ONE STEP HYBRID BLOCK METHODS WITH GENERALISED OFF-STEP POINTS FOR SOLVING DIRECTLY HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS.

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Abstrak

Permasalahan kehidupan nyata terutamanya dalam sains dan kejuruteraan boleh diungkapkan dalam persamaan pembeza untuk tujuan menganalisis dan memahami fenomena fizikal. Persamaan pembeza ini melibatkan kadar perubahan satu atau lebih pembolehubah tak bersandar. Masalah nilai awal persamaan pembeza biasa peringkat tinggi diselesaikan secara konvensional dengan menukarkan persamaan tersebut ke sistem persamaan pembeza biasa peringkat pertama yang setara terlebih dahulu. Kaedah berangka bersesuaian yang sedia ada kemudiannya digunakan untuk menyelesai persamaan yang terhasil. Walau bagaimanapun, pendekatan ini akan menambah bilangan persamaan. Akibatnya, kekompleksan pengiraan akan bertambah dan ianya boleh menjejaskan kejituan penyelesaian. Bagi mengatasi kelemahan ini, kaedah langsung digunakan. Namun, kebanyakan kaedah ini menganggar penyelesaian berangka pada satu titik pada satu masa. Oleh itu, beberapa kaedah blok diperkenalkan bertujuan untuk menganggar penyelesaian berangka pada beberapa titik serentak. Seterusnya, kaedah blok hibrid diperkenalkan bagi mengatasi sawar kestabilan-sifar yang berlaku dalam kaedah blok. Walau bagaimanapun, kaedah blok hibrid satu langkah sedia ada hanya tertumpu kepada titik pinggir-langkah yang spesifik. Oleh yang demikian, kajian ini mencadangkan beberapa kaedah blok hibrid satu langkah dengan titik pinggir-langkah teritlak bagi menyelesaikan persamaan pembeza biasa peringkat tinggi secara langsung. Dalam pembangunan kaedah ini, siri kuasa telah digunakan sebagai penyelesaian hampir kepada permasalahan persamaan pembeza biasa peringkat γ. Siri kuasa diinterpolasi pada γ titik sementara terbitannya yang tertinggi dikolokasi pada semua titik dalam selang terpilih. Sifat bagi kaedah baharu seperti peringkat, pemalar ralat, kestabilan-sifar, ketekalan, penumpuan dan rantau kestabilan mutlak juga turut dikaji. Beberapa masalah nilai awal persamaan pembeza biasa peringkat tinggi kemudiannya diselesaikan dengan menggunakan kaedah baharu yang telah dibangunkan. Keputusan berangka mendedahkan kaedah baharu menghasilkan penyelesaian yang lebih jitu berbanding dengan kaedah yang sedia ada apabila menyelesaikan masalah yang sama. Oleh itu, kaedah baharu adalah alternatif berdaya saing dalam menyelesaikan masalah nilai awal persamaan pembeza biasa peringkat tinggi secara langsung.

Kata kunci: Interpolasi, kolokasi, kaedah blok hibrid satu langkah, penyelesaian langsung masalah nilai awal peringkat tinggi, titik pinggir-langkah teritlak.

Abstract

Real life problems particularly in sciences and engineering can be expressed in differential equations in order to analyse and understand the physical phenomena. These differential equations involve rates of change of one or more independent variables. Initial value problems of higher order ordinary differential equations are conventionally solved by first converting them into their equivalent systems of first order ordinary differential equations. Appropriate existing numerical methods will then be employed to solve the resulting equations. However, this approach will enlarge the number of equations. Consequently, the computational complexity will increase and thus may jeopardise the accuracy of the solution. In order to overcome these setbacks, direct methods were employed. Nevertheless, most of these methods approximate numerical solutions at one point at a time. Therefore, block methods were then introduced with the aim of approximating numerical solutions at many points simultaneously. Subsequently, hybrid block methods were introduced to overcome the zero-stability barrier occurred in the block methods. However, the existing one step hybrid block methods only focus on the specific off-step point(s). Hence, this study proposed new one step hybrid block methods with generalised off-step point(s) for solving higher order ordinary differential equations. In developing these methods, a power series was used as an approximate solution to the problems of ordinary differential equations of order γ. The power series was interpolated at γ points while its highest derivative was collocated at all points in the selected interval. The properties of the new methods such as order, error constant, zero-stability, consistency, convergence and region of absolute stability were also investigated. Several initial value problems of higher order ordinary differential equations were then solved using the new developed methods. The numerical results revealed that the new methods produced more accurate solutions than the existing methods when solving the same problems. Hence, the new methods are viable alternatives for solving initial value problems of higher order ordinary differential equations directly.

Keywords: Interpolation, collocation, one step hybrid block method, direct solution, higher order initial value problems, generalised off-step point(s).

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CHAPTER ONE
INTRODUCTION

1.1 Background of the Study

Mathematicians develop mathematical models to help them understanding the physical phenomena in real life problems. These models frequently lead to equations involving some derivatives of an unknown function of single or several variables, which are called differential equations. Differential equations have vast application in many fields such as engineering, medicine, economics, operation research, psychology and anthropology.

There are two types of differential equation namely Ordinary Differential Equation (ODE) and Partial Differential Equation (PDE). ODE is a differential equation that has single independent variable, while PDE is differential equation with two or more variables (Omar & Suleiman, 1999). The general form of ODE on the interval $[a,b]$ is denoted as

$$y^{\gamma} = f(x, y, y^{'}, y^{''}, \ldots, y^{\gamma-1}). \tag{1.1}$$

In order to solve the equation (1.1), the conditions stated below need to be imposed.

$$y(a) = \eta_0, \quad y^{'}(a) = \eta_1, \ldots, y^{\gamma-1}(a) = \eta_{\gamma-1} \tag{1.2}$$

Equation (1.1) and equation (1.2) are called initial value problem(IVP). If there is another condition at the different value of x such as b, then it is called boundary value problem(BVP) (Lambert, 1973).

1.2 Uniqueness and Existence Theorem

The first order initial value problems of ODE is generally denoted as

$$y^{'} = f(x,y), \quad y(a) = \eta \tag{1.3}$$

If we face a system of first order ordinary differential equations in m dependent variables, i.e $y_1, y_2, \ldots, y_m$, where the given conditions are satisfied at the same initial point for each of these equations, then we have (IVPs) for first order system as follows

$$\begin{aligned}
y_1^{'} &= f_1(x,y_1,y_2,\ldots,y_m),\, y_1(a) = \eta_1, \\
y_2^{'} &= f_2(x,y_1,y_2,\ldots,y_m),\, y_2(a) = \eta_2, \\
&\vdots \\
y_m^{'} &= f_m(x,y_1,y_2,\ldots,y_m),\, y_m(a) = \eta_m,
\end{aligned} \tag{1.4}$$

For simplicity, system (1.4) can also be expressed in the following vector form

$$\boldsymbol{y}^{'} = \boldsymbol{f}(x,\boldsymbol{y}), \;\; \boldsymbol{y}(a) = \boldsymbol{\eta}$$

where $$\boldsymbol{y}^{'} = \begin{bmatrix} y_1^{'} \\ y_2^{'} \\ \vdots \\ y_m^{'} \end{bmatrix}, \quad \boldsymbol{f}(x,\boldsymbol{y}) = \begin{bmatrix} f_1(x,y_1,y_2,\ldots,y_m) \\ f_2(x,y_1,y_2,\ldots,y_m) \\ \vdots \\ f_m(x,y_1,y_2,\ldots,y_m) \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\boldsymbol{y}(a) = \begin{bmatrix} y_1(a) \\ y_2(a) \\ \vdots \\ y_m(a) \end{bmatrix}, \quad \boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix}$$

In this work, two theorems are considered. Theorem1.1 focuses on existence and uniqueness solution of first order ODEs while Theorem (1.2) discusses the existence and uniqueness solution of higher order ODEs.

Theorem 1.1 (Lambert, 1991)**.** *Let $f(x,y)$, where f be defined and continuous for (x,y) in the region D defined by $a \leq x \leq b$, $-\infty < y < \infty$, a and b are finite, and let there exist a constant L such that*

$$|f(x,y) - f(x,y^*)| \leq L|y - y^*| \tag{1.5}$$

holds for every (x,y), $(x,y^*) \in D$. Then for any $\delta \in R$, there exists a unique solution $y(x)$ of the problem (1.3) where $y(x)$ is continuous and differentiable for all $(x,y) \in D$.

The requirement in Theorem1.1 is known as Lipschitz condition and the constant L is called a Lipischitz constant. In addition, if $f(x,y)$ is differentiable with respect to y, then from the Mean Value Theorem,

$$f(x,y) - f(x,y^*) = \frac{\partial f(x,\bar{y})}{\partial y}(y - y^*) \tag{1.6}$$

where $\bar{y}$ is a point in the interior of the interval whose end points are y and y^*, and both (x,y) and (x,y^*) are in the region D.

Therefore, if we choose

$$L = sup_{(x,y)\in D}\left|\frac{\partial f(x,\bar{y})}{\partial y}\right|,$$

the condition (1.5) of Theorem1.1 is then satisfied (Lambert, 1991).

In this part, we adopted the existence and uniqueness of theorem stated by Wend (1969) to guarantee the existence and uniqueness of the solutions of (1.1) as stated below.

Theorem 1.2 (Wend,1969)**.** *Let R be a region defined by the inequalities,*

$0 \leq x - x_0 \leq a$, $|s_k - \eta_k| < b$, $k = 0, 1, \ldots, \gamma - 1$, *where* $a, b > 0$.

Suppose $f(x, s_0, s_1, \ldots, s_{\gamma-1})$ *is defined in R and in addition:*

1- f *is nonnegative and nondecreasing in each of* $x, s_0, \cdots, s_{\gamma-1}$ *in R.*

2- $f(x, \eta_0, \eta_1, \ldots, \eta_{\gamma-1}) > 0$ *for* $0 \leq x - x_0 \leq a$.

3- $\eta_k \geq 0$, $k = 1, 2, \cdots, \gamma - 1$.

Then, the initial value problem has a unique solution in R.

There are two ways of solving higher order ODEs; reduction method and direct method.

i. Reduction method.

In this approach, a higher Order ODE is converted to its equivalent system of first order ODEs (Awoyemi, 2003). Then the existing numerical methods for solving first ODE are employed to solve the problem. This is demonstrated in the following example.

Example 1.1. Convert the following third order differential equation to a system of first order IVPs.

$$3y''' + 5y'' - y' + 7y = 0,\ y(1) = 0,\ y'(1) = 0,\ y''(1) = 0 \tag{1.7}$$

Solution: Equation (1.7) can be written as a system of IVPs by changing the variables as follows

$$z_1(t) = y(t)$$
$$z_2(t) = y'(t)$$
$$z_3(t) = y''(t)$$

Differentiating both sides of the above equations gives

$$\begin{aligned}
z_1'(t) &= y'(t) = z_2(t) \\
z_2'(t) &= y''(t) = z_3(t) \\
z_3'(t) &= y'''(t) = -\frac{5}{3}y'' + \frac{1}{3}y' - \frac{7}{3}y = -\frac{5}{3}z_3(t) + \frac{1}{3}z_2(t) - \frac{7}{3}z_1(t)
\end{aligned}$$

The initial conditions must also be converted to the new functions as follows

$$z_1(1) = y(1) = 0$$
$$z_2(1) = y'(1) = 0$$
$$z_3(1) = y''(1) = 0$$

As a result, the following system of first order differential equations is obtained.

$$\begin{aligned}
z_1'(t) &= z_2(t), & z_1(1) = 0 \\
z_2'(t) &= z_3(t), & z_2(1) = 0 \\
z_3'(t) &= -\frac{5}{3}z_3(t) + \frac{1}{3}z_2(t) - \frac{7}{3}z_1(t), & z_3(1) = 0
\end{aligned}$$

The existing numerical methods for solving first order ODEs are then applied to solve Problem (1.7).

ii. Direct Method

This method solve higher order ODEs directly without converting them to the equivalent system of first order ODEs. Generally, some problems of IVPs can be solved analytically and there is one analytical solution for IVP, and *m* exact solution for systems of *m* equations of IVPs.

It is observed that, on the other hand, some differential equations do not have analytically solution. Therefore, obtaining approximate numerical solution becomes important, and this requires for the development of numerical methods. Some of these method methods can be found in Henrici (1962), Milne (1970), Lambert (1973), Lambert and Watson (1976), Stetter (1973), Sand and Osterby (1979), Fatunla (1988), Hairer and Wanner (1975), Hairer, Norsett and Wanner (2008), Iserles, Norsett and Ew (1997) and Butcher (2003).

In general, there are two types of numerical methods namely single step method and multistep method. Single-step method uses the data from the previous point to find the approximation solution at the current point. On the other hand, multistep method uses the information from several previous points to find the approximate solution at the current point (Omar & Suleiman, 1999). The details of these methods is described in the following sections.

1.3 Single Step Method

Single step method is also known as one step method. Many scholars have developed single-step method for solving ODEs.

A single step explicit method is generally represented as below

$$y_{n+1} = y_n + h\vartheta(x_n, y_n; h) \tag{1.8}$$

where h is the step size of the interval $[x_n, x_{n+1}]$ and $\vartheta(x_n, y_n; h)$ is the increment function. If the increment function in (1.8) is defined by $\vartheta(x_n, y_n, y_{n+1}; h)$ then it is called implicit method.

Among well-known methods are Forward and Backward Euler methods, Theta method, Runge Kutta method, Trapezoid method, Modified Trapezoid methods. These methods are suitable for solution of first order IVPs of ODEs. However, the fourth order Runge Kutta is the most famous among all one step methods (Anake, 2011).

1.4 Multistep Method

As we have mentioned earlier, this method uses information at the previous points for the approximation of numerical solution at the current point. For example a method of five step length makes use of information at the previous five points, *i.e.* $x_n, \ldots, x_{n+4}$ to approximate a numerical solution at x_{n+5}.

Generally, the k -step linear multistep method is given as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^{\gamma} \sum_{j=0}^{k} \beta_j f_{n+j}. \tag{1.9}$$

where the coefficients $\alpha_0, \ldots, \alpha_k$ and $\beta_0, \ldots, \beta_k$ are real constants and y_{n+j}=$y(x_{n+j})$ and $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j}, \ldots, y^{\gamma-1}_{n+j})$ (Lambert, 1991).

Remark 1.4

i. The values $\alpha_k \neq 0$ and that β_0 and α_0 in Equation(1.9) are not both equal zero.

ii. If $\beta_k = 0$, then Equation (1.9) is said to be explicit.

iii. On the other hand, if $\beta_k \neq 0$ that is y_{n+k} appears on the both sides of the equation (1.9), then it is known implicit method.

iv. The k-step linear multistep method is considered linear because it includes only linear terms of the f_{n+j} and y_{n+j}.

1.5 Block Method

According to Olabode (2007), block method was first adopted by Milne in 1953. This block method can be defined as a class of linear multistep methods which are applied concurrently to initial value problems of ODEs to give an efficient approximate solution in term of accuracy. On the other hand, a set of new approximate derived points when the method is applied to the equation is known as block. There are two types of block method; single step and multistep block methods. In a single step block method, the information at the grid point x_n is used to compute y_{n+i}, $i = 1, 2, \ldots, k$. Multistep block method, on the other hand, uses information at the previous block for the computation of the next block (Omar & Suleiman, 1999).

Extending Chu and Hamilton (1987), r-point, k-block method for solving (1.1) can be represented in the following form:

$$Y_m^{\gamma} = D_0 Y_0 + \sum_{i=1}^{k} A_i Y_{m-i} + h^{\gamma} \sum_{i=0}^{k} B_i F_{m-i}. \tag{1.10}$$

where D_0 is $r \times \gamma$ matrix and A_i and B_i are all $r \times r$ square matrices and Y_m, Y_0, Y_{m-i} and F_{m-i} are defined by

$$\begin{aligned}
Y_m &= [(y_n, y_{n+1}, \ldots, y_{n+r-1})]^T \\
Y_{m-i} &= [(y_{n-ir}, y_{n-ir+1}, \ldots, y_{n-(i-1)r-1})]^T \\
F_{m-i} &= [(f_{n-ir}, f_{n-ir+1}, \ldots, f_{n-(i-1)r-1})]^T \\
Y_0 &= [(y_{n-kr-1}, y'_{n-kr-1}, \ldots, y^{\gamma-1}_{n-kr-1})]^T
\end{aligned}$$

1.6 Hybrid Method

Hybrid method introduced by Gragg and Stetter (1964) incorporates the evaluation of function at off-step points(these are points that are not on the grid) to overcome the zero stability barrier (Lambert, 1973). The beauty of hybrid method is that it possesses certain characteristics of continuous linear multistep method. It also shares the same property of Range-Kutta such that the information at off-step points are also utilized (Gear, 1965).

The general form of hybrid method of k -step is denoted as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h\beta_s f_{n+s} \tag{1.11}$$

where $\alpha_k = 1, \alpha_0$ and β_0 are both not zero, $s \notin \{0, 1, 2, \ldots, k\}$, $y_{n+j} = y(x_n + jh)$, $f_{n+j} = f(x_{n+j}, y_{n+j})$ and $f_{n+s} = f(x_{n+s}, y_{n+s})$ (Lambert, 1973).

1.7 Problem Statement

Reduction of higher order ODEs to the its equivalent system of first order has been found having some drawbacks which include computation burden and complication in writing computer program which affects the accuracy of the method in terms of error (Awoyemi, 1992). The reduction method does not fully utilize information associated with certain ODEs like oscillatory nature of the solution (Vigo-Aguiar & Ramos, 2006).

The implementation of linear multistep method can be done by using two approaches namely block method and predictor corrector method. Furthermore, the implementation of linear multistep method in predictor-corrector mode have been discovered to be very expensive to implement in terms of function to be evaluated per step. In addition, the development of predictor consume a lot of human efforts. The accuracy of predictors always found to be very low than the corrector especially when all the step within the interval are considered for interpolation and collocation. Subroutine to supply the starting values are needed in predictor- corrector method which leads to inefficiency of the method in terms of error (Anake, Awoyemi & Adesanya, 2012b).

On the other hand, in block method the development of separate predictors is not needed. Furthermore, applying the method as parallel integration is possible and thus requires less computational burden and human effort which resulted in high accuracy of the method (Anake, Awoyemi & Adesanya, 2012a).

The idea of hybrid method which involves the use off-step points was introduced to overcome the zero stability barrier in linear multistep method. This barrier implies that the highest order of zero stability of linear multistep method when steplength k is odd is $k+1$ and $k+2$ when k is even (Lambert, 1973).

In order to overcome the setbacks mentioned above, hybrid block method was introduced. Anake et al. (2012a) proposed hybrid block method in which two off-step points ($x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}$) were considered in the development the method but the accuracy of the method are low.

Adesanya, Fasansi and Odekunle (2013) developed one step with three off-step points ($x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}$) for third order ODEs but the implementation was through predictor-corrector mode which usually bring computational burden that contribute to lower accuracy of the method. Subsequently, one step with one off-step point ($x_{n+\frac{1}{2}}$) was developed by Anake et al. (2012b). However this method is of lower accuracy when it was applied to solve second order ODEs. Subsequently, Kayode, Duromola and Bolaji (2014) proposed a one step method having four off-step points , $x_{n+\frac{1}{5}}, x_{n+\frac{2}{5}}, x_{n+\frac{3}{5}}, x_{n+\frac{4}{5}}$

for solving fourth order ODEs but the accuracy of the method is not encouraging.

Hybrid block method depends on the off-step points chosen. So far one step hybrid block methods has been developed to specific point(s). This study, therefore, attempts to generalise off-step points of one step hybrid block methods which gives better approximate solution.

1.8 Objectives of the Research

The main aim of this research is to develop one step hybrid block method with generalized off-step points for solving IVPs of higher order ODEs. To realize this goal, the following sub-objectives should be achieved:

1. To derive continuous implicit hybrid one step method with generalised off-step points by collocating and interpolating technique at both the off-step and step points.
2. To derive one step hybrid block method with generalised one, two and three off-step points for solving second, third and fourth order ODEs.
3. To establish the basic important properties of the method with generalised off-step points which involve order, convergence, zero stability, consistency and region of absolute stability.
4. To compare the new developed methods with the existing methods in term of error.

1.9 Significance of the Study

This study has add the following contribution:

1. New classes of generalised continuous implicit hybrid one step methods for the

direct solutions of IVPs of ODEs have been derived.

2. New generalised one step hybrid methods for the direct solution of IVPs of ODEs have been developed.
3. Flexible computer programs to solve ODEs have been developed.
4. Numerical properties of the one step hybrid block method at several off-step points have been established.

1.10 Limitation of the Study

The development of hybrid block method using multistep collocation interpolation technique to solve any n-th order ODEs is impossible because it is the order of differential equation that determines the points of interpolation. Due to this constraint and for the purpose of comparing with the existing methods, the scope of this research work are limited to the second, third and fourth order IVPs of ODEs.

CHAPTER TWO

LITERATURE REVIEW

The need to find more accurate approximate solutions for ODEs that are emerging from engineering, science and other physical phenomena which do not have exact solutions led many mathematicians to develop new numerical methods. In this chapter, some of numerical methods which are closely related to our work were reviewed. The review is divided into three parts namely block methods for second, third and fourth order ODEs.

The method of reduction of higher order differential equation to first order initial value problem is a popular approach. This approach is widely discussed by some memorable scholars such as, Lambert and Watson (1976), Ixaru (1984), Jain, Kambo and Goel (1984), Fatunla (1988), Sarafyan (1990), Awoyemi (1992), Bun and Vasil'yev (1992), Hairer et al. (2008), Hull, Enright, Fellen and Sedgwick (1972), Omar and Suleiman (2006), Ngwane and Jator (2012), Omar and Sulaiman (2004) among others. Despite of the effectiveness and success of this method, there are also some inevitable setbacks. For instance, the development of computer programs for these methods require more computer time and human effort (Awoyemi, 1992). Moreover, these methods do not use additional data associated with particular ODEs as appear in oscillatory equations (Vigo-Aguiar & Ramos, 2006). Thus, this approach is not suitable and not efficient for general applications.

According to Anake (2011), the direct solution of (1.1) and its equivalent system of first order ODEs was initially investigated by Rutishauser (1960). Since then, many other researchers such as Anake, Adesanya, Oghonyon and Agarana (2013), Awoyemi and Idowu (2005), Henrici (1962), Gear (1971), Fatunla (1988), Omar and Suleiman (2005), Omar, Sulaiman, Saman and Evans (2002), Omar and Sulaiman (2004), Majid, Suleiman and Omar (2006), Anake et al. (2012a), Yahaya, Sagir and Tech (2013), James, Adesanya and Joshua (2013), Olabode and Yusuph (2009), Olabode (2013), Adesanya, Udoh and Ajileye (2013) Olabode and Yusuph (2009), Yap, Ismail and Senu (2014), Mohammed, Hamza and Mohammed (2013), Adesanya, Fasansi and Ajileye (2013), Adesanya, Udoh and Ajileye (2013), Krishnaiah (1987), Ademiluyi, Duromola and Bolaji (2014), Jator (2010) and Olabode and Alabi (2013) proposed direct methods to approximate solutions of higher order instead of going through the reduction process to the first order IVPs.

2.1 Block Methods for Second Order ODEs

In literature, Adeniyi and Adeyefa (2013) advocated the use of chebyshev polynomial instead of power series for development of methods through collocation and interpolation approach for solving second order ODEs. Anake (2011) developed a one step implicit hybrid method to solve second order initial value problem directly. The accuracy of the method was tested by solving some different problems. The zero-stability and consistency of the method assured that the method is convergent. Adesanya, Ibrahim, Abdulkadi and Anake (2014) proposed two step hybrid block method with four off-step points which have order four. The basic properties of the method was investigated and found to be zero stable, consistent and convergent. The method was tested on some numerical examples and the accuracy of the method can still be improved.

Yahaya et al. (2013) developed a 3-Step Implicit block hybrid method to solve second order IVPs. They stated consumption of computer time and computational difficulties as the major setbacks for reduction of higher order equations to the first order system. Furthermore, stiff and non-stiff equation are examined and the efficiency of the method was tested by comparing with the existing method and it was found better.

Sagir (2012) proposed self-starting block hybrid method of order five for the solution of second order ODEs with associated boundary or initial conditions. The continuous hybrid formulations method enables someone to differentiate and evaluate at some grid and off– grid points to obtain four discrete schemes, which were applied in block form for sequential or parallel solutions of the problems. The computational burden and computer time wastage involved in the usual reduction of second order problem into system of first order equations are avoided by this approach. Furthermore, efficiency of the block method are examined on stiff ODEs, and the results were compared favorably with the existing method. A two step with two Off-step point was proposed byKayode and Adeyeye (2013). They used Chebyshev polynomials as basis function for the development of the methods in predictor-corrector approach. The results show a better performance when the method was tested with linear and non-linear problems. However, the accuracy of the method can be improved. Four step self starting having order five was proposed by Jator and Li (2009) for solving second order ordinary differential equation directly. The method was developed through collocation and interpolation approach. Numerical properties which involve order, zero stability, convergence, consistency were discussed. Recently, Kayode and Obarhua (2015) considered three step implicit hybrid method to present direct solution of second ODEs. The predictor was developed to evaluate the implicit scheme and some numerical properties were also established.

2.2 Block Methods for Third Order ODEs

An accurate hybrid method was proposed by Mohammed and Adeniyi (2014) for direct solution of third order ODEs. The steplength K=3, with one off-step points and power series approximate solution were used in the development of the method. The derivation of one step block method having three hybrid points which was implemented in predictor corrector mode was carried out by Adesanya, Fasansi and Odekunle (2013). The method was developed via linear multistep collocation approach which was applied to solve third order IVPs of ODEs. Kuboye and Omar (2015b) developed an accurate block method for solving third order ODEs. In this method, seven-step was used but the accuracy of the method is still low when the method is applied to solve third order IVPs. Furthermore, a four step with order six block predictor-corrector method was proposed by Adesanya, Udo and Alkali (2012). Bolarinwa, Ademiluyi, Awoyemi and Ogundele (2012) developed one step block method with two off-step points, which was implemented by collocation interpolation method. The method solves third order ODEs directly but the accuracy of the method is low.

Omar and Kuboye (2015) introduced multistep method to approximate the solution of general third order IVPs directly by using the collocation and interpolation method. The numerical properties of the method were established through some problems that were tested. **?** solved particular third order differential equations by developing a direct 6-steps block method using constant step size. He concluded that the hybrid method gives better approximation results than the traditional predictor corrector method. While Awoyemi, Kayode and Adoghe (2014) proposed four step continuous linear multistep method by using collocation and interpolation approach to derive seven order scheme to solve general third order ODE equation. Taylor series was adopted to implement the method, where it was used to predict the initial evaluations. The numerical results were presented to show good performance of the method by examining some problems and comparing these results with traditional methods. Recently, Gbenga, Olaoluwa and Olayemi (2015) proposed hybrid and non-hybrid Implicit Schemes for Solving Third (IVPs) through block approach. It was found that the hybrid scheme is more superior to the non-hybrid scheme in terms of accuracy and efficiency.

2.3 Block Methods for Fourth Order ODEs

In literature, Olabode (2009) also developed a six-step scheme for solving fourth order ODEs. The approach used in the development of this method is collocation approxi-

mation. This approach made the method more efficient than when it was applied over predictor corrector in which the result overlapped with each other.
An implicit hybrid block method with four off-step points for solving fourth order IVPs examined by Kayode et al. (2014). The collocation approximation to produce the corrector with continuous coefficient was adopted in developing the method.

Kayode (2008b) and Adesanya, Alkali, Adamu and Tahir (2012) presented new block methods to solve fourth order ODEs. However, the accuracy of these methods were low. Similarly, Kayode (2008a) used collocation interpolation strategy to derive five step linear multistep method through predictor corrector mode to solve general fourth order IVP. The performance of method was tested by examining some problems.

Adesanya, Alkali, et al. (2012) considered method of collocation of the differential system and interpolation of the approximate solution to generate a continuous linear multistep method which was solved for the independent solution to yield a continuous block method. The resultant method was evaluated at selected grid points to generate discrete block method. The basic properties of the method was investigated and found to be zero stable, consistent and convergent. The method was tested on numerical examples solved by the traditional method and it was found to give better approximation.

A six step continuous multistep method for the solution of general fourth order IVPs was proposed by Awoyemi, Kayode and Adoghe (2015) where the implementation was carried out via collocation and interpolation approach. The method was tested by solving some of differential problems and it found that some results were still of low accuracy. In the same area, Olabode and Omole (2015) proposed one, two and three step implicit hybrid block methods in Numerov-Type through collocation and interpolation for the direct solution of fourth order IVPs. The results was compared favourably with the numerical values when the same schemes were implemented in predictor-corrector mode. Recently, Kuboye and Omar (2015a) developed six step block method for solving fourth order IVPs directly. Collocation and interpolation method was applied for deriving the methods. New strategy was adopted in selecting the interpolation points. Although, this method performs better than some existing methods in terms of accuracy, can still be improved in terms of accuracy.

Table 2.1
Highlight of Literature Review on Block Collocation Method for Second Order ODEs

Authors	Methods	Advantages	Disadvantages
Sagir (2012)	3-step self-starting block hybrid method of order five for solving second order ODEs with one hybrid point $x_{n+\frac{4}{3}}$.	The method reduces human effort since It solves second order ODEs directly.	The accuracy of method is low. the method can be improved by increasing number of off-step point.
Anake (2011)	One-step implicit hybrid method having three off-step points at $x_{n+\frac{1}{4}}$, $x_{n+\frac{1}{2}}$ and $x_{n+\frac{3}{4}}$ for solving second order ODEs	The method solves second order ODEs directly. This overcomes the setbacks in reduction method	The error is too large. The method can be improved by choosing another off-step points
Adeniyi & Adeyefa(2013)	One step hybrid block method with one off-step points for solving second order ODEs using collocation and interpolation approach.	The method is applied to solve second order ODEs directly. This overcomes computational burden in reduction method	The method is of lower accuracy.
Yahaya, Sagir & Tech (2013)	3-Step Implicit block hybrid method with one step point at $x_{n+\frac{4}{3}}$ for solving second order IVPs.	The method requires less computational burden compared to predictor-corrector method. It also solves second order ODEs directly.	The error of the method is not encouraging.
Adesanya et al. (2014)	2-step hybrid block method having four off-step pints through collocation and interpolation approach	The method solves second order ODEs directly.	The numerical results is not efficient in terms of error. it can be improved by selecting different off-step points.

Table 2.1
(Continued)

Authors	Methods	Advantages	Disadvantages
Kayode & Adeyeye (2013)	2-step hybrid block method for solving second order ODEs directly using corrector predictor mode.	The method reduces human effort since It solves second order ODEs directly.	The numerical results is not efficient in terms of error. The accuracy of the method can be improved.
Jator & Li(2009)	Four-step linear multiste method for solving second order ODEs using collocation interpolation approach	The method solves second order ODEs directly. This overcomes the setbacks in reduction method	The error is too large.
Kayode & Obarhua(2015)	3- step hybrid method for solving second order ODEs using collocation interpolation approach	The method solves second order ODEs directly. This overcomes the setbacks in reduction method	The method is not efficient in terms of error.

Table 2.2
Highlight of Literature Review on Block Collocation Method for Third Order ODES.

Authors	Methods	Advantages	Disadvantages
Omar & Kuboye (2015b)	Five step block method for solving third order ODEs using multistep collocation approach.	The method solves third order ODEs directly. This overcomes the setbacks in reduction method	The accuracy of the method can be improved.
Olabode(2014)	Block method with steplingh six for solving third order ODEs using collocation and interpolation method.	The method reduces human effort since it solves third ODEs directly.	The method can only solve third order and the accuracy is not efficient in terms of error.
Mohammed & Adeniyi(2014)	Three step hybrid method with one Off-step point for solving third order ODEs using collocation interpolation approach.	The method solves third ODEs directly.	The accuracy is not
Adesanya et al.(2012)	Four step block method for solving third order ODE which was implemented by predictor corrector mode.	The method generates numerical solution for solving third order ODEs directly.	The method associated with much computational burden which affects the accuracy of the method in terms of error.
Gbenga et al.(2015)	Three step block method with one Off-step points which was implemented by collocation interpolation method.	The method applied to solve third order ODEs directly. This overcomes computational burden in reduction method	The accuracy of method is low. It can still be improved .

Table 2.2
(Continued)

Awoyemi et al. (2014)	Four step linear multistep method having order seven for solving third order (ODEs).	The numerical results of method are good but it can be improved.	The method associated with much computational burden which affects on the efficiency in terms of error.
Anake et al.(2013)	Five step Block method for solving third ODEs using multistep collocation method	The method requires less computational burden compared to predictor corrector method it also solves third order ODEs directly.	The accuracy is low and can still be improved.
Kuboye & Omar (2015b)	Seven step block method having using collocation interpolation approach for solving third order ODEs.	The method generates numerical solution for third order ODEs.	The method can only solve third order ODEs.

Table 2.3
Highlight of Literature Review on Block Collocation Method for Fourth Order ODES.

Authors	Methods	Advantages	Disadvantages
Olabode (2009)	6-step block method for solving fourth order through collocation and interpolation approach.	The method reduces human effort since it solves fourth order ODEs directly.	The accuracy of method is low.
Adesanya et al (2012)	5-step implicit block method using collocation and interpolation method for solving fourth ODEs.	The method solves fourth order ODEs directly. This overcomes the disadvantage of reduction method which including computational burden and human effort	The error of the developed is not encouraging in terms of error.
Kayode, Duromda & Bolarinwa(2014)	One-step fourth order hybrid block method with four off-step points at $x_{n+\frac{1}{5}}$, $x_{n+\frac{2}{5}}$, $x_{n+\frac{3}{5}}$ and $x_{n+\frac{4}{5}}$ for solving fourth order ODEs using collocation and interpolation method.	The method solves fourth order ODEs directly.	The error of method is large. It can be reduced by changing the off-step points.
Olabode & Omole Ezekiel(2015)	One and two step hybrid Block Numerov- type methods for the direct solution of fourth order IVPs which was implemented by collocation interpolation method.	The method applied to solve general fourth order ODEs directly. It can be improved by developing the method with different hybrid points	The method is of low accuracy in terms of error.

Table 3.3
(Continued)

Authors	Methods	Advantages	Disadvantages
Kayode(2008a)	5-step block method for solving fourth ODEs	The method generates numerical results for solving fourth order ODEs directly. This overcome problems of reduction method	The method is not efficiency in terms of error.
Awoyemi, Kyode & Adoghe(2015)	A Six-Step Continuous Multistep Method For The direct Solution Of General Fourth Order IVPs for which was implemented by collocation interpolation method.	The method applied to solve general fourth order ODEs directly	The method is of low accuracy.
Kayode (2008b)	5-step block predictor corrector method for solving fourth ODEs	The method generates numerical results for solving fourth order ODEs directly. This overcome problems of reduction method	The method associated with much computational burden and the error is too large .
Kuboye & Omar(2015a)	6-step block block methods for solving fourth ODE	The method generates numerical results for solving fourth order ODEs directly. This overcome problems of reduction method	The error of the methods can be improved.

CHAPTER THREE

ONE STEP HYBRID BLOCK METHODS FOR SOLVING SECOND ORDER ODEs DIRECTLY

In this chapter, the development of one step hybrid block methods with one, two and three generalised off-step points using collocation and interpolation method for solving second order initial value problems of ODEs is described.

The following power series of the form

$$y(x) = \sum_{i=0}^{v+m-1} a_i \left(\frac{x-x_n}{h}\right)^i \tag{3.1}$$

is used to approximate the solution of general second order initial value problem

$$y^{''} = f(x, y, y^{'}) \tag{3.2}$$

with initial conditions

$$y(a) = \eta_0, y^{'}(a) = \eta_1$$

on the integration interval $[a, b]$, where $x \in [x_n, x_{n+1}]$ for $n = 0, 1, 2, ..., N-1$, v denotes of the number of interpolation points which is equal to the order of differential equation, m represents the number of collocation points and $h = x_n - x_{n-1}$ is constant step size of partition of interval $[a, b]$ which is given by $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$.

The first and second derivatives of equation (3.1) are

$$y^{'}(x) = \sum_{i=1}^{v+m-1} \frac{i}{h} a_i \left(\frac{x-x_n}{h}\right)^{i-1} \tag{3.3}$$

and

$$y^{''}(x) = \sum_{i=2}^{v+m-1} \frac{i(i-1)}{h^2} a_i \left(\frac{x-x_n}{h}\right)^{i-2}. \tag{3.4}$$

Substituting equation (3.4) into equation (3.2) gives

$$\sum_{i=2}^{v+m-1} \frac{i(i-1)}{h^2} a_i \left(\frac{x-x_n}{h}\right)^{i-2} = f(x,y,y^{'}). \tag{3.5}$$

3.1 Derivation of One Step Hybrid Block Method with Generalised One Off-Step Points for Second Order ODEs

To derive this method, Equation (3.1) is interpolated at x_n and x_{n+s} while Equation (3.5) is collocated at all points in the selected interval, where s is defined previously in section 1.6. This is illustrated clearly in Figure 3.1 below where I and C represent interpolation and collocation points respectively.

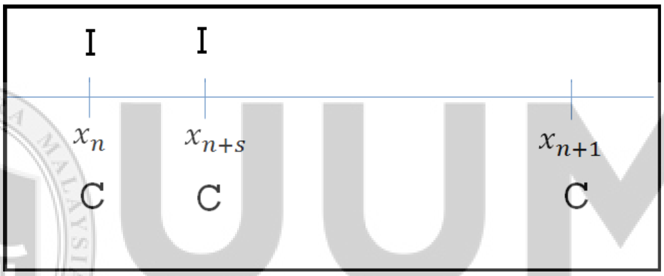

Figure 3.1. One step hybrid block method with generalised one off-step point for solving second ODEs.

From the above figure, $v = 2$ and $m = 3$. This interpolation and collocation strategy produces the following equations

$$\begin{aligned}
y_n &= a_0. \\
y_{n+s} &= a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4. \\
f_n &= \frac{2}{h^2} a_2. \\
f_{n+s} &= \frac{2}{h^2} a_2 + \frac{6s}{h^2} a_3 + \frac{12s^2}{h^2} a_4. \\
f_{n+1} &= \frac{2}{h^2} a_2 + \frac{6}{h^2} a_3 + \frac{12}{h^2} a_4.
\end{aligned} \tag{3.6}$$

which can be written in a matrix form as below

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & s & s^2 & s^3 & s^4 \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{6s}{h^2} & \frac{12s^2}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+s} \\ f_n \\ f_{n+s} \\ f_{n+1} \end{pmatrix} \tag{3.7}$$

Applying Gaussian elimination method on (3.7) leads to

$$\begin{aligned} a_0 &= y_n \\ a_1 &= \frac{-(12s-12)}{12s(s-1)}y_n + \frac{(12s-12)}{12s(s-1)}y_{n+s} + \frac{(h^2s^4-5h^2s^3+4h^2s^2)}{12s(s-1)}f_n \\ &\quad + \frac{(2h^2s^2-h^2s^3)}{12s(s-1)}f_{n+s} - \frac{(h^2s^3)}{12(s-1)}f_{n+1} \\ a_2 &= \frac{(f_nh^2)}{2} \\ a_3 &= \frac{h^2s}{6(s-1)}f_{n+1} - \frac{h^2(s^2-1)}{6s(s-1)}f_n - \frac{h^2}{6s(s-1)}f_{n+s} \\ a_4 &= \frac{-h^2}{12(s-1)}f_{n+1} + \frac{h^2}{12s}f_n + \frac{h^2}{12s(s-1)}f_{n+s} \end{aligned}$$

The values of $a_i's, i = 0(1)4$ are then substituted into Equation (3.1) to give a continuous implicit scheme of the form

$$y(x) = \alpha_0(x)y_n + \alpha_s(x)y_{n+s} + \sum_{i=0}^{1}\beta_i(x)f_{n+i} + \beta_s(x)f_{n+s} \tag{3.8}$$

Differentiating (3.8) once gives

$$y'(x) = \frac{\partial}{\partial x}\alpha_0(x)y_n + \frac{\partial}{\partial x}\alpha_s(x)y_{n+s} + \sum_{i=0}^{1}\frac{\partial}{\partial x}\beta_i(x)f_{n+i} + \frac{\partial}{\partial x}\beta_s(x)f_{n+s} \tag{3.9}$$

where

$$
\begin{aligned}
\alpha_0 &= \frac{1-(x-x_n)}{hs} \\
\alpha_s &= \frac{x-x_n}{hs} \\
\beta_0 &= \frac{(x-x_n)(x_n-x+hs)(h^2s^2-4h^2s+hsx-hsx_n+2hx-2hx_n-x^2+2xx_n-x_n^2)}{12h^2s} \\
\beta_s &= \frac{(x_n-x)(x_n-x+hs)}{12h^2s(s-1)}(h^2s^2-2h^2s+hsx-hsx_n-2hx+2hx_n+x^2-2xx_n+x_n^2) \\
\beta_1 &= \frac{-(x-x_n)(x_n-x+hs)}{12h^2(s-1)}(h^2s^2+hsx-hsx_n-x^2+2xx_n-x_n^2)
\end{aligned}
$$

Evaluating Equation (3.8) at the non-interpolating point x_{n+1} yields

$$
\begin{aligned}
y_{n+1}-\frac{y_{n+s}}{s} &= \frac{(12s-12)}{12s}y_n+\frac{(h^2s^3-4h^2s^2+4h^2s-h^2)}{12s}f_n \\
&+\frac{(-h^2s^2+h^2s+h^2)}{12s}f_{n+s}-\frac{(h^2s^3+h^2s^2-h^2s)}{12s}f_{n+1} \qquad (3.10)
\end{aligned}
$$

Equation (3.9) is then evaluated at all points i.e x_n, x_{n+s} and x_{n+1} to produce

$$
\begin{aligned}
y_n^{'}-\frac{(12s-12)}{12hs(s-1)}y_{n+s} &= -\frac{(12s-12)}{12hs(s-1)}yn+\frac{(h^2s^4-5h^2s^3+4h^2s^2)}{12hs(s-1)}f_n \\
&+\frac{2h^2s^2-h^2s^3}{12hs(s-1)}f_{n+s}-\frac{hs^3}{12(s-1)}f_{n+1} \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+s}^{'}-\frac{(12s-12)}{12hs(s-1)}y_{n+s} &= \frac{-(12s-12)}{12hs(s-1)}y_n-\frac{(h^2s^4-3h^2s^3+2h^2s^2)}{12hs(s-1)}f_n \\
&-\frac{(4h^2s^2-3h^2s^3)}{12hs(s-1)}f_{n+s}+\frac{hs^3}{12(s-1)}f_{n+1} \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+1}^{'}-\frac{(12s-12)}{12hs(s-1)}y_{n+s} &= \frac{-12(s-1)}{12hs(s-1)}y_n+\frac{h^2s^4-5h^2s^3+10h^2s^2-8h^2s+2h^2}{12hs(s-1)}f_n \\
&-\frac{(h^3s^3-2h^2s^2+2h^2)}{12hs(s-1)}f_{n+s} \\
&-\frac{h^2s^4-6h^2s^2+4h^2s}{12hs(s-1)}f_{n+1} \qquad (3.13)
\end{aligned}
$$

Combining (3.10) and (3.11) produces a block of the form

$$A^{[1]_2}Y_m^{[1]_2} = B_1^{[1]_2}R_1^{[1]_2} + B_2^{[1]_2}R_2^{[1]_2} + h^2\left[D^{[1]_2}R_3^{[1]_2} + E^{[1]_2}R_4^{[1]_2}\right] \quad (3.14)$$

where

$$A^{[1]_2} = \begin{pmatrix} \frac{-1}{s} & 1 \\ \frac{-1}{hs} & 0 \end{pmatrix}, \ Y_m^{[1]_2} = \begin{pmatrix} y_{n+s} \\ y_{n+1} \end{pmatrix}, \ B_1^{[1]_2} = \begin{pmatrix} 0 & \frac{s-1}{s} \\ 0 & \frac{-1}{hs} \end{pmatrix}$$

$$R_1^{[1]_2} = \begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix}, \quad B_2^{[1]_2} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad R_2^{[1]_2} = \begin{pmatrix} y'_{n-1} \\ y'_n \end{pmatrix}$$

$$D^{[1]_2} = \begin{pmatrix} 0 & \frac{(s-1)(s^2-3s+1)}{(12s)} \\ 0 & \frac{(s(s-4))}{12h} \end{pmatrix}, R_3^{[1]_2} = \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix},$$

$$E^{[1]_2} = \begin{pmatrix} \frac{(-s^2+s+1)}{(12s)} & \frac{-(s^2+s-1)}{12} \\ -\frac{(s(s-2))}{(12s-12)h} & \frac{-(s^3)}{(12s-12)h} \end{pmatrix} \text{ and } R_4^{[1]_2} = \begin{pmatrix} f_{n+s} \\ f_{n+1} \end{pmatrix}$$

Equation (3.14) is multiplied by the inverse of $A^{[1]_2}$ to give

$$I^{[1]_2}Y_m^{[1]_2} = \bar{B}_1^{[1]_2}R_1^{[1]_2} + \bar{B}_2^{[1]_2}R_2^{[1]_2} + h^2\left[\bar{D}^{[1]_2}R_3^{[1]_2} + \bar{E}^{[1]_2}R_4^{[1]_2}\right] \quad (3.15)$$

where

$$I^{[1]_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \bar{B}_1^{[1]_2} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \bar{B}_2^{[1]_2} = \begin{pmatrix} 0 & hs \\ 0 & h \end{pmatrix},$$

$$\bar{D}^{[1]_2} = \begin{pmatrix} 0 & \frac{-(s^2(s-4))}{(12)} \\ 0 & \frac{((4s-1))}{12s} \end{pmatrix} \text{ and } \bar{E}^{[1]_2} = \begin{pmatrix} \frac{(s^2(s-2))}{(12(s-1))} & \frac{s^4}{(12(s-1))} \\ \frac{-1}{12s(s-1)} & \frac{(2s-1)}{(12s-12)} \end{pmatrix}$$

From (3.15), the following equations are obtained

$$y_{n+s} = y_n + hsy'_n - \frac{h^2s^2(s-4)}{12}f_n + \frac{h^2s^2(s-2)}{(12s-12)}f_{n+s} + \frac{h^2s^4}{(12s-12)}f_{n+1} \quad (3.16)$$

$$
\begin{aligned}
y_{n+1} &= y_n + hy_n^{'} - \frac{(h^2(4s-1))}{12s} f_n - \frac{h^2}{(12s(s-1))} f_{n+s} \\
&\quad + \frac{(h^2(2s-1))}{(12(s-1))} f_{n+1}
\end{aligned}
\tag{3.17}
$$

Substituting Equation (3.16) into (3.12) and (3.13) gives the first derivative of the block as below

$$
y_{n+s}^{'} = y_n^{'} - \frac{hs(s-3)}{6} f_n + \frac{(hs(2s-3))}{(6(s-1))} f_{n+s} + \frac{(hs^3)}{(6(s-1))} f_{n+1} \tag{3.18}
$$

$$
y_{n+1}^{'} = y_n^{'} + \frac{h(3s-1)}{(6s)} f_n - \frac{h}{(6s(s-1))} f_{n+s} + \frac{(h(3s-2))}{(6(s-1))} f_{n+1} \tag{3.19}
$$

In order to find properties of the derivative of the method shown above, (3.18) and (3.19) are written in a block form

$$
\acute{Y}_m^{[1]_2} = \acute{B}_2^{[1]_2} R_2^{[1]_2} + h\left[\acute{D}^{[1]_2} R_3^{[1]_2} + \acute{E}^{[1]_2} R_4^{[1]_2}\right] \tag{3.20}
$$

where

$$
\acute{Y}_m^{[1]_2} = \begin{pmatrix} y_{n+s}^{'} \\ y_{n+1}^{'} \end{pmatrix}, \ \acute{B}_2^{[1]_2} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \ \acute{D}^{[1]_2} = \begin{pmatrix} 0 & \frac{-s(s-3)}{6} \\ 0 & \frac{(3s-1)}{(6s)} \end{pmatrix}
$$

$$
\text{and } \acute{E}^{[1]_2} = \begin{pmatrix} \frac{(s(2s-3))}{(6(s-1))} & \frac{(s^3)}{(6(s-1))} \\ \frac{-1}{(6s(s-1))} & \frac{(3s-2)}{(6(s-1))} \end{pmatrix}
$$

3.1.1 Establishing Properties of One Step Hybrid Block Method with Generalised One Off-Step Point for Second Order ODEs

In finding the order of the method, we have extended the methods proposed by Lambert (1973) and Fatunla (1988).

The equation of one step hybrid method for solving (1.1) is given by

$$\sum_{i=0}^{1}\alpha_i y_{n+i}+\sum_{i=s_1,s_2,\cdots}\alpha_i y_{n+i}=h^{\gamma}\sum_{i=0}^{1}\beta_i f_{n+i}+h^{\gamma}\sum_{i=s_1,s_2,\cdots}\beta_i f_{n+i} \tag{3.21}$$

and its associated linear difference operator L associated with (3.21) is defined by

$$\begin{aligned} L[y(x),h] &= \sum_{i=0}^{1}\alpha_i y(x+ih)+\sum_{i=s_1,s_2,\cdots}\alpha_i y(x+ih) \\ &\quad -h^{\gamma}\left(\sum_{i=0}^{1}\beta_i y^{\gamma}(x+ih)+\sum_{i=s_1,s_2,\cdots}\beta_i y^{\gamma}(x+ih)\right) \end{aligned} \tag{3.22}$$

Expanding the functions $y(x+ih)$ and $y^{\gamma}(x+ih)$ in Taylor series about x and collecting like terms gives

$$L[y(x),h]=c_0 y(x)+c_1 hy^{'}(x)+c_2 hy^{''}(x)+\cdots \tag{3.23}$$

Definition 3.1.1. The linear difference operator (3.22) and the associated linear hybrid method (3.21) are said to be of order P, if $c_0=c_1=c_2=\cdots=c_{p+\gamma-1}=0$ and $c_{p+\gamma}\neq 0$ with error constant $c_{p+\gamma}$.

Transforming equation (3.21) to one step hybrid block method gives

$$Y_m^{[q]_\gamma}=\sum_{i=1}^{\gamma}\bar{B}_i^{[q]_\gamma}R_i^{[q]_\gamma}+h^{\gamma}\left[\bar{D}^{[q]_\gamma}R_{\gamma+1}^{[q]_\gamma}+\bar{E}^{[q]_\gamma}R_{\gamma+2}^{[q]_\gamma}\right] \tag{3.24}$$

where $\bar{B}_i^{[q]_\gamma}, i=1,2\cdots\gamma$, $\bar{E}^{[q]_\gamma}$ and $\bar{D}^{[q]_\gamma}$ are $(q+1)\times(q+1)$ matrices of coefficients, q denotes the number of off-step points, γ represents the order of differential equation and

$$
\begin{aligned}
Y_m^{[q]\gamma} &= [y_{n+s_1}, y_{n+s_2}, \cdots, y_{n+1}]^T \\
R_1^{[q]\gamma} &= [y_{n-q}, y_{n-(q-1)}, \cdots, y_{n-1}, y_n]^T \\
&\vdots \\
R_\gamma^{[q]\gamma} &= [y_{n-q}^{\gamma-1}, y_{n-(q-1)}^{\gamma-1}, \cdots, y_{n-1}^{\gamma-1}, y_n^{\gamma-1}]^T \\
R_{\gamma+1}^{[q]\gamma} &= [f_{n-q}, f_{n-(q-1)}, \cdots, f_{n-1}, f_n]^T \\
R_{\gamma+2}^{[q]\gamma} &= [f_{n+s_1}, f_{n+s_2}, \cdots, f_{n+s_q}, f_{n+1}]^T
\end{aligned}
$$

The linear difference operator L associated with equation(3.24) is defined as

$$
L[y(x),h] = Y_m^{[q]\gamma}(X) - \left[\sum_{i=1}^{\gamma} \bar{B}_i^{[q]\gamma} R_i^{[q]\gamma}(X) + h^\gamma \left[\bar{E}^{[q]\gamma} R_{\gamma+1}^{[q]\gamma}(X) + \bar{D}^{[q]\gamma} R_{\gamma+2}^{[q]\gamma}(X)\right]\right] \tag{3.25}
$$

where

$$
\begin{aligned}
Y_m^{[q]\gamma}(X) &= [y(x+s_1h), y(x+s_2h), \cdots, y(x+h)]^T \\
R_1^{[q]\gamma}(X) &= [y(x_{n-q}), y(x_{n-(q-1)}), \cdots, y(x_{n-1}), y(x_n)]^T \\
&\vdots \\
R_\gamma^{[q]\gamma}(X) &= [y^{\gamma-1}(x_{n-q}), y^{\gamma-1}(x_{n-(q-1)}), \cdots, y^{\gamma-1}(x_{n-1}), y^{\gamma-1}(x_n)]^T \\
R_{\gamma+1}^{[q]\gamma}(X) &= [y^\gamma(x_{n-q}), y^\gamma(x_{n-(q-1)}), \cdots, y^\gamma(x_{n-1}), y^\gamma(x_n)]^T \\
R_{\gamma+2}^{[q]\gamma}(X) &= [y^\gamma(x_+s_1h), y^\gamma(x+s_2h), \cdots, y^\gamma(x_+s_qh), y^\gamma(x+h)]^T
\end{aligned}
$$

Expanding components of $Y_m^{[q]\gamma}$ and $R_{\gamma+2}^{[q]\gamma}$ in Taylor's series about x and collecting terms in powers of h gives

$$
L[y(x),h] = \bar{C}_0 y(x) + \bar{C}_1 hy^{'}(x) + \bar{C}_2 hy^{''}(x) + \cdots
$$

Definition 3.1.2. One step hybrid block method (3.24) and associated linear operator (3.25) are said to be of order P, if $\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \cdots = \bar{C}_{P+\gamma-1} = 0$ and $\bar{C}_{P+\gamma} \neq 0$ with error vector constants $\bar{C}_{P+\gamma}$.

Definition 3.1.3. The one step hybrid block method (3.24) is said to be zero stable as $h \to 0$ if its first characteristic polynomial

$$\pi(x,h) = |zI^{[q]_\gamma} - B_1^{[q]_\gamma}| \tag{3.26}$$

having roots z_r such that $|z_r| \leq 1$, and if $|z_r| = 1$ then the multiplicity of z_r must not greater than γ. where $I^{[q]_\gamma}$ is $(q+1)\times(q+1)$ identity matrix and $B_1^{[q]_\gamma}$ is $(q+1)\times(q+1)$ coefficients matrix of y_n

Definition 3.1.4. A numerical method is said to be consistent, if it has order greater than one.

Theorem 3.1 (Henrici, 1962)**.** *The necessary and sufficient conditions for a linear multistep method to be convergent are that it must be consistent and zero stable.*

Region of Absolute Stability

In order to find the region stability of hybrid block (3.24), the method known as boundary locus method proposed by Lambert (1973) and Henrici (1962) is adopted. This is given by $h\bar{(z)} = \frac{\rho(z)}{\sigma(z)}$ where $\rho(z)$ is the first characteristics function and $\sigma(z)$ is the second characteristics function. The test problem of the form $y^\gamma = \lambda^\gamma y$ is substituted into the block (3.24) to give

$$Y_m^{[q]_\gamma} = \sum_{i=1}^{\gamma} \bar{B}_i^{[q]_\gamma} R_i^{[q]_\gamma} + h^\gamma \lambda^\gamma \left[\bar{D}^{[q]_\gamma} Y_{R_{\gamma+1}^{[q]_\gamma}} + \bar{E}^{[q]_\gamma} Y_{R_{\gamma+2}^{[q]_\gamma}} \right] \tag{3.27}$$

This leads to

$$\bar{h}(z,h) = \frac{I^{[q]\gamma} Y_m^{[q]\gamma}(z) - \bar{B}_1^{[q]\gamma} R_1^{[q]\gamma}}{\left[\bar{D}^{[q]\gamma} Y_{R_{\gamma+1}^{[q]\gamma}}(z) + \bar{E}^{[q]\gamma} Y_{R_{\gamma+2}^{[q]\gamma}}(z)\right]} \tag{3.28}$$

where $\bar{h} = \lambda^{\gamma} h^{\gamma}$. Equation (3.28) can be written in Euler's form as

$$\bar{h}(\theta,h) = \frac{I^{[q]\gamma} Y_m^{[q]\gamma}(\theta) - \bar{B}_1^{[q]\gamma} R_1^{[q]\gamma}(\theta)}{[\bar{D} Y_{R_{\gamma+1}^{[q]\gamma}}(\theta) + \bar{E} Y_{R_{\gamma+2}^{[q]\gamma}}(\theta)]} \tag{3.29}$$

where $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Equation (3.29) is called the characteristics matrix. The stability function of the method is obtained by finding the determinant of (3.29).

The following subsections examine the numerical properties of one step hybrid block method with one generalised off-step point which include order, error constant, zero stability, consistency and region of absolute stability.

3.1.1.1 Order of One Step Hybrid Block Method with Generalised One Off-Step Point for Second Order ODEs

In finding the order of the main block (3.15), Definition (3.1.2) is applied by expanding y and f- function in Taylor series about x_n. This is demonstrated below.

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{(s)^j h^j}{j!} y_n^j - y_n - (sh)y_n' + \frac{s^2(s-4)h^2}{12} y_n'' - \frac{s^4}{12s-12} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\ -\frac{s^2(s-2)}{12s-12} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+2}}{j!} y_n^{j+2} \\ \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j - y_n - hy_n' - \frac{(4s-1)h^2}{12s} y_n'' - \frac{2s-1}{12s-12} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\ +\frac{1}{s(12s-12)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+2}}{j!} y_n^{j+2} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$

Comparing the coefficients of h^j and y^j gives

$$\bar{C}_0 = \begin{bmatrix} 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} s-s \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} \frac{s^2}{2!} + \frac{s^2(s-4)}{12} - \frac{s^4}{12s-12} - \frac{s^2(s-2)}{12s-12} \\ \frac{1}{2!} - \frac{(4s-1)}{12s} - \frac{2s-1}{12s-12} + \frac{1}{s(12s-12)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} \frac{s^3}{3!} - \frac{s^4}{12s-12} - \frac{s^2(s-2)}{12s-12}\frac{s}{1!} \\ \frac{1}{3!} - \frac{2s-1}{12s-12} + \frac{1}{s(12s-12)}\frac{s}{1!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_4 = \begin{bmatrix} \frac{s^4}{4!} - \frac{s^4}{12s-12}\frac{1}{2!} - \frac{s^2(s-2)}{12s-12}\frac{s^2}{2!} \\ \frac{1}{4!} - \frac{2s-1}{12s-12}\frac{1}{2!} + \frac{1}{s(12s-12)}\frac{s^2}{2!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_5 = \begin{bmatrix} \frac{s^5}{5!} - \frac{s^4}{12s-12}\frac{1}{3!} - \frac{s^2(s-2)}{12s-12}\frac{s^3}{3!} \\ \frac{1}{5!} - \frac{2s-1}{12s-12}\frac{1}{3!} + \frac{1}{s(12s-12)}\frac{s^3}{3!} \end{bmatrix} = \begin{bmatrix} \frac{120s^4-48s^3}{8640} \\ \frac{120s-48}{8640} \end{bmatrix}$$

Hence, by comparing the coefficient of h, it is discovered that the block method (3.15) has order $[3,3]^T$ together with the error constants $\left[\frac{120s^4-48s^5}{8640}, \frac{120s-48}{8640}\right]^T$ for all $s \in (0,1) \setminus \{s = \frac{48}{120}\}$

Expanding $y^{'}$ and f function in Taylor series in finding the order of the derivative block yields

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{(s)^j h^j}{j!} y_n^{j+1} - y_n^{'} + \frac{s(s-3)h}{6} y_n^{''} - \frac{2s^3}{12s-12} \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+2} \\ -\frac{s(2s-3)}{6(s-1)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+1}}{j!} y_n^{j+2} \\ \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{j+1} - y_n^{'} - \frac{(3s-1)h}{6s} y_n^{''} - \frac{3s-2}{6(s-1)} \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+2} \\ + \frac{1}{6s(s-1)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+1}}{j!} y_n^{j+2} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \\ 0 \end{bmatrix}$$

Comparing the coefficients of h^j and y^j. This gives

$$\bar{C}_0 = \begin{bmatrix} 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} \frac{s}{1!} + \frac{s(s-3)}{6} - \frac{2s^3}{12s-12} - \frac{s(2s-3)}{6(s-1)} \\ \frac{1}{1!} - \frac{(3s-1)}{6s} - \frac{3s-2}{6(s-1)} + \frac{1}{6s(s-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} \frac{s^2}{2!} - \frac{2s^3}{12s-12} - \frac{s(2s-3)}{6(s-1)}\frac{s}{1!} \\ \frac{1}{2!} - \frac{3s-2}{6(s-1)} + \frac{1}{6s(s-1)}\frac{s}{1!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_4 = \begin{bmatrix} \frac{s^3}{3!} - \frac{2s^3}{12s-12}\frac{1}{2!} - \frac{s(2s-3)}{6(s-1)}\frac{s^2}{2!} \\ \frac{1}{3!} - \frac{3s-2}{6(s-1)}\frac{1}{2!} + \frac{1}{6s(s-1)}\frac{s^2}{2!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_5 = \begin{bmatrix} \frac{s^4}{4!} - \frac{2s^3}{12s-12}\frac{1}{3!} - \frac{s(2s-3)}{6(s-1)}\frac{s^3}{3!} \\ \frac{1}{4!} - \frac{3s-2}{6(s-1)}\frac{1}{3!} + \frac{1}{6s(s-1)}\frac{s^3}{3!} \end{bmatrix} = \begin{bmatrix} \frac{2s^3-s^4}{72} \\ \frac{2s-1}{72} \end{bmatrix}$$

Therefore, the derivative block (3.20) has order $[3,3]^T$ with the error constants vector $\left[\frac{2s^3-s^4}{72}, \frac{2s-1}{72}\right]^T$ for all $s \in (0,1) \setminus \{s = \frac{1}{2}\}$.

3.1.1.2 Zero Stability of One Step Hybrid Block Method with Generalised One Off-Step Point for Second Order ODEs

In order to find the zero-stability of the block (3.15), we only put into consideration the first characteristic polynomial according to Definition (3.1.3), that is

$$
\begin{aligned}
\Pi(z) &= |z\, I^{[1]_2} - \bar{B}_1^{[1]_2}| \\
&= \left| z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right| \\
&= z(z-1) = 0
\end{aligned}
$$

which implies $z = 0, 1$. The characteristic polynomial for the derivative block (3.20) is given by

$$
\begin{aligned}
\Pi(z) &= |z\, I^{[1]_2} - \acute{B}_2^{[1]_2}| \\
&= \left| z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right| \\
&= z(z-1) = 0
\end{aligned}
$$

which also implies $z = 0, 1$. Hence, the conditions in Definition (3.1.3) are satisfied. Therefore, the block method and its derivative are zero stable.

3.1.1.3 Consistency and Convergent of One Step Hybrid Block Method with Generalised One Off-Step Point for Second Order ODEs

The block method (3.15) and its derivatives (3.20) are consistent and convergent as stated in Definition (3.1.4) and Theorem (3.1)

3.1.1.4 Region of Absolute Stability One Step Hybrid Block Method with Generalised One Off-Step Point for Second Order ODEs

Applying (3.29) for one step hybrid block with one generalised off-step points(3.15), it gives

$$\bar{h}(\theta,h)=\left|\frac{I^{[1]_2}Y_m^{[1]_2}(\theta)-\bar{B}_1^{[1]_2}R_1^{[1]_2}(\theta)}{[\bar{D}^{[1]_2}Y_{R_3^{[1]_2}}(\theta)+\bar{E}^{[1]_2}Y_{R_4^{[1]_2}}(\theta)]}\right| \tag{3.30}$$

where

$$I^{[1]_2}Y_m^{[1]_2}(\theta)=\begin{bmatrix} e^{si\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

$$\bar{B}_1^{[1]_2}R_1^{[1]_2}(\theta)=\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\bar{D}^{[1]_2}Y_{R_3^{[1]_2}}(\theta)=\begin{bmatrix} 0 & \frac{-(s^2(s-4))}{(12)} \\ 0 & \frac{(4s-1)}{(12s)} \end{bmatrix}$$

$$\bar{E}^{[1]_2}Y_{R_4^{[1]_2}}(\theta)=\begin{bmatrix} \frac{s^2(s-2)}{(12(s-1))}e^{si\theta} & \frac{(s^4)}{(12(s-1))}e^{i\theta} \\ \frac{-1}{(12s(s-1))}e^{si\theta} & \frac{(2s-1)}{(12(s-1))}e^{i\theta} \end{bmatrix}$$

Simplifying (3.30) and then finding its determinant yields

$$\bar{h}(\theta,h)=\frac{(72e^{i\theta}-72)}{(s^2e^{i\theta}-3s+2s^2)}$$

Some time it is not possible to quantify the region of stability in meaning full way other than by presenting diagrams showing the boundary of region plotted in the complex plane, if such diagram is not available and we are not prepared to spend computing efforts in finding it, the knowledge of interval of stability which is the intersection of the boundary of region stability with real line, can still give some indication of safe choice for h (Lambert, 1973). In order to obtain this interval, expanding $\bar{h}(\theta,h)$ trigonometrically and equating the imaginary part to zero, the equation of the absolute stability region for (3.15) is obtained as below

$$\bar{h}(\theta,h)=\frac{(72\cos(\theta)-72)}{(s^2\cos(\theta)-3s+2s^2)} \tag{3.31}$$

3.2 Derivation of One Step Hybrid Block Method with Generalised Two Off-Step Points for Second Order ODEs

In deriving this method, Equation (3.1) is interpolated at $x_{n+s},\ x_{n+r}$ and Equation (3.5) is collocated at all points in the selected interval i.e $x_n,\ x_{n+s},\ x_{n+r},\ x_{n+1}$ as illustrated in Figure (3.2) below.

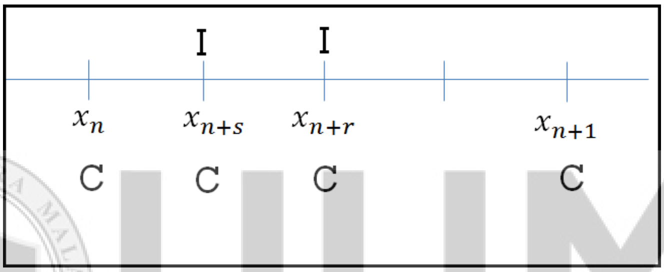

Figure 3.2. One step hybrid block method with generalised two off-step points for solving second order ODEs.

