(-6)	5.8349(-6)	1.3402(-5)	7.2357(-7)
0.125	7.1452(-5)	6.4590(-6)	4.7674(-6)	3.6920(-6)	9.2247(-6)
0.15	6.6460(-5)	6.5306(-6)	4.2580(-6)	4.5511(-6)	1.5264(-5)
0.175	6.5840(-5)	5.3102(-6)	3.7809(-6)	4.2105(-6)	1.0682(-5)
0.225	7.0906(-5)	1.4828(-6)	2.9730(-6)	4.4395(-6)	1.4215(-5)
0.25	7.0895(-5)	4.7905(-7)	2.6575(-6)	5.0754(-6)	2.1024(-5)
0.275	6.7069(-5)	7.8425(-7)	2.4052(-6)	3.8681(-6)	1.4936(-5)
0.325	5.2751(-5)	3.6534(-6)	1.9914(-6)	2.3634(-6)	1.5737(-5)
0.35	4.9364(-5)	4.0862(-6)	1.8001(-6)	2.7662(-6)	2.2475(-5)
0.375	5.0169(-5)	3.2163(-6)	1.6256(-6)	2.2498(-6)	1.6042(-5)
0.425	5.6507(-5)	2.4815(-7)	1.3256(-6)	1.7421(-6)	1.5267(-5)
0.45	5.7640(-5)	1.0429(-6)	1.2003(-6)	2.0168(-6)	2.1439(-5)
0.475	5.4849(-5)	5.3165(-7)	1.0940(-6)	1.6014(-6)	1.5326(-5)
0.525	4.1727(-5)	2.5518(-6)	9.1244(-7)	1.1447(-6)	1.3605(-5)
0.55	3.9189(-5)	3.1093(-6)	8.3009(-7)	1.3270(-6)	1.8910(-5)
0.575	4.0774(-5)	2.3623(-6)	7.5569(-7)	1.0687(-6)	1.3513(-5)
0.625	4.8131(-5)	9.7389(-7)	6.2570(-7)	7.9269(-7)	1.1204(-5)
0.65	4.9918(-5)	1.692(-6)	5.6899(-7)	9.1240(-7)	1.5455(-5)
0.675	4.7735(-5)	1.1048(-6)	5.1898(-7)	7.3374(-7)	1.1019(-5)
0.725	3.5483(-5)	2.0624(-6)	4.3098(-7)	5.4024(-7)	8.3343(-6)
0.75	3.3465(-5)	2.6733(-6)	3.9108(-7)	6.2067(-7)	1.1408(-5)
0.775	3.5539(-5)	1.9764(-6)	3.5501(-7)	5.0260(-7)	8.0874(-6)
0.825	4.3646(-5)	1.3477(-6)	2.9131(-7)	3.4432(-7)	5.1561(-6)
0.85	4.5858(-5)	2.0780(-6)	2.6268(-7)	3.7913(-7)	6.9737(-6)
0.875	4.4079(-5)	1.4753(-6)	2.3647(-7)	3.0242(-7)	4.8687(-6)
0.925	3.2478(-5)	2.1095(-6)	1.8818(-7)	2.7767(-7)	1.7722(-6)
0.95	3.0818(-5)	3.1757(-6)	1.6616(-7)	3.4700(-7)	2.2783(-6)
0.975	3.3235(-5)	2.6607(-6)	1.4669(-7)	2.9500(-7)	1.4576(-6)

