NEW SPLINE METHODS FOR SOLVING FIRST AND SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstrak

Banyak permasalahan yang timbul daripada pelbagai aplikasi kehidupan nyata boleh menjurus kepada model matematik yang dapat diungkapkan sebagai masalah nilai awal (MNA) dan masalah nilai sempadan (MNS) untuk persamaan pembeza biasa (PPB) peringkat pertama dan kedua. Masalah ini mungkin tidak mempunyai penyelesaian analitik, dengan itu kaedah berangka diperlukan bagi menganggarkan penyelesaian. Apabila sesuatu persamaan pembeza diselesaikan secara berangka, selang pengamiran dibahagikan kepada subselang. Akibatnya, penyelesaian berangka pada titik grid dapat ditentukan melalui pengiraan berangka, manakala penyelesaian antara titik grid masih tidak diketahui. Bagi mencari penyelesaian hampir antara dua titik grid, kaedah splin diperkenalkan. Walau bagaimanapun, kebanyakan keadah splin yang sedia ada digunakan untuk menganggar penyelesaian bagi MNA dan MNS yang tertentu sahaja. Oleh itu, kajian ini membangunkan beberapa kaedah splin baharu yang berasaskan fungsi splin polynomial dan bukan polynomial bagi menyelesaikan MNA dan MNS umum yang berperingkat pertama dan kedua. Analisis penumpuan bagi setiap kaedah splin baharu turut dibincangkan. Dari segi pelaksanaan, kaedah Runge-Kutta tersurat bertahap empat dan berperingkat keempat digunakan bagi mendapat penyelesaian pada titik grid, manakala kaedah splin baharu digunakan untuk memperoleh penyelesaian antara titik grid. Prestasi kaedah splin yang baharu kemudiannya dibandingkan dengan beberapa kaedah splin yang sedia ada dalam menyelesaikan 12 masalah ujian. Secara umumnya, keputusan berangka menunjukkan bahawa kaedah splin baharu memberikan kejituan yang lebih baik daripada kaedah splin yang sedia ada. Oleh itu, kaedah splin baharu adalah alternatif yang berdaya saing dalam menyelesaikan MNA dan MNS berperingkat pertama dan kedua.

Kata kunci: Interpolasi, Keadah splin, Masalah nilai awal, Masalah nilai sempadan, Persamaan pembeza biasa.

Abstract

Many problems arise from various real life applications may lead to mathematical models which can be expressed as initial value problems (IVPs) and boundary value problems (BVPs) of first and second ordinary differential equations (ODEs).These problems might not have analytical solutions, thus numerical methods are needed inapproximating the solutions. When a differential equation is solved numerically, the interval of integration is divided into subintervals.Consequently, numerical solutions at the grid pointscan be determined through numerical computations, whereas the solutions between the grid points still remain unknown. In order to find the approximate solutions between any two grid points, spline methods are introduced. However, most of the existing spline methods are used to approximate the solutions of specific cases of IVPs and BVPs. Therefore, this study develops new spline methods based on polynomial and non-polynomial spline functions for solving general cases of first and second order IVPs and BVPs. The convergence analysis for each new spline method is also discussed. In terms of implementation, the 4-stage fourth order explicit Runge-Kutta method is employed to obtain the solutions at the grid points, while the new spline methods are used to obtain the solutions between the grid points. The performance of the new spline methods are then compared with the existing spline methods in solving12 test problems. Generally, the numerical results indicate that the new spline methods provide better accuracy than their counterparts. Hence, the new spline methods are viable alternatives for solving first and second order IVPs and BVPs.

Keywords: Interpolation, Spline method, Initial value problem, Boundary value problem, Ordinary differential equation.

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CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Spline functions have been rapidly developed as a result of their applications usefulness. Spline functions with their various categories have many high quality approximation powers as well as structural properties such as zero properties, sign change properties and determinental properties (Dold & Eckmann, 1976). There are many applications of spline functions in applied mathematics and engineering. Some of these applications are data fitting, approximating functions, optimal control problems, integro-differential equation and Computer-Aided Geometric Design (CAGD). It is important to note that programmes based on spline functions have been embedded in various computer applications.

A common consensus is that, Schoenberg (1946) made the first mathematical reference to spline in his interesting article, and this probably was the first time that 'spline' was used in connection with smooth piecewise polynomial approximation. However, it is important to note that the ideas of developing splines were originated from shipbuilding and aircraft industries earlier than computer modeling was available (Dermoune & Preda, 2014). Then, naval architects faced the necessity to draw a smooth curve through a set of points. The answer to this challenge was to put metal weights (called ***knots***) at the points of control so that a thin metal or wooden beam (called a ***spline***) would be bent through the weights (see Figure 1.1). Bending splines from physicist's point of view was important as the weight has some greatest

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