From above figure, $v=2$ and $m=4$. As a result, we get the following equations

$$\begin{aligned}
y_{n+s} &= a_0+a_1s+a_2s^2+a_3s^3+a_4s^4+a_5s^5.\\
y_{n+r} &= a_0+a_1r+a_2r^2+a_3r^3+a_4r^4+a_5r^5.\\
f_n &= \frac{2}{h^2}a_2\\
f_{n+s} &= \frac{2}{h^2}a_2+\frac{6s}{h^2}a_3+\frac{12s^2}{h^2}a_4+\frac{20s^3}{h^2}a_5\\
f_{n+r} &= \frac{2}{h^2}a_2+\frac{6r}{h^2}a_3+\frac{12r^2}{h^2}a_4+\frac{20r^3}{h^2}a_5\\
f_{n+1} &= \frac{2}{h^2}a_2+\frac{6}{h^2}a_3+\frac{12}{h^2}a_4+\frac{20}{h^2}a_5
\end{aligned} \tag{3.32}$$

Re-writing Equations (3.32) in a matrix form, we obtain

$$\begin{pmatrix} 1 & s & s^2 & s^3 & s^4 & s^5 \\ 1 & r & r^2 & r^3 & r^4 & r^5 \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{6s}{h^2} & \frac{12s^2}{h^2} & \frac{20s^3}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6r}{h^2} & \frac{12r^2}{h^2} & \frac{20r^3}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_{n+s} \\ y_{n+r} \\ f_n \\ f_{n+s} \\ f_{n+r} \\ f_{n+1} \end{pmatrix} \tag{3.33}$$

Applying Gaussian elimination method on (3.33) gives the values of $a_i's, i = 0(1)5$ as follows

$$\begin{aligned} a_0 &= \frac{r}{(r-s)} y_{n+s} - \frac{s}{(r-s)} y_{n+r} \\ &- \frac{h^2 rs(r-2s)(2r-s)(r+s)}{60(r-1)(s-1)} f_{n+1} \\ &+ \frac{h^2(2r^3 - 3r^2 s - 5r^2 - 3rs^2 + 15rs + 2s^3 - 5s^2)}{60} f_n \\ &+ \frac{h^2 r(2r^3 + 2r^2 s - 5r^2 + 2rs^2 - 5rs - 3s^3 + 5s^2)}{60(r-s)(s-1)} f_{n+s} \\ &+ \frac{h^2 s(3r^3 - 2r^2 s - 5r^2 - 2rs^2 + 5rs - 2s^3 + 5s^2)}{(60(r-s)(r-1))} f_{n+r} \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{1}{(r-s)} y_{n+r} + \frac{h^2(2r^4 - 3r^3 s - 3r^2 s^2 - 3rs^3 + 2s^4)}{(60(r-1)(s-1))} f_{n+1} - \frac{1}{(r-s)} y_{n+s} \\ &- \frac{h^2(2r^4 - 3r^3 s - 5r^3 - 3r^2 s^2 + 15r^2 s - 3rs^3 + 15rs^2 + 2s^4 - 5s^3)}{(60rs)} f_n \\ &- \frac{h^2(3r^4 - 2r^3 s - 5r^3 - 2r^2 s^2 + 5r^2 s - 2rs^3 + 5rs^2 - 2s^4 + 5s^3)}{(60r(r-s)(r-1))} f_{n+r} \\ &- \frac{h^2(2r^4 + 2r^3 s - 5r^3 + 2r^2 s^2 - 5r^2 s + 2rs^3 - 5rs^2 - 3s^4 + 5s^3)}{(60s(r-s)(s-1))} f_{n+s} \\ a_2 &= \frac{h^2}{2} f_n \\ a_3 &= \frac{-(h^2(r^2 s - r^2 - rs^2 + r + s^2 - s))}{(20rs(r-s)(r-1)(s-1))} f_n + \frac{(h^2(r - r^2))}{(20rs(r-s)(r-1)(s-1))} f_{n+s} \\ &+ \frac{-(h^2(rs^2 - r^2 s))}{(20rs(r-s)(r-1)(s-1))} f_{n+1} + \frac{-(h^2(s - s^2))}{(20rs(r-s)(r-1)(s-1))} f_{n+r} \end{aligned}$$

$$a_4 = \frac{(h^2(rs^3 - r^3 s))}{(12rs(r-s)(r-1)(s-1))} f_{n+1} + \frac{(h^2(r^3 s - r^3 - rs^3 + r + s^3 - s))}{(12rs(r-s)(r-1)(s-1))} f_n$$
$$+ \frac{(h^2(s - s^3)}{(12rs(r-s)(r-1)(s-1))} f_{n+r} + \frac{-(h^2(r - r^3))}{(12rs(r-s)(r-1)(s-1))} f_{n+s}$$

$$a_5 = \frac{-(h^2(rs^2 - r^2 s))}{(20rs(r-s)(r-1)(s-1))} f_{n+1} + \frac{(h^2(r - r^2))}{(20rs(r-s)(r-1)(s-1))} f_{n+s}$$
$$+ \frac{-(h^2(r^2 s - r^2 - rs^2 + r + s^2 - s))}{(20rs(r-s)(r-1)(s-1))} f_n + \frac{-(h^2(s - s^2))}{(20rs(r-s)(r-1)(s-1))} f_{n+r}$$

The values of $a_i' s$ are then substituted back into equation (3.1) to produce a continuous implicit hybrid method of the form

$$y(x) = \sum_{i=s,r} \alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \beta_i(x) f_{n+i} + \sum_{i=s,r} \beta_i(x) f_{n+i} \tag{3.34}$$

Differentiating (3.34) once yields

$$y'(x) = \sum_{i=s,r} \frac{\partial}{\partial x} \alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \frac{\partial}{\partial x} \beta_i(x) f_{n+i} + \sum_{i=s,r} \frac{\partial}{\partial x} \beta_i(x) f_{n+i} \tag{3.35}$$

where

$$\alpha_r = \frac{-(x_n - x + hs)}{h(r-s)}$$

$$\alpha_s = \frac{(x_n - x + hr)}{h(r-s)}$$

$$\beta_0 = \frac{(x_n - x + hs)(x_n - x + hr)}{(60rsh^3)} (2h^3 r^3 - 3h^3 r^2 s - 5h^3 r^2 - 3h^3 rs^2 + 15h^3 rs + 3x_n^3$$
$$-5h^3 s^2 + 2h^2 r^2 x - 2h^2 r^2 x_n - 3h^2 rsx + 3h^2 rsx_n - 5h^2 rx + 5h^2 rx_n + 2h^2 s^2 x + 5hx^2$$
$$-5h^2 sx + 5h^2 sx_n + 2hrx_n^2 + 2hsx^2 - 4hsxx_n + 2hsx_n^2 + 5hx_n^2 - 3x^3 + 2h^3 s^3 + 2hrx^2$$
$$-4hrxx_n + 9x^2 x_n - 9xx_n^2 - 2h^2 s^2 x_n - 10hxx_n)$$

$$\beta_s = \frac{(x_n - x + hs)(x_n - x + hr)}{(60h^3 s(s-1)(r-s))}(2h^3r^3 + 2h^3r^2s - 5h^3r^2 + 2h^3rs^2 - 5h^3rs - 3h^3s^3$$

$$+3x_n^3 + 5h^3s^2 + 2h^2r^2x - 2h^2r^2x_n + 2h^2rsx - 2h^2rsx_n - 5h^2rx + 5h^2rx_n - 3h^2s^2x$$

$$-3x^3 + 5h^2sx + 2hrx^2 - 4hrxx_n + 2hrx_n^2 - 3hsx^2 + 6hsxx_n - 3hsx_n^2 + 5hx^2 + 9x^2x_n$$

$$-5h^2sx_n - 10hxx_n + 3h^2s^2x_n + 5hx_n^2 - 9xx_n^2)$$

$$\beta_r = \frac{(x_n - x + hs)(x_n - x + hr)}{(60h^3 r(r-1)(r-s))}(3h^3r^3 - 2h^3r^2s - 5h^3r^2 - 2h^3rs^2 + 5h^3rs - 2h^3s^3$$

$$+5h^3s^2 + 3h^2r^2x - 3h^2r^2x_n - 2h^2rsx + 2h^2rsx_n - 5h^2rx + 5h^2rx_n - 2h^2s^2x + 3x^3$$

$$+5h^2sx - 5h^2sx_n + 3hrx^2 - 6hrxx_n + 3hrx_n^2 - 2hsx^2 + 4hsxx_n - 2hsx_n^2 - 5hx^2$$

$$-5hx_n^2 + 10hxx_n + 2h^2s^2x_n - 9x^2x_n + 9xx_n^2 - 3x_n^3)$$

$$\beta_1 = -\frac{(x_n - x + hs)(x_n - x + hr)}{(60h^3(s-1)(r-1))}(2h^3r^3 - 3h^3r^2s - 3h^3rs^2 + 2h^3s^3 + 2h^2r^2x$$

$$-2h^2r^2x_n - 3h^2rsx + 3h^2rsx_n + 2h^2s^2x - 2h^2s^2x_n + 2hrx^2 - 4hrxx_n + 2hrx_n^2$$

$$-4hsxx_n + 2hsx_n^2 + 2hsx^2 + 9x^2x_n - 9xx_n^2 + 3x_n^3 - 3x^3)$$

Evaluating (3.34) at non-interpolating points x_n and x_{n+1} produces the following schemes

$$\begin{aligned}
& y_n - \frac{r}{(r-s)}y_{n+s} + \frac{s}{(r-s)}y_{n+r} \\
&= \frac{h^2(2r^3 - 3r^2s - 5r^2 - 3rs^2 + 15rs + 2s^3 - 5s^2)}{60}f_n \\
&+ \frac{h^2r(2r^3 + 2r^2s - 5r^2 + 2rs^2 - 5rs - 3s^3 + 5s^2)}{(60(r-s)(s-1))}f_{n+s} \\
&+ \frac{h^2s(3r^3 - 2r^2s - 5r^2 - 2rs^2 + 5rs - 2s^3 + 5s^2)}{(60(r-s)(r-1))}f_{n+r} \\
&- \frac{h^2rs(r-2s)(2r-s)(r+s))}{(60(r-1)(s-1))}f_{n+1} \qquad (3.36)
\end{aligned}$$

$$
\begin{aligned}
&y_{n+1}-\frac{(r-1)}{(r-s)}y_{n+s}+\frac{(s-1)}{(r-s)}y_{n+r}\\
&=\frac{h^2(s-1)(r-1)(r+s-2)(2r-s-1)(r-2s+1)}{(60rs)}f_n\\
&+\frac{h^2(r-1)(2r^3+2r^2s-3r^2+2rs^2-3rs-3r-3s^3+2s^2+2s+2)}{(60s(r-s))}f_{n+s}\\
&+\frac{h^2(s-1)(3r^3-2r^2s-2r^2-2rs^2+3rs-2r-2s^3+3s^2+3s-2)}{(60r(r-s))}f_{n+r}\\
&-\frac{h^2(2r^3-3r^2s+2r^2-3rs^2-3rs+2r+2s^3+2s^2+2s-3)}{60}f_{n+1} \qquad (3.37)
\end{aligned}
$$

Similarly, evaluating (3.35) at all points i.e x_n, x_{n+s}, x_{n+r} and x_{n+1} gives

$$
\begin{aligned}
&y_n^{'}+\frac{1}{(h(r-s))}y_{n+s}-\frac{1}{(h(r-s))}y_{n+r}\\
&=-\frac{h(2r^4-3r^3s-5r^3-3r^2s^2+15r^2s-3rs^3+15rs^2+2s^4-5s^3)}{(60rs)}f_n\\
&-\frac{h(2r^4+2r^3s-5r^3+2r^2s^2-5r^2s+2rs^3-5rs^2-3s^4+5s^3)}{(60s(r-s)(s-1))}f_{n+s}\\
&-\frac{h(3r^4-2r^3s-5r^3-2r^2s^2+5r^2s-2rs^3+5rs^2-2s^4+5s^3)}{(60r(r-s)(r-1))}f_{n+r}\\
&\frac{h(2r^4-3r^3s-3r^2s^2-3rs^3+2s^4)}{(60(r-1)(s-1))}f_{n+1} \qquad (3.38)
\end{aligned}
$$

$$
\begin{aligned}
&y_{n+s}^{'}+\frac{1}{(h(r-s))}y_{n+s}-\frac{1}{(h(r-s))}y_{n+r}\\
&=-\frac{(h(2r+3s-5)(r-s)^3)}{(60rs)}f_n\\
&-\frac{(h(r-s)(3r^2+4rs-5r+3s^2-5s))}{(60r(r-1))}f_{n+r}\\
&-\frac{h(r-s)(2r^2+6rs-5r+12s^2-15s)}{(60s(s-1))}f_{n+s}\\
&+\frac{(h(2r+3s)(r-s)^3)}{(60(r-1)(s-1))}f_{n+1} \qquad (3.39)
\end{aligned}
$$

$$
\begin{aligned}
& y'_{n+r} + \frac{1}{(h(r-s))} y_{n+s} - \frac{1}{(h(r-s))} y_{n+r} \\
& = \frac{h(3r+2s-5)(r-s)^3}{(60rs)} f_n \\
& + \frac{h(r-s)(3r^2+4rs-5r+3s^2-5s)}{(60s(s-1))} f_{n+s} \\
& + \frac{h(r-s)(12r^2+6rs-15r+2s^2-5s)}{60r(r-1)} f_{n+r} \\
& - \frac{h(3r+2s)(r-s)^3}{(60(r-1)(s-1))} f_{n+1}
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
& y'_{n+1} + \frac{1}{h(r-s)} y_{n+s} - \frac{1}{h(r-s)} y_{n+r} \\
& = \frac{-h}{60rs}(2r^4 - 3r^3 s - 5r^3 - 3r^2 s^2 + 15r^2 s - 3rs^3 + 15rs^2 - 30rs + 10r + 2s^4 \\
& -5s^3 + 10s - 5) f_n \\
& - \frac{h(2r^4 + 2r^3 s - 5r^3 + 2r^2 s^2 - 5r^2 s + 2rs^3 - 5rs^2 + 10r - 3s^4 + 5s^3 - 5)}{(60s(r-s)(s-1))} f_{n+s} \\
& - \frac{h(3r^4 - 2r^3 s - 5r^3 - 2r^2 s^2 + 5r^2 s - 2rs^3 + 5rs^2 - 2s^4 + 5s^3 - 10s + 5)}{60r(r-s)(r-1)} f_{n+r} \\
& + \frac{h(2r^4 - 3r^3 s - 3r^2 s^2 - 3rs^3 + 30rs - 20r + 2s^4 - 20s + 15)}{60(r-1)(s-1)} f_{n+1}
\end{aligned}
\tag{3.41}
$$

Using the same procedure as mentioned in section 3.1, we combine (3.36) and (3.38) to form a block method one step with two off-step pointsfor second ODEs as below

$$
A^{[2]_2} Y_m^{[2]_2} \;=\; B_1^{[2]_2} R_1^{[2]_2} + B_2^{[2]_2} R_2^{[2]_2} + h^2 \left[D^{[2]_2} R_3^{[2]_2} + E^{[2]_2} R_4^{[2]_2} \right] \tag{3.42}
$$

where

$$
A^{[2]_2} = \begin{pmatrix} \frac{-r}{r-s} & \frac{s}{r-s} & 0 \\ \frac{1-r}{r-s} & \frac{s-1}{r-s} & 1 \\ \frac{1}{h(r-s)} & \frac{-1}{h(r-s)} & 0 \end{pmatrix}, \; Y_m^{[2]_2} = \begin{pmatrix} y_{n+s} \\ y_{n+r} \\ y_{n+1} \end{pmatrix}, \; B_1^{[2]_2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
R_1^{[2]_2} = \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}, B_2^{[2]_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, R_2^{[2]_2} = \begin{pmatrix} y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}
$$

$$
D^{[2]_2} = \begin{pmatrix} 0 & 0 & D_{13}^{[2]_2} \\ 0 & 0 & D_{23}^{[2]_2} \\ 0 & 0 & D_{33}^{[2]_2} \end{pmatrix}, \quad R_3^{[2]_2} = \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}, E^{[2]_2} = \begin{pmatrix} E_{11}^{[2]_2} & E_{12}^{[2]_2} & E_{13}^{[2]_2} \\ E_{21}^{[2]_2} & E_{22}^{[2]_2} & E_{23}^{[2]_2} \\ E_{31}^{[2]_2} & E_{32}^{[2]_2} & E_{33}^{[2]_2} \end{pmatrix}
$$

and $R_4^{[2]_2} = \begin{pmatrix} f_{n+s} \\ f_{n+r} \\ f_{n+1} \end{pmatrix}$

The elements of of $D^{[2]_2}$ and $E^{[2]_2}$ are given by

$$
\begin{aligned}
D_{13}^{[2]_2} &= \frac{(2r^3-3r^2s-5r^2-3rs^2+15rs+2s^3-5s^2)}{60} \\
D_{23}^{[2]_2} &= \frac{(r-1)(s-1)(r+s-2)(r-2s+1)(2r-s-1)}{60rs} \\
D_{33}^{[2]_2} &= \frac{-(2r^4-3r^3s-5r^3-3r^2s^2+15r^2s-3rs^3+15rs^2+2s^4-5s^3)}{60hrs} \\
E_{11}^{[2]_2} &= \frac{r(2r^3+2r^2s-5r^2+2rs^2-5rs-3s^3+5s^2)}{60(r-s)(s-1)} \\
E_{12}^{[2]_2} &= \frac{s(3r^3-2r^2s-5r^2-2rs^2+5rs-2s^3+5s^2)}{60(r-s)(s-1)} \\
E_{13}^{[2]_2} &= \frac{-rs(2r-s)(r+s)(r-2s)}{60(r-1)(s-1)} \\
E_{21}^{[2]_2} &= \frac{(r-1)(2r^3+2r^2s-3r^2+2rs^2-3rs\breve{}3r-3s^3+2s^2+2s+2)}{60s(r-s)} \\
E_{22}^{[2]_2} &= \frac{(s-1)(3r^3-2r^2s-2r^2-2rs^2+3rs-2r-2s^3+3s^2+3s-2)}{60r(r-s)} \\
E_{23}^{[2]_2} &= \frac{-(2r^3-3r^2s+2r^2-3rs^2-3rs+2r+2s^3+2s^2+2s-3)}{60} \\
E_{31}^{[2]_2} &= \frac{-(2r^4+2r^3s-5r^3+2r^2s^2-5r^2s+2rs^3-5rs^2-3s^4+5s^3)}{60hs(r-s)(s-1)} \\
E_{32}^{[2]_2} &= \frac{-(3r^4-2r^3s-5r^3-2r^2s^2+5r^2s-2rs^3+5rs^2-2s^4+5s^3)}{60hr(r-s)(s-1)} \\
E_{33}^{[2]_2} &= \frac{(2r^4-3r^3s-3r^2s^2-3rs^3+2s^4)}{60h(r-1)(s-1)}
\end{aligned}
$$

Multiplying Equation (3.42) by inverse of $A^{[2]_2}$ yields

$$I^{[2]_2} Y_m^{[2]_2} = \bar{B}_1^{[2]_2} R_1^{[2]_2} + \bar{B}_2^{[2]_2} R_2^{[2]_2} + h^2 \left[\bar{D}^{[2]_2} R_3^{[2]_2} + \bar{E}^{[2]_2} R_4^{[2]_2} \right] \tag{3.43}$$

where

$$\bar{B}_1^{[2]_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \bar{B}_2^{[2]_2} = \begin{pmatrix} 0 & 0 & sh \\ 0 & 0 & rh \\ 0 & 0 & h \end{pmatrix},$$

$$\bar{D}^{[2]_2} = \begin{pmatrix} 0 & 0 & \frac{-s^2(5s-20r+5rs-2s^2)}{60r} \\ 0 & 0 & \frac{r^2(20s-5r-5rs+2r^2)}{60s} \\ 0 & 0 & \frac{(20rs-5s-5r+2)}{60rs} \end{pmatrix},$$

$$\bar{E}^{[2]_2} = \begin{pmatrix} \frac{s^2(5s-10r+5rs-3s^2)}{60(r-1)(s-1)} & \frac{-s^4(2s-5)}{60r(r-1)(r-s)} & \frac{s^4(5r-2s)}{60(r-1)(s-1)} \\ \frac{r^4(2r-5)}{60s(s-1)(r-s)} & \frac{r^2(10s-5r-5rs+3r^2)}{60(r-s)(r-1)} & \frac{-r^4(2r-5s)}{60(r-1)(s-1)} \\ \frac{-(5r-2)}{60s(s-1)(r-s)} & \frac{(5s-2)}{60r(r-1)(r-s)} & \frac{(10rs-5s-5r+3)}{60(s-1)(r-1)} \end{pmatrix}$$

This gives

$$\begin{aligned} y_{n+s} &= y_n + hsy_n' - \frac{h^2 s^2(5s-20r+5rs-2s^2)}{60r} f_n + \frac{h^2 s^2(5s-10r+5rs-3s^2)}{60(r-s)(s-1)} f_{n+s} \\ &\quad - \frac{h^2 s^4(2s-5)}{60r(r-s)(r-1)} f_{n+r} + \frac{h^2 s^4(5r-2s)}{60(r-1)(s-1)} f_{n+1} \end{aligned} \tag{3.44}$$

$$\begin{aligned} y_{n+r} &= y_n + hry_n' + \frac{h^2 r^2(20s-5r-5rs+2r^2)}{60s} f_n + \frac{h^2 r^4(2r-5)}{60s(r-s)(s-1)} f_{n+s} \\ &\quad + \frac{h^2 r^2(10s-5r-5rs+3r^2)}{60(r-s)(r-1)} f_{n+r} - \frac{h^2 r^4(2r-5s)}{60(r-1)(s-1)} f_{n+1} \end{aligned} \tag{3.45}$$

$$\begin{aligned} y_{n+1} &= y_n + hy_n' + \frac{h^2(20rs-5s-5r+2)}{60sr} f_n - \frac{h^2(5r-2)}{60s(r-s)(s-1)} f_{n+s} \\ &\quad + \frac{h^2(5s-2)}{60r(r-s)(r-1)} f_{n+r} + \frac{h^2(10sr-5s-5r+3)}{60(r-1)(s-1)} f_{n+1} \end{aligned} \tag{3.46}$$

Substituting equation (3.44) and (3.46) into (3.39) – (3.41) produces the block derivative as below

$$
\begin{aligned}
y'_{n+s} &= y'_n - \frac{hs(2s-6r+2rs-s^2)}{12r} f_n + \frac{hs(4s-6r+4rs-3s^2)}{12(r-s)(s-1)} f_{n+s} \\
&\quad - \frac{hs^3(s-2)}{12r(r-s)(r-1)} f_{n+r} + \frac{hs^3(2r-s)}{12(r-1)(s-1)} f_{n+1}
\end{aligned} \tag{3.47}
$$

$$
\begin{aligned}
y'_{n+r} &= y'_n + \frac{hr(6s-2r-2rs+r^2)}{12s} f_n + \frac{hr^3(r-2)}{12s(r-s)(s-1)} f_{n+s} \\
&\quad + \frac{hr(6s-4r-4rs+3r^2)}{12(r-s)(r-1)} f_{n+r} - \frac{hr^3(r-2s)}{12(r-1)(s-1)} f_{n+1}
\end{aligned} \tag{3.48}
$$

$$
\begin{aligned}
y'_{n+1} &= y'_n + \frac{h(6rs-2s-2r+1)}{12rs} f_n - \frac{h(2r-1)}{12s(r-s)(s-1)} f_{n+s} \\
&\quad + \frac{h(2s-1)}{12r(r-s)(r-1)} f_{n+r} + \frac{h(6rs-4s-4r+3)}{12(r-1)(s-1)} f_{n+1}
\end{aligned} \tag{3.49}
$$

The derivative of the block can be represent in equation of the form

$$
\acute{Y}_m^{[2]_2} = \acute{B}_2^{[2]_2} R_2^{[2]_2} + h\left[\acute{D}^{[2]_2} R_3^{[2]_2} + \acute{E}^{[2]_2} R_4^{[2]_2}\right] \tag{3.50}
$$

where

$$
\acute{Y}_m^{[2]_2} = \begin{pmatrix} y'_{n+s} \\ y'_{n+r} \\ y'_{n+1} \end{pmatrix}, \; \acute{B}_2^{[2]_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; \acute{D}^{[2]_2} = \begin{pmatrix} 0 & 0 & \frac{-s(2s-6r+2sr-s^2)}{12r} \\ 0 & 0 & \frac{r(6s-2r-2sr+r^2)}{12s} \\ 0 & 0 & \frac{(6rs-2s-2r+1)}{12rs} \end{pmatrix},
$$

$$
\text{and } \acute{E}^{[2]_2} = \begin{pmatrix} \frac{s(4s-6r+4rs-3s^2)}{12(r-s)(s-1)} & \frac{-s^3(s-2)}{12r(r-1)(r-s)} & \frac{s^3(2r-s)}{12(r-1)(s-1)} \\ \frac{r^3(r-2)}{12s(s-1)(r-s)} & \frac{r(6s-4r-4rs+3r^2)}{12(r-1)(r-s)} & \frac{-r^3(r-2s)}{12(r-1)(s-1)} \\ \frac{-(2r-1)}{12s(r-s)(s-1)} & \frac{(2s-1)}{12r(r-s)(s-1)} & \frac{(6rs-4s-4r+3)}{12(r-1)(s-1)} \end{pmatrix}
$$

3.2.1 Establishing Properties of One step Hybrid Block Method with Generalised Two Off-Step Points for Second Order ODEs

The order, error constant, zero stability, consistency, convergence and region of absolute stability of the method are examined in this section.

3.2.1.1 Order of One Step Hybrid Block Method with Generalised Two Off-Step Points for Second Order ODEs

The method used in section (3.1.1.1) is applied in finding the order of hybrid block method with two generalised off-step points(3.43) as demonstrated below.

$$\begin{bmatrix} \begin{array}{c} \sum_{j=0}^{\infty}\frac{(s)^j h^j}{j!}y_n^j - y_n - shy_n' + \frac{h^2s^2(5s-20r+5rs-2s^2)}{60r}y_n'' \\ -\frac{s^4(5r-2s)}{60(r-1)(s-1)}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} + \frac{s^4(2s-5)}{60r(r-s)(r-1)}\sum_{j=0}^{\infty}\frac{(r^j)h^{j+2}}{j!}y_n^{j+2} \\ -\frac{s^2(5s-10r+5rs-3s^2)}{60(r-s)(s-1)}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+2}}{j!}y_n^{j+2} \end{array} \\ \begin{array}{c} \sum_{j=0}^{\infty}\frac{(r)^j h^j}{j!}y_n^j - y_n - rhy_n' - \frac{h^2r^2(20s-5r-5rs+2r^2)}{60s}y_n'' \\ +\frac{r^4(2r-5s)}{60(r-1)(s-1)}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} - \frac{r^2(10s-5r-5rs+3r^2)}{60(r-s)(r-1)}\sum_{j=0}^{\infty}\frac{(r)^j h^{j+2}}{j!}y_n^{j+2} \\ -\frac{r^4(2r-5)}{60s(r-s)(s-1)}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+2}}{j!}y_n^{j+2} \end{array} \\ \begin{array}{c} \sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j - y_n - hy_n' - \frac{h^2(20rs-5s-5r+2)}{60sr}y_n'' \\ -\frac{(10sr-5s-5r+3)}{60(r-1)(s-1)}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} - \frac{(5s-2)}{60r(r-s)(r-1)}\sum_{j=0}^{\infty}\frac{(r)^j h^{j+2}}{j!}y_n^{j+2} \\ +\frac{(5r-2)}{60s(r-s)(s-1)}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+2}}{j!}y_n^{j+2} \end{array} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Comparing the coefficients of h^j and y^j yields

$$\bar{C}_0 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} s-s \\ r-r \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} \frac{(s)^2}{2!} + \frac{h^2s^2(5s-20r+5rs-2s^2)}{60r} - \frac{s^4(5r-2s)}{60(r-1)(s-1)} + \frac{s^4(2s-5)}{60r(r-s)(r-1)} \\ -\frac{s^2(5s-10r+5rs-3s^2)}{60(r-s)(s-1)} \\ \frac{(r)^2}{2!} - \frac{r^2(20s-5r-5rs+2r^2)}{60s} + \frac{r^4(2r-5s)}{60(r-1)(s-1)} - \frac{r^2(10s-5rs+3r^2)}{60(r-s)(r-1)} \\ -\frac{r^4(2r-5)}{60s(r-s)(s-1)} \\ \frac{1}{2!} - \frac{h^2(20rs-5s-5r+2)}{60sr} - \frac{(10sr-5s-5r+3)}{60(r-1)(s-1)} - \frac{(5s-2)}{60r(r-s)(r-1)} \\ +\frac{(5r-2)}{60s(r-s)(s-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} \frac{(s)^3}{3!} - \frac{s^4(5r-2s)}{60(r-1)(s-1)} + \frac{s^4(2s-5)}{60r(r-s)(r-1)}\frac{r}{1!} - \frac{s^2(5s-10r+5rs-3s^2)}{60(r-s)(s-1)}\frac{s}{1!} \\ \frac{(r)^3}{3!} + \frac{r^4(2r-5s)}{60(r-1)(s-1)} - \frac{r^2(10s-5r-5rs+3r^2)}{60(r-s)(r-1)}\frac{r}{1!} - \frac{r^4(2r-5)}{60s(r-s)(s-1)}\frac{s}{1!} \\ \frac{1}{3!} - \frac{(10sr-5s-5r+3)}{60(r-1)(s-1)} - \frac{(5s-2)}{60r(r-s)(r-1)}\frac{r}{1!} + \frac{(5r-2)}{60s(r-s)(s-1)}\frac{s}{1!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_4 = \begin{bmatrix} \frac{(s)^4}{4!} - \frac{s^4(5r-2s)}{60(r-1)(s-1)}\frac{1}{2!} + \frac{s^4(2s-5)}{60r(r-s)(r-1)}\frac{r^2}{2!} - \frac{s^2(5s-10r+5rs-3s^2)}{60(r-s)(s-1)}\frac{s^2}{2!} \\ \frac{(r)^4}{4!} + \frac{r^4(2r-5s)}{60(r-1)(s-1)}\frac{1}{2!} - \frac{r^2(10s-5r-5rs+3r^2)}{60(r-s)(r-1)}\frac{r^2}{2!} - \frac{r^4(2r-5)}{60s(r-s)(s-1)}\frac{s^2}{2!} \\ \frac{1}{4!} - \frac{(10sr-5s-5r+3)}{60(r-1)(s-1)}\frac{1}{2!} - \frac{(5s-2)}{60r(r-s)(r-1)}\frac{r^2}{2!} + \frac{(5r-2)}{60s(r-s)(s-1)}\frac{s^2}{2!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_5 = \begin{bmatrix} \frac{(s)^5}{5!} - \frac{s^4(5r-2s)}{60(r-1)(s-1)}\frac{1}{3!} + \frac{s^4(2s-5)}{60r(r-s)(r-1)}\frac{r^3}{3!} - \frac{s^2(5s-10r+5rs-3s^2)}{60(r-s)(s-1)}\frac{s^3}{3!} \\ \frac{(r)^5}{5!} + \frac{r^4(2r-5s)}{60(r-1)(s-1)}\frac{1}{3!} - \frac{r^2(10s-5r-5rs+3r^2)}{60(r-s)(r-1)}\frac{r^3}{3!} - \frac{r^4(2r-5)}{60s(r-s)(s-1)}\frac{s^3}{3!} \\ \frac{1}{5!} - \frac{(10sr-5s-5r+3)}{60(r-1)(s-1)}\frac{1}{3!} - \frac{(5s-2)}{60r(r-s)(r-1)}\frac{r^3}{3!} + \frac{(5r-2)}{60s(r-s)(s-1)}\frac{s^3}{3!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_6 = \begin{bmatrix} \frac{(s)^6}{6!} - \frac{s^4(5r-2s)}{60(r-1)(s-1)}\frac{1}{4!} + \frac{s^4(2s-5)}{60r(r-s)(r-1)}\frac{r^4}{4!} - \frac{s^2(5s-10r+5rs-3s^2)}{60(r-s)(s-1)}\frac{s^4}{4!} \\ \frac{(r)^6}{6!} + \frac{r^4(2r-5s)}{60(r-1)(s-1)}\frac{1}{4!} - \frac{r^2(10s-5r-5rs+3r^2)}{60(r-s)(r-1)}\frac{r^4}{4!} - \frac{r^4(2r-5)}{60s(r-s)(s-1)}\frac{s^4}{4!} \\ \frac{1}{6!} - \frac{(10sr-5s-5r+3)}{60(r-1)(s-1)}\frac{1}{4!} - \frac{(5s-2)}{60r(r-s)(r-1)}\frac{r^4}{4!} + \frac{(5r-2)}{60s(r-s)(s-1)}\frac{s^4}{4!} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s^4(2s-5r+2rs-s^2)}{1440} \\ \frac{-r^4(5s-2r-2rs+r^2)}{1440} \\ \frac{-(5rs-2s-2r+1)}{1440} \end{bmatrix}$$

Hence, by comparing the coefficient of h, the block method has order $[4,4,4]^T$ with the error constants vector

$$\left[\frac{s^4(2s-5r+2rs-s^2)}{1440}, \frac{-r^4(5s-2r-2rs+r^2)}{1440}, \frac{-(5rs-2s-2r+1)}{1440}\right]^T$$

for all $s,r \in (0,1) \setminus \{\{r = \frac{(s^2-2s)}{(-5+2s)}\} \cup \{s = \frac{(2r+r^2)}{(5-2r)}\} \cup \{s = \frac{(1-2r)}{(2-5r)}\}\}$

In finding the order of derivative block (3.50), $y_n^{'}$ and f_n- functions are expanded in spirit of Definition (3.1.2) as demonstrated below.

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{(s)^j h^j}{j!} y_n^{j+1} - y_n^{'} + \frac{hs(2s-6r+2rs-s^2)}{12r} y_n^{''} - \frac{s^3(2r-s)}{12(r-1)(s-1)} \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+2} \\ + \frac{s^3(s-2)}{12r(r-s)(r-1)} \sum_{j=0}^{\infty} \frac{(r)^j h^{j+1}}{j!} y_n^{j+2} - \frac{s(4s-6r+4rs-3s^2)}{12(r-s)(s-1)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+1}}{j!} y_n^{j+2} \\ \sum_{j=0}^{\infty} \frac{(r)^j h^j}{j!} y_n^{j+1} - y_n^{'} - \frac{hr(6s-2r-2rs+r^2)}{12s} y_n^{''} + \frac{r^3(r-2s)}{12(r-1)(s-1)} \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+2} \\ - \frac{r(6s-4r-4rs+3r^2)}{12(r-s)(r-1)} \sum_{j=0}^{\infty} \frac{(r)^j h^{j+1}}{j!} y_n^{j+2} - \frac{r^3(r-2)}{12s(r-s)(s-1)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+1}}{j!} y_n^{j+2} \\ \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{j+1} - y_n^{'} - \frac{h(6rs-2s-2r+1)}{12rs} y_n^{''} - \frac{(6rs-4s-4r+3)}{12(r-1)(s-1)} \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+2} \\ - \frac{(2s-1)}{12r(r-s)(r-1)} \sum_{j=0}^{\infty} \frac{(r)^j h^{j+1}}{j!} y_n^{j+2} + \frac{(2r-1)}{12s(r-s)(s-1)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+1}}{j!} y_n^{j+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Comparing the coefficients of h^j and y^j gives

$$\bar{C}_0 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{C}_1 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} \frac{(s)^1}{1!} + \frac{s(2s-6r+2rs-s^2)}{12r} - \frac{s^3(2r-s)}{12(r-1)(s-1)} + \frac{hs^3(s-2)}{12r(r-s)(r-1)} \\ - \frac{s(4s-6r+4rs-3s^2)}{12(r-s)(s-1)} \\ \frac{(r)^1}{1!} - \frac{r(6s-2r-2rs+r^2)}{12s} + \frac{1}{144} - \frac{r(6s-4r-4rs+3r^2)}{12(r-s)(r-1)} \\ - \frac{r^3(r-2)}{12s(r-s)(s-1)} \\ \frac{1}{1!} - \frac{(6rs-2s-2r+1)}{12rs} - \frac{(6rs-4s-4r+3)}{12(r-1)(s-1)} - \frac{(2s-1)}{12r(r-s)(r-1)} \\ + \frac{(2r-1)}{12s(r-s)(s-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} \frac{(s)^2}{2!} - \frac{s^3(2r-s)}{12(r-1)(s-1)} + \frac{s^3(s-2)}{12r(r-s)(r-1)}\frac{r}{1!} - \frac{s(4s-6r+4rs-3s^2)}{12(r-s)(s-1)}\frac{s}{1!} \\ \frac{(r)^2}{2!} + \frac{1}{144} - \frac{r(6s-4r-4rs+3r^2)}{12(r-s)(r-1)}\frac{r}{1!} - \frac{r^3(r-2)}{12s(r-s)(s-1)}\frac{s}{1!} \\ \frac{1}{2!} - \frac{(6rs-4s-4r+3)}{12(r-1)(s-1)} - \frac{(2s-1)}{12r(r-s)(r-1)}\frac{r}{1!} + \frac{(2r-1)}{12s(r-s)(s-1)}\frac{s}{1!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_4 = \begin{bmatrix} \frac{(s)^3}{3!} - \frac{s^3(2r-s)}{12(r-1)(s-1)}\frac{1}{2!} + \frac{s^3(s-2)}{12r(r-s)(r-1)}\frac{r^2}{2!} - \frac{s(4s-6r+4rs-3s^2)}{12(r-s)(s-1)}\frac{s^2}{2!} \\ \frac{(r)^3}{3!} + \frac{1}{144}\frac{1}{2!} - \frac{r(6s-4r-4rs+3r^2)}{12(r-s)(r-1)}\frac{r^2}{2!} - \frac{r^3(r-2)}{12s(r-s)(s-1)}\frac{s^2}{2!} \\ \frac{1}{3!} - \frac{(6rs-4s-4r+3)}{12(r-1)(s-1)}\frac{1}{2!} - \frac{(2s-1)}{12r(r-s)(r-1)}\frac{r^2}{2!} + \frac{(2r-1)}{12s(r-s)(s-1)}\frac{s^2}{2!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_5 = \begin{bmatrix} \frac{(s)^4}{4!} - \frac{s^3(2r-s)}{12(r-1)(s-1)}\frac{1}{3!} + \frac{s^3(s-2)}{12r(r-s)(r-1)}\frac{r^3}{3!} - \frac{s(4s-6r+4rs-3s^2)}{12(r-s)(s-1)}\frac{s^3}{3!} \\ \frac{(r)^4}{4!} + \frac{1}{144}\frac{1}{3!} - \frac{r(6s-4r-4rs+3r^2)}{12(r-s)(r-1)}\frac{r^3}{3!} - \frac{r^3(r-2)}{12s(r-s)(s-1)}\frac{s^3}{3!} \\ \frac{1}{4!} - \frac{(6rs-4s-4r+3)}{12(r-1)(s-1)}\frac{1}{3!} - \frac{(2s-1)}{12r(r-s)(r-1)}\frac{r^3}{3!} + \frac{(2r-1)}{12s(r-s)(s-1)}\frac{s^3}{3!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_6 = \begin{bmatrix} \frac{(s)^5}{5!} - \frac{s^3(2r-s)}{12(r-1)(s-1)}\frac{1}{4!} + \frac{s^3(s-2)}{12r(r-s)(r-1)}\frac{r^4}{4!} - \frac{s(4s-6r+4rs-3s^2)}{12(r-s)(s-1)}\frac{s^3}{3!} \\ \frac{(r)^4}{4!} + \frac{1}{144}\frac{1}{3!} - \frac{r(6s-4r-4rs+3r^2)}{12(r-s)(r-1)}\frac{r^4}{4!} - \frac{r^3(r-2)}{12s(r-s)(s-1)}\frac{s^4}{4!} \\ \frac{1}{5!} - \frac{(6rs-4s-4r+3)}{12(r-1)(s-1)}\frac{1}{4!} - \frac{(2s-1)}{12r(r-s)(r-1)}\frac{r^4}{4!} + \frac{(2r-1)}{12s(r-s)(s-1)}\frac{s^4}{4!} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s^3(5s-10r+5rs-3s^2)}{1440} \\ \frac{r^3(5r-10s+5rs-3r^2)}{1440} \\ \frac{-(10sr-5s-5r+3)}{1440} \end{bmatrix}$$

Hence, by comparing the coefficient of h, the block of derivative has order $[4,4,4]^T$ with the error constants

$$\left[\frac{s^3(5s-10r+5rs-3s^2)}{1440}, \frac{r^3(5r-10s+5rs-3r^2)}{1440}, \frac{-(10sr-5s-5r+3)}{1440}\right]^T$$

which is true for all $s, r \in (0,1) \setminus \{\{r = \frac{(3s^2-5s)}{(5s-10)}\} \cup \{s = \frac{(3r^2-5r)}{(5r-10)}\} \cup \{r = \frac{(5s-3)}{(10s-5)}\}\}$.

3.2.1.2 Zero Stability of One Step Hybrid Block Method with Generalised Two Off-Step Points for Second Order ODEs

In finding the zero-stability of the block (3.43), we only put into consideration the roots of first characteristic polynomial according to Definition (3.1.3), that is

$$
\begin{aligned}
\Pi(z) &= |z\, I^{[2]_2} - \bar{B}_1^{[2]_2}| \\
&= \left| z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^2(z-1)
\end{aligned}
$$

which gives $z = 0, 0, 1$. Similarly, the characteristic polynomial of derivative of a block (3.50) is

$$
\begin{aligned}
\Pi(z) &= |z\, I^{[2]_2} - \acute{B}_2^{[2]_2}| \\
&= \left| z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^2(z-1)
\end{aligned}
$$

which also implies $z = 0, 0, 1$. Hence , the conditions in Definition (3.1.3) are satisfied. Therefore, the block method(3.43) and its derivative(3.50) are zero stable.

3.2.1.3 Consistency and Convergent of One Step Hybrid Block Method with Generalised Two Off-Step Points for Second Order ODEs

The block method (3.43) and its derivatives (3.50) are consistent and convergent as stated in Definition (3.1.4) and Theorem (3.1)

3.2.1.4 Region of Absolute Stability of One Step Hybrid Block Method with Generalised Two Off-Step Points for Second Order ODEs

Applying (3.29) for one step hybrid block with two generalised off-step points(3.43) gives

$$\bar{h}(\theta,h)=\frac{I^{[2]_2}Y_m^{[2]_2}(\theta)-\bar{B}_1^{[2]_2}R_1^{[2]_2}(\theta)}{[\bar{D}^{[2]_2}Y_{R_3^{[2]_2}}(\theta)+\bar{E}^{[2]_2}Y_{R_4^{[2]_2}}(\theta)]} \tag{3.51}$$

where

$$I^{[2]_2}Y_m^{[2]_2}(\theta)=\begin{bmatrix} e^{si\theta} & 0 & 0 \\ 0 & e^{ri\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{bmatrix}$$

$$\bar{B}_1^{[2]_2}R_1^{[2]_2}(\theta)=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{D}^{[2]_2}Y_{R_3^{[2]_2}}(\theta)=\begin{bmatrix} 0 & 0 & \frac{-s^2(5s-20r+5rs-2s^2)}{(60r)} \\ 0 & 0 & \frac{r^2(20s-5r-5rs+2r^2)}{(60s)} \\ 0 & 0 & \frac{(20rs-5s-5r+2)}{(60rs)} \end{bmatrix}$$

$$\bar{E}^{[2]_2}Y_{R_4^{[2]_2}}(\theta)=\begin{bmatrix} \frac{(s^2(5s-10r+5rs-3s^2))}{(60(r-s)(s-1))}e^{si\theta} & \frac{(-s^4(2s-5))}{(60r(r-1)(r-s)))}e^{ri\theta} & \frac{(s^4(5r-2s))}{(60(r-1)(s-1))}e^{i\theta} \\ \frac{(r^4(2r-5))}{(60s(r-1)(r-s))}e^{si\theta} & \frac{(r^2(10s-5r-5rs+3r^2))}{(60(r-s)(r-1))}e^{ri\theta} & \frac{(-r^4(2r-5s))}{(60(r-1)(s-1))}e^{i\theta} \\ \frac{-(5r-2)}{(60s(s-1)(r-s))}e^{si\theta} & \frac{(5s-2)}{(60r(r-1)(r-s))}e^{ri\theta} & \frac{(10rs-5s-5r+3)}{(60(s-1)(r-1))}e^{i\theta} \end{bmatrix}$$

The above matrix is then simplified and after finding the determinant we have

$$\bar{h}(\theta,h)=\frac{(1440e^{i\theta}-1440)}{(2r^2s^2+4rs-3rs^2-3r^2s+r^2s^2e^{i\theta})}$$

The above equation is expanded trigonometrically and the imaginary part are equated to zero. This gives the equation of region stability as below

$$\bar{h}(\theta,h)=\frac{(1440\cos(\theta)-1440)}{(2r^2s^2+4rs-3rs^2-3r^2s+r^2s^2\cos(\theta))} \tag{3.52}$$

3.3 Derivation of One Step Hybrid Block Method with Generalised Three Off-Step Points for Second Order ODEs

To derive this method, Equation (3.1) is interpolated at the last two off-step points i.e at $x_{n+s_2},\ x_{n+s_3}$ and Equation (3.5) is collocated at all points in selected interval $[x_n,x_{n+1}]$. This strategy is clearly illustrated in the Figure 3.3.

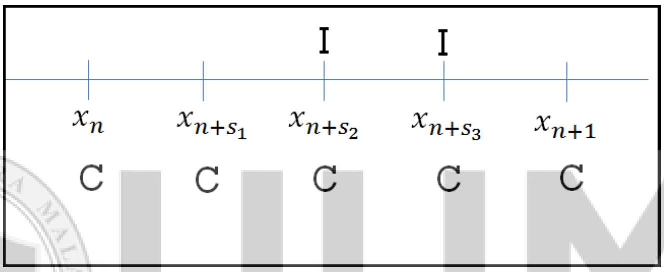

Figure 3.3. One step hybrid block method with generalised three off-step points for solving second order ODEs.

Based in Figure 3.3, $v=2$ and $m=5$ which produces the following equations

$$\begin{aligned}
y_{n+s_2} &= a_0+a_1s_2+a_2s_2^2+a_3s_2^3+a_4s_2^4+a_5s_2^5+a_6s_2^6.\\
y_{n+s_3} &= a_0+a_1s_3+a_2s_3^2+a_3s_3^3+a_4s_3^4+a_5s_3^5+a_6s_3^6.\\
f_n &= \frac{2}{h^2}a_2\\
.f_{n+s_1} &= a_2\frac{2}{h^2}a_2+\frac{6s_1}{h^2}a_3+\frac{12s_1^2}{h^2}a_4+\frac{20s_1^3}{h^2}a_5+\frac{30s_1^4}{h^2}a_6\\
.f_{n+s_2} &= \frac{2}{h^2}a_2+\frac{6s_2}{h^2}a_3+\frac{12s_2^2}{h^2}a_4+\frac{20s_2^3}{h^2}a_5+\frac{30s_2^4}{h^2}a_6.\\
f_{n+s_3} &= \frac{2}{h^2}a_2+\frac{6s_3}{h^2}a_3+\frac{12s_3^2}{h^2}a_4+\frac{20s_3^3}{h^2}a_5+\frac{30s_3^4}{h^2}a_6.\\
f_{n+1} &= \frac{2}{h^2}a_2+\frac{6}{h^2}a_3+\frac{12}{h^2}a_4+\frac{20}{h^2}a_5+\frac{30}{h^2}a_6.
\end{aligned} \tag{3.53}$$

which can be written in a matrix form

$$\begin{pmatrix} 1 & s_2 & s_2^2 & s_2^3 & s_2^4 & s_2^5 & s_2^6 \\ 1 & s_3 & s_3^2 & s_3^3 & s_3^4 & s_3^5 & s_3^6 \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2}s_1 & \frac{12}{h^2}s_1^2 & \frac{20}{h^2}s_1^3 & \frac{30}{h^2}s_1^4 \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2}s_2 & \frac{12}{h^2}s_2^2 & \frac{20}{h^2}s_2^3 & \frac{30}{h^2}s_2^4 \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2}s_3 & \frac{12}{h^2}s_3^2 & \frac{20}{h^2}s_3^3 & \frac{30}{h^2}s_3^4 \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_{n+s_2} \\ y_{n+s_3} \\ f_n \\ f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix} \tag{3.54}$$

Gaussian Elimination method is applied to (3.54) in order to find the unknown coefficients $a_i{}'s$ for $i = 0(1)6$ as below

$$\begin{aligned} a_0 = {} & \frac{s_2}{(s_2-s_3)} y_{n+s_3} - \frac{s_3}{(s_2-s_3)} y_{n+s_2} + \frac{h^2 s_2}{60(s_1-s_3)(s_2-s_3)(s_3-1)} \, (2s_3^4 - 3s_3^3 \\ & +2s_1s_2^3 - 5s_1s_2^2 - s_2^2s_3^2 + 5s_1s_3^2 - 3s_1s_3^3 + 2s_2s_3^2 + 2s_2^2s_3 - s_2s_3^3 - s_2^3s_3 + 2s_2^3 - s_2^4 \\ & +2s_1s_2s_3^2 + 2s_1s_2^2s_3 - 5s_1s_2s_3) f_{n+s_3} + \frac{h^2 s_3}{(60(s_1-s_2)(s_2-s_3)(s_2-1))} (s_2^2s_3^2 - 5s_1s_2^2 \\ & +3s_1s_2^3 + 5s_1s_3^2 - 2s_1s_3^3 - 2s_2s_3^2 - 2s_2^2s_3 + s_2s_3^3 + s_2^3s_3 + 3s_2^3 - 2s_2^4 - 2s_3^3 + s_3^4 \\ & -2s_1s_2s_3^2 - 2s_1s_2^2s_3 + 5s_1s_2s_3) f_{n+s_2} - \frac{h^2 s_2 s_3}{(60(s_1-1)(s_2-1)(s_3-1))} \, (-s_2^4 + s_2^3s_3 \\ & +2s_1s_2^3 + s_2^2s_3^2 - 3s_1s_2^2s_3 + s_2s_3^3 - 3s_1s_2s_3^2 - s_3^4 + 2s_1s_3^3) f_{n+1} \\ & - \frac{h^2 s_2 s_3}{(60s_1(s_1-s_2)(s_1-s_3)(s_1-1))} (-s_2s_3^3 + s_2^4 - s_2^3s_3 - 2s_2^3 - s_2^2s_3^2 + 3s_2^2s_3 + 3s_2s_3^2 \\ & + s_3^4 - 2s_3^3) f_{n+s_1} \end{aligned}$$

$$a_1 = \frac{-1}{(s_2-s_3)} y_{n+s_3} + \frac{1}{(s_2-s_3)} y_{n+s_2} - \frac{h^2}{(60 s_1 s_2 s_3)} \; (s_2^2 s_3^3 + s_2^3 s_3^2 - 5 s_1 s_2^3 - s_3^5$$
$$-3s_2^2s_3^2 + 2s_1s_2^4 - 5s_1s_3^3 + 2s_1s_3^4 - 3s_2s_3^3 - 3s_2^3s_3 + s_2s_3^4 + s_2^4s_3 + 2s_2^4 - s_2^5 + 2s_3^4$$
$$+15s_1s_2s_3^2 + 15s_1s_2^2s_3 - 3s_1s_2s_3^3 - 3s_1s_2^3s_3 - 3s_1s_2^2s_3^2) f_n$$
$$+\frac{h^2}{(60(s_1-1)(s_2-1)(s_3-1))} \; (-s_2^5 + s_2^4s_3 + 2s_1s_2^4 + s_2^3s_3^2 - 3s_1s_2^3s_3 + s_2^2s_3^3$$
$$-3s_1s_2^2s_3^2 + s_2s_3^4 - 3s_1s_2s_3^3 - s_3^5 + 2s_1s_3^4) f_{n+1}$$
$$+\frac{h^2}{60s_1(s_1-s_2)(s_1-s_3)(s_1-1)} (s_2^5 - s_2^4s_3 - 2s_2^4 - s_2^3s_3^2 + 3s_2^3s_3 - s_2^2s_3^3 + 3s_2^2s_3^2$$
$$-s_2s_3^4 + 3s_2s_3^3 + s_3^5 - 2s_3^4) f_{n+s_1} - \frac{h^2}{60s_3(s_1-s_3)(s_2-s_3)(s_3-1)} (2s_2^2s_3^2 - s_2^2s_3^3$$
$$-s_2^3s_3^2 - 5s_1s_2^3 + 2s_1s_2^4 + 5s_1s_3^3 - 3s_1s_3^4 + 2s_2s_3^3 + 2s_2^3s_3 - s_2s_3^4 - s_2^4s_3 + 2s_2^4 - s_2^5$$
$$-3s_3^4 + 2s_3^5 - 5s_1s_2s_3^2 - 5s_1s_2^2s_3 + 2s_1s_2s_3^3 + 2s_1s_2^3s_3 + 2s_1s_2^2s_3^2) f_{n+s_3}$$
$$-\frac{h^2}{(60s_2(s_1-s_2)(s_2-s_3)(s_2-1))} (s_2^2s_3^3 - 2s_2^2s_3^2 + s_2^3s_3^2 - 5s_1s_2^3 + 3s_1s_2^4 + 5s_1s_3^3$$
$$-2s_1s_3^4 - 2s_2s_3^3 - 2s_2^3s_3 + s_2s_3^4 + s_2^4s_3 + 3s_2^4 - 2s_2^5 - 2s_3^4 + 5s_1s_2s_3^2 + 5s_1s_2^2s_3$$
$$-2s_1s_2s_3^3 - 2s_1s_2^3s_3 - 2s_1s_2^2s_3^2 + s_3^5) f_{n+s_2}$$

$$a_2 = \frac{h^2}{2} f_n$$

$$a_3 = \frac{-h^2(s_1s_2 + s_1s_3 + s_2s_3 + s_1s_2s_3)}{(6s_2s_3s_1)} f_n - \frac{(h^2s_2s_1)}{(6s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3}$$
$$+\frac{(h^2s_2s_3s_1)}{6(s_3-1)(s_1-1)(s_2-1)} f_{n+1} \quad + \quad \frac{(h^2s_3s_1)}{(6s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2}$$
$$-\frac{(s_2s_3h^2)}{(6s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1}$$

$$a_4 = \frac{h^2(s_1+s_2+s_3+s_1s_2+s_1s_3+s_2s_3)}{(12s_2s_3s_1)} f_n - \frac{h^2(s_1+s_3+s_1s_3)}{12s_2(s_2-1)(s_2-s_3)(s_1-s_2)} f_{n+s_2}$$
$$-\frac{h^2(s_1s_2+s_1s_3+s_2s_3)}{12(s_3-1)(s_1-1)(s_2-1)} f_{n+1} \quad + \quad \frac{h^2(s_1+s_2+s_1s_2)}{12s_3(s_3-1)(s_2-s_3)(s_1-s_3)} f_{n+s_3}$$
$$+\frac{h^2(s_2+s_3+s_2s_3)}{12s_1(s_1-1)(s_1-s_3)(s_1-s_2)} f_{n+s_1}$$

$$
\begin{aligned}
a_5 =& \frac{h^2(s_1+s_3+1)}{(20s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} - \frac{h^2(s_2+s_3+1)}{(20s_1(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&-\frac{(h^2(s_1+s_2+1))}{(20s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} + \frac{(h^2(s_1+s_2+s_3))}{(20(s_3-1)(s_1-1)(s_2-1))}f_{n+1} \\
&-\frac{(h^2(s_1+s_2+s_3+1))}{(20s_2s_3s_1)}f_n
\end{aligned}
$$

$$
\begin{aligned}
a_6 =& \frac{h^2}{30s_3(s_1-s_3)(s_2-s_3)(s_3-1)}f_{n+s_3} + \frac{h^2}{30s_1(s_1-s_2)(s_1-s_3)(s_1-1)}f_{n+s_1} \\
&-\frac{h^2}{30s_2(s_1-s_2)(s_2-s_3)(s_2-1)}f_{n+s_2} - \frac{h^2}{30(s_1-1)(s_2-1)(s_3-1)}f_{n+1} \\
&+\frac{h^2}{(30s_1s_2s_3)}f_n
\end{aligned}
$$

The values of $a_i^{\prime}s$ are then substituted into Equation (3.1) and simplified, this gives a continuous hybrid one step method of the form

$$
y(x) = \sum_{i=2}^{3}\alpha_{s_i}y_{n+s_i} + \sum_{i=0}^{1}\beta_i f_{n+i} + \sum_{i=1}^{3}\beta_{s_i}f_{n+s_i} \tag{3.55}
$$

The first derivative of Equation (3.55) is

$$
y^{'}(x) = \sum_{i=2}^{3}\frac{\partial}{\partial x}\alpha_{s_i}(x)y_{n+s_i} + \sum_{i=0}^{1}\frac{\partial}{\partial x}\beta_i(x)f_{n+i} + \sum_{i=1}^{3}\frac{\partial}{\partial x}\beta_{s_i}(x)f_{n+s_i} \tag{3.56}
$$

where

$$
\alpha_{s_3} = \frac{(x_n - x + hs_2)}{(h(s_2-s_3))}
$$

$$
\alpha_{s_2} = \frac{(x - x_n - hs_3)}{(h(s_2-s_3))}
$$

$$
\begin{aligned}
\beta_0 = \; & \frac{(x_n - x + hs_3)(x_n - x + hs_2)}{(60 s_1 s_2 s_3 h^4)} (12x^2x_n^2 - 3hx^3 + 3hx_n^3 - 8xx_n^3 - 8x^3x_n + 2x^4 \\
& + 2x_n^4 + 2h^4s_2^3 - h^4s_2^4 + 2h^4s_3^3 - h^4s_3^4 - 3hs_1x^3 - hs_2x^3 - hs_3x^3 + 3hs_1x_n^3 + hs_2x_n^3 \\
& + hs_3x_n^3 - 9hxx_n^2 + 9hx^2x_n - 5h^4s_1s_2^2 + 2h^4s_1s_2^3 - 5h^4s_1s_3^2 + 2h^4s_1s_3^3 - 3h^4s_2s_3^2 \\
& - 3h^4s_2^2s_3 + h^4s_2s_3^3 + h^4s_2^3s_3 + 5h^2s_1x^2 + 2h^2s_2x^2 + 2h^3s_2^2x - h^3s_2^3x + 2h^2s_3x^2 \\
& + 2h^3s_3^2x - h^3s_3^3x + 5h^2s_1x_n^2 + 2h^2s_2x_n^2 - 2h^3s_2^2x_n + h^3s_2^3x_n + 2h^2s_3x_n^2 - 2h^3s_3^2x_n \\
& + h^3s_3^3x_n + h^4s_2^2s_3^2 - h^2s_2^2x^2 - h^2s_3^2x^2 - h^2s_2^2x_n^2 - h^2s_3^2x_n^2 + 15h^4s_1s_2s_3 - 5h^3s_1s_2x \\
& - 5h^3s_1s_3x - 3h^3s_2s_3x + 5h^3s_1s_2x_n + 5h^3s_1s_3x_n + 3h^3s_2s_3x_n - 9hs_1xx_n^2 + 9hs_1x^2x_n \\
& - 10h^2s_1xx_n - 3hs_2xx_n^2 + 3hs_2x^2x_n - 4h^2s_2xx_n - 3hs_3xx_n^2 + 3hs_3x^2x_n - 4h^2s_3xx_n \\
& - 3h^4s_1s_2s_3^2 - 3h^4s_1s_2^2s_3 + 2h^2s_1s_2x^2 + 2h^3s_1s_2^2x + 2h^2s_1s_3x^2 + 2h^2s_1s_2x_n^2 \\
& + 2h^3s_1s_3^2x + h^3s_2s_3^2x + h^3s_2^2s_3x - 2h^3s_1s_2^2x_n + 2h^2s_1s_3x_n^2 - 2h^3s_1s_3^2x_n + h^2s_2s_3x_n^2 \\
& - h^3s_2s_3^2x_n - h^3s_2^2s_3x_n + 2h^2s_2^2xx_n + 2h^2s_3^2xx_n - 3h^3s_1s_2s_3x + 3h^3s_1s_2s_3x_n \\
& - 4h^2s_1s_2xx_n - 4h^2s_1s_3xx_n - 2h^2s_2s_3xx_n + h^2s_2s_3x^2)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{s_1} = \; & \frac{-(x_n - x + hs_3)(x_n - x + hs_2)}{60h^4s_1(s_1 - 1)(s_1 - s_3)(s_1 - s_2)} (h^4s_2^4 - h^4s_2^3s_3 - 2h^4s_2^3 - h^4s_2^2s_3^2 + 3h^4s_2^2s_3 \\
& - h^4s_2s_3^3 + 3h^4s_2s_3^2 + h^4s_3^4 - 2h^4s_3^3 + h^3s_2^3x - h^3s_2^3x_n - h^3s_2^2s_3x + h^3s_2^2s_3x_n - 2h^3s_2^2x \\
& + 2h^3s_2^2x_n - h^3s_2s_3^2x + h^3s_2s_3^2x_n + 3h^3s_2s_3x - 3h^3s_2s_3x_n + h^3s_3^3x - h^3s_3^3x_n - 2h^3s_3^2x \\
& + 2h^3s_3^2x_n + h^2s_2^2x^2 - 2h^2s_2^2xx_n + h^2s_2^2x_n^2 - h^2s_2s_3x^2 + 2h^2s_2s_3xx_n - h^2s_2s_3x_n^2 \\
& + 4h^2s_2xx_n - 2h^2s_2x_n^2 + h^2s_3^2x^2 - 2h^2s_3^2xx_n + h^2s_3^2x_n^2 - 2h^2s_3x^2 + 4h^2s_3xx_n - hs_3x_n^3 \\
& - 2h^2s_3x_n^2 + hs_2x^3 - 3hs_2x^2x_n + 3hs_2xx_n^2 - hs_2x_n^3 + hs_3x^3 - 3hs_3x^2x_n + 3hs_3xx_n^2 \\
& - 2h^2s_2x^2 + 3hx^3 - 9hx^2x_n + 9hxx_n^2 - 3hx_n^3 - 2x^4 + 8x^3x_n - 12x^2x_n^2 + 8xx_n^3 - 2x_n^4)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{s_2} = {} & \frac{(x_n - x + hs_3)(x_n - x + hs_2)}{60h^4 s_2(s_2-1)(s_2-s_3)(s_1-s_2)} \; (3hx^3 - 12x^2x_n^2 + 8xx_n^3 + 8x^3x_n + h^2s_3^2x^2 \\
& + 3h^4s_2^3 - 2h^4s_2^4 - 2h^4s_3^3 + h^4s_3^4 + 3hs_1x^3 - 2hs_2x^3 + hs_3x^3 - 3hs_1x_n^3 + 2hs_2x_n^3 \\
& - 2h^2s_2s_3xx_n - 2x_n^4 + 9hxx_n^2 - 9hx^2x_n - 5h^4s_1s_2^2 + 3h^4s_1s_2^3 + 5h^4s_1s_3^2 - 2h^4s_1s_3^3 \\
& + h^4s_2s_3^3 + h^4s_2^3s_3 - 5h^2s_1x^2 + 3h^2s_2x^2 + 3h^3s_2^2x - 2h^3s_2^3x - 2h^2s_3x^2 - 2h^3s_3^2x \\
& - 5h^2s_1x_n^2 + 3h^2s_2x_n^2 - 3h^3s_2^2x_n + 2h^3s_2^3x_n - 2h^2s_3x_n^2 + 2h^3s_3^2x_n - h^3s_3^3x_n + h^4s_2^2s_3^2 \\
& - 2h^2s_2^2x^2 - 2h^2s_2^2x_n^2 + h^2s_3^2x_n^2 + 5h^4s_1s_2s_3 - 5h^3s_1s_2x + 5h^3s_1s_3x - 2h^3s_2s_3x \\
& - 5h^3s_1s_3x_n + 2h^3s_2s_3x_n + 9hs_1xx_n^2 - 9hs_1x^2x_n + 10h^2s_1xx_n - 6hs_2xx_n^2 + 6hs_2x^2x_n \\
& - 6h^2s_2xx_n + 3hs_3xx_n^2 - 3hs_3x^2x_n + 4h^2s_3xx_n - 2h^4s_1s_2s_3^2 - 2h^4s_1s_2^2s_3 + 3h^2s_1s_2x^2 \\
& + 3h^3s_1s_2^2x - 2h^2s_1s_3x^2 - 2h^3s_1s_3^2x + h^2s_2s_3x^2 + h^3s_2s_3^2x + h^3s_2^2s_3x + 3h^2s_1s_2x_n^2 \\
& - 3h^3s_1s_2^2x_n - 2h^2s_1s_3x_n^2 + 2h^3s_1s_3^2x_n + h^2s_2s_3x_n^2 - h^3s_2s_3^2x_n - h^3s_2^2s_3x_n - 2h^4s_2^2s_3 \\
& + 4h^2s_2^2xx_n - hs_3x_n^3 - 2x^4 - 2h^2s_3^2xx_n - 2h^3s_1s_2s_3x + 2h^3s_1s_2s_3x_n - 6h^2s_1s_2xx_n \\
& + 4h^2s_1s_3xx_n - 2h^4s_2s_3^2 + h^3s_3^3x - 3hx_n^3 + 5h^3s_1s_2x_n)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{s_3} = {} & \frac{(x_n - x + hs_3)(x_n - x + hs_2)}{(60h^4 s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \; (12x^2x_n^2 - 3hx^3 + 3hx_n^3 - 8xx_n^3 - 8x^3x_n \\
& + 6h^2s_1s_3xx_n + 2h^4s_2^3 - h^4s_2^4 - 3h^4s_3^3 + 2h^4s_3^4 - 3hs_1x^3 + 2hs_3x^3 + 3hs_1x_n^3 + hs_2x_n^3 \\
& - 2hs_3x_n^3 - 9hxx_n^2 + 9hx^2x_n - 5h^4s_1s_2^2 + 2h^4s_1s_2^3 + 5h^4s_1s_3^2 + 2h^4s_2s_3^2 + 2h^4s_2^2s_3 \\
& - h^4s_2s_3^3 - h^4s_2^3s_3 + 5h^2s_1x^2 + 2h^2s_2x^2 + 2h^3s_2^2x - h^3s_2^3x - 3h^2s_3x^2 - 3h^3s_3^2x \\
& + 5h^2s_1x_n^2 + 2h^2s_2x_n^2 - 2h^3s_2^2x_n + h^3s_2^3x_n - 3h^2s_3x_n^2 + 3h^3s_3^2x_n - 2h^3s_3^3x_n - h^4s_2^2s_3^2 \\
& - h^2s_2^2x^2 + 2h^2s_3^2x^2 - h^2s_2^2x_n^2 + 2h^2s_3^2x_n^2 - 5h^4s_1s_2s_3 - 5h^3s_1s_2x + 5h^3s_1s_3x \\
& + 5h^3s_1s_2x_n - 5h^3s_1s_3x_n - 2h^3s_2s_3x_n - 9hs_1xx_n^2 + 9hs_1x^2x_n - 10h^2s_1xx_n + 2x_n^4 \\
& + 3hs_2x^2x_n - 4h^2s_2xx_n + 6hs_3xx_n^2 - 6hs_3x^2x_n + 6h^2s_3xx_n + 2h^4s_1s_2s_3^2 + 2h^4s_1s_2^2s_3 \\
& + 2h^2s_1s_2x^2 + 2h^3s_1s_2^2x - 3h^2s_1s_3x^2 - 3h^3s_1s_3^2x - h^2s_2s_3x^2 - h^3s_2s_3^2x + h^3s_2^2s_3x_n \\
& + 2h^2s_1s_2x_n^2 - 2h^3s_1s_2^2x_n - 3h^2s_1s_3x_n^2 + 3h^3s_1s_3^2x_n - h^2s_2s_3x_n^2 - 4h^2s_1s_2xx_n + 2x^4 \\
& - 2h^3s_1s_2s_3x_n + 2h^2s_2^2xx_n - 4h^2s_3^2xx_n - 3h^4s_1s_3^3 + 2h^3s_3^3x - 3hs_2xx_n^2 + 2h^3s_2s_3x \\
& + h^3s_2^2s_3x + h^3s_2s_3^2x_n + 2h^3s_1s_2s_3x + 2h^2s_2s_3xx_n - hs_2x^3)
\end{aligned}
$$

$$
\begin{aligned}
\beta_1 &= -\frac{(x_n-x+hs_3)(x_n-x+hs_2)}{(60h^4(s_3-1)(s_2-1)(s_1-1))}(2s_1h^4s_2^3 + h^4s_2^2s_3^2 - 3s_1h^4s_2^2s_3 - h^3s_2^2s_3x_n \\
&+h^4s_2^3s_3 + h^4s_2s_3^3 - 3s_1h^4s_2s_3^2 - h^4s_3^4 + 2s_1h^4s_3^3 - h^3s_2^3x + h^3s_2^3x_n + h^3s_2^2s_3x - h^4s_2^4 \\
&+2s_1h^3s_2^2x - 2s_1h^3s_2^2x_n + h^3s_2s_3^2x - h^3s_2s_3^2x_n - 3s_1h^3s_2s_3x + 3s_1h^3s_2s_3x_n - h^3s_3^3x \\
&+h^3s_3^3x_n + 2s_1h^3s_3^2x - 2s_1h^3s_3^2x_n - h^2s_2^2x^2 + 2h^2s_2^2xx_n - h^2s_2^2x_n^2 + h^2s_2s_3x^2 - hs_3x^3 \\
&+h^2s_2s_3x_n^2 + 2s_1h^2s_2x^2 - 4s_1h^2s_2xx_n + 2s_1h^2s_2x_n^2 - h^2s_3^2x^2 + 2h^2s_3^2xx_n - h^2s_3^2x_n^2 \\
&-8xx_n^3 + 2s_1h^2s_3x^2 - 4s_1h^2s_3xx_n + 2s_1h^2s_3x_n^2 - hs_2x^3 + 3hs_2x^2x_n + hs_2x_n^3 + 2x^4 \\
&+3hs_3x^2x_n - 3hs_3xx_n^2 + hs_3x_n^3 - 3s_1hx^3 + 9s_1hx^2x_n - 9s_1hxx_n^2 + 3s_1hx_n^3 - 8x^3x_n \\
&-3hs_2xx_n^2 - 2h^2s_2s_3xx_n + 2x_n^4 + 12x^2x_n^2)
\end{aligned}
$$

Evaluating (3.55) at non-interpolating points i.e x_n, x_{n+s_1} and x_{n+1} gives the following schemes

$$
\begin{aligned}
&y_n + \frac{s_3}{(s_2-s_3)}y_{n+s_2} - \frac{s_2}{(s_2-s_3)}y_{n+s_3} = \frac{h^2}{(60s_1)}\,(s_2^2s_3^2 - 5s_1s_2^2 + 2s_1s_2^3 - 5s_1s_3^2 \\
&+2s_1s_3^3 - 3s_2s_3^2 - 3s_2^2s_3 + s_2s_3^3 + s_2^3s_3 + 2s_2^3 - s_2^4 + 2s_3^3 - s_3^4 - 3s_1s_2s_3^2 - 3s_1s_2^2s_3 \\
&+15s_1s_2s_3)f_n + \frac{h^2s_2}{(60(s_3-1)(s_1-s_3)(s_2-s_3))}\,(2s_1s_2^3 - 5s_1s_2^2 - s_2^2s_3^2 + 5s_1s_3^2 \\
&-3s_1s_3^3 + 2s_2s_3^2 + 2s_2^2s_3 - s_2s_3^3 - s_2^3s_3 + 2s_2^3 - s_2^4 - 3s_3^3 + 2s_3^4 + 2s_1s_2s_3^2 \\
&+2s_1s_2^2s_3 - 5s_1s_2s_3)f_{n+s_3} \\
&-\frac{h^2s_2s_3}{(60(s_3-1)(s_1-1)(s_2-1))}\,(-s_2^4 + s_2^3s_3 + 2s_1s_2^3 + s_2^2s_3^2 - 3s_1s_2^2s_3 + s_2s_3^3 \\
&-3s_1s_2s_3^2 - s_3^4 + 2s_1s_3^3)f_{n+1} - \frac{h^2s_2s_3}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}(s_2^4 - s_2^3s_3 - 2s_2^3 \\
&-s_2^2s_3^2 + 3s_2^2s_3 - s_2s_3^3 + 3s_2s_3^2 + s_3^4 - 2s_3^3)f_{n+s_1} \\
&+\frac{h^2s_3}{(60(s_2-1)(s_1-s_2)(s_2-s_3))}(s_2^2s_3^2 - 5s_1s_2^2 + 3s_1s_2^3 + 5s_1s_3^2 - 2s_1s_3^3 - 2s_2s_3^2 \\
&-2s_2^2s_3 + s_2s_3^3 + s_2^3s_3 + 3s_2^3 - 2s_2^4 - 2s_3^3 + s_3^4 - 2s_1s_2s_3^2 - 2s_1s_2^2s_3 + 5s_1s_2s_3)f_{n+s_2}
\end{aligned}
\tag{3.57}
$$

$$y_{n+s_1}\frac{-(s_1-s_3)}{(s_2-s_3)}y_{n+s_2}+\frac{(s_1-s_2)}{(s_2-s_3)}y_{n+s_3}=-\frac{h^2(s_1-s_3)(s_1-s_2)}{(60s_1s_2s_3)}\ (s_1^4-s_1^3s_2-s_1^3s_3$$
$$-2s_1^3-s_1^2s_2^2+2s_1^2s_2s_3+3s_1^2s_2-s_1^2s_3^2+3s_1^2s_3-s_1s_2^3+2s_1s_2^2s_3+3s_1s_2^2+2s_1s_2s_3^2$$
$$-12s_1s_2s_3-s_1s_3^3+3s_1s_3^2+s_2^4-s_2^3s_3-2s_2^3-s_2^2s_3^2+3s_2^2s_3-s_2s_3^3+3s_2s_3^2$$
$$+s_3^4-2s_3^3)f_n+\frac{h^2(s_1-s_3)(s_1-s_2)}{(60(s_1-1)(s_2-1)(s_3-1))}\ (s_1^4-s_1^3s_2-s_1^3s_3-s_1^2s_2^2+2s_1^2s_2s_3$$
$$-s_1^2s_3^2-s_1s_2^3+2s_1s_2^2s_3+2s_1s_2s_3^2-s_1s_3^3+s_2^4-s_2^3s_3-s_2^2s_3^2-s_2s_3^3+s_3^4)f_{n+1}$$
$$+\frac{h^2(s_1-s_3)}{60s_2(s_2-s_3)(s_2-1)}\ (s_1^3s_2-s_1^3s_3-2s_1^3+s_1^2s_2^2-s_1^2s_2s_3-2s_1^2s_2-s_1^2s_3^2+3s_2^3$$
$$s_1^4+3s_1^2s_3+s_1s_2^3-s_1s_2^2s_3-2s_1s_2^2-s_1s_2s_3^2+3s_1s_2s_3-s_1s_3^3+3s_1s_3^2-2s_2^4+s_2^3s_3$$
$$+s_2^2s_3^2-2s_2^2s_3+s_2s_3^3-2s_2s_3^2+s_3^4-2s_3^3)f_{n+s_2}+\frac{h^2}{60s_1(s_1-1)}\ (2s_1^4-s_1^3s_2-s_1^3s_3$$
$$-3s_1^3-s_1^2s_2^2+s_1^2s_2s_3+2s_1^2s_2-s_1^2s_3^2+2s_1^2s_3-s_1s_2^3+s_1s_2^2s_3+2s_1s_2^2+s_1s_2s_3^2+2s_2^3$$
$$-3s_1s_2s_3-s_1s_3^3+2s_1s_3^2-s_2^4+s_2^3s_3+s_2^2s_3^2-3s_2^2s_3+s_2s_3^3-3s_2s_3^2-s_3^4+2s_3^3)f_{n+s_1}$$
$$-\frac{h^2(s_1-s_2)}{60s_3(s_2-s_3)(s_3-1)}\ (s_1^4-s_1^3s_2+s_1^3s_3-2s_1^3-s_1^2s_2^2-s_1^2s_2s_3+3s_1^2s_2+s_1^2s_3^2$$
$$-2s_1^2s_3-s_1s_2^3-s_1s_2^2s_3+3s_1s_2^2-s_1s_2s_3^2+3s_1s_2s_3+s_1s_3^3-2s_1s_3^2+s_2^4+s_2^3s_3-2s_2^3$$
$$+s_2^2s_3^2-2s_2^2s_3+s_2s_3^3-2s_2s_3^2-2s_3^4+3s_3^3)f_{n+s_3} \tag{3.58}$$

$$y_{n+1}-\frac{(s_3-1)}{(s_2-s_3)}y_{n+s_2}+\frac{(s_2-1)}{(s_2-s_3)}y_{n+s_3}=\frac{h^2(s_3-1)(s_2-1)}{(60s_1s_2s_3)}\ (2s_1+s_2+s_3$$
$$+s_2^2s_3^2-3s_1s_2-3s_1s_3-2s_2s_3-3s_1s_2^2+2s_1s_2^3-3s_1s_3^2+2s_1s_3^3-2s_2s_3^2-2s_2^2s_3$$
$$+s_2s_3^3s_2^3s_3+s_2^2+s_2^3-s_2^4+s_3^2+s_3^3-s_3^4-3s_1s_2s_3^2-3s_1s_2^2s_3+12s_1s_2s_3-1)\ f_n$$
$$+\frac{h^2(s_2-1)}{(60s_3(s_1-s_3)(s_2-s_3)}\ (2s_1+s_2-s_3-s_2^2s_3^2-3s_1s_2+2s_1s_3+s_2s_3-3s_1s_2^2$$
$$+2s_1s_2^3+2s_1s_3^2-3s_1s_3^3+s_2s_3^2+s_2^2s_3-s_2s_3^3-s_2^3s_3+s_2^2+s_2^3-s_2^4-s_3^2-s_3^3+2s_3^4$$
$$+2s_1s_2s_3^2+2s_1s_2^2s_3-3s_1s_2s_3-1)f_{n+s_3}+\frac{h^2(s_3-1)}{(60s_2(s_1-s_2)(s_2-s_3)}(s_2-2s_1-s_3$$
$$+s_2^2s_3^2-2s_1s_2+3s_1s_3-s_2s_3-2s_1s_2^2+3s_1s_2^3+3s_1s_3^2-2s_1s_3^3-s_2s_3^2-s_2^2s_3+s_2^3$$
$$+s_2s_3^3+s_2^3s_3+s_2^2-2s_2^4-s_3^2-s_3^3+s_3^4-2s_1s_2s_3^2-2s_1s_2^2s_3+3s_1s_2s_3+1)\ f_{n+s_2}$$
$$-\frac{h^2(s_3-1)(s_2-1)}{(60s_1(s_1-s_2)(s_1-s_3)(s_1-1))}\ (s_2^4-s_2^3s_3-s_2^3-s_2^2s_3^2+2s_2^2s_3-s_2^2-s_2s_3^3$$
$$+2s_2s_3^2+2s_2s_3-s_2+s_3^4-s_3^3-s_3^2-s_3+1)f_{n+s_1}$$

$$-\frac{h^2}{(60(s_1-1))}(s_2^2s_3^2-s_2-s_3-3s_1+2s_1s_2+2s_1s_3+s_2s_3+2s_1s_2^2+2s_1s_2^3$$
$$+2s_1s_3^2+2s_1s_3^3+s_2s_3^2+s_2^2s_3+s_2s_3^3+s_2^3s_3-s_2^2-s_2^3-s_2^4-s_3^2-s_3^3-s_3^4$$
$$-3s_1s_2s_3^2-3s_1s_2^2s_3-3s_1s_2s_3+2)f_{n+1} \tag{3.59}$$

Equation (3.56) is evaluated at all points in selected interval i.e x_n, x_{n+s_1}, x_{n+s_2}, x_{n+s_3}, and x_{n+1}. This produces following schemes

$$y_n' - \frac{1}{h(s_2-s_3)}y_{n+s_2} + \frac{1}{h(s_2-s_3)}y_{n+s_3} = \frac{h}{60(s_1-1)(s_2-1)(s_3-1)}(-s_2^5+s_2^4s_3$$
$$+2s_1s_2^4+s_2^3s_3^2-3s_1s_2^3s_3+s_2^2s_3^3-3s_1s_2^2s_3^2+s_2s_3^4-3s_1s_2s_3^3-s_3^5+2s_1s_3^4)f_{n+1}$$
$$+\frac{h}{60s_1(s_1-s_2)(s_1-s_3)(s_1-1)}\ (s_2^5-s_2^4s_3-2s_2^4-s_2^3s_3^2+3s_2^3s_3-s_2^2s_3^3+3s_2^2s_3^2$$
$$-s_2s_3^4+3s_2s_3^3+s_3^5-2s_3^4)f_{n+s_1}-\frac{h}{(60s_1s_2s_3)}(s_2^2s_3^3-3s_2^2s_3^2+s_2^3s_3^2-5s_1s_2^3+s_2^4s_3$$
$$+2s_1s_2^4-5s_1s_3^3+2s_1s_3^4-3s_2s_3^3-3s_2^3s_3+s_2s_3^4+2s_2^4+2s_3^4-3s_1s_2^3s_3+15s_1s_2s_3^2$$
$$-s_2^5-s_3^5+15s_1s_2^2s_3-3s_1s_2s_3^3-3s_1s_2^2s_3^2)f_n$$
$$-\frac{h}{60s_3(s_1-s_3)(s_2-s_3)(s_3-1)}\ (2s_2^2s_3^2-s_2^2s_3^3-s_2^3s_3^2-5s_1s_2^3+2s_1s_2^4+5s_1s_3^3$$
$$-3s_1s_3^4+2s_2s_3^3+2s_2^3s_3-s_2s_3^4-s_2^4s_3+2s_2^4-s_2^5-3s_3^4+2s_3^5-5s_1s_2s_3^2+2s_1s_2^3s_3$$
$$-5s_1s_2^2s_3+2s_1s_2s_3^3+2s_1s_2^2s_3^2)f_{n+s_3}$$
$$-\frac{h}{60s_2(s_1-s_2)(s_2-s_3)(s_2-1)}(s_2^2s_3^3-2s_2^2s_3^2+s_2^3s_3^2-5s_1s_2^3+3s_1s_2^4+5s_1s_3^3+s_3^5$$
$$-2s_2s_3^3-2s_2^3s_3+s_2s_3^4+s_2^4s_3+3s_2^4-2s_3^4-2s_1s_3^4+5s_1s_2s_3^2+5s_1s_2^2s_3-2s_1s_2^2s_3^2$$
$$-2s_1s_2s_3^3-2s_1s_2^3s_3-2s_2^5)f_{n+s_2} \tag{3.60}$$

$$
\begin{aligned}
& y'_{n+s_1} - \frac{1}{h(s_2-s_3)}y_{n+s_2} + \frac{1}{h(s_2-s_3)}y_{n+s_3} \\
&= \frac{-h}{60s_3(s_1-s_3)(s_2-s_3)(s_3-1)}(-5s_1^4s_2 + 3s_1^5 + 10s_1^3s_2 + 2s_1s_2^4 + 2s_1s_2^3s_3 - 5s_1s_2^2s_3 \\
&-5s_1^4 - 5s_1s_2^3 + 2s_1s_2^2s_3^2 + 2s_1s_2s_3^3 - 5s_1s_2s_3^2 - 3s_1s_3^4 + 5s_1s_3^3 - s_2^4s_3 + 2s_2^4 + 2s_2^3s_3 \\
&-3s_3^4 - s_2^5 - s_2^3s_3^2 - s_2^2s_3^3 + 2s_2^2s_3^2 - s_2s_3^4 + 2s_2s_3^3 + 2s_3^5)f_{n+s_3} \\
&+\frac{h}{60s_1(s_1-s_2)(s_1-s_3)(s_1-1)}(12s_1^5 - 15s_1^4s_2 - 15s_1^4s_3 + 20s_1^3s_2s_3 + 20s_1^3s_2 + s_3^5 \\
&-15s_1^4 + 20s_1^3s_3 - 30s_1^2s_2s_3 + s_2^5 - s_2^4s_3 - 2s_2^4 - s_2^3s_3^2 + 3s_2^3s_3 + 3s_2^2s_3^2 - s_2^2s_3^3 - s_2s_3^4 \\
&+3s_2s_3^3 - 2s_3^4)f_{n+s_1} + \frac{h}{60(s_1-1)(s_2-1)(s_3-1)}(3s_1^5 - 5s_1^4s_3 + 10s_1^3s_2s_3 - 3s_1s_2^2s_3^2 \\
&+2s_1s_2^4 - 3s_1s_2^3s_3 - 5s_1^4s_2 - 3s_1s_2s_3^3 + 2s_1s_3^4 - s_2^5 + s_2^4s_3 + s_2^3s_3^2 + s_2^2s_3^3 + s_2s_3^4 - s_3^5)f_{n+1} \\
&+\frac{h}{60s_2(s_1-s_2)(s_2-s_3)(s_2-1)}(3s_1^5 - 5s_1^4s_3 - 5s_1^4 + 10s_1^3s_3 \\
&-3s_1s_2^4 + 2s_1s_2^3s_3 - 3s_2^4 + 5s_1s_2^3 + 2s_1s_2^2s_3^2 - 5s_1s_2^2s_3 + 2s_1s_2s_3^3 - 5s_1s_2s_3^2 + 2s_1s_3^4 \\
&+2s_2^5 - 5s_1s_3^3 - s_2^4s_3 - s_2^3s_3^2 + 2s_2^3s_3 - s_2^2s_3^3 + 2s_2^2s_3^2 - s_2s_3^4 + 2s_2s_3^3 - s_3^5 + 2s_3^4)f_{n+s_2} \\
&-\frac{h}{(60s_1s_2s_3)}(-5s_1^4s_2 - 5s_1^4s_3 3s_1^5 + 10s_1^3s_2s_3 + 10s_1^3s_2 + 10s_1^3s_3 - 30s_1^2s_2s_3 + 2s_1s_2^4 \\
&-5s_1^4 - 3s_1s_2^3s_3 - 5s_1s_2^3 - 3s_1s_2^2s_3^2 + 15s_1s_2^2s_3 - 3s_1s_2s_3^3 + 15s_1s_2s_3^2 + 2s_1s_3^4 + s_2^4s_3 \\
&+2s_2^4 - s_2^5 + s_2^3s_3^2 - 3s_2^3s_3 + s_2^2s_3^3 - 3s_2^2s_3^2 + s_2s_3^4 - 3s_2s_3^3 - s_3^5 + 2s_3^4 - 5s_1s_3^3)f_n \qquad (3.61)
\end{aligned}
$$

$$
\begin{aligned}
& y'_{n+s_2} - \frac{1}{h(s_2-s_3)}y_{n+s_2} + \frac{1}{h(s_2-s_3)}y_{n+s_3} = \frac{-h(s_2-s_3)^3}{60s_1(s_1-s_2)(s_1-s_3)(s_1-1)}(2s_2^2 \\
&+2s_2s_3 - 3s_2 + s_3^2 - 2s_3)\, f_{n+s_1} + \frac{h(s_2-s_3)}{60s_2(s_1-s_2)(s_2-1)}(15s_1s_2 + 5s_1s_3 - 6s_2s_3 \\
&-12s_1s_2^2 - 2s_1s_3^2 + 3s_2s_3^2 + 6s_2^2s_3 - 12s_2^2 + 10s_2^3 - 2s_3^2 + s_3^3 - 6s_1s_2s_3)f_{n+s_2} \\
&-\frac{h(s_2-s_3)}{60s_3(s_1-s_3)(s_3-1)}(5s_1s_2 + 5s_1s_3 - 4s_2s_3 - 3s_1s_2^2 - 3s_1s_3^2 + 3s_2s_3^2 + 3s_2^2s_3 \\
&-3s_2^2 + 2s_2^3 - 3s_3^2 + 2s_3^3 - 4s_1s_2s_3)f_{n+s_3} \\
&+\frac{h(s_2-s_3)^3}{(60s_1-60)(s_2-1)(s_3-1)}(2s_2^2 + 2s_2s_3 - 3s_1s_2 + s_3^2 - 2s_1s_3)f_{n+1} \\
&-\frac{h(s_2-s_3)^3}{(60s_1s_2s_3)}(5s_1 - 3s_2 - 2s_3 - 3s_1s_2 - 2s_1s_3 + 2s_2s_3 + 2s_2^2 + s_3^2)f_n \qquad (3.62)
\end{aligned}
$$

$$
\begin{aligned}
y'_{n+s_3} - \frac{1}{h(s_2-s_3)}y_{n+s_2} + \frac{1}{h(s_2-s_3)}y_{n+s_3} &= \frac{h(s_2-s_3)^3}{60s_1(s_1-s_2)(s_1-s_3)(s_1-1)}\,(s_2^2 \\
&-2s_2+2s_2s_3+2s_3^2-3s_3)\,f_{n+s_1} + \frac{h(s_2-s_3)}{60s_2(s_1-s_2)(s_2-1)}\,(5s_1s_2+5s_1s_3-4s_2s_3 \\
&-3s_3^2-3s_1s_2^2-3s_1s_3^2+3s_2s_3^2+3s_2^2s_3-3s_2^2+2s_2^3+2s_3^3-4s_1s_2s_3)f_{n+s_2} \\
&+\frac{h(s_2-s_3)}{60s_3(s_1-s_3)(s_3-1)}(5s_1s_2+15s_1s_3-6s_2s_3-2s_1s_2^2-12s_1s_3^2+6s_2s_3^2+3s_2^2s_3 \\
&-2s_2^2+s_2^3-12s_3^2+10s_3^3-6s_1s_2s_3)f_{n+s_3} \\
&-\frac{h(s_2-s_3)^3}{60(s_1-1)(s_2-1)(s_3-1)}(s_2^2+2s_2s_3-2s_1s_2+2s_3^2-3s_1s_3)f_{n+1} \\
&+\frac{h(s_2-s_3)^3}{(60s_1s_2s_3)}(5s_1-2s_2-3s_3-2s_1s_2-3s_1s_3+2s_2s_3+s_2^2+2s_3^2)f_n \qquad (3.63)
\end{aligned}
$$

$$
\begin{aligned}
y'_{n+1} - \frac{1}{h(s_2-s_3)}y_{n+s_2} + \frac{1}{h(s_2-s_3)}y_{n+s_3} &= \frac{-h}{60s_1(s_1-s_2)(s_1-s_3)(s_1-1)}(-s_2^5 \\
&+s_2^4s_3+2s_2^4+s_2^3s_3^2-3s_2^3s_3+s_2^2s_3^3-3s_2^2s_3^2+s_2s_3^4-3s_2s_3^3+10s_2s_3-5s_2-s_3^5 \\
&+2s_3^4-5s_3+3)f_{n+s_1} - \frac{h}{60s_2(s_1-s_2)(s_2-s_3)(s_2-1)}\,(5s_1+5s_3-2s_2^2s_3^2+s_2^2s_3^3 \\
&+s_2^3s_3^2-10s_1s_3-5s_1s_2^3+3s_1s_2^4+5s_1s_3^3-2s_1s_3^4-2s_2s_3^3+s_2^4s_3-2s_1s_2^3s_3-2s_2^3s_3 \\
&+s_2s_3^4+3s_2^4-2s_2^5-2s_3^4+s_3^5+5s_1s_2s_3^2+5s_1s_2^2s_3-2s_1s_2s_3^3-2s_1s_2^2s_3^2-3)f_{n+s_2} \\
&+\frac{h}{60s_3(s_1-s_3)(s_2-s_3)(s_3-1)}\,(5s_1+5s_2-2s_2^2s_3^2+s_2^2s_3^3+s_2^3s_3^2-10s_1s_2 \\
&+5s_1s_2^3-2s_1s_2^4-5s_1s_3^3+3s_1s_3^4-2s_2s_3^3-2s_2^3s_3+s_2s_3^4+s_2^4s_3-2s_2^4+s_2^5+3s_3^4 \\
&-2s_3^5+5s_1s_2s_3^2+5s_1s_2^2s_3-2s_1s_2s_3^3-2s_1s_2^3s_3-2s_1s_2^2s_3^2-3)f_{n+s_3} \\
&+\frac{h}{(60s_1-60)(s_2-1)(s_3-1)}\,(15s_1+15s_3+s_2^2s_3^3+s_2^3s_3^2-20s_1s_2-20s_1s_3 \\
&+15s_2-20s_2s_3+2s_1s_2^4+2s_1s_3^4+s_2s_3^4+s_2^4s_3-s_2^5-s_3^5-3s_1s_2s_3^3-3s_1s_2^3s_3 \\
&-3s_1s_2^2s_3^2+30s_1s_2s_3-12)f_{n+1}+\frac{h}{(60s_1s_2s_3)}(5s_1+5s_2+5s_3+3s_2^2s_3^2-s_2^2s_3^3 \\
&+5s_1s_2^3-2s_1s_2^4+5s_1s_3^3-2s_1s_3^4+3s_2s_3^3+3s_2^3s_3-s_2s_3^4-s_2^4s_3-2s_2^4+s_2^5-2s_3^4 \\
&-s_2^3s_3^2-10s_1s_2-10s_1s_3-10s_2s_3+s_3^5-15s_1s_2s_3^2-15s_1s_2^2s_3+3s_1s_2s_3^3+3s_1s_2^3s_3 \\
&+3s_1s_2^2s_3^2+30s_1s_2s_3-3)f_n \qquad (3.64)
\end{aligned}
$$