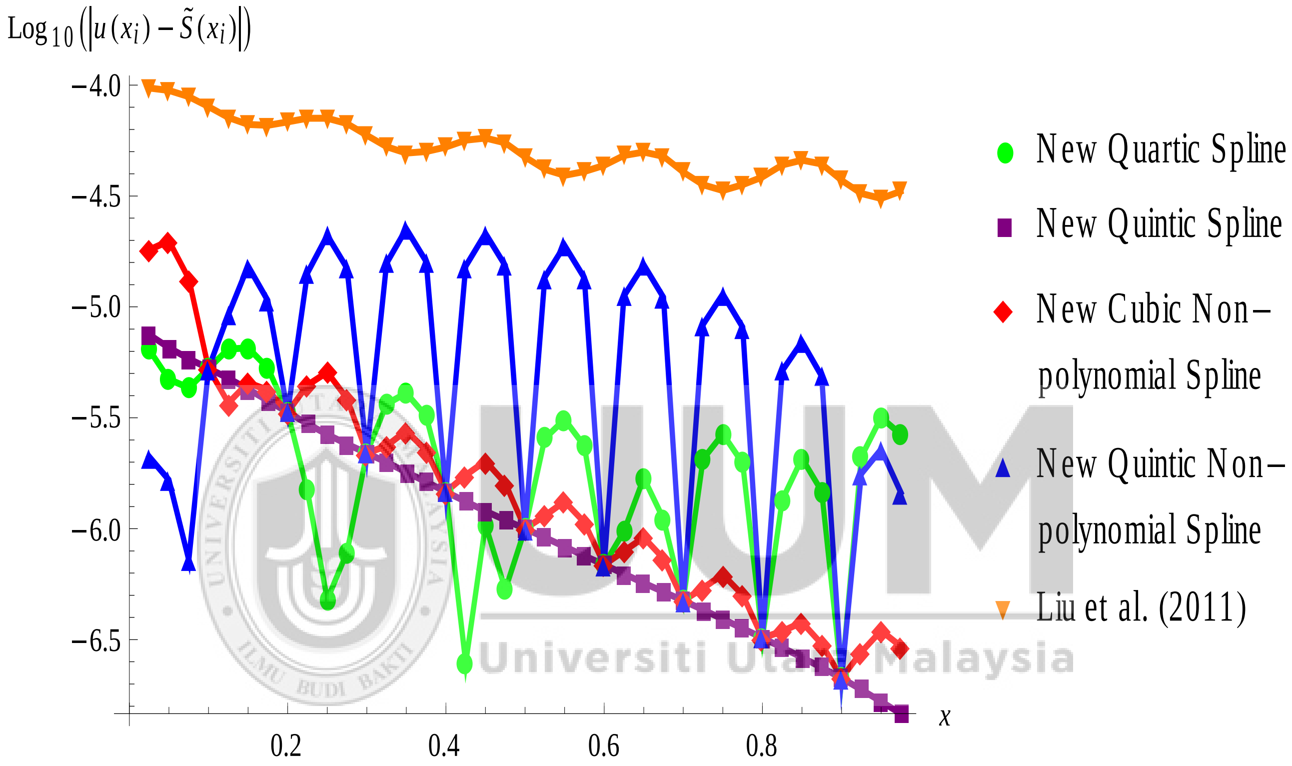

Figure 5.12. Graph of $\log_{10}[|u(x_i)-\tilde{S}(x_i)|]$ vs. x_i for the new spline methods proposed in Chapter Three and Chapter Four, and the spline method from Liu et al. (2011), correspond to ***Problem* 12**

In this section, we examine 6 BVPs from the literatures using the new proposed spline methods and the existing spline methods of Al-Said et al. (2011), Al-Towaiq and Ala'yed (2014), Rashidinia and Sharifi (2015) and Liu et al. (2011). These 6 test problems can be sorted out into three groups:

i. ***Problem*** **7** and ***Problem*** **8** are first order BVPs.
ii. ***Problem*** **9** and ***Problem*** **10** are second order BVPs subject to Dirichlet boundary conditions.
iii. ***Problem*** **11** and ***Problem*** **12** are second order BVPs subject to Neumann boundary conditions.

The numerical results in Table 5.9 and Table 5.10 confirm that the new quartic spline, quintic spline and cubic non-polynomial spline methods are more accurate than the quartic B-spline method in Rashidinia and Sharifi (2015), the quartic spline method in Al-Said et al. (2011), the cubic spline method in Al-Towaiq and Ala'yed (2014), the new quintic non-polynomial spline methods in solving ***Problem*** **9** and ***Problem*** **10.** We note that the quartic spline method in Al-Said et al. (2011) is derived for solving specific linear second order BVPs where the term $u'(x)$ is absent. Therefore, the quartic spline method in Al-Said et al. (2011) is not applicable to ***Problem*** **10**, and hence there are no numerical results presented in Table 5.10. Moreover, the results from Table 5.11 and Table 5.12 showed that the proposed methods are generally more accurate than the quartic spline method in Liu et al. (2011) in solving ***Problem*** **11** and ***Problem*** **12**, respectively.

Furthermore, the numerical results in Table 5.9 and Table 5.12 for the new quintic spline method stated favorably over the new quartic, cubic non-polynomial, and quintic non-polynomial spline methods for solving ***Problem* 9** and ***Problem* 12**. On the other hand, the cubic non-polynomial spline method is more accurate than other new proposed methods when solving ***Problem* 10**. In addition, the numerical results of the new quintic non-polynomial method considered preferable over other new developed methods in solving ***Problem* 11**. This can be noticed in Table 5.11. Finally, the results in Table 5.7 and Table 5.8 seem to indicate that the new proposed spline methods are found to have comparable accuracy in solving ***Problem* 7** and ***Problem* 8**, respectively.