Combining Equations (3.57)-(3.60) of discrete schemes to form a block

$$A^{[3]_2} Y_m^{[3]_2} \;=\; B_1^{[3]_2} R_1^{[3]_2} + B_2^{[3]_2} R_2^{[3]_2} + h^2 \left[D^{[3]_2} R_3^{[3]_2} + E^{[3]_2} R_4^{[3]_2} \right] \tag{3.65}$$

where

$$A^{[3]_2} = \begin{pmatrix} 0 & \frac{s_3}{s_2-s_3} & \frac{s_2}{s_2-s_3} & 0 \\ 1 & \frac{-(s_1-s_3)}{(s_2-s_3)} & \frac{(s_1-s_2)}{(s_2-s_3)} & 0 \\ 0 & \frac{(s_3-1)}{(s_2-s_3)} & \frac{-(s_2-1)}{(s_2-s_3)} & 1 \\ 0 & \frac{-1}{(h(s_2-s_3))} & \frac{1}{(h(s_2-s_3))} & 0 \end{pmatrix}, \; Y_m^{[3]_2} = \begin{pmatrix} y_{n+s_1} \\ y_{n+s_2} \\ y_{n+s_3} \\ y_{n+1} \end{pmatrix},$$

$$B_1^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; R_1^{[3]_2} = \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix},$$

$$B_2^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; R_2^{[3]_2} = \begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix},$$

$$D^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & D_{14}^{[3]_2} \\ 0 & 0 & 0 & D_{24}^{[3]_2} \\ 0 & 0 & 0 & D_{34}^{[3]_2} \\ 0 & 0 & 0 & D_{44}^{[3]_2} \end{pmatrix}, \; R_3^{[3]_2} = \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix},$$

$$E^{[3]_2} = \begin{pmatrix} E_{11}^{[3]_2} & E_{12}^{[3]} & E_{13}^{[3]_2} & E_{14}^{[3]_2} \\ E_{21}^{[3]_2} & E_{22}^{[3]_2} & E_{23}^{[3]_2} & E_{24}^{[3]_2} \\ E_{31}^{[3]_2} & E_{32}^{[3]_2} & E_{33}^{[3]_2} & E_{34}^{[3]_2} \\ E_{41}^{[3]_2} & E_{42}^{[3]_2} & E_{43}^{[3]_2} & E_{44}^{[3]_2} \end{pmatrix} \text{ and } R_4^{[3]_2} = \begin{pmatrix} f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix}$$

and the non-zero terms of $D^{[3]_2}$ and $E^{[3]_2}$ are given by

$$
\begin{aligned}
D^{[3]_2}_{14} &= \frac{1}{(60s_1)}(s_2^2s_3^2 - 5s_1s_2^2 + 2s_1s_2^3 - 5s_1s_3^2 + 2s_1s_3^3 - 3s_2s_3^2 - 3s_2^2s_3 + s_2s_3^3 + s_2^3s_3 \\
&\quad + 2s_2^3 - s_2^4 + 2s_3^3 - s_3^4 - 3s_1s_2s_3^2 - 3s_1s_2^2s_3 + 15s_1s_2s_3) \\
D^{[3]_2}_{24} &= -\frac{1}{(60s_1s_2s_3)}\,(s_1 - s_2)\,(s_1 - s_3)\,(s_1^4 - s_1^3s_2 - s_1^3s_3 - 2s_1^3 - s_1^2s_2^2 + 2s_1^2s_2s_3 \\
&\quad + 3s_1^2s_2 - s_1^2s_3^2 + 3s_1^2s_3 - s_1s_2^3 + 2s_1s_2^2s_3 + 3s_1s_2^2 + 2s_1s_2s_3^2 - 12s_1s_2s_3 - 2s_2^3 \\
&\quad - s_1s_3^3 + 3s_1s_3^2 + s_2^4 - s_2^3s_3 - s_2^2s_3^2 + 3s_2^2s_3 - s_2s_3^3 + 3s_2s_3^2 + s_3^4 - 2s_3^3) \\
D^{[3]_2}_{34} &= \frac{1}{(60s_1s_2s_3)}\,(s_2 - 1)\,(s_3 - 1)\,(2s_1 + s_2 + s_3 + s_2^2s_3^2 - 3s_1s_2 - 3s_1s_3 - 2s_2s_3 \\
&\quad - 3s_1s_2^2 + 2s_1s_2^3 - 3s_1s_3^2 + 2s_1s_3^3 - 2s_2s_3^2 - 2s_2^2s_3 + s_2s_3^3 + s_2^3s_3 + s_2^2 + s_2^3 \\
&\quad - s_2^4 + s_3^2 + s_3^3 - s_3^4 - 3s_1s_2s_3^2 - 3s_1s_2^2s_3 + 12s_1s_2s_3 - 1) \\
D^{[3]_2}_{44} &= -\frac{1}{h(60s_1s_2s_3)}(s_2^2s_3^3 - 3s_2^2s_3^2 + s_2^3s_3^2 - 5s_1s_2^3 + 2s_1s_2^4 - 5s_1s_3^3 + 2s_1s_3^4 - 3s_2s_3^3 \\
&\quad - 3s_2^3s_3 + s_2s_3^4 + s_2^4s_3 + 2s_2^4 + 2s_3^4 - s_3^5 + 15s_1s_2s_3^2 + 15s_1s_2^2s_3 - 3s_1s_2s_3^3 \\
&\quad - 3s_1s_2^3s_3 - 3s_1s_2^2s_3^2 - s_2^5) \\
E^{[3]_2}_{11} &= \frac{s_2s_3}{60s_1(s_1 - s_2)(s_1 - s_3)(s_1 - 1)}(-s_2^4 + s_2^3s_3 + 2s_2^3 + s_2^2s_3^2 - 3s_2^2s_3 + s_2s_3^3 - s_3^4 \\
&\quad - 3s_2s_3^2 + 2s_3^3) \\
E^{[3]}_{12} &= \frac{s_3}{60(s_1 - s_2)(s_2 - s_3)(s_2 - 1)}\,(s_2^2s_3^2 - 5s_1s_2^2 + 3s_1s_2^3 + 5s_1s_3^2 - 2s_1s_3^3 - 2s_2s_3^2 \\
&\quad - 2s_2^2s_3 + s_2s_3^3 + s_2^3s_3 + 3s_2^3 - 2s_2^4 - 2s_3^3 + s_3^4 - 2s_1s_2s_3^2 - 2s_1s_2^2s_3 + 5s_1s_2s_3) \\
E^{[3]_2}_{13} &= \frac{-s_2}{60(s_1 - s_3)(s_2 - s_3)(s_3 - 1)}(s_2^2s_3^2 + 5s_1s_2^2 - 2s_1s_2^3 - 5s_1s_3^2 + 3s_2s_3^3 - 2s_2s_3^2 \\
&\quad - 2s_2^2s_3 + s_2s_3^3 + s_2^3s_3 - 2s_2^3 + s_2^4 + 3s_3^3 - 2s_3^4 - 2s_1s_2s_3^2 - 2s_1s_2^2s_3 + 5s_1s_2s_3) \\
E^{[3]_2}_{14} &= -\frac{s_2s_3}{(60s_1 - 60)(s_2 - 1)(s_3 - 1)}\,(-s_2^4 + s_2^3s_3 + 2s_1s_2^3 + s_2^2s_3^2 - 3s_1s_2^2s_3 + s_2s_3^3 \\
&\quad - 3s_1s_2s_3^2 - s_3^4 + 2s_1s_3^3) \\
E^{[3]_2}_{21} &= \frac{1}{(60s_1(s_1 - 1))}(2s_1^4 - s_1^3s_2 - s_1^3s_3 - 3s_1^3 - s_1^2s_2^2 + s_1^2s_2s_3 + 2s_1^2s_2 - s_1^2s_3^2 - s_2^4 \\
&\quad + 2s_1^2s_3 - s_1s_2^3 + s_1s_2^2s_3 + 2s_1s_2^2 + s_1s_2s_3^2 - 3s_1s_2s_3 - s_1s_3^3 + 2s_1s_3^2 + s_2s_3^3 \\
&\quad + s_2^3s_3 + 2s_2^3 + s_2^2s_3^2 - 3s_2^2s_3 - 3s_2s_3^2 - s_3^4 + 2s_3^3)
\end{aligned}
$$

$$
\begin{aligned}
E_{22}^{[3]_2} &= \frac{-(s_1-s_3)}{(60s_2(s_2-s_3)(s_2-1))}\,(-s_1^4-s_1^3s_2+s_1^3s_3+2s_1^3-s_1^2s_2^2+s_1^2s_2s_3+2s_1^2s_2 \\
&\quad +s_1^2s_3^2-3s_1^2s_3-s_1s_2^3+s_1s_2^2s_3+2s_1s_2^2+s_1s_2s_3^2-3s_1s_2s_3+s_1s_3^3-3s_1s_3^2 \\
&\quad +2s_2^4-s_2^3s_3-3s_2^3-s_2^2s_3^2+2s_2^2s_3-s_2s_3^3+2s_2s_3^2-s_3^4+2s_3^3) \\
E_{23}^{[3]_2} &= \frac{(s_1-s_2)}{(60s_3(s_2-s_3)(s_3-1))}\,(-s_1^4+s_1^3s_2-s_1^3s_3+2s_1^3+s_1^2s_2^2+s_1^2s_2s_3-3s_1^2s_2 \\
&\quad -s_1^2s_3^2+2s_1^2s_3+s_1s_2^3+s_1s_2^2s_3-3s_1s_2^2+s_1s_2s_3^2-3s_1s_2s_3-s_1s_3^3+2s_1s_3^2 \\
&\quad -s_2^4-s_2^3s_3+2s_2^3-s_2^2s_3^2+2s_2^2s_3-s_2s_3^3+2s_2s_3^2+2s_3^4-3s_3^3) \\
E_{24}^{[3]_2} &= \frac{-(s_1-s_2)(s_1-s_3)}{(60(s_1-1)(s_2-1)(s_3-1))}\,(-s_1^4+s_1^3s_2+s_1^3s_3+s_1^2s_2^2-2s_1^2s_2s_3+s_1^2s_3^2 \\
&\quad +s_1s_2^3-2s_1s_2^2s_3-2s_1s_2s_3^2+s_1s_3^3-s_2^4+s_2^3s_3+s_2^2s_3^2+s_2s_3^3-s_3^4) \\
E_{31}^{[3]_2} &= \frac{(s_2-1)(s_3-1)}{(60s_1(s_1-s_2)(s_1-s_3)(s_1-1))}\,(-s_2^4+s_2^3s_3+s_2^3+s_2^2s_3^2-2s_2^2s_3+s_2^2+s_2s_3^3 \\
&\quad -2s_2s_3^2-2s_2s_3+s_2-s_3^4+s_3^3+s_3^2+s_3-1) \\
E_{32}^{[3]_2} &= \frac{-(s_3-1)}{(60s_2(s_1-s_2)(s_2-s_3))}(2s_1-s_2+s_3-s_2^2s_3^2+2s_1s_2-3s_1s_3+s_2s_3+2s_1s_2^2 \\
&\quad -3s_1s_2^3-3s_1s_3^2+2s_1s_3^3+s_2s_3^2+s_2^2s_3-s_2s_3^3-s_2^3s_3-s_2^2-s_2^3+2s_2^4+s_3^2+s_3^3 \\
&\quad +2s_1s_2s_3^2+2s_1s_2^2s_3-s_3^4-3s_1s_2s_3-1) \\
E_{33}^{[3]_2} &= \frac{(s_2-1)}{(60s_3(s_1-s_3)(s_2-s_3))}(2s_1+s_2-s_3-s_2^2s_3^2-3s_1s_2+2s_1s_3+s_2s_3-3s_1s_2^2 \\
&\quad +2s_1s_2^3+2s_1s_3^2-3s_1s_3^3+s_2s_3^2+s_2^2s_3-s_2s_3^3-s_2^3s_3+s_2^2+s_2^3-s_2^4-s_3^2-s_3^3 \\
&\quad +2s_3^4+2s_1s_2s_3^2+2s_1s_2^2s_3-3s_1s_2s_3-1) \\
E_{34}^{[3]_2} &= \frac{-1}{(60s_1-60)}\,(s_2^2s_3^2-s_2-s_3-3s_1+2s_1s_2+2s_1s_3+s_2s_3+2s_1s_2^2+2s_1s_2^3 \\
&\quad +2s_1s_3^2+2s_1s_3^3+s_2s_3^2+s_2^2s_3+s_2s_3^3+s_2^3s_3-s_2^2-s_2^3-s_2^4-s_3^2-s_3^3-s_3^4 \\
&\quad -3s_1s_2s_3^2-3s_1s_2^2s_3-3s_1s_2s_3+2) \\
E_{41}^{[3]_2} &= \frac{-1}{(60hs_1(s_1-s_2)(s_1-s_3)(s_1-1))}\,(-s_2^5+s_2^4s_3+2s_2^4+s_2^3s_3^2-3s_2^3s_3+s_2^2s_3^3 \\
&\quad -3s_2^2s_3^2+s_2s_3^4-3s_2s_3^3-s_3^5+2s_3^4) \\
E_{42}^{[3]_2} &= \frac{1}{(60hs_2(s_1-s_2)(s_2-s_3)(s_2-1))}(2s_2^2s_3^2-s_2^2s_3^3-s_2^3s_3^2+5s_1s_2^3 3s_1s_2^4-5s_1s_3^3 \\
&\quad +2s_1s_3^4+2s_2s_3^3+2s_2^3s_3-s_2s_3^4-s_2^4s_3-3s_2^4+2s_3^4-s_3^5-5s_1s_2s_3^2-5s_1s_2^2s_3 \\
&\quad +2s_2^5+2s_1s_2s_3^3+2s_1s_2^3s_3+2s_1s_2^2s_3^2)
\end{aligned}
$$

$$
\begin{aligned}
E_{43}^{[3]_2} &= \frac{-1}{(60hs_3(s_1-s_3)(s_2-s_3)(s_3-1))}(2s_2^2s_3^2 - s_2^2s_3^3 - s_2^3s_3^2 - 5s_1s_2^3 + 2s_1s_2^4 - s_2^5 \\
&\quad -3s_1s_3^4 + 2s_2s_3^3 + 2s_2^3s_3 - s_2s_3^4 - s_2^4s_3 + 2s_2^4 - 3s_3^4 + 2s_3^5 - 5s_1s_2s_3^2 + 5s_1s_3^3 \\
&\quad -5s_1s_2^2s_3 + 2s_1s_2s_3^3 + 2s_1s_2^3s_3 + 2s_1s_2^2s_3^2) \\
E_{44}^{[3]_2} &= \frac{1}{(h(60s_1-60)(s_2-1)(s_3-1))}(-s_2^5 + s_2^4s_3 + 2s_1s_2^4 + s_2^3s_3^2 - 3s_1s_2^3s_3 + s_2^2s_3^3 \\
&\quad -3s_1s_2^2s_3^2 + s_2s_3^4 - 3s_1s_2s_3^3 - s_3^5 + 2s_1s_3^4)
\end{aligned}
$$

Multiplying Equation (3.65) by inverse of $A^{[3]_2}$ gives one step hybrid block methods as below

$$
I^{[3]_2}Y_m^{[3]_2} = \bar{B}_1^{[3]_2}R_1^{[3]_2} + \bar{B}_2^{[3]_2}R_2^{[3]_2} + h^2\bar{D}^{[3]_2}R_3^{[3]_2} + h^2\bar{E}^{[3]_2}R_4^{[3]_2} \tag{3.66}
$$

where

$$
\bar{B}_1^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \bar{B}_2^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & s_1h \\ 0 & 0 & 0 & s_2h \\ 0 & 0 & 0 & s_3h \\ 0 & 0 & 0 & h \end{pmatrix}
$$

$$
\bar{D}^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & \bar{D}_{14}^{[3]_2} \\ 0 & 0 & 0 & \bar{D}_{24}^{[3]_2} \\ 0 & 0 & 0 & \bar{D}_{34}^{[3]_2} \\ 0 & 0 & 0 & \bar{D}_{44}^{[3]_2} \end{pmatrix}, \bar{E}^{[3]_2} = \begin{pmatrix} \bar{E}_{11}^{[3]_2} & \bar{E}_{12}^{[3]} & \bar{E}_{13}^{[3]_2} & \bar{E}_{14}^{[3]_2} \\ \bar{E}_{21}^{[3]_2} & \bar{E}_{22}^{[3]_2} & \bar{E}_{23}^{[3]_2} & \bar{E}_{24}^{[3]_2} \\ \bar{E}_{31}^{[3]_2} & \bar{E}_{32}^{[3]_2} & \bar{E}_{33}^{[3]_2} & \bar{E}_{34}^{[3]_2} \\ \bar{E}_{41}^{[3]_2} & \bar{E}_{42}^{[3]_2} & \bar{E}_{43}^{[3]_2} & \bar{E}_{44}^{[3]_2} \end{pmatrix}
$$

with

$$
\begin{aligned}
\bar{D}_{14}^{[3]_2} &= \frac{-s_1^2(5s_1s_2 + 5s_1s_3 - 20s_2s_3 - 2s_1^2s_2 - 2s_1^2s_3 - 2s_1^2 + s_1^3 + 5s_1s_2s_3)}{(60s_2s_3)} \\
\bar{D}_{24}^{[3]_2} &= \frac{s_2^2(20s_1s_3 - 5s_1s_2 - 5s_2s_3 + 2s_1s_2^2 + 2s_2^2s_3 + 2s_2^2 - s_2^3 - 5s_1s_2s_3)}{(60s_1s_3)} \\
\bar{D}_{34}^{[3]_2} &= \frac{-s_3^2(5s_1s_3 - 20s_1s_2 + 5s_2s_3 - 2s_1s_3^2 - 2s_2s_3^2 - 2s_3^2 + s_3^3 + 5s_1s_2s_3)}{(60s_1s_2)} \\
\bar{D}_{44}^{[3]_2} &= \frac{(2s_1 + 2s_2 + 2s_3 - 5s_1s_2 - 5s_1s_3 - 5s_2s_3 + 20s_1s_2s_3 - 1)}{(60s_1s_2s_3)}
\end{aligned}
$$

$$
\begin{aligned}
\bar{E}^{[3]_2}_{11} &= \frac{s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\\
\bar{E}^{[3]_2}_{12} &= \frac{s_1^4(5s_3-2s_1-2s_1s_3+s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\\
\bar{E}^{[3]_2}_{13} &= \frac{-s_1^4(5s_2-2s_1-2s_1s_2+s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\bar{E}^{[3]_2}_{14} &= \frac{s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\\
\bar{E}^{[3]_2}_{21} &= \frac{-s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\\
\bar{E}^{[3]_2}_{22} &= \frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\\
\bar{E}^{[3]_2}_{23} &= \frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\bar{E}^{[3]_2}_{24} &= \frac{-s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\\
\bar{E}^{[3]_2}_{31} &= \frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\\
\bar{E}^{[3]_2}_{32} &= \frac{-s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\\
\bar{E}^{[3]_2}_{33} &= \frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\bar{E}^{[3]_2}_{34} &= \frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\\
\bar{E}^{[3]_2}_{41} &= \frac{-(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\\
\bar{E}^{[3]_2}_{42} &= \frac{(5s_1s_3-2s_3-2s_1+1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\\
\bar{E}^{[3]_2}_{43} &= \frac{-(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\bar{E}^{[3]_2}_{44} &= \frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}
\end{aligned}
$$

Equation (3.66) can also be written as

$$
\begin{aligned}
y_{n+s_1} &= y_n + hs_1 y_n^{'} \\
&+ \frac{-h^2 s_1^2(5s_1s_2 + 5s_1s_3 - 20s_2s_3 - 2s_1^2s_2 - 2s_1^2s_3 - 2s_1^2 + s_1^3 + 5s_1s_2s_3)}{(60s_2s_3)} f_n \\
&+ \frac{h^2 s_1^2(5s_1s_2 + 5s_1s_3 - 10s_2s_3 - 3s_1^2s_2 - 3s_1^2s_3 - 3s_1^2 + 2s_1^3 + 5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{h^2 s_1^4(5s_3 - 2s_1 - 2s_1s_3 + s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{-h^2 s_1^4(5s_2 - 2s_1 - 2s_1s_2 + s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{h^2 s_1^4(5s_2s_3 - 2s_1s_3 - 2s_1s_2 + s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} \qquad (3.67)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+s_2} &= y_n + hs_2 y_n^{'} \\
&+ \frac{h^2 s_2^2(20s_1s_3 - 5s_1s_2 - 5s_2s_3 + 2s_1s_2^2 + 2s_2^2s_3 + 2s_2^2 - s_2^3 - 5s_1s_2s_3)}{(60s_1s_3)} f_n \\
&+ \frac{-h^2 s_2^4(5s_3 - 2s_2 - 2s_2s_3 + s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{h^2 s_2^2(10s_1s_3 - 5s_1s_2 - 5s_2s_3 + 3s_1s_2^2 + 3s_2^2s_3 + 3s_2^2 - 2s_2^3 - 5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{h^2 s_2^4(2s_2 - 5s_1 + 2s_1s_2 - s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{-h^2 s_2^4(2s_1s_2 - 5s_1s_3 + 2s_2s_3 - s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))} f_{n+1} \qquad (3.68)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+s_3} &= y_n + hs_3 y_n^{'} \\
&+ \frac{-h^2 s_3^2(5s_1s_3 - 20s_1s_2 + 5s_2s_3 - 2s_1s_3^2 - 2s_2s_3^2 - 2s_3^2 + s_3^3 + 5s_1s_2s_3)}{(60s_1s_2)} f_n \\
&+ \frac{h^2 s_3^4(2s_3 - 5s_2 + 2s_2s_3 - s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{-h^2 s_3^4(2s_3 - 5s_1 + 2s_1s_3 - s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{h^2 s_3^2(5s_1s_3 - 10s_1s_2 + 5s_2s_3 - 3s_1s_3^2 - 3s_2s_3^2 - 3s_3^2 + 2s_3^3 + 5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{h^2 s_3^4(5s_1s_2 - 2s_1s_3 - 2s_2s_3 + s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))} f_{n+1} \qquad (3.69)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+1} &= y_n + hy_n^{'} \\
&+\frac{h^2(2s_1+2s_2+2s_3-5s_1s_2-5s_1s_3-5s_2s_3+20s_1s_2s_3-1)}{(60s_1s_2s_3)}f_n \\
&+\frac{-h^2(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&+\frac{h^2(5s_1s_3-2s_3-2s_1+1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&+\frac{-h^2(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \\
&+\frac{h^2(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}f_{n+1}
\end{aligned}
\tag{3.70}
$$

Substituting equations (3.68) and (3.69) into (3.61) – (3.64) gives the derivative of the block

$$
\begin{aligned}
y_{n+s_1}^{'} &= y_n^{'} \\
&+\frac{-hs_1(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3)}{60s_2s_3}f_n \\
&+\frac{hs_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&+\frac{s_1^3h(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&+\frac{-hs_1^3(10s_2-5s_1-5s_1s_2+3s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \\
&+\frac{hs_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}f_{n+1}
\end{aligned}
\tag{3.71}
$$

$$
\begin{aligned}
y_{n+s_2}^{'} &= y_n^{'} \\
&+\frac{hs_2(30s_1s_3-10s_1s_2-10s_2s_3+5s_1s_2^2+5s_2^2s_3+5s_2^2-3s_2^3-10s_1s_2s_3)}{(60s_1s_3)}f_n \\
&+\frac{-hs_2^3(10s_3-5s_2-5s_2s_3+3s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&+\frac{hs_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&+\frac{hs_2^3(5s_2-10s_1+5s_1s_2-3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \\
&+\frac{-hs_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}f_{n+1}
\end{aligned}
\tag{3.72}
$$

$$
\begin{aligned}
y'_{n+s_3} &= y'_n \\
&+ \frac{-hs_3(10s_1s_3 - 30s_1s_2 + 10s_2s_3 - 5s_1s_3^2 - 5s_2s_3^2 - 5s_3^2 + 3s_3^3 + 10s_1s_2s_3)}{(60s_1s_2)} f_n \\
&+ \frac{hs_3^3(5s_3 - 10s_2 + 5s_2s_3 - 3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{-hs_3^3(5s_3 - 10s_1 + 5s_1s_3 - 3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{hs_3(20s_1s_3 - 30s_1s_2 + 20s_2s_3 - 15s_1s_3^2 - 15s_2s_3^2 - 15s_3^2 + 12s_3^3 + 20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{hs_3^3(10s_1s_2 - 5s_1s_3 - 5s_2s_3 + 3s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))} f_{n+1}
\end{aligned} \tag{3.73}
$$

$$
\begin{aligned}
y'_{n+1} &= y'_n \\
&+ \frac{h(5s_1 + 5s_2 + 5s_3 - 10s_1s_2 - 10s_1s_3 - 10s_2s_3 + 30s_1s_2s_3 - 3)}{(60s_1s_2s_3)} f_n \\
&- \frac{h(10s_2s_3 - 5s_3 - 5s_2 + 3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{h(10s_1s_3 - 5s_3 - 5s_1 + 3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{-h(10s_1s_2 - 5s_2 - 5s_1 + 3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{h(15s_1 + 15s_2 + 15s_3 - 20s_1s_2 - 20s_1s_3 - 20s_2s_3 + 30s_1s_2s_3 - 12)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1}
\end{aligned} \tag{3.74}
$$

The derivative of the block can be represented in matrix form

$$
\acute{Y}_m^{[3]_2} = \acute{B}_2^{[3]_2} R_2^{[3]_2} + h\left[\acute{D}^{[3]_2} R_3^{[3]_2} + h\acute{E}^{[3]_2} R_4^{[3]_2}\right] \tag{3.75}
$$

where

$$
\acute{Y}_m^{[3]_2} = \begin{pmatrix} y'_{n+s_1} \\ y'_{n+s_2} \\ y'_{n+s_3} \\ y'_{n+1} \end{pmatrix}, \quad \acute{B}_2^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$\acute{D}^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & \acute{D}^{[3]_2}_{14} \\ 0 & 0 & 0 & \acute{D}^{[3]_2}_{24} \\ 0 & 0 & 0 & \acute{D}^{[3]_2}_{34} \\ 0 & 0 & 0 & \acute{D}^{[3]_2}_{44} \end{pmatrix}, \; \acute{E}^{[3]_2} = \begin{pmatrix} \acute{E}^{[3]_2}_{11} & \acute{E}^{[3]}_{12} & \acute{E}^{[3]_2}_{13} & \acute{E}^{[3]_2}_{14} \\ \acute{E}^{[3]_2}_{21} & \acute{E}^{[3]_2}_{22} & \acute{E}^{[3]_2}_{23} & \acute{E}^{[3]_2}_{24} \\ \acute{E}^{[3]_2}_{31} & \acute{E}^{[3]_2}_{32} & \acute{E}^{[3]_2}_{33} & \acute{E}^{[3]_2}_{34} \\ \acute{E}^{[3]_2}_{41} & \acute{E}^{[3]_2}_{42} & \acute{E}^{[3]_2}_{43} & \acute{E}^{[3]_2}_{44} \end{pmatrix}$$

and the non-zero terms of $\acute{D}^{[3]_2}$ and $\acute{E}^{[3]_2}$ are given by

$$\begin{aligned}
\acute{D}^{[3]_2}_{14} &= \frac{-s_1(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3)}{(60s_2s_3)} \\
\acute{D}^{[3]_2}_{24} &= \frac{s_2(30s_1s_3-10s_1s_2-10s_2s_3+5s_1s_2^2+5s_2^2s_3+5s_2^2-3s_2^3-10s_1s_2s_3)}{(60s_1s_3)} \\
\acute{D}^{[3]_2}_{34} &= \frac{-s_3(10s_1s_3-30s_1s_2+10s_2s_3-5s_1s_3^2-5s_2s_3^2-5s_3^2+3s_3^3+10s_1s_2s_3)}{(60s_1s_2)} \\
\acute{D}^{[3]_2}_{44} &= \frac{(5s_1+5s_2+5s_3-10s_1s_2-10s_1s_3-10s_2s_3+30s_1s_2s_3-3)}{(60s_1s_2s_3)} \\
\acute{E}^{[3]_2}_{11} &= \frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))} \\
\acute{E}^{[3]_2}_{12} &= \frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \\
\acute{E}^{[3]_2}_{13} &= \frac{-s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \\
\acute{E}^{[3]_2}_{14} &= \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))} \\
\acute{E}^{[3]_2}_{21} &= \frac{-s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \\
\acute{E}^{[3]_2}_{22} &= \frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))} \\
\acute{E}^{[3]_2}_{23} &= \frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \\
\acute{E}^{[3]_2}_{24} &= \frac{-s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}
\end{aligned}$$

$$
\begin{aligned}
\acute{E}_{31}^{[3]_2} &= \frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\\
\acute{E}_{32}^{[3]_2} &= \frac{-s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\\
\acute{E}_{33}^{[3]_2} &= \frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\acute{E}_{34}^{[3]_2} &= \frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\\
\acute{E}_{41}^{[3]_2} &= \frac{-(10s_2s_3-5s_3-5s_2+3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\\
\acute{E}_{42}^{[3]_2} &= \frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\\
\acute{E}_{43}^{[3]_2} &= \frac{-(10s_1s_2-5s_2-5s_1+3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\acute{E}_{44}^{[3]_2} &= \frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))}
\end{aligned}
$$

3.3.1 Establishing Properties of One Step Hybrid Block Method with Generalised Three Off-Step Points for Second Order ODEs

The order, error constant, zero stability, convergence, consistency, and region of absolute stability are examined in this section.

3.3.1.1 Order of One Step Hybrid Block Method with Generalised Three Off-Step Points for Second Order ODEs

In order to find the order of the block (3.66), strategy in section (3.1.1.1) is employed by expanding y and f- function in Taylor series as illustrated below.

$$
\begin{bmatrix}
\begin{array}{l}
\sum_{j=0}^{\infty}\frac{(s_1)^jh^j}{j!}y_n^j-y_n-s_1hy_n'+\frac{h^2s_1^2(5s_1s_2+5s_1s_3-20s_2s_3-2s_1^2s_2-2s_1^2s_3-2s_1^2+s_1^3+5s_1s_2s_3)}{(60s_2s_3)}y_n'' \\
-\frac{s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\
-\frac{s_1^4(5s_3-2s_1-2s_1s_3+s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+2}}{j!}y_n^{j+2} \\
-\frac{s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+2}}{j!}y_n^{j+2} \\
+\frac{s_1^4(5s_2-2s_1-2s_1s_2+s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+2}}{j!}y_n^{j+2}
\end{array} \\
\begin{array}{l}
\sum_{j=0}^{\infty}\frac{(s_2)^jh^j}{j!}y_n^j-y_n-s_2hy_n'-\frac{h^2s_2^2(20s_1s_3-5s_1s_2-5s_2s_3+2s_1s_2^2+2s_2^2s_3+2s_2^2-s_2^3-5s_1s_2s_3)}{(60s_1s_3)}y_n'' \\
+\frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+2}}{j!}y_n^{j+2} \\
+\frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+2}}{j!}y_n^{j+2} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+2}}{j!}y_n^{j+2}
\end{array} \\
\begin{array}{l}
\sum_{j=0}^{\infty}\frac{(s_3)^jh^j}{j!}y_n^j-y_n-s_3hy_n'+\frac{h^2s_3^2(5s_1s_3-20s_1s_2+5s_2s_3-2s_1s_3^2-2s_2s_3^2-2s_3^2+s_3^3+5s_1s_2s_3)}{(60s_1s_2)}y_n'' \\
-\frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\
+\frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+2}}{j!}y_n^{j+2} \\
-\frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+2}}{j!}y_n^{j+2} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+2}}{j!}y_n^{j+2}
\end{array} \\
\begin{array}{l}
\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j-y_n-hy_n'-\frac{h^2(2s_1+2s_2+2s_3-5s_1s_2-5s_1s_3-5s_2s_3+20s_1s_2s_3-1)}{(60s_1s_2s_3)}y_n'' \\
-\frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\
-\frac{(5s_1s_3-2s_3-2s_1+1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+2}}{j!}y_n^{j+2} \\
+\frac{(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+2}}{j!}y_n^{j+2} \\
+\frac{(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+2}}{j!}y_n^{j+2}
\end{array}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Comparing the coefficients of h^j and y^j yields

$$
\bar{C}_0 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; \bar{C}_1 = \begin{bmatrix} s_1-s_1 \\ s_2-s_2 \\ s_3-s_3 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_2 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^2}{2!} + \frac{s_1^2(5s_1s_2+5s_1s_3-20s_2s_3-2s_1^2s_2-2s_1^2s_3-2s_1^2+s_1^3+5s_1s_2s_3)}{(60s_2s_3)} \\
-\frac{s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!} \\
-\frac{s_1^4(5s_3-2s_1-2s_1s_3+s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{s_2^0}{0!} \\
-\frac{s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
+\frac{s_1^4(5s_2-2s_1-2s_1s_2+s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_2)^2}{2!} - \frac{h^2s_2^2(20s_1s_3-5s_1s_2-5s_2s_3+2s_1s_2^2+2s_2^2s_3+2s_2^2-s_2^3-5s_1s_2s_3)}{(60s_1s_3)} \\
+\frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{0!} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_3)^2}{2!} + \frac{s_3^2(5s_1s_3-20s_1s_2+5s_2s_3-2s_1s_3^2-2s_2s_3^2-2s_3^2+s_3^3+5s_1s_2s_3)}{(60s_1s_2)} \\
-\frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{0!} \\
+\frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{1}{2!} - \frac{(2s_1+2s_2+2s_3-5s_1s_2-5s_1s_3-5s_2s_3+20s_1s_2s_3-1)}{(60s_1s_2s_3)} \\
-\frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!} \\
-\frac{(5s_1s_3-2s_3-2s_1+1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+\frac{(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
+\frac{(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{s_3^0}{0!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_3 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^3}{3!} - \frac{s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!} \\
-\frac{s_1^4(5s_3-2s_1-2s_1s_3+s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
-\frac{s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
+\frac{s_1^4(5s_2-2s_1-2s_1s_2+s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!}
\end{gathered} \\[3em]
\begin{gathered}
\frac{(s_2)^3}{3!} + \frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{1!} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+\frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!}
\end{gathered} \\[3em]
\begin{gathered}
\frac{(s_3)^3}{3!} - \frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{1!} \\
+\frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
-\frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!}
\end{gathered} \\[3em]
\begin{gathered}
\frac{1}{3!} - \frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!} \\
-\frac{(5s_1s_3-2s_3-2s_1+1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+\frac{(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
+\frac{(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_4 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^4}{4!} - \frac{s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!} \\
-\frac{s_1^4(5s_3-2s_1-2s_1s_3+s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
-\frac{s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
+\frac{s_1^4(5s_2-2s_1-2s_1s_2+s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_2)^4}{4!} + \frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{2!} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+\frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_3)^4}{4!} - \frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{2!} \\
+\frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
-\frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{1}{4!} - \frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!} \\
-\frac{(5s_1s_3-2s_3-2s_1+1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+\frac{(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
+\frac{(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}
\end{gathered}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_5 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^5}{5!} - \frac{s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \\
-\frac{s_1^4(5s_3-2s_1-2s_1s_3+s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
-\frac{s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
+\frac{s_1^4(5s_2-2s_1-2s_1s_2+s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}
\end{array} \\[2em]
\begin{array}{c}
\frac{(s_2)^5}{5!} + \frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{3!} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
+\frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}
\end{array} \\[2em]
\begin{array}{c}
\frac{(s_3)^5}{5!} - \frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{3!} \\
+\frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
-\frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}
\end{array} \\[2em]
\begin{array}{c}
\frac{1}{5!} - \frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \\
-\frac{(5s_1s_3-2s_3-2s_1+1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
+\frac{(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
+\frac{(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}
\end{array}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_6 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^6}{6!} - \frac{s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!} \\
-\frac{s_1^4(5s_3-2s_1-2s_1s_3+s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
-\frac{s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
+\frac{s_1^4(5s_2-2s_1-2s_1s_2+s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
\\
\frac{(s_2)^6}{6!} + \frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{4!} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+\frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
\\
\frac{(s_3)^6}{6!} - \frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{4!} \\
+\frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
-\frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
\\
\frac{1}{6!} - \frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!} \\
-\frac{(5s_1s_3-2s_3-2s_1+1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+\frac{(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
+\frac{(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!}
\end{array}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_7 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^7}{7!} - \frac{s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \\
-\frac{s_1^4(5s_3-2s_1-2s_1s_3+s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
-\frac{s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
+\frac{s_1^4(5s_2-2s_1-2s_1s_2+s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!}
\end{array} \\ \\
\begin{array}{c}
\frac{(s_2)^7}{7!} + \frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{5!} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
+\frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!}
\end{array} \\ \\
\begin{array}{c}
\frac{(s_3)^7}{7!} - \frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{5!} \\
+\frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
-\frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!}
\end{array} \\ \\
\begin{array}{c}
\frac{1}{7!} - \frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \\
-\frac{(5s_1s_3-2s_3-2s_1+1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
+\frac{(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
+\frac{(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!}
\end{array}
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Comparing the coefficients of h^j and y^j gives the order of the method to be $[5,5,5,5]^T$ with error constant vector

$$
C_7 = \begin{bmatrix}
-\frac{s_1^4(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{50400} \\
\frac{s_2^4(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3)}{50400} \\
\frac{-s_3^4(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3)}{50400} \\
\frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{50400}
\end{bmatrix}
$$

for all $s_1, s_2, s_3 \in (0,1) \setminus \{\{s_2 = \frac{-14s_1s_3+7s_1^2s_3+7s_1^2-4s_1^3}{14s_1-35s_3-7s_1^2+14s_1s_3}\} \cup \{s_1 = \frac{14s_3-7s_2^2s_3-7s_2^2+4s_2^3}{35s_3-14s_1+7s_2^2-14s_2s_3}\}$ $\cup\{s_1 = \frac{-14s_2s_3+7s_2s_3^2+7s_3^2-4s_3^3}{14s_3-35s_2-7s_3^2+14s_2s_3}\} \cup \{s_1 = \frac{-7s_2-7s_3+14s_2s_3+4}{7-14s_2-14s_3+35s_2s_3}\}\}$.

The same strategy as mention earlier is also applied in finding the order of the derivatives of the block(3.75). Expanding $y^{'}$ and f in Taylor series yields.

$$
\left[\begin{array}{l}
\sum_{j=0}^{\infty}\frac{(s_1)^j h^j}{j!}y_n^{j+1} - y_n^{'} + \frac{s_1h(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3)}{(60s_2s_3)}y_n^{''} \\
\quad - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+2} \\
\quad - \frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+1}}{j!}y_n^{j+2} \\
\quad - \frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+1}}{j!}y_n^{j+2} \\
\quad + \frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+1}}{j!}y_n^{j+2} \\
\\
\sum_{j=0}^{\infty}\frac{(s_2)^j h^j}{j!}y_n^{j+1} - y_n^{'} - \frac{s_2h(30s_1s_3-10s_1s_2-10s_2s_3+5s_1s_2^2+5s_2^2s_3+5s_2^2-3s_2^3-10s_1s_2s_3)}{(60s_1s_3)}y_n^{''} \\
\quad + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+2} \\
\quad - \frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+1}}{j!}y_n^{j+2} \\
\quad + \frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+1}}{j!}y_n^{j+2} \\
\quad - \frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+1}}{j!}y_n^{j+2} \\
\\
\sum_{j=0}^{\infty}\frac{(s_3)^j h^j}{j!}y_n^{j+1} - y_n^{'} + \frac{s_3h(10s_1s_3-30s_1s_2+10s_2s_3-5s_1s_3^2-5s_2s_3^2-5s_3^2+3s_3^3+10s_1s_2s_3)}{(60s_1s_2)}y_n^{''} \\
\quad - \frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+2} \\
\quad + \frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+1}}{j!}y_n^{j+2} \\
\quad - \frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+1}}{j!}y_n^{j+2} \\
\quad - \frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+1}}{j!}y_n^{j+2} \\
\\
\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j} - y_n^{'} - \frac{h(5s_1+5s_2+5s_3-10s_1s_2-10s_1s_3-10s_2s_3+30s_1s_2s_3-3)}{(60s_1s_2s_3)}y_n^{''} \\
\quad + \frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+2} \\
\quad - \frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+1}}{j!}y_n^{j+2} \\
\quad + \frac{(10s_2s_3-5s_3-5s_2+3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+1}}{j!}y_n^{j+2} \\
\quad + \frac{(10s_1s_2-5s_2-5s_1+3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+1}}{j!}y_n^{j+2}
\end{array}\right]
=
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]
$$

By comparing the coefficients of the h, we have

$$\bar{C}_0 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\bar{C}_2 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^1}{1!} + \frac{s_1(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3)}{(60s_2s_3)} \\
-\frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!} \\
-\frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{s_2^0}{0!} \\
-\frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
+\frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!}
\end{array} \\[2ex]
\begin{array}{c}
\frac{(s_2)^1}{1!} - \frac{s_2(30s_1s_3-10s_1s_2-10s_2s_3+5s_1s_2^2+5s_2^2s_3+5s_2^2-3s_2^3-10s_1s_2s_3)}{(60s_1s_3)} \\
+\frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{0!} \\
-\frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!}
\end{array} \\[2ex]
\begin{array}{c}
\frac{(s_3)^1}{1!} + \frac{s_3(10s_1s_3-30s_1s_2+10s_2s_3-5s_1s_3^2-5s_2s_3^2-5s_3^2+3s_3^3+10s_1s_2s_3)}{(60s_1s_2)} \\
-\frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{0!} \\
+\frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!}
\end{array} \\[2ex]
\begin{array}{c}
\frac{1}{1!} - \frac{(5s_1+5s_2+5s_3-10s_1s_2-10s_1s_3-10s_2s_3+30s_1s_2s_3-3)}{(60s_1s_2s_3)} \\
+\frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!} \\
-\frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{(10s_2s_3-5s_3-5s_2+3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
+\frac{(10s_1s_2-5s_2-5s_1+3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!}
\end{array}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_3 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^2}{2!} - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_3^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!} \\
-\frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{s_2^1}{1!} \\
-\frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
+\frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_3^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!}
\end{array} \\
\\
\begin{array}{c}
\frac{(s_2)^2}{2!} + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{1!} \\
-\frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+\frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!}
\end{array} \\
\\
\begin{array}{c}
\frac{(s_3)^2}{2!} - \frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{1!} \\
+\frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
-\frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!}
\end{array} \\
\\
\begin{array}{c}
\frac{1}{2!} - \frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!} \\
-\frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+\frac{(10s_2s_3-5s_3-5s_2+3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
+\frac{(10s_1s_2-5s_2-5s_1+3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!}
\end{array}
\end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{bmatrix}
$$

$$
\bar{C}_4 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^3}{3!} - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_3^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!} \\
-\frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{s_2^2}{2!} \\
-\frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
+\frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_3^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}
\end{gathered} \\
\\
\begin{gathered}
\frac{(s_2)^3}{3!} + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{2!} \\
-\frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+\frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}
\end{gathered} \\
\\
\begin{gathered}
\frac{(s_3)^3}{3!} - \frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{2!} \\
+\frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
-\frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}
\end{gathered} \\
\\
\begin{gathered}
\frac{1}{3!} - \frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!} \\
-\frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+\frac{(10s_2s_3-5s_3-5s_2+3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
+\frac{(10s_1s_2-5s_2-5s_1+3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}
\end{gathered}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_5 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^4}{4!} - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_3^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \\
-\frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{s_2^3}{3!} \\
-\frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
+\frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_3^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}
\end{gathered} \\
\\
\begin{gathered}
\frac{(s_2)^4}{4!} + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{3!} \\
-\frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
+\frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
-\frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}
\end{gathered} \\
\\
\begin{gathered}
\frac{(s_3)^4}{4!} - \frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{3!} \\
+\frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
-\frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
-\frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}
\end{gathered} \\
\\
\begin{gathered}
\frac{1}{4!} - \frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \\
-\frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
+\frac{(10s_2s_3-5s_3-5s_2+3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
+\frac{(10s_1s_2-5s_2-5s_1+3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}
\end{gathered}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_6 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^5}{5!} - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_3^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!} \\
- \frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{s_2^4}{4!} \\
- \frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
+ \frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_3^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!}
\end{array} \\[2em]
\begin{array}{c}
\frac{(s_2)^5}{5!} + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{4!} \\
- \frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+ \frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
- \frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!}
\end{array} \\[2em]
\begin{array}{c}
\frac{(s_3)^5}{5!} - \frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{4!} \\
+ \frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
- \frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
- \frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!}
\end{array} \\[2em]
\begin{array}{c}
\frac{1}{5!} - \frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!} \\
- \frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+ \frac{(10s_2s_3-5s_3-5s_2+3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
+ \frac{(10s_1s_2-5s_2-5s_1+3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!}
\end{array}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$\bar{C}_7 = \begin{bmatrix} \begin{array}{c} \frac{(s_1)^6}{6!} - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_3^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \\ - \frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{s_2^5}{5!} \\ - \frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ + \frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_3^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\ \\ \frac{(s_2)^6}{6!} + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{5!} \\ - \frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ + \frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ - \frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\ \\ \frac{(s_3)^6}{6!} - \frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{5!} \\ + \frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ - \frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ - \frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\ \\ \frac{1}{6!} - \frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \\ - \frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ + \frac{(10s_2s_3-5s_3-5s_2+3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ + \frac{(10s_1s_2-5s_2-5s_1+3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \end{array} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, by comparing the coefficient of h, the derivative block has order $[5,5,5,5]^T$ with error constants vector

$$\begin{bmatrix} \frac{-s_1^3(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{7200} \\ \frac{s_2^3(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{7200} \\ \frac{-s_3^3(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{7200} \\ \frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{7200} \end{bmatrix}$$

which is true for all $s_1, s_2, s_3 \in (0,1) \setminus \{\{s_2 = \frac{-5s_1s_3+3s_1^2s_3+3s_1^2-2s_1^3}{5s_1-10s_3-3s_1^2+5s_1s_3}\} \cup \{s_1 = \frac{5s_2s_3-3s_2^2s_3-3s_2^2+2s_2^3}{10s_3-5s_2+3s_2^2-5s_2s_3}\}$ $\cup\{s_1 = \frac{-5s_2s_3+3s_2s_3^2+3s_3^2-2s_3^3}{5s_3-10s_2-3s_3^2+5s_2s_3}\} \cup \{s_1 = \frac{-3s_2-3s_3+5s_2s_3+2}{3-5s_2-5s_3+10s_2s_3}\}\}$.

3.3.1.2 Zero Stability of One Step Hybrid Block Method with Generalised Three Off-Step Points for Second Order ODEs

In order to find the zero-stability of the block (3.66), we only put into consideration the first characteristic polynomial according to definition (3.1.3), that is

$$
\begin{aligned}
\Pi(z) &= |z\, I^{[3]_2} - \bar{B}_1^{[3]_2}| \\
&= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^3(z-1) = 0
\end{aligned}
$$

whose solution is $z = 0,0,0,1$. Similarly, the characteristic polynomial of the derivative block (3.75) is given by

$$
\begin{aligned}
\Pi(z) &= |z\, I^{[3]_2} - \acute{B}_2^{[3]_2}| \\
&= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^3(z-1)
\end{aligned}
$$

which yields $z = 0,0,0,1$. Hence, the conditions in Definition (3.1.3) are satisfied. Therefore, the block method (3.66) and its derivative(3.75) are zero stable.

3.3.1.3 Consistency and Convergent of One Step Hybrid Block Method with Generalised Three Off-Step Points for Second Order ODEs

According to Definition (3.1.4) and Theorem (3.1) the block method (3.66) and its derivatives (3.75) are consistent and convergent.

3.3.1.4 Region of Absolute Stability of One Step Hybrid Block Method with Generalised Three Off-Step Points for Second Order ODEs

Applying (3.29) for one step hybrid block with three generalised off-step points(3.66), it gives

$$\bar{h}(\theta,h)=\frac{I^{[3]_2}Y_m^{[3]_2}(\theta)-B_1^{[3]_2}R_1^{[3]_2}(\theta)}{[\bar{D}^{[3]_2}Y_{R_3^{[3]_2}}(\theta)+\bar{E}^{[3]_2}Y_{R_4^{[3]_2}}(\theta)]} \tag{3.76}$$

where

$$I^{[3]_2}Y_m^{[3]_2}(\theta)=\begin{bmatrix} e^{is_1\theta} & 0 & 0 & 0 \\ 0 & e^{is_2\theta} & 0 & 0 \\ 0 & 0 & e^{is_3\theta} & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{bmatrix}$$

$$\bar{B}_1^{[3]_2}R_1^{[3]_2}(\theta)=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{D}^{[3]_2}Y_{R_3^{[3]_2}}(\theta)=\begin{bmatrix} 0 & 0 & 0 & \bar{D}_{14}^{[3]_2} \\ 0 & 0 & 0 & \bar{D}_{24}^{[3]_2} \\ 0 & 0 & 0 & \bar{D}_{34}^{[3]_2} \\ 0 & 0 & 0 & \bar{D}_{44}^{[3]_2} \end{bmatrix}$$

$$\bar{E}^{[3]_2}Y_{R_4^{[3]_2}}(\theta) = \begin{bmatrix} \bar{E}_{11}^{[3]_2}e^{is_1\theta} & \bar{E}_{12}^{[3]}e^{is_2\theta} & \bar{E}_{13}^{[3]_2}e^{is_3\theta} & \bar{E}_{14}^{[3]_2}e^{i\theta} \\ \bar{E}_{21}^{[3]_2}e^{is_1\theta} & \bar{E}_{22}^{[3]_2}e^{is_2\theta} & \bar{E}_{23}^{[3]_2}e^{is_3\theta} & \bar{E}_{24}^{[3]_2}e^{i\theta} \\ \bar{E}_{31}^{[3]_2}e^{is_1\theta} & \bar{E}_{32}^{[3]_2}e^{is_2\theta} & \bar{E}_{33}^{[3]_2}e^{is_3\theta} & \bar{E}_{34}^{[3]_2}e^{i\theta} \\ \bar{E}_{41}^{[3]_2}e^{is_1\theta} & \bar{E}_{42}^{[3]_2}e^{is_2\theta} & \bar{E}_{43}^{[3]_2}e^{is_3\theta} & \bar{E}_{44}^{[3]_2}e^{i\theta} \end{bmatrix}$$

Simplifying the above matrix and finding its determinant, we have

$$\bar{h}(\theta,h) = \frac{43200(e^{i\theta}-1)}{(s_1s_2s_3(4s_1+4s_2+4s_3-3s_1s_2-3s_1s_3-3s_2s_3+2s_1s_2s_3+s_1s_2s_3e^{i\theta}-5))}$$

Expanding the above equation trigonometrically and equating the imaginary part a to zero, the equation of absolute stability region for one step hybrid block method with three generalised off-step points is obtained as below

$$\bar{h}(\theta,h) = \frac{43200(\cos(\theta)-1)}{(s_1s_2s_3(4s_1+4s_2+4s_3-3s_1s_2-3s_1s_3-3s_2s_3+2s_1s_2s_3+s_1s_2s_3\cos(\theta)-5))} \tag{3.77}$$

3.4 Numerical Results for Solving Second Order ODEs

This section considers some specific numerical method for one, two and three hybrid points.

Substituting $s = \frac{1}{3}$ into Equations (3.16)-(3.17) and (3.18)-(3.19), the following block of one step with one hybrid points and its derivative are obtained

$$\begin{aligned} y_{n+\frac{1}{3}} &= y_n + \frac{h}{3}y_n' + \left[\frac{11}{324}f_n - \frac{1}{648}f_{n+1} + \frac{5}{216}f_{n+\frac{1}{3}}\right] \\ y_{n+1} &= y_n + hy_n' + h^2\left[\frac{1}{12}f_n + \frac{1}{24}f_{n+1} + \frac{3}{8}f_{n+\frac{1}{3}}\right] \end{aligned} \tag{3.78}$$

with derivative

$$
\begin{aligned}
y'_{n+\frac{1}{3}} &= y'_n + h\left[\tfrac{4}{27}f_n - \tfrac{1}{108}f_{n+1} + \tfrac{7}{36}f_{n+\frac{1}{3}}\right] \\
y'_{n+1} &= y'_n + h\left[\tfrac{1}{4}f_{n+1} + \tfrac{3}{4}f_{n+\frac{1}{3}}\right]
\end{aligned}
\tag{3.79}
$$

Based on the approach used in section (3.1.1.1), the block and its derivative above are of order $[3,3]^T$ and $[3,3]^T$ with error constant $[1.486054e^{-4}, -9.259259e^{-4}]^T$ and $[8.573388e^{-4}, -4.629630e^{-3}]^T$ respectively.

In order to find the region of absolute stability of (3.78), $s = \frac{1}{3}$ is substituted into Equation (3.31), this gives

$$
\bar{h}(\theta,h) = \frac{(648(\cos(t)-1))}{(\cos(t)-7)} \tag{3.80}
$$

Equation (3.80) is evaluated at intervals of 30° , this produces tabulated results below.

θ	0	30°	60°	90°	120°	150°	180°
$\bar{h}(\theta,h)$	0	14.1532	49.8462	92.5714	129.6	153.7224	162

Hence, the interval of absolute stability is $(0,162)$ as demonstrated in Figure 3.4 as a region in polar coordinate system.

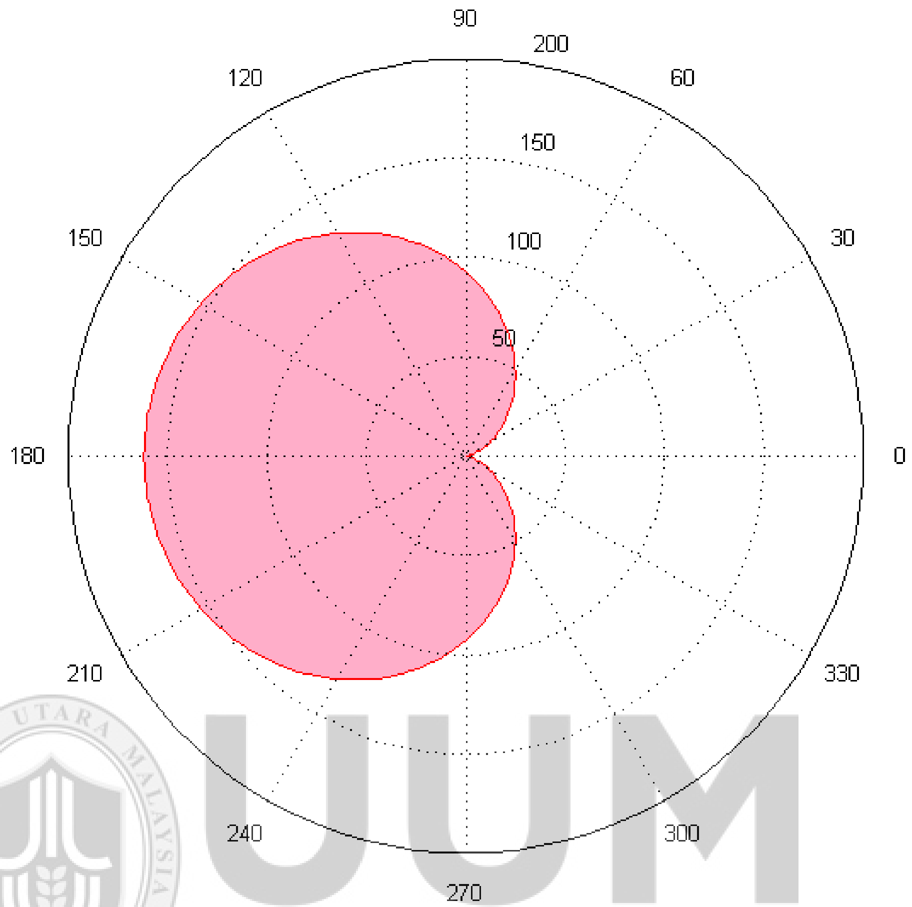

Figure 3.4. Region stability of one step hybrid block method with one off-step point $s = \frac{1}{3}$ for second order ODEs.

The values $s = \frac{1}{10}$ and $r = \frac{1}{5}$ are substituted into Equations (3.44)-(3.46) and (3.47)-(3.49) to give the block and derivative below

$$\begin{aligned}
y_{n+\frac{1}{10}} &= y_n + \frac{h}{10}y_n' + h^2\left[\frac{57}{20000}f_n + \frac{1}{540000}f_{n+1} - \frac{1}{2000}f_{n+\frac{1}{5}} + \frac{143}{54000}f_{n+\frac{1}{10}}\right] \\
y_{n+\frac{1}{5}} &= y_n + \frac{h}{5}y_n' + h^2\left[\frac{49}{7500}f_n + \frac{1}{270000}f_{n+1} - \frac{1}{6000}f_{n+\frac{1}{5}} + \frac{46}{3375}f_{n+\frac{1}{10}}\right] \\
y_{n+1} &= y_n + hy_n' + h^2\left[\frac{3}{4}f_n - \frac{17}{432}f_{n+1} + \frac{25}{16}f_{n+\frac{1}{5}} - \frac{50}{27}f_{n+\frac{1}{10}}\right]
\end{aligned} \tag{3.81}$$

$$\begin{aligned}
y'_{n+\frac{1}{10}} &= y_n' + h\left[\frac{97}{2400}f_n + \frac{1}{28800}f_{n+1} - \frac{19}{1920}f_{n+\frac{1}{5}} + \frac{5}{72}f_{n+\frac{1}{10}}\right] \\
y'_{n+\frac{1}{5}} &= y_n' + h\left[\frac{1}{30}f_n + \frac{1}{30}f_{n+\frac{1}{5}} + \frac{2}{15}f_{n+\frac{1}{10}}\right] \\
y'_{n+1} &= y_n' + h\left[\frac{13}{6}f_n + \frac{2}{9}f_{n+1} + \frac{25}{6}f_{n+\frac{1}{5}} - \frac{50}{9}f_{n+\frac{1}{10}}\right]
\end{aligned} \tag{3.82}$$

Methods (3.81)and (3.82) are of order $[4,4,4]^T$ and $[4,4,4]^T$
with error constant $[5.347222e^{-8}, -2.222222e^{-8}, -3.472222e^{-4}]^T$
and $[-9.930556e^{-7}, -1.111111e^{-7}, -1.180556e^{-3}]^T$ respectively.
Substituting $s = \frac{1}{10}$ and $r = \frac{1}{5}$ into equation (3.81), this yields

$$\bar{h}(\theta,h) = \frac{(3600000(\cos(\theta)-1))}{(\cos(\theta)+157)} \tag{3.83}$$

Evaluating (3.83) at intervals of 30°, this gives the results displayed below.

θ	0	30°	60°	90°	120°	150°	180°
$\bar{h}(\theta,h)$	0	-3055	-11428	-22929.9	-22929.8	-43025	-46153

Hence, the interval of stability gives (0,-46153)as illustrated in Figure 3.5 as a region in coordinate system.

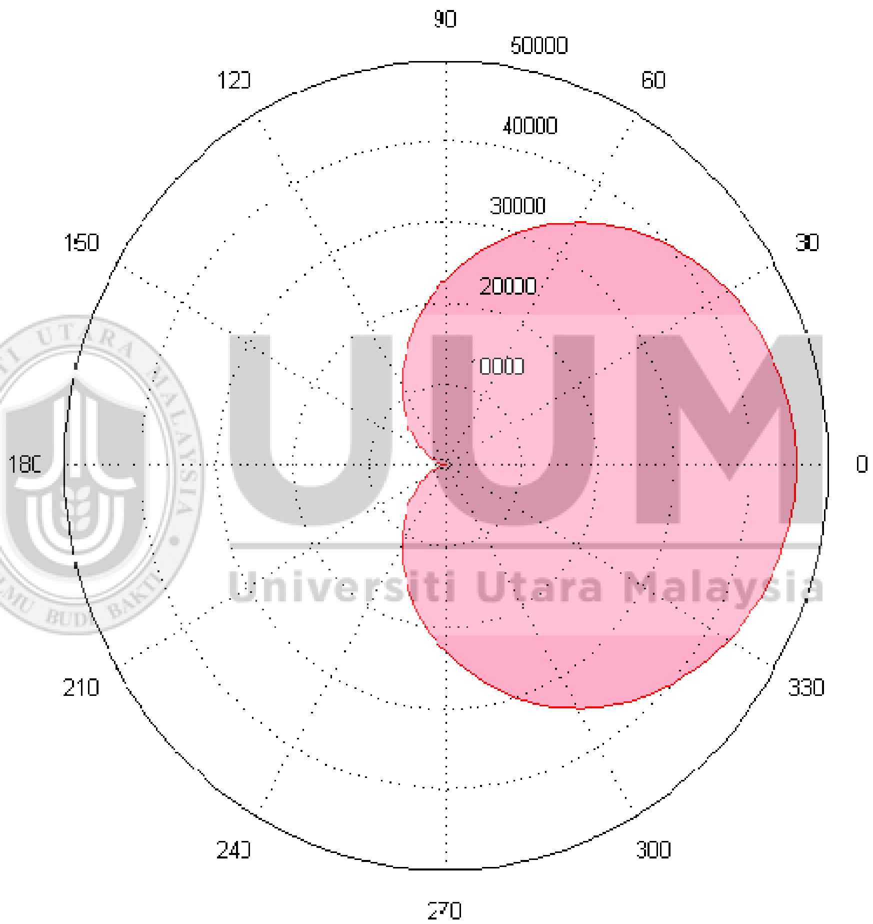

Figure 3.5. Region stability of one step hybrid block method with two off-step points $s = \frac{1}{10}$ and $r = \frac{1}{5}$ for second order ODEs.

The values $s_1 = \frac{1}{8}$, $s_2 = \frac{1}{4}$ and $s_3 = \frac{1}{2}$ are substituted into Equations (3.67) - (3.70) and (3.71) - (3.74) to produce the block and its derivatives in below

$$
\begin{aligned}
y_{n+\frac{1}{8}} &= y_n + \frac{h}{8}y_n^{'} + h^2\Big[\frac{1027}{245760}f_n - \frac{29}{5160960}f_{n+1} + \frac{5}{1008}f_{n+\frac{1}{8}} \\
&\qquad - \frac{137}{92160}f_{n+\frac{1}{4}} + \frac{61}{368640}f_{n+\frac{1}{2}}\Big] \\
y_{n+\frac{1}{4}} &= y_n + \frac{h}{4}y_n^{'} + h^2\Big[+\frac{37}{3840}f_n - \frac{1}{80640}f_{n+1} + \frac{29}{1260}f_{n+\frac{1}{8}} \\
&\qquad - \frac{1}{567}f_{n+\frac{1}{4}} + \frac{1}{2880}f_{n+\frac{1}{2}}\Big] \\
y_{n+\frac{1}{2}} &= y_n + \frac{h}{2}y_n^{'} + h^2\Big[\frac{11}{480}f_n - \frac{1}{10080}f_{n+1} + \frac{16}{315}f_{n+\frac{1}{8}} \\
&\qquad + \frac{2}{45}f_{n+\frac{1}{4}} + \frac{1}{144}f_{n+\frac{1}{2}}\Big] \\
y_{n+1} &= y_n + y_n^{'}h + h^2\Big[\frac{-1}{30}f_n + \frac{1}{63}f_{n+1} + \frac{128}{315}f_{n+\frac{1}{8}} \\
&\qquad - \frac{8}{45}f_{n+\frac{1}{4}} + \frac{13}{45}f_{n+\frac{1}{2}}\Big]
\end{aligned}
\tag{3.84}
$$

$$
\begin{aligned}
y_{n+\frac{1}{8}}^{'} &= y_n^{'} + h\Big[\frac{1427}{30720}f_n - \frac{53}{645120}f_{n+1} + \frac{499}{5040}f_{n+\frac{1}{8}} \\
&\qquad - \frac{263}{11520}f_{n+\frac{1}{4}} + \frac{113}{46080}f_{n+\frac{1}{2}}\Big] \\
y_{n+\frac{1}{4}}^{'} &= y_n^{'} + h\Big[\frac{79}{1920}f_n - \frac{1}{40320}f_{n+1} + \frac{53}{315}f_{n+\frac{1}{8}} \\
&\qquad - \frac{29}{720}f_{n+\frac{1}{4}} + \frac{1}{2880}f_{n+\frac{1}{2}}\Big] \\
y_{n+\frac{1}{2}}^{'} &= y_n^{'} + h\Big[\frac{1}{15}f_n - \frac{1}{1260}f_{n+1} + \frac{16}{315}f_{n+\frac{1}{8}} \\
&\qquad + \frac{13}{45}f_{n+\frac{1}{4}} + \frac{17}{180}f_{n+\frac{1}{2}}\Big] \\
y_{n+1}^{'} &= y_n^{'} + h\Big[\frac{-11}{30}f_n + \frac{89}{630}f_{n+1} + \frac{512}{315}f_{n+\frac{1}{8}} \\
&\qquad - \frac{64}{45}f_{n+\frac{1}{4}} + \frac{46}{45}f_{n+\frac{1}{2}}\Big]
\end{aligned}
\tag{3.85}
$$

Referred on the strategy used in section (3.3.1.1) the block (3.84) and its derivative are of order $[5,5,5,5]^T$ and $[5,5,5,5]^T$ with errors constants $[1.466463e^{-8}, 3.330291e^{-8}, 1.937624e^{-7}, -7.750496e^{-6}]^T$ and $[2.119276e^{-7}, 8.477105e^{-8}, 1.356337e^{-6}, -4.340278e^{-5}]^T$ respectively. Substituting s_1, s_2, s_3 into equation (3.77), this yields

$$\bar{h}(\theta,h) = \frac{(176947200(\cos(\theta)-1)}{(cos(\theta)-136)} \tag{3.86}$$

Evaluating (3.86) at intervals of 30°, the following tabulation are obtained

θ	0	30°	60°	90°	120°	150°	180°
$\bar{h}(\theta,h)$	0	175429	652941	1301082	1301082	2412490	2583200

Therefore, the interval of absolute stability is (0, 2583200)as shown in Figure 3.6 as a region in polar coordinate system.

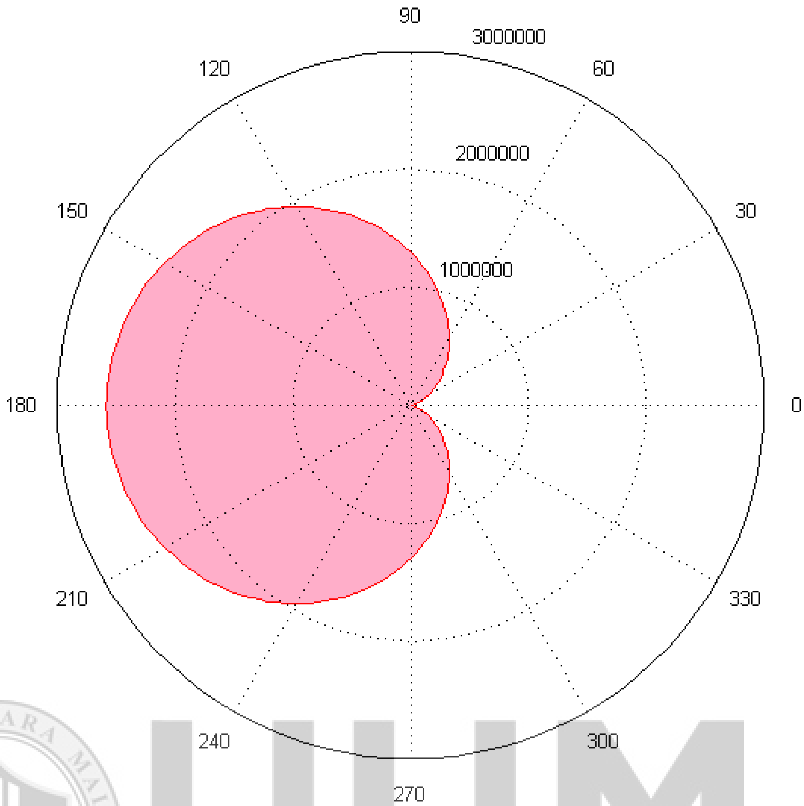

Figure 3.6. Region stability of one step hybrid block method with three off-step points $s_1 = \frac{1}{8}$, $s_2 = \frac{1}{4}$ and $s_3 = \frac{1}{2}$ for second order ODEs

3.4.1 Implementation of Method

Our method is implemented more efficiently by combining the developed hybrid block as simultaneous integrators for IVPs. We proceed by explicitly obtaining initial conditions at x_{n+1}, $n = 0, 1 \ldots N-1$ using the computed value over sub interval $[x_0, x_1], \ldots, [x_{N-1}, x_N]$. For instance, using (3.66), $n = 0$, $[y_{s_1}, y_{s_2}, y_{s_3}, y_1]^T$ are simultaneously obtained over the sub-interval $[x_0, x_1]$, as y_0 is known from the initial value of IVP. For $n = 1$, $[y_{1+s_1}, y_{1+s_2}, y_{1+s_3}, y_2]^T$ are simultaneously obtained over the sub-interval interval $[x_1, x_2]$, as y_1 is known from the previous block, and so on. Hence, the sub-intervals do not over-lap and the solutions obtained in this manner are more accurate than those obtained in the conventional fashion.

In order to compare the performance of our proposed methods with the existing ones, we tested the same problems as used in previous work. It is also observed that most of the problems considered involve small intervals. For the sake of comparison, the same intervals were considered, even though our methods work for larger intervals as well.

Problem 1: $y'' - x(y')^2 = 0, \quad y(0) = 1, \quad y'(0) = \frac{1}{2}, \quad h = \frac{1}{320}.$

Exact solution: $y(x) = 1 + \frac{1}{2} ln\left(\frac{2+x}{2-x}\right) \qquad 0 \leq x \leq 1$

Source: *Adesanya et al.*, 2014

Problem 2: $y'' + (\frac{6}{x})y' + (\frac{4}{x^2})y = 0, \quad y(1) = 1, \quad y'(1) = 1, \quad h = \frac{1}{320}$

Exact solution: $y(x) = \frac{5}{3x} - \frac{2}{3x^4} \qquad 1 \leq x \leq 1.0313$

Source: *Anake*, 2011

Problem 3: $y'' = 2y^3, \quad y(1) = 1, \quad y'(1) = -1, \quad h = \frac{1}{10}$

Exact solution: $y(x) = \frac{1}{x} \qquad 1 \leq x \leq 2$

Source: *Yahaya et al.*, 2013

Problem 4: $y'' - y' = 0, \quad y(0) = 0, \quad y'(0) = -1, \quad h = \frac{1}{10}$

Exact solution: $y(x) = 1 - e^x \qquad 0 \leq x \leq 1$

Source: *Adeniyi and Adeyefa*, 2013

Problem 5: $2yy'' - (y')^2 + 4y^2 = 0, \quad y(\frac{\pi}{6}) = \frac{1}{4}, \quad y'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}, \quad h = \frac{1}{320}$

Exact solution: $y(x) = \sin^2 x \qquad 0.544 \leq x \leq 0.624$

Source: *Kayode and Adeyeye*, 2013

Problem 6: $y'' - 4y' + 8y = x^3, \quad y(0) = 2, \quad y'(0) = 4, \quad h = \frac{1}{10}$

Exact solution: $y(x) = e^{2x}(2\cos(2x) - \frac{3}{64}\sin(2x)) + \frac{3}{32}x + \frac{3}{16}x^2 + \frac{1}{8}x^3$

Source: *Jator*, 2009 $\qquad 0 \leq x \leq 1$

Problem 7: $y'' - y = 0, \qquad y(0) = 1, \quad y'(0) = 1, \qquad h = 0.1$

Exact solution: $y(x) = e^x \qquad 0 \leq x \leq 1$

Source: *Sagir*, 2012

Table 3.1

Comparison of the New Methods with Two Step Hybrid Block Method (Adesanya et al.,2014) for Solving Problem 1 where $h = \frac{1}{100}$

x		$s=\frac{1}{4}, P=3$	$s=\frac{1}{10}, r=\frac{1}{5}, P=4$	$s_1=\frac{1}{5}, s_2=\frac{2}{5}, s_3=\frac{3}{5}, P=5$	Adesanya(2014), $P=4$
0.1	Exact solution	1.050041729278491400	1.050041729278491400	1.050041729278491400	1.050041729278491
	Computed solution	1.050041729303426100	1.050041729278490000	1.050041729278490900	1.05004172927851
	Error	$2.493472e^{-11}$	$1.332268e^{-15}$	$4.440892e^{-16}$	$2.3314e^{-14}$
0.2	Exact solution	1.100335347731075600	1.100335347731075600	1.100335347731075600	1.100335347731076
	Computed Solution	1.100335347831791400	1.100335347731064500	1.100335347731076000	1.10033534773126
	Error	$1.007159e^{-10}$	$1.110223e^{-14}$	$4.440892e^{-16}$	$1.8918e^{-13}$
0.3	Exact solution	1.151140435936467000	1.151140435936467000	1.151140435936467000	1.151144043593646
	Computed solution	1.151140436159892300	1.151140435936421000	1.151140435936466300	1.15114043593712
	Error	$2.234253e^{-10}$	$4.596323e^{-14}$	$6.661338e^{-16}$	$6.5836e^{-13}$
0.4	Exact solution	1.202732554054082300	1.202732554054082300	1.202732554054082300	1.202732554054082
	Computed solution	1.202732554438952200	1.202732554053943100	1.202732554054081000	1.20273255405572
	Error	$3.848699e-10$	$1.392220e^{-13}$	$1.332268e^{-15}$	$1.6406e^{-12}$
0.5	Exact Solution	1.255412811882995500	1.255412811882995500	1.255412811882995500	1.255412811882995
	Computed solution	1.255412812452661600	1.255412811882627800	1.255412811882991300	1.25541281188643
	Error	$5.696661e^{-10}$	$3.677059e^{-13}$	$4.218847e^{-15}$	$3.4350e^{-12}$
0.6	Exact Solution	1.309519604203111900	1.309519604203111900	1.309519604203111900	1.309519604203112
	Computed solution	1.309519604952894300	1.309519604202227200	1.309519604203101000	1.30951960420960
	Error	$7.497825e^{-10}$	$8.846257e^{-13}$	$1.088019e^{-14}$	$6.4921e^{-12}$
0.7	Exact solution	1.365443754271396400	1.365443754271396400	1.365443754271396400	1.365443754271396
	Computed solution	1.365443755145816100	1.365443754269402700	1.365443754271372400	1.36544375428292
	Error	$8.744196e^{-10}$	$1.993739e^{-12}$	$2.398082e^{-14}$	$1.1529e^{-11}$
0.8	Exact solution	1.423648930193602200	1.423648930193602200	1.423648930193602200	1.423648930193602
	Computed solution	1.423648931044523100	1.423648930189312700	1.423648930193550200	1.42364893021330
	Error	$8.509209e^{-10}$	$4.289458e^{-12}$	$5.195844e^{-14}$	$1.9728e^{-11}$
0.9	Exact solution	1.484700278594052000	1.484700278594052000	1.484700278594052000	1.484700278594052
	Computed Solution	1.484700279101696100	1.484700278585092900	1.484700278593942700	1.48470027862716
	Error	$5.076441e^{-10}$	$8.959056e^{-12}$	$1.092459e^{-13}$	$3.3111e^{-11}$
1.0	Exact solution	1.549306144334055400	1.549306144334055400	1.549306144334055400	1.549306144334055
	Computed solution	1.549306143852831100	1.549306144315616400	1.549306144333828700	1.54930614438934
	Error	$4.812244e^{-10}$	$1.843903e^{-11}$	$2.267075e^{-13}$	$5.5286e^{-11}$

Table 3.2

Comparison of the New Methods with One Step Hybrid Block Method (Anake, 2011) for Solving Problem 2 where $h = \frac{1}{320}$

x		$s=\frac{1}{4}, P=3$	$s=\frac{1}{10}, r=\frac{1}{5}, P=4$	$s_1=\frac{1}{8}, s_2=\frac{1}{4}, s_3=\frac{1}{2}, P=5$	Anake(2011), $P=5$
1.0031	Exact solution	1.003076525857696100	1.003076525857696100	1.003076525857696100	$0.10030765e^{+1}$
	Computed solution	1.003076525860357300	1.003076525857683700	1.003076525857696100	$0.10030765e^{+1}$
	Error	$2.661205e^{-12}$	$1.243450e^{-14}$	$0.000000e^{00}$	$0.77009510e^{-11}$
1.0063	Exact solution	1.006057503083516400	1.006057503083516400	1.006057503083516400	$0.10060575e^{+1}$
	Computed Solution	1.006057503053025200	1.006057503083672900	1.006057503083712200	$0.10060575e^{+1}$
	Error	$3.049117e^{-11}$	$1.565414e^{-13}$	$1.958433e^{-13}$	$0.71779915e^{-9}$
1.0094	Exact solution	1.008944995088837600	1.008944995088837600	1.008944995088837600	$0.10089450e^{+1}$
	Computed solution	1.008944994991059600	1.008944995089336300	1.008944995089416000	$0.10089450e^{+1}$
	Error	$9.777801e^{-11}$	$4.987122e^{-13}$	$5.784262e^{-13}$	$0.19176578e^{-8}$
1.0125	Exact solution	1.011741018167988700	1.011741018167988700	1.011741018167988700	$0.10117410e^{+1}$
	Computed solution	1.011741017970402500	1.011741018168993600	1.011741018169126400	$0.10117410e^{+1}$
	Error	$1.975862e^{-10}$	$1.004974e^{-12}$	$1.137757e^{-12}$	$0.35708470e^{-8}$
1.0156	Exact Solution	1.014447542686413900	1.014447542686413900	1.014447542686413900	$0.10144475e^{+1}$
	Computed solution	1.014447542358049500	1.014447542688081200	1.014447542688279300	$0.10144475e^{+1}$
	Error	$3.283644e^{-10}$	$1.667333e^{-12}$	$1.865397e^{-12}$	$0.56571030e^{-8}$
1.0188	Exact Solution	1.017066494235672900	1.017066494235672900	1.017066494235672900	$0.10170665e^{+1}$
	Computed solution	1.017066493747049500	1.017066494238150000	1.017066494238424900	$0.10170665e^{+1}$
	Error	$4.886234e^{-10}$	$2.477130e^{-12}$	$2.752021e^{-12}$	$0.81569016e^{-8}$
1.0219	Exact solution	1.019599754756288100	1.019599754756288100	1.019599754756288100	$0.10195998e^{+1}$
	Computed solution	1.019599754079357800	1.019599754759715500	1.019599754760078100	$0.10195998e^{+1}$
	Error	$6.769303e^{-10}$	$3.427481e^{-12}$	$3.790079e^{-12}$	$0.11051428e^{-7}$
1.0250	Exact solution	1.022049163629432200	1.022049163629432200	1.022049163629432200	$0.10220492e^{+1}$
	Computed solution	1.022049162737523700	1.022049163633943300	1.022049163634403800	$0.10220492e^{+1}$
	Error	$8.919085e^{-10}$	$4.511058e^{-12}$	$4.971579e^{-12}$	$0.14322554e^{-7}$
1.0281	Exact solution	1.024416518738402900	1.024416518738402900	1.024416518738402900	$0.10244165e^{+1}$
	Computed Solution	1.024416517606167500	1.024416518744123900	1.024416518744692300	$0.10244165e^{+1}$
	Error	$1.132235e^{-9}$	$5.720979e^{-12}$	$6.289413e^{-12}$	$0.17952815e^{-7}$
1.0313	Exact solution	1.026703577500806200	1.026703577500806200	1.026703577500806200	$0.10267036e^{+1}$
	Computed solution	1.026703576104165600	1.026703577507856100	1.026703577508541800	$0.10267036e^{+1}$
	Error	$1.396641e^{-9}$	$7.049916e^{-12}$	$7.735590e^{-12}$	$0.21925381e^{-7}$

Table 3.3

Comparison of the New Methods with Three Step Hybrid Block Method (Yahaya et al., 2013) for Solving Problem 3 where $h = \frac{1}{10}$

x		$s=\frac{51}{100}, P=3$	$s=\frac{1}{5}, r=\frac{3}{5}, P=4$	$s_1=\frac{1}{5}, s_2=\frac{2}{5}, s_3=\frac{3}{5}, P=5$	Yahaya et al.(2013), $P=5$
	Exact solution	0.909090909090909060	0.909090909090909060	0.909090909090909060	0.909090109
1.1	Computed solution	0.909092388468811570	0.909090908583565780	0.909090908598411910	0.9090914826
	Error	$1.479378e^{-6}$	$5.073433e^{-10}$	$4.924972e^{-10}$	$1.37360e^{-6}$
	Exact solution	0.833333333333333260	0.833333333333333260	0.833333333333333260	0.833333333
1.2	Computed Solution	0.833335168388707470	0.833333483910495660	0.833333406122353400	0.8333348875
	Error	$1.835055e^{-6}$	$1.505772e^{-7}$	$7.278902e^{-8}$	$1.55450e^{-6}$
	Exact solution	0.769230769230769050	0.769230769230769050	0.769230769230769050	0.769230769
1.3	Computed solution	0.769232417519202790	0.769231152957930870	0.769230953379247300	0.7692330259
	Error	$1.648288e^{-6}$	$3.837272e^{-7}$	$1.841485e^{-7}$	$2.25690e^{-6}$
	Exact solution	0.714285714285714080	0.714285714285714080	0.714285714285714080	0.714285714
1.4	Computed solution	0.714286896719551030	0.714286385052843030	0.714286034061476950	0.7142880945
	Error	$1.182434e^{-6}$	$6.707671e^{-7}$	$3.197758e^{-7}$	$2.38050e^{-6}$
	Exact Solution	0.666666666666666520	0.666666666666666520	0.666666666666666520	0.666666667
1.5	Computed solution	0.666667226784077300	0.666667667522848890	0.666667141320006600	0.6666693006
	Error	$5.601174e^{-7}$	$1.000856e^{-6}$	$4.746533e^{-7}$	$2.63360e^{-6}$
	Exact Solution	0.625	0.625	0.625	0.625
1.6	Computed solution	0.624999838482378610	0.625001370784780800	0.625000647472223500	0.6250029040
	Error	$1.615176e^{-7}$	$1.370785e^{-6}$	$6.474722e^{-7}$	$2.90400e^{-6}$
	Exact solution	0.588235294117646860	0.588235294117646860	0.588235294117646860	0.588235294
1.7	Computed solution	0.588234337134730280	0.588237074932675070	0.588236132653105040	0.5882382492
	Error	$9.569829e^{-7}$	$1.780815e^{-6}$	$8.385355e^{-7}$	$2.95520e^{-6}$
	Exact solution	0.555555555555555360	0.555555555555555360	0.555555555555555360	0.555555556
1.8	Computed solution	0.555553739038619040	0.555557788368472580	0.555555378041913790	0.5555586357
	Error	$1.816517e^{-6}$	$2.232813e^{-6}$	$1.775136e^{-7}$	$3.07970e^{-6}$
	Exact solution	0.526315789473683960	0.526315789473683960	0.526315789473683960	0.526315789
1.9	Computed Solution	0.526313051174458700	0.526318518853825190	0.526317069209899850	0.5263190397
	Error	$2.738299e^{-6}$	$2.729380e^{-6}$	$1.279736e^{-6}$	$3.25070e^{-6}$
	Exact solution	0.5	0.5	0.5	0.5
2.0	Computed solution	0.499996275468830630	0.500003273441148590	0.500001532616930370	0.5000032814
	Error	$3.724531e^{-6}$	$3.273441e^{-6}$	$1.532600e^{-6}$	$3.28140e^{-6}$

Table 3.4

Comparison of the New Methods with One Step Hybrid Block Method (Adeniyi and Adeyefa, 2013) for Solving Problem 4 where $h = \frac{1}{10}$

x		$s=\frac{2}{5}, P=3$	$s=\frac{1}{5}, r=\frac{3}{5}, P=4$	$s_1=\frac{1}{5}, s_2=\frac{2}{5}, s_3=\frac{3}{5}, P=5$	Adeniyi & Adeyefa(2013), $P=3$
0.1	Exact solution	-0.105170918075647710	-0.105170918075647710	-0.105170918075647710	-0.105170918
	Computed solution	-0.105170916666666670	-0.105170918055555580	-0.105170918075555580	-0.105170902
	Error	$1.408981e^{-9}$	$2.009214e^{-11}$	$9.213463e^{-14}$	$0.160756e^{-7}$
0.2	Exact solution	-0.221402758160169850	-0.221402758160169850	-0.221402758160169850	-0.221402758
	Computed Solution	-0.221402707329405870	-0.221402749205445590	-0.221402758011792180	-0.221402723
	Error	$5.083076e^{-8}$	$8.954724e^{-9}$	$1.483777e^{-10}$	$0.351602e^{-7}$
0.3	Exact solution	-0.349858807576003180	-0.349858807576003180	-0.349858807576003180	-0.349858807
	Computed solution	-0.349858649227159590	-0.349858778897228880	-0.349858807099976080	-0.34985857
	Error	$1.583488e^{-7}$	$2.867877e^{-8}$	$4.760271e^{-10}$	$0.237576e^{-6}$
0.4	Exact solution	-0.491824697641270350	-0.491824697641270350	-0.491824697641270350	-0.491824697
	Computed solution	-0.491824362004643470	-0.491824636278388410	-0.491824696622142730	-0.491824433
	Error	$3.356366e^{-7}$	$6.136288e^{-8}$	$1.019128e^{-9}$	$0.2646413e^{-6}$
0.5	Exact Solution	-0.648721270700128190	-0.648721270700128190	-0.648721270700128190	-0.64872127
	Computed solution	-0.648720674519802420	-0.648721161185205620	-0.648721268880754320	-0.648720974
	Error	$5.961803e^{-7}$	$1.095149e^{-7}$	$1.819374e^{-9}$	$0.2967001e^{-6}$
0.6	Exact Solution	-0.822118800390508890	-0.822118800390508890	-0.822118800390508890	-0.8221188
	Computed solution	-0.822117844859362350	-0.822118624363576860	-0.822118797465661970	-0.822118466
	Error	$9.555311e^{-7}$	$1.760269e^{-7}$	$2.924847e^{-9}$	$0.3343905e^{-6}$
0.7	Exact solution	-1.013752707470476600	-1.013752707470476600	-1.013752707470476600	-1.013752707
	Computed solution	-1.013751275880001600	-1.013752443242303500	-1.013752703079578500	-1.013752329
	Error	$1.431590e^{-6}$	$2.642282e^{-7}$	$4.390898e^{-9}$	$0.3784705e^{-6}$
0.8	Exact solution	-1.225540928492467400	-1.225540928492467400	-1.225540928492467400	-1.225540928
	Computed solution	1.225538883560262300	-1.225540550547354900	-1.225540922211324500	-1.225540498
	Error	$-2.044932e^{-6}$	$3.779451e^{-7}$	$6.281143e^{-9}$	$0.4304925e^{-6}$
0.9	Exact solution	-1.459603111156949400	-1.459603111156949400	-1.459603111156949400	-1.459603111
	Computed Solution	-1.459600291990074600	-1.459602589587767800	-1.459603102488357100	-1.45960262
	Error	$2.819167e^{-6}$	$5.215692e^{-7}$	$8.668592e^{-9}$	$0.4911569e^{-6}$
1.0	Exact solution	-1.718281828459045100	-1.718281828459045100	-1.718281828459045100	-1.718281828
	Computed solution	-1.718278047106213300	-1.718281128325760600	-1.718281816822123900	-1.718281267
	Error	$3.781353e^{-6}$	$7.001333e^{-7}$	$1.163692e^{-8}$	$0.561459e^{-6}$

Table 3.5

Comparison of the New Methods with Two Step Hybrid Block Method (Kayode and Adeyeye, 2013) for Solving Problem 5 where $h = \frac{1}{100}$

x		$s=\frac{1}{5}, P=3$	$s=\frac{1}{5}, r=\frac{3}{5}, P=4$	$s_1=\frac{1}{4}, s_2=\frac{1}{2}, s_3=\frac{3}{4}, P=5$	Kayode and Adeyeye(2013), $P=6$
0.544	Exact solution	0.267515862977780850	0.267515862977780850	0.267515862977780850	0.26751586298
	Computed solution	0.267515863065587330	0.267515862977377780	0.267515862977780130	0.26751586348
	Error	$8.780648e^{-11}$	$4.030665e^{-13}$	$7.216450e^{-16}$	$4.04e^{-10}$
0.554	Exact solution	0.276415041478145830	0.276415041478145830	0.276415041478145830	0.27641504148
	Computed Solution	0.276415041728347910	0.276415041476929190	0.276415041478143550	0.27641504257
	Error	$2.502021e^{-10}$	$1.216638e^{-12}$	$2.275957e^{-15}$	$1.10e^{-9}$
0.564	Exact solution	0.285403650980826370	0.285403650980826370	0.285403650980826370	0.28540365098
	Computed solution	0.285403651470813420	0.285403650978377500	0.285403650980821990	0.28540365300
	Error	$4.899871e^{-10}$	$2.448874e^{-12}$	$4.385381e^{-15}$	$2.02e^{-9}$
0.574	Exact solution	0.294478096161868210	0.294478096161868210	0.294478096161868210	0.29447809616
	Computed solution	0.294478096967724480	0.294478096157760990	0.294478096161860880	0.29447809933
	Error	$8.058563e^{-10}$	$4.107215e^{-12}$	$7.327472e^{-15}$	$3.17e^{-9}$
0.584	Exact Solution	0.303634747364189770	0.303634747364189770	0.303634747364189770	0.30363474736
	Computed solution	0.303634748560621340	0.303634747357991620	0.303634747364178780	0.30363475191
	Error	$1.196432e^{-9}$	$6.198153e^{-12}$	$1.099121e^{-14}$	$4.55e^{-9}$
0.594	Exact Solution	0.312869942049397220	0.312869942049397220	0.312869942049397220	0.31286994205
	Computed solution	0.312869943709664590	0.312869942040669650	0.312869942049382010	0.31286994820
	Error	$1.660267e^{-9}$	$8.727574e^{-12}$	$1.521006e^{-14}$	$6.15e^{-9}$
0.604	Exact solution	0.322179986262750740	0.322179986262750740	0.322179986262750740	0.32217998626
	Computed solution	0.322179988458606510	0.322179986251049880	0.322179986262730590	0.32217999423
	Error	$2.195856e^{-9}$	$1.170086e^{-11}$	$2.015055e^{-14}$	$7.97e^{-9}$
0.614	Exact solution	0.331561156110697200	0.331561156110697200	0.331561156110697200	0.33156115611
	Computed solution	0.331561158912329180	0.331561156095574460	0.331561156110671610	0.33156116611
	Error	$2.801632e^{-9}$	$1.512274e^{-11}$	$2.559064e^{-14}$	$9.99e^{-9}$
0.624	Exact solution	0.341009699250378160	0.341009699250378160	0.341009699250378160	0.34100969925
	Computed Solution	0.341009702726356780	0.341009699231380640	0.341009699250346300	0.34100971149
	Error	$3.475979e^{-9}$	$1.899753e^{-11}$	$3.186340e^{-14}$	$1.22e^{-8}$

Table 3.6

Comparison of the New Methods with Four Step Linear Multistep Method(Jator,2009) for Solving Problem 6 where $h = \frac{1}{100}$

x		$s=\frac{1}{5}, r=\frac{4}{5}, P=4$	$s_1=\frac{1}{5}, s_2=\frac{2}{5}, s_3=\frac{3}{5}, P=5$	Jator(2009), $P=5$
0.1	Exact solution	2.394112576996395800	2.394112576996395800	2.39411257699639579
	Computed solution	2.394112595491555000	2.394112577175466300	2.39410746995856893
	Error	$1.849516e^{-8}$	$1.790705e^{-10}$	$5.10704e^{-6}$
0.2	Exact solution	2.748141332426423200	2.748141332426423200	2.74814133242642366
	Computed Solution	2.748150713506311200	2.748141299565079800	2.74812637387718794
	Error	$9.3.81080e^{-6}$	$3.286134e^{-8}$	$14.9586e^{-6}$
0.3	Exact solution	3.007866940511068500	3.007866940511068500	3.00786694051106806
	Computed solution	3.007897909662759200	3.007866653657193200	3.00783908734558069
	Error	$30.96915e^{-6}$	$2.868539e^{-7}$	$27.8532e^{-6}$
0.4	Exact solution	3.101762405774207900	3.101762405774207900	3.10176240577420792
	Computed solution	3.101829065044717100	3.101761319932946100	3.10171951500341869
	Error	$66.65927e^{-6}$	$1.085841e^{-6}$	$42.8908e^{-6}$
0.5	Exact Solution	2.939543100745262400	2.939543100745262400	2.93954310074526183
	Computed solution	2.939659340690647900	2.939540166412848000	2.9394760700463145
	Error	$11.62399e^{-5}$	$2.934332e^{-6}$	$67.0307e^{-6}$
0.6	Exact Solution	2.939543100745262400	2.411836534415714900	2.41183653441571488
	Computed solution	2.939659340690647900	2.411829971821199500	2.41173389792982373
	Error	$11.62399e^{-5}$	$6.562595e^{-6}$	$102.637e^{-6}$
0.7	Exact solution	1.391554830489844200	1.391554830489844200	1.39155483048984374
	Computed solution	1.391791521707820900	1.391541863294833800	1.39140992309112743
	Error	$236.6912e^{-6}$	$12.96720e^{-6}$	$144.907e^{-6}$
0.8	Exact solution	−0.262326758334356260	−0.262326758334356260	−0.262326758334358522
	Computed solution	−0.262045020204241140	−0.262350196302984010	−0.262517663191533134
	Error	$281.7381e^{-6}$	$23.43797e^{-6}$	$190.905e^{-6}$
0.9	Exact solution	−2.697771160773068100	−2.697771160773068100	−2.69777116077307121
	Computed Solution	−2.697486782192122200	−2.697810719833169100	−2.69801089333982879
	Error	$284.3786e^{-6}$	$39.55906e^{-6}$	$239.733e^{-6}$
1.0	Exact solution	−6.058560720845662200	−6.058560720845662200	−6.05856072084566754
	Computed solution	−6.058355394529937300	−6.058623888533274400	−6.05885539046894461
	Error	$205.3263e^{-6}$	$63.16769e^{-6}$	$294.670e^{-6}$

Table 3.7

Comparison of the New Methods with Three Step Hybrid Block Method (Sagir, 2012) for Solving Problem 7 where $h = \frac{1}{10}$

x		$s=\frac{2}{5}, P=3$	$s=\frac{1}{5}, r=\frac{3}{5}, P=4$	$s_1=\frac{1}{5}, s_2=\frac{2}{5}, s_3=\frac{3}{5}, P=5$	Sagir(2012), $P=5$
0.1	Exact solution	1.105170918075647700	1.105170918075647700	1.105170918075647700	1.105170918
	Computed solution	1.105170918210185200	1.105170918075807600	1.105170918075815800	1.105170918
	Error	$1.345375e^{-10}$	$1.598721e^{-13}$	$1.680878e^{-13}$	$00e^{0}$
0.2	Exact solution	1.221402758160169900	1.221402758160169900	1.221402758160169900	1.221402758
	Computed Solution	1.221402777779802700	1.221402758158494500	1.221402758160181400	1.221402758
	Error	$1.961963e^{-8}$	$1.675327e^{-12}$	$1.154632e^{-14}$	$0000e^{0}$
0.3	Exact solution	1.349858807576003200	1.349858807576003200	1.349858807576003200	1.349858808
	Computed solution	1.349858868261282000	1.349858807570269300	1.349858807575497800	1.349858807
	Error	$6.068528e^{-8}$	$5.733858e^{-12}$	$5.053735e^{-13}$	$5.76e^{-10}$
0.4	Exact solution	1.491824697641270300	1.491824697641270300	1.491824697641270300	1.491824698
	Computed solution	1.491824823632744900	1.491824697628982200	1.491824697639844600	1.491824696
	Error	$1.259915e^{-7}$	$1.228817e^{-11}$	$1.425748e^{-12}$	1.6413^{-9}
0.5	Exact Solution	1.648721270700128200	1.648721270700128200	1.648721270700128200	1.648721271
	Computed solution	1.648721489377484000	1.648721270678467700	1.648721270697327500	1.648721269
	Error	$2.186774e^{-7}$	$2.166045e^{-11}$	$2.800649e^{-12}$	$1.7001e^{-9}$
0.6	Exact Solution	1.822118800390508900	1.822118800390508900	1.822118800390508900	1.822118800
	Computed solution	1.822119142807998400	1.822118800356280900	1.822118800385818400	1.822118798
	Error	$3.424175e^{-7}$	$3.422795e^{-11}$	$4.690470e^{-12}$	$2.3905e^{-9}$
0.7	Exact solution	2.013752707470476600	2.013752707470476600	2.013752707470476600	2.013752707
	Computed solution	2.013753208956616400	2.013752707420045900	2.013752707463311700	2.013752704
	Error	$5.014861e^{-7}$	$5.043077e^{-11}$	$7.164935e^{-12}$	3.4705^{-9}
0.8	Exact solution	2.225540928492467400	2.225540928492467400	2.225540928492467400	2.225540928
	Computed solution	2.225541629322902000	2.225540928421692500	2.225540928482161900	2.225540924
	Error	$7.008304e^{-7}$	$7.077494e^{-11}$	$1.030553e^{-11}$	$4.4925e^{-9}$
0.9	Exact solution	2.459603111156949400	2.459603111156949400	2.459603111156949400	2.459603111
	Computed Solution	2.459604057310402300	2.459603111061100700	2.459603111142743000	2.459603107
	Error	$9.461535e^{-7}$	$9.584866e^{-11}$	$1.420641e^{-11}$	$4.1569e^{-9}$
1.0	Exact solution	2.718281828459045100	2.718281828459045100	2.718281828459045100	2.718281828
	Computed solution	2.718283072467460000	2.718281828332719900	2.718281828440069200	2.718281824
	Error	$1.244008e^{-6}$	$1.263252e^{-10}$	$1.897593e^{-11}$	$4.4590e^{-9}$

Table 3.8

Comparison of the New Methods with Three Step Hybrid Method (Kayode and Obarhua, 2015) for Solving Problem 1 where $h = \frac{1}{100}$

x		$s = \frac{1}{10}, r = \frac{1}{2}, P = 4$	$s_1 = \frac{1}{4}, s_2 = \frac{2}{4}, s_3 = \frac{3}{4}, P = 5$	Kayode and Obarhua(2015), $P = 7$
0.1	Exact solution	1.05004172927849140	1.050041729278491400	1.050041729278
	Computed solution	1.050041729278490000	1.050041729278490700	1.050041729281
	Error	$1.332268e^{-15}$	$6.661338e^{-16}$	$2.312595e^{-12}$
0.2	Exact solution	1.10033534773107530	1.100335347731075600	1.100335347731
	Computed Solution	1.100335347731064500	1.100335347731075800	1.100335347742
	Error	$1.110223e^{-14}$	$2.220446e^{-16}$	$1.088329e^{-11}$
0.3	Exact solution	1.15114043593646650	1.151140435936467000	1.151140435936
	Computed solution	1.151140435936421000	1.151140435936466100	1.151140435961
	Error	$4.596323e^{-14}$	$8.881784e^{-16}$	$2.430833e^{-11}$
0.4	Exact solution	1.20273255405408160	1.202732554054082300	1.202732554054
	Computed solution	1.202732554053943100	1.202732554054079900	1.202732554094
	Error	$1.392220e^{-13}$	$2.442491e^{-15}$	$4.018186e^{-11}$
0.5	Exact Solution	1.25541281188299460	1.255412811882995500	1.255412811883
	Computed solution	1.255412811882627800	1.255412811882989700	1.255412811937
	Error	$3.677059e^{-13}$	$5.773160e^{-15}$	$5.422818e^{-11}$
0.6	Exact Solution	1.30951960420311190	1.309519604203111900	1.309519604203
	Computed solution	1.309519604202227200	1.309519604203098500	1.309519604262
	Error	$8.846257e^{-13}$	$1.332268e^{-14}$	$5.901679e^{-11}$
0.7	Exact solution	1.36544375427139710	1.365443754271396400	1.365443754271
	Computed solution	1.365443754269402700	1.365443754271367800	1.365443754313
	Error	$1.993739e^{-12}$	$2.864375e^{-14}$	$4.161738e^{-11}$
0.8	Exact solution	1.42364893019360350	1.423648930193602200	1.423648930194
	Computed solution	1.423648930189312700	1.423648930193543100	1.423648930173
	Error	$4.289458e^{-12}$	$5.906386e^{-14}$	$2.077827e^{-11}$
0.9	Exact solution	1.48470027859405460	1.484700278594052000	1.484700278594
	Computed Solution	1.484700278585092900	1.484700278593932300	1.484700278425
	Error	$8.959056e^{-12}$	$1.196820e^{-13}$	$1.692806e^{-10}$
1.0	Exact solution	1.54930614433405860	1.549306144334055400	1.549306144334
	Computed solution	1.549306144315616400	1.549306144333811600	1.549306143854
	Error	$1.843903e^{-11}$	$2.438050e^{-13}$	$4.802496e^{-10}$

3.5 Comments on the Results

In general, the results from Tables 3.1-3.8 show that the performance of one-step hybrid block methods for solving second order ODEs directly using three hybrids points performs better in terms of error than using one or two hybrid points. Increasing the number of hybrid points will increase the order of the methods. It is discovered that the order of one-step hybrid block method using three hybrid points is five while the order of the methods using one and two hybrid points are three and four respectively. It can also be observed that the accuracy of the new one-step block methods is superior to the existing methods when solving the same IVPs of second order ODEs.

3.6 Conclusion

In this chapter, we have successfully developed a new single-step hybrid block methods with generalised one, two and three off-step points for solving second order ODEs. The zero stability, consistency, convergence, order, region of absolute stability and error constant of the developed methods are examined. These block methods are convergent because they are consistent and zero stable. The proposed methods not only possess good properties of a numerical methods, they have also been proven to be superior than the existing methods in term of accuracy when solving the same initial value problems of second order ODEs. Hence, these methods should be considered as a viable alternative for solving initial value problems of second order ODEs. Furthermore, the developed methods can be extended to solve a system of initial value problems of higher ODEs directly.

CHAPTER FOUR

ONE STEP HYBRID BLOCK METHODS FOR SOLVING THIRD ORDER ODEs DIRECTLY

In this chapter, the development of one step hybrid block methods with generalised two and three off step points using collocation and interpolation method for solving third order IVPs of ODEs is considered.

Equation (3.1) is also used to approximate the solution of the general third order IVPs of the form

$$y'''(x) = f(x, y, y', y'') \tag{4.1}$$

with three initial conditions

$$y(a) = \eta_0,\ y'(a) = \eta_1,\ y''(a) = \eta_2$$

on the integration interval $[a, b]$.

The first, second and third derivatives of equation (3.1) are

$$y'(x) = \sum_{i=1}^{v+m-1} \frac{i}{h} a_i \left(\frac{x - x_n}{h}\right)^{i-1}. \tag{4.2}$$

$$y''(x) = \sum_{i=2}^{v+m-1} \frac{i(i-1)}{h^2} a_i \left(\frac{x - x_n}{h}\right)^{i-2}. \tag{4.3}$$

and

$$y'''(x) = \sum_{i=3}^{v+m-1} \frac{i(i-1)(i-2)}{h^3} a_i \left(\frac{x - x_n}{h}\right)^{i-3}. \tag{4.4}$$

Substituting Equation (4.1) into Equation (4.4) yields

$$\sum_{i=3}^{v+m-1} \frac{i(i-1)(i-2)}{h^3} a_i \left(\frac{x - x_n}{h}\right)^{i-3} = f(x, y, y', y''). \tag{4.5}$$

4.1 Derivation of One Step Hybrid Block Method with Generalised Two Off-Step Points for Third Order ODEs

In order to derive this method, Equation (3.1) is interpolated at x_n, x_{n+s} x_{n+r} while Equation (4.5) is collocated at all points i.e x_n, x_{n+s}, x_{n+r} and x_{n+1} in the selected interval. Hence, in this case $v = 3$ and $m = 4$. This is clearly demonstrated in Figure 4.1.

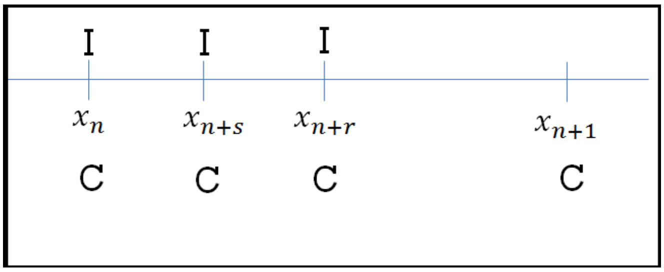

Figure 4.1. One step hybrid block method with generalised two off-step points for solving third order ODEs.

This strategy above produces the following collocation and interpolation equations

$$
\begin{aligned}
y_n &= a_0. \\
y_{n+s} &= a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6. \\
y_{n+r} &= a_0 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 + a_5 r^5 + a_6 r^6. \\
f_n &= \frac{6}{h^3} a_3 \\
f_{n+s} &= \frac{6}{h^3} a_3 + \frac{24s}{h^3} a_4 + \frac{60s^2}{h^3} a_5 + \frac{120s^3}{h^3} a_6. \\
f_{n+r} &= \frac{6}{h^3} a_3 + \frac{24r}{h^3} a_4 + \frac{60r^2}{h^3} a_5 + \frac{120r^3}{h^3} a_6. \\
f_{n+1} &= \frac{6}{h^3} a_3 + \frac{24}{h^3} a_4 + \frac{60}{h^3} a_5 + \frac{120}{h^3} a_6.
\end{aligned}
\tag{4.6}
$$

which can be written in a matrix form as below

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & s & s^2 & s^3 & s^4 & s^5 & s^6 \\ 1 & r & r^2 & r^3 & r^4 & r^5 & r^6 \\ 0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24s}{h^3} & \frac{60s^2}{h^3} & \frac{120s^3}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24r}{h^3} & \frac{60r^2}{h^3} & \frac{120r^3}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+s} \\ y_{n+r} \\ f_n \\ f_{n+s} \\ f_{n+r} \\ f_{n+1} \end{pmatrix} \tag{4.7}$$

In order to find the values of $a_i{}'s, i = 0(1)7$ in(4.7) Gaussian elimination method is employed. This gives the values of $a_i{}'s$ as below

$$\begin{aligned} a_0 =& \; y_n \\ a_1 =& \; \frac{r}{s(r-s)} y_{n+s} - \frac{s}{r(r-s)} y_{n+r} - \frac{r+s}{(rs)} y_n \\ &+ \frac{(h^3(r^3 - 2r^2s - 3r^2 - 2rs^2 + 12rs + s^3 - 3s^2))}{120} f_n \\ &+ \frac{(h^3r(r^3 + r^2s - 3r^2 + rs^2 - 3rs - s^3 + 2s^2))}{(120(r-s)(s-1))} f_{n+s} \\ &+ \frac{(h^3s(r^3 - r^2s - 2r^2 - rs^2 + 3rs - s^3 + 3s^2))}{(120(r-s)(r-1))} f_{n+r} \\ &- \frac{(h^3rs(r+s)(r^2 - 3rs + s^2))}{(120(r-1)(s-1))} f_{n+1} \end{aligned}$$

$$\begin{aligned} a_2 =& \; \frac{1}{(r(r-s))} y_{n+r} + \frac{1}{(rs)} y_n - \frac{1}{(s(r-s))} y_{n+s} \\ &- \frac{(h^3(r^4 - 2r^3s - 3r^3 - 2r^2s^2 + 12r^2s - 2rs^3 + 12rs^2 + s^4 - 3s^3))}{(120rs)} f_n \\ &- \frac{(h^3(r^4 - r^3s - 2r^3 - r^2s^2 + 3r^2s - rs^3 + 3rs^2 - s^4 + 3s^3))}{(120r(r-s)(r-1))} f_{n+r} \\ &+ \frac{(h^3(r^4 - 2r^3s - 2r^2s^2 - 2rs^3 + s^4))}{(120(r-1)(s-1))} f_{n+1} \\ &- \frac{(h^3(r^4 + r^3s - 3r^3 + r^2s^2 - 3r^2s + rs^3 - 3rs^2 - s^4 + 2s^3))}{(120s(r-s)(s-1))} f_{n+s} \end{aligned}$$

$$a_3 = \frac{6}{h^3} f_n$$

$$a_4 = -\frac{(h^3(r+s+rs))}{(24rs)} f_n - \frac{(h^3 r)}{(24s(r-s)(s-1))} f_{n+s} + \frac{(h^3 rs)}{(24(r-1)(s-1))} f_{n+1} + \frac{(h^3 s)}{(24r(r-s)(r-1))} f_{n+r}$$

$$a_5 = \frac{h^3(r+s+1)}{(60rs)} f_n - \frac{h^3(r+s))}{60(r-1)(s-1)} f_{n+1} - \frac{h^3(s+1)}{(60r(r-s)(r-1))} f_{n+r} + \frac{h^3(r+1)}{(60s(r-s)(s-1))} f_{n+s}$$

$$a_6 = -\frac{h^3}{120rs} f_n - \frac{h^3}{120s(r-s)(s-1)} f_{n+s} + \frac{h^3}{120r(r-s)(r-1)} f_{n+r} + \frac{h^3}{120(r-1)(s-1)} f_{n+1}$$

Substituting the values of $a_i{}'s$ in Equation (3.1) gives a continuous implicit scheme of the form as below

$$y(x) = \sum_{i=0,s,r} \alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \beta_i(x) f_{n+i} + \sum_{i=s,r} \beta_i(x) f_{n+i} \tag{4.8}$$

Differentiating (4.8) twice produces

$$y^{'}(x) = \sum_{i=0,s,r} \frac{\partial}{\partial x} \alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \frac{\partial}{\partial x} \beta_i(x) f_{n+i} + \sum_{i=s,r} \frac{\partial}{\partial x} \beta_i(x) f_{n+i} \tag{4.9}$$

$$y^{''}(x) = \sum_{i=0,s,r} \frac{\partial^2}{\partial x^2} \alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \frac{\partial^2}{\partial x^2} \beta_i(x) f_{n+i} + \sum_{i=s,r} \frac{\partial^2}{\partial x^2} \beta_i(x) f_{n+i} \tag{4.10}$$

where

$$\alpha_0 = \frac{(x_n - x + hs)(x_n - x + hr)}{(h^2 rs)}$$
$$\alpha_s = \frac{(x - x_n)(x_n - x + hr)}{(h^2 s(r - s))}$$
$$\alpha_r = \frac{(x - x_n)(x - x_n - hs)}{h^2 r(r - s)}$$

$$\beta_0 = \frac{(x - x_n)(x_n - x + hs)(x_n - x + hr)}{(120h^3 rs)}(h^3r^3 - 2h^3r^2s - 3h^3r^2 - 2h^3rs^2 - 3xx_n^2$$
$$+12h^3rs + h^3s^3 - 3h^3s^2 + h^2r^2x - h^2r^2x_n - 2h^2rsx + 2h^2rsx_n - 3h^2rx + 3h^2rx_n$$
$$+h^2s^2x - h^2s^2x_n - 3h^2sx + 3h^2sx_n + hrx^2 - 2hrxx_n + hrx_n^2 + hsx^2 - 2hsxx_n + x_n^3$$
$$+hsx_n^2 + 2hx^2 - 4hxx_n + 2hx_n^2 - x^3 + 3x^2x_n)$$

$$\beta_s = \frac{(x - x_n)(x_n - x + hs)(x_n - x + hr)}{(120h^3 s(s-1)(r - s))}(h^3r^3 + h^3r^2s - 3h^3r^2 + h^3rs^2 - 3h^3rs$$
$$+2h^3s^2 + h^2r^2x - h^2r^2x_n + h^2rsx - h^2rsx_n - 3h^2rx + 3h^2rx_n - h^2s^2x - hsx^2$$
$$+2h^2sx - 2h^2sx_n + hrx^2 - 2hrxx_n + hrx_n^2 + 2hsxx_n - hsx_n^2 + 2hx^2 - 4hxx_n$$
$$-h^3s^3 + 2hx_n^2 - x^3 + 3x^2x_n - 3xx_n^2 + h^2s^2x_n + x_n^3)$$

$$\beta_r = \frac{(x - x_n)(x_n - x + hs)(x_n - x + hr)}{120h^3 r(r-1)(r - s)}(h^3r^3 - h^3r^2s - 2h^3r^2 - h^3rs^2 + 3h^3rs + x^3$$
$$+3h^3s^2 + h^2r^2x - h^2r^2x_n - h^2rsx + h^2rsx_n - 2h^2rx + 2h^2rx_n - h^2s^2x + h^2s^2x_n$$
$$-3h^2sx_n + hrx^2 - 2hrxx_n + hrx_n^2 - hsx^2 + 2hsxx_n - hsx_n^2 - 2hx^2 + 4hxx_n - 2hx_n^2$$
$$-3x^2x_n + 3xx_n^2 - h^3s^3 + 3h^2sx - x_n^3)$$

$$\beta_1 = -\frac{(x - x_n)(x_n - x + hs)(x_n - x + hr)}{(120h^3(s-1)(r-1))}(h^3r^3 - 2h^3r^2s - 2h^3rs^2 + h^3s^3 + h^2r^2x$$
$$-h^2r^2x_n - 2h^2rsx + 2h^2rsx_n + h^2s^2x - h^2s^2x_n + hrx^2 - 2hrxx_n + hrx_n^2 + hsx^2 + x_n^3$$
$$-2hsxx_n + hsx_n^2 - x^3 + 3x^2x_n - 3xx_n^2)$$

Evaluating (4.8) at the non-interpolating point x_{n+1} produces the following scheme

$$
\begin{aligned}
&y_{n+1}-\frac{(r-1)}{s(r-s)}y_{n+s}+\frac{(s-1)}{r(r-s)}y_{n+r}=\frac{(s-1)(r-1)}{(rs)}y_n\\
&+\frac{h^3(s-1)(r-1)(r^3-2r^2s-2r^2-2rs^2+10rs-2r+s^3-2s^2-2s+1)}{120rs}f_n\\
&+\frac{h^3(r-1)(r^3+r^2s-2r^2+rs^2-2rs-2r-s^3+s^2+s+1)}{120s(r-s)}f_{n+s}\\
&+\frac{h^3(s-1)(r^3-r^2s-r^2-rs^2+2rs-r-s^3+2s^2+2s-1)}{120r(r-s)}f_{n+r}\\
&-\frac{h^3(r^3-2r^2s+r^2-2rs^2-2rs+r+s^3+s^2+s-1)}{120}f_{n+1}
\end{aligned}
\tag{4.11}
$$

Equation (4.9) is then evaluated at all points i.e, at x_n, x_{n+s}, x_{n+r} and x_{n+1} to give the following schemes

$$
\begin{aligned}
&y'_n-\frac{r}{hs(r-s)}y_{n+s}+\frac{s}{hr(r-s)}y_{n+r}=-\frac{h^2rs(r+s)(r^2-3rs+s^2)}{120(r-1)(s-1)}f_{n+1}\\
&+\frac{h^2(r^3-2r^2s-3r^2-2rs^2+12rs+s^3-3s^2)}{120}f_n-\frac{(r+s)}{hrs}y_n\\
&+\frac{h^2r(r^3+r^2s-3r^2+rs^2-3rs-s^3+2s^2)}{120(r-s)(s-1)}f_{n+s}\\
&+\frac{h^2s(r^3-r^2s-2r^2-rs^2+3rs-s^3+3s^2)}{120(r-s)(r-1)}f_{n+r}
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
&y'_{n+s}-\frac{(r-2s)}{hs(r-s)}y_{n+s}-\frac{s}{hr(r-s)}y_{n+r}=\frac{h^2s(r-2s)(r-s)(r^2+rs-s^2)}{120(r-1)(s-1)}f_{n+1}\\
&-\frac{(r-s)}{hrs}y_n-\frac{h^2s}{(120r(r-1))}(r^3-2r^2-rs^2+rs-2s^3+4s^2))f_{n+r}\\
&-\frac{h^2(r-s)(r^3-r^2s-3r^2-3rs^2+9rs+2s^3-4s^2)}{120r}f_n\\
&-\frac{h^2(r^3+2r^2s-3r^2+3rs^2-6rs-4s^3+6s^2)}{120(s-1)}f_{n+s}
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
y'_{n+r} &+ \frac{r}{hs(r-s)}y_{n+s} - \frac{(2r-s)}{hr(r-s)}y_{n+r} = -\frac{h^2r(r-s)(2r-s)(r^2-rs-s^2)}{120(r-1)(s-1)}f_{n+1} \\
&+ \frac{(r-s)}{(hrs)}y_n + \frac{h^2r(2r^3+r^2s-4r^2-rs-s^3+2s^2)}{120s(s-1)}f_{n+s} \\
&+ \frac{h^2(4r^3-3r^2s-6r^2-2rs^2+6rs-s^3+3s^2)}{120(r-1)}f_{n+r} \\
&+ \frac{h^2(r-s)(2r^3-3r^2s-4r^2-rs^2+9rs+s^3-3s^2)}{120s}f_n
\end{aligned} \tag{4.14}
$$

$$
\begin{aligned}
y'_{n+1} &- \frac{(r-2)}{hs(r-s)}y_{n+s} + \frac{(s-2)}{hr(r-s)}y_{n+r} = \frac{-h^2}{120(r-1)(s-1)}(r^4s-2r^4-2r^3s^2-6 \\
&-2s^4+4r^3s-2r^2s^3+4r^2s^2+rs^4+4rs^3-20rs+10r+10s)f_{n+1} - \frac{(r+s-2)}{(hrs)}y_n \\
&+ \frac{h^2(s-2)(r^4-r^3s-2r^3-r^2s^2+3r^2s-rs^3+3rs^2+2s^3-2s^2-4s+2)}{120r(r-s)(r-1)}f_{n+r} \\
&+ \frac{h^2(r-2)(r^3s-2r^3+r^2s^2-3r^2s+2r^2+rs^3-3rs^2+4r-s^4+2s^3-2)}{120s(r-s)(s-1)}f_{n+s} \\
&+ \frac{h^2(r+s-2)}{120rs}(r^3s-2r^3-3r^2s^2+5r^2s+2r^2+rs^3+5rs^2-16rs+4r-2s^3 \\
&+2s^2+4s-2)f_n
\end{aligned} \tag{4.15}
$$

Similarly, evaluating (4.10) at all points. This produces the following schemes

$$
\begin{aligned}
y''_n &+ \frac{2}{h^2s(r-s)}y_{n+s} - \frac{2}{h^2r(r-s)}y_{n+r} = \frac{h(r^4-2r^3s-2r^2s^2-2rs^3+s^4)}{60(r-1)(s-1)}f_{n+1} \\
&+ \frac{2}{(h^2rs)}y_n - \frac{h(r^4+r^3s-3r^3+r^2s^2-3r^2s+rs^3-3rs^2-s^4+2s^3)}{60s(r-s)(s-1)}f_{n+s} \\
&- \frac{h(r^4-r^3s-2r^3-r^2s^2+3r^2s-rs^3+3rs^2-s^4+3s^3)}{60r(r-s)(r-1)}f_{n+r} \\
&- \frac{h(r^4-2r^3s-3r^3-2r^2s^2+12r^2s-2rs^3+12rs^2+s^4-3s^3)}{(60rs)}f_n
\end{aligned} \tag{4.16}
$$

$$y''_{n+s}+\frac{2}{(h^2s(r-s)}y_{n+s}-\frac{2}{h^2r(r-s)}y_{n+r}=\frac{h(r^4-2r^3s-2r^2s^2+8rs^3-4s^4)}{60(r-1)(s-1)}f_{n+1}$$
$$+\frac{2}{(h^2rs)}y_n-\frac{h(r^4+r^3s-3r^3+r^2s^2-3r^2s-19rs^3+27rs^2+14s^4-18s^3)}{60s(r-s)(s-1)}f_{n+s}$$
$$-\frac{h(r^4-2r^3s-3r^3-2r^2s^2+12r^2s+8rs^3-18rs^2-4s^4+7s^3)}{60rs}f_n$$
$$-\frac{h(r^4-r^3s-2r^3-r^2s^2+3r^2s-rs^3+3rs^2+4s^4-7s^3)}{60r(r-s)(r-1)}f_{n+r} \tag{4.17}$$

$$y''_{n+r}+\frac{2}{h^2s(r-s)}y_{n+s}-\frac{2}{h^2r(r-s)}y_{n+r}=\frac{2}{(h^2rs)}y_n$$
$$+\frac{h(4r^4-r^3s-7r^3-r^2s^2+3r^2s-rs^3+3rs^2+s^4-2s^3)}{60s(r-s)(s-1)}f_{n+s}$$
$$+\frac{h(14r^4-19r^3s-18r^3+r^2s^2+27r^2s+rs^3-3rs^2+s^4-3s^3)}{60r(r-s)(r-1)}f_{n+r}$$
$$+\frac{h}{(60rs)}(4r^4-8r^3s-7r^3+2r^2s^2+18r^2s+2rs^3-12rs^2-s^4+3s^3)f_n$$
$$\frac{-h(4r^4-8r^3s+2r^2s^2+2rs^3-s^4)}{60(r-1)(s-1)}f_{n+1} \tag{4.18}$$

$$y''_{n+1}+\frac{2}{h^2s(r-s)}y_{n+s}-\frac{2}{h^2r(r-s)}y_{n+r}=-\frac{h}{(60rs)}(r^4-2r^3s-3r^3-2r^2s^2$$
$$+12r^2s-2rs^3+12rs^2-30rs+10r+s^4-3s^3+10s-5)f_n+\frac{2}{h^2rs}y_n$$
$$+\frac{h}{60(r-1)(s-1)}(r^4-2r^3s-2r^2s^2-2rs^3+30rs-20r+s^4-20s+15)f_{n+1}$$
$$-\frac{h(r^4+r^3s-3r^3+r^2s^2-3r^2s+rs^3-3rs^2+10r-s^4+2s^3-5)}{60s(r-s)(s-1)}f_{n+s}$$
$$-\frac{h(r^4-r^3s-2r^3-r^2s^2+3r^2s-rs^3+3rs^2-s^4+3s^3-10s+5)}{60r(r-s)(r-1)}f_{n+r} \tag{4.19}$$

Combining equations (4.11) ,(4.13) and (4.16) produces a block form as below

$$A^{[2]_3}Y_m^{[2]_3} = B_1^{[2]_3}R_1^{[2]_3}+B_2^{[2]_3}R_2^{[2]_3}+B_3^{[2]_3}R_3^{[2]_3}+h^3D^{[2]_3}R_4^{[2]_3}+h^3E^{[2]_3}R_5^{[2]_3} \tag{4.20}$$

where

$$A^{[2]_3} = \begin{pmatrix} \frac{-(r-1)}{(s(r-s))} & \frac{(s-1)}{(r(r-s))} & 1 \\ \frac{-r}{hs(r-s)} & \frac{s}{hr(r-s)} & 0 \\ \frac{2}{h^2(rs-s^2)} & \frac{2}{h^2(r^2-rs)} & 0 \end{pmatrix}, Y_m^{[2]_3} = \begin{pmatrix} y_{n+s} \\ y_{n+r} \\ y_{n+1} \end{pmatrix}, B_1^{[2]_3} = \begin{pmatrix} 0 & 0 & \frac{(r-1)(s-1)}{rs} \\ 0 & 0 & \frac{-(r+s)}{(hrs)} \\ 0 & 0 & \frac{2}{(h^2rs)} \end{pmatrix}$$

$$R_1^{[2]_3} = \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}, \qquad B_2^{[2]_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad R_2^{[2]_3} = \begin{pmatrix} y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix},$$

$$B_3^{[2]_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad R_3^{[2]_3} = \begin{pmatrix} y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix}, \qquad D^{[2]_3} = \begin{pmatrix} 0 & 0 & D_{13}^{[2]_3} \\ 0 & 0 & D_{23}^{[2]_3} \\ 0 & 0 & D_{33}^{[2]_3} \end{pmatrix},$$

$$R_4^{[2]_3} = \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}, \qquad E^{[2]_3} = \begin{pmatrix} E_{11}^{[2]_3} & E_{12}^{[2]_3} & E_{13}^{[2]_3} \\ E_{21}^{[2]_3} & E_{21}^{[2]_3} & E_{23}^{[2]_3} \\ E_{31}^{[2]_3} & E_{32}^{[2]_3} & E_{33}^{[2]_3} \end{pmatrix}, \qquad R_5^{[2]_3} = \begin{pmatrix} f_{n+s} \\ f_{n+r} \\ f_{n+1} \end{pmatrix}$$

with

$$
\begin{aligned}
D_{13}^{[2]_3} &= \frac{(r-1)(s-1)}{120rs}(r^3-2r^2s-2r^2-2rs^2+10rs-2r+s^3-2s^2-2s+1)\\
D_{23}^{[2]_3} &= \frac{(r^3-2r^2s-3r^2-2rs^2+12rs+s^3-3s^2)}{120h}\\
D_{33}^{[2]_3} &= -\frac{(r^4-2r^3s-3r^3-2r^2s^2+12r^2s-2rs^3+12rs^2+s^4-3s^3)}{60rsh^2}\\
E_{11}^{[2]_3} &= \frac{(r-1)(r^3+r^2s-2r^2+rs^2-2rs-2r-s^3+s^2+s+1)}{120s(r-s)}\\
E_{12}^{[2]_3} &= \frac{(s-1)(r^3-r^2s-r^2-rs^2+2rs-r-s^3+2s^2+2s-1)}{120r(r-s)}\\
E_{13}^{[2]_3} &= -\frac{(r^3-2r^2s+r^2-2rs^2-2rs+r+s^3+s^2+s-1)}{120}\\
E_{21}^{[2]_3} &= \frac{r(r^3+r^2s-3r^2+rs^2-3rs-s^3+2s^2)}{120h(r-s)(s\breve{}1)}\\
E_{21}^{[2]_3} &= \frac{s(r^3-r^2s-2r^2-rs^2+3rs-s^3+3s^2)}{120h(r-s)(r-1)}\\
E_{23}^{[2]_3} &= -\frac{rs(r+s)(r^2-3rs+s^2)}{120h(r-1)(s-1)}\\
E_{31}^{[2]_3} &= -\frac{(r^4+r^3s-3r^3+r^2s^2-3r^2s+rs^3-3rs^2-s^4+2s^3)}{60sh^2(r-s)(s-1)}\\
E_{32}^{[2]_3} &= -\frac{(r^4-r^3s-2r^3-r^2s^2+3r^2s-rs^3+3rs^2-s^4+3s^3)}{60rh^2(r-s)(r-1)}\\
E_{33}^{[2]_3} &= \frac{(r^4-2r^3s-2r^2s^2-2rs^3+s^4)}{60h^2(r-1)(s-1)}
\end{aligned}
$$

Multiplying (4.20) by $(A^{[2]_3})^{-1}$ gives one step hybrid block method with two generalized off step points for third order ODEs as following.

$$
I^{[2]_3}Y_m^{[2]_3} = \bar{B}_1^{[2]_3}R_1^{[2]_3} + \bar{B}_2^{[2]_3}R_2^{[2]_3} + \bar{B}_3^{[2]_3}R_3^{[2]_3} + h^3\bar{D}^{[2]_3}R_4^{[2]_3} + h^3\bar{E}^{[2]_3}R_5^{[2]_3} \tag{4.21}
$$

where,

$$
I^{[2]_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \bar{B}_1^{[2]_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$\bar{B}_2^{[2]_3} = \begin{pmatrix} 0 & 0 & hs \\ 0 & 0 & hr \\ 0 & 0 & h \end{pmatrix}, \qquad \bar{B}_3^{[2]_3} = \begin{pmatrix} 0 & 0 & \frac{h^2s^2}{2} \\ 0 & 0 & \frac{h^2r^2}{2} \\ 0 & 0 & \frac{h^2}{2} \end{pmatrix}$$

$$\bar{D}^{[2]_3} = \begin{pmatrix} 0 & 0 & \bar{D}_{13}^{[2]_3} \\ 0 & 0 & \bar{D}_{23}^{[2]_3} \\ 0 & 0 & \bar{D}_{33}^{[2]_3} \end{pmatrix}, \quad \bar{E}^{[2]_3} = \begin{pmatrix} \bar{E}_{11}^{[2]_3} & \bar{E}_{12}^{[2]_3} & \bar{E}_{13}^{[2]_3} \\ \bar{E}_{21}^{[2]_3} & \bar{E}_{21}^{[2]_3} & \bar{E}_{23}^{[2]_3} \\ \bar{E}_{31}^{[2]_3} & \bar{E}_{32}^{[2]_3} & \bar{E}_{33}^{[2]_3} \end{pmatrix}$$

The non-zero elements of $\bar{D}^{[2]_3}$and $\bar{E}^{[2]_3}$ are

$$\begin{aligned}
\bar{D}_{13}^{[2]_3} &= -\frac{s^3(3s-15r+3rs-s^2)}{120r} \\
\bar{D}_{23}^{[2]_3} &= \frac{r^3(15s-3r-3rs+r^2)}{120s} \\
\bar{D}_{33}^{[2]_3} &= \frac{(15rs-3s-3r+1)}{120rs} \\
\bar{E}_{11}^{[2]_3} &= \frac{s^3(2s-5r+2rs-s^2)}{120(s-1)(r-s)} \\
\bar{E}_{12}^{[2]_3} &= -\frac{s^5(s-3)}{120r(r-1)(r-s)} \\
\bar{E}_{13}^{[2]_3} &= \frac{s^5(3r-s)}{120(r-1)(s-1)} \\
\bar{E}_{21}^{[2]_3} &= \frac{r^5(r-3)}{120s(s-1)(r-s)} \\
\bar{E}_{22}^{[2]_3} &= \frac{r^3(5s-2r-2rs+r^2)}{120(r-1)(r-s)} \\
\bar{E}_{23}^{[2]_3} &= -\frac{r^5(r-3s)}{120(s-1)(r-1)} \\
\bar{E}_{31}^{[2]_3} &= -\frac{(3r-1)}{120s(s-1)(r-s)} \\
\bar{E}_{32}^{[2]_3} &= \frac{(3s-1)}{120r(r-1)(r-s)} \\
\bar{E}_{33}^{[2]_3} &= \frac{(5rs-2s-2r+1)}{120(s-1)(r-1)}
\end{aligned}$$

Equation (4.21) can also be written as

$$\begin{aligned} y_{n+s} &= y_n + hsy'_n + \frac{h^2s^2}{2}y''_n - \frac{h^3s^3(3s-15r+3rs-s^2)}{120r}f_n - \frac{h^3s^5(s-3)}{120r(r-1)(r-s)}f_{n+r} \\ &\quad + \frac{h^3s^3(2s-5r+2rs-s^2)}{120(s-1)(r-s)}f_{n+s} + \frac{h^3s^5(3r-s)}{120(r-1)(s-1)}f_{n+1} \end{aligned} \tag{4.22}$$

$$\begin{aligned} y_{n+r} &= y_n + hry'_n + \frac{h^2r^2}{2}y''_n + \frac{h^3r^3(15s-3r-3rs+r^2)}{120s}f_n + \frac{h^3r^5(r-3)}{120s(s-1)(r-s)}f_{n+s} \\ &\quad + \frac{h^3r^3(5s-2r-2rs+r^2)}{120(r-1)(r-s)}f_{n+r} - \frac{h^3r^5(r-3s)}{120(s-1)(r-1)}f_{n+1} \end{aligned} \tag{4.23}$$

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3(15rs-3s-3r+1)}{120rs}f_n - \frac{h^3(3r-1)}{120s(s-1)(r-s)}f_{n+s} \\ &\quad + \frac{h^3(3s-1)}{120r(r-1)(r-s)}f_{n+r} + \frac{h^3(5rs-2s-2r+1)}{120(s-1)(r-1)}f_{n+1} \end{aligned} \tag{4.24}$$

Substituting (4.22) and (4.23) into Equations (4.13) to (4.15) yields the first derivative of the block as below

$$\begin{aligned} y'_{n+s} &= y'_n + hsy''_n - \frac{h^2s^2(5s-20r+5rs-2s^2)f_n}{60r} + \frac{h^2s^4(5r-2s)}{60(s-1)(r-1)}f_{n+1} \\ &\quad - \frac{h^2s^4(2s-5)}{60r(r-1)(r-s)}f_{n+r} + \frac{h^2s^2(5s-10r+5rs-3s^2)}{60(s-1)(r-s)}f_{n+s} \end{aligned} \tag{4.25}$$

$$\begin{aligned} y'_{n+r} &= y'_n + hry''_n + \frac{h^2r^2(20s-5r-5rs+2r^2)}{60s}f_n + \frac{h^2r^4(2r-5)}{60s(s-1)(r-s)}f_{n+s} \\ &\quad + \frac{h^2r^2(10s-5r-5rs+3r^2)}{60(r-1)(r-s)}f_{n+r} - \frac{h^2r^4(2r-5s)}{60(r-1)(s-1)}f_{n+1} \end{aligned} \tag{4.26}$$

$$
\begin{aligned}
y'_{n+1} &= y'_n + hy''_n + \frac{h^2(20rs-5s-5r+2)}{60rs} f_n - \frac{h^2(5r-2)}{60s(s-1)(r-s)} f_{n+s} \\
&\quad + \frac{h^2(5s-2)}{60r(r-1)(r-s)} f_{n+r} + \frac{h^2(10rs-5s-5r+3)}{60(s-1)(r-1)} f_{n+1}
\end{aligned} \tag{4.27}
$$

The first derivative of the derived method can be represented in equation of the form

$$
\acute{Y}_m^{[2]_3} = \acute{B}_2^{[2]_3} R_2^{[2]_3} + \acute{B}_3^{[2]_3} R_3^{[2]_3} + h^2 \acute{D}^{[2]_3} R_4^{[2]_3} + h^2 \acute{E}^{[2]_3} R_5^{[2]_3} \tag{4.28}
$$

where

$$
\acute{B}_2^{[2]_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; \acute{B}_3^{[2]_3} = \begin{pmatrix} 0 & 0 & hs \\ 0 & 0 & hr \\ 0 & 0 & h \end{pmatrix},
$$

$$
\acute{D}^{[2]_3} = \begin{pmatrix} 0 & 0 & \acute{D}_{13}^{[2]_3} \\ 0 & 0 & \acute{D}_{23}^{[2]_3} \\ 0 & 0 & \acute{D}_{33}^{[2]_3} \end{pmatrix}, \; \acute{E}^{[2]_3} = \begin{pmatrix} \acute{E}_{11}^{[2]_3} & \acute{E}_{12}^{[2]_3} & \acute{E}_{13}^{[2]_3} \\ \acute{E}_{21}^{[2]_3} & \acute{E}_{21}^{[2]_3} & \acute{E}_{23}^{[2]_3} \\ \acute{E}_{31}^{[2]_3} & \acute{E}_{32}^{[2]_3} & \acute{E}_{33}^{[2]_3} \end{pmatrix}
$$

the elements of $\acute{D}^{[2]_3}$ and $\acute{E}^{[2]_3}$ are given as below

$$
\begin{aligned}
\acute{D}_{13}^{[2]_3} &= -\frac{s^2(5s-20r+5rs-2s^2)}{60r} \\
\acute{D}_{23}^{[2]_3} &= \frac{r^2(20s-5r-5rs+2r^2)}{60s} \\
\acute{D}_{33}^{[2]_3} &= \frac{(20rs-5s-5r+2)}{60rs} \\
\acute{E}_{11}^{[2]_3} &= \frac{s^2(5s-10r+5rs-3s^2)}{60(s-1)(r-s)} \\
\acute{E}_{12}^{[2]_3} &= -\frac{s^4(2s-5)}{60r(r-1)(r-s)} \\
\acute{E}_{13}^{[2]_3} &= \frac{s^4(5r-2s)}{60(s-1)(r-1)} \\
\acute{E}_{21}^{[2]_3} &= \frac{r^4(2r-5)}{60s(s-1)(r-s)} \\
\acute{E}_{22}^{[2]_3} &= \frac{r^2(10s-5r-5rs+3r^2)}{60(r-1)(r-s)}
\end{aligned}
$$

$$
\begin{aligned}
\acute{E}_{23}^{[2]_3} &= -\frac{r^4(2r-5s)}{60(r-1)(s-1)} \\
\acute{E}_{31}^{[2]_3} &= -\frac{(5r-2)}{60s(s-1)(r-s)} \\
\acute{E}_{32}^{[2]_3} &= \frac{(5s-2)}{60r(r-1)(r-s)} \\
\acute{E}_{33}^{[2]_3} &= \frac{(10rs-5s-5r+3)}{60(s-1)(r-1)}
\end{aligned}
$$

Similarly, substituting (4.22) and (4.23) into equations (4.17) to (4.19) gives block of second derivative as the following

$$
\begin{aligned}
y''_{n+s} &= y''_n + \frac{hs^3(2r-s)}{12(s-1)(r-1)} f_{n+1} - \frac{hs(2s-6r+2rs-s^2)}{(12r)} f_n \\
&\quad - \frac{hs^3(s-2)}{12r(r-1)(r-s)} f_{n+r} + \frac{hs(4s-6r+4rs-3s^2)}{(12(s-1)(r-s))} f_{n+s}
\end{aligned} \tag{4.29}
$$

$$
\begin{aligned}
y''_{n+r} &= y''_n + \frac{hr(6s-2r-2rs+r^2)}{(12s)} f_n + \frac{hr^3(r-2)}{12s(s-1)(r-s)} f_{n+s} \\
&\quad + \frac{hr(6s-4r-4rs+3r^2)}{12(r-1)(r-s)} f_{n+r} - \frac{hr^3(r-2s)}{12(s-1)(r-1)} f_{n+1}
\end{aligned} \tag{4.30}
$$

$$
\begin{aligned}
y''_{n+1} &= y''_n + \frac{h(6rs-2s-2r+1)}{(12rs)} f_n - \frac{h(2r-1)}{12s(s-1)(r-s)} f_{n+s} \\
&\quad + \frac{h(2s-1)}{12r(r-1)(r-s)} f_{n+r} + \frac{h(6rs-4s-4r+3)}{12(s-1)(r-1)} f_{n+1}
\end{aligned} \tag{4.31}
$$

The second derivative of the derived block method can be represented in equation of the form

$$
\acute{\acute{Y}}_m^{[2]_3} = \acute{\acute{B}}_3^{[2]_3} R_3^{[2]_3} + h\acute{\acute{D}}^{[2]_3} R_4^{[2]_3} + h\acute{\acute{E}}^{[2]_3} R_5^{[2]_3} \tag{4.32}
$$

where

$$\acute{\acute{B}}_3^{[2]_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ \acute{\acute{D}}^{[2]_3} = \begin{pmatrix} 0 & 0 & \acute{\acute{D}}_{13}^{[2]_3} \\ 0 & 0 & \acute{\acute{D}}_{23}^{[2]_3} \\ 0 & 0 & \acute{\acute{D}}_{33}^{[2]_3} \end{pmatrix}, \quad \acute{\acute{E}}^{[2]_3} = \begin{pmatrix} \acute{\acute{E}}_{11}^{[2]_3} & \acute{\acute{E}}_{12}^{[2]_3} & \acute{\acute{E}}_{13}^{[2]_3} \\ \acute{\acute{E}}_{21}^{[2]_3} & \acute{\acute{E}}_{21}^{[2]_3} & \acute{\acute{E}}_{23}^{[2]_3} \\ \acute{\acute{E}}_{31}^{[2]_3} & \acute{\acute{E}}_{32}^{[2]_3} & \acute{\acute{E}}_{33}^{[2]_3} \end{pmatrix}$$

The non-zero terms of $\acute{\acute{D}}^{[2]_3}$ and $\acute{\acute{E}}^{[2]_3}$ are

$$\begin{aligned}
\acute{\acute{D}}_{13}^{[2]_3} &= -\frac{s(2s-6r+2rs-s^2)}{(12r)} \\
\acute{\acute{D}}_{23}^{[2]_3} &= \frac{r(6s-2r-2rs+r^2)}{(12s)} \\
\acute{\acute{D}}_{33}^{[2]_3} &= \frac{(6rs-2s-2r+1)}{(12rs)} \\
\acute{\acute{E}}_{11}^{[2]_3} &= +\frac{s(4s-6r+4rs-3s^2)}{(12(s-1)(r-s))} \\
\acute{\acute{E}}_{12}^{[2]_3} &= -\frac{s^3(s-2)}{12r(r-1)(r-s)} \\
\acute{\acute{E}}_{13}^{[2]_3} &= +\frac{s^3(2r-s)}{12(s-1)(r-1)} \\
\acute{\acute{E}}_{21}^{[2]_3} &= +\frac{r^3(r-2)}{12s(s-1)(r-s)} \\
\acute{\acute{E}}_{22}^{[2]_3} &= \frac{r(6s-4r-4rs+3r^2)}{12(r-1)(r-s)} \\
\acute{\acute{E}}_{23}^{[2]_3} &= -\frac{r^3(r-2s)}{(12(s-1)(r-1)} \\
\acute{\acute{E}}_{31}^{[2]_3} &= -\frac{(2r-1)}{12s(s-1)(r-s)} \\
\acute{\acute{E}}_{32}^{[2]_3} &= \frac{(2s-1)}{12r(r-1)(r-s)} \\
\acute{\acute{E}}_{33}^{[2]_3} &= \frac{(6rs-4s-4r+3)}{12(s-1)(r-1)}
\end{aligned}$$

4.1.1 Establishing Properties of One Step Hybrid Block Method with Generalised Two Off-Step Points for Third Order ODEs

This section examines the order, error constant zero stability, consistency and region of absolute stability.