CHAPTER SIX
CONCLUSION AND FUTURE RESEARCH

6.1 Conclusion

In this research work, after a short introduction in Chapter One and some literature review to support the rationale of our studies, the main contributions of this thesis begin with the developments of four new spline methods for solving first and second order IVPs and BVPs proposed in Chapter Three and Chapter Four. In Chapter Three, two new methods based on polynomial spline functions, namely: quartic and quintic spline methods are developed. On the other hand, two new methods based on non-polynomial spline functions, namely: cubic non-polynomial and quintic non-polynomial spline methods are developed in Chapter Four.

Every spline function of order *m* should have a continuous (*m*-1)-th derivatives on the interval of integration $[a,b]$. In order to develop a polynomial spline method, we defined the (*m*-1)-th derivative at the subinterval $[x_i, x_{i+1}]$ as a linear polynomial by using Lagrange polynomial and then integrated it (*m*-1) times to obtain a polynomial spline function of order *m* with unknown coefficients. While for a non-polynomial spline method, we directly defined the spline function as a linear combination of non-polynomial functions with unknown coefficients over the subinterval $[x_i, x_{i+1}]$. For both polynomial and non-polynomial splines, the continuity conditions are then imposed at the interior points to gain sufficient conditions, so that the unknown coefficients for all spline functions are uniquely determined. On applying Gaussian elimination method to the equations formed by the continuity conditions, the values of the unknown coefficients are determined. These coefficients are substituted back into the original spline functions to form continuous schemes.

The basic idea used in the convergence analysis for each developed spline method, is to investigate whether a spline method approximates the solutions of first and second order IVPs and BVPs. A spline method is said to be convergent if the generated approximate solution approaches the exact solution as the step-size goes to zero. This can be visualized through the error sustained by a spline method in all steps, and the error is being bounded as the step-size goes to zero. Each newly developed spline method in Chapter Three and Chapter Four is proven to be convergent, as the error is being bounded. Hence, the convergence properties for all proposed spline methods are stated in four new theorems of convergence. These four theorems showed that the order of convergence for these proposed spline methods are four.

For the implementations, we first divide the interval of integration $[a,b]$ into small subintervals, then the 4-stage fourth order explicit Runge-Kutta method is applied to obtain the approximate solutions at the grid points, while the new derived spline methods are used to obtain the approximate solutions between the grid points. The proposed new spline methods are validated through 12 test problems chosen from the literature. These test problems can be classified into three categories:

i. First and second order IVPs.
ii. First and second order BVPs subject to Dirichlet boundary conditions.
iii. Second order BVPs subject to Neumann boundary conditions.

Moreover, to check the accuracy of the new developed spline methods, the numerical results of the new spline methods are compared with those numerical results generated by:

i. The cubic spline method of order $O(h^4)$ in Tung (2013) for the test problems involving first, second and third order IVPs.

ii. The quartic spline method of order $O(h^4)$ in Al-Said et al. (2011), the cubic spline method of order $O(h^4)$ in Al-Towaiq and Ala'yed (2014) and the quartic B-spline method of order $O(h^5)$ in Rashidinia and Sharifi (2015) for the test problems involving second order BVPs with Dirichlet boundary conditions.

iii. The quartic spline method of order $O(h^4)$ in Liu et al. (2011) for the test problems involving second order BVPs with Neumann boundary conditions.

Generally, the new proposed spline methods are more accurate in terms of error and these new methods can be implemented on a computer without any difficulties.

Last but not least, all objectives stated in Chapter One have been achieved.

6.2 Future Research

This research considered the derivations of new spline methods to obtain numerical solutions of first and second order IVPs and BVPs. This study has spawned some open research problems as shown below:

i. New spline methods which solve IVPs and BVPs directly without the assistance of other methods such as the 4-stage fourth order explicit Runge-Kutta method or reduction approach.
ii. New spline methods to solve problems involving partial differential equations, fuzzy differential equations, fractional differential equations, differential-algebric equations, integro-differential equations and delay differential equations, to name a few.
iii. New spline methods which implement in variable step-size approach in order to control the error introduced at each individual step.

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