4.1.1.1 Order of One Step Hybrid Block Method with Generalised Two Off-Step Points for Third Order ODEs

In finding the order of the block (4.21), Definition (3.1.2) is also applied. y and f-function are expanded in Taylor series as below

$$
\begin{bmatrix}
\begin{array}{c}
\sum_{j=0}^{\infty}\frac{(s)^j h^j}{j!}y_n^j - y_n - shy_n' - \frac{h^2s^2}{2}y_n'' + \frac{s^3h^3(3s-15r+3rs-s^2)}{120r}y_n''' \\
-\frac{s^5(3r-s)}{120(r-1)(s-1)}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3} + \frac{s^5(s-3)}{120r(r-1)(r-s)}\sum_{j=0}^{\infty}\frac{r^j h^{j+3}}{j!}y_n^{j+3} \\
-\frac{s^3(2s-5r+2rs-s^2)}{120(s-1)(r-s)}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+3}}{j!}y_n^{j+3} \\
\sum_{j=0}^{\infty}\frac{(r)^j h^j}{j!}y_n^j - y_n - rhy_n' - \frac{h^2r^2}{2}y_n'' - \frac{r^3h^3(15s-3r-3rs+r^2)}{120s}y_n''' \\
+\frac{r^5(r-3s)}{120(s-1)(r-1)}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3} - \frac{r^3(5s-2r-2rs+r^2)}{120(r-1)(r-s)}\sum_{j=0}^{\infty}\frac{(r)^j h^{j+3}}{j!}y_n^{j+3} \\
-\frac{r^5(r-3)}{120s(s-1)(r-s)}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+3}}{j!}y_n^{j+3} \\
\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j - y_n - hy_n' - \frac{h^2}{2}y_n'' - \frac{h^3(15rs-3s-3r+1)}{120rs}y_n''' \\
-\frac{(5rs-2s-2r+1)}{120(s-1)(r-1)}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3} - \frac{(3s-1)}{120r(r-1)(r-s)}\sum_{j=0}^{\infty}\frac{(r)^j h^{j+3}}{j!}y_n^{j+3} \\
+\frac{(3r-1)}{120s(s-1)(r-s)}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+3}}{j!}y_n^{j+3}
\end{array}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Comparing the coefficients of h^j and y^j yields

$$
\bar{C}_0 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_1 = \begin{bmatrix} s-s \\ r-r \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_2 = \begin{bmatrix} \frac{s^2}{2}-\frac{s^2}{2} \\ \frac{r^2}{2}-\frac{r^2}{2} \\ \frac{1}{2}-\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$\bar{C}_3 = \begin{bmatrix} \frac{(s)^3}{3!} + \frac{s^3h^3(3s-15r+3rs-s^2)}{120r} - \frac{s^5(3r-s)}{120(r-1)(s-1)}\frac{1}{0!} + \frac{s^5(s-3)}{120r(r-1)(r-s)}\frac{(r)^0}{0!} \\ -\frac{s^3(2s-5r+2rs-s^2)}{120(s-1)(r-s)}\frac{(s)^0}{0!} \\ \frac{(r)^3}{3!} - \frac{r^3h^3(15s-3r-3rs+r^2)}{120s} + \frac{r^5(r-3s)}{120(s-1)(r-1)}\frac{1}{0!} - \frac{r^3(5s-2r-2rs+r^2)}{120(r-1)(r-s)}\frac{(r)^0}{0!} \\ -\frac{r^5(r-3)}{120s(s-1)(r-s)}\frac{(s)^0}{0!} \\ \frac{(1)^3}{3!} - \frac{h^3(15rs-3s-3r+1)}{120rs} - \frac{(5rs-2s-2r+1)}{120(s-1)(r-1)}\frac{1}{0!} - \frac{(3s-1)}{120r(r-1)(r-s)}\frac{(r)^0}{0!} \\ +\frac{(3r-1)}{120s(s-1)(r-s)}\frac{(s)^0}{0!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_4 = \begin{bmatrix} \frac{(s)^4}{4!} - \frac{s^5(3r-s)}{120(r-1)(s-1)}\frac{1}{1!} + \frac{s^5(s-3)}{120r(r-1)(r-s)}\frac{(r)^1}{1!} - \frac{s^3(2s-5r+2rs-s^2)}{120(s-1)(r-s)}\frac{(s)^1}{1!} \\ \frac{(r)^4}{4!} + \frac{r^5(r-3s)}{120(s-1)(r-1)}\frac{1}{1!} - \frac{r^3(5s-2r-2rs+r^2)}{120(r-1)(r-s)}\frac{(r)^1}{1!} - \frac{r^5(r-3)}{120s(s-1)(r-s)}\frac{(s)^1}{1!} \\ \frac{(1)^4}{4!} - \frac{(5rs-2s-2r+1)}{120(s-1)(r-1)}\frac{1}{1!} - \frac{(3s-1)}{120r(r-1)(r-s)}\frac{(r)^1}{1!} + \frac{(3r-1)}{120s(s-1)(r-s)}\frac{(s)^1}{1!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_5 = \begin{bmatrix} \frac{(s)^5}{5!} - \frac{s^5(3r-s)}{120(r-1)(s-1)}\frac{1}{2!} + \frac{s^5(s-3)}{120r(r-1)(r-s)}\frac{(r)^2}{2!} - \frac{s^3(2s-5r+2rs-s^2)}{120(s-1)(r-s)}\frac{(s)^2}{2!} \\ \frac{(r)^5}{5!} + \frac{r^5(r-3s)}{120(s-1)(r-1)}\frac{1}{2!} - \frac{r^3(5s-2r-2rs+r^2)}{120(r-1)(r-s)}\frac{(r)^2}{2!} - \frac{r^5(r-3)}{120s(s-1)(r-s)}\frac{(s)^2}{2!} \\ \frac{(1)^5}{5!} - \frac{(5rs-2s-2r+1)}{120(s-1)(r-1)}\frac{1}{2!} - \frac{(3s-1)}{120r(r-1)(r-s)}\frac{(r)^2}{2!} + \frac{(3r-1)}{120s(s-1)(r-s)}\frac{(s)^2}{2!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_6 = \begin{bmatrix} \frac{(s)^6}{6!} - \frac{s^5(3r-s)}{120(r-1)(s-1)}\frac{1}{3!} + \frac{s^5(s-3)}{120r(r-1)(r-s)}\frac{(r)^3}{3!} - \frac{s^3(2s-5r+2rs-s^2)}{120(s-1)(r-s)}\frac{(s)^3}{3!} \\ \frac{(r)^6}{6!} + \frac{r^5(r-3s)}{120(s-1)(r-1)}\frac{1}{3!} - \frac{r^3(5s-2r-2rs+r^2)}{120(r-1)(r-s)}\frac{(r)^3}{3!} - \frac{r^5(r-3)}{120s(s-1)(r-s)}\frac{(s)^3}{3!} \\ \frac{(1)^6}{6!} - \frac{(5rs-2s-2r+1)}{120(s-1)(r-1)}\frac{1}{3!} - \frac{(3s-1)}{120r(r-1)(r-s)}\frac{(r)^3}{3!} + \frac{(3r-1)}{120s(s-1)(r-s)}\frac{(s)^3}{3!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_7 = \begin{bmatrix} \frac{(s)^7}{7!} - \frac{s^5(3r-s)}{120(r-1)(s-1)}\frac{1}{4!} + \frac{s^5(s-3)}{120r(r-1)(r-s)}\frac{(r)^4}{4!} - \frac{s^3(2s-5r+2rs-s^2)}{120(s-1)(r-s)}\frac{(s)^4}{4!} \\ \frac{(r)^7}{7!} + \frac{r^5(r-3s)}{120(s-1)(r-1)}\frac{1}{4!} - \frac{r^3(5s-2r-2rs+r^2)}{120(r-1)(r-s)}\frac{(r)^4}{4!} - \frac{r^5(r-3)}{120s(s-1)(r-s)}\frac{(s)^4}{4!} \\ \frac{(1)^7}{7!} - \frac{(5rs-2s-2r+1)}{120(s-1)(r-1)}\frac{1}{4!} - \frac{(3s-1)}{120r(r-1)(r-s)}\frac{(r)^4}{4!} + \frac{(3r-1)}{120s(s-1)(r-s)}\frac{(s)^4}{4!} \end{bmatrix}$$

$$
= \begin{bmatrix} \frac{s^5(7s-21r+7rs-3s^2)}{20160} \\ -\frac{r^5(21s-7r-7rs+3r^2)}{20160} \\ -\frac{(21rs-7s-7r+3)}{20160} \end{bmatrix}
$$

Therefore, the block method has order $[4,4,4]^T$ for all $s, r \in (0,1) \setminus \{\{r = \frac{(-7s+3s^2)}{(-21+7s)}\}$ $\cup \{s = \frac{(7r-3r^2)}{(21-7r)}\} \cup \{s = \frac{(7r-3)}{(21r-7)}\}\}$ together with the following error constants

$$
\left[\frac{s^5(7s-21r+7rs-3s^2)}{20160}, -\frac{r^5(21s-7r-7rs+3r^2)}{20160}, -\frac{(21rs-7s-7r+3)}{20160}\right]^T
$$

The same method as use earlier is also employed in finding the order of block first derivative (4.28). Expanding y' and f- function Taylor series leads to

$$
\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{(s)^j h^j}{j!}y_n^{j+1} - y_n' - hsy_n'' - \frac{s^2h^2(5s-20r+5rs-2s^2)}{60r}y_n''' \\
-\frac{s^4(5r-2s)}{60(s-1)(r-1)}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3} + \frac{s^4(2s-5)}{60r(r-1)(r-s)}\sum_{j=0}^{\infty}\frac{r^j h^{j+3}}{j!}y_n^{j+3} \\
-\frac{s^2(5s-10r+5rs-3s^2)}{60(s-1)(r-s)}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+3}}{j!}y_n^{j+3} \\
\sum_{j=0}^{\infty}\frac{(r)^j h^j}{j!}y_n^{j+1} - y_n' - hry_n'' + \frac{r^2(20s-5r-5rs+2r^2)}{60s}y_n''' \\
+\frac{r^4(2r-5s)}{60(r-1)(s-1)}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3} + \frac{r^2(10s-5r-5rs+3r^2)}{60(r-1)(r-s)}\sum_{j=0}^{\infty}\frac{(r)^j h^{j+3}}{j!}y_n^{j+3} \\
-\frac{r^4(2r-5)}{60s(s-1)(r-s)}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+3}}{j!}y_n^{j+3} \\
\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{j+1} - y_n' - hy_n'' + \frac{(20rs-5s-5r+2)}{60rs}y_n''' \\
+\frac{(10rs-5s-5r+3)}{60(s-1)(r-1)}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+3} + \frac{(5s-2)}{60r(r-1)(r-s)}\sum_{j=0}^{\infty}\frac{(r)^j h^{j+3}}{j!}y_n^{j+3} \\
+\frac{h^2(5r-2)}{60s(s-1)(r-s)}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+3}}{j!}y_n^{j+3}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Comparing the coefficients of h^j and y^j. This gives

$$\bar{C}_0 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} s-s \\ r-r \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} \frac{(s)^2}{2!} - \frac{s^2h^2(5s-20r+5rs-2s^2)}{60r} - \frac{s^4(5r-2s)}{60(s-1)(r-1)}\frac{1}{0!} + \frac{s^4(2s-5)}{60r(r-1)(r-s)}\frac{(r)^0}{0!} \\ -\frac{s^2(5s-10r+5rs-3s^2)}{60(s-1)(r-s)}\frac{(s)^0}{0!} \\ \frac{(r)^2}{2!} + \frac{r^2(20s-5r-5rs+2r^2)}{60s} + \frac{r^4(2r-5s)}{60(r-1)(s-1)}\frac{1}{0!} + \frac{r^2(10s-5r-5rs+3r^2)}{60(r-1)(r-s)}\frac{(r)^0}{0!} \\ -\frac{r^4(2r-5)}{60s(s-1)(r-s)}\frac{(s)^0}{0!} \\ \frac{(1)^2}{2!} + \frac{(20rs-5s-5r+2)}{60rs} + \frac{(10rs-5s-5r+3)}{60(s-1)(r-1)}\frac{1}{0!} + \frac{(5s-2)}{60r(r-1)(r-s)}\frac{(r)^0}{0!} \\ +\frac{h^2(5r-2)}{60s(s-1)(r-s)}\frac{(s)^0}{0!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_4 = \begin{bmatrix} \frac{(s)^3}{3!} - \frac{s^4(5r-2s)}{60(s-1)(r-1)}\frac{1}{1!} + \frac{s^4(2s-5)}{60r(r-1)(r-s)}\frac{(r)^1}{1!} - \frac{s^2(5s-10r+5rs-3s^2)}{60(s-1)(r-s)}\frac{(s)^1}{1!} \\ \frac{(r)^3}{3!} + \frac{r^4(2r-5s)}{60(r-1)(s-1)}\frac{1}{1!} + \frac{r^2(10s-5r-5rs+3r^2)}{60(r-1)(r-s)}\frac{(r)^1}{1!} - \frac{r^4(2r-5)}{60s(s-1)(r-s)}\frac{(s)^1}{1!} \\ \frac{(1)^3}{3!} + \frac{(10rs-5s-5r+3)}{60(s-1)(r-1)}\frac{1}{1!} + \frac{(5s-2)}{60r(r-1)(r-s)}\frac{(r)^1}{1!} + \frac{h^2(5r-2)}{60s(s-1)(r-s)}\frac{(s)^1}{1!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_5 = \begin{bmatrix} \frac{(s)^4}{4!} - \frac{s^4(5r-2s)}{60(s-1)(r-1)}\frac{1}{2!} + \frac{s^4(2s-5)}{60r(r-1)(r-s)}\frac{(r)^2}{2!} - \frac{s^2(5s-10r+5rs-3s^2)}{60(s-1)(r-s)}\frac{(s)^2}{2!} \\ \frac{(r)^4}{4!} + \frac{r^4(2r-5s)}{60(r-1)(s-1)}\frac{1}{2!} + \frac{r^2(10s-5r-5rs+3r^2)}{60(r-1)(r-s)}\frac{(r)^2}{2!} - \frac{r^4(2r-5)}{60s(s-1)(r-s)}\frac{(s)^2}{2!} \\ \frac{(1)^4}{4!} + \frac{(10rs-5s-5r+3)}{60(s-1)(r-1)}\frac{1}{2!} + \frac{(5s-2)}{60r(r-1)(r-s)}\frac{(r)^2}{2!} + \frac{h^2(5r-2)}{60s(s-1)(r-s)}\frac{(s)^2}{2!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_6 = \begin{bmatrix} \frac{(s)^5}{5!} - \frac{s^4(5r-2s)}{60(s-1)(r-1)}\frac{1}{3!} + \frac{s^4(2s-5)}{60r(r-1)(r-s)}\frac{(r)^3}{3!} - \frac{s^2(5s-10r+5rs-3s^2)}{60(s-1)(r-s)}\frac{(s)^3}{3!} \\ \frac{(r)^5}{5!} + \frac{r^4(2r-5s)}{60(r-1)(s-1)}\frac{1}{3!} + \frac{r^2(10s-5r-5rs+3r^2)}{60(r-1)(r-s)}\frac{(r)^3}{3!} - \frac{r^4(2r-5)}{60s(s-1)(r-s)}\frac{(s)^3}{3!} \\ \frac{(1)^5}{5!} + \frac{(10rs-5s-5r+3)}{60(s-1)(r-1)}\frac{1}{3!} + \frac{(5s-2)}{60r(r-1)(r-s)}\frac{(r)^3}{3!} + \frac{h^2(5r-2)}{60s(s-1)(r-s)}\frac{(s)^3}{3!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_7 = \begin{bmatrix} \frac{(s)^6}{6!} - \frac{s^4(5r-2s)}{60(s-1)(r-1)}\frac{1}{4!} + \frac{s^4(2s-5)}{60r(r-1)(r-s)}\frac{(r)^4}{4!} - \frac{s^2(5s-10r+5rs-3s^2)}{60(s-1)(r-s)}\frac{(s)^4}{4!} \\ \frac{(r)^6}{6!} + \frac{r^4(2r-5s)}{60(r-1)(s-1)}\frac{1}{4!} - \frac{r^2(10s-5r-5rs+3r^2)}{60(r-1)(r-s)}\frac{(r)^4}{4!} - \frac{r^4(2r-5)}{60s(s-1)(r-s)}\frac{(s)^4}{4!} \\ \frac{(1)^6}{6!} - \frac{(10rs-5s-5r+3)}{60(s-1)(r-1)}\frac{1}{4!} - \frac{(5s-2)}{60r(r-1)(r-s)}\frac{(r)^4}{4!} + \frac{h^2(5r-2)}{60s(s-1)(r-s)}\frac{(s)^4}{4!} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s^4(2s-5r+2rs-s^2)}{1440} \\ \frac{-(r^4(5s-2r-2rs+r^2))}{1440} \\ \frac{-(5rs-2s-2r+1)}{1440} \end{bmatrix}$$

Hence, by comparing the coefficient of h, the block of first derivative has order $[4,4,4]^T$ together with the following error constants vector

$$\left[\frac{s^4(2s-5r+2rs-s^2)}{1440}, \frac{-(r^4(5s-2r-2rs+r^2))}{1440}, \frac{-(5rs-2s-2r+1)}{1440}\right]^T$$

This is true for all $s, r \in (0,1) \setminus \{\{r = \frac{(-2s+s^2)}{(-5+2s)}\} \cup \{s = \frac{(2r-r^2)}{(5-2r)}\} \cup \{s = \frac{(2r-1)}{(5r-2)}\}\}$

In order to find order of second derivative block (4.35), strategy above is also applied. Such that, y'' and f- function are expanded in Taylor series. This is illustrated below.

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{(s)^j h^j}{j!}y_n^{j+2} - y_n'' + \frac{s(2s-6r+2rs-s^2)}{(12r)}y_n''' - \frac{s^3(2r-s)}{12(s-1)(r-1)}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3} \\ + \frac{s^3(s-2)}{12r(r-1)(r-s)}\sum_{j=0}^{\infty}\frac{r^j h^{j+3}}{j!}y_n^{j+3} - \frac{s(4s-6r+4rs-3s^2)}{(12(s-1)(r-s))}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+3}}{j!}y_n^{j+3} \\ \sum_{j=0}^{\infty}\frac{(r)^j h^j}{j!}y_n^{j+2} - y_n'' - \frac{r(6s-2r-2rs+r^2)}{(12s)}y_n''' + \frac{r^3(r-2s)}{(12(s-1)(r-1)}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3} \\ - \frac{r(6s-4r-4rs+3r^2)}{12(r-1)(r-s)}\sum_{j=0}^{\infty}\frac{(r)^j h^{j+3}}{j!}y_n^{j+2} - \frac{r^3(r-2)}{12s(s-1)(r-s)}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+3}}{j!}y_n^{j+3} \\ \sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{j+2} - y_n'' - \frac{(6rs-2s-2r+1)}{(12rs)}y_n''' - \frac{(6rs-4s-4r+3)}{12(s-1)(r-1)}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+3} \\ - \frac{(2s-1)}{12r(r-1)(r-s)}\sum_{j=0}^{\infty}\frac{(r)^j h^{j+3}}{j!}y_n^{j+2} + \frac{(2r-1)}{12s(s-1)(r-s)}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+3}}{j!}y_n^{j+3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Comparing the coefficients of h^j and y^j . This gives

$$\bar{C}_0 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} \frac{(s)^1}{1!} + \frac{s(2s-6r+2rs-s^2)}{(12r)} - \frac{s^3(2r-s)}{12(s-1)(r-1)}\frac{1}{0!} + \frac{s^3(s-2)}{12r(r-1)(r-s)}\frac{r^0}{0!} \\ -\frac{s(4s-6r+4rs-3s^2)}{(12(s-1)(r-s))}\frac{(s)^0}{0!} \\ \frac{(r)^1}{1!} - \frac{r(6s-2r-2rs+r^2)}{(12s)} + \frac{r^3(r-2s)}{(12(s-1)(r-1)}\frac{1}{0!} - \frac{r(6s-4r-4rs+3r^2)}{12(r-1)(r-s)}\frac{(r)^0}{0!} \\ -\frac{r^3(r-2)}{12s(s-1)(r-s)}\frac{(s)^0}{0!} \\ \frac{1}{1!} - \frac{(6rs-2s-2r+1)}{(12rs)} - \frac{(6rs-4s-4r+3)}{12(s-1)(r-1)}\frac{1}{0!} - \frac{(2s-1)}{12r(r-1)(r-s)}\frac{(r)^0}{0!} \\ +\frac{(2r-1)}{12s(s-1)(r-s)}\frac{(s)^0}{0!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_4 = \begin{bmatrix} \frac{(s)^2}{2!} - \frac{s^3(2r-s)}{12(s-1)(r-1)}\frac{1}{1!} + \frac{s^3(s-2)}{12r(r-1)(r-s)}\frac{r^1}{1!} - \frac{s(4s-6r+4rs-3s^2)}{(12(s-1)(r-s))}\frac{(s)^1}{1!} \\ \frac{(r)^2}{2!} + \frac{r^3(r-2s)}{(12(s-1)(r-1)}\frac{1}{0!} - \frac{r(6s-4r-4rs+3r^2)}{12(r-1)(r-s)}\frac{(r)^1}{1!} - \frac{r^3(r-2)}{12s(s-1)(r-s)}\frac{(s)^1}{1!} \\ \frac{1}{2!} - \frac{(6rs-4s-4r+3)}{12(s-1)(r-1)}\frac{1}{1!} - \frac{(2s-1)}{12r(r-1)(r-s)}\frac{(r)^1}{1!} + \frac{(2r-1)}{12s(s-1)(r-s)}\frac{(s)^1}{1!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_5 = \begin{bmatrix} \frac{(s)^3}{3!} - \frac{s^3(2r-s)}{12(s-1)(r-1)}\frac{1}{2!} + \frac{s^3(s-2)}{12r(r-1)(r-s)}\frac{r^2}{2!} - \frac{s(4s-6r+4rs-3s^2)}{(12(s-1)(r-s))}\frac{(s)^2}{2!} \\ \frac{(r)^3}{3!} + \frac{r^3(r-2s)}{(12(s-1)(r-1)}\frac{1}{2!} - \frac{r(6s-4r-4rs+3r^2)}{12(r-1)(r-s)}\frac{(r)^2}{2!} - \frac{r^3(r-2)}{12s(s-1)(r-s)}\frac{(s)^2}{2!} \\ \frac{1}{3!} - \frac{(6rs-4s-4r+3)}{12(s-1)(r-1)}\frac{1}{2!} - \frac{(2s-1)}{12r(r-1)(r-s)}\frac{(r)^2}{2!} + \frac{(2r-1)}{12s(s-1)(r-s)}\frac{(s)^2}{2!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_6 = \begin{bmatrix} \frac{(s)^4}{4!} - \frac{s^3(2r-s)}{12(s-1)(r-1)}\frac{1}{3!} + \frac{s^3(s-2)}{12r(r-1)(r-s)}\frac{r^3}{3!} - \frac{s(4s-6r+4rs-3s^2)}{(12(s-1)(r-s))}\frac{(s)^3}{3!} \\ \frac{(r)^4}{4!} + \frac{r^3(r-2s)}{(12(s-1)(r-1)}\frac{1}{3!} - \frac{r(6s-4r-4rs+3r^2)}{12(r-1)(r-s)}\frac{(r)^3}{3!} - \frac{r^3(r-2)}{12s(s-1)(r-s)}\frac{(s)^3}{3!} \\ \frac{1}{4!} - \frac{(6rs-4s-4r+3)}{12(s-1)(r-1)}\frac{1}{3!} - \frac{(2s-1)}{12r(r-1)(r-s)}\frac{(r)^3}{3!} + \frac{(2r-1)}{12s(s-1)(r-s)}\frac{(s)^3}{3!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_7 = \begin{bmatrix} \frac{(s)^5}{5!} - \frac{s^3(2r-s)}{12(s-1)(r-1)}\frac{1}{4!} + \frac{s^3(s-2)}{12r(r-1)(r-s)}\frac{r^4}{4!} - \frac{s(4s-6r+4rs-3s^2)}{(12(s-1)(r-s))}\frac{(s)^4}{4!} \\ \frac{(r)^5}{5!} + \frac{r^3(r-2s)}{(12(s-1)(r-1)}\frac{1}{4!} - \frac{r(6s-4r-4rs+3r^2)}{12(r-1)(r-s)}\frac{(r)^4}{4!} - \frac{r^3(r-2)}{12s(s-1)(r-s)}\frac{(s)^4}{4!} \\ \frac{1}{5!} - \frac{(6rs-4s-4r+3)}{12(s-1)(r-1)}\frac{1}{4!} - \frac{(2s-1)}{12r(r-1)(r-s)}\frac{(r)^4}{4!} + \frac{(2r-1)}{12s(s-1)(r-s)}\frac{(s)^4}{4!} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s^3(5s-10r+5rs+33s^2)}{5760} \\ \frac{r^3(5r-10s+5rs+33r^2)}{5760} \\ -\frac{10rs-5s-5r\breve{\ }33}{5760} \end{bmatrix}$$

Therefore, by comparing the coefficient of h, the block of second derivative has order $[4,4,4]^T$ for all $s, r \in (0,1) \setminus \{\{r = \frac{(-5s-33s^2)}{(-10+5s)}\} \cup \{s = \frac{(-5r-33r^2)}{(5-2r)}\} \cup \{s = \frac{(5r+33)}{(10r-5)}\}\}$ with the following error constants vector

$$\left[\frac{s^3(5s-10r+5rs+33s^2)}{5760}, \frac{r^3(5r-10s+5rs+33r^2)}{5760}, -\frac{(10rs-5s-5r-33)}{5760}\right]^T$$

4.1.1.2 Zero Stability of One Step Hybrid Block Method with Generalised Two Off-Step Points for Third Order ODEs

In finding the zero-stability of the block (4.21), we only put into consideration the first characteristic function according to Definition (3.1.3), that is

$$\begin{aligned} \Pi(z) &= |zI^{[2]_3} - \bar{B}_1^{[2]_3}| \\ &= \left| z\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| \\ &= z^2(z-1) = 0. \end{aligned}$$

This implies $z = 0,0,1$. In order to find zero-stability of the block of first derivative (4.28), Definition (3.1.3) is also applied, that is

$$
\begin{aligned}
\Pi(z) &= |zI^{[2]_3} - \acute{B}_2^{[2]_3}| \\
&= \left| z\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^2(z-1) = 0
\end{aligned}
$$

whose zeros are $z = 0,0,1$. The characteristic polynomial for the second derivative block (4.35) is given as

$$
\begin{aligned}
\Pi(z) &= |zI^{[2]_3} - \acute{\acute{B}}_3^{[2]_3}| \\
&= \left| z\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^2(z-1) = 0
\end{aligned}
$$

which implies that $z = 0,0,1$. Hence, the conditions in Definition (3.1.3) are satisfied. Therefore, the block method and its derivatives are zero stable.

4.1.1.3 Consistency and Convergent of One Step Hybrid Block Method with Generalised Two Off-Step Points for Third Order ODEs

The block method (4.21) and its derivatives (4.28),(4.35) are consistent and convergent as stated in Definition (3.1.4) and theorem (3.1)

4.1.1.4 Region of Absolute Stability of One Step Hybrid Block Method with Generalised Two Off-Step Points for Third Order ODEs

Applying (3.29) for one step hybrid block with two generalised off step points (4.21), it gives

$$\bar{h}(\theta,h)=\frac{I^{[2]_3}Y_m^{[2]_3}(\theta)-B_1^{[2]_3}R_1^{[2]_3}(\theta)}{[\bar{D}^{[2]_3}Y_{R_4^{[2]_3}}(\theta)+\bar{E}^{[2]_3}Y_{R_5^{[2]_3}}(\theta)]} \tag{4.33}$$

where

$$I^{[2]_3}Y_m^{[2]_3}(\theta)=\begin{bmatrix} e^{is\theta} & 0 & 0 \\ 0 & e^{ir\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{bmatrix}$$

$$\bar{B}_1^{[2]_3}R_1^{[2]_3}(\theta)=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{D}^{[2]_3}Y_{R_4^{[2]_3}}(\theta)=\begin{bmatrix} 0 & 0 & \frac{-(s^3(3s-15r+3rs-s^2))}{(120r)} \\ 0 & 0 & \frac{(r^3(15s-3r-3rs+r^2))}{(120s)} \\ 0 & 0 & \frac{((15rs-3s-3r+1))}{(120rs)} \end{bmatrix}$$

$$\bar{E}^{[2]_3}Y_{R_5^{[2]_3}}(\theta)=\begin{bmatrix} \frac{(s^3(2s-5r+2rs-s^2))}{(120(s-1)(r-s))}e^{si\theta} & \frac{-(s^5(s-3))}{(120r(r-1)(r-s))}e^{ri\theta} & \frac{(s^5(3r-s))}{(120(r-1)(s-1))}e^{i\theta} \\ \frac{(r^5(r-3))}{(120s(s-1)(r-s))}e^{si\theta} & \frac{(r^3(5s-2r-2rs+r^2))}{(120(r-1)(r-s))}e^{ri\theta} & \frac{-(r^5(r-3s))}{(120(s-1)(r-1))}e^{i\theta} \\ \frac{-(3r-1)}{(120s(s-1)(r-s))}e^{si\theta} & \frac{(3s-1)}{(120r(r-1)(r-s))}e^{ri\theta} & \frac{(5rs-2s-2r+1)}{(120(s-1)(r-1))}e^{i\theta} \end{bmatrix}$$

The above matrix is simplified and after finding the determinant, we have

$$\bar{h}(\theta,h)=\frac{(172800e^{i\theta}-172800)}{(10r^2s^2-6r^2s^3-6r^3s^2+3r^3s^3+r^3s^3e^{i\theta})}$$

The above equation is expanded trigonometrically and the imaginary part are equated to zero. This produces the equation of absolute stability region for one step hybrid block method with two generalised off-step points for third order ODEs as below

$$\bar{h}(\theta,h) = \frac{(172800\cos(\theta) - 172800)}{(10r^2s^2 - 6r^2s^3 - 6r^3s^2 + 3r^3s^3 + r^3s^3\cos(\theta))} \tag{4.34}$$

4.2 Derivation of One Step Hybrid Block Method with Generalised Three Off-Step Points for Third Order ODEs

In order to derive this method, Equation (4.5) is collocated at all points i.e x_n, x_{n+s_1}, x_{n+s_2}, x_{n+s_3} and x_{n+1}. while (3.1) is interpolated at points x_n, x_{n+s_1}, x_{n+s_2}. This is illustrated in the Figure 4.2 below.

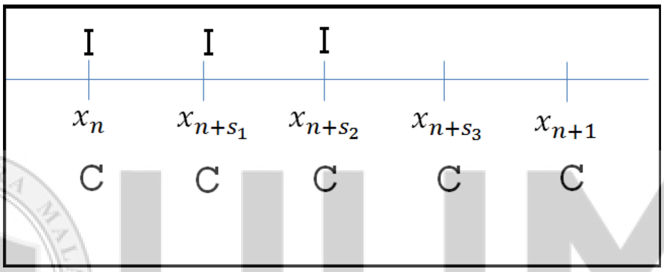

Figure 4.2. One step hybrid block method with generalised three off-step points for solving third order ODEs.

From above figure, $v = 3$ and $m = 5$. As a result, we get

$$\begin{aligned}
y_n &= a_0. \\
y_{n+s_1} &= a_0 + a_1s_1 + a_2s_1^2 + a_3s_1^3 + a_4s_1^4 + a_5s_1^5 + a_6s_1^6 + a_7s_1^7 \\
y_{n+s_2} &= a_0 + a_1s_2 + a_2s_2^2 + a_3s_2^3 + a_4s_2^4 + a_5s_2^5 + a_6s_2^6 + a_7s_2^7. \\
f_n &= \frac{6}{h^3}a_3 \\
f_{n+s_1} &= \frac{6}{h^3}a_3 + \frac{24s_1}{h^3}a_4 + \frac{60s_1^2}{h^3}a_5 + \frac{120s_1^3}{h^3}a_6 + \frac{210s_1^4}{h^3}a_7 \\
f_{n+s_2} &= \frac{6}{h^3}a_3 + \frac{24s_2}{h^3}a_4 + \frac{60s_2^2}{h^3}a_5 + \frac{120s_2^3}{h^3}a_6 + \frac{210s_2^4}{h^3}a_7 \\
f_{n+s_3} &= \frac{6}{h^3}a_3 + \frac{24s_3}{h^3}a_4 + \frac{60s_3^2}{h^3}a_5 + \frac{120s_3^3}{h^3}a_6 + \frac{210s_3^4}{h^3}a_7 \\
f_{n+1} &= \frac{6}{h^3}a_3 + \frac{24}{h^3}a_4 + \frac{60}{h^3}a_5 + \frac{120}{h^3}a_6 + \frac{210}{h^3}a_7
\end{aligned} \tag{4.35}$$

Equations (4.35) can also be written in a matrix form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & s_1 & s_1^2 & s_1^3 & s_1^4 & s_1^5 & s_1^6 & s_1^7 \\ 1 & s_2 & s_2^2 & s_2^3 & s_2^4 & s_2^5 & s_2^6 & s_2^7 \\ 0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24s_1}{h^3} & \frac{60s_1^2}{h^3} & \frac{120s_1^3}{h^3} & \frac{210s_1^4}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24s_2}{h^3} & \frac{60s_2^2}{h^3} & \frac{120s_2^3}{h^3} & \frac{120s_2^4}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24s_3}{h^3} & \frac{60s_3^2}{h^3} & \frac{120s_3^3}{h^3} & \frac{120s_3^4}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} & \frac{210}{h^3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+s_1} \\ y_{n+s_2} \\ f_n \\ f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix} \qquad (4.36)$$

Employing the Gaussian elimination method in finding the unknown values of $a_i{}'s$, $i = 0(1)8$ in (4.36) yields

$$\begin{aligned}
&a_0 = y_n \\
&a_1 = -\frac{(s_1+s_2)}{(s_1 s_2)}\, y_n - \frac{s_2}{(s_1(s_1-s_2))}\, y_{n+s_1} + \frac{s_1}{(s_2(s_1-s_2))}\, y_{n+s_2} \\
&-\frac{h^3}{840s_3}(14s_1s_2^2 - 4s_1^2s_2^2 + 14s_1^2s_2 - 4s_1s_2^3 - 4s_1^3s_2 + 21s_1^2s_3 - 7s_1^3s_3 + 21s_2^2s_3 \\
&-7s_2^3s_3 - 7s_1^3 + 3s_1^4 - 7s_2^3 + 3s_2^4 + 14s_1s_2^2s_3 + 14s_1^2s_2s_3 - 84s_1s_2s_3)f_n \\
&+\frac{h^3s_2}{840(s_1-s_2)(s_1-s_3)(s_1-1)}(7s_1s_2^2 - 3s_1^2s_2^2 + 7s_1^2s_2 - 3s_1s_2^3 - 3s_1^3s_2 \\
&+ 14s_1^2s_3 + 4s_1^4 - 7s_1^3s_3 - 21s_2^2s_3 + 7s_2^3s_3 - 7s_1^3 + 7s_2^3 - 3s_2^4 + 7s_1s_2^2s_3 \\
&+7s_1^2s_2s_3 - 21s_1s_2s_3)f_{n+s_1} \\
&+\frac{h^3s_1}{(840(s_1-s_2)(s_2-s_3)(s_2-1))}(3s_1^2s_2^2 - 7s_1s_2^2 - 7s_1^2s_2 + 3s_1s_2^3 + 3s_1^3s_2 \\
&+ 21s_1^2s_3 - 4s_2^4 - 7s_1^3s_3 - 14s_2^2s_3 + 7s_2^3s_3 - 7s_1^3 + 3s_1^4 + 7s_2^3 - 7s_1s_2^2s_3 \\
&-7s_1^2s_2s_3 + 21s_1s_2s_3)f_{n+s_2} - \frac{h^3s_1s_2}{(840s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(3s_1^4 - 4s_1^3s_2 \\
&-7s_1^3 - 4s_1^2s_2^2 + 14s_1^2s_2 - 4s_1s_2^3 + 14s_1s_2^2 + 3s_2^4 - 7s_2^3)f_{n+s_3} \\
&+\frac{h^3s_1s_2}{(840(s_1-1)(s_2-1)(s_3-1))}(3s_1^4 - 4s_1^3s_2 - 7s_3s_1^3 - 4s_1^2s_2^2 + 14s_3s_1^2s_2 \\
&-4s_1s_2^3 + 14s_3s_1s_2^2 + 3s_2^4 - 7s_3s_2^3)f_{n+1}
\end{aligned}$$

$$a_2=\frac{1}{(s_1s_2)}y_n+\frac{1}{(s_1(s_1-s_2))}y_{n+s_1}-\frac{1}{(s_2(s_1-s_2))}y_{n+s_2}=\ \frac{h^3}{(840s_1s_2s_3)}(\,14s_1^2s_2^2s_3$$
$$-4s_1^2s_2^3-4s_1^3s_2^2+14s_1s_2^3+14s_1^3s_2-4s_1s_2^4-4s_1^4s_2+21s_1^3s_3-7s_1^4s_3+21s_2^3s_3-7s_1^4$$
$$+3s_1^5-7s_2^4+3s_2^5-84s_1s_2^2s_3-84s_1^2s_2s_3+14s_1s_2^3s_3+14s_1^3s_2s_3+14s_1^2s_2^2-7s_2^4s_3)f_n$$
$$+\frac{h^3}{(840s_3(s_1-s_3)(s_2-s_3)(s_3-1))}\ (3s_1^5-4s_1^4s_2-7s_1^4-4s_1^3s_2^2+14s_1^3s_2-4s_1^2s_2^3$$
$$+14s_1^2s_2^2-4s_1s_2^4+14s_1s_2^3+3s_2^5-7s_2^4)f_{n+s_3}$$
$$-\frac{(h^3}{(840(s_1-1)(s_2-1)(s_3-1))}\ (3s_1^5-4s_1^4s_2-7s_3s_1^4-4s_1^3s_2^2+14s_3s_1^3s_2-4s_1^2s_2^3$$
$$+14s_3s_1^2s_2^2-4s_1s_2^4+14s_3s_1s_2^3+3s_2^5-7s_3s_2^4)f_{n+1}$$
$$-\frac{(h^3}{(840s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(3s_1^2s_2^3-7s_1^2s_2^2+3s_1^3s_2^2-7s_1s_2^3-7s_1^3s_2+3s_1s_2^4$$
$$+3s_1^4s_2+21s_1^3s_3-7s_1^4s_3-14s_2^3s_3+7s_2^4s_3-7s_1^4+3s_1^5+7s_2^4+21s_1s_2^2s_3+21s_1^2s_2s_3$$
$$-7s_1s_2^3s_3-7s_1^3s_2s_3-4s_2^5-7s_1^2s_2^2s_3)f_{n+s_2}$$
$$-\frac{h^3}{(840s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(7s_1^2s_2^2-3s_1^2s_2^3-3s_1^3s_2^2+7s_1s_2^3+7s_1^3s_2-3s_1s_2^4$$
$$-3s_1^4s_2+14s_1^3s_3-7s_1^4s_3-21s_2^3s_3+7s_2^4s_3-7s_1^4+4s_1^5+7s_2^4-21s_1s_2^2s_3+7s_1^2s_2^2s_3$$
$$-21s_1^2s_2s_3+7s_1s_2^3s_3+7s_1^3s_2s_3-3s_2^5)f_{n+s_1}$$

$$a_4=-\frac{h^3(s_1s_2+s_1s_3+s_2s_3+s_1s_2s_3)}{(24s_1s_2s_3)}f_n\ +\ \frac{(h^3s_1s_2s_3)}{(24(s_1-1)(s_2-1)(s_3-1))}f_{n+1}$$
$$+\frac{h^3s_1s_3}{(24s_2(s_1-s_2)(s_2-s_3)(s_2-1))}f_{n+s_2}-\frac{h^3s_1s_2}{(24s_3(s_1-s_3)(s_2-s_3)(s_3-1))}f_{n+s_3}$$
$$-\frac{h^3s_2s_3}{(24s_1(s_1-s_2)(s_1-s_3)(s_1-1))}f_{n+s_1}$$

$$a_5=-\frac{h^3(s_1s_2+s_1s_3+s_2s_3)}{(60(s_1-1)(s_2-1)(s_3-1))}f_{n+1}+\frac{h^3(s_1+s_2+s_1s_2)}{(60s_3(s_1-s_3)(s_2-s_3)(s_3-1))}f_{n+s_3}$$
$$+\frac{h^3(s_2+s_3+s_2s_3)}{(60s_1(s_1-s_2)(s_1-s_3)(s_1-1))}f_{n+s_1}\ -\ \frac{h^3(s_1+s_3+s_1s_3)}{(60s_2(s_1-s_2)(s_2-s_3)(s_2-1))}f_{n+s_2}$$
$$+\frac{h^3(s_1+s_2+s_3+s_1s_2+s_1s_3+s_2s_3)}{(60s_1s_2s_3)}f_n$$

$$a_6 = \frac{h^3(s_1+s_3+1))}{(120s_2(s_1-s_2)(s_2-s_3)(s_2-1))}f_{n+s_2} + \frac{h^3(s_1+s_2+s_3))}{(120(s_1-1)(s_2-1)(s_3-1))}f_{n+1}$$
$$-\frac{h^3(s_1+s_2+1))}{(120s_3(s_1-s_3)(s_2-s_3)(s_3-1))}f_{n+s_3} - \frac{(h^3(s_2+s_3+1))}{(120s_1(s_1-s_2)(s_1-s_3)(s_1-1))}f_{n+s_1}$$
$$-\frac{h^3(s_1+s_2+s_3+1)}{(120s_1s_2s_3)}f_n$$

$$a_7 = \frac{h^3}{(210s_3(s_1-s_3)(s_2-s_3)(s_3-1))}f_{n+s_3} - \frac{h^3}{(210(s_1-1)(s_2-1)(s_3-1))}f_{n+1}$$
$$-\frac{h^3}{(210s_2(s_1-s_2)(s_2-s_3)(s_2-1))}f_{n+s_2} + \frac{h^3}{(210s_1(s_1-s_2)(s_1-s_3)(s_1-1))}f_{n+s_1}$$
$$+\frac{(h^3)}{(210s_1s_2s_3)}f_n$$

Substituting the values of a's into equation (3.1) and simplifying, this gives a continuous linear multistep method of the form:

$$y(x) = \sum_{i=0,s_1,s_2} \alpha_i y_{n+i} + \sum_{i=0}^{1} \beta_i f_{n+i} + \sum_{i=1}^{3} \beta_{s_i} f_{n+s_i} \tag{4.37}$$

The first and second derivatives of equation (4.37) are given by

$$y^{'}(x) = \sum_{i=0,s_1,s_2} \frac{\partial}{\partial x}\alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \frac{\partial}{\partial x}\beta_i(x) f_{n+i} + \sum_{i=1}^{3} \frac{\partial}{\partial x}\beta_{s_i}(x) f_{n+s_i} \tag{4.38}$$

$$y^{''}(x) = \sum_{i=0,s_1,s_2} \frac{\partial^2}{\partial x^2}\alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \frac{\partial^2}{\partial x^2}\beta_i(x) f_{n+i} + \sum_{i=1}^{3} \frac{\partial^2}{\partial x^2}\beta_{s_i}(x) f_{n+s_i} \tag{4.39}$$

where

$$\begin{aligned}
\alpha_0 &= \frac{(x_n-x+hs_2)(x_n-x+hs_1)}{(h^2s_1s_2)} \\
\alpha_{s_1} &= \frac{((x-x_n)(x-x_n-hs_2))}{(h^2s_1(s_1-s_2))} \\
\alpha_{s_2} &= \frac{(x-x_n)(x_n-x+hs_1)}{(h^2s_2(s_1-s_2))}
\end{aligned}$$

$$\beta_0 = -\frac{(x-x_n)(x_n-x+hs_2)(x_n-x+hs_1)}{(840h^4s_1s_2s_3)}\,(7hx^3 + 3h^3s_1^3x - 24x^2x_n^2 + 7hs_3x^3$$

$$+16x^3x_n - 4x^4 - 4x_n^4 - 7h^4s_1^3 + 3h^4s_1^4 - 7h^4s_2^3 + 3h^4s_2^4 + 3hs_1x^3 + 3hs_2x^3 + 16xx_n^3$$

$$-3hs_1x_n^3 - 3hs_2x_n^3 - 7hs_3x_n^3 + 21hxx_n^2 - 21hx^2x_n + 14h^4s_1s_2^2 + 14h^4s_1^2s_2 - 4h^4s_1s_2^3$$

$$-4h^4s_1^3s_2 + 21h^4s_1^2s_3 - 7h^4s_1^3s_3 + 21h^4s_2^2s_3 - 7h^4s_2^3s_3 -,\ 7h^2s_1x^2 - 7h^3s_1^2x$$

$$-7hx_n^3 - 7h^2s_2x^2 - 7h^3s_2^2x + 3h^3s_2^3x - 14h^2s_3x^2 - 7h^2s_1x_n^2 + 7h^3s_1^2x_n - 3h^3s_1^3x_n$$

$$+7h^3s_2^2x_n - 3h^3s_2^3x_n - 14h^2s_3x_n^2, -4h^4s_1^2s_2^2 + 3h^2s_1^2x^2 - 14h^3s_1s_2x_n - 21h^3s_1s_3x_n$$

$$+3h^2s_2^2x^2 + 3h^2s_1^2x_n^2 + 3h^2s_2^2x_n^2 - 84h^4s_1s_2s_3 + 14h^3s_1s_2x + 21h^3s_1s_3x + 9hs_1xx_n^2$$

$$+21h^3s_2s_3x - 21h^3s_2s_3x_n + 14h^2s_1xx_n + 9hs_2xx_n^2 - 9hs_2x^2x_n + 14h^2s_2xx_n$$

$$+21hs_3xx_n^2 + 28h^2s_3xx_n + 14h^4s_1s_2^2s_3 + 14h^4s_1^2s_2s_3 - 4h^2s_1s_2x^2 - 4h^3s_1s_2^2x$$

$$-4h^3s_1^2s_2x\ ,-7h^2s_1s_3x^2 - 7h^3s_1^2s_3x - 7h^2s_2s_3x^2 - 4h^2s_1s_2x_n^2 + 4h^3s_1s_2^2x_n$$

$$+4h^3s_1^2s_2x_n - 7h^2s_1s_3x_n^2 + 7h^3s_1^2s_3x_n - 7h^2s_2s_3x_n^2 + 7h^3s_2^2s_3x_n + 14h^2s_2s_3xx_n$$

$$-6h^2s_2^2xx_n + 14h^3s_1s_2s_3x - 14h^3s_1s_2s_3x_n + 8h^2s_1s_2xx_n + 14h^2s_1s_3xx_n - 6h^2s_1^2xx_n$$

$$-9hs_1x^2x_n - 21hs_3x^2x_n - 7h^3s_2^2s_3x - 7h^2s_2x_n^2)$$

$$\beta_{s_1} = \frac{(x-x_n)(x_n-x+hs_2)(x_n-x+hs_1)}{(840h^4s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\,(24x^2x_n^2 - 14h^2s_2xx_n + 4h^3s_1^3x + 4h^4s_1^4$$

$$+6h^2s_2^2xx_n - 3h^3s_1s_2^2x - 16x^3x_n + 4x_n^4 + 7h^4s_2^3 - 3h^4s_2^4 + 4hs_1x^3 - 14h^3s_1s_3x_n$$

$$-4hs_1x_n^3 + 3hs_2x_n^3 + 7hs_3x_n^3 - 21hxx_n^2 + 21hx^2x_n + 7h^4s_1s_2^2 + 7h^4s_1^2s_2 + 3h^3s_1s_2^2x_n$$

$$-3h^4s_1^3s_2 + 14h^4s_1^2s_3 - 7h^4s_1^3s_3 - 21h^4s_2^2s_3 + 7h^4s_2^3s_3 - 7h^2s_1x^2 - 7h^3s_1^2x - 16xx_n^3$$

$$+7h^2s_2x^2 + 7h^3s_2^2x - 3h^3s_2^3x + 14h^2s_3x^2 - 7h^2s_1x_n^2 + 7h^3s_1^2x_n - 4h^3s_1^3x_n + 7h^2s_2x_n^2$$

$$-7h^3s_2^2x_n + 3h^3s_2^3x_n + 14h^2s_3x_n^2 - 3h^4s_1^2s_2^2 + 4h^2s_1^2x^2 - 3h^2s_2^2x^2 + 4h^2s_1^2x_n^2 + 7hx_n^3$$

$$-21h^4s_1s_2s_3 + 7h^3s_1s_2x + 14h^3s_1s_3x - 21h^3s_2s_3x - 7h^3s_1s_2x_n - 3hs_2x^3 - 3h^2s_2^2x_n^2$$

$$+21h^3s_2s_3x_n + 12hs_1xx_n^2 - 12hs_1x^2x_n + 14h^2s_1xx_n - 9hs_2xx_n^2 + 9hs_2x^2x_n - 7hs_3x^3$$

$$-21hs_3xx_n^2 + 21hs_3x^2x_n - 28h^2s_3xx_n + 7h^4s_1s_2^2s_3 + 7h^4s_1^2s_2s_3 - 3h^2s_1s_2x^2 - 7hx^3$$

$$-3h^3s_1^2s_2x - 7h^2s_1s_3x^2 - 7h^3s_1^2s_3x + 7h^2s_2s_3x^2 + 7h^3s_2^2s_3x - 3h^2s_1s_2x_n^2 - 3h^4s_1s_2^3$$

$$+3h^3s_1^2s_2x_n - 7h^2s_1s_3x_n^2 + 7h^3s_1^2s_3x_n + 7h^2s_2s_3x_n^2 - 7h^3s_2^2s_3x_n - 8h^2s_1^2xx_n - 7h^4s_1^3$$

$$+7h^3s_1s_2s_3x - 7h^3s_1s_2s_3x_n + 6h^2s_1s_2xx_n + 14h^2s_1s_3xx_n - 14h^2s_2s_3xx_n + 4x^4)$$

$$
\begin{aligned}
\beta_{s_2} = {} & \frac{(x-x_n)(x_n-x+hs_2)(x_n-x+hs_1)}{(840h^4 s_2(s_2-1)(s_2-s_3)(s_1-s_2))}(7hx^3 - 24x^2x_n^2 + 16xx_n^3 + 3h^3s_1^3x - 4x^4 \\
& +16x^3x_n - 4x_n^4 - 7h^4s_1^3 + 3h^4s_1^4 + 7h^4s_2^3 - 4h^4s_2^4 + 3hs_1x^3 - 4hs_2x^3 + 21h^4s_1s_2s_3 \\
& -3hs_1x_n^3 + 4hs_2x_n^3 - 7hs_3x_n^3 + 21hxx_n^2 - 21hx^2x_n - 7h^4s_1s_2^2 - 7h^4s_1^2s_2 + 3h^4s_1s_2^3 \\
& +3h^4s_1^3s_2 + 21h^4s_1^2s_3 - 7h^4s_1^3s_3 - 14h^4s_2^2s_3 + 7h^4s_2^3s_3 - 7h^2s_1x^2 - 7h^3s_1^2x - 7hx_n^3 \\
& +7h^3s_2^2x - 4h^3s_2^3x - 14h^2s_3x^2 - 7h^2s_1x_n^2 + 7h^3s_1^2x_n - 3h^3s_1^3x_n + 7h^2s_2x_n^2 - 7h^3s_2^2x_n \\
& +4h^3s_2^3x_n - 14h^2s_3x_n^2 + 3h^4s_1^2s_2^2 + 3h^2s_1^2x^2 - 4h^2s_2^2x^2 + 3h^2s_1^2x_n^2 - 7h^3s_2^2s_3x_n \\
& +28h^2s_3xx_n + 21h^3s_1s_3x - 14h^3s_2s_3x + 7h^3s_1s_2x_n - 21h^3s_1s_3x_n + 14h^3s_2s_3x_n \\
& +14h^2s_1xx_n - 12hs_2xx_n^2 + 12hs_2x^2x_n - 14h^2s_2xx_n + 21hs_3xx_n^2 - 21hs_3x^2x_n \\
& -7h^4s_1^2s_2s_3 + 3h^2s_1s_2x^2 + 3h^3s_1s_2^2x + 3h^3s_1^2s_2x - 7h^2s_1s_3x^2 - 7h^3s_1^2s_3x + 7hs_3x^3 \\
& +3h^2s_1s_2x_n^2 - 3h^3s_1s_2^2x_n - 3h^3s_1^2s_2x_n - 7h^2s_1s_3x_n^2 + 7h^3s_1^2s_3x_n + 7h^2s_2s_3x_n^2 \\
& +8h^2s_2^2xx_n - 7h^3s_1s_2s_3x + 7h^3s_1s_2s_3x_n - 6h^2s_1s_2xx_n + 14h^2s_1s_3xx_n - 14h^2s_2s_3xx_n \\
& -6h^2s_1^2xx_n + 7h^3s_2^2s_3x - 7h^4s_1s_2^2s_3 - 9hs_1x^2x_n + 7h^2s_2x^2 + 9hs_1xx_n^2 + 7h^2s_2s_3x^2 \\
& -4h^2s_2^2x_n^2 - 7h^3s_1s_2x)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{s_3} = {} & -\frac{(x-x_n)(x_n-x+hs_2)(x_n-x+hs_1)}{(840h^4 s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(3h^4s_1^4 - 4h^4s_1^3s_2 - 7h^4s_1^3 - 4h^4s_1^2s_2^2 \\
& +14h^4s_1^2s_2 - 4h^4s_1s_2^3 + 14h^4s_1s_2^2 + 3h^4s_2^4 - 7h^4s_2^3 + 3h^3s_1^3x - 3h^3s_1^3x_n - 4h^3s_1^2s_2x \\
& +4h^3s_1^2s_2x_n - 7h^3s_1^2x + 7h^3s_1^2x_n - 4h^3s_1s_2^2x + 4h^3s_1s_2^2x_n + 14h^3s_1s_2x - 14h^3s_1s_2x_n \\
& -3h^3s_2^3x_n - 7h^3s_2^2x + 7h^3s_2^2x_n + 3h^2s_1^2x^2 - 6h^2s_1^2xx_n + 3h^2s_1^2x_n^2 - 4h^2s_1s_2x^2 - 4x_n^4 \\
& -4h^2s_1s_2x_n^2 - 7h^2s_1x^2 + 14h^2s_1xx_n - 7h^2s_1x_n^2 + 3h^2s_2^2x^2 - 6h^2s_2^2xx_n + 3h^2s_2^2x_n^2 \\
& -7h^2s_2x^2 + 14h^2s_2xx_n - 7h^2s_2x_n^2 + 3hs_1x^3 + 9hs_1xx_n^2 - 3hs_1x_n^3 + 3hs_2x^3 + 16xx_n^3 \\
& -9hs_2x^2x_n + 9hs_2xx_n^2 - 3hs_2x_n^3 + 7hx^3 - 21hx^2x_n + 21hxx_n^2 - 7hx_n^3 - 4x^4 + 16x^3x_n \\
& +3h^3s_2^3x - 24x^2x_n^2 - 9hs_1x^2x_n + 8h^2s_1s_2xx_n)
\end{aligned}
$$

$$\beta_1 = \frac{(x-x_n)(x_n-x+hs_2)(x_n-x+hs_1)}{(840h^4(s_3-1)(s_2-1)(s_1-1))}(3h^4s_1^4 - 4h^4s_1^3s_2 - 7s_3h^4s_1^3 - 4h^4s_1^2s_2^2$$
$$-4x^4 + 14s_3h^4s_1^2s_2 - 4h^4s_1s_2^3 + 14s_3h^4s_1s_2^2 + 3h^4s_2^4 - 7s_3h^4s_2^3 + 3h^3s_1^3x + 16x^3x_n$$
$$-4h^3s_1^2s_2x + 4h^3s_1^2s_2x_n - 7s_3h^3s_1^2x + 7s_3h^3s_1^2x_n - 4h^3s_1s_2^2x + 4h^3s_1s_2^2x_n + 3hs_2x^3$$
$$+14s_3h^3s_1s_2x - 14s_3h^3s_1s_2x_n + 3h^3s_2^3x - 3h^3s_2^3x_n - 7s_3h^3s_2^2x + 7s_3h^3s_2^2x_n - 4x_n^4$$
$$+3h^2s_1^2x^2 - 6h^2s_1^2xx_n + 3h^2s_1^2x_n^2 - 4h^2s_1s_2x^2 + 8h^2s_1s_2xx_n - 4h^2s_1s_2x_n^2 + 16xx_n^3$$
$$-7s_3h^2s_1x^2 + 14s_3h^2s_1xx_n - 7s_3h^2s_1x_n^2 + 3h^2s_2^2x^2 - 6h^2s_2^2xx_n - 3hs_1x_n^3 - 3hs_2x_n^3$$
$$-7s_3h^2s_2x^2 + 14s_3h^2s_2xx_n - 7s_3h^2s_2x_n^2 + 3hs_1x^3 - 9hs_1x^2x_n + 9hs_1xx_n^2 - 7s_3hx_n^3$$
$$-9hs_2x^2x_n + 9hs_2xx_n^2 + 3h^2s_2^2x_n^2 + 7s_3hx^3 - 21s_3hx^2x_n + 21s_3hxx_n^2 - 24x^2x_n^2$$
$$-3h^3s_1^3x_n)$$

Evaluating Equation (4.37) at non-interpolating points i e, at x_{n+s_3} and x_{n+1} produces the following schemes

$$y_{n+1} + \frac{(s_2-1)}{(s_1(s_1-s_2))}y_{n+s_1} - \frac{(s_1-1)}{(s_2(s_1-s_2))}y_{n+s_2} = \frac{(s_1-1)(s_2-1)}{(s_1s_2)}y_n$$
$$-\frac{(h^3(s_2-1)(s_1-1)}{(840s_1s_2s_3)}(10s_1s_2 - 4s_2 - 7s_3 - 4s_1^2s_2^2 - 4s_1 + 14s_1s_3 + 14s_2s_3 - 4s_1^2$$
$$+10s_1s_2^2 + 10s_1^2s_2 - 4s_1s_2^3 - 4s_1^3s_2 + 14s_1^2s_3 - 7s_1^3s_3 + 14s_2^2s_3 - 7s_2^3s_3 + 14s_1s_2^2s_3$$
$$-4s_1^3 + 3s_1^4 - 4s_2^2 - 4s_2^3 + 3s_2^4 + 14s_1^2s_2s_3 - 70s_1s_2s_3 + 3)f_n$$
$$+\frac{h^3(s_2-1)}{(840s_1(s_1-s_2)(s_1-s_3))}(4s_2 - 3s_1 + 7s_3 - 3s_1^2s_2^2 + 4s_1s_2 + 7s_1s_3 - 14s_2s_3$$
$$+4s_1s_2^2 + 4s_1^2s_2 - 3s_1s_2^3 - 3s_1^3s_2 + 7s_1^2s_3 - 7s_1^3s_3 - 14s_2^2s_3 + 7s_2^3s_3 + 7s_1s_2^2s_3 - 3s_1^2$$
$$-3s_1^3 + 4s_1^4 + 4s_2^2 + 4s_2^3 - 3s_2^4 + 7s_1^2s_2s_3 - 14s_1s_2s_3 - 3)f_{n+s_1}$$
$$+\frac{h^3(s_1-1)}{840s_2(s_1-s_2)(s_2-s_3)}(4s_1 - 3s_2 + 7s_3 - 3s_1^2s_2^2 + 4s_1s_2 - 14s_1s_3 + 7s_2s_3$$
$$+4s_1s_2^2 + 4s_1^2s_2 - 3s_1s_2^3 - 3s_1^3s_2 - 14s_1^2s_3 + 7s_1^3s_3 + 7s_2^2s_3 - 7s_2^3s_3 + 4s_1^2 + 7s_1^2s_2s_3$$
$$-3s_1^4 - 3s_2^2 - 3s_2^3 + 4s_2^4 + 7s_1s_2^2s_3 - 14s_1s_2s_3 + 4s_1^3 - 3)f_{n+s_2}$$
$$+\frac{h^3}{(840s_1s_2s_3(s_1-s_2)(s_1-s_3)(s_2-s_3)(s_3-1))}(-3s_1^7s_2^2 + 3s_1^7s_2 + 7s_1^6s_2^3 - 7s_1^6s_2$$
$$-21s_1^5s_2^3 + 21s_1^5s_2^2 - 7s_1s_2^6 + 21s_1^3s_2^5 - 21s_1^3s_2^2 + 7s_1^3s_2 + 3s_1^2s_2^7 - 21s_1^2s_2^5 + 21s_1^2s_2^3$$

$$-3s_1^2s_2 - 3s_1s_2^7 + 7s_1s_2^6 - 7s_1s_2^3 + 3s_1s_2^2)f_{n+s_3} - \frac{h^3}{(840s_3 - 840)}(4s_1^2s_2^2 - 3s_2 - 7s_3$$
$$+4s_1^2s_2 + 4s_1s_2^3 + 4s_1^3s_2 + 7s_1^2s_3 + 7s_1^3s_3 + 7s_2^2s_3 + 7s_2^3s_3 - 3s_1^2 - 3s_1^3 - 14s_1s_2s_3$$
$$-3s_1 + 4s_1s_2 + 7s_1s_3 + 7s_2s_3 + 4s_1s_2^2 - 3s_1^4 - 3s_2^2 - 3s_2^3 - 3s_2^4 - 14s_1s_2^2s_3$$
$$-14s_1^2s_2s_3 + 4)f_{n+1} \tag{4.40}$$

$$y_{n+s_3} + \frac{(s_3(s_2-s_3))}{(s_1(s_1-s_2))}y_{n+s_1} - \frac{(s_3(s_1-s_3))}{(s_2(s_1-s_2))}y_{n+s_2} = \frac{(s_1-s_3)(s_2-s_3)}{(s_1s_2)}y_n$$
$$+\frac{h^3(s_1-s_3)(s_2-s_3)}{(840s_1s_2)}(-3s_1^4 + 4s_1^3s_2 + 4s_1^3s_3 + 7s_1^3 + 4s_1^2s_2^2 - 10s_1^2s_2s_3 - 14s_1^2s_2$$
$$-14s_1^2s_3 + 4s_1s_2^3 - 10s_1s_2^2s_3 - 14s_1s_2^2 - 10s_1s_2s_3^2 + 70s_1s_2s_3 + 4s_1s_3^3 - 14s_1s_3^2$$
$$-3s_2^4 + 4s_2^3s_3 + 7s_2^3 + 4s_2^2s_3^2 - 14s_2^2s_3 + 4s_2s_3^3 - 14s_2s_3^2 - 3s_3^4 + 7s_3^3 + 4s_1^2s_3^2)f_n$$
$$+\frac{h^3s_3(s_2-s_3)}{(840s_1(s_1-s_2)(s_1-1))}(4s_1^4 - 3s_1^3s_2 - 3s_1^3s_3 - 7s_1^3 - 3s_1^2s_2^2 + 4s_1^2s_2s_3 + 7s_1^2s_2$$
$$-3s_1^2s_3^2 + 7s_1^2s_3 - 3s_1s_2^3 + 4s_1s_2^2s_3 + 7s_1s_2^2 + 4s_1s_2s_3^2 - 14s_1s_2s_3 - 3s_1s_3^3 + 7s_1s_3^2$$
$$-3s_2^4 + 4s_2^3s_3 + 7s_2^3 + 4s_2^2s_3^2 - 14s_2^2s_3 + 4s_2s_3^3 - 14s_2s_3^2 - 3s_3^4 + 7s_3^3)f_{n+s_1}$$
$$-\frac{h^3s_3(s_1-s_3)}{(840s_2(s_1-s_2)(s_2-1))}(-3s_1^4 - 3s_1^3s_2 + 4s_1^3s_3 + 7s_1^3 - 3s_1^2s_2^2 + 4s_1^2s_2s_3 + 7s_1^2s_2$$
$$+4s_1^2s_3^2 - 14s_1^2s_3 - 3s_1s_2^3 + 4s_1s_2^2s_3 + 7s_1s_2^2 + 4s_1s_2s_3^2 - 14s_1s_2s_3 + 4s_1s_3^3 - 14s_1s_3^2$$
$$+4s_2^4 - 3s_2^3s_3 - 7s_2^3 - 3s_2^2s_3^2 + 7s_2^2s_3 - 3s_2s_3^3 + 7s_2s_3^2 - 3s_3^4 + 7s_3^3)f_{n+s_2}$$
$$+\frac{h^3}{(840s_3-840)}(-3s_1^4 + 4s_1^3s_2 - 3s_1^3s_3 + 7s_1^3 + 4s_1^2s_2^2 + 4s_1^2s_2s_3 - 14s_1^2s_2 - 3s_1^2s_3^2$$
$$+7s_1^2s_3 + 4s_1s_2^3 + 4s_1s_2^2s_3 - 14s_1s_2^2 + 4s_1s_2s_3^2 - 14s_1s_2s_3 - 3s_1s_3^3 + 7s_1s_3^2 - 3s_2^4$$
$$-3s_2^3s_3 + 7s_2^3 - 3s_2^2s_3^2 + 7s_2^2s_3 - 3s_2s_3^3 + 7s_2s_3^2 + 4s_3^4 - 7s_3^3)f_{n+s_3}$$
$$-\frac{h^3}{840s_1s_2(s_1-s_2)(s_1-1)(s_2-1)(s_3-1)}(-21s_1^3s_2^2s_3^5 - 21s_1^5s_2^3s_3^2 + 21s_1^3s_2^5s_3^2$$
$$+3s_1^7s_2s_3^2 - 3s_1^7s_2^2s_3 + 7s_1^6s_2^3s_3 - 7s_1^6s_2s_3^3 + 21s_1^5s_2^2s_3^3 - 7s_1^3s_2^6s_3 + 7s_1^3s_2s_3^6 + 3s_1^2s_2^7s_3$$
$$-21s_1^2s_2^5s_3^3 + 21s_1^2s_2^3s_3^5 - 3s_1^2s_2s_3^7 - 3s_1s_2^7s_3^2 + 7s_1s_2^6s_3^3 - 7s_1s_2^3s_3^6 + 3s_1s_2^2s_3^7)f_{n+1}$$
$$\tag{4.41}$$

Evaluating (4.38) at all points i.e x_{n+}, x_{n+s_1}, x_{n+s_2}, x_{n+s_3} and x_{n+1} yields

$$
\begin{aligned}
&y_n^{'}+\frac{s_2}{(hs_1(s_1-s_2))}y_{n+s_1}-\frac{s_1}{(hs_2(s_1-s_2))}y_{n+s_2}=-\frac{(s_1+s_2)}{(hs_1s_2)}y_n-\frac{h^2}{(840s_3)}(14s_1s_2^2\\
&-4s_1^2s_2^2+14s_1^2s_2-4s_1s_2^3-4s_1^3s_2+21s_1^2s_3-7s_1^3s_3+21s_2^2s_3-7s_2^3s_3-7s_1^3+3s_1^4\\
&-7s_2^3+3s_2^4+14s_1s_2^2s_3+14s_1^2s_2s_3-84s_1s_2s_3)f_n\\
&-\frac{h^2s_2}{(840s_1-840s_2)(s_1-s_3)(s_1-1))}(3s_1^2s_2^2-7s_1s_2^2-7s_1^2s_2+3s_1s_2^3+3s_1^3s_2\\
&-14s_1^2s_3+7s_1^3s_3+21s_2^2s_3-7s_2^3s_3+7s_1^3-4s_1^4-7s_2^3+3s_2^4+21s_1s_2s_3\\
&-7s_1s_2^2s_3-7s_1^2s_2s_3)f_{n+s_1}+\frac{h^2s_1}{840(s_1-s_2)(s_2-s_3)(s_2-1)}(3s_1^2s_2^2-7s_1s_2^2-7s_1^2s_2\\
&+3s_1s_2^3+3s_1^3s_2+21s_1^2s_3-7s_1^3s_3-14s_2^2s_3+7s_2^3s_3-7s_1^3+3s_1^4+7s_2^3-4s_2^4\\
&-7s_1s_2^2s_3-7s_1^2s_2s_3+21s_1s_2s_3)f_{n+s_2}\\
&\frac{h^2s_1s_2}{840s_3(s_1-s_3)(s_2-s_3)(s_3-1)}(-3s_1^4+4s_1^3s_2+7s_1^3+4s_1^2s_2^2-14s_1^2s_2+4s_1s_2^3\\
&-14s_1s_2^2-3s_2^4+7s_2^3)f_{n+s_3}-\frac{h^2s_1s_2}{(840s_1-840)(s_2-1)(s_3-1))}(4s_1^3s_2+7s_3s_1^3\\
&+4s_1^2s_2^2-14s_3s_1^2s_2+4s_1s_2^3-14s_3s_1s_2^2-3s_2^4+7s_3s_2^3-3s_1^4)f_{n+1}
\end{aligned}
\tag{4.42}
$$

$$
\begin{aligned}
&y_{n+s_1}^{'}-\frac{(2s_1-s_2)}{(hs_1(s_1-s_2))}y_{n+s_1}+\frac{s_1}{(hs_2(s_1-s_2))}y_{n+s_2}=\frac{(s_1-s_2)}{(hs_1s_2)}y_n\\
&-\frac{h^2(s_1-s_2)}{(840s_2s_3)}(7s_1s_2^2-5s_1^2s_2^2+21s_1^2s_2-s_1s_2^3-9s_1^3s_2+28s_1^2s_3-14s_1^3s_3+21s_2^2s_3\\
&-7s_2^3s_3-14s_1^3+8s_1^4-7s_2^3+3s_2^4+7s_1s_2^2s_3+21s_1^2s_2s_3-63s_1s_2s_3)f_n\\
&+\frac{h^2}{(840(s_1-1)(s_1-s_3))}(14s_1s_2^2-9s_1^2s_2^2+21s_1^2s_2-6s_1s_2^3-12s_1^3s_2+42s_1^2s_3\\
&-28s_1^3s_3-21s_2^2s_3+7s_2^3s_3-28s_1^3+20s_1^4+7s_2^3-3s_2^4+14s_1s_2^2s_3+21s_1^2s_2s_3\\
&-42s_1s_2s_3)f_{n+s_1}+\frac{h^2s_1}{(840s_2(s_2-s_3)(s_2-1)}(2s_1^2s_2^2-7s_1^2s_2-s_1s_2^3+5s_1^3s_2+28s_1^2s_3\\
&-14s_1^3s_3-14s_2^2s_3+7s_2^3s_3-14s_1^3+8s_1^4+7s_2^3-4s_2^4-7s_1^2s_2s_3+7s_1s_2s_3)f_{n+s_2}\\
&-\frac{h^2s_1}{840s_3(s_1-s_3)(s_2-s_3)(s_3-1)}(8s_1^5-17s_1^4s_2-14s_1^4+4s_1^3s_2^2+35s_1^3s_2+4s_1^2s_2^3\\
&-14s_1^2s_2^2+4s_1s_2^4-14s_1s_2^3-3s_2^5+7s_2^4)f_{n+s_3}\\
&-\frac{h^2s_1}{(840s_1-840)(s_2-1)(s_3-1)}(s_1-s_2)(-8s_1^4+9s_1^3s_2+14s_3s_1^3+5s_1^2s_2^2+s_1s_2^3\\
&-21s_3s_1^2s_2-7s_3s_1s_2^2-3s_2^4+7s_3s_2^3)f_{n+1}
\end{aligned}
\tag{4.43}
$$

$$y'_{n+s_2} - \frac{s_2}{(hs_1(s_1-s_2))}y_{n+s_1} - \frac{(s_1-2s_2)}{(hs_2(s_1-s_2))}y_{n+s_2} = -\frac{(s_1-s_2)}{(hs_1s_2)}y_n$$
$$+\frac{h^2(s_1-s_2)}{(840s_1s_3)}(21s_1s_2^2 - 5s_1^2s_2^2 + 7s_1^2s_2 - 9s_1s_2^3 - s_1^3s_2 + 21s_1^2s_3 - 7s_1^3s_3 + 28s_2^2s_3$$
$$-14s_2^3s_3 - 7s_1^3 + 3s_1^4 - 14s_2^3 + 8s_2^4 + 21s_1s_2^2s_3 + 7s_1^2s_2s_3 - 63s_1s_2s_3)f_n$$
$$+\frac{h^2s_2}{840s_1(s_1-s_3)(s_1-1)}(2s_1^2s_2^2 - 7s_1s_2^2 + 5s_1s_2^3 - s_1^3s_2 - 14s_1^2s_3 + 7s_1^3s_3 + 28s_2^2s_3$$
$$-14s_2^3s_3 + 7s_1^3 - 4s_1^4 - 14s_2^3 + 8s_2^4 - 7s_1s_2^2s_3 + 7s_1s_2s_3)f_{n+s_1}$$
$$+\frac{-(h^2)}{(840(s_2-1)(s_2-s_3))}(9s_1^2s_2^2 - 21s_1s_2^2 - 14s_1^2s_2 + 12s_1s_2^3 + 6s_1^3s_2 + 21s_1^2s_3$$
$$-7s_1^3s_3 - 42s_2^2s_3 + 28s_2^3s_3 - 7s_1^3 + 3s_1^4 + 28s_2^3 - 20s_2^4 - 21s_1s_2^2s_3 - 14s_1^2s_2s_3$$
$$+42s_1s_2s_3)f_{n+s_2}$$
$$-\frac{h^2s_2}{(840s_3(s_1-s_3)(s_2-s_3)(s_3-1)}(-3s_1^5 + 4s_1^4s_2 + 7s_1^4 + 4s_1^3s_2^2 - 14s_1^3s_2 + 4s_1^2s_2^3$$
$$-14s_1^2s_2^2 - 17s_1s_2^4 + 35s_1s_2^3 + 8s_2^5 - 14s_2^4)f_{n+s_3}$$
$$+\frac{h^2s_2(s_1-s_2)}{(840s_1-840)(s_2-1)(s_3-1))}(-3s_1^4 + s_1^3s_2 + 7s_3s_1^3 + 5s_1^2s_2^2 - 7s_3s_1^2s_2 + 9s_1s_2^3$$
$$-21s_3s_1s_2^2 - 8s_2^4 + 14s_3s_2^3)f_{n+1} \tag{4.44}$$

$$y'_{n+s_3} + \frac{(s_2-2s_3)}{(hs_1(s_1-s_2))}y_{n+s_1} - \frac{(s_1-2s_3)}{(hs_2(s_1-s_2))}y_{n+s_2} = -\frac{(s_1+s_2-2s_3)}{(hs_1s_2)}y_n$$
$$-\frac{h^2}{(840s_1s_2s_3)}(3s_1^5s_2 - 6s_1^5s_3 - 4s_1^4s_2^2 + s_1^4s_2s_3 - 7s_1^4s_2 + 14s_1^4s_3^2 + 14s_1^4s_3 + 14s_2^4s_3$$
$$-4s_1^3s_2^3 + 22s_1^3s_2^2s_3 + 14s_1^3s_2^2 - 28s_1^3s_2s_3^2 - 7s_1^3s_2s_3 - 42s_1^3s_3^2 - 4s_1^2s_2^4 + 22s_1^2s_2^3s_3 - 42s_2^3s_3^2$$
$$+14s_1^2s_2^3 - 28s_1^2s_2^2s_3^2 - 112s_1^2s_2^2s_3 + 168s_1^2s_2s_3^2 + 3s_1s_2^5 + s_1s_2^4s_3 - 7s_1s_2^4 - 28s_1s_2^3s_3^2$$
$$-7s_1s_2^3s_3 + 168s_1s_2^2s_3^2 + 70s_1s_2s_3^4 - 280s_1s_2s_3^3 - 28s_1s_3^5 + 70s_1s_3^4 - 6s_2^5s_3 + 14s_2^4s_3^2$$
$$-28s_2s_3^5 + 70s_2s_3^4 + 14s_3^6 - 28s_3^5)f_n$$
$$-\frac{h^2}{(840s_1(s_1-s_2)(s_1-s_3)(s_1-1)}(-4s_1^5s_2 + 8s_1^5s_3 + 3s_1^4s_2^2 + s_1^4s_2s_3 + 7s_1^4s_2 - 14s_1^4s_3^2$$
$$-14s_1^4s_3 + 3s_1^3s_2^3 - 13s_1^3s_2^2s_3 - 7s_1^3s_2^2 + 14s_1^3s_2s_3^2 + 28s_1^3s_3^2 + 3s_1^2s_2^4 - 13s_1^2s_2^3s_3 - 7s_1^2s_2^3$$
$$+14s_1^2s_2^2s_3^2 + 35s_1^2s_2^2s_3 - 42s_1^2s_2s_3^2 + 3s_1s_2^5 - 13s_1s_2^4s_3 - 7s_1s_2^4 + 14s_1s_2^3s_3^2 + 35s_1s_2^3s_3$$
$$-42s_1s_2^2s_3^2 - 6s_2^5s_3 + 14s_2^4s_3^2 + 14s_2^4s_3 - 42s_2^3s_3^2 - 28s_2s_3^5 + 70s_2s_3^4 + 14s_3^6 - 28s_3^5)f_{n+s_1}$$

$$
\begin{aligned}
&+\frac{h^2}{(840s_2(s_1-s_2)(s_2-s_3)(s_2-1)}\left(3s_1^5s_2-6s_1^5s_3+3s_1^4s_2^2-13s_1^4s_2s_3-7s_1^4s_2+14s_1^4s_3^2\right.\\
&+14s_1^4s_3+3s_1^3s_2^3-13s_1^3s_2^2s_3-7s_1^3s_2^2+14s_1^3s_2s_3^2+35s_1^3s_2s_3-42s_1^3s_3^2+3s_1^2s_2^4-42s_1^2s_2s_3^2\\
&-13s_1^2s_2^3s_3-7s_1^2s_2^3+14s_1^2s_2^2s_3^2+35s_1^2s_2^2s_3-4s_1s_2^5+s_1s_2^4s_3+7s_1s_2^4+14s_1s_2^3s_3^2+28s_2^3s_3^2\\
&\left.-42s_1s_2^2s_3^2-28s_1s_3^5+70s_1s_3^4+8s_2^5s_3-14s_2^4s_3^2-14s_2^4s_3+14s_3^6-28s_3^5\right)f_{n+s_2}\\
&-\frac{h^2}{(840s_3(s_1-s_3)(s_2-s_3)(s_3-1))}\left(3s_1^5s_2-6s_1^5s_3-4s_1^4s_2^2+8s_1^4s_2s_3-7s_1^4s_2+14s_1^4s_3\right.\\
&-4s_1^3s_2^3+8s_1^3s_2^2s_3+14s_1^3s_2^2-28s_1^3s_2s_3-4s_1^2s_2^4+8s_1^2s_2^3s_3+14s_1^2s_2^3-28s_1^2s_2^2s_3+3s_1s_2^5\\
&+8s_1s_2^4s_3-7s_1s_2^4-28s_1s_2^3s_3-70s_1s_2s_3^4+140s_1s_2s_3^3+42s_1s_3^5-70s_1s_3^4-6s_2^5s_3-28s_3^6\\
&\left.+14s_2^4s_3+42s_2s_3^5-70s_2s_3^4+42s_3^5\right)f_{n+s_3}+\frac{h^2}{(840(s_3-1)(s_2-1)(s_1-1))}\left(3s_1^5s_2-6s_1^5s_3\right.\\
&-4s_1^4s_2^2+s_1^4s_2s_3+14s_1^4s_3^2-4s_1^3s_2^3+22s_1^3s_2^2s_3-28s_1^3s_2s_3^2-4s_1^2s_2^4+22s_1^2s_2^3s_3-28s_1^2s_2^2s_3^2\\
&\left.+3s_1s_2^5+s_1s_2^4s_3-28s_1s_2^3s_3^2+70s_1s_2s_3^4-28s_1s_3^5-6s_2^5s_3+14s_2^4s_3^2-28s_2s_3^5+14s_3^6\right)f_{n+1}
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
&y'_{n+1}+\frac{(s_2-2)}{(hs_1(s_1-s_2))}y_{n+s_1}-\frac{(s_1-2)}{(hs_2(s_1-s_2))}y_{n+s_2}=-\frac{(s_1+s_2-2)}{(hs_1s_2)}y_n\\
&-\frac{h^2}{(840s_1s_2s_3)}\left(22s_1^2s_2^3-28s_2-28s_3-28s_1^2s_2^2-28s_1+22s_1^3s_2^2-4s_1^2s_2^4-4s_1^3s_2^3-4s_1^4s_2^2\right.\\
&+70s_1s_2+70s_1s_3+70s_2s_3-28s_1s_2^3-28s_1^3s_2+s_1s_2^4+s_1^4s_2+3s_1s_2^5+3s_1^5s_2-42s_1^3s_3\\
&+14s_1^4s_3-42s_2^3s_3+14s_2^4s_3+14s_1^4-6s_1^5+14s_2^4-6s_2^5+168s_1s_2^2s_3+168s_1^2s_2s_3-7s_1s_2^3s_3\\
&\left.-7s_1^3s_2s_3-7s_1s_2^4s_3-7s_1^4s_2s_3-112s_1^2s_2^2s_3+14s_1^2s_2^3s_3+14s_1^3s_2^2s_3-280s_1s_2s_3+14\right)f_n\\
&-\frac{h^2}{(840s_1(s_1-s_2)(s_1-s_3)(s_1-1))}\left(14s_1^2s_2^2-28s_3-28s_2-13s_1^2s_2^3-13s_1^3s_2^2+3s_1^2s_2^4\right.\\
&+3s_1^3s_2^3+3s_1^4s_2^2+70s_2s_3+14s_1s_2^3+14s_1^3s_2-13s_1s_2^4+s_1^4s_2+3s_1s_2^5-4s_1^5s_2+28s_1^3s_3\\
&-14s_1^4s_3-42s_2^3s_3+14s_2^4s_3-14s_1^4+8s_1^5+14s_2^4-6s_2^5-42s_1s_2^2s_3-42s_1^2s_2s_3+35s_1s_2^3s_3\\
&\left.-7s_1s_2^4s_3+7s_1^4s_2s_3+35s_1^2s_2^2s_3-7s_1^2s_2^3s_3-7s_1^3s_2^2s_3+14\right)f_{n+s_1}\\
&+\frac{h^2}{(840s_2(s_1-s_2)(s_2-s_3)(s_2-1))}\left(14s_1^2s_2^2-28s_3-28s_1-13s_1^2s_2^3-13s_1^3s_2^2+3s_1^2s_2^4\right.\\
&+3s_1^3s_2^3+3s_1^4s_2^2+70s_1s_3+14s_1s_2^3+14s_1^3s_2+s_1s_2^4-13s_1^4s_2-4s_1s_2^5+3s_1^5s_2-42s_1^3s_3
\end{aligned}
$$

$$+14s_1^4s_3+28s_2^3s_3-14s_2^4s_3+14s_1^4-6s_1^5-14s_2^4+8s_2^5-42s_1s_2^2s_3-42s_1^2s_2s_3+35s_1^3s_2s_3$$
$$+7s_1s_2^4s_3-7s_1^4s_2s_3+35s_1^2s_2^2s_3-7s_1^2s_2^3s_3-7s_1^3s_2^2s_3+14)f_{n+s_2}$$
$$-\frac{h^2}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(3s_1^5s_2-6s_1^5-4s_1^4s_2^2+s_1^4s_2+14s_1^4-4s_1^3s_2^3+22s_1^3s_2^2$$
$$-28s_1^3s_2-4s_1^2s_2^4+22s_1^2s_2^3-28s_1^2s_2^2+3s_1s_2^5+s_1s_2^4-28s_1s_2^3+70s_1s_2-28s_1-6s_2^5+14s_2^4$$
$$-28s_2+14)f_{n+s_3}+\frac{h^2}{((840s_1-840)(s_2-1)(s_3-1))}(42s_1+42s_2+42s_3+8s_1^2s_2^3+8s_1^3s_2^2$$
$$-4s_1^2s_2^4-4s_1^3s_2^3-4s_1^4s_2^2-70s_1s_2-70s_1s_3-70s_2s_3+8s_1s_2^4+8s_1^4s_2+3s_1s_2^5+3s_1^5s_2$$
$$+14s_1^4s_3+14s_2^4s_3-6s_1^5-6s_2^5-28s_1s_2^3s_3-28s_1^3s_2s_3-7s_1s_2^4s_3-7s_1^4s_2s_3-28s_1^2s_2^2s_3$$
$$+14s_1^2s_2^3s_3+14s_1^3s_2^2s_3+140s_1s_2s_3-28)f_{n+1} \tag{4.46}$$

Evaluating (4.39) at all pints in selected interval produces the following schemes

$$y_n'' - \frac{2}{(h^2s_1^2-h^2s_1s_2)}y_{n+s_1} - \frac{2}{(h^2s_2^2-h^2s_1s_2)}y_{n+s_2} = \frac{2}{(h^2s_1s_2)}y_n$$
$$+\frac{h}{(420s_1s_2s_3)}(14s_1^2s_2^2-4s_1^2s_2^3-4s_1^3s_2^2+14s_1s_2^3+14s_1^3s_2-4s_1s_2^4-4s_1^4s_2+21s_1^3s_3$$
$$-7s_1^4s_3+21s_2^3s_3-7s_2^4s_3-7s_1^4+3s_1^5-7s_2^4+3s_2^5-84s_1s_2^2s_3-84s_1^2s_2s_3+14s_1s_2^3s_3$$
$$+14s_1^3s_2s_3+14s_1^2s_2^2s_3)f_n-\frac{h}{420s_1(s_1-s_2)(s_1-s_3)(s_1-1)}(7s_1^2s_2^2-3s_1^2s_2^3-3s_1^3s_2^2$$
$$+7s_1s_2^3+7s_1^3s_2-3s_1s_2^4-3s_1^4s_2+14s_1^3s_3-7s_1^4s_3-21s_2^3s_3+7s_2^4s_3-7s_1^4+4s_1^5$$
$$+7s_2^4-3s_2^5-21s_1s_2^2s_3-21s_1^2s_2s_3+7s_1s_2^3s_3+7s_1^3s_2s_3+7s_1^2s_2^2s_3)f_{n+s_1}$$
$$+\frac{h}{(420s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(7s_1^2s_2^2-3s_1^2s_2^3-3s_1^3s_2^2+7s_1s_2^3+7s_1^3s_2-3s_1s_2^4$$
$$-3s_1^4s_2-21s_1^3s_3+7s_1^4s_3+14s_2^3s_3-7s_2^4s_3+7s_1^4-3s_1^5-7s_2^4+4s_2^5-21s_1s_2^2s_3$$
$$-21s_1^2s_2s_3+7s_1s_2^3s_3+7s_1^3s_2s_3+7s_1^2s_2^2s_3)f_{n+s_2}$$
$$+\frac{h}{(420s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(3s_1^5-4s_1^4s_2-7s_1^4-4s_1^3s_2^2+14s_1^3s_2-4s_1^2s_2^3$$
$$+14s_1^2s_2^2-4s_1s_2^4+14s_1s_2^3+3s_2^5-7s_2^4)f_{n+s_3}$$
$$+\frac{h}{(420s_1-420)(s_2-1)(s_3-1)}(-3s_1^5+4s_1^4s_2+7s_3s_1^4+4s_1^3s_2^2-14s_3s_1^3s_2+4s_1^2s_2^3$$
$$-14s_3s_1^2s_2^2+4s_1s_2^4-14s_3s_1s_2^3-3s_2^5+7s_3s_2^4)f_{n+1} \tag{4.47}$$

$$y''_{n+s_1} - \frac{2}{(h^2s_1^2-h^2s_1s_2)}y_{n+s_1} - \frac{2}{(h^2s_2^2-h^2s_1s_2)}y_{n+s_2} = \frac{2}{(h^2s_1s_2)}y_n$$
$$-\frac{h}{(420s_1s_2s_3)}(4s_1^2s_2^3 - 14s_1^2s_2^2 + 4s_1^3s_2^2 - 14s_1s_2^3 + 56s_1^3s_2 + 4s_1s_2^4 - 31s_1^4s_2 + 18s_1^5$$
$$+49s_1^3s_3 - 28s_1^4s_3 - 21s_2^3s_3 + 7s_2^4s_3 - 28s_1^4 + 7s_2^4 - 3s_2^5 + 84s_1s_2^2s_3 - 126s_1^2s_2s_3$$
$$-14s_1s_2^3s_3 + 56s_1^3s_2s_3 - 14s_1^2s_2^2s_3)f_n$$
$$-\frac{h}{(420s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(7s_1^2s_2^2 - 3s_1^2s_2^3 - 3s_1^3s_2^2 + 7s_1s_2^3 - 133s_1^3s_2 - 3s_2^5$$
$$-3s_1s_2^4 + 102s_1^4s_2 - 126s_1^3s_3 + 98s_1^4s_3 - 21s_2^3s_3 + 7s_2^4s_3 + 98s_1^4 - 80s_1^5 + 7s_1^2s_2^2s_3$$
$$+7s_2^4 - 21s_1s_2^2s_3 + 189s_1^2s_2s_3 + 7s_1s_2^3s_3 - 133s_1^3s_2s_3)f_{n+s_1}$$
$$+\frac{h}{(420s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(7s_1^2s_2^2 - 3s_1^2s_2^3 - 3s_1^3s_2^2 + 7s_1s_2^3 + 7s_1^3s_2 - 3s_1s_2^4$$
$$-3s_1^4s_2 + 49s_1^3s_3 - 28s_1^4s_3 + 14s_2^3s_3 - 7s_2^4s_3 - 28s_1^4 + 18s_1^5 - 7s_2^4 + 4s_2^5 - 21s_1s_2^2s_3$$
$$-21s_1^2s_2s_3 + 7s_1s_2^3s_3 + 7s_1^3s_2s_3 + 7s_1^2s_2^2s_3)f_{n+s_2}$$
$$-\frac{h}{(420s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(18s_1^5 - 31s_1^4s_2 - 28s_1^4 + 4s_1^3s_2^2 + 56s_1^3s_2 + 7s_2^4$$
$$+4s_1^2s_2^3 - 14s_1^2s_2^2 + 4s_1s_2^4 - 14s_1s_2^3 - 3s_2^5)f_{n+s_3}$$
$$+\frac{h}{((420s_1-420)(s_2-1)(s_3-1))}(18s_1^5 - 31s_1^4s_2 - 28s_3s_1^4 + 4s_1^3s_2^2 + 56s_3s_1^3s_2$$
$$+4s_1^2s_2^3 - 14s_3s_1^2s_2^2 + 4s_1s_2^4 - 14s_3s_1s_2^3 - 3s_2^5 + 7s_3s_2^4)f_{n+1} \tag{4.48}$$

$$y''_{n+s_2} - \frac{2}{(h^2s_1^2-h^2s_1s_2)}y_{n+s_1} - \frac{2}{(h^2s_2^2-h^2s_1s_2)}y_{n+s_2} = \frac{2}{(h^2s_1s_2)}y_n$$
$$+\frac{h}{(420s_1s_2s_3)}(14s_1^2s_2^2 - 4s_1^2s_2^3 - 4s_1^3s_2^2 - 56s_1s_2^3 + 14s_1^3s_2 + 31s_1s_2^4 - 4s_1^4s_2 + 21s_1^3s_3$$
$$-7s_1^4s_3 - 49s_2^3s_3 + 28s_2^4s_3 - 7s_1^4 + 3s_1^5 + 28s_2^4 - 18s_2^5 + 126s_1s_2^2s_3 - 84s_1^2s_2s_3$$
$$-56s_1s_2^3s_3 + 14s_1^3s_2s_3 + 14s_1^2s_2^2s_3)f_n - \frac{h}{420s_1(s_1-s_2)(s_1-s_3)(s_1-1)}(7s_1^2s_2^2 - 7s_1^4$$
$$-3s_1^3s_2^2 + 7s_1s_2^3 + 7s_1^3s_2 - 3s_1s_2^4 - 3s_1^4s_2 + 14s_1^3s_3 - 7s_1^4s_3 + 49s_2^3s_3 - 28s_2^4s_3 - 3s_1^2s_2^3$$
$$+4s_1^5 - 28s_2^4 + 18s_2^5 - 21s_1s_2^2s_3 - 21s_1^2s_2s_3 + 7s_1s_2^3s_3 + 7s_1^3s_2s_3 + 7s_1^2s_2^2s_3)f_{n+s_1}$$
$$+\frac{h}{(420s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(7s_1^2s_2^2 - 3s_1^2s_2^3 - 3s_1^3s_2^2 - 133s_1s_2^3 + 7s_1^3s_2 + 7s_1^4$$
$$+102s_1s_2^4 - 3s_1^4s_2 - 21s_1^3s_3 + 7s_1^4s_3 - 126s_2^3s_3 + 98s_2^4s_3 + 98s_2^4 - 80s_2^5 + 189s_1s_2^2s_3$$
$$-3s_1^5 - 21s_1^2s_2s_3 - 133s_1s_2^3s_3 + 7s_1^3s_2s_3 + 7s_1^2s_2^2s_3)f_{n+s_2}$$

$$
\begin{aligned}
&+\frac{h}{(420s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\,(3s_1^5-4s_1^4s_2-7s_1^4-4s_1^3s_2^2+14s_1^3s_2-4s_1^2s_2^3\\
&-18s_2^5+14s_1^2s_2^2+31s_1s_2^4-56s_1s_2^3+28s_2^4)f_{n+s_3}\\
&+\frac{h}{(420s_1-420)(s_2-1)(s_3-1))}(-3s_1^5+4s_1^4s_2+7s_3s_1^4+4s_1^3s_2^2-14s_3s_1^3s_2+4s_1^2s_2^3\\
&-14s_3s_1^2s_2^2-31s_1s_2^4+56s_3s_1s_2^3+18s_2^5-28s_3s_2^4)f_{n+1} \qquad (4.49)
\end{aligned}
$$

$$
\begin{aligned}
&y''_{n+s_3}-\frac{2}{(h^2s_1^2-h^2s_1s_2)}y_{n+s_1}-\frac{2}{(h^2s_2^2-h^2s_1s_2)}y_{n+s_2}=\frac{2}{(h^2s_1s_2)}y_n\\
&+\frac{h}{(420s_1s_2s_3)}\,(3s_1^5-4s_1^4s_2-7s_1^4s_3-7s_1^4-4s_1^3s_2^2+14s_1^3s_2s_3+14s_1^3s_2+21s_1^3s_3\\
&-4s_1^2s_2^3+14s_1^2s_2^2s_3+14s_1^2s_2^2-84s_1^2s_2s_3-4s_1s_2^4+14s_1s_2^3s_3+14s_1s_2^3-84s_1s_2^2s_3\\
&-70s_1s_2s_3^3+210s_1s_2s_3^2+35s_1s_3^4-70s_1s_3^3+3s_2^5-7s_2^4s_3-7s_2^4+21s_2^3s_3+35s_2s_3^4\\
&-70s_2s_3^3-21s_3^5+35s_3^4)f_n\\
&+\frac{h}{(420s_1(s_1-s_2)(s_1-s_3)(s_1-1))}\,(4s_1^5-3s_1^4s_2-7s_1^4s_3-7s_1^4-3s_1^3s_2^2+7s_1^3s_2s_3\\
&+7s_1^3s_2+14s_1^3s_3-3s_1^2s_2^3+7s_1^2s_2^2s_3+7s_1^2s_2^2-21s_1^2s_2s_3-3s_1s_2^4+7s_1s_2^3s_3+7s_1s_2^3\\
&-21s_1s_2^2s_3-3s_2^5+7s_2^4s_3+7s_2^4-21s_2^3s_3-35s_2s_3^4+70s_2s_3^3+21s_3^5-35s_3^4)f_{n+s_1}\\
&+\frac{h}{(420s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(-3s_1^5-3s_1^4s_2+7s_1^4s_3+7s_1^4-3s_1^3s_2^2+7s_1^3s_2s_3\\
&+7s_1^3s_2-21s_1^3s_3-3s_1^2s_2^3+7s_1^2s_2^2s_3+7s_1^2s_2^2-21s_1^2s_2s_3-3s_1s_2^4+7s_1s_2^3s_3+7s_1s_2^3\\
&-21s_1s_2^2s_3-35s_1s_3^4+70s_1s_3^3+4s_2^5-7s_2^4s_3-7s_2^4+14s_2^3s_3+21s_3^5-35s_3^4)f_{n+s_2}\\
&+\frac{h}{(420s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\,(3s_1^5-4s_1^4s_2-7s_1^4-4s_1^3s_2^2+14s_1^3s_2-4s_1^2s_2^3\\
&+14s_1^2s_2^2-4s_1s_2^4+14s_1s_2^3+140s_1s_2s_3^3-210s_1s_2s_3^2-105s_1s_3^4+140s_1s_3^3+3s_2^5\\
&-7s_2^4-105s_2s_3^4+140s_2s_3^3+84s_3^5-105s_3^4)f_{n+s_3}\\
&-\frac{h}{(420(s_3-1)(s_2-1)(s_1-1))}(3s_1^5-4s_1^4s_2-7s_1^4s_3-4s_1^3s_2^2+14s_1^3s_2s_3+14s_1^2s_2^2s_3\\
&-4s_1^2s_2^3-4s_1s_2^4+14s_1s_2^3s_3-70s_1s_2s_3^3+35s_1s_3^4+3s_2^5-7s_2^4s_3+35s_2s_3^4-21s_3^5)f_{n+1}
\end{aligned}
$$

(4.50)

$$y''_{n+1} - \frac{2}{(h^2s_1^2 - h^2s_1s_2)}y_{n+s_1} - \frac{2}{(h^2s_2^2 - h^2s_1s_2)}y_{n+s_2} = \frac{2}{(h^2s_1s_2)}y_n$$
$$+\frac{h}{(420s_1s_2s_3)}(35s_1 + 35s_2 + 35s_3 + 14s_1^2s_2^2 - 4s_1^2s_2^3 - 4s_1^3s_2^2 - 70s_1s_2$$
$$+14s_1s_2^3 + 14s_1^3s_2 - 4s_1s_2^4 - 4s_1^4s_2 + 21s_1^3s_3 - 7s_1^4s_3 + 21s_2^3s_3 - 7s_2^4s_3$$
$$-70s_1s_3 - 70s_2s_3 + 3s_2^5 - 84s_1^2s_2s_3 + 14s_1s_2^3s_3 + 14s_1^3s_2s_3 + 14s_1^2s_2^2s_3$$
$$-7s_1^4 + 3s_1^5 - 84s_1s_2^2s_3 + 210s_1s_2s_3 - 7s_2^4 - 21)f_n$$
$$+\frac{h}{(420s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(35s_2 + 35s_3 - 7s_1^2s_2^2 + 3s_1^2s_2^3 + 3s_1^3s_2^2$$
$$-7s_1s_2^3 - 7s_1^3s_2 + 3s_1s_2^4 + 3s_1^4s_2 - 14s_1^3s_3 + 7s_1^4s_3 + 21s_2^3s_3 - 7s_2^4s_3 + 7s_1^4$$
$$-4s_1^5 - 70s_2s_3 - 7s_2^4 + 3s_2^5 + 21s_1s_2^2s_3 + 21s_1^2s_2s_3 - 7s_1s_2^3s_3 - 7s_1^3s_2s_3$$
$$-7s_1^2s_2^2s_3 - 21)f_{n+s_1}$$
$$-\frac{h}{(420s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(35s_1 + 35s_3 - 7s_1^2s_2^2 + 3s_1^2s_2^3 + 3s_1^5$$
$$-7s_1s_2^3 - 7s_1^3s_2 + 3s_1s_2^4 + 3s_1^4s_2 + 21s_1^3s_3 - 7s_1^4s_3 - 14s_2^3s_3 + 7s_2^4s_3 - 7s_1^4$$
$$+3s_1^3s_2^2 - 70s_1s_3 + 7s_2^4 - 4s_2^5 + 21s_1s_2^2s_3 + 21s_1^2s_2s_3 - 7s_1s_2^3s_3 - 7s_1^3s_2s_3$$
$$-7s_1^2s_2^2s_3 - 21)f_{n+s_2}$$
$$+\frac{h}{(420s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(3s_1^5 - 4s_1^4s_2 - 4s_1^3s_2^2 + 14s_1^3s_2 - 4s_1^2s_2^3$$
$$-7s_1^4 + 14s_1^2s_2^2 - 4s_1s_2^4 + 14s_1s_2^3 - 70s_1s_2 + 35s_1 + 3s_2^5 - 7s_2^4 + 35s_2 - 21)f_{n+s_3}$$
$$+\frac{h}{((420s_1-420)(s_2-1)(s_3-1))}(105s_1 + 105s_2 + 105s_3 + 4s_1^2s_2^3 + 4s_1^3s_2^2$$
$$-140s_1s_2 - 140s_1s_3 - 140s_2s_3 + 4s_1s_2^4 + 4s_1^4s_2 + 7s_1^4s_3 + 7s_2^4s_3 - 3s_1^5 - 3s_2^5$$
$$-14s_1s_2^3s_3 - 14s_1^3s_2s_3 - 14s_1^2s_2^2s_3 + 210s_1s_2s_3 - 84)f_{n+1} \qquad (4.51)$$

Applying the same strategy as mention earlier in section 4.1, we join (4.40), (4.41), (4.42) and (4.47) of discrete schemes to form a block as below

$$A^{[3]_3}Y_m^{[3]_3} = B_1^{[3]_3}R_1^{[3]_3} + B_2^{[3]_3}R_2^{[3]_3} + B_3^{[3]_3}R_3^{[3]_3} + h^3D^{[3]_3}R_4^{[3]_3} + h^3E^{[3]_3}R_5^{[3]_3} \qquad (4.52)$$

where

$$A^{[3]_3} = \begin{pmatrix} \frac{(s_2-1)}{(s_1(s_1-s_2))} & -\frac{(s_1-1)}{(s_2(s_1-s_2))} & 0 & 1 \\ \frac{(s_3(s_2-s_3))}{(s_1(s_1-s_2))} & -\frac{(s_3(s_1-s_3))}{(s_2(s_1-s_2))} & 1 & 0 \\ \frac{s_2}{(hs_1(s_1-s_2))} & -\frac{s_1}{(hs_2(s_1-s_2))} & 0 & 0 \\ -\frac{2}{h^2(s_1^2-s_1s_2)} & -\frac{2}{h^2(s_2^2-s_1s_2)} & 0 & 0 \end{pmatrix}, \; Y_m^{[3]_3} = \begin{pmatrix} y_{n+s_1} \\ y_{n+s_2} \\ y_{n+s_3} \\ y_{n+1} \end{pmatrix},$$

$$B_1^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & \frac{(s_1-1)(s_2-1)}{(s_1s_2)} \\ 0 & 0 & 0 & \frac{(s_1-s_3)(s_2-s_3)}{(s_1s_2)} \\ 0 & 0 & 0 & -\frac{(s_1+s_2)}{(hs_1s_2)} \\ 0 & 0 & 0 & \frac{2}{(h^2s_1s_2)} \end{pmatrix}, R_1^{[3]_3} = \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix},$$

$$B_2^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_2^{[3]_3} = \begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}, B_3^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$R_3^{[3]_3} = \begin{pmatrix} y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix}, \quad D^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & D_{14}^{[3]_3} \\ 0 & 0 & 0 & D_{24}^{[3]_3} \\ 0 & 0 & 0 & D_{34}^{[3]_3} \\ 0 & 0 & 0 & D_{44}^{[3]_3} \end{pmatrix}, \quad R_4^{[3]_3} \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$

$$E^{[3]_3} = \begin{pmatrix} E_{11}^{[3]_3} & E_{12}^{[3]_3} & E_{13}^{[3]_3} & E_{14}^{[3]_3} \\ E_{21}^{[3]_3} & E_{21}^{[3]_3} & E_{23}^{[3]_3} & E_{24}^{[3]_3} \\ E_{31}^{[3]_3} & E_{32}^{[3]_3} & E_{33}^{[3]_3} & E_{34}^{[3]_3} \\ E_{41}^{[3]_3} & E_{42}^{[3]_3} & E_{43}^{[3]_3} & E_{44}^{[3]_3} \end{pmatrix} \text{ and } R_5^{[3]_3} = \begin{pmatrix} f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix}$$

The elements of $D^{[3]_3}$ and $E^{[3]_3}$ are given by

$$
\begin{aligned}
D_{14}^{[3]_3} &= -\frac{(s_2-1)(s_1-1)}{(840s_1s_2s_3)}(10s_1s_2-4s_2-7s_3-4s_1^2s_2^2-4s_1+14s_1s_3+14s_2s_3-4s_1^3\\
&\quad +10s_1s_2^2+10s_1^2s_2-4s_1s_2^3-4s_1^3s_2+14s_1^2s_3-7s_1^3s_3+14s_2^2s_3-7s_2^3s_3-4s_1^2\\
&\quad +3s_1^4-4s_2^2-4s_2^3+3s_2^4+14s_1s_2^2s_3+14s_1^2s_2s_3-70s_1s_2s_3+3)\\
D_{24}^{[3]_3} &= -\frac{(s_1-s_3)(s_2-s_3)}{(840s_1s_2)}(3s_1^4-4s_1^3s_2-4s_1^3s_3-7s_1^3-4s_1^2s_2^2+10s_1^2s_2s_3\\
&\quad +14s_1^2s_2-4s_1^2s_3^2+14s_1^2s_3-4s_1s_2^3+10s_1s_2^2s_3+14s_1s_2^2+10s_1s_2s_3^2\\
&\quad -70s_1s_2s_3-4s_1s_3^3+14s_1s_3^2+3s_2^4-4s_2^3s_3-7s_2^3-4s_2^2s_3^2+14s_2^2s_3\\
&\quad -4s_2s_3^3+14s_2s_3^2+3s_3^4-7s_3^3)\\
D_{34}^{[3]_3} &= \frac{-1}{(840s_3h)}(14s_1s_2^2-4s_1^2s_2^2+14s_1^2s_2-4s_1s_2^3-4s_1^3s_2+21s_1^2s_3-7s_1^3s_3+3s_2^4\\
&\quad +21s_2^2s_3-7s_2^3s_3-7s_1^3+3s_1^4-7s_2^3+14s_1s_2^2s_3+14s_1^2s_2s_3-84s_1s_2s_3)\\
D_{44}^{[3]_3} &= \frac{1}{(420h^2s_1s_2s_3)}(14s_1^2s_2^2-4s_1^2s_2^3-4s_1^3s_2^2+14s_1s_2^3+14s_1^3s_2-4s_1s_2^4-4s_1^4s_2\\
&\quad +21s_1^3s_3-7s_1^4s_3+21s_2^3s_3-7s_2^4s_3-7s_1^4+3s_1^5-7s_2^4+3s_2^5-84s_1s_2^2s_3\\
&\quad -84s_1^2s_2s_3+14s_1s_2^3s_3+14s_1^3s_2s_3+14s_1^2s_2^2s_3)\\
E_{11}^{[3]_3} &= \frac{(s_2-1)}{840s_1(s_1-s_2)(s_1-s_3)}(4s_2-3s_1+7s_3-3s_1^2s_2^2+4s_1s_2+7s_1s_3-3s_1^2\\
&\quad -14s_2s_3+4s_1s_2^2+4s_1^2s_2-3s_1s_2^3-3s_1^3s_2+7s_1^2s_3-7s_1^3s_3-14s_2^2s_3+4s_1^4\\
&\quad +7s_2^3s_3-3s_1^3+4s_2^2+4s_2^3-3s_2^4+7s_1s_2^2s_3+7s_1^2s_2s_3-14s_1s_2s_3-3)\\
E_{12}^{[3]_3} &= -\frac{(s_1-1)}{840s_2(s_1-s_2)(s_2-s_3)}(4s_1-3s_2+7s_3-3s_1^2s_2^2+4s_1s_2-14s_1s_3-3s_2^2\\
&\quad +7s_2s_3+4s_1s_2^2+4s_1^2s_2-3s_1s_2^3-3s_1^3s_2-14s_1^2s_3+7s_1^3s_3+7s_2^2s_3-3s_2^3\\
&\quad -7s_2^3s_3+4s_1^2+4s_1^3-3s_1^4+4s_2^4+7s_1s_2^2s_3+7s_1^2s_2s_3-14s_1s_2s_3-3)\\
E_{13}^{[3]_3} &= -\frac{(s_2-1)(s_1-1)}{(840s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(3s_1^4-4s_1^3s_2-4s_1^3-4s_1^2s_2^2+10s_1^2s_2\\
&\quad -4s_1^2-4s_1s_2^3+10s_1s_2^2+10s_1s_2-4s_1+3s_2^4-4s_2^3-4s_2^2-4s_2+3)\\
E_{14}^{[3]_3} &= -\frac{1}{(840s_3-840)}(4s_1^2s_2^2-3s_2-7s_3-3s_1+4s_1s_2+7s_1s_3+7s_2s_3+4s_1s_2^2\\
&\quad +4s_1^2s_2+4s_1s_2^3+4s_1^3s_2+7s_1^2s_3+7s_1^3s_3+7s_2^2s_3+7s_2^3s_3-3s_1^2-3s_1^3-3s_1^4\\
&\quad -3s_2^2-3s_2^3-3s_2^4-14s_1s_2^2s_3-14s_1^2s_2s_3-14s_1s_2s_3+4)
\end{aligned}
$$

$$
\begin{aligned}
E_{21}^{[3]_3} &= \frac{s_3(s_2-s_3)}{(840s_1(s_1-s_2)(s_1-1))}(4s_1^4-3s_1^3s_2-3s_1^3s_3-7s_1^3-3s_1^2s_2^2+4s_1^2s_2s_3-3s_3^4\\
&\quad +7s_1^2s_2-3s_1^2s_3^2+7s_1^2s_3-3s_1s_2^3+4s_1s_2^2s_3+7s_1s_2^2+4s_1s_2s_3^2-14s_1s_2s_3+7s_3^3\\
&\quad -3s_1s_3^3+7s_1s_3^2-3s_2^4+4s_2^3s_3+7s_2^3+4s_2^2s_3^2-14s_2^2s_3+4s_2s_3^3-14s_2s_3^2)\\
E_{22}^{[3]_3} &= \frac{-s_3(s_1-s_3)}{840s_2(s_1-s_2)(s_2-1)}(-3s_1^4-3s_1^3s_2+4s_1^3s_3+7s_1^3-3s_1^2s_2^2+4s_1^2s_2s_3+4s_2^4\\
&\quad +7s_1^2s_2+4s_1^2s_3^2-14s_1^2s_3-3s_1s_2^3+4s_1s_2^2s_3+7s_1s_2^2+4s_1s_2s_3^2-14s_1s_2s_3\\
&\quad -3s_3^4+4s_1s_3^3-14s_1s_3^2-3s_2^3s_3-7s_2^3-3s_2^2s_3^2+7s_2^2s_3-3s_2s_3^3+7s_2s_3^2+7s_3^3)\\
E_{23}^{[3]_3} &= \frac{1}{(840s_3-840)}(-3s_1^4+4s_1^3s_2-3s_1^3s_3+7s_1^3+4s_1^2s_2^2+4s_1^2s_2s_3-14s_1^2s_2\\
&\quad -3s_1^2s_3^2+7s_1^2s_3+4s_1s_2^3+4s_1s_2^2s_3-14s_1s_2^2+4s_1s_2s_3^2-14s_1s_2s_3-3s_1s_3^3\\
&\quad +7s_1s_3^2-3s_2^4-3s_2^3s_3+7s_2^3-3s_2^2s_3^2+7s_2^2s_3-3s_2s_3^3+7s_2s_3^2+4s_3^4-7s_3^3)\\
E_{24}^{[3]_3} &= \frac{s_3(s_2-s_3)(s_1-s_3)}{840(s_1-1)(s_2-1)(s_3-1)}(3s_1^4-4s_1^3s_2-4s_1^3s_3-4s_1^2s_2^2+10s_1^2s_2s_3+3s_2^4\\
&\quad -4s_1^2s_3^2-4s_1s_2^3+10s_1s_2^2s_3+10s_1s_2s_3^2-4s_1s_3^3-4s_2^3s_3-4s_2^2s_3^2-4s_2s_3^3+3s_3^4)\\
E_{31}^{[3]_3} &= -\frac{s_2}{h(840s_1-840s_2)(s_1-s_3)(s_1-1)}(3s_1^2s_2^2-7s_1s_2^2-7s_1^2s_2+3s_1s_2^3+3s_1^3s_2\\
&\quad -14s_1^2s_3+7s_1^3s_3+21s_2^2s_3-7s_2^3s_3+7s_1^3-4s_1^4-7s_2^3+3s_2^4-7s_1s_2^2s_3\\
&\quad -7s_1^2s_2s_3+21s_1s_2s_3)\\
E_{32}^{[3]_3} &= \frac{s_1}{h(840s_1-840s_2)(s_2-s_3)(s_2-1)}(3s_1^2s_2^2-7s_1s_2^2-7s_1^2s_2+3s_1s_2^3+3s_1^3s_2\\
&\quad -7s_1^3+21s_1^2s_3-7s_1^3s_3-14s_2^2s_3+7s_2^3s_3+3s_1^4+7s_2^3-4s_2^4-7s_1s_2^2s_3\\
&\quad -7s_1^2s_2s_3+21s_1s_2s_3)\\
E_{33}^{[3]_3} &= \frac{s_1s_2}{(840hs_3(s_1-s_3)(s_2-s_3)(s_3-1))}(-3s_1^4+4s_1^3s_2+7s_1^3+4s_1^2s_2^2-14s_1^2s_2\\
&\quad +4s_1s_2^3-14s_1s_2^2-3s_2^4+7s_2^3)\\
E_{34}^{[3]_3} &= -\frac{s_1s_2}{(h(840s_1-840)(s_2-1)(s_3-1))}(-3s_1^4+4s_1^3s_2+7s_3s_1^3+4s_1^2s_2^2-14s_3s_1^2s_2\\
&\quad +4s_1s_2^3-14s_3s_1s_2^2-3s_2^4+7s_3s_2^3)\\
E_{41}^{[3]_3} &= -\frac{1}{(420h^2s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(7s_1^2s_2^2-3s_1^2s_2^3-3s_1^3s_2^2+7s_1s_2^3-3s_2^5\\
&\quad -3s_1s_2^4-3s_1^4s_2+14s_1^3s_3-7s_1^4s_3-21s_2^3s_3+7s_2^4s_3-7s_1^4+4s_1^5+7s_2^4+7s_1^3s_2\\
&\quad -21s_1s_2^2s_3-21s_1^2s_2s_3+7s_1s_2^3s_3+7s_1^3s_2s_3+7s_1^2s_2^2s_3)
\end{aligned}
$$

$$E_{42}^{[3]_3} = \frac{1}{(420h^2 s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(7s_1^2s_2^2 - 3s_1^2s_2^3 - 3s_1^3s_2^2 + 7s_1s_2^3 + 7s_1^3s_2$$
$$-3s_1s_2^4 - 3s_1^4s_2 - 21s_1^3s_3 + 7s_1^4s_3 + 14s_2^3s_3 - 7s_2^4s_3 + 7s_1^4 - 3s_1^5 - 7s_2^4 + 4s_2^5$$
$$-21s_1s_2^2s_3 - 21s_1^2s_2s_3 + 7s_1s_2^3s_3 + 7s_1^3s_2s_3 + 7s_1^2s_2^2s_3)$$

$$E_{43}^{[3]_3} = \frac{1}{(420h^2 s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(3s_1^5 - 4s_1^4s_2 - 7s_1^4 - 4s_1^3s_2^2 + 14s_1^3s_2$$
$$-4s_1^2s_2^3 + 14s_1^2s_2^2 - 4s_1s_2^4 + 14s_1s_2^3 + 3s_2^5 - 7s_2^4)$$

$$E_{44}^{[3]_3} = \frac{1}{420h^2(s_1-1)(s_2-1)(s_3-1)}(-3s_1^5 + 4s_1^4s_2 + 7s_3s_1^4 + 4s_1^3s_2^2 - 14s_3s_1^3s_2$$
$$+4s_1^2s_2^3 - 14s_3s_1^2s_2^2 + 4s_1s_2^4 - 14s_3s_1s_2^3 - 3s_2^5 + 7s_3s_2^4)$$

Multiplying Equation (4.52) by $(A^{[3]_3})^{-1}$, this produces the following one step hybrid block method with three generalised off-step points.

$$I^{[3]_3}Y_m^{[3]_3} = \bar{B}_1^{[3]_3}R_1^{[3]_3} + \bar{B}_2^{[3]_3}R_2^{[3]_3} + \bar{B}_3^{[3]_3}R_3^{[3]_3} + h^3\bar{D}^{[3]_3}R_4^{[3]_3} + h^3\bar{E}^{[3]_3}R_5^{[3]_3} \tag{4.53}$$

where

$$I^{[3]_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \bar{B}_1^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\bar{B}_2^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & s_1h \\ 0 & 0 & 0 & s_2h \\ 0 & 0 & 0 & s_3h \\ 0 & 0 & 0 & h \end{pmatrix}, \qquad \bar{B}_3^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & \frac{h^2s_1^2}{2} \\ 0 & 0 & 0 & \frac{h^2s_2^2}{2} \\ 0 & 0 & 0 & \frac{h^2s_3^2}{2} \\ 0 & 0 & 0 & \frac{h^2}{2} \end{pmatrix}$$

$$\bar{D}^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & \bar{D}_{14}^{[3]_3} \\ 0 & 0 & 0 & \bar{D}_{24}^{[3]_3} \\ 0 & 0 & 0 & \bar{D}_{34}^{[3]_3} \\ 0 & 0 & 0 & \bar{D}_{44}^{[3]_3} \end{pmatrix}, \qquad \bar{E}^{[3]_3} = \begin{pmatrix} \bar{E}_{11}^{[3]_3} & E_{12}^{[3]_3} & E_{13}^{[3]_3} & E_{14}^{[3]_3} \\ E_{21}^{[3]_3} & E_{21}^{[3]_3} & E_{23}^{[3]_3} & E_{24}^{[3]_3} \\ E_{31}^{[3]_3} & E_{32}^{[3]_3} & E_{33}^{[3]_3} & E_{34}^{[3]_3} \\ E_{41}^{[3]_3} & E_{42}^{[3]_3} & E_{43}^{[3]_3} & E_{44}^{[3]_3} \end{pmatrix},$$

The elements of $\bar{D}^{[3]_3}$ and $\bar{E}^{[3]_3}$ are

$$\begin{aligned}
\bar{D}_{13}^{[3]_3} &= -\frac{s_1^3}{(840s_2s_3)}(21s_1s_2 + 21s_1s_3 - 105s_2s_3 - 7s_1^2s_2 - 7s_1^2s_3 \\
&\quad -7s_1^2 + 3s_1^3 + 21s_1s_2s_3) \\
\bar{D}_{23}^{[3]_3} &= \frac{s_2^3}{(840s_1s_3)}(105s_1s_3 - 21s_1s_2 - 21s_2s_3 + 7s_1s_2^2 + 7s_2^2s_3 \\
&\quad +7s_2^2 - 3s_2^3 - 21s_1s_2s_3) \\
\bar{D}_{33}^{[3]_3} &= -\frac{s_3^3}{(840s_1s_2)}(21s_1s_3 - 105s_1s_2 + 21s_2s_3 - 7s_1s_3^2 - 7s_2s_3^2 \\
&\quad -7s_3^2 + 3s_3^3 + 21s_1s_2s_3) \\
\bar{D}_{43}^{[3]_3} &= \frac{1}{(840s_1s_2s_3)}(7s_1 + 7s_2 + 7s_3 - 21s_1s_2 - 21s_1s_3 - 21s_2s_3 \\
&\quad +105s_1s_2s_3 - 3) \\
\bar{E}_{11}^{[3]_3} &= \frac{s_1^3}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}(14s_1s_2 + 14s_1s_3 - 35s_2s_3 - 7s_1^2s_2 \\
&\quad -7s_1^2s_3 - 7s_1^2 + 4s_1^3 + 14s_1s_2s_3) \\
\bar{E}_{12}^{[3]_3} &= \frac{s_1^5}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}(21s_3 - 7s_1 - 7s_1s_3 + 3s_1^2) \\
\bar{E}_{13}^{[3]_3} &= -\frac{s_1^5}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(21s_2 - 7s_1 - 7s_1s_2 + 3s_1^2) \\
\bar{E}_{14}^{[3]_3} &= \frac{s_1^5}{(840(s_3-1)(s_2-1)(s_1-1))}(21s_2s_3 - 7s_1s_3 - 7s_1s_2 + 3s_1^2) \\
\bar{E}_{21}^{[3]_3} &= -\frac{s_2^5}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}(21s_3 - 7s_2 - 7s_2s_3 + 3s_2^2) \\
\bar{E}_{22}^{[2]_3} &= \frac{s_2^3}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}(35s_1s_3 - 14s_1s_2 - 14s_2s_3 + 7s_1s_2^2 \\
&\quad +7s_2^2s_3 + 7s_2^2 - 4s_2^3 - 14s_1s_2s_3)
\end{aligned}$$

$$
\begin{aligned}
\bar{E}^{[3]_3}_{23} &= \frac{s_2^5}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(7s_2-21s_1+7s_1s_2-3s_2^2)\\
\bar{E}^{[3]_3}_{24} &= -\frac{s_2^5}{(840(s_3-1)(s_1-1)(s_2-1))}(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2)\\
\bar{E}^{[3]_3}_{31} &= \frac{s_3^5}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}(7s_3-21s_2+7s_2s_3-3s_3^2)\\
\bar{E}^{[3]_3}_{32} &= -\frac{s_3^5}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}(7s_3-21s_1+7s_1s_3-3s_3^2)\\
\bar{E}^{[3]_3}_{33} &= \frac{s_3^3}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2\\
&\quad -7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3)\\
\bar{E}^{[3]_3}_{34} &= \frac{s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2)}{(840(s_2-1)(s_1-1)(s_3-1))}\\
\bar{E}^{[3]_3}_{41} &= -\frac{(21s_2s_3-7s_3-7s_2+3)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\\
\bar{E}^{[3]_3}_{42} &= \frac{(21s_1s_3-7s_3-7s_1+3)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\\
\bar{E}^{[3]_3}_{43} &= -\frac{(21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\bar{E}^{[3]_3}_{44} &= \frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{(840(s_3-1)(s_2-1)(s_1-1))}
\end{aligned}
$$

Equation (4.53) can also be represented as

$$
\begin{aligned}
&y_{n+s_1} = y_n + hs_1y_n' + \frac{h^2s_1^2}{2}y_n''\\
&-\frac{h^3s_1^3(21s_1s_2+21s_1s_3-105s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+3s_1^3+21s_1s_2s_3)}{840s_2s_3}f_n\\
&+\frac{h^3s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1}\\
&+\frac{h^3s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2}\\
&-\frac{h^3s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3}\\
&+\frac{h^3s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}f_{n+1} \qquad (4.54)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+s_2} &= y_n + hs_2 y_n^{'} + \frac{h^2 s_2^2}{2} y_n^{''} \\
&+ \frac{h^3 s_2^3 (105 s_1 s_3 - 21 s_1 s_2 - 21 s_2 s_3 + 7 s_1 s_2^2 + 7 s_2^2 s_3 + 7 s_2^2 - 3 s_2^3 - 21 s_1 s_2 s_3)}{(840 s_1 s_3)} f_n \\
&- \frac{h^3 s_2^5 (21 s_3 - 7 s_2 - 7 s_2 s_3 + 3 s_2^2)}{(840 s_1 (s_1 - 1)(s_1 - s_3)(s_1 - s_2))} f_{n+s_1} \\
&+ \frac{h^3 s_2^3 (35 s_1 s_3 - 14 s_1 s_2 - 14 s_2 s_3 + 7 s_1 s_2^2 + 7 s_2^2 s_3 + 7 s_2^2 - 4 s_2^3 - 14 s_1 s_2 s_3)}{(840 (s_2 - 1)(s_2 - s_3)(s_1 - s_2))} f_{n+s_2} \\
&+ \frac{h^3 s_2^5 (7 s_2 - 21 s_1 + 7 s_1 s_2 - 3 s_2^2)}{(840 s_3 (s_3 - 1)(s_2 - s_3)(s_1 - s_3))} f_{n+s_3} \\
&- \frac{h^3 s_2^5 (7 s_1 s_2 - 21 s_1 s_3 + 7 s_2 s_3 - 3 s_2^2)}{(840 (s_3 - 1)(s_1 - 1)(s_2 - 1))} f_{n+1} \qquad (4.55)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+s_3} &= y_n + hs_3 y_n^{'} + \frac{h^2 s_3^2}{2} y_n^{''} \\
&- \frac{h^3 s_3^3 (21 s_1 s_3 - 105 s_1 s_2 + 21 s_2 s_3 - 7 s_1 s_3^2 - 7 s_2 s_3^2 - 7 s_3^2 + 3 s_3^3 + 21 s_1 s_2 s_3)}{(840 s_1 s_2)} f_n \\
&+ \frac{h^3 s_3^5 (7 s_3 - 21 s_2 + 7 s_2 s_3 - 3 s_3^2)}{(840 s_1 (s_1 - 1)(s_1 - s_3)(s_1 - s_2))} f_{n+s_1} \\
&- \frac{h^3 s_3^5 (7 s_3 - 21 s_1 + 7 s_1 s_3 - 3 s_3^2)}{(840 s_2 (s_2 - 1)(s_2 - s_3)(s_1 - s_2))} f_{n+s_2} \\
&+ \frac{h^3 s_3^3 (14 s_1 s_3 - 35 s_1 s_2 + 14 s_2 s_3 - 7 s_1 s_3^2 - 7 s_3^2 - 7 s_2 s_3^2 + 4 s_3^3 + 14 s_1 s_2 s_3)}{(840 (s_3 - 1)(s_2 - s_3)(s_1 - s_3))} f_{n+s_3} \\
&+ \frac{h^3 s_3^5 (21 s_1 s_2 - 7 s_1 s_3 - 7 s_2 s_3 + 3 s_3^2)}{(840 (s_2 - 1)(s_1 - 1)(s_3 - 1))} f_{n+1} \qquad (4.56)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+1} &= y_n + hy_n^{'} + \frac{h^2}{2}y_n^{''} \\
&+ \frac{h^3(7s_1+7s_2+7s_3-21s_1s_2-21s_1s_3-21s_2s_3+105s_1s_2s_3-3)}{(840s_1s_2s_3)}f_n \\
&- \frac{h^3(21s_2s_3-7s_3-7s_2+3)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} + \\
&\frac{h^3(21s_1s_3-7s_3-7s_1+3)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&- \frac{h^3(21s_1s_2-7s_2-7s_1+3)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \\
&+ \frac{h^3(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{(840(s_2-1)(s_1-1)(s_3-1))}f_{n+1}
\end{aligned} \tag{4.57}
$$

Substituting (4.55) and (4.56) into (14.43) – (4.46) gives the block of first derivative

$$
\begin{aligned}
y_{n+s_1}^{'} &= y_n^{'} + s_1hy_n^{''} \\
&- \frac{h^2s_1^2(5s_1s_2+5s_1s_3-20s_2s_3-2s_1^2s_2-2s_1^2s_3-2s_1^2+s_1^3+5s_1s_2s_3)}{(60s_2s_3)}f_n \\
&+ \frac{(h^2s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&+ \frac{(h^2s_1^4(5s_3-2s_1-2s_1s_3+s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&- \frac{(h^2s_1^4(5s_2-2s_1-2s_1s_2+s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \\
&+ \frac{(h^2s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}f_{n+1}
\end{aligned} \tag{4.58}
$$

$$
\begin{aligned}
y_{n+s_2}^{'} &= y_n^{'} + s_2hy_n^{''} \\
&+ \frac{(h^2s_2^2(20s_1s_3-5s_1s_2-5s_2s_3+2s_1s_2^2+2s_2^2s_3+2s_2^2-s_2^3-5s_1s_2s_3))}{(60s_1s_3)}f_n \\
&- \frac{h^2s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&+ \frac{(h^2s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&+ \frac{(h^2s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \\
&- \frac{(h^2s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}f_{n+1}
\end{aligned} \tag{4.59}
$$

$$
\begin{aligned}
y'_{n+s_3} &= y'_n + s_3 h y''_n \\
&- \frac{h^2 s_3^2(5s_1s_3 - 20s_1s_2 + 5s_2s_3 - 2s_1s_3^2 - 2s_2s_3^2 - 2s_3^2 + s_3^3 + 5s_1s_2s_3)}{(60s_1s_2)} f_n \\
&+ \frac{(h^2 s_3^4(5s_1s_2 - 2s_1s_3 - 2s_2s_3 + s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} \\
&+ \frac{h^2 s_3^4(2s_3 - 5s_2 + 2s_2s_3 - s_3^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)} f_{n+s_1} \\
&- \frac{h^2 s_3^4(2s_3 - 5s_1 + 2s_1s_3 - s_3^2)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)} f_{n+s_2} \\
&+ \frac{h^2 s_3^2(5s_1s_3 - 10s_1s_2 + 5s_2s_3 - 3s_1s_3^2 - 3s_2s_3^2 - 3s_3^2 + 2s_3^3 + 5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3}
\end{aligned}
\tag{4.60}
$$

$$
\begin{aligned}
y'_{n+1} &= y'_n + h y''_n \\
&+ \frac{h^2(2s_1 + 2s_2 + 2s_3 - 5s_1s_2 - 5s_1s_3 - 5s_2s_3 + 20s_1s_2s_3 - 1))}{(60s_1s_2s_3)} f_n \\
&- \frac{h^2(5s_2s_3 - 2s_3 - 2s_2 + 1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{h^2(5s_1s_3 - 2s_3 - 2s_1 + 1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&- \frac{h^2(5s_1s_2 - 2s_2 - 2s_1 + 1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} + \\
&+ \frac{h^2(3s_1 + 3s_2 + 3s_3 - 5s_1s_2 - 5s_1s_3 - 5s_2s_3 + 10s_1s_2s_3 - 2)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1}
\end{aligned}
\tag{4.61}
$$

Block of first derivative can be represented as below

$$
\acute{Y}_m^{[3]_3} = \acute{B}_2^{[3]_3} R_2^{[3]_3} + \acute{B}_3^{[3]_3} R_3^{[3]_3} + h^2 \acute{D}^{[3]_3} R_4^{[3]_3} + h^2 \acute{E}^{[3]_3} R_5^{[3]_3} \tag{4.62}
$$

where

$$
\acute{B}_2^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \acute{B}_3^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & s_1h \\ 0 & 0 & 0 & s_2h \\ 0 & 0 & 0 & s_3h \\ 0 & 0 & 0 & h \end{pmatrix}
$$

$$\acute{D}^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & \acute{D}^{[3]_3}_{14} \\ 0 & 0 & 0 & \acute{D}^{[3]_3}_{24} \\ 0 & 0 & 0 & \acute{D}^{[3]_3}_{34} \\ 0 & 0 & 0 & \acute{D}^{[3]_3}_{44} \end{pmatrix} \text{ and } \acute{E}^{[3]_3} = \begin{pmatrix} \acute{E}^{[3]_3}_{11} & \acute{E}^{[3]_3}_{12} & \acute{E}^{[3]_3}_{13} & \acute{E}^{[3]_3}_{14} \\ \acute{E}^{[3]_3}_{21} & \acute{E}^{[3]_3}_{21} & \acute{E}^{[3]_3}_{23} & \acute{E}^{[3]_3}_{24} \\ \acute{E}^{[3]_3}_{31} & \acute{E}^{[3]_3}_{32} & \acute{E}^{[3]_3}_{33} & \acute{E}^{[3]_3}_{34} \\ \acute{E}^{[3]_3}_{41} & \acute{E}^{[3]_3}_{42} & \acute{E}^{[3]_3}_{43} & \acute{E}^{[3]_3}_{44} \end{pmatrix}$$

with

$$\begin{aligned}
\acute{D}^{[3]_3}_{14} &= -\frac{s_1^2(5s_1s_2+5s_1s_3-20s_2s_3-2s_1^2s_2-2s_1^2s_3-2s_1^2+s_1^3+5s_1s_2s_3)}{(60s_2s_3)} \\
\acute{D}^{[3]_3}_{24} &= \frac{s_2^2(20s_1s_3-5s_1s_2-5s_2s_3+2s_1s_2^2+2s_2^2s_3+2s_2^2-s_2^3-5s_1s_2s_3)}{(60s_1s_3)} \\
\acute{D}^{[3]_3}_{34} &= -\frac{s_3^2(5s_1s_3-20s_1s_2+5s_2s_3-2s_1s_3^2-2s_2s_3^2-2s_3^2+s_3^3+5s_1s_2s_3)}{(60s_1s_2)} \\
\acute{D}^{[3]_3}_{44} &= \frac{(2s_1+2s_2+2s_3-5s_1s_2-5s_1s_3-5s_2s_3+20s_1s_2s_3-1)}{(60s_1s_2s_3)} \\
\acute{E}^{[3]_3}_{11} &= \frac{s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))} \\
\acute{E}^{[3]_3}_{12} &= \frac{s_1^4(5s_3-2s_1-2s_1s_3+s_1^2)}{(60hs_2(s_2-1)(s_2-s_3)(s_1-s_2))} \\
\acute{E}^{[3]_3}_{13} &= -\frac{(s_1^4(5s_2-2s_1-2s_1s_2+s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \\
\acute{E}^{[3]_3}_{14} &= +\frac{(s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))} \\
\acute{E}^{[3]_3}_{21} &= -\frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \\
\acute{E}^{[3]_3}_{22} &= \frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))} \\
\acute{E}^{[3]_3}_{23} &= \frac{(s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}
\end{aligned}$$

$$
\begin{aligned}
\acute{E}^{[3]_3}_{24} &= -\frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}\\
\acute{E}^{[3]_3}_{31} &= +\frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)}\\
\acute{E}^{[3]_3}_{32} &= -\frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)}\\
\acute{E}^{[3]_3}_{33} &= \frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\acute{E}^{[3]_3}_{34} &= \frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{(60(s_3-1)(s_2-1)(s_1-1))}\\
\acute{E}^{[3]_3}_{41} &= -\frac{(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\\
\acute{E}^{[3]_3}_{42} &= +\frac{(5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\\
\acute{E}^{[3]_3}_{43} &= -\frac{(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\acute{E}^{[3]_3}_{44} &= \frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}
\end{aligned}
$$

Substituting (4.55) and (4.56) into (4.48) – (4.51) to give the block of second derivative

$$
\begin{aligned}
y''_{n+s_1} &= y''_n\\
&-\frac{hs_1(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3)}{(60s_2s_3)}f_n\\
&+\frac{hs_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1}\\
&+\frac{hs_1^3(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2)}f_{n+s_2}\\
&-\frac{hs_1^3(10s_2-5s_1-5s_1s_2+3s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3}\\
&+\frac{hs_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}f_{n+1} \qquad (4.63)
\end{aligned}
$$

$$
\begin{aligned}
y''_{n+s_2} &= y''_n \\
&+ \frac{hs_2(30s_1s_3 - 10s_1s_2 - 10s_2s_3 + 5s_1s_2^2 + 5s_2^2s_3 + 5s_2^2 - 3s_2^3 - 10s_1s_2s_3)}{(60s_1s_3)} f_n \\
&- \frac{hs_2^3(10s_3 - 5s_2 - 5s_2s_3 + 3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{hs_2(30s_1s_3 - 20s_1s_2 - 20s_2s_3 + 15s_1s_2^2 + 15s_2^2s_3 + 15s_2^2 - 12s_2^3 - 20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2)} f_{n+s_2} \\
&+ \frac{hs_2^3(5s_2 - 10s_1 + 5s_1s_2 - 3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&- \frac{hs_2^3(5s_1s_2 - 10s_1s_3 + 5s_2s_3 - 3s_2^2)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} \qquad (4.64)
\end{aligned}
$$

$$
\begin{aligned}
y''_{n+s_3} &= y''_n \\
&- \frac{hs_3(10s_1s_3 - 30s_1s_2 + 10s_2s_3 - 5s_1s_3^2 - 5s_2s_3^2 - 5s_3^2 + 3s_3^3 + 10s_1s_2s_3)}{(60s_1s_2)} f_n \\
&+ \frac{hs_3^3(5s_3 - 10s_2 + 5s_2s_3 - 3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&- \frac{hs_3^3(5s_3 - 10s_1 + 5s_1s_3 - 3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{hs_3(20s_1s_3 - 30s_1s_2 + 20s_2s_3 - 15s_1s_3^2 - 15s_2s_3^2 - 15s_3^2 + 12s_3^3 + 20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{hs_3^3(10s_1s_2 - 5s_1s_3 - 5s_2s_3 + 3s_3^2)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} \qquad (4.65)
\end{aligned}
$$

$$
\begin{aligned}
y''_{n+1} &= y''_n \\
&+ \frac{h(5s_1 + 5s_2 + 5s_3 - 10s_1s_2 - 10s_1s_3 - 10s_2s_3 + 30s_1s_2s_3 - 3)}{(60s_1s_2s_3)} f_n \\
&- \frac{h(10s_2s_3 - 5s_3 - 5s_2 + 3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{h(10s_1s_3 - 5s_3 - 5s_1 + 3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&- \frac{h(10s_1s_2 - 5s_2 - 5s_1 + 3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3)} f_{n+s_3} \\
&+ \frac{h(15s_1 + 15s_2 + 15s_3 - 20s_1s_2 - 20s_1s_3 - 20s_2s_3 + 30s_1s_2s_3 - 12)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} \qquad (4.66)
\end{aligned}
$$

Block of second derivative can be represented as following

$$\acute{\acute{Y}}_m^{[3]_3} \quad = \quad \acute{\acute{B}}_3^{[3]_3} R_3^{[3]_3} + h\acute{\acute{D}}^{[3]_3} R_4^{[3]_3} + h\acute{\acute{E}}^{[3]_3} R_5^{[3]_3} \tag{4.67}$$

where

$$\acute{\acute{B}}_3^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \acute{\acute{D}}^{[3]_3} = \begin{pmatrix} 0 & 0 & 0 & \acute{\acute{D}}_{14}^{[3]_3} \\ 0 & 0 & 0 & \acute{\acute{D}}_{24}^{[3]_3} \\ 0 & 0 & 0 & \acute{\acute{D}}_{34}^{[3]_3} \\ 0 & 0 & 0 & \acute{\acute{D}}_{44}^{[3]_3} \end{pmatrix}$$

$$\text{and } \acute{\acute{E}}^{[3]_3} = \begin{pmatrix} \acute{\acute{E}}_{11}^{[3]_3} & \acute{\acute{E}}_{12}^{[3]_3} & \acute{\acute{E}}_{13}^{[3]_3} & \acute{\acute{E}}_{14}^{[3]_3} \\ \acute{\acute{E}}_{21}^{[3]_3} & \acute{\acute{E}}_{21}^{[3]_3} & \acute{\acute{E}}_{23}^{[3]_3} & \acute{\acute{E}}_{24}^{[3]_3} \\ \acute{\acute{E}}_{31}^{[3]_3} & \acute{\acute{E}}_{32}^{[3]_3} & \acute{\acute{E}}_{33}^{[3]_3} & \acute{\acute{E}}_{34}^{[3]_3} \\ \acute{\acute{E}}_{41}^{[3]_3} & \acute{\acute{E}}_{42}^{[3]_3} & \acute{\acute{E}}_{43}^{[3]_3} & \acute{\acute{E}}_{44}^{[3]_3} \end{pmatrix}$$

The elements of $\acute{\acute{E}}^{[3]_3}$ and $\acute{\acute{D}}^{[3]_3}$ are below

$$\begin{aligned}
\acute{\acute{D}}_{14}^{[3]_3} &= -\frac{s_1(10s_1s_2 + 10s_1s_3 - 30s_2s_3 - 5s_1^2s_2 - 5s_1^2s_3 - 5s_1^2 + 3s_1^3 + 10s_1s_2s_3)}{(60s_2s_3)} \\
\acute{\acute{D}}_{24}^{[3]_3} &= \frac{s_2(30s_1s_3 - 10s_1s_2 - 10s_2s_3 + 5s_1s_2^2 + 5s_2^2s_3 + 5s_2^2 - 3s_2^3 - 10s_1s_2s_3)}{(60s_1s_3)} \\
\acute{\acute{D}}_{34}^{[3]_3} &= -\frac{s_3(10s_1s_3 - 30s_1s_2 + 10s_2s_3 - 5s_1s_3^2 - 5s_2s_3^2 - 5s_3^2 + 3s_3^3 + 10s_1s_2s_3)}{(60s_1s_2)} \\
\acute{\acute{D}}_{44}^{[3]_3} &= \frac{(5s_1 + 5s_2 + 5s_3 - 10s_1s_2 - 10s_1s_3 - 10s_2s_3 + 30s_1s_2s_3 - 3)}{(60s_1s_2s_3)} \\
\acute{\acute{E}}_{11}^{[3]_3} &= \frac{s_1(20s_1s_2 + 20s_1s_3 - 30s_2s_3 - 15s_1^2s_2 - 15s_1^2s_3 - 15s_1^2 + 12s_1^3 + 20s_1s_2s_3))}{(60(s_1 - 1)(s_1 - s_3)(s_1 - s_2))} \\
\acute{\acute{E}}_{12}^{[3]_3} &= \frac{s_1^3(10s_3 - 5s_1 - 5s_1s_3 + 3s_1^2))}{(60s_2(s_2 - 1)(s_2 - s_3)(s_1 - s_2))} \\
\acute{\acute{E}}_{13}^{[3]_3} &= -\frac{s_1^3(10s_2 - 5s_1 - 5s_1s_2 + 3s_1^2))}{(60s_3(s_3 - 1)(s_2 - s_3)(s_1 - s_3))}
\end{aligned}$$

$$
\begin{aligned}
\acute{E}_{14}^{[3]_3} &= \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\\
\acute{E}_{21}^{[3]_3} &= -\frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\\
\acute{E}_{21}^{[3]_3} &= \frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\\
\acute{E}_{23}^{[3]_3} &= \frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\acute{E}_{24}^{[3]_3} &= -\frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\\
\acute{E}_{31}^{[3]_3} &= \frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\\
\acute{E}_{32}^{[3]_3} &= -\frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\\
\acute{E}_{33}^{[3]_3} &= \frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\acute{E}_{34}^{[3]_3} &= \frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\\
\acute{E}_{41}^{[3]_3} &= -\frac{(10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\\
\acute{E}_{42}^{[3]_3} &= \frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\\
\acute{E}_{43}^{[3]_3} &= -\frac{(10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\\
\acute{E}_{44}^{[3]_3} &= \frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))}
\end{aligned}
$$

4.2.1 Establishing the Properties of One Step Hybrid Block Method with Generalised Three Off-Step Points for Third Order ODEs

The zero stability, order, error constant, consistency, convergence and region of absolute stability are considered in this section.

4.2.1.1 Order of One Step Hybrid Block Method with Three Generalised Off-Step Points for Third Order ODEs

In order to find the order of the block (4.53), Definition (3.1.2) is employed. Expanding y and f- function in Taylor series. This is illustrated below.

$$
\begin{bmatrix}
\begin{array}{c}
\sum_{j=0}^{\infty} \frac{(s_1)^j h^j}{j!} y_n^j - y_n - s_1 h y_n^{\prime} - \frac{h^2 s_1^2}{2} y_n^{\prime\prime} \\
+ \frac{s_1^3 h^3 (21s_1s_2+21s_1s_3-105s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+3s_1^3+21s_1s_2s_3)}{(840s_2s_3)} y_n^{\prime\prime\prime} \\
- \frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{s_2^j h^{j+3}}{j!} y_n^{j+3} \\
+ \frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{s_2^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \\
\sum_{j=0}^{\infty} \frac{(s_2)^j h^j}{j!} y_n^j - y_n - s_2 h y_n^{\prime} - \frac{h^2 s_2^2}{2} y_n^{\prime\prime} \\
- \frac{s_2^3 h^3 (105s_1s_3-21s_1s_2-21s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-3s_2^3-21s_1s_2s_3)}{(840s_1s_3)} y_n^{\prime\prime\prime} \\
+ \frac{s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3)}{(840(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+3}}{j!} y_n^{j+3} \\
+ \frac{s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2)}{(840(s_3-1)(s_1-1)(s_2-1))} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \\
\sum_{j=0}^{\infty} \frac{(s_3)^j h^j}{j!} y_n^j - y_n - s_3 h y_n^{\prime} - \frac{h^2 s_3^2}{2} y_n^{\prime\prime} \\
+ \frac{s_3^3 h^3 (21s_1s_3-105s_1s_2+21s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+3s_3^3+21s_1s_2s_3)}{(840s_1s_2)} y_n^{\prime\prime\prime} \\
- \frac{s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+3}}{j!} y_n^{j+3} \\
+ \frac{s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3)}{(840(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2)}{(840(s_2-1)(s_1-1)(s_3-1))} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j - y_n - h y_n^{\prime} - \frac{h^2}{2} y_n^{\prime\prime} \\
- \frac{h^3(7s_1+7s_2+7s_3-21s_1s_2-21s_1s_3-21s_2s_3+105s_1s_2s_3-3)}{(840s_1s_2s_3)} y_n^{\prime\prime\prime} \\
+ \frac{(21s_2s_3-7s_3-7s_2+3)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{(21s_1s_3-7s_3-7s_1+3)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+3}}{j!} y_n^{j+3} \\
+ \frac{(21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{(840(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3}
\end{array}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

Comparing the coefficients of h^j and y^j yields

$$\bar{C}_0 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} s_1-s_1 \\ s_2-s_2 \\ s_3-s_3 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} \frac{s_1^2}{2}-\frac{s_1^2}{2} \\ \frac{s_2^2}{2}-\frac{s_2^2}{2} \\ \frac{s_3^2}{2}-\frac{s_3^2}{2} \\ \frac{1}{2}-\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\bar{C}_3 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^3}{3!} + \frac{(s_1^3(21s_1s_2+21s_1s_3-105s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+3s_1^3+21s_1s_2s_3)}{(840s_2s_3)} \\
-\frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))} \\
-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \\
+\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \\
-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_1)^3}{3!} - \frac{(s_2^3(105s_1s_3-21s_1s_2-21s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-3s_2^3-21s_1s_2s_3)}{(840s_1s_3)} \\
+\frac{s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \\
-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3)}{(840(s_2-1)(s_2-s_3)(s_1-s_2))} \\
-\frac{s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \\
+\frac{s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2)}{(840(s_3-1)(s_1-1)(s_2-1))}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_1)^3}{3!} + \frac{s_3^3(21s_1s_3-105s_1s_2+21s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+3s_3^3+21s_1s_2s_3)}{(840s_1s_2)} \\
-\frac{s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \\
+\frac{s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \\
-\frac{s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3)}{(840(s_3-1)(s_2-s_3)(s_1-s_3))} \\
-\frac{s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2)}{(840(s_2-1)(s_1-1)(s_3-1))}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_1)^3}{3!} - \frac{(7s_1+7s_2+7s_3-21s_1s_2-21s_1s_3-21s_2s_3+105s_1s_2s_3-3)}{(840s_1s_2s_3)} \\
+\frac{(21s_2s_3-7s_3-7s_2+3)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \\
-\frac{(21s_1s_3-7s_3-7s_1+3)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \\
+\frac{(21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \\
-\frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{(840(s_3-1)(s_2-1)(s_1-1))}
\end{gathered}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C_4} = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^4}{4!} - \frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)}{1!} \\
-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)}{1!} \\
+\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)}{1!} \\
-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{(1)}{1!}
\end{gathered} \\[4em]
\begin{gathered}
\frac{(s_2)^4}{4!} + \frac{s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)}{1!} \\
-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3)}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)}{1!} \\
-\frac{s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)}{1!} \\
+\frac{s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2)}{(840(s_3-1)(s_1-1)(s_2-1))}\frac{(1)}{1!}
\end{gathered} \\[4em]
\begin{gathered}
\frac{(s_3)^4}{4!} - \frac{s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)}{1!} \\
+\frac{s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)}{1!} \\
-\frac{s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3)}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)}{1!} \\
-\frac{s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2)}{(840(s_2-1)(s_1-1)(s_3-1))}\frac{(1)}{1!}
\end{gathered} \\[4em]
\begin{gathered}
\frac{(1)^4}{4!} + \frac{(21s_2s_3-7s_3-7s_2+3)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)}{1!} \\
-\frac{(21s_1s_3-7s_3-7s_1+3)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)}{1!} \\
+\frac{(21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)}{1!} \\
-\frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{(1)}{1!}
\end{gathered}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_5 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^5}{5!} - \frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!}\\
-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!}\\
+\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}\\
-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{(1)}{2!}
\end{gathered}\\[2ex]
\begin{gathered}
\frac{(s_2)^5}{5!} + \frac{s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!}\\
-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3)}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!}\\
-\frac{s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}\\
+\frac{s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2)}{(840(s_3-1)(s_1-1)(s_2-1))}\frac{(1)}{2!}
\end{gathered}\\[2ex]
\begin{gathered}
\frac{(s_3)^5}{5!} - \frac{s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!}\\
+\frac{s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!}\\
-\frac{s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3)}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}\\
-\frac{s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2)}{(840(s_2-1)(s_1-1)(s_3-1))}\frac{(1)}{2!}
\end{gathered}\\[2ex]
\begin{gathered}
\frac{(1)^5}{5!} + \frac{(21s_2s_3-7s_3-7s_2+3)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!}\\
-\frac{(21s_1s_3-7s_3-7s_1+3)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!}\\
+\frac{(21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!}\\
-\frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{(1)}{2!}
\end{gathered}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_6 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^6}{6!} - \frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!}\\
-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!}\\
+\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}\\
-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{(1)}{3!}
\end{gathered}\\[2em]
\begin{gathered}
\frac{(s_2)^6}{6!} + \frac{s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!}\\
-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3)}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!}\\
-\frac{s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}\\
+\frac{s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2)}{(840(s_3-1)(s_1-1)(s_2-1))}\frac{(1)}{3!}
\end{gathered}\\[2em]
\begin{gathered}
\frac{(s_3)^6}{6!} - \frac{s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!}\\
+\frac{s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!}\\
-\frac{s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3)}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}\\
-\frac{s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2)}{(840(s_2-1)(s_1-1)(s_3-1))}\frac{(1)}{3!}
\end{gathered}\\[2em]
\begin{gathered}
\frac{(1)^6}{6!} + \frac{(21s_2s_3-7s_3-7s_2+3)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!}\\
-\frac{(21s_1s_3-7s_3-7s_1+3)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!}\\
+\frac{(21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!}\\
-\frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{(1)}{3!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$\bar{C}_7 = \begin{bmatrix} \begin{gathered} \frac{(s_1)^7}{7!} - \frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\ -\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\ +\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\ -\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{(1)}{4!} \end{gathered} \\ \\ \begin{gathered} \frac{(s_2)^7}{7!} + \frac{s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\ -\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3)}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\ -\frac{s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\ +\frac{s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2)}{(840(s_3-1)(s_1-1)(s_2-1))}\frac{(1)}{4!} \end{gathered} \\ \\ \begin{gathered} \frac{(s_3)^7}{7!} - \frac{s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\ +\frac{s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\ -\frac{s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3)}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\ -\frac{s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2)}{(840(s_2-1)(s_1-1)(s_3-1))}\frac{(1)}{4!} \end{gathered} \\ \\ \begin{gathered} \frac{(1)^7}{7!} + \frac{(21s_2s_3-7s_3-7s_2+3)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\ -\frac{(21s_1s_3-7s_3-7s_1+3)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\ +\frac{(21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\ -\frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{(1)}{4!} \end{gathered} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{bmatrix}$$

$$
\bar{C}_8 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^8}{8!} - \frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
+\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{(1)}{5!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_2)^8}{8!} + \frac{s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3)}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
-\frac{s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
+\frac{s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2)}{(840(s_3-1)(s_1-1)(s_2-1))}\frac{(1)}{5!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_3)^8}{8!} - \frac{s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
+\frac{s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
-\frac{s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3)}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
-\frac{s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2)}{(840(s_2-1)(s_1-1)(s_3-1))}\frac{(1)}{5!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(1)^8}{8!} + \frac{(21s_2s_3-7s_3-7s_2+3)}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
-\frac{(21s_1s_3-7s_3-7s_1+3)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
+\frac{(21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
-\frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{(1)}{5!}
\end{gathered}
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

By comparing the coefficients of h, we conclude that the main block has order $[5,5,5,5]^T$ together with the following error constants vector

$$
\begin{bmatrix}
\frac{-(s_1^5(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{201600} \\
\frac{(s_2^5(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{201600} \\
\frac{-(s_3^5(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{201600} \\
\frac{(6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3)}{201600}
\end{bmatrix}
$$

This is true for all $s_1, s_2, s_3 \in (0,1) \setminus \{s_2 = \frac{-14s_1s_3+6s_1^2s_3+6s_1^2-3s_1^3}{14s_1-42s_3-6s_1^2+14s_1s_3}\} \cup \{s_1 = \frac{14s_2s_3-6s_2^2s_3-6s_2^2+3s_2^3}{42s_3-14s_2+6s_2^2-14s_2s_3}\}$ $\cup \{s_2 = \frac{-14s_1s_3+6s_1s_3^2+6s_3^2-3s_3^3}{-42s_1+14s_3-6s_3^2+14s_1s_3}\} \cup \{s_1 = \frac{-6s_2-6s_3+14s_2s_3+3}{6-14s_2-14s_3+42s_2s_3}\}$. Expanding y' and f function in Taylor series in order to find the order of the first derivative block. This yields

$$\begin{bmatrix}
\begin{gathered}
\sum_{j=0}^{\infty}\frac{(s_1)^j h^j}{j!}y_n^{j+1} - y_n' - s_1 h y_n'' \\
+\frac{h^2 s_1^2(5s_1s_2+5s_1s_3-20s_2s_3-2s_1^2s_2-2s_1^2s_3-2s_1^2+s_1^3+5s_1s_2s_3)}{(60s_2s_3)}y_n''' \\
-\frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+2}}{j!}y_n^{j+3} \\
-\frac{s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{s_2^j h^{j+2}}{j!}y_n^{j+3} \\
+\frac{s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{s_2^j h^{j+2}}{j!}y_n^{j+3} \\
-\frac{s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+3}
\end{gathered} \\
\begin{gathered}
\sum_{j=0}^{\infty}\frac{(s_2)^j h^j}{j!}y_n^{j+1} - y_n' - s_2 h y_n'' \\
-\frac{h^2 s_2^2(20s_1s_3-5s_1s_2-5s_2s_3+2s_1s_2^2+2s_2^2s_3+2s_2^2-s_2^3-5s_1s_2s_3)}{(60s_1s_3)}y_n''' \\
+\frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+2}}{j!}y_n^{j+3} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+2}}{j!}y_n^{j+3} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+2}}{j!}y_n^{j+3} \\
+\frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+3}
\end{gathered} \\
\begin{gathered}
\sum_{j=0}^{\infty}\frac{(s_3)^j h^j}{j!}y_n^{j+1} - y_n' - s_3 h y_n'' \\
+\frac{h^2 s_3^2(5s_1s_3-20s_1s_2+5s_2s_3-2s_1s_3^2-2s_2s_3^2-2s_3^2+s_3^3+5s_1s_2s_3)}{(60s_1s_2)}y_n''' \\
-\frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+2}}{j!}y_n^{j+3} \\
+\frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+2}}{j!}y_n^{j+3} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+2}}{j!}y_n^{j+3} \\
-\frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+3}
\end{gathered} \\
\begin{gathered}
\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{j+1} - y_n' - h y_n'' \\
-\frac{h^2(2s_1+2s_2+2s_3-5s_1s_2-5s_1s_3-5s_2s_3+20s_1s_2s_3-1))}{(60s_1s_2s_3)}y_n''' \\
+\frac{(5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+2}}{j!}y_n^{j+3} \\
-\frac{(5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+2}}{j!}y_n^{j+3} \\
+\frac{(5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+2}}{j!}y_n^{j+3} \\
-\frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+3}
\end{gathered}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Comparing the coefficients of h^j and y^j. This gives

$$\bar{C}_0 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} s_1-s_1 \\ s_2-s_2 \\ s_3-s_3 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\bar{C}_3 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^2}{2!} + \frac{s_1^2(5s_1s_2+5s_1s_3-20s_2s_3-2s_1^2s_2-2s_1^2s_3-2s_1^2+s_1^3+5s_1s_2s_3)}{(60s_2s_3)} \\
-\frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} - \frac{s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{(1)^0}{0!}
\end{gathered} \\ \\
\begin{gathered}
\frac{(s_2)^2}{2!} - \frac{s_2^2(20s_1s_3-5s_1s_2-5s_2s_3+2s_1s_2^2+2s_2^2s_3+2s_2^2-s_2^3-5s_1s_2s_3)}{(60s_1s_3)} \\
+\frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} + \frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{(1)^0}{0!}
\end{gathered} \\ \\
\begin{gathered}
\frac{(s_3)^2}{2!} + \frac{s_3^2(5s_1s_3-20s_1s_2+5s_2s_3-2s_1s_3^2-2s_2s_3^2-2s_3^2+s_3^3+5s_1s_2s_3)}{(60s_1s_2)} \\
-\frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)}\frac{(s_1)^0}{0!} + \frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)}\frac{(s_2)^0}{0!} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{(1)^0}{0!}
\end{gathered} \\ \\
\begin{gathered}
\frac{1}{2!} - \frac{(2s_1+2s_2+2s_3-5s_1s_2-5s_1s_3-5s_2s_3+20s_1s_2s_3-1))}{(60s_1s_2s_3)} \\
+\frac{(5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{(5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} + \frac{(5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{(s_3)^0}{0!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{bmatrix}
$$

$$
\bar{C}_4 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^3}{3!} - \frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+\frac{s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} - \frac{s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}
\end{gathered} \\[3ex]
\begin{gathered}
\frac{(s_2)^3}{3!} + \frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} + \frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{1!}
\end{gathered} \\[3ex]
\begin{gathered}
\frac{(s_3)^3}{3!} - \frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)}\frac{(s_1)^1}{1!} + \frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)}\frac{(s_2)^1}{1!} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
-\frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}
\end{gathered} \\[3ex]
\begin{gathered}
\frac{1}{3!} + \frac{(5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{(5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} + \frac{(5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
-\frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}
\end{gathered}
\end{bmatrix}
= \begin{bmatrix} 0 \\[3ex] 0 \\[3ex] 0 \\[3ex] 0 \end{bmatrix}
$$

$$
\bar{C}_5 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^4}{4!} - \frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+\frac{s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} - \frac{s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{array} \\
\\
\begin{array}{c}
\frac{(s_2)^4}{4!} + \frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} + \frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{2!}
\end{array} \\
\\
\begin{array}{c}
\frac{(s_3)^4}{4!} - \frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)}\frac{(s_1)^2}{2!} + \frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)}\frac{(s_2)^2}{2!} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
-\frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{array} \\
\\
\begin{array}{c}
\frac{1}{4!} + \frac{(5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{(5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} + \frac{(5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
-\frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{array}
\end{bmatrix}
= \begin{bmatrix} 0 \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{bmatrix}
$$

$$\bar{C}_6 = \begin{bmatrix} \begin{gathered} \frac{(s_1)^5}{5!} - \frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ +\frac{s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} - \frac{s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \end{gathered} \\ \\ \begin{gathered} \frac{(s_2)^5}{5!} + \frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ -\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} + \frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{3!} \end{gathered} \\ \\ \begin{gathered} \frac{(s_3)^5}{5!} - \frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)}\frac{(s_1)^3}{3!} + \frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)}\frac{(s_2)^3}{3!} \\ -\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ -\frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \end{gathered} \\ \\ \begin{gathered} \frac{1}{5!} + \frac{(5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{(5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} + \frac{(5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ -\frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \end{gathered} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{bmatrix}$$

$$\bar{C}_7 = \begin{bmatrix} \frac{(s_1)^6}{6!} - \frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\ -\frac{s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\ +\frac{s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} - \frac{s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!} \\ \\ \frac{(s_2)^6}{6!} + \frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\ -\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\ -\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} + \frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{4!} \\ \\ \frac{(s_3)^6}{6!} - \frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)}\frac{(s_1)^4}{4!} + \frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)}\frac{(s_2)^4}{4!} \\ -\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\ -\frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!} \\ \\ \frac{1}{6!} + \frac{(5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\ -\frac{(5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} + \frac{(5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\ -\frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{bmatrix}$$

$$\bar{C}_8 = \begin{bmatrix} \frac{(s_1)^7}{7!} - \frac{s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3)}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ -\frac{s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2)}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ +\frac{s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2)}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} - \frac{s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2)}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \\ \\ \frac{(s_2)^7}{7!} + \frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ -\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ -\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} + \frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{5!} \\ \\ \frac{(s_3)^7}{7!} - \frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)}\frac{(s_1)^5}{5!} + \frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)}\frac{(s_2)^5}{5!} \\ -\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\ -\frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \\ \\ \frac{1}{7!} + \frac{(5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ -\frac{(5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} + \frac{(5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\ -\frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, by comparing the coefficient of h, the block of first derivative has order $[5,5,5,5]^T$ together with the following error constants vector

$$\begin{bmatrix} \frac{-(s_1^4(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3))}{50400} \\ \frac{(s_2^4(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3))}{50400} \\ \frac{-(s_3^4(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3))}{50400} \\ \frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{50400} \end{bmatrix}$$

which is true for all $s_1, s_2, s_3 \in (0,1) \setminus \{s_2 = \frac{-14s_1s_3+7s_1^2s_3+7s_1^2-4s_1^3}{14s_1-35s_3-7s_1^2+14s_1s_3}\} \cup \{s_1 = \frac{14s_2s_3-7s_2^2s_3-7s_2^2+4s_2^3}{35s_3-14s_2+7s_2^2-14s_2s_3}\}$ $\cup \{s_2 = \frac{-14s_1s_3+7s_1s_3^2+7s_3^2-4s_3^3}{-35s_1+14s_3-7s_3^2+14s_1s_3}\} \cup \{s_1 = \frac{-7s_2-7s_3+14s_2s_3+4}{7-14s_2-14s_3+35s_2s_3}\}$.

Expanding y'' and f function in Taylor series in finding the order of the second derivative block. This is demonstrated below.

$$
\begin{bmatrix}
\sum_{j=0}^{\infty} \frac{(s_1)^j h^j}{j!} y_n^{j+2} - y_n'' \\
+\frac{hs_1(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3)}{(60s_2s_3)} y_n''' \\
-\frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+1}}{j!} y_n^{j+3} \\
-\frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{s_2^j h^{j+1}}{j!} y_n^{j+3} \\
+\frac{s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{s_3^j h^{j+1}}{j!} y_n^{j+3} \\
-\frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+3} \\
\sum_{j=0}^{\infty} \frac{(s_2)^j h^j}{j!} y_n^{j+2} - y_n'' \\
-\frac{hs_2(30s_1s_3-10s_1s_2-10s_2s_3+5s_1s_2^2+5s_2^2s_3+5s_2^2-3s_2^3-10s_1s_2s_3)}{(60s_1s_3)} y_n''' \\
+\frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+1}}{j!} y_n^{j+3} \\
-\frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+1}}{j!} y_n^{j+3} \\
-\frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+1}}{j!} y_n^{j+3} \\
+\frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+3} \\
\sum_{j=0}^{\infty} \frac{(s_3)^j h^j}{j!} y_n^{j+2} - y_n'' \\
+\frac{hs_3(10s_1s_3-30s_1s_2+10s_2s_3-5s_1s_3^2-5s_2s_3^2-5s_3^2+3s_3^3+10s_1s_2s_3)}{(60s_1s_2)} y_n''' \\
-\frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+1}}{j!} y_n^{j+3} \\
+\frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+1}}{j!} y_n^{j+3} \\
-\frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+1}}{j!} y_n^{j+3} \\
-\frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+3} \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{j+2} - y_n'' \\
-\frac{h(5s_1+5s_2+5s_3-10s_1s_2-10s_1s_3-10s_2s_3+30s_1s_2s_3-3)}{(60s_1s_2s_3)} y_n''' \\
+\frac{(10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+1}}{j!} y_n^{j+3} \\
-\frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+1}}{j!} y_n^{j+3} \\
+\frac{(10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+1}}{j!} y_n^{j+3} \\
-\frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} -\frac{h^{j+3}}{j!} y_n^{j+3}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

Comparing the coefficients of h^j and y^j gives

$$\bar{C}_0 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\bar{C}_3 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^1}{1!} + \frac{s_1(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3)}{(60s_2s_3)} \\
-\frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_2)^1}{1!} - \frac{s_2(30s_1s_3-10s_1s_2-10s_2s_3+5s_1s_2^2+5s_2^2s_3+5s_2^2-3s_2^3-10s_1s_2s_3)}{(60s_1s_3)} \\
+\frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_3)^1}{1!} + \frac{s_3(10s_1s_3-30s_1s_2+10s_2s_3-5s_1s_3^2-5s_2s_3^2-5s_3^2+3s_3^3+10s_1s_2s_3)}{(60s_1s_2)} \\
-\frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} + \frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{1}{1!} - \frac{(5s_1+5s_2+5s_3-10s_1s_2-10s_1s_3-10s_2s_3+30s_1s_2s_3-3)}{(60s_1s_2s_3)} \\
+\frac{(10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} + \frac{(10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))} - \frac{1}{0!}
\end{gathered}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_4 = \begin{bmatrix}
\begin{gathered}\frac{(s_1)^2}{2!} - \frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\ -\frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\ +\frac{s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}\end{gathered} \\[4ex]
\begin{gathered}\frac{(s_2)^2}{2!} + \frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\ -\frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\ -\frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}\end{gathered} \\[4ex]
\begin{gathered}\frac{(s_3)^2}{2!} - \frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} + \frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\ -\frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\ -\frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}\end{gathered} \\[4ex]
\begin{gathered}\frac{1}{2!} + \frac{(10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\ -\frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} + \frac{(10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\ -\frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))} - \frac{1}{1!}\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\[8ex] 0 \\[8ex] 0 \\[8ex] 0 \end{bmatrix}
$$

$$
\bar{C}_5 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^3}{3!} - \frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+\frac{s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{gathered} \\ \\
\begin{gathered}
\frac{(s_2)^3}{3!} + \frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
-\frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{gathered} \\ \\
\begin{gathered}
\frac{(s_3)^3}{3!} - \frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} + \frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
-\frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
-\frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{gathered} \\ \\
\begin{gathered}
\frac{1}{3!} + \frac{(10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} + \frac{(10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
-\frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))} - \frac{1}{2!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$\bar{C}_6 = \begin{bmatrix} \frac{(s_1)^4}{4!} - \frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ +\frac{s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \\ \\ \frac{(s_2)^4}{4!} + \frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ -\frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \\ \\ \frac{(s_3)^4}{4!} - \frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} + \frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ -\frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ -\frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \\ \\ \frac{1}{4!} + \frac{(10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} + \frac{(10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ -\frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))} - \frac{1}{3!} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{bmatrix}$$

$$
\bar{C}_7 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^5}{5!} - \frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+\frac{s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!}
\end{gathered} \\
\\
\begin{gathered}
\frac{(s_2)^5}{5!} + \frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
-\frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!}
\end{gathered} \\
\\
\begin{gathered}
\frac{(s_3)^5}{5!} - \frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} + \frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
-\frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
-\frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!}
\end{gathered} \\
\\
\begin{gathered}
\frac{1}{5!} + \frac{(10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} + \frac{(10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
-\frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))} - \frac{1}{4!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$\bar{C}_8 = \begin{bmatrix} \frac{(s_1)^6}{6!} - \frac{s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ -\frac{s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ +\frac{s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} - \frac{s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \\ \\ \frac{(s_2)^6}{6!} + \frac{s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ -\frac{s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ -\frac{s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} + \frac{s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \\ \\ \frac{(s_3)^6}{6!} - \frac{s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} + \frac{s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ -\frac{s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\ -\frac{s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \\ \\ \frac{1}{6!} + \frac{(10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ -\frac{(10s_1s_3-5s_3-5s_1+3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} + \frac{(10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\ -\frac{(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12)}{(60(s_3-1)(s_2-1)(s_1-1))} - \frac{1}{5!} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, by comparing the coefficient of h, the block of second derivative has order $[5,5,5,5]^T$ together with the following error constants vector

$$= \begin{bmatrix} \frac{-(s_1^4(2s_1^3-s_1^2-11s_1^2+2s_1s_2+2s_1s_3-5s_2s_3-s_1^2s_2-s_1^2s_3+2s_1s_2s_3))}{7200} \\ \frac{(s_2^4(5s_1s_3-2s_1s_2-2s_2s_3+s_1s_2^2+s_2^2s_3+11s_2^2-2s_2^3-2s_1s_2s_3))}{7200} \\ \frac{-(s_3^4(2s_1s_3-5s_1s_2+2s_2s_3-s_1s_3^2-s_2s_3^2-11s_3^2+2s_3^3+2s_1s_2s_3))}{7200} \\ \frac{(s_1+s_2+s_3-2s_1s_2-2s_1s_3-2s_2s_3+5s_1s_2s_3+8)}{7200} \end{bmatrix}$$

This is true for all $s_1,s_2,s_3 \in (0,1) \setminus \{s_2 = \frac{11s_1^2-2s_1^3-2s_1s_3-s_1^2s_3}{2s_1-5s_3-s_1^2+2s_1s_3}\} \cup \{s_1 = \frac{2s_2s_3-s_2^2s_3-11s_2^2+2s_2^3}{5s_3-2s_2+s_2^2-2s_2s_3}\}$ $\cup\{s_2 = \frac{-2s_1s_3+s_1s_3^2+11s_3^2-2s_3^3}{-5s_1+2s_3-s_3^2+2s_1s_3}\} \cup \{s_1 = \frac{-s_2-s_3+2s_2s_3+8}{1-2s_2-2s_3+5s_2s_3}\}$.

4.2.1.2 Zero Stability of One Step Hybrid Block Method with Generalised Three Off-Step Points for Third Order ODEs

In finding the zero-stability of the block (4.53), we only consider the first characteristic function according to Definition (3.1.3), that is

$$
\begin{aligned}
\Pi(z) &= |zI^{[3]_3} - \bar{B}_1^{[3]_3}| \\
&= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^3(z-1) = 0
\end{aligned}
$$

which implies $z = 0, 0, 0, 1$.

In order to find zero-stability of the block of first derivative (4.62), Definition (3.1.3) is also applied, that is

$$
\begin{aligned}
\Pi(z) &= |zI^{[3]_3} - \acute{B}_2^{[3]_3}| \\
&= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^3(z-1) = 0
\end{aligned}
$$

whose solution is $z = 0,0,0,1$.

The characteristic polynomial for the second derivative block (4.67) is given as

$$
\begin{aligned}
\Pi(z) &= |zI^{[3]_3} - \acute{\acute{B}}_3^{[3]_3}| \\
&= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^3(z-1) = 0
\end{aligned}
$$

which gives $z = 0,0,0,1$. Therefore, the conditions in Definition (3.1.3) are satisfied. Hence, the block method and its derivatives are zero stable.

4.2.1.3 Consistency and Convergent of One Step Hybrid Block Method with Generalised Three Off-Step Points for Third Order ODEs

According to Theorem (3.1) and Definition (3.1.4) the block method (4.53) and its derivatives (4.62),(4.67) are consistent and convergent.

4.2.1.4 Region of Absolute Stability of One Step Block Method with Generalised Three Off-Step Points for Third Order ODEs

Employing (3.29) for one step hybrid block with three generalised off-step points (4.53) yields

$$
\bar{h}(\theta, h) = \frac{I^{[3]_3} Y_m^{[3]_3}(\theta) - B_1^{[3]_3} R_1^{[3]_3}(\theta)}{[\bar{D}^{[3]_3} Y_{R_4^{[3]_3}}(\theta) + \bar{E}^{[3]_3} Y_{R_5^{[3]_3}}(\theta)]} \tag{4.68}
$$

where

$$I^{[3]_3}Y_m^{[3]_3}(\theta) = \begin{bmatrix} e^{is_1\theta} & 0 & 0 & 0 \\ 0 & e^{is_2\theta} & 0 & 0 \\ 0 & 0 & e^{is_3\theta} & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{bmatrix}$$

$$B_1^{[3]_3}R_1^{[3]_3}(\theta) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{D}^{[3]_3}Y_{R_4^{[3]_3}}(\theta) = \begin{bmatrix} 0 & 0 & 0 & \bar{D}_{14}^{[3]_3} \\ 0 & 0 & 0 & \bar{D}_{24}^{[3]_3} \\ 0 & 0 & 0 & \bar{D}_{34}^{[3]_3} \\ 0 & 0 & 0 & \bar{D}_{44}^{[3]_3} \end{bmatrix}$$

$$\bar{E}^{[3]_3}Y_{R_5^{[3]_3}}(\theta) = \begin{bmatrix} \bar{E}_{11}^{[3]_3}e^{is_1\theta} & \bar{E}_{12}^{[3]}e^{is_2\theta} & \bar{E}_{13}^{[3]_3}e^{is_3\theta} & \bar{E}_{14}^{[3]_3}e^{i\theta} \\ \bar{E}_{21}^{[3]_3}e^{is_1\theta} & \bar{E}_{22}^{[3]_3}e^{is_2\theta} & \bar{E}_{23}^{[3]_3}e^{is_3\theta} & \bar{E}_{24}^{[3]_3}e^{i\theta} \\ \bar{E}_{31}^{[3]_3}e^{is_1\theta} & \bar{E}_{32}^{[3]_3}e^{is_2\theta} & \bar{E}_{33}^{[3]_3}e^{is_3\theta} & \bar{E}_{34}^{[3]_3}e^{i\theta} \\ \bar{E}_{41}^{[3]_3}e^{is_1\theta} & \bar{E}_{42}^{[3]_3}e^{is_2\theta} & \bar{E}_{43}^{[3]_3}e^{is_3\theta} & \bar{E}_{44}^{[3]_3}e^{i\theta} \end{bmatrix}$$

The above matrix is simplified and after finding the determinant, we have

$$\bar{h}(\theta,h) = \frac{36288000(e^{i\theta}-1)}{(s_1^2s_2^2s_3^2(10s_1+10s_2+10s_3-6s_1s_2-6s_1s_3-6s_2s_3+3s_1s_2s_3+s_1s_2s_3e^{i\theta}-15))}$$

The above equation is expanded trigonometrically and the imaginary part are equated to zero. This produces the equation of absolute stability region for one step hybrid block method with three generalised off-step points for third order ODE as below

$$\bar{h}(\theta,h)=\frac{36288000(\cos(\theta)-1)}{s_1^2s_2^2s_3^2(10s_1+10s_2+10s_3-6s_1s_2-6s_1s_3-6s_2s_3+3s_1s_2s_3+s_1s_2s_3\cos(\theta)-15)} \tag{4.69}$$

4.3 Numerical Results for Solving Third Order ODEs

This section considers some specific numerical method for two and three hybrid points. Substituting $s=\frac{1}{5}$, $r=\frac{3}{5}$ into equation (4.22)-(4.24), (4.25)-(4.27) and (4.29)-(4.31), the following block of one step with two hybrid points and its derivatives are obtained

$$\begin{aligned}
y_{n+\frac{1}{5}} &= y_n+\frac{h}{5}y_n^{'}+\frac{h^2}{50}y_n^{''}+\frac{101h^3}{112500}f_n-\frac{7h^3}{90000}f_{n+\frac{3}{5}}+\frac{h^3}{2000}f_{n+\frac{1}{5}}\\
&\quad+\frac{3022314549036572875h^3}{226673591177742970257408}f_{n+1}.\\
y_{n+\frac{3}{5}} &= y_n+\frac{3h}{5}y_n^{'}+\frac{9h^2}{50}y_n^{''}+\frac{27h^3}{2500}f_n+\frac{9h^3}{10000}f_{n+\frac{3}{5}}+\frac{243h^3}{10000}f_{n+\frac{1}{5}}\\
&\quad-\frac{125h^3}{75557863725914323419136}f_{n+1}.\\
y_{n+1} &= y_n+hy_n^{'}+\frac{h^2}{2}y_n^{''}+\frac{h^3}{36}f_n+\frac{5h^3}{144}f_{n+\frac{3}{5}}+\frac{5h^3}{48}f_{n+\frac{1}{5}}\\
&\quad-\frac{588233h^3}{1416709944860893564108800000}f_{n+1}.
\end{aligned} \tag{4.70}$$

$$\begin{aligned}
y_{n+\frac{1}{5}}^{'} &= y_n^{'}+\frac{h}{5}y_n^{''}+\frac{131h^2}{11250}f_n-\frac{23h^2}{18000}f_{n+\frac{3}{5}}+\frac{113h^2}{12000}f_{n+\frac{1}{5}}\\
&\quad+\frac{16370870473948104375h^2}{75557863725914323419136}f_{n+1}.\\
y_{n+\frac{3}{5}}^{'} &= y_n^{'}+\frac{3h}{5}y_n^{''}+\frac{21h^2}{625}f_n+\frac{39h^2}{2000}f_{n+\frac{3}{5}}+\frac{513h^2}{4000}f_{n+\frac{1}{5}}\\
&\quad-\frac{306009348089952798125h^2}{226673591177742970257408}f_{n+1}.\\
y_{n+1}^{'} &= y_n^{'}+hy_n^{''}+\frac{h^2}{18}f_n+\frac{25h^2}{144}f_{n+\frac{3}{5}}+\frac{25h^2}{96}f_{n+\frac{1}{5}}\\
&\quad+\frac{1475739525896764128197747h^2}{1416709944860893564108800000}f_{n+1}.
\end{aligned} \tag{4.71}$$

$$
\begin{aligned}
y'_{n+\frac{1}{5}} &= y''_n + \frac{h}{12}f_n - \frac{h}{80}f_{n+\frac{3}{5}} + \frac{61h}{480}f_{n+\frac{1}{5}} \\
&+ \frac{19676527011956853125h}{94447329657392904273920}f_{n+1}. \\
y''_{n+\frac{3}{5}} &= y''_n + \frac{3h}{100}f_n - \frac{3h}{16}f_{n+\frac{3}{5}} + \frac{63h}{160}f_{n+\frac{1}{5}} \\
&- \frac{106253245864567046875h}{94447329657392904273920}f_{n+1}. \\
y''_{n+1} &= y''_n + \frac{h}{12}f_n + \frac{25h}{48}f_{n+\frac{3}{5}} + \frac{25h}{96}f_{n+\frac{1}{5}} \\
&+ \frac{159871781972149447330433h}{1180591620717411303424000}f_{n+1}.
\end{aligned}
\tag{4.72}
$$

Using strategy in section(4.1.1.1) methodss (4.70), (4.71) and (4.72) are of order $[4,4,4]^T$, $[4,4,5]^T$ and $[4,4,4]^T$ with error constant $[-1.663492e^{-7}, -9.257143e^{-7}, 3.968254e^{-6}]^T$, $[-2.666667e^{-6}, 7.200000e^{-6}, -1.58730e^{-6}]^T$ and $[-2.511111e^{-5}, 7.200000e^{-5}, -1.388889e^{-4}]^T$ respectively.

Substituting $s = \frac{1}{5}$ and $r = \frac{3}{5}$ into equation 4.34), this yields

$$
\bar{h}(\theta, h) = \frac{(300000000(\cos(\theta) - 1))}{(3\cos(\theta) + 139)} \tag{4.73}
$$

Evaluating (4.73) at intervals of 30°, this gives the results displayed below.

θ	0	30°	60°	90°	120°	150°	180°
$\bar{h}(\theta,h) \simeq$	0	−283848	−1067615	−2158273	−3272727	−4104103	−4411764

Hence, the interval of stability gives (-4411764,0) as illustrated in Figure 4.3 as a region in polar coordinate system.

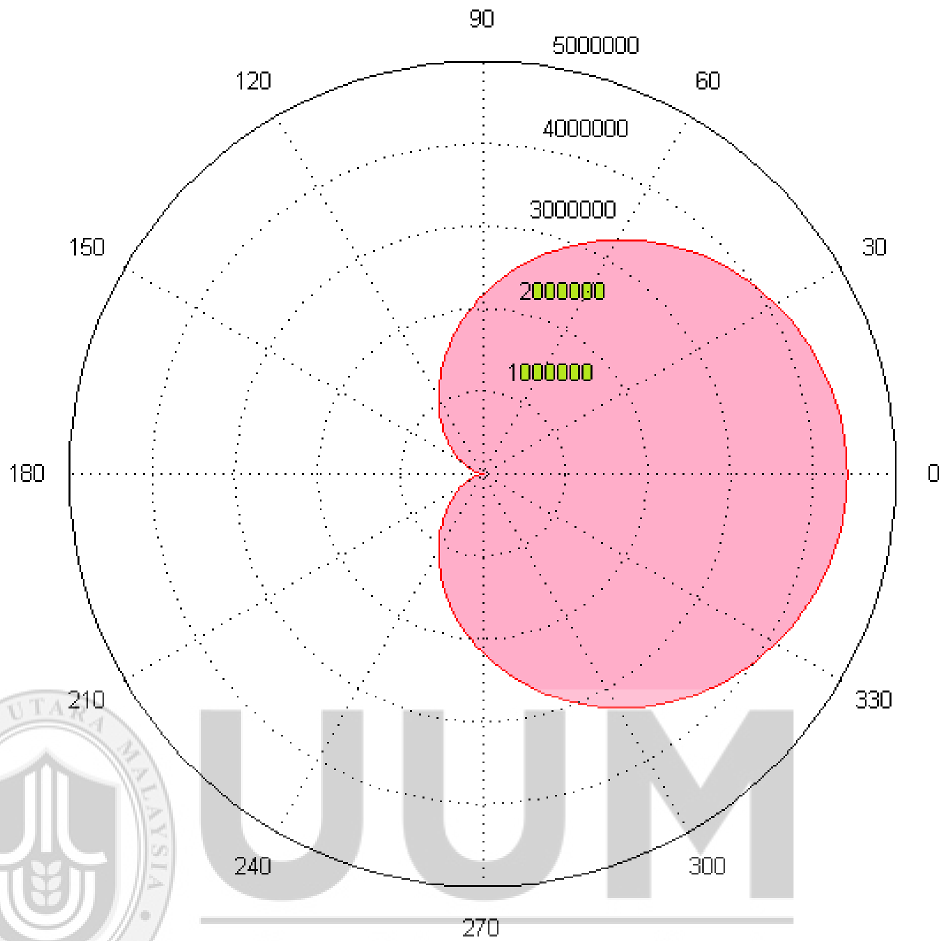

Figure 4.3. Region stability of one step hybrid block method with two off-step points $s = \frac{1}{5}$ and $r = \frac{3}{5}$ for third order ODEs.

The values $s_1 = \frac{1}{12}$, $s_2 = \frac{2}{5}$ and $s_3 = \frac{9}{10}$ are substituted into equation (4.54) - (4.57) , (4.58) - (4.61) and (4.63) - (4.66) to produce the block and derivatives as below

$$
\begin{aligned}
y_{n+\frac{1}{12}} &= y_n + \frac{h}{12}y_n^{'} + \frac{h^2}{288}y_n^{''} + \frac{6842396601826321h^3}{102141639548762849280}f_n \\
&\quad - \frac{129317375663869h^3}{217902164370694078464}f_{n+1} + \frac{180492881271309h^3}{5765170472987656192}f_{n+\frac{1}{12}} \\
&\quad - \frac{72537225443555h^3}{3234485252377490 2272}f_{n+\frac{2}{5}} + \frac{979789414400009h^3}{98974650236581969 9200}f_{n+\frac{9}{10}} \\
y_{n+\frac{2}{5}} &= y_n + \frac{2h}{5}y_n^{'} + \frac{2h^2}{25}y_n^{''} + \frac{1461808391049431h^3}{1418633882621706240}f_n \\
&\quad + \frac{483074110430269h^3}{3120994541767753728}f_{n+1} + \frac{3907786264096887h^3}{450403943202160640}f_{n+\frac{1}{12}} \\
&\quad + \frac{3851718593307377h^3}{3593872502641655808}f_{n+\frac{2}{5}} - \frac{1022074594562633h^3}{383551316951040000 0}f_{n+\frac{9}{10}} \\
y_{n+\frac{9}{10}} &= y_n + \frac{9h}{10}y_n^{'} + \frac{81h^2}{200}y_n^{''} - \frac{1089997210613231h^3}{1008806316530991104}f_n \\
&\quad - \frac{1219005479274 79h^3}{1122060206080000 00}f_{n+1} + \frac{60553952663901h^3}{846623953387520}f_{n+\frac{1}{12}} \\
&\quad + \frac{3663504837911501h^3}{748723438050344 96}f_{n+\frac{2}{5}} + \frac{4259767891525633h^3}{13258564042752000 00}f_{n+\frac{9}{10}} \\
y_{n+1} &= y_n + hy_n^{'} + \frac{h^2}{2}y_n^{''} - \frac{3002399751580217h^3}{45396284243894599680}f_n \\
&\quad - \frac{45206587168681h^3}{11821949021847552}f_{n+1} + \frac{3467419869354393h^3}{39410345030189056}f_{n+\frac{1}{12}} \\
&\quad + \frac{8121725890505389h^3}{112308515707551744}f_{n+\frac{2}{5}} + \frac{95h^3}{9261}f_{n+\frac{9}{10}}
\end{aligned}
\tag{4.74}
$$

$$
\begin{aligned}
y'_{n+\frac{1}{12}} = {} & y'_n + \frac{h}{12}y''_n + \frac{191746004642677h^2}{91197892454252544}f_n \\
& - \frac{870378834234533h^2}{3745193450121304736}f_{n+1} + \frac{3748113940480003h^2}{2597849892126720000}f_{n+\frac{1}{12}} \\
& - \frac{1433369510308721h^2}{1617242626188745136}f_{n+\frac{2}{5}} + \frac{1142640257706671h^2}{29456741141839872000}f_{n+\frac{9}{10}}
\end{aligned}
$$

$$
\begin{aligned}
y'_{n+\frac{2}{5}} = {} & y'_n + \frac{2h}{5}y''_n - \frac{18700954480079807h^2}{5319877059831398400}f_n \\
& + \frac{15112078749621h^2}{7740561859543040}f_{n+1} + \frac{6681765587148237h^2}{98525862575472640}f_{n+\frac{1}{12}} \\
& - \frac{8457773294341327h^2}{555236192157696000}f_{n+\frac{2}{5}} - \frac{928h^2}{275625}f_{n+\frac{9}{10}}
\end{aligned}
$$

(4.75)

$$
\begin{aligned}
y'_{n+\frac{9}{10}} = {} & y'_n + \frac{9h}{10}y''_n + \frac{5297704720662499h^2}{834941642342400000}f_n \\
& - \frac{2840471467248847h^2}{133938555125760000}f_{n+1} + \frac{6326298051856627h^2}{39410345030189056}f_{n+\frac{1}{12}} \\
& + \frac{730527180317629h^2}{3509641115860992}f_{n+\frac{2}{5}} - \frac{1090772979941377h^2}{21308406497280000}f_{n+\frac{9}{10}}
\end{aligned}
$$

$$
\begin{aligned}
y'_{n+1} = {} & y'_n + hy''_n + \frac{2189249818860673h^2}{157625986957967360}f_n \\
& - \frac{7318349394477097h^2}{222928181554839552}f_{n+1} + \frac{3799912185593849h^2}{22520197160108032}f_{n+\frac{1}{12}} \\
& + \frac{2428699017798545h^2}{9359042975629312}f_{n+\frac{2}{5}} - \frac{40h^2}{441}f_{n+\frac{9}{10}}
\end{aligned}
$$

$$
\begin{aligned}
y''_{n+\frac{1}{12}} &= y''_n + \frac{1154803940909427h}{31525197391593472} f_n \\
&- \frac{4494043378161185h}{8322652111380676608} f_{n+1} + \frac{694033396499425h}{14331034556432384} f_{n+\frac{1}{12}} \\
&- \frac{1876635586913975h}{898468125660413952} f_{n+\frac{2}{5}} + \frac{1375h}{1524096} f_{n+\frac{9}{10}}
\end{aligned}
$$

$$
\begin{aligned}
y''_{n+\frac{2}{5}} &= y''_n - \frac{4948755430538141h}{94575592174780416} f_n \\
&+ \frac{619294988759303h}{43347146413441024} f_{n+1} + \frac{3021690169984235h}{9852586257547264} f_{n+\frac{1}{12}} \\
&+ \frac{2191657993661405h}{14038564463443968} f_{n+\frac{2}{5}} - \frac{4096h}{165375} f_{n+\frac{9}{10}}
\end{aligned}
$$

$$(4.76)$$

$$
\begin{aligned}
y''_{n+\frac{9}{10}} &= y''_n + \frac{604692692467501h}{7881299347898368} f_n \\
&- \frac{5222325145514803h}{40181566537728000} f_{n+1} + \frac{2291657949819191h}{28662069112864768} f_{n+\frac{1}{12}} \\
&+ \frac{344485789075243h}{668503069687808} f_{n+\frac{2}{5}} + \frac{17541h}{49000} f_{n+\frac{9}{10}}
\end{aligned}
$$

$$
\begin{aligned}
y''_{n+1} &= y''_n + \frac{250199979298361h}{3377699720527872} f_n \\
&- \frac{290004521459467h}{3377699720527872} f_{n+1} + \frac{6649846324789317h}{78820690060378112} f_{n+\frac{1}{12}} \\
&+ \frac{2394491989378845h}{4679521487814656} f_{n+\frac{2}{5}} + \frac{550h}{1323} f_{n+\frac{9}{10}}
\end{aligned}
$$

Based on the approach used in section (4.2.1.1), the block and its derivatives above are of order $[5,5,5,5]^T$, $[5,5,5,5]^T$ and $[5,5,5,5]^T$ with error constant

$[2.646777e^{-10}, -5.407831e^{-8}, -2.577536e^{-7}, 1.653439e^{-8}]^T$,

$[1.030799e^{-8}, -6.787725e^{-7}, 1.967866e^{-6}, 3.505291e^{-6}]^T$ and

$[1.076759e^{-8}, 4.685037e^{-6}, 6.451802e^{-4}, 1.193981e^{-3}]^T$

In order to find the region of absolute stability of (4.74), $s_1 = \frac{1}{12}$, $s_2 = \frac{2}{5}$ and $s_3 = \frac{9}{10}$

are substituted into equation (4.69), this gives

$$\bar{h}(\theta,h) = \frac{(12096000000000(\cos(\theta)-1))}{(9\cos(\theta)-1166)} \tag{4.77}$$

Equation (4.77) is evaluated at intervals of 30°, this produces tabulated results below.

θ	0	30°	60°	90°	120°	150°	180°
$\bar{h}(\theta,h)$	0	$1.3992e^{9}$	$5.2071e^{9}$	$1.0374e^{10}$	$1.5501e^{10}$	$1.9229e^{10}$	$2.0589e^{10}$

Hence, the interval of absolute stability is $(0, 20589000000)$ as demonstrated in Figure 4.4 as a region in polar coordinate system.

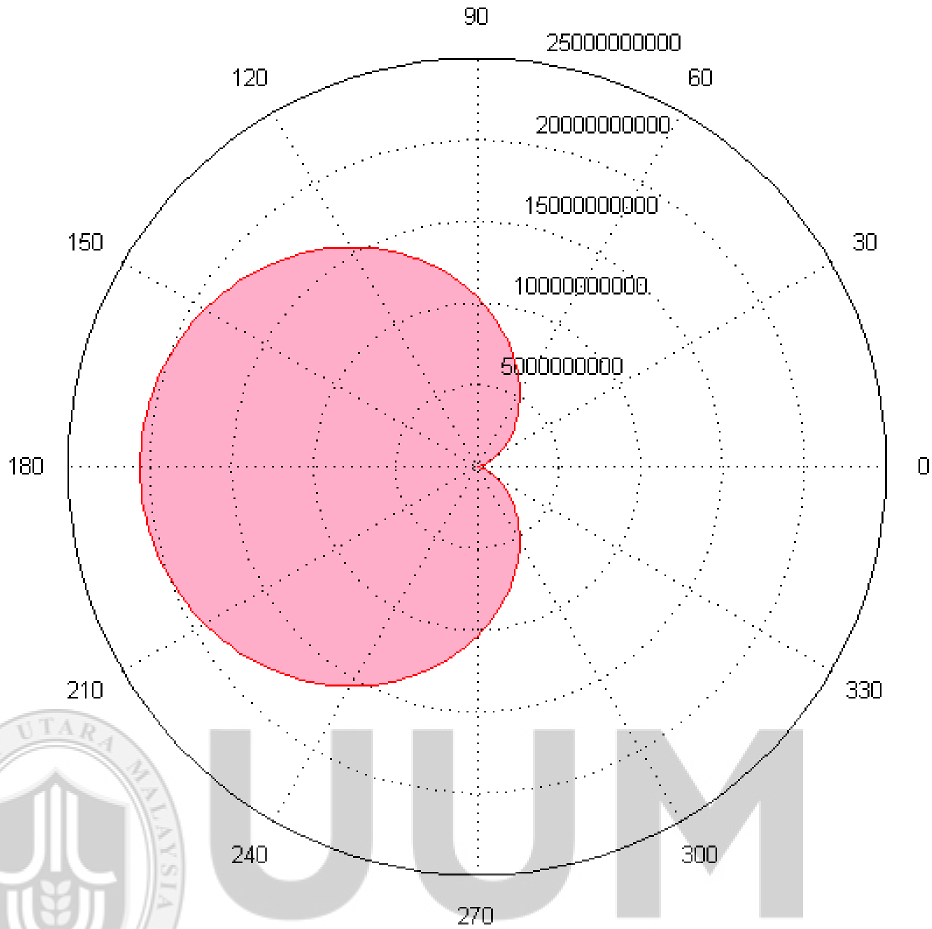

Figure 4.4. Region stability of one step hybrid block method with three off step points $s_1 = \frac{1}{12}$, $s_2 = \frac{2}{5}$ and $s_3 = \frac{9}{10}$ for third order ODEs.

In order to find the accuracy of our methods, the following third order ODEs are tasted. The new block methods solved the same problems the existing methods solved in order to compare results in terms of error.

Problem 8: $y^{'''} + y = 0,\ y(0) = 1,\ y^{'}(0) = -1,\ y^{''}(0) = 1,\ h = \dfrac{1}{10}$

Exact solution: $y(x) = e^{-x} \qquad 0 \leq x \leq 1$

Source: *Kuboye and Omar*, 2015*b*

Problem 9: $y^{'''} = 3\sin x, \quad y(0) = 1,\ y^{'}(0) = 0,\ y^{''}(0) = -2,\ h = \dfrac{1}{10}$

Exact solution: $y(x) = 3\cos x + \dfrac{x^2}{2} - 2 \qquad 0 \leq x \leq 1$

Source: *Olabode*, 2013

Problem 10: $y^{'''} + y^{'} = 0,\ y(0) = 0,\ y^{'}(0) = 1,\ y^{''}(0) = 2,\ h = \dfrac{1}{10}$

Exact solution: $y(x) = \sin x - 2\cos x + 2 \qquad 0 \leq x \leq 1$

Source: *Anake et al*., 2013

Problem 11: $y^{'''} - 2xy^{'}y^{''} - (y^{'})^2 = 0,\ y(0) = 1,\ y^{'}(0) = \dfrac{1}{2},\ y^{''}(0) = 0,\ h = \dfrac{1}{100}.$

Exact solution: $y(x) = 1 + \dfrac{1}{2}\log\dfrac{(2+x)}{(2-x)} \qquad 0 \leq x \leq 1$

Source: *Gbenga et al*., 2015

Problem 12: $y^{'''} + y^{''} + 3y^{'} - 5y = 6x - 5x^2 + 2, \quad y(0) = -1,\ y^{'}(0) = 1,$

$y^{''}(0) = -3,\ h = \dfrac{1}{10}.$

Exact solution: $y(x) = x^2 - e^x + e^{-x}\sin(2x) \qquad 0 \leq x \leq 1$

Source: *Awoyemi et al*., 2014

Problem 13: $y''' + 4y' - x = 0,\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 1,\ h = \dfrac{1}{10}$

Exact solution: $y(x) = x^2 - e^x + e^{-x}\sin(2x) \qquad 0 \le x \le 1$

Source: *Mohammed and Adeniyi*, 2014

Problem 14: $y''' = e^x,\ y(0) = 3,\ y'(0) = 1,\ y''(0) = 5, \quad h = \dfrac{1}{10}$

Exact solution: $y(x) = 2 + 2x^2 + e^x \qquad 0 \le x \le 1$

Source: *Olabode and Yusuph*, 2009

Problem 15: $y''' - 2y'' - 3y' + 10y = 34xe^{-2x} - 16e^{-2x} - 10x^2 + 6x + 4,$

$$y(0) = 3,\ y'(0) = 0,\ y''(0) = 0.$$

Exact solution: $y(x) = x^2e^{-2x} - x^2 + 3 \qquad 0 \le x \le 1$

Source: *Awoyemi*, 2005

Problem 16: $y''' - y'' + y' + y = 0,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -1,\ h = 0.01$

Exact solution: $y(x) = \cos x \qquad 0 \le x \le 1$

Source: *Mohammed and Adeniyi*, 2014

Table 4.1

Comparison of the New Method with both Seven Step Block Method (Kuboye and Omar, 2015b) and Five Step Block Method (Omar and Kuboye, 2015) for Solving Problem 8 where $h = \frac{1}{10}$

x		$s=\frac{3}{10}, r=\frac{17}{20}, P=4$	$s_1=\frac{1}{10}, s_2=\frac{2}{5}, s_3=\frac{9}{10}, P=5$	Omar and kuboye(2015), $P=6$	Kuboye and Omar(2015b),$P=8$
0.1	Exact solution	0.904837418035959520	0.904837418035959520	0.904837418035959520	0.904837418035959520
	Computed solution	0.904837418034494580	0.904837418035957190	0.904837418033777150	0.904837418033821120
	Error	$1.464939e^{-12}$	$2.331468e^{-15}$	$2.182365e^{-12}$	$2.138401e^{-12}$
0.2	Exact solution	0.818730753077981820	0.818730753077981820	0.818730753077981820	0.818730753077981820
	Computed Solution	0.818730753075065930	0.818730753077983710	0.818730753073947600	0.818730753077376310
	Error	$2.915890e^{-12}$	$1.887379e^{-15}$	$4.034217e^{-12}$	$6.055156e^{-13}$
0.3	Exact solution	0.740818220681717770	0.740818220681717770	0.740818220681717770	0.740818220681717770
	Computed solution	0.740818220677561420	0.740818220681773170	0.740818220676328960	0.740818220674322010
	Error	$4.156342e^{-12}$	$5.540013e^{-14}$	$5.388801e^{-12}$	$7.395751e^{-12}$
0.4	Exact solution	0.670320046035639330	0.670320046035639330	0.670320046035639330	0.670320046035639330
	Computed solution	0.670320046030631780	0.670320046035835840	0.670320045970895120	0.670320046037797490
	Error	$5.007550e^{-12}$	$1.965095e^{-13}$	$6.474421e^{-11}$	$2.158163e^{-12}$
0.5	Exact Solution	0.606530659712633420	0.606530659712633420	0.606530659712633420	0.606530659712633420
	Computed solution	0.606530659707327670	0.606530659713093500	0.606530659653555240	0.606530659727479220
	Error	$5.305756e^{-12}$	$4.600764e^{-13}$	$5.907819e^{-11}$	$1.484579e^{-11}$
0.6	Exact Solution	0.548811636094026500	0.548811636094026500	0.548811636094026390	0.548811636094026390
	Computed solution	0.548811636089124310	0.548811636094903690	0.548811636082354610	0.548811636105011600
	Error	$4.902190e^{-12}$	$8.771872e^{-13}$	$1.167177e^{-11}$	$1.098521e^{-11}$
0.7	Exact solution	0.496585303791409530	0.496585303791409530	0.496585303791409470	0.496585303791409470
	Computed solution	0.496585303787748230	0.496585303792885290	0.496585303768886100	0.496585303822838330
	Error	$3.661293e^{-12}$	$1.475764e^{-12}$	$2.252337e^{-11}$	$3.142886e^{-11}$
0.8	Exact solution	0.449328964117221620	0.449328964117221620	0.449328964117221560	0.449328964117221560
	Computed solution	0.449328964115760400	0.449328964119501960	0.449328964078100250	0.449328964140316870
	Error	$1.461220e^{-12}$	$2.280343e^{-12}$	$3.912132e^{-11}$	$2.309530e^{-11}$
0.9	Exact solution	0.406569659740599170	0.406569659740599170	0.406569659740599110	0.406569659740599110
	Computed Solution	0.406569659742406500	0.406569659743911680	0.406569659679422270	0.406569659792140600
	Error	$1.807332e^{-12}$	$3.312517e^{-12}$	$6.117684e^{-11}$	$5.154149e^{-11}$
1.0	Exact solution	0.367879441171442330	0.367879441171442330	0.367879441171442330	0.367879441171442330
	Computed solution	0.367879441177683120	0.367879441176032940	0.367879441122222760	0.367879441253447680
	Error	$6.240786e^{-12}$	$4.590606e^{-12}$	$4.921957e^{-11}$	$8.200535e^{-11}$

Table 4.2
Comparison of the New Method with Five Step Block Method (Olabode, 2009) and Six Step Block Method (Olabode, 2014) for Solving Problem 9 where $h = \frac{1}{10}$

x		$s = \frac{3}{10}, r = \frac{17}{20}, P = 4$	$s_1 = \frac{1}{10}, s_2 = \frac{2}{5}, s_3 = \frac{9}{10}, P = 5$	Olabode(2009), $P = 7$	Olabode(2014), $P = 8$
0.1	Exact solution	0.990012495834077020	0.990012495834077020	0.99001249583	0.990012496
	Computed solution	0.990012495834223240	0.990012495834069580	0.99001249580	0.990012496
	Error	$1.462164e^{-13}$	$7.438494e^{-15}$	$3.4077519e^{-11}$	$1.65922e^{-10}$
0.2	Exact solution	0.960199733523725120	0.960199733523725120	0.96019973352	0.960199734
	Computed Solution	0.960199733522700270	0.960199733523361850	0.96019973340	0.960199734
	Error	$1.024847e^{-12}$	$3.632650e^{-13}$	$1.2372514e^{-10}$	$4.76275e^{-10}$
0.3	Exact solution	0.911009467376818090	0.911009467376818090	0.91100946738	0.911009467
	Computed solution	0.911009467368195770	0.911009467375297980	0.91100946720	0.911009468
	Error	$8.622325e^{-12}$	$1.520117e^{-12}$	$1.7681812e^{-10}$	$6.23182e^{-10}$
0.4	Exact solution	0.843182982008655380	0.843182982008655380	0.84318298201	0.843182982
	Computed solution	0.843182981976797750	0.843182982004730630	0.84318298160	0.843182984
	Error	$3.185763e^{-11}$	$3.924749e^{-12}$	$4.0865533e^{-10}$	$19.9134e^{-10}$
0.5	Exact Solution	0.757747685671118280	0.757747685671118280	0.75774768567	0.757747686
	Computed solution	0.757747685587168870	0.757747685663108350	0.75774768530	0.757747686
	Error	$8.394940e^{-11}$	$8.009926e^{-12}$	$3.7111825e^{-10}$	$3.28882e^{-10}$
0.6	Exact Solution	0.656006844729035250	0.656006844729035250	0.65600684473	0.656006845
	Computed solution	0.656006844547042280	0.656006844714842610	0.65600684480	0.656006846
	Error	$1.819930e^{-10}$	$1.419265e^{-11}$	$7.0964790e^{-10}$	$1.27096e^{-9}$
0.7	Exact solution	0.539526561853465480	0.539526561853465480	0.53952656185	0.539526562
	Computed solution	0.539526561506677220	0.539526561830597990	0.53952656260	0.539526567
	Error	$3.467883e^{-10}$	$2.286749e^{-11}$	$7.4653450e^{-10}$	$4.84653e^{-9}$
0.8	Exact solution	0.410120128041496110	0.410120128041496110	0.41012012804	0.410120128
	Computed solution	0.410120127438862720	0.410120128007091080	0.41012013000	0.410120139
	Error	$6.026334e^{-10}$	$3.440503e^{-11}$	$1.9585035e^{-9}$	$1.09585e^{-8}$
0.9	Exact solution	0.269829904811993430	0.269829904811993430	0.26982990481	0.269829905
	Computed Solution	0.269829903834911570	0.269829904762845680	0.26982990870	0.269829925
	Error	$9.770819e^{-10}$	$4.914774e^{-11}$	$3.8880070e^{-9}$	$2.0188e^{-8}$
1.0	Exact solution	0.120906917604419300	0.120906917604419300	0.12090691760	0.120906918
	Computed solution	0.120906916103754060	0.120906917537014950	0.12090692400	0.120906953
	Error	$1.500665e^{-9}$	$6.740435e^{-11}$	$6.3955807e^{-9}$	$3.53956e^{-8}$

Table 4.3

Comparison of the new method with Five Step Block Method (Anake et al., 2013) for Solving Problem 10 where $h = \frac{1}{10}$

x		$s = \frac{1}{5}, r = \frac{3}{5}, P = 4$	$s_1 = \frac{1}{5}, s_2 = \frac{3}{5}, s_3 = \frac{4}{5}\ P = 5$	Anake et al.(2013), $P = 5$
0.1	Exact solution	0.109825086090776790	0.109825086090774850	0.1098250860907
	Computed solution	0.109825086091211120	0.109825086090774830	0.109825087699
	Error	$4.343331e^{-13}$	$1.942890e^{-15}$	$1.6088e^{-9}$
0.2	Exact solution	0.238536175112578070	0.238536175231475130	0.2385361751125
	Computed Solution	0.238536175686210580	0.238536175231475270	0.2385361885
	Error	$5.736325e^{-10}$	$1.188971e^{-10}$	$1.0387e^{-8}$
0.3	Exact solution	0.384847228410127640	0.384847228410127640	0.3847228410128
	Computed solution	0.384847231091219710	0.384847229014260380	0.38484725778
	Error	$2.681092e^{-9}$	$6.041327e^{-10}$	$2.9572e^{-8}$
0.4	Exact solution	0.547296354302880370	0.547296354302880370	0.5472963543028
	Computed solution	0.547296361725598150	0.547296356038821100	0.54729658572
	Error	$7.422718e^{-9}$	$1.735941e^{-9}$	$2.3147e^{-7}$
0.5	Exact Solution	0.724260414823457490	0.724260414823457490	0.7242604148234
	Computed solution	0.724260430838088550	0.724260418645165100	0.724259960
	Error	$1.601463e^{-8}$	$3.821708e^{-9}$	$4.5420e^{-7}$
0.6	Exact Solution	0.913971243575678830	0.913971243575678830	0.9139712435767
	Computed solution	0.913971273340073110	0.913971250765393980	0.9139697789
	Error	$2.976439e^{-8}$	$7.189715e^{-9}$	$1.4746e^{-6}$
0.7	Exact solution	1.114533312668714000	1.114533312668714000	1.1145333126687
	Computed solution	1.114533362712738700	1.114533324851086900	1.114530439
	Error	$5.004402e^{-8}$	$1.218237e^{-8}$	$2.8734e^{-6}$
0.8	Exact solution	1.323942672205191700	1.323942672205191700	1.3239426722051
	Computed solution	1.323942750466393900	1.323942691354229800	1.323937959
	Error	$7.826120e^{-8}$	$1.914904e^{-8}$	$4.6826e^{-6}$
0.9	Exact solution	1.540106973086154300	1.540106973086154300	1.5401069730861
	Computed Solution	1.540107088915270600	1.540107001524690700	1.540100051
	Error	$1.158291e^{-7}$	$2.843854e^{-8}$	$6.9217e^{-6}$
1.0	Exact solution	1.760866373071616800	1.760866373071616800	1.7608663707162
	Computed solution	1.760866537207026000	1.760866413463112900	1.760856775
	Error	$1.641354e^{-7}$	$4.039150e^{-8}$	$9.5974e^{-6}$

Table 4.4

Comparison of the New Method with Three Step Hybrid Block Method (Gbenga et al., 2015) for Solving Problem 11 where $h = \frac{1}{100}$

x		$s_1 = \frac{1}{12}, s_2 = \frac{1}{6}, s_3 = \frac{1}{4}, P = 5$	$s_1 = \frac{1}{6}, s_2 = \frac{2}{6}, s_3 = \frac{3}{6}, P = 5$	Gbenga et al.(2015), $P = 6$
	Exact solution	1.105388447838498800	1.105388447838498800	1.105388447838498800
0.21	Computed solution	1.105388447838495900	1.105388447838495700	1.105388447838482300
	Error	$2.886580e^{-15}$	$3.108624e^{-15}$	$1.643130e^{-14}$
	Exact solution	1.156259497799360100	1.156259497799360100	1.156259497799360100
0.31	Computed Solution	1.156259497799337200	1.156259497799336300	1.156259497799276400
	Error	$2.287059e^{-14}$	$2.375877e^{-14}$	$8.371082e^{-14}$
	Exact solution	1.207946365635211800	1.207946365635211800	1.207946365635211800
0.41	Computed solution	1.207946365635111900	1.207946365635110600	1.207946365634930500
	Error	$9.992007e^{-14}$	$1.012523e^{-13}$	$2.813305e^{-13}$
	Exact solution	1.260753316593162600	1.260753316593162600	1.260753316593162600
0.51	Computed solution	1.260753316592835500	1.260753316592833500	1.260753316592395900
	Error	$3.270717e^{-13}$	$3.290701e^{-13}$	$7.667200e^{-13}$
	Exact Solution	1.315023237096001100	1.315023237096001100	1.315023237096001100
0.61	Computed solution	1.315023237095107200	1.315023237095104000	1.315023237094148600
	Error	$8.939516e^{-13}$	$8.970602e^{-13}$	$1.852518e^{-12}$
	Exact Solution	1.371153208259014500	1.371153208259014500	1.371153208259014500
0.71	Computed solution	1.371153208256838100	1.371153208256831200	1.371153208254851700
	Error	$2.176481e^{-12}$	$2.183365e^{-12}$	$4.162892e^{-12}$
	Exact solution	1.429615588111108300	1.429615588111108300	1.429615588111108300
0.81	Computed solution	1.429615588106190300	1.429615588106178300	1.429615588102143500
	Error	$4.918066e^{-12}$	$4.930056e^{-12}$	$8.964829e^{-12}$

Table 4.5

Comparison of the New Methods with Four Step Linear Multistep (Awoyemi et al., 2014) for Solving Problem 12 where $h = \frac{1}{10}$

x		$s_1 = \frac{1}{12}, s_2 = \frac{1}{6}, s_3 = \frac{1}{4}, P = 5$	Awoyemi et al(2014), $P = 7$
0.1	Exact solution	-0.915407473756112530	$-0.915407e^{0}$
	Computed solution	-0.915407473759437980	$-0.915407e^{0}$
	Error	$3.325451e^{-12}$	$8.547820e^{-11}$
0.2	Exact solution	-0.862573985499428990	$-0.862574e^{0}$
	Computed Solution	-0.862574010213353560	$-0.862574e^{0}$
	Error	$2.318680e^{-8}$	$2.232510e^{-9}$
0.3	Exact solution	-0.841561375114168730	$-0.841561e^{0}$
	Computed solution	-0.862574008686228580	$-0.841561e^{0}$
	Error	$8.681931e^{-8}$	$5.824412e^{-8}$
0.4	Exact solution	-0.850966529765555760	$-0.850967e^{0}$
	Computed solution	-0.850966716276783440	$-0.850967e^{0}$
	Error	$1.865112e^{-7}$	$1.226405e^{-6}$
0.5	Exact Solution	-0.888343319155555420	$-0.888343e^{0}$
	Computed solution	-0.888343621222831390	$-0.888341e^{0}$
	Error	$3.020673e^{-7}$	$2.811820e^{-6}$
0.6	Exact Solution	-0.950604904717254340	$-0.950606e^{0}$
	Computed solution	-0.950605307970210920	$-0.950599e^{0}$
	Error	$4.032530e^{-7}$	$6.295841e^{-6}$
0.7	Exact solution	-1.034392853932994700	$-0.103439e^{1}$
	Computed solution	-1.034393309015891900	$-0.103438e^{1}$
	Error	$4.550829e^{-7}$	$1.695782e^{-5}$
0.8	Exact solution	-1.136403556878909000	$-0.113640e^{1}$
	Computed solution	-1.136403979231666000	$-0.113636e^{1}$
	Error	$4.223528e^{-7}$	$4.765221e^{-5}$
0.9	Exact solution	-1.253666211231613000	$-0.125367e^{1}$
	Computed solution	-1.253666484474872600	$-0.125353e^{1}$
	Error	$2.732433e^{-7}$	$1.316541E^{-4}$
1.0	Exact solution	-1.383769999219782900	$-0.138377E^{1}$
	Computed solution	-1.383769981140660500	$-0.138343E^{1}$
	Error	$1.807912e^{-8}$	$3.417856e^{-4}$

Table 4.6

Comparison of the New Methods with Three step hybrid Method (Mohammed and Adeniyi, 2014) and Four Step Linear Multistep (Awoyemi et al, 2014) for Solving Problem 13 where $h = \frac{1}{10}$

x		$s=\frac{1}{5}, r=\frac{3}{5}, P=4$	$s_1=\frac{1}{6}, s_2=\frac{2}{6}, s_3=\frac{3}{6}, P=5$	Mohammed and Adeniyi(2014), $P=5$	Awoyemi et al(2014), $P=7$
0.1	Exact solution	0.004987516654767182	0.004987516654767182	0.004987516700	$4.98752e^{-3}$
	Computed solution	0.004987516655733334	0.004987516654497357	0.004987517661	$4.98752e^{-3}$
	Error	$9.661516e^{-13}$	$2.698258e^{-13}$	$9.61000e^{-10}$	$1.1899e^{-11}$
0.2	Exact solution	0.019801063624459048	0.019801063624459048	0.01980106360	$1.98011e^{-2}$
	Computed Solution	0.019801064054750038	0.019801064050468446	0.01980107010	$1.98011e^{-9}$
	Error	$4.302910e^{-10}$	$4.260094e^{-10}$	$6.50000e^{-9}$	$3.0422e^{-9}$
0.3	Exact solution	0.043999572204435344	0.043999572204435344	0.04399957220	$4.39996e^{-2}$
	Computed solution	0.043999576766315253	0.043999574119394910	0.04399958817	$4.39997e^{-2}$
	Error	$4.561880e^{-9}$	$1.914960e^{-9}$	$1.59700e^{-8}$	$7.7796e^{-8}$
0.4	Exact solution	0.076867491997406487	0.076867491997406487	0.07686749200	$7.68675e^{-2}$
	Computed solution	0.076867497024167147	0.076867497024167147	0.07686750864	$7.68676e^{-2}$
	Error	$5.026761e^{-9}$	$5.026761e^{-9}$	$1.66400e^{-8}$	$1.5559e^{-7}$
0.5	Exact Solution	0.117443317649723790	0.117443317649723790	0.1174433176	$1.174433e^{0}$
	Computed solution	0.117443372485769800	0.117443327848342760	0.1174433379	$1.174437e^{0}$
	Error	$5.483605e^{-7}$	$1.019862e^{-8}$	$2.03000e^{-8}$	$3.0541e^{-7}$
0.6	Exact Solution	0.164557921035623690	0.164557921035623690	0.1645579210	$1.645579e^{-1}$
	Computed solution	0.164558044423877030	0.164557938740584940	0.1645579476	$1.64558e^{-1}$
	Error	$1.233883e^{-7}$	$1.770496e^{-8}$	$2.66000e^{-8}$	$4.6102e^{-7}$
0.7	Exact solution	0.216881160706204810	0.216881160706204810	0.2168811607	$2.16881e^{-1}$
	Computed solution	0.216881399301494220	0.216881188333538640	0.2168811874	$2.16881e^{-1}$
	Error	$2.385953e^{-7}$	$2.762733e^{-8}$	$2.67000e^{-8}$	$3.138Ee^{-7}$
0.8	Exact solution	0.272974910431491580	0.272974910431491580	0.2729749104	$2.72975e^{-1}$
	Computed solution	0.272975324537504060	0.272974950267461710	0.2729749375	$2.72976e^{-1}$
	Error	$4.141060e^{-7}$	$3.983597e^{-8}$	$2.71000e^{-8}$	$7.0374e^{-7}$
0.9	Exact solution	0.331350392754953760	0.331350392754953760	0.3313503928	$3.31350e^{-1}$
	Computed Solution	0.331351055106514410	0.331350446739321010	0.3313504205	$3.31352e^{-1}$
	Error	$6.623516e^{-7}$	$5.398437e^{-8}$	$2.77000e^{-8}$	$1.0177e^{-6}$
1.0	Exact solution	0.390527531852589150	0.390527531852589150	0.3905275319	$3.905280e^{-1}$
	Computed solution	0.390528525127819770	0.390527601370098790	0.3905275591	$3.90531e^{-1}$
	Error	$9.932752e^{-7}$	$6.951751e^{-8}$	$2.72000e^{-8}$	$1.6528e^{-6}$

Table 4.7

Comparison of the New Methods with Seven Step Block Method (Kuboye and Omar,2015b) and Three Step Block Method (Olabode and Yusuph, 2009) for Solving Problem 14 where $h = \frac{1}{10}$

x		$s=\frac{1}{10}, r=\frac{3}{5}, P=4$	$s_1=\frac{1}{10},\ s_2=\frac{2}{5},\ s_3=\frac{4}{5},\ P=5$	Kuboye and Zurni(2015b), $P=8$	Olabode and Yusuph (2009), $P=5$
0.1	Exact solution	3.125170918075647700	3.125170918075647700	3.125170918075647700	3.125170918
	Computed solution	3.125170918072344200	3.125170918075648200	3.125170918075673000	3.125170918
	Error	$3.303580e^{-12}$	$4.440892e^{-16}$	$2.531308e^{-14}$	$-7.56479e^{-11}$
0.2	Exact solution	3.301402758160169700	3.301402758160169700	3.301402758160169700	3.301402758
	Computed Solution	3.301402758149756300	3.301402758160143900	3.301402758160330900	3.301402760
	Error	$1.041345e^{-11}$	$2.575717e^{-14}$	$1.612044e^{-13}$	$1.83983e^{-9}$
0.3	Exact solution	3.529858807576003300	3.529858807576003300	3.529858807576003300	3.529858808
	Computed solution	3.529858807561749000	3.529858807575891900	3.529858807576405700	3.529858812
	Error	$1.425438e^{-11}$	$1.114664e^{-13}$	$4.023448e^{-13}$	$4.42400e^{-9}$
0.4	Exact solution	3.811824697641270600	3.811824697641270600	3.811824697641270600	3.811824698
	Computed solution	3.811824697634265600	3.811824697640978900	3.811824697642024300	3.811824708
	Error	$7.005063e^{-12}$	$2.917666e^{-13}$	$7.536194e^{-13}$	$1.03587e^{-8}$
0.5	Exact Solution	4.148721270700128200	4.148721270700128200	4.148721270700128200	4.148721271
	Computed solution	4.148721270720105100	4.148721270699523300	4.148721270701340600	4.148721282
	Error	$1.997691e^{-11}$	$6.048495e^{-13}$	$1.212364e^{-12}$	$1.12999e^{-8}$
0.6	Exact Solution	4.542118800390508900	4.542118800390508900	4.542118800390509700	4.542118800
	Computed solution	4.542118800466751000	4.542118800389414600	4.542118800392290500	4.542118815
	Error	$7.624212e^{-11}$	$1.094236e^{-12}$	$1.780798e^{-12}$	$1.46095e^{-8}$
0.7	Exact solution	4.993752707470476600	4.993752707470476600	4.993752707470477500	4.993752707
	Computed solution	4.993752707642824100	4.993752707468667400	4.993752707472934200	4.993752728
	Error	$1.723475e^{-10}$	$1.809219e^{-12}$	$2.456702e^{-12}$	$2.05295e^{-8}$
0.8	Exact solution	5.505540928492466800	5.505540928492466800	5.505540928492468600	5.505540928
	Computed solution	5.505540928812426800	5.505540928489666400	5.505540928470347600	5.505540948
	Error	$3.199601e^{-10}$	$2.800427e^{-12}$	$2.212097e^{-11}$	$1.95075e^{-8}$
0.9	Exact solution	6.079603111156949100	6.079603111156949100	6.079603111156950800	6.079603111
	Computed Solution	6.079603111688920900	6.079603111152821700	6.079603111104630900	6.079603122
	Error	$5.319718e^{-10}$	$4.127365e^{-12}$	$5.231993e^{-11}$	$1.08431e^{-8}$
1.0	Exact solution	6.718281828459044600	6.718281828459044600	6.718281828459045500	6.718281828
	Computed solution	6.718281829281677300	6.718281828453189800	6.718281828370444400	6.718281830
	Error	$8.226326e^{-10}$	$5.854872e^{-12}$	$8.860113e^{-11}$	$0.154095e^{-8}$

Table 4.8

Comparison of the New Methods with Three Step Predictor-Corrector Method (Awoyemi, 2005) for Solving Problem 15

x		Awoyemi(2005), $P=5$			$s_1=\frac{1}{5}, s_2=\frac{3}{5}, s_3=\frac{4}{5}, P=5$		
		$h=0.1$	$h=0.025$	$h=0.0125$	$h=0.1$	$h=0.025$	$h=0.0125$
0.2	Error	$1.51e^{-5}$	$3.65e^{-7}$	$4.83e^{-8}$	$5.51e^{-7}$	$3.63e^{-9}$	$2.40e^{-10}$
0.4	Error	$1.68e^{-4}$	$3.13e^{-6}$	$4.01e^{-7}$	$6.71e^{-6}$	$3.13e^{-8}$	$2.00e^{-9}$
0.6	Error	$6.41e^{-4}$	$1.12e^{-5}$	$1.42e^{-6}$	$2.61e^{-5}$	$1.12e^{-7}$	$7.10e^{-9}$
0.8	Error	$1.68e^{-3}$	$2.84e^{-5}$	$3.58e^{-6}$	$6.89e^{-5}$	$2.85e^{-7}$	$1.79e^{-8}$
1.0	Error	$3.64e^{-3}$	$6.03e^{-5}$	$7.60e^{-6}$	$1.49e^{-4}$	$6.07e^{-7}$	$3.81e^{-8}$

Table 4.9

Comparison of the new methods with Three step hybrid Method (Mohammed and Adeniyi, 2014) for solving Problem 16 where $h = \frac{1}{100}$

x		$s_1 = \frac{1}{12}, s_2 = \frac{2}{12}, s_3 = \frac{3}{12}, P = 5$	$s_1 = \frac{1}{6}, s_2 = \frac{2}{6}, s_3 = \frac{3}{6}, P = 5$	Mohammed and Adeniyi(2015), $P = 5$
0.1	Exact solution	0.999950000416665260	0.999950000416665260	0.9999500004
	Computed solution	0.999950000416665710	0.999950000416669150	0.9999506724
	Error	$4.440892e^{-16}$	$3.885781e-015e^{-15}$	$6.72000e^{-7}$
0.2	Exact solution	0.999800006666577760	0.999800006666577760	0.9998000067
	Computed Solution	0.999800006666578870	0.999800006666585530	0.9998013508
	Error	$1.110223e^{-15}$	$7.771561e^{-15}$	$1.34410e^{-6}$
0.3	Exact solution	0.999550033748987540	0.999550033748987540	0.9995500337
	Computed solution	0.999550033748989320	0.999550033748999200	0.9995520507
	Error	$1.776357e^{-15}$	$1.165734e^{14}$	$2.01700e^{-6}$
0.4	Exact solution	0.999200106660977920	0.999200106660977920	0.9992001067
	Computed solution	0.999200106660980580	0.999200106660993900	0.9992027951
	Error	$2.664535e^{-15}$	$1.598721e^{-14}$	$2.68840e^{-6}$
0.5	Exact Solution	0.998750260394966280	0.998750260394966280	0.9987502604
	Computed solution	0.998750260394970060	0.998750260394986710	0.9987536198
	Error	$3.774758e^{-15}$	$2.042810e^{-14}$	$23.35940e^{-6}$

Table 4.10

Comparison of the New Methods with Four Step Block Method (Adesanya et al., 2012) for Solving Problem 9 where $h = \frac{1}{100}$

x		$s = \frac{1}{10}, r = \frac{3}{5}, P = 4$	$s_1 = \frac{1}{10}, s_2 = \frac{2}{5}, s_3 = \frac{9}{10}, P = 5$	Adesanya et al.(2012), $P = 6$
0.1	Exact solution	0.990012495834077470	0.990012495834077470	0.990012495834077
	Computed solution	0.990012495834077020	0.990012495834077690	0.990012495834077
	Error	$4.440892e^{-16}$	$2.220446e^{-16}$	$0.0000e^{-0}$
0.2	Exact solution	0.960199733523725120	0.960199733523725120	0.960199733523725
	Computed Solution	0.960199733523724670	0.960199733523725340	0.960199733523724
	Error	$4.440892e^{-16}$	$2.220446e^{-16}$	$9.99200e^{-16}$
0.3	Exact solution	0.911009467376818090	0.911009467376818090	0.911009467376818
	Computed solution	0.911009467376818320	0.911009467376818320	0.911009467376816
	Error	$2.220446e^{-16}$	$2.220446e^{-16}$	1.55431^{-15}
0.4	Exact solution	0.843182982008654940	0.843182982008654940	0.843182982008655
	Computed solution	0.843182982008657040	0.843182982008655050	0.843182982008652
	Error	$2.109424e^{-15}$	$1.110223e^{-16}$	$3.10862e^{-15}$
0.5	Exact Solution	0.757747685671117830	0.757747685671117830	0.757747685671118
	Computed solution	0.757747685671122830	0.757747685671117940	0.757747685671113
	Error	$4.996004e^{-15}$	$1.110223e^{-16}$	4.66293^{-15}
0.6	Exact Solution	0.656006844729034370	0.656006844729034370	0.656006844729035
	Computed solution	0.656006844729045250	0.656006844729034370	0.656006844729028
	Error	$1.088019e^{-14}$	0.00000000	$6.88338e^{-15}$
0.7	Exact solution	0.539526561853464590	0.539526561853464590	0.539526561853465
	Computed solution	0.539526561853484690	0.539526561853464370	0.539526561853456
	Error	$2.009504e^{-14}$	$2.220446e^{-16}$	$9.10382e^{-15}$
0.8	Exact solution	0.410120128041495670	0.410120128041495670	0.410120128041496
	Computed solution	0.410120128041528860	0.410120128041495280	0.410120128041484
	Error	$3.319567e^{-14}$	$3.885781e^{-16}$	1.14908^{-14}
0.9	Exact solution	0.269829904811992540	0.269829904811992540	0.269829904811992
	Computed Solution	0.269829904812045500	0.269829904811992540	0.269829904811978
	Error	$5.295764e^{-14}$	0.00000000	$1.42108e^{-14}$
1.0	Exact solution	0.120906917604417960	0.120906917604417960	0.120906917604418
	Computed solution	0.120906917604498470	0.120906917604418160	0.120906917604401
	Error	$8.050505e^{-14}$	$1.942890e^{-16}$	1.74582^{-14}

4.4 Comments on the Results

In all the tested problems, the results demonstrate the advantages of using three hybrid points over two hybrid points for solving IVPs of third order ODEs. The accuracy of the methods using the former points is better than the later points. This is due to the fact that the more off-step points utilized the better accuracy will be as the order of the methods increases. When solving the same IVPs of third ODEs, the numerical results also reveal that the developed methods outperform the existing methods.

4.5 Conclusion

This chapter has presented a new classes of one step hybrid block methods for numerically approximating the solution of third order initial value problems. These hybrid block methods have satisfied possessing properties that will confirm its convergence when applied to solve third order ODEs as seen in the numerical results. Also, it is worth noting that the block methods presented is in a generalised two and three off-step points form and hence can take varying hybrid point values which gives room for flexibility. Hence, future research can consider the other classes of hybrid points.

CHAPTER FIVE

ONE STEP HYBRID BLOCK METHODS FOR SOLVING FOURTH ORDER ODEs DIRECTLY

In this chapter, the development of one step hybrid block methods with three generalised off-step points using collocation and interpolation method for solving forth order IVPs of ODEs is considered.

Similarly, Equation (3.1) is employed to approximate the solution of general fourth order initial value problem

$$y^{iv} = f(x, y, y^{'}, y^{''}, y^{'''}), \; x \in [a, b]. \tag{5.1}$$

with four initial conditions

$$y(a) = \eta_0, y^{'}(a) = \eta_1, y^{''}(a) = \eta_2, y^{'''}(a) = \eta_3.$$

The first, second, third and fourth derivative of equation (3.1) are given below

$$y^{'}(x) = \sum_{i=1}^{v+m-1} \frac{i}{h} a_i \left(\frac{x - x_n}{h} \right)^{i-1}. \tag{5.2}$$

$$y^{''}(x) = \sum_{i=2}^{v+m-1} \frac{i(i-1)}{h^2} a_i \left(\frac{x - x_n}{h} \right)^{i-2}. \tag{5.3}$$

$$y^{'''}(x) = \sum_{i=3}^{v+m-1} \frac{i(i-1)(i-2)}{h^3} a_i \left(\frac{x - x_n}{h} \right)^{i-3}. \tag{5.4}$$

$$y^{iv}(x) = \sum_{i=4}^{v+m-1} \frac{i(i-1)(i-2)(i-3)}{h^4} a_i \left(\frac{x - x_n}{h} \right)^{i-4}. \tag{5.5}$$

Substituting (5.5) into (5.1) yields

$$f(x,y,y^{'},y^{''},y^{'''}) = \sum_{i=4}^{v+m-1} \frac{i(i-1)(i-2)(i-3)}{h^4} a_i \left(\frac{x-x_n}{h}\right)^{i-4}. \tag{5.6}$$

5.1 Derivation of One Step Hybrid Block Method with Generalised Three Off-Step Points for Fourth Order ODEs

To derive this method, Equation (3.1) is interpolated at x_n, x_{n+s_1}, x_{n+s_2} and x_{n+s_3} and Equation (5.6) is collocated at all point in the selected interval i.e, x_n, x_{n+s_1}, x_{n+s_2}, x_{n+s_3} and x_{n+1}. This strategy is clearly demonstrates in the following Figure 5.1.

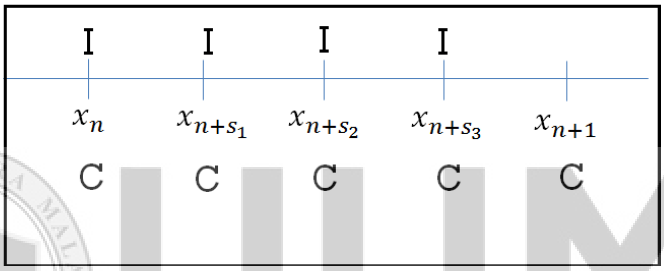

Figure 5.1. One step hybrid block method with generalised three off-step points for solving fourth order ODEs.

Based on Figure 5.1, $v = 4$ and $m = 5$ which produces following interpolation and collocation equations

$$\begin{aligned}
y_n &= a_0 \\
y_{n+s_1} &= a_0 + a_1 s_1 + a_2 s_1^2 + a_3 s_1^3 + a_4 s_1^4 + a_5 s_1^5 + a_6 s_1^6 + a_7 s_1^7 + a_8 s_1^8. \\
y_{n+s_2} &= a_0 + a_1 s_2 + a_2 s_2^2 + a_3 s_2^3 + a_4 s_2^4 + a_5 s_2^5 + a_6 s_2^6 + a_7 s_2^7 + a_8 s_2^8. \\
y_{n+s_3} &= a_0 + a_1 s_3 + a_2 s_3^2 + a_3 s_3^3 + a_4 s_3^4 + a_5 s_3^5 + a_6 s_3^6 + a_7 s_3^7 + a_8 s_3^8. \\
f_n &= \frac{24}{h^4} a_4 \\
.f_{n+s_1} &= \frac{24}{h^4} a_4 + \frac{120 s_1}{h^4} a_5 + \frac{360 s_1^2}{h^4} a_6 + \frac{840 s_1^3}{h^4} a_7 + \frac{1680 s_1^4}{h^4} a_8 \\
.f_{n+s_2} &= \frac{24}{h^4} a_4 + \frac{120 s_2}{h^4} a_5 + \frac{360 s_2^2}{h^4} a_6 + \frac{840 s_2^3}{h^4} a_7 + \frac{1680 s_2^4}{h^4} a_8
\end{aligned} \tag{5.7}$$

$$
\begin{aligned}
f_{n+s_3} &= \frac{24}{h^4}a_4 + \frac{120s_3}{h^4}a_5 + \frac{360s_3^2}{h^4}a_6 + \frac{840s_3^3}{h^4}a_7 + \frac{1680s_3^4}{h^4}a_8 \\
f_{n+s_3} &= \frac{24}{h^4}a_4 + \frac{120}{h^4}a_5 + \frac{360}{h^4}a_6 + \frac{840}{h^4}a_7 + \frac{1680}{h^4}a_8
\end{aligned}
$$

which can be written in a matrix form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & s_1 & s_1^2 & s_1^3 & s_1^4 & s_1^5 & s_1^6 & s_1^7 & s_1^8 \\
1 & s_2 & s_2^2 & s_2^3 & s_2^4 & s_2^5 & s_2^6 & s_2^7 & s_2^8 \\
1 & s_3 & s_3^2 & s_3^3 & s_3^4 & s_3^5 & s_3^6 & s_2^7 & s_3^8 \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120s_1}{h^4} & \frac{360s_1^2}{h^4} & \frac{840s_1^3}{h^4} & \frac{1680s_1^4}{h^4} \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120s_2}{h^4} & \frac{360s_2^2}{h^4} & \frac{840s_2^3}{h^4} & \frac{1680s_2^4}{h^4} \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120s_3}{h^4} & \frac{360s_3^2}{h^4} & \frac{840s_3^3}{h^4} & \frac{1680s_3^4}{h^4} \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120}{h^4} & \frac{360}{h^4} & \frac{840}{h^4} & \frac{1680}{h^4}
\end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix}
=
\begin{pmatrix} y_n \\ y_{n+s_1} \\ y_{n+s_2} \\ y_{n+s_3} \\ f_n \\ f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix}
\qquad (5.8)
$$

Gaussian Elimination method is employed to solve (5.8) in order to find the unknown coefficients $a_i{}'s$ for $i = 0(1)8$ as below

$$
\begin{aligned}
a_1 \quad = \quad & \frac{(s_1\ s_2)}{(s_3\ (s_1-s_3)\ \ (s_2-s_3))}\ \ y_{n+s_3} \quad + \quad \frac{(s_2\ s_3)}{(s_1\ \ (s_1-s_2)\ \ (s_1-s_3))}\ \ y_{n+s_1} \\
- \quad & \frac{(s_1 s_3)}{(s_2\ (s_1-s_2)\ \ (s_2-s_3))}\ \ y_{n+s_2} \quad - \quad \frac{(s_1s_2+s_1s_3+s_2s_3))}{(s_1s_2s_3)}\ \ y_n \\
& + \frac{h^4}{5040}(3s_1^4-5s_1^3s_2-5s_1^3s_3-8s_1^3-5s_1^2s_2^2+15s_1^2s_2s_3+20s_1^2s_2-5s_1^2s_3^2+20s_1^2s_3 \\
& -5s_1s_2^3+15s_1s_2^2s_3+20s_1s_2^2+15s_1s_2s_3^2-120s_1s_2s_3-5s_1s_3^3+20s_1s_3^2-5s_2^2s_3^2 \\
& -5s_2^3s_3-8s_2^3+3s_2^4+20s_2^2s_3-5s_2s_3^3+20s_2s_3^2+3s_3^4-8s_3^3)f_n \\
& + \frac{h^4s_1s_2}{(5040(s_1-s_3)(s_2-s_3)(s_3-1))}(3s_1^4-5s_1^3s_2+3s_1^3s_3-8s_1^3-5s_1^2s_2^2-5s_1^2s_2s_3 \\
& +20s_1^2s_2+3s_1^2s_3^2-8s_1^2s_3-5s_1s_2^3-5s_1s_2^2s_3+20s_1s_2^2-5s_1s_2s_3^2+20s_1s_2s_3-3s_3^4 \\
& +3s_1s_3^3-8s_1s_3^2+3s_2^4+3s_2^3s_3-8s_2^3+3s_2^2s_3^2-8s_2^2s_3+3s_2s_3^3-8s_2s_3^2+6s_3^3)f_{n+s_3} \\
& - \frac{(h^4s_2s_3}{(5040(s_1-s_2)(s_1-s_3)(s_1-1))}(3s_1^4-3s_1^3s_2-3s_1^3s_3-6s_1^3-3s_1^2s_2^2+5s_1^2s_2s_3 \\
& +8s_1^2s_2-3s_1^2s_3^2+8s_1^2s_3-3s_1s_2^3+5s_1s_2^2s_3+8s_1s_2^2+5s_1s_2s_3^2-20s_1s_2s_3-3s_1s_3^3 \\
& +8s_1s_3^2-3s_2^4+5s_2^3s_3+8s_2^3+5s_2^2s_3^2-20s_2^2s_3+5s_2s_3^3-20s_2s_3^2-3s_3^4+8s_3^3)f_{n+s_1} \\
& - \frac{h^4s_1s_3}{(5040(s_1-s_2)(s_2-s_3)(s_2-1))}(3s_1^4+3s_1^3s_2-5s_1^3s_3-8s_1^3+3s_1^2s_2^2-5s_1^2s_2s_3 \\
& -8s_1^2s_2-5s_1^2s_3^2+20s_1^2s_3+3s_1s_2^3-5s_1s_2^2s_3-8s_1s_2^2-5s_1s_2s_3^2+20s_1s_2s_3-5s_1s_3^3 \\
& +20s_1s_3^2-3s_2^4+3s_2^3s_3+6s_2^3+3s_2^2s_3^2-8s_2^2s_3+3s_2s_3^3-8s_2s_3^2+3s_3^4-8s_3^3)f_{n+s_2} \\
& - \frac{h^4s_1s_2s_3}{5040(s_1-1)(s_2-1)(s_3-1)}(3s_1^4-5s_1^3s_2-5s_1^3s_3-5s_1^2s_2^2+15s_1^2s_2s_3-5s_1^2s_3^2 \\
& -5s_1s_2^3+15s_1s_2^2s_3+15s_1s_2s_3^2-5s_1s_3^3+3s_2^4-5s_2^3s_3-5s_2^2s_3^2-5s_2s_3^3+3s_3^4)f_{n+1}
\end{aligned}
$$

$$a_2 = \frac{(s_1+s_2+s_3)}{(s_1 s_2 s_3)} y_n - \frac{(s_1+s_2)}{(s_3(s_1-s_3)(s_2-s_3))} y_{n+s_3} - \frac{(s_2+s_3)}{(s_1(s_1-s_2)(s_1-s_3))} y_{n+s_1}$$

$$+\frac{(s_1+s_3)}{(s_2(s_1-s_2)(s_2-s_3))} y_{n+s_2} + \frac{h^4}{(5040(s_1-1)(s_2-1)(s_3-1))} (3s_1^5 s_2 + 3s_1^5 s_3$$

$$-5s_1^4 s_2^2 - 10s_1^4 s_2 s_3 - 5s_1^4 s_3^2 - 5s_1^3 s_2^3 + 10s_1^3 s_2^2 s_3 + 10s_1^3 s_2 s_3^2 - 5s_1^3 s_3^3 - 5s_1^2 s_2^4 - 5s_2^2 s_3^4$$

$$+10s_1^2 s_2^3 s_3 + 30s_1^2 s_2^2 s_3^2 + 10s_1^2 s_2 s_3^3 - 5s_1^2 s_3^4 + 3s_1 s_2^5 - 10s_1 s_2^4 s_3 + 10s_1 s_2^3 s_3^2 - 5s_2^4 s_3^2$$

$$+10s_1 s_2^2 s_3^3 - 10s_1 s_2 s_3^4 + 3s_1 s_3^5 + 3s_2^5 s_3 - 5s_2^3 s_3^3 + 3s_2 s_3^5) f_{n+1}$$

$$+\frac{h^4}{(5040 s_2(s_1-s_2)(s_2-s_3)(s_2-1))} (3s_1^5 s_2 + 3s_1^5 s_3 + 3s_1^4 s_2^2 - 2s_1^4 s_2 s_3 - 10s_1^3 s_2 s_3^2$$

$$-8s_1^4 s_2 - 5s_1^4 s_3^2 - 8s_1^4 s_3 + 3s_1^3 s_2^3 - 2s_1^3 s_2^2 s_3 - 8s_1^3 s_2^2 + 12s_1^3 s_2 s_3 - 5s_1^3 s_3^3 - 10s_1^2 s_2 s_3^3$$

$$+20s_1^3 s_3^2 + 3s_1^2 s_2^4 - 2s_1^2 s_2^3 s_3 - 8s_1^2 s_2^3 - 10s_1^2 s_2^2 s_3^2 + 12s_1^2 s_2^2 s_3 + 6s_2^4 s_3 + 3s_2^3 s_3^3 + 3s_2^2 s_3^4$$

$$+40s_1^2 s_2 s_3^2 - 5s_1^2 s_3^4 + 20s_1^2 s_3^3 - 3s_1 s_2^5 + 6s_1 s_2^4 s_3 + 6s_1 s_2^4 - 2s_1 s_2^3 s_3^2 - 16s_1 s_2^3 s_3$$

$$-2s_1 s_2^2 s_3^3 + 12s_1 s_2^2 s_3^2 - 2s_1 s_2 s_3^4 + 12s_1 s_2 s_3^3 + 3s_1 s_3^5 - 8s_1 s_3^4 - 3s_2^5 s_3 + 3s_2^4 s_3^2$$

$$-8s_2^3 s_3^2 - 8s_2^2 s_3^3 + 3s_2 s_3^5 - 8s_2 s_3^4) f_{n+s_2}$$

$$+\frac{h^4}{(5040 s_1(s_1-s_2)(s_1-s_3)(s_1-1))} (3s_1^5 s_2 + 3s_1^5 s_3 - 3s_1^4 s_2^2 - 6s_1^4 s_2 s_3 - 6s_1^4 s_2$$

$$-3s_1^4 s_3^2 - 6s_1^4 s_3 - 3s_1^3 s_2^3 + 2s_1^3 s_2^2 s_3 + 8s_1^3 s_2^2 + 2s_1^3 s_2 s_3^2 + 16s_1^3 s_2 s_3 - 3s_1^3 s_3^3 + 8s_1^3 s_3^2$$

$$-3s_1^2 s_2^4 + 2s_1^2 s_2^3 s_3 + 8s_1^2 s_2^3 + 10s_1^2 s_2^2 s_3^2 - 12s_1^2 s_2^2 s_3 + 2s_1^2 s_2 s_3^3 - 12s_1^2 s_2 s_3^2 - 3s_1^2 s_3^4$$

$$+8s_1^2 s_3^3 - 3s_1 s_2^5 + 2s_1 s_2^4 s_3 + 8s_1 s_2^4 + 10s_1 s_2^3 s_3^2 - 12s_1 s_2^3 s_3 + 10s_1 s_2^2 s_3^3 - 40s_1 s_2^2 s_3^2$$

$$+2s_1 s_2 s_3^4 - 12s_1 s_2 s_3^3 - 3s_1 s_3^5 + 8s_1 s_3^4 - 3s_2^5 s_3 + 5s_2^4 s_3^2 + 8s_2^4 s_3 + 5s_2^3 s_3^3 - 20s_2^3 s_3^2$$

$$+5s_2^2 s_3^4 - 20s_2^2 s_3^3 - 3s_2 s_3^5 + 8s_2 s_3^4) f_{n+s_1} - \frac{h^4}{(5040 s_1 s_2 s_3)} (3s_1^5 s_2 + 3s_1^5 s_3 - 5s_1^4 s_2^2$$

$$-10s_1^4 s_2 s_3 - 8s_1^4 s_2 - 5s_1^4 s_3^2 - 8s_1^4 s_3 - 5s_1^3 s_2^3 + 10s_1^3 s_2^2 s_3 + 20s_1^3 s_2^2 + 10s_1^3 s_2 s_3^2$$

$$+40s_1^3 s_2 s_3 - 5s_1^3 s_3^3 + 20s_1^3 s_3^2 - 5s_1^2 s_2^4 + 10s_1^2 s_2^3 s_3 + 20s_1^2 s_2^3 + 30s_1^2 s_2^2 s_3^2 - 8s_2 s_3^4$$

$$-100s_1^2 s_2^2 s_3 + 10s_1^2 s_2 s_3^3 - 100s_1^2 s_2 s_3^2 - 5s_1^2 s_3^4 + 20s_1^2 s_3^3 + 3s_1 s_2^5 - 10s_1 s_2^4 s_3 + 3s_2 s_3^5$$

$$-8s_1 s_2^4 + 10s_1 s_2^3 s_3^2 + 40s_1 s_2^3 s_3 + 10s_1 s_2^2 s_3^3 - 100s_1 s_2^2 s_3^2 - 10s_1 s_2 s_3^4 + 40s_1 s_2 s_3^3$$

$$+3s_1 s_3^5 - 8s_1 s_3^4 + 3s_2^5 s_3 - 5s_2^4 s_3^2 - 8s_2^4 s_3 - 5s_2^3 s_3^3 + 20s_2^3 s_3^2 - 5s_2^2 s_3^4 + 20s_2^2 s_3^3) f_n$$

$$-\frac{h^4}{(5040 s_3(s_1-s_3)(s_2-s_3)(s_3-1))} (3s_1^5 s_2 + 3s_1^5 s_3 - 5s_1^4 s_2^2 - 2s_1^4 s_2 s_3 - 8s_1^4 s_2$$

$$+3s_1^4s_3^2-8s_1^4s_3-5s_1^3s_2^3-10s_1^3s_2^2s_3+20s_1^3s_2^2-2s_1^3s_2s_3^2+12s_1^3s_2s_3+3s_1^3s_3^3$$
$$-8s_1^3s_3^2-5s_1^2s_2^4-10s_1^2s_2^3s_3+20s_1^2s_2^3-10s_1^2s_2^2s_3^2+40s_1^2s_2^2s_3-2s_1^2s_2s_3^3+6s_2s_3^4$$
$$+12s_1^2s_2s_3^2+3s_1^2s_3^4-8s_1^2s_3^3+3s_1s_2^5-2s_1s_2^4s_3-8s_1s_2^4-2s_1s_2^3s_3^2+12s_1s_2^3s_3$$
$$-2s_1s_2^2s_3^3+12s_1s_2^2s_3^2+6s_1s_2s_3^4-16s_1s_2s_3^3-3s_1s_3^5+6s_1s_3^4+3s_2^5s_3+3s_2^4s_3^2$$
$$-8s_2^4s_3+3s_2^3s_3^3-8s_2^3s_3^2+3s_2^2s_3^4-8s_2^2s_3^3-3s_2s_3^5)f_{n+s_3}$$

$$a_3=\frac{1}{(s_3(s_1-s_3)(s_2-s_3))}y_{n+s_3}+\frac{1}{(s_1(s_1-s_2)(s_1-s_3))}y_{n+s_1}-\frac{1}{(s_1s_2s_3)}y_n$$
$$-\frac{1}{(s_2(s_1-s_2)(s_2-s_3))}y_{n+s_2}+\frac{h^4}{(5040s_1s_2s_3)}(3s_1^5-5s_1^4s_2-5s_1^4s_3-8s_1^4-5s_1^3s_2^2$$
$$+15s_1^3s_2s_3+20s_1^3s_2-5s_1^3s_3^2+20s_1^3s_3-5s_1^2s_2^3+15s_1^2s_2^2s_3+20s_1^2s_2^2+15s_1^2s_2s_3^2$$
$$-120s_1^2s_2s_3-5s_1^2s_3^3+20s_1^2s_3^2-5s_1s_2^4+15s_1s_2^3s_3+20s_1s_2^3+15s_1s_2^2s_3^2+20s_2^3s_3$$
$$-120s_1s_2^2s_3+15s_1s_2s_3^3-120s_1s_2s_3^2-5s_1s_3^4+20s_1s_3^3+3s_2^5-5s_2^4s_3-8s_2^4-5s_2^3s_3^2$$
$$-5s_2^2s_3^3+20s_2^2s_3^2-5s_2s_3^4+20s_2s_3^3+3s_3^5-8s_3^4)f_n$$
$$+\frac{h^4}{(5040s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(3s_1^5-5s_1^4s_2+3s_1^4s_3-8s_1^4-5s_1^3s_2^2-5s_1^3s_2s_3$$
$$+20s_1^3s_2+3s_1^3s_3^2-8s_1^3s_3-5s_1^2s_2^3-5s_1^2s_2^2s_3+20s_1^2s_2^2-5s_1^2s_2s_3^2+20s_1^2s_2s_3+3s_1^2s_3^3$$
$$-8s_1^2s_3^2-5s_1s_2^4-5s_1s_2^3s_3+20s_1s_2^3-5s_1s_2^2s_3^2+20s_1s_2^2s_3-5s_1s_2s_3^3+20s_1s_2s_3^2$$
$$+3s_1s_3^4-8s_1s_3^3+3s_2^5+3s_2^4s_3-8s_2^4+3s_2^3s_3^2-8s_2^3s_3+3s_2^2s_3^3-8s_2^2s_3^2+3s_2s_3^4$$
$$-8s_2s_3^3-3s_3^5+6s_3^4)f_{n+s_3}-\frac{h^4}{(5040(s_1-1)(s_2-1)(s_3-1))}(3s_1^5-5s_1^4s_2-5s_1^4s_3$$
$$-5s_1^3s_2^2+15s_1^3s_2s_3-5s_1^3s_3^2-5s_1^2s_2^3+15s_1^2s_2^2s_3+15s_1^2s_2s_3^2-5s_1^2s_3^3-5s_1s_2^4-5s_2s_3^4$$
$$+15s_1s_2^3s_3+15s_1s_2^2s_3^2+15s_1s_2s_3^3-5s_1s_3^4+3s_2^5-5s_2^4s_3-5s_2^3s_3^2-5s_2^2s_3^3+3s_3^5)f_{n+1}$$
$$-\frac{h^4}{(5040s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(3s_1^5+3s_1^4s_2-5s_1^4s_3-8s_1^4+3s_2^2s_3^3-8s_2^2s_3^2$$
$$+3s_1^3s_2^2-5s_1^3s_2s_3-8s_1^3s_2-5s_1^3s_3^2+20s_1^3s_3+3s_1^2s_2^3-5s_1^2s_2^2s_3-8s_1^2s_2^2-5s_1^2s_2s_3^2$$
$$+20s_1^2s_2s_3-5s_1^2s_3^3+20s_1^2s_3^2+3s_1s_2^4-5s_1s_2^3s_3-8s_1s_2^3-5s_1s_2^2s_3^2+20s_1s_2^2s_3+3s_3^5$$
$$-5s_1s_2s_3^3+20s_1s_2s_3^2-5s_1s_3^4+20s_1s_3^3-3s_2^5+3s_2^4s_3+6s_2^4+3s_2^3s_3^2-8s_2^3s_3-8s_3^4$$
$$+3s_2s_3^4-8s_2s_3^3)f_{n+s_2}$$

$$
-\frac{h^4}{(5040s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(3s_1^5-3s_1^4s_2-3s_1^4s_3-6s_1^4-3s_1^3s_2^2+5s_1^3s_2s_3
$$

$$
+8s_1^3s_2-3s_1^3s_3^2+8s_1^3s_3-3s_1^2s_2^3+5s_1^2s_2^2s_3+8s_1^2s_2^2+5s_1^2s_2s_3^2-20s_1^2s_2s_3-3s_1^2s_3^3
$$

$$
+8s_1^2s_3^2-3s_1s_2^4+5s_1s_2^3s_3+8s_1s_2^3+5s_1s_2^2s_3^2-20s_1s_2^2s_3+5s_1s_2s_3^3-20s_1s_2s_3^2
$$

$$
-3s_1s_3^4+8s_1s_3^3-3s_2^5+5s_2^4s_3+8s_2^4+5s_2^3s_3^2-20s_2^3s_3+5s_2^2s_3^3-20s_2^2s_3^2+5s_2s_3^4
$$

$$
-20s_2s_3^3-3s_3^5+8s_3^4)f_{n+s_1}
$$

$$
a_4=\frac{h^4}{24}
$$

$$
\begin{aligned}
a_5 = &-\frac{h^4(s_1s_2+s_1s_3+s_2s_3+s_1s_2s_3))}{(120s_1s_2s_3)}f_n+\frac{h^4s_1s_2s_3}{(120(s_1-1)(s_2-1)(s_3-1))}f_{n+1}\\
&+\frac{h^4s_1s_3}{(120s_2(s_1-s_2)(s_2-s_3)(s_2-1))}f_{n+s_2}-\frac{h^4s_1s_2}{(120s_3(s_1-s_3)(s_2-s_3)(s_3-1))}f_{n+s_3}\\
&-\frac{h^4s_2s_3}{(120s_1(s_1-s_2)(s_1-s_3)(s_1-1))}f_{n+s_1}
\end{aligned}
$$

$$
\begin{aligned}
a_6 = &\frac{h^4(s_1+s_2+s_3+s_1s_2+s_1s_3+s_2s_3)}{(360s_1s_2s_3)}f_n-\frac{h^4(s_1s_2+s_1s_3+s_2s_3)}{(360(s_1-1)(s_2-1)(s_3-1))}f_{n+1}\\
&+\frac{h^4(s_2+s_3+s_2s_3)}{(360s_1(s_1-s_2)(s_1-s_3)(s_1-1))}f_{n+s_1}-\frac{h^4(s_1+s_3+s_1s_3)}{(360s_2(s_1-s_2)(s_2-s_3)(s_2-1))}f_{n+s_2}\\
&+\frac{h^4(s_1+s_2+s_1s_2)}{(360s_3(s_1-s_3)(s_2-s_3)(s_3-1))}f_{n+s_3}
\end{aligned}
$$

$$
\begin{aligned}
a_7 = &\frac{h^4(s_1+s_2+s_3)}{(840(s_1-1)(s_2-1)(s_3-1))}f_{n+1}+\frac{h^4(s_1+s_3+1)}{(840s_2(s_1-s_2)(s_2-s_3)(s_2-1))}f_{n+s_2}\\
&-\frac{h^4(s_1+s_2+s_3+1)}{(840s_1s_2s_3)}f_n-\frac{h^4(s_1+s_2+1)}{(840s_3(s_1-s_3)(s_2-s_3)(s_3-1))}f_{n+s_3}\\
&-\frac{h^4(s_2+s_3+1)}{(840s_1(s_1-s_2)(s_1-s_3)(s_1-1))}f_{n+s_1}
\end{aligned}
$$

$$a_8 = -\frac{h^4}{(1680(s_1-1)(s_2-1)(s_3-1))}f_{n+1} + \frac{h^4}{(1680s_3(s_1-s_3)(s_2-s_3)(s_3-1))}f_{n+s_3}$$
$$+\frac{h^4}{(1680s_1(s_1-s_2)(s_1-s_3)(s_1-1))}f_{n+s_1} - \frac{h^4}{(1680s_2(s_1-s_2)(s_2-s_3)(s_2-1))}f_{n+s_2}$$
$$+\frac{h^4}{(1680s_1s_2s_3)}f_n$$

Substituting the values of $a_i{}'s$ into Equation (3.1) and then simplifying yields a continuous hybrid one step method of the form:

$$y(x) = \alpha_0(x) + \sum_{i=1}^{3}\alpha_{s_i}(x)y_{n+s_i} + \sum_{i=0}^{1}\beta_i(x)f_{n+i} + \sum_{i=1}^{3}\beta_{s_i}(X)f_{n+s_i} \tag{5.9}$$

The first, second and third derivatives of Equation(5.9) are

$$y^{'}(x) = \frac{\partial}{\partial x}\alpha_0(x) + \sum_{i=1}^{3}\frac{\partial}{\partial x}\alpha_{s_i}(x)y_{n+s_i} + \sum_{i=0}^{1}\frac{\partial}{\partial x}\beta_i(x)f_{n+i} + \sum_{i=1}^{3}\frac{\partial}{\partial x}\beta_{s_i}(x)f_{n+s_i} \tag{5.10}$$

$$y^{''}(x) = \frac{\partial^2}{\partial x^2}\alpha_0(x) + \sum_{i=1}^{3}\frac{\partial^2}{\partial x^2}\alpha_{s_i}(x)y_{n+s_i} + \sum_{i=0}^{1}\frac{\partial^2}{\partial x^2}\beta_i(x)f_{n+i} + \sum_{i=1}^{3}\frac{\partial^2}{\partial x^2}\beta_{s_i}(x)f_{n+s_i} \tag{5.11}$$

$$y^{'''}(x) = \frac{\partial^3}{\partial x^3}\alpha_0(x) + \sum_{i=1}^{3}\frac{\partial^3}{\partial x^3}\alpha_{s_i}(x)y_{n+s_i} + \sum_{i=0}^{1}\frac{\partial^3}{\partial x^3}\beta_i(x)f_{n+i} + \sum_{i=1}^{3}\frac{\partial^3}{\partial x^3}\beta_{s_i}(x)f_{n+s_i} \tag{5.12}$$

respectively, where

$$\alpha_0 = \frac{(x_n-x+hs_3)(x_n-x+hs_1)(x_n-x+hs_2)}{(h^3s_1s_2s_3)}$$
$$\alpha_{s_1} = \frac{(x-x_n)(x_n-x+hs_3)(x_n-x+hs_2)}{(h^3s_1(s_1-s_3)(s_1-s_2))}$$
$$\alpha_{s_2} = \frac{(x-x_n)(x-x_n-hs_3)(x_n-x+hs_1)}{(h^3s_2(s_2-s_3)(s_1-s_2))}$$
$$\alpha_{s_3} = \frac{(x-x_n)(x_n-x+hs_2)(x_n-x+hs_1)}{(h^3s_3(s_2-s_3)(s_1-s_3))}$$

$$
\begin{aligned}
\beta_0 = &-\frac{(x-x_n)(x_n-x+hs_1)(x_n-x+hs_2)(x_n-x+hs_3)}{(5040h^4s_1s_2s_3)}(3h^4s_1^4-5h^4s_1^3s_2\\
&-5h^4s_1^3s_3-8h^4s_1^3-5h^4s_1^2s_2^2+15h^4s_1^2s_2s_3+20h^4s_1^2s_2-5h^4s_1^2s_3^2+20h^4s_1^2s_3\\
&-5h^4s_1s_2^3+15h^4s_1s_2^2s_3+20h^4s_1s_2^2+15h^4s_1s_2s_3^2-120h^4s_1s_2s_3-5h^4s_1s_3^3\\
&+20h^4s_1s_3^2+3h^4s_2^4-5h^4s_2^3s_3-8h^4s_2^3-5h^4s_2^2s_3^2+20h^4s_2^2s_3-5h^4s_2s_3^3\\
&+20h^4s_2s_3^2-8h^4s_3^3+3h^3s_1^3x-3h^3s_1^3x_n-5h^3s_1^2s_2x+5h^3s_1^2s_2x_n+12x^3x_n\\
&-5h^3s_1^2s_3x+5h^3s_1^2s_3x_n-8h^3s_1^2x+8h^3s_1^2x_n-5h^3s_1s_2^2x+5h^3s_1s_2^2x_n+3h^4s_3^4\\
&-15h^3s_1s_2s_3x_n+20h^3s_1s_2x-20h^3s_1s_2x_n-5h^3s_1s_3^2x+5h^3s_1s_3^2x_n+3hs_3x^3\\
&-20h^3s_1s_3x_n+3h^3s_2^3x-3h^3s_2^3x_n-5h^3s_2^2s_3x+5h^3s_2^2s_3x_n-8h^3s_2^2x-3x_n^4\\
&-5h^3s_2s_3^2x+5h^3s_2s_3^2x_n+20h^3s_2s_3x-20h^3s_2s_3x_n+3h^3s_3^3x-3h^3s_3^3x_n-3x^4\\
&+8h^3s_3^2x_n+3h^2s_1^2x^2-6h^2s_1^2xx_n+3h^2s_1^2x_n^2-5h^2s_1s_2x^2+10h^2s_1s_2xx_n\\
&-5h^2s_1s_3x^2+10h^2s_1s_3xx_n-5h^2s_1s_3x_n^2-8h^2s_1x^2+16h^2s_1xx_n-8h^2s_1x_n^2\\
&-6h^2s_2^2xx_n+3h^2s_2^2x_n^2-5h^2s_2s_3x^2+10h^2s_2s_3xx_n-5h^2s_2s_3x_n^2-8h^2s_2x^2\\
&-8h^2s_2x_n^2+3h^2s_3^2x^2-6h^2s_3^2xx_n+3h^2s_3^2x_n^2-8h^2s_3x^2+16h^2s_3xx_n-8h^2s_3x_n^2\\
&+3hs_1x^3-9hs_1x^2x_n+9hs_1xx_n^2-3hs_1x_n^3+3hs_2x^3-9hs_2x^2x_n+9hs_2xx_n^2\\
&+3h^2s_2^2x^2-9hs_3x^2x_n+9hs_3xx_n^2-3hs_3x_n^3+6hx^3-18hx^2x_n+18hxx_n^2-6hx_n^3\\
&+20h^3s_1s_3x-3hs_2x_n^3-5h^2s_1s_2x_n^2+16h^2s_2xx_n+15h^3s_1s_2s_3x-8h^3s_3^2x\\
&+8h^3s_2^2x_n-18x^2x_n^2+12xx_n^3)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{s_1} = & -\frac{((x-x_n)(x_n-x+hs_1)(x_n-x+hs_2)(x_n-x+hs_3))}{(5040h^4s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(3h^4s_1^4-3h^4s_1^3s_2\\
& -3h^4s_1^3s_3-6h^4s_1^3-3h^4s_1^2s_2^2+5h^4s_1^2s_2s_3+8h^4s_1^2s_2-3h^4s_1^2s_3^2+8h^4s_1^2s_3-3h^4s_1s_2^3\\
& +5h^4s_1s_2^2s_3+8h^4s_1s_2^2+5h^4s_1s_2s_3^2-20h^4s_1s_2s_3-3h^4s_1s_3^3+8h^4s_1s_3^2-3h^4s_2^4\\
& +5h^4s_2^3s_3+8h^4s_2^3+5h^4s_2^2s_3^2-20h^4s_2^2s_3+5h^4s_2s_3^3-20h^4s_2s_3^2-3h^4s_3^4+8h^4s_3^3\\
& +3h^3s_1^3x-3h^3s_1^3x_n-3h^3s_1^2s_2x+3h^3s_1^2s_2x_n-3h^3s_1^2s_3x+3h^3s_1^2s_3x_n-6h^3s_1^2x\\
& +6h^3s_1^2x_n-3h^3s_1s_2^2x+3h^3s_1s_2^2x_n+5h^3s_1s_2s_3x-5h^3s_1s_2s_3x_n+8h^3s_1s_2x-12xx_n^3\\
& -3h^3s_1s_3^2x+3h^3s_1s_3^2x_n+8h^3s_1s_3x-8h^3s_1s_3x_n-3h^3s_2^3x+3h^3s_2^3x_n+5h^3s_2^2s_3x\\
& +5h^3s_2s_3^2x-5h^3s_2s_3^2x_n-20h^3s_2s_3x+20h^3s_2s_3x_n-3h^3s_3^3x+3h^3s_3^3x_n+8h^3s_3^2x\\
& -8h^3s_3^2x_n+3h^2s_1^2x^2-6h^2s_1^2xx_n+3h^2s_1^2x_n^2-3h^2s_1s_3x^2+8h^2s_2x_n^2-8h^3s_1s_2x_n\\
& +6h^2s_1s_3xx_n-3h^2s_1s_3x_n^2-6h^2s_1x^2+12h^2s_1xx_n-6h^2s_1x_n^2-3h^2s_2^2x^2+6h^2s_2^2xx_n\\
& -3h^2s_2^2x_n^2+5h^2s_2s_3x^2-10h^2s_2s_3xx_n+5h^2s_2s_3x_n^2+8h^2s_2x^2-16h^2s_2xx_n+3x_n^4\\
& +6h^2s_3^2xx_n-3h^2s_3^2x_n^2+8h^2s_3x^2-16h^2s_3xx_n+8h^2s_3x_n^2+3hs_1x^3-9hs_1x^2x_n\\
& -3hs_2x^3+9hs_2x^2x_n-9hs_2xx_n^2+3hs_2x_n^3-3hs_3x^3+9hs_3x^2x_n-9hs_3xx_n^2+3hs_3x_n^3\\
& +9hs_1xx_n^2-6hx^3-3h^2s_3^2x^2-3h^2s_1s_2x^2+6h^2s_1s_2xx_n-3h^2s_1s_2x_n^2-3hs_1x_n^3+3x^4\\
& -5h^3s_2^2s_3x_n+8h^3s_2^2x-8h^3s_2^2x_n+18hx^2x_n-18hxx_n^2+6hx_n^3-12x^3x_n+18x^2x_n^2)
\end{aligned}
$$

$$
\beta_{s_2}=\frac{(x-x_n)(x_n-x+hs_1)(x_n-x+hs_2)(x_n-x+hs_3)}{(5040h^4s_2(s_1-s_2)(s_2-s_3)(s_2-1))}\,(3h^2s_1s_2x_n^2
$$

$$
-5h^4s_1^3s_3-8h^4s_1^3+3h^4s_1^2s_2^2-5h^4s_1^2s_2s_3-8h^4s_1^2s_2-5h^4s_1^2s_3^2+3h^4s_3^4
$$

$$
+3h^4s_1s_2^3-5h^4s_1s_2^2s_3-8h^4s_1s_2^2-5h^4s_1s_2s_3^2+20h^4s_1s_2s_3-5h^4s_1s_3^3
$$

$$
+20h^4s_1s_3^2-3h^4s_2^4+3h^4s_2^3s_3+6h^4s_2^3+3h^4s_2^2s_3^2-8h^4s_2^2s_3+3h^4s_2s_3^3
$$

$$
-8h^4s_2s_3^2-8h^4s_3^3+3h^3s_1^3x-3h^3s_1^3x_n+3h^3s_1^2s_2x-3h^3s_1^2s_2x_n-3x^4
$$

$$
+5h^3s_1^2s_3x_n-8h^3s_1^2x+8h^3s_1^2x_n+3h^3s_1s_2^2x-3h^3s_1s_2^2x_n-5h^3s_1s_2s_3x
$$

$$
+5h^3s_1s_2s_3x_n-8h^3s_1s_2x+8h^3s_1s_2x_n-5h^3s_1s_3^2x+5h^3s_1s_3^2x_n-6hx_n^3
$$

$$
-20h^3s_1s_3x_n-3h^3s_2^3x+3h^3s_2^3x_n+3h^3s_2^2s_3x-3h^3s_2^2s_3x_n+6h^3s_2^2x
$$

$$
+3h^3s_2s_3^2x-3h^3s_2s_3^2x_n-8h^3s_2s_3x+8h^3s_2s_3x_n+3h^3s_3^3x-3h^3s_3^3x_n
$$

$$
+8h^3s_3^2x_n+3h^2s_1^2x^2-6h^2s_1^2xx_n+3h^2s_1^2x_n^2+3h^2s_1s_2x^2-6h^2s_1s_2xx_n
$$

$$
-5h^2s_1s_3x^2+10h^2s_1s_3xx_n-5h^2s_1s_3x_n^2-8h^2s_1x^2+16h^2s_1xx_n-3x_n^4
$$

$$
-3h^2s_2^2x^2+6h^2s_2^2xx_n-3h^2s_2^2x_n^2+3h^2s_2s_3x^2-6h^2s_2s_3xx_n+3h^2s_2s_3x_n^2
$$

$$
+6h^2s_2x^2-12h^2s_2xx_n+6h^2s_2x_n^2+3h^2s_3^2x^2-6h^2s_3^2xx_n+3h^2s_3^2x_n^2+6hx^3
$$

$$
+16h^2s_3xx_n-8h^2s_3x_n^2+3hs_1x^3-9hs_1x^2x_n+9hs_1xx_n^2-3hs_1x_n^3-3hs_2x^3
$$

$$
+9hs_2x^2x_n-9hs_2xx_n^2+3hs_2x_n^3+3hs_3x^3-9hs_3x^2x_n+9hs_3xx_n^2-3hs_3x_n^3
$$

$$
-8h^2s_3x^2-8h^2s_1x_n^2+3h^4s_1^4-8h^3s_3^2x-6h^3s_2^2x_n+20h^3s_1s_3x-5h^3s_1^2s_3x
$$

$$
+20h^4s_1^2s_3+3h^4s_1^3s_2-18hx^2x_n+18hxx_n^2+12x^3x_n-18x^2x_n^2+12xx_n^3)
$$

$$
\begin{aligned}
\beta_{s_3} = {} & \frac{-(x-x_n)(x_n-x+hs_1)(x_n-x+hs_2)(x_n-x+hs_3))}{(5040h^4 s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(3h^4s_1^4 - 5h^4s_1^3s_2 \\
& +3h^4s_1^3s_3 - 8h^4s_1^3 - 5h^4s_1^2s_2^2 - 5h^4s_1^2s_2s_3 + 20h^4s_1^2s_2 + 3h^4s_1^2s_3^2 - 8h^4s_1^2s_3 \\
& -5h^4s_1s_2^3 - 5h^4s_1s_2^2s_3 + 20h^4s_1s_2^2 - 5h^4s_1s_2s_3^2 + 20h^4s_1s_2s_3 + 3h^4s_1s_3^3 - 6hx_n^3 \\
& -8h^4s_1s_3^2 + 3h^4s_2^4 + 3h^4s_2^3s_3 - 8h^4s_2^3 + 3h^4s_2^2s_3^2 - 8h^4s_2^2s_3 + 3h^4s_2s_3^3 - 8h^4s_2s_3^2 \\
& -3h^4s_3^4 + 6h^4s_3^3 + 3h^3s_1^3x - 3h^3s_1^3x_n - 5h^3s_1^2s_2x + 5h^3s_1^2s_2x_n + 3h^3s_1^2s_3x - 3x^4 \\
& -3h^3s_1^2s_3x_n - 8h^3s_1^2x + 8h^3s_1^2x_n - 5h^3s_1s_2^2x + 5h^3s_1s_2^2x_n - 5h^3s_1s_2s_3x - 3x_n^4 \\
& +5h^3s_1s_2s_3x_n + 20h^3s_1s_2x - 20h^3s_1s_2x_n + 3h^3s_1s_3^2x - 3h^3s_1s_3^2x_n - 8h^3s_1s_3x \\
& +8h^3s_1s_3x_n + 3h^3s_2^3x - 3h^3s_2^3x_n + 3h^3s_2^2s_3x - 3h^3s_2^2s_3x_n - 8h^3s_2^2x + 8h^3s_2^2x_n \\
& +3h^3s_2s_3^2x - 3h^3s_2s_3^2x_n - 8h^3s_2s_3x + 8h^3s_2s_3x_n - 3h^3s_3^3x + 3h^3s_3^3x_n + 6h^3s_3^2x \\
& -6h^3s_3^2x_n + 3h^2s_1^2x^2 - 6h^2s_1^2xx_n + 3h^2s_1^2x_n^2 - 5h^2s_1s_2x^2 + 10h^2s_1s_2xx_n + 6hx^3 \\
& -5h^2s_1s_2x_n^2 + 3h^2s_1s_3x^2 - 6h^2s_1s_3xx_n + 3h^2s_1s_3x_n^2 - 8h^2s_1x^2 + 16h^2s_1xx_n \\
& -8h^2s_1x_n^2 + 3h^2s_2^2x^2 - 6h^2s_2^2xx_n + 3h^2s_2^2x_n^2 + 3h^2s_2s_3x^2 - 6h^2s_2s_3xx_n + 12xx_n^3 \\
& +3h^2s_2s_3x_n^2 - 8h^2s_2x^2 + 16h^2s_2xx_n - 8h^2s_2x_n^2 - 3h^2s_3^2x^2 + 6h^2s_3^2xx_n - 3h^2s_3^2x_n^2 \\
& +6h^2s_3x^2 - 12h^2s_3xx_n + 6h^2s_3x_n^2 + 3hs_1x^3 - 9hs_1x^2x_n + 9hs_1xx_n^2 - 3hs_1x_n^3 \\
& +3hs_2x^3 - 9hs_2x^2x_n + 9hs_2xx_n^2 - 3hs_2x_n^3 - 3hs_3x^3 + 9hs_3x^2x_n - 9hs_3xx_n^2 \\
& +3hs_3x_n^3 - 18hx^2x_n + 18hxx_n^2 + 12x^3x_n - 18x^2x_n^2)
\end{aligned}
$$

$$
\begin{aligned}
\beta_1 = &-\frac{(x-x_n)(x_n-x+hs_3)(x_n-x+hs_1)(x_n-x+hs_2)}{(5040h^4(s_3-1)(s_2-1)(s_1-1))}\,(3h^4s_1^4-5h^4s_1^3s_2\\
&-5h^4s_1^3s_3-5h^4s_1^2s_2^2+15h^4s_1^2s_2s_3-5h^4s_1^2s_3^2-5h^4s_1s_2^3+15h^4s_1s_2^2s_3-3hs_3x_n^3\\
&-5h^4s_1s_3^3+3h^4s_2^4-5h^4s_2^3s_3-5h^4s_2^2s_3^2-5h^4s_2s_3^3+3h^4s_3^4+3h^3s_1^3x-3h^3s_1^3x_n\\
&-5h^3s_1^2s_2x+5h^3s_1^2s_2x_n-5h^3s_1^2s_3x+5h^3s_1^2s_3x_n-5h^3s_1s_2^2x+5h^3s_1s_2^2x_n-3x_n^4\\
&+15h^3s_1s_2s_3x-15h^3s_1s_2s_3x_n-5h^3s_1s_3^2x+5h^3s_1s_3^2x_n+3h^3s_2^3x-3h^3s_2^3x_n\\
&+5h^3s_2^2s_3x_n-5h^3s_2s_3^2x+5h^3s_2s_3^2x_n+3h^3s_3^3x-3h^3s_3^3x_n+3h^2s_1^2x^2-6h^2s_1^2xx_n\\
&+3h^2s_1^2x_n^2-5h^2s_1s_2x^2+10h^2s_1s_2xx_n-5h^2s_1s_2x_n^2-5h^2s_1s_3x^2+10h^2s_1s_3xx_n\\
&+3h^2s_2^2x^2-6h^2s_2^2xx_n+3h^2s_2^2x_n^2-5h^2s_2s_3x^2+10h^2s_2s_3xx_n-5h^2s_2s_3x_n^2\\
&-6h^2s_3^2xx_n+3h^2s_3^2x_n^2+3hs_1x^3-9hs_1x^2x_n+9hs_1xx_n^2-3hs_1x_n^3+3hs_2x^3\\
&-9hs_2x^2x_n+3h^2s_3^2x^2-5h^2s_1s_3x_n^2-5h^3s_2^2s_3x+9hs_2xx_n^2-3hs_2x_n^3+3hs_3x^3\\
&+15h^4s_1s_2s_3^2-9hs_3x^2x_n+9hs_3xx_n^2-3x^4+12x^3x_n-18x^2x_n^2+12xx_n^3)
\end{aligned}
$$

Evaluating Equation (5.9) at the non-interpolating point x_{n+1} produces

$$
\begin{aligned}
& y_{n+1} - \frac{((s_2-1)(s_3-1))}{(s_1(s_1-s_2)(s_1-s_3))}\, y_{n+s_1} + \frac{((s_1-1)(s_3-1))}{(s_2(s_1-s_2)(s_2-s_3))}\, y_{n+s_2} \\
& - \frac{((s_1-1)\ (s_2-1))}{(s_3(s_1-s_3)\ (s_2-s_3))}\, y_{n+s_3} = \frac{(s_1-1)\ (s_2-1)\ (s_3-1)}{(s_1 s_2 s_3)}\, y_n \\
& + \frac{-h^4(s_1-1)(s_2-1)(s_3-1)}{(5040 s_1 s_2 s_3)} (5s_1^3 s_2 + 5s_1^3 s_3 + 5s_1 - 15s_1 s_2^2 s_3 + 5s_3^2 + 5s_3 \\
& -3s_1^4 + 5s_1^3 + 5s_1^2 s_2^2 - 15s_1^2 s_2 s_3 - 15s_1^2 s_2 + 5s_1^2 s_3^2 - 15s_1^2 s_3 + 5s_1^2 + 5s_1 s_2^3 - 3s_2^4 \\
& +5s_2^3 - 15s_1 s_2^2 - 15s_1 s_2 s_3^2 - 15s_1 s_2 + 5s_1 s_3^3 - 15s_1 s_3^2 - 15s_1 s_3 + 5s_3^3 - 3s_3^4 + 5s_2 \\
& +105 s_1 s_2 s_3 + 5s_2^3 s_3 + 5s_2^2 s_3^2 - 15s_2^2 s_3 + 5s_2^2 + 5s_2 s_3^3 - 15s_2 s_3^2 - 15s_2 s_3 - 3)\, f_n \\
& - \frac{h^4(s_2-1)(s_3-1)}{5040 s_1(s_1-s_2)(s_1-s_3)} (3s_1^4 - 3s_1^3 s_2 - 3s_1^3 s_3 - 3s_1^3 - 3s_1^2 s_2^2 + 5s_1^2 s_2 s_3 + 5s_1 s_2 \\
& +5s_1^2 s_2 - 3s_1^2 s_3^2 + 5s_1^2 s_3 - 3s_1^2 - 3s_1 s_2^3 + 5s_1 s_2^2 s_3 + 5s_1 s_2^2 + 5s_1 s_2 s_3^2 - 15s_1 s_2 s_3 \\
& -3s_1 s_3^3 + 5s_1 s_3^2 + 5s_1 s_3 - 3s_1 - 3s_2^4 + 5s_2^3 s_3 + 5s_2^3 + 5s_2^2 s_3^2 - 15s_2^2 s_3 + 5s_2^2 + 5s_2^2 \\
& +5s_2 s_3^3 - 15s_2 s_3^2 - 15s_2 s_3 + 5s_2 - 3s_3^4 + 5s_3^3 + 5s_3^2 + 5s_3 - 3)\, f_{n+s_1} \\
& + \frac{h^4(s_1-1)(s_3-1)}{5040 s_2(s_1-s_2)(s_2-s_3)} (5s_1^3 - 3s_1^4 - 3s_1^3 s_2 + 5s_1^3 s_3 - 3s_1^2 s_2^2 + 5s_1^2 s_2 s_3 + 5s_1^2 s_2 \\
& +5s_1^2 s_3^2 - 15s_1^2 s_3 + 5s_1^2 - 3s_1 s_2^3 + 5s_1 s_2^2 s_3 + 5s_1 s_2^2 + 5s_1 s_2 s_3^2 - 15s_1 s_2 s_3 + 5s_1 s_2 \\
& +5s_1 s_3^3 - 15s_1 s_3^2 - 15s_1 s_3 + 5s_1 + 3s_2^4 - 3s_2^3 s_3 - 3s_2^3 - 3s_2^2 s_3^2 + 5s_2^2 s_3 - 3s_2^2 + 5s_3 \\
& -3s_2 s_3^3 + 5s_2 s_3^2 + 5s_2 s_3 - 3s_2 - 3s_3^4 + 5s_3^3 + 5s_3^2 - 3)\, f_{n+s_2} \\
& - \frac{h^4(s_1-1)(s_2-1)}{(5040 s_3(s_1-s_3)(s_2-s_3))} (-3s_1^4 + 5s_1^3 s_2 - 3s_1^3 s_3 + 5s_1^3 + 5s_1^2 s_2^2 + 5s_1^2 s_2 s_3 - 3s_3 \\
& -15s_1^2 s_2 - 3s_1^2 s_3^2 + 5s_1^2 s_3 + 5s_1^2 + 5s_1 s_2^3 + 5s_1 s_2^2 s_3 - 15s_1 s_2^2 + 5s_1 s_2 s_3^2 - 15s_1 s_2 s_3 \\
& -15s_1 s_2 - 3s_1 s_3^3 + 5s_1 s_3^2 + 5s_1 s_3 + 5s_1 - 3s_2^4 - 3s_2^3 s_3 + 5s_2^3 - 3s_2^2 s_3^2 + 5s_2^2 s_3 + 5s_2^2 \\
& -3s_2 s_3^3 + 5s_2 s_3^2 + 5s_2 s_3 + 5s_2 + 3s_3^4 - 3s_3^3 - 3s_3^2 - 3)\, f_{n+s_3} \\
& + \frac{h^4}{5040} (-3s_1^4 + 5s_1^3 s_2 + 5s_1^3 s_3 - 3s_1^3 + 5s_1^2 s_2^2 - 15s_1^2 s_2 s_3 + 5s_1^2 s_2 + 5s_1^2 s_3^2 + 5s_1^2 s_3 \\
& -3s_1^2 + 5s_1 s_2^3 - 15s_1 s_2^2 s_3 + 5s_1 s_2^2 - 15s_1 s_2 s_3^2 - 15s_1 s_2 s_3 + 5s_1 s_2 + 5s_1 s_3^3 + 5s_1 s_3^2 \\
& +5s_1 s_3 - 3s_1 - 3s_2^4 + 5s_2^3 s_3 - 3s_2^3 + 5s_2^2 s_3^2 + 5s_2^2 s_3 - 3s_2^2 + 5s_2 s_3^3 + 5s_2 s_3^2 + 5s_2 s_3 \\
& -3s_2 - 3s_3^4 - 3s_3^3 - 3s_3^2 - 3s_3 + 3)\, f_{n+1} \qquad (5.13)
\end{aligned}
$$

Equation (5.10) is evaluated at all points i.e, x_n, x_{n+s_1}, x_{n+s_2}, x_{n+s_3} and x_{n+1}. This gives the following schemes

$$
\begin{aligned}
&y_n^{'} - \frac{(s_2\ s_3)}{(h\,s_1\,(s_1-s_2)\,(s_1-s_3))}\ y_{n+s_1} + \frac{(s_1\ s_3)}{(hs_2\,(s_1-s_2)\,(s_2-s_3))}\ y_{n+s_2} \\
&-\frac{(s_1s_2)}{(hs_3\,(s_1-s_3)(s_2-s_3))}y_{n+s_3} = -\frac{(s_1s_2+s_1s_3+s_2s_3)}{(hs_1s_2s_3)}+\frac{h^3}{5040}(3s_1^4-5s_1^3s_2 \\
&-8s_1^3-5s_1^2s_2^2+15s_1^2s_2s_3+20s_1^2s_2-5s_1^2s_3^2+20s_1^2s_3-5s_1s_2^3+15s_1s_2^2s_3+20s_1s_2^2 \\
&+15s_1s_2s_3^2-120s_1s_2s_3-5s_1s_3^3+20s_1s_3^2+3s_2^4-5s_2^3s_3-8s_2^3-5s_1^3s_3-5s_2^2s_3^2 \\
&+3s_3^4+20s_2^2s_3-5s_2s_3^3+20s_2s_3^2-8s_3^3)f_n \\
&-\frac{h^3s_2s_3}{5040(s_1-s_2)(s_1-s_3)(s_1-1)}(3s_1^4-3s_1^3s_2-3s_1^3s_3-3s_1^2s_2^2+8s_3^3+5s_2^2s_3^2 \\
&+5s_1^2s_2s_3+8s_1^2s_2-3s_1^2s_3^2+8s_1^2s_3-3s_1s_2^3+5s_1s_2^2s_3+8s_1s_2^2+5s_1s_2s_3^2-20s_1s_2s_3 \\
&-6s_1^3-3s_1s_3^3+8s_1s_3^2-3s_2^4+5s_2^3s_3+8s_2^3-20s_2^2s_3+5s_2s_3^3-20s_2s_3^2-3s_3^4)f_{n+s_1} \\
&+\frac{h^3s_1s_3}{5040(s_1-s_2)(s_2-s_3)(s_2-1)}(-3s_1^4-3s_1^3s_2+5s_1^3s_3-3s_1^2s_2^2-3s_3^4+5s_1^2s_2s_3 \\
&+8s_1^3+8s_1^2s_2+5s_1^2s_3^2-20s_1^2s_3-3s_1s_2^3+5s_1s_2^2s_3+8s_1s_2^2+5s_1s_2s_3^2-20s_1s_2s_3 \\
&+5s_1s_3^3-20s_1s_3^2+3s_2^4-3s_2^3s_3-6s_2^3-3s_2^2s_3^2+8s_2^2s_3-3s_2s_3^3+8s_2s_3^2+8s_3^3)f_{n+s_2} \\
&-\frac{h^3s_1s_2}{5040(s_1-s_3)(s_2-s_3)(s_3-1)}(3s_3^4+5s_1^3s_2-3s_1^3s_3+8s_1^3+5s_1^2s_2^2+5s_1^2s_2s_3 \\
&-20s_1^2s_2-3s_1^2s_3^2+8s_1^2s_3+5s_1s_2^3+5s_1s_2^2s_3-20s_1s_2^2+5s_1s_2s_3^2-20s_1s_2s_3-3s_1s_3^3 \\
&+8s_1s_3^2-3s_2^4-3s_2^3s_3+8s_2^3-3s_2^2s_3^2+8s_2^2s_3-3s_2s_3^3+8s_2s_3^2-6s_3^3-3s_1^4)\ f_{n+s_3} \\
&+\frac{h^3s_1s_2s_3}{5040(s_1-1)(s_2-1)(s_3-1)}(5s_2^2s_3^25s_2^3s_3-3s_1^4+5s_1^3s_2+5s_1^3s_3+5s_1^2s_2^2-3s_3^4 \\
&-15s_1^2s_2s_3+5s_1^2s_3^2+5s_1s_2^3-15s_1s_2^2s_3-15s_1s_2s_3^2+5s_1s_3^3-3s_2^4+5s_2s_3^3)f_{n+1}
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
&y'_{n+s_1} - \frac{(s_2s_3-2s_1s_3-2s_1s_2+3s_1^2)}{(hs_1(s_1-s_2)(s_1-s_3))} y_{n+s_1} + \frac{s_1(s_1-s_3)}{(hs_2(s_1-s_2)(s_2-s_3))} y_{n+s_2}\\
&- \frac{(s_1 \quad (s_1-s_2))}{(hs_3 \quad (s_1-s_3) \quad (s_2-s_3))} y_{n+s_3} = -\frac{((s_1-s_2) \quad (s_1-s_3))}{(hs_1s_2s_3)} y_n\\
&-\frac{h^3(s_1-s_2)(s_1-s_3)}{(5040s_2s_3)}(20s_2s_3^2+9s_1^4-12s_1^3s_2-12s_1^3s_3-18s_1^3-7s_1^2s_2^2+25s_1^2s_2s_3\\
&+32s_1^2s_2-7s_1^2s_3^2+32s_1^2s_3-2s_1s_2^3+10s_1s_2^2s_3+12s_1s_2^2+10s_1s_2s_3^2+3s_3^4-8s_3^3\\
&-100s_1s_2s_3-2s_1s_3^3+12s_1s_3^2+3s_2^4-5s_2^3s_3-8s_2^3-5s_2^2s_3^2+20s_2^2s_3-5s_2s_3^3)\ f_n\\
&+\frac{h^3}{5040(s_1-1)} \quad (15s_1^4-12s_1^3s_2-12s_1^3s_3-24s_1^3+5s_2s_3^3-20s_2s_3^2-3s_3^4+8s_3^3\\
&-9s_1^2s_2^2+15s_1^2s_2s_3+24s_1^2s_2-9s_1^2s_3^2+24s_1^2s_3-6s_1s_2^3+10s_1s_2^2s_3+16s_1s_2^2+8s_2^3\\
&+10s_1s_2s_3^2-40s_1s_2s_3-6s_1s_3^3+16s_1s_3^2-3s_2^4+5s_2^3s_3+5s_2^2s_3^2-20s_2^2s_3)\ f_{n+s_1}\\
&-\frac{(h^3s_1(s_1-s_3)}{(5040s_2(s_2-s_3) \quad (s_2-1))} \quad (-9s_1^4-6s_1^3s_2-3s_2s_3^3+8s_2s_3^2-3s_3^4+8s_3^3\\
&+12s_1^3s_3+18s_1^3-3s_1^2s_2^2+7s_1^2s_2s_3+10s_1^2s_2+7s_1^2s_3^2-32s_1^2s_3+2s_1s_2^2s_3+2s_1s_2^2\\
&+2s_1s_2s_3^2-12s_1s_2s_3+2s_1s_3^3-12s_1s_3^2+3s_2^4-3s_2^3s_3-6s_2^3-3s_2^2s_3^2+8s_2^2s_3)f_{n+s_2}\\
&+\frac{h^3s_1(s_1-s_2)}{(5040s_3(s_2-s_3)(s_3-1))} \quad (-9s_1^4+12s_1^3s_2-6s_1^3s_3+18s_1^3+7s_1^2s_2^2+7s_1^2s_2s_3\\
&-32s_1^2s_2-3s_1^2s_3^2+10s_1^2s_3+2s_1s_2^3+2s_1s_2^2s_3-12s_1s_2^2+2s_1s_2s_3^2-12s_1s_2s_3-3s_2^4\\
&+2s_1s_3^2-3s_2^3s_3+8s_2^3-3s_2^2s_3^2+8s_2^2s_3-3s_2s_3^3+8s_2s_3^2+3s_3^4-6s_3^3+2s_1s_3^2)f_{n+s_3}\\
&-\frac{(h^3s_1(s_1-s_2)(s_1-s_3)}{5040(s_1-1)(s_2-1)(s_3-1)}(12s_1^3s_2-9s_1^4+12s_1^3s_3-25s_1^2s_2s_3+7s_1^2s_3^2-3s_3^4\\
&+7s_1^2s_2^2+2s_1s_2^3-10s_1s_2^2s_3-10s_1s_2s_3^2+2s_1s_3^3-3s_2^4+5s_2^3s_3+5s_2^2s_3^2+5s_2s_3^3)f_{n+1}
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
& y'_{n+s_2} - \frac{(s_2(s_2-s_3))}{(hs_1(s_1-s_2)(s_1-s_3))}\, y_{n+s_1} + \frac{(s_1s_3-2s_1s_2-2s_2s_3+3s_2^2)}{(hs_2(s_1-s_2)(s_2-s_3))}\, y_{n+s_2} \\
& + \frac{(s_2(s_1-s_2))}{(hs_3(s_1-s_3)(s_2-s_3))}\, y_{n+s_3} = \frac{(s_1-s_2)(s_2-s_3)}{(hs_1s_2s_3)}\, y_n \\
& + \frac{h^3(s_1-s_2)(s_2-s_3)}{(5040s_1s_3)}\,(3s_1^4-2s_1^3s_2-5s_1^3s_3-8s_1^3-7s_1^2s_2^2+10s_1^2s_2s_3+12s_1^2s_2 \\
& -5s_1^2s_3^2+20s_1^2s_3-12s_1s_2^3+25s_1s_2^2s_3+32s_1s_2^2+10s_1s_2s_3^2-100s_1s_2s_3-5s_1s_3^3 \\
& +20s_1s_3^2+9s_2^4-12s_2^3s_3-18s_2^3-7s_2^2s_3^2+32s_2^2s_3-2s_2s_3^3+12s_2s_3^2+3s_3^4-8s_3^3)f_n \\
& -\frac{h^3s_2(s_2-s_3)}{(5040s_1(s_1-s_3)(s_1-1))}(3s_1^4-3s_1^3s_3-6s_1^3-3s_1^2s_2^2+2s_1^2s_2s_3+2s_1^2s_2-3s_1^2s_3^2 \\
& +8s_1^2s_3-6s_1s_2^3+7s_1s_2^2s_3+10s_1s_2^2+2s_1s_2s_3^2-12s_1s_2s_3-3s_1s_3^3+8s_1s_3^2-9s_2^4 \\
& +12s_2^3s_3+18s_2^3+7s_2^2s_3^2-32s_2^2s_3+2s_2s_3^3-12s_2s_3^2-3s_3^4+8s_3^3)f_{n+s_1} \\
& +\frac{h^3}{5040(s_2-1)}\,(8s_3^3-3s_1^4-6s_1^3s_2+5s_1^3s_3+8s_1^3-9s_1^2s_2^2+10s_1^2s_2s_3+16s_1^2s_2 \\
& +5s_1^2s_3^2-20s_1^2s_3-12s_1s_2^3+15s_1s_2^2s_3+24s_1s_2^2+10s_1s_2s_3^2-40s_1s_2s_3+5s_1s_3^3 \\
& -20s_1s_3^2+15s_2^4-12s_2^3s_3-24s_2^3-9s_2^2s_3^2+24s_2^2s_3-6s_2s_3^3+16s_2s_3^2-3s_3^4)\, f_{n+s_2} \\
& -\frac{h^3s_2(s_1-s_2)}{(5040s_3(s_1-s_3)(s_3-1))}(-3s_1^4+2s_1^3s_2-3s_1^3s_3+8s_1^3+7s_1^2s_2^2+2s_1^2s_2s_3-6s_3^3 \\
& -12s_1^2s_2-3s_1^2s_3^2+8s_1^2s_3+12s_1s_2^3+7s_1s_2^2s_3-32s_1s_2^2+2s_1s_2s_3^2-12s_1s_2s_3+3s_3^4 \\
& -3s_1s_3^3+8s_1s_3^2-9s_2^4-6s_2^3s_3+18s_2^3-3s_2^2s_3^2+10s_2^2s_3+2s_2s_3^2)f_{n+s_3} \\
& +\frac{h^3s_2(s_1-s_2)(s_2-s_3)}{5040(s_1-1)(s_2-1)(s_3-1)}\,(2s_2s_3^3-3s_1^4+2s_1^3s_2+5s_1^3s_3+7s_1^2s_2^2-10s_1^2s_2s_3 \\
& +5s_1^2s_3^2+12s_1s_2^3-25s_1s_2^2s_3-10s_1s_2s_3^2+5s_1s_3^3-9s_2^4+12s_2^3s_3+7s_2^2s_3^2-3s_3^4)f_{n+1}
\end{aligned}
\tag{5.16}
$$

$$
\begin{aligned}
& y'_{n+s_3} + \frac{(s_3(s_2-s_3))}{(hs_1(s_1-s_2)(s_1-s_3))} y_{n+s_1} - \frac{s_3(s_1-s_3)}{(hs_2(s_1-s_2)(s_2-s_3))} y_{n+s_2} \\
& -\frac{(s_1s_2-2s_1s_3-2s_2s_3+3s_3^2)}{(hs_3(s_1-s_3)(s_2-s_3))} y_{n+s_3} = -\frac{(s_1-s_3)(s_2-s_3)}{(hs_1s_2s_3)} y_n \\
& -\frac{h^3(s_1-s_3)(s_2-s_3)}{(5040s_1s_2)} (3s_1^4 - 5s_1^3s_2 - 2s_1^3s_3 - 8s_1^3 - 5s_1^2s_2^2 + 10s_1^2s_2s_3 + 20s_1^2s_2 \\
& -7s_1^2s_3^2 + 12s_1^2s_3 - 5s_1s_2^3 + 10s_1s_2^2s_3 + 20s_1s_2^2 + 25s_1s_2s_3^2 - 100s_1s_2s_3 - 12s_1s_3^3 \\
& +32s_1s_3^2 + 3s_2^4 - 2s_2^3s_3 - 8s_2^3 - 7s_2^2s_3^2 + 12s_2^2s_3 - 12s_2s_3^3 + 32s_2s_3^2 + 9s_3^4 - 18s_3^3) f_n \\
& +\frac{h^3s_3(s_2-s_3)}{(5040s_1(s_1-s_2)(s_1-1))}(3s_1^4 - 3s_1^3s_2 - 6s_1^3 - 3s_1^2s_2^2 + 2s_1^2s_2s_3 + 8s_1^2s_2 - 3s_1^2s_3^2 \\
& +2s_1^2s_3 - 3s_1s_2^3 + 2s_1s_2^2s_3 + 8s_1s_2^2 + 7s_1s_2s_3^2 - 12s_1s_2s_3 - 6s_1s_3^3 + 10s_1s_3^2 - 3s_2^4 \\
& +2s_2^3s_3 + 8s_2^3 + 7s_2^2s_3^2 - 12s_2^2s_3 + 12s_2s_3^3 - 32s_2s_3^2 - 9s_3^4 + 18s_3^3) f_{n+s_1} \\
& -\frac{h^3s_3(s_1-s_3)}{5040s_2(s_1-s_2)(s_2-1)}(-3s_1^4 - 3s_1^3s_2 + 2s_1^3s_3 + 8s_1^3 - 3s_1^2s_2^2 + 2s_1^2s_2s_3 + 8s_1^2s_2 \\
& +7s_1^2s_3^2 - 12s_1^2s_3 - 3s_1s_2^3 + 2s_1s_2^2s_3 + 8s_1s_2^2 + 7s_1s_2s_3^2 - 12s_1s_2s_3 + 12s_1s_3^3 + 18s_3^3 \\
& -32s_1s_3^2 + 3s_2^4 - 6s_2^3 - 3s_2^2s_3^2 + 2s_2^2s_3 - 6s_2s_3^3 + 10s_2s_3^2 - 9s_3^4) f_{n+s_2} \\
& +\frac{h^3}{5040(s_3-1)} (15s_1s_2s_3^2 + 5s_1^3s_2 - 6s_1^3s_3 + 8s_1^3 + 5s_1^2s_2^2 + 10s_1^2s_2s_3 - 20s_1^2s_2 \\
& -24s_3^3 - 9s_1^2s_3^2 + 16s_1^2s_3 + 5s_1s_2^3 + 10s_1s_2^2s_3 - 20s_1s_2^2 - 40s_1s_2s_3 - 12s_1s_3^3 - 3s_1^4 \\
& +24s_1s_3^2 - 3s_2^4 - 6s_2^3s_3 + 8s_2^3 - 9s_2^2s_3^2 + 16s_2^2s_3 - 12s_2s_3^3 + 24s_2s_3^2 + 15s_3^4) f_{n+s_3} \\
& -\frac{h^3s_3(s_1-s_3)(s_2-s_3)}{5040(s_1-1)(s_2-1)(s_3-1)}(7s_1^2s_3^2 + 2s_1^3s_3 + 5s_1^2s_2^2 - 10s_1^2s_2s_3 + 12s_1s_3^3 - 3s_1^4 \\
& +5s_1^3s_2 + 5s_1s_2^3 - 10s_1s_2^2s_3 - 25s_1s_2s_3^2 - 3s_2^4 + 2s_2^3s_3 + 7s_2^2s_3^2 + 12s_2s_3^3 - 9s_3^4) f_{n+1}
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
&y'_{n+1} + \frac{(2s_2+2s_3-s_2s_3-3)}{(hs_1(s_1-s_2)(s_1-s_3))}\, y_{n+s_1} + \frac{(s_1s_3-2s_3-2s_1+3)}{(hs_2(s_1-s_2)(s_2-s_3))}\, y_{n+s_2}\\
&+\frac{(2s_1+2s_2-s_1s_2-3)}{(hs_3(s_1-s_3)(s_2-s_3))}\, y_{n+s_3} = -\frac{(s_1s_2-2s_2-2s_3-2s_1+s_1s_3+s_2s_3+3)}{(hs_1s_2s_3)}\, y_n\\
&+\frac{h^3}{(5040s_1s_2s_3)}\,(3s_1^5s_2s_3-6s_1^5s_2-6s_1^5s_3+9s_1^5-5s_1^4s_2^2s_3+10s_1^4s_2^2-5s_1^4s_2s_3^2\\
&+12s_1^4s_2s_3+s_1^4s_2+10s_1^4s_3^2+s_1^4s_3-24s_1^4-5s_1^3s_2^3s_3+10s_1^3s_2^3+15s_1^3s_2^2s_3^2-55s_1^3s_2^2\\
&-5s_1^3s_2s_3^3-35s_1^3s_2s_3+60s_1^3s_2+10s_1^3s_3^3-55s_1^3s_3^2+60s_1^3s_3-5s_1^2s_2^4s_3+10s_1^2s_2^4\\
&+15s_1^2s_2^3s_3^2-55s_1^2s_2^3+15s_1^2s_2^2s_3^3-180s_1^2s_2^2s_3^2+245s_1^2s_2^2s_3+60s_1^2s_2^2-5s_1^2s_2s_3^4\\
&+245s_1^2s_2s_3^2-360s_1^2s_2s_3+10s_1^2s_3^4-55s_1^2s_3^3+60s_1^2s_3^2+3s_1s_2^5s_3-6s_1s_2^5-126s_2s_3\\
&-5s_1s_2^4s_3^2+12s_1s_2^4s_3+s_1s_2^4-5s_1s_2^3s_3^3-35s_1s_2^3s_3+60s_1s_2^3-5s_1s_2^2s_3^4+245s_1s_2^2s_3^2\\
&-360s_1s_2^2s_3+3s_1s_2s_3^5+12s_1s_2s_3^4-35s_1s_2s_3^3-360s_1s_2s_3^2+630s_1s_2s_3-126s_1s_2\\
&+60s_1s_3^3-126s_1s_3-6s_2^5s_3+9s_2^5+10s_2^4s_3^2+s_2^4s_3-24s_2^4+10s_2^3s_3^3-55s_2^3s_3^2\\
&-6s_1s_3^5+s_1s_3^4+60s_2^3s_3+10s_2^2s_3^4-55s_2^2s_3^3+60s_2^2s_3^2-6s_2s_3^5+s_2s_3^4+60s_2s_3^3\\
&+42s_2+9s_3^5-24s_3^4+42s_1+42s_3-18)f_n\\
&-\frac{h^3}{5040s_1(s_1-s_2)(s_1-s_3)(s_1-1)}\,(3s_1^5s_2s_3-6s_1^5s_2-6s_1^5s_3+9s_1^5-3s_1^4s_2^2s_3\\
&+6s_1^4s_2^2-3s_1^4s_2s_3^2+6s_1^4s_2s_3+3s_1^4s_2+6s_1^4s_3^2+3s_1^4s_3-18s_1^4-3s_1^3s_2^3s_3+6s_1^3s_2^3\\
&+5s_1^3s_2^2s_3^2+4s_1^3s_2^2s_3-25s_1^3s_2^2-3s_1^3s_2s_3^3+4s_1^3s_2s_3^2-17s_1^3s_2s_3+24s_1^3s_2+6s_1^3s_3^3\\
&+6s_1^2s_2^4+5s_1^2s_2^3s_3^2+4s_1^2s_2^3s_3-25s_1^2s_2^3+5s_1^2s_2^2s_3^3-40s_1^2s_2^2s_3^2+39s_1^2s_2^2s_3+24s_1^2s_2^2\\
&-3s_1^2s_2s_3^4+4s_1^2s_2s_3^3+39s_1^2s_2s_3^2-60s_1^2s_2s_3+6s_1^2s_3^4-25s_1^2s_3^3+24s_1^2s_3^2-3s_1s_2^5s_3\\
&-25s_1^3s_3^2+24s_1^3s_3-3s_1^2s_2^4s_3+6s_1s_2^5+5s_1s_2^4s_3^2+4s_1s_2^4s_3-25s_1s_2^4+5s_1s_2^3s_3^3\\
&-40s_1s_2^3s_3^2+39s_1s_2^3s_3+24s_1s_2^3+5s_1s_2^2s_3^4-40s_1s_2^2s_3^3+95s_1s_2^2s_3^2-60s_1s_2^2s_3\\
&+4s_1s_2s_3^4+39s_1s_2s_3^3-60s_1s_2s_3^2+6s_1s_3^5-25s_1s_3^4+24s_1s_3^3+6s_2^5s_3-9s_2^5+6s_2s_3^5\\
&-10s_2^4s_3^2-s_2^4s_3+24s_2^4-10s_2^3s_3^3+55s_2^3s_3^2-60s_2^3s_3-10s_2^2s_3^4+55s_2^2s_3^3-60s_2^2s_3^2\\
&-3s_1s_2s_3^5-s_2s_3^4-60s_2s_3^3+126s_2s_3-42s_2-9s_3^5+24s_3^4-42s_3+18)f_{n+s_1}\\
&+\frac{h^3}{(5040s_2(s_1-s_2)(s_2-s_3)(s_2-1))}\,(-3s_1^5s_2s_3+6s_1^5s_2+6s_1^5s_3-9s_1^5-3s_1^4s_2^2s_3\\
&-10s_1^4s_3^2-s_1^4s_3+24s_1^4-3s_1^3s_2^3s_3+6s_1^3s_2^3+5s_1^3s_2^2s_3^2+4s_1^3s_2^2s_3-25s_1^3s_2^2+5s_1^3s_2s_3^3
\end{aligned}
$$

$$-40s_1^3s_2s_3^2+39s_1^3s_2s_3+24s_1^3s_2-10s_1^3s_3^3+55s_1^3s_3^2-60s_1^3s_3-3s_1^2s_2^4s_3+6s_1^2s_2^4$$
$$+4s_1^2s_2^3s_3-25s_1^2s_2^3+5s_1^2s_2^2s_3^3-40s_1^2s_2^2s_3^2+39s_1^2s_2^2s_3+24s_1^2s_2^2+5s_1^2s_2s_3^4-40s_1^2s_2s_3^3$$
$$+5s_1^2s_2^3s_3^2+6s_1^4s_2^2+5s_1^4s_2s_3^2+4s_1^4s_2s_3-25s_1^4s_2+95s_1^2s_2s_3^2-60s_1^2s_2s_3-10s_1^2s_3^4$$
$$+55s_1^2s_3^3-60s_1^2s_3^2+3s_1s_2^5s_3-6s_1s_2^5-3s_1s_2^4s_3^2+6s_1s_2^4s_3+3s_1s_2^4-3s_1s_2^3s_3^3$$
$$-17s_1s_2^3s_3+24s_1s_2^3-3s_1s_2^2s_3^4+4s_1s_2^2s_3^3+39s_1s_2^2s_3^2-60s_1s_2^2s_3-3s_1s_2s_3^5-9s_3^5$$
$$-60s_1s_2s_3^2+6s_1s_3^5-s_1s_3^4-60s_1s_3^3+126s_1s_3-42s_1-6s_2^5s_3+9s_2^5+6s_2^4s_3^2$$
$$-18s_2^4+6s_2^3s_3^3-25s_2^3s_3^2+24s_2^3s_3+6s_2^2s_3^4-25s_2^2s_3^3+24s_2^2s_3^2+6s_2s_3^5-25s_2s_3^4$$
$$+3s_2^4s_3+4s_1s_2^3s_3^2+4s_1s_2s_3^4+24s_2s_3^3+39s_1s_2s_3^3+24s_3^4-42s_3+18)\quad f_{n+s_2}$$
$$-\frac{h^3}{(5040s_3(s_1-s_3)(s_2-s_3)(s_3-1))}\ (-3s_1^5s_2s_3+6s_1^5s_2+6s_1^5s_3-9s_1^5+5s_1^4s_2^2s_3$$
$$-10s_1^4s_2^2-3s_1^4s_2s_3^2+4s_1^4s_2s_3-s_1^4s_2+6s_1^4s_3^2-25s_1^4s_3+24s_1^4+5s_1^3s_2^3s_3-10s_1^3s_2^3$$
$$+5s_1^3s_2^2s_3^2-40s_1^3s_2^2s_3+55s_1^3s_2^2-3s_1^3s_2s_3^3+4s_1^3s_2s_3^2+39s_1^3s_2s_3-60s_1^3s_2+6s_1^3s_3^3$$
$$+24s_1^3s_3+5s_1^2s_2^4s_3-10s_1^2s_2^4+5s_1^2s_2^3s_3^2-40s_1^2s_2^3s_3+55s_1^2s_2^3+5s_1^2s_2^2s_3^3-40s_1^2s_2^2s_3^2$$
$$-60s_1^2s_2^2-3s_1^2s_2s_3^4+4s_1^2s_2s_3^3+39s_1^2s_2s_3^2-60s_1^2s_2s_3+6s_1^2s_3^4-25s_1^2s_3^3+24s_1^2s_3^2$$
$$+6s_1s_2^5-3s_1s_2^4s_3^2+4s_1s_2^4s_3-s_1s_2^4-3s_1s_2^3s_3^3+4s_1s_2^3s_3^2+39s_1s_2^3s_3-60s_1s_2^3$$
$$+4s_1s_2^2s_3^3+39s_1s_2^2s_3^2-60s_1s_2^2s_3+3s_1s_2s_3^5+6s_1s_2s_3^4-17s_1s_2s_3^3-60s_1s_2s_3^2$$
$$+3s_1s_3^4+24s_1s_3^3-42s_1+6s_2^5s_3-9s_2^5+6s_2^4s_3^2-25s_2^4s_3+24s_2^4+6s_2^3s_3^3-25s_1^3s_3^2$$
$$-6s_1s_3^5-3s_1s_2^5s_3-3s_1s_2^2s_3^4+95s_1^2s_2^2s_3-25s_2^3s_3^2+24s_2^3s_3+6s_2^2s_3^4-25s_2^2s_3^3$$
$$+126s_1s_2+24s_2^2s_3^2-6s_2s_3^5+3s_2s_3^4+24s_2s_3^3-42s_2+9s_3^5-18s_3^4+18)\quad f_{n+s_3}$$
$$+\frac{h^3}{5040(s_1-1)(s_2-1)(s_3-1)}(6s_1^5s_2-3s_1^5s_2s_3+6s_1^5s_3-9s_1^5+5s_1^4s_2^2s_3-10s_1^4s_2^2$$
$$+5s_1^4s_2s_3^2-20s_1^4s_2s_3+15s_1^4s_2-10s_1^4s_3^2+15s_1^4s_3+5s_1^3s_2^3s_3-10s_1^3s_2^3-15s_1^3s_2^2s_3^2$$
$$+20s_1^3s_2^2s_3+15s_1^3s_2^2+5s_1^3s_2s_3^3+20s_1^3s_2s_3^2-45s_1^3s_2s_3-10s_1^3s_3^3+15s_1^3s_3^2+5s_1^2s_2^4s_3$$
$$-10s_1^2s_2^4-15s_1^2s_2^3s_3^2+20s_1^2s_2^3s_3+15s_1^2s_2^3-15s_1^2s_2^2s_3^3+60s_1^2s_2^2s_3^2-45s_1^2s_2^2s_3-9s_3^5$$
$$+20s_1^2s_2s_3^3-45s_1^2s_2s_3^2-10s_1^2s_3^4+15s_1^2s_3^3-3s_1s_2^5s_3+6s_1s_2^5+5s_1s_2^4s_3^2-20s_1s_2^4s_3$$
$$+15s_1s_2^4+5s_1s_2^3s_3^3+20s_1s_2^3s_3^2-45s_1s_2^3s_3+5s_1s_2^2s_3^4+20s_1s_2^2s_3^3-45s_1s_2^2s_3^2+42s_1$$
$$-84s_2s_3-20s_1s_2s_3^4-45s_1s_2s_3^3+210s_1s_2s_3-84s_1s_2+6s_1s_3^5+15s_1s_3^4-84s_1s_3$$

$$+6s_2^5s_3-9s_2^5-3s_1s_2s_3^5+5s_1^2s_2s_3^4-10s_2^4s_3^2+15s_2^4s_3-10s_2^3s_3^3+15s_2^3s_3^2+6s_2s_3^5$$

$$-10s_2^2s_3^4+15s_2^2s_3^3+15s_2s_3^4+42s_2+42s_3-24)f_{n+1} \tag{5.18}$$

Evaluating Equation (5.11) at all points gives

$$y_n'' + \frac{2(s_2+s_3)}{h^2\, s_1\, (s_1-s_2)\, (s_1-s_3)}\, y_{n+s_1} - \frac{2(s_1+s_3)}{h^2\, s_2\, (s_1-s_2)\, (s_2-s_3)}\, y_{n+s_2}$$

$$+\frac{2(s_1+s_2)}{h^2s_3(s_1-s_3)(s_2-s_3)}y_{n+s_3} = \frac{(2s_1+2s_2+2s_3)}{(h^2s_1s_2s_3)}y_n$$

$$-\frac{h^2}{(2520s_1s_2s_3)}(3s_1^5s_2+3s_1^5s_3-5s_1^4s_2^2-10s_1^4s_2s_3-8s_1^4s_2-5s_1^4s_3^2+3s_1s_3^5-8s_1s_3^4$$

$$-8s_1^4s_3-5s_1^3s_2^3+10s_1^3s_2^2s_3+20s_1^3s_2^2+10s_1^3s_2s_3^2+40s_1^3s_2s_3-5s_1^3s_3^3+3s_2^5s_3-5s_2^4s_3^2$$

$$+20s_1^3s_3^2-5s_1^2s_2^4+10s_1^2s_2^3s_3+20s_1^2s_2^3+30s_1^2s_2^2s_3^2-100s_1^2s_2^2s_3-8s_2^4s_3-100s_1s_2^2s_3^2$$

$$+10s_1^2s_2s_3^3-100s_1^2s_2s_3^2-5s_1^2s_3^4+20s_1^2s_3^3+3s_1s_2^5-10s_1s_2^4s_3-5s_2^3s_3^3+40s_1s_2s_3^3$$

$$+10s_1s_2^3s_3^2+40s_1s_2^3s_3+10s_1s_2^2s_3^3-10s_1s_2s_3^4+20s_2^3s_3^2-5s_2^2s_3^4+20s_2^2s_3^3+3s_2s_3^5$$

$$-8s_2s_3^4-8s_1s_2^4)f_n$$

$$+\frac{h^2}{(2520s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(3s_1^5s_2+3s_1^5s_3-3s_1^4s_2^2-6s_1^4s_2s_3-6s_1^4s_3$$

$$-6s_1^4s_2-3s_1^3s_2^3+2s_1^3s_2^2s_3+8s_1^3s_2^2+2s_1^3s_2s_3^2+16s_1^3s_2s_3-3s_1^3s_3^3+8s_1^3s_3^2-3s_1^4s_3^2$$

$$-3s_1^2s_2^4+2s_1^2s_2^3s_3+8s_1^2s_2^3+10s_1^2s_2^2s_3^2-12s_1^2s_2^2s_3+2s_1^2s_2s_3^3-12s_1^2s_2s_3^2+5s_2^4s_3^2$$

$$-3s_1^2s_3^4+8s_1^2s_3^3-3s_1s_2^5+2s_1s_2^4s_3+8s_1s_2^4+10s_1s_2^3s_3^2-12s_1s_2^3s_3-3s_2s_3^5+8s_2s_3^4$$

$$+10s_1s_2^2s_3^3-40s_1s_2^2s_3^2+2s_1s_2s_3^4-12s_1s_2s_3^3-3s_1s_3^5+8s_1s_3^4-3s_2^5s_3-20s_2^3s_3^2$$

$$+8s_2^4s_3+5s_2^3s_3^3+5s_2^2s_3^4-20s_2^2s_3^3)f_{n+s_1}$$

$$-\frac{h^2}{(2520s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(-3s_1^5s_2-3s_1^5s_3-3s_1^4s_2^2+2s_1^4s_2s_3-3s_2^2s_3^4$$

$$+8s_1^4s_2+5s_1^4s_3^2+8s_1^4s_3-3s_1^3s_2^3+2s_1^3s_2^2s_3+8s_1^3s_2^2+10s_1^3s_2s_3^2-3s_2^3s_3^3+8s_2^3s_3^2$$

$$-12s_1^3s_2s_3+5s_1^3s_3^3-20s_1^3s_3^2-3s_1^2s_2^4+2s_1^2s_2^3s_3+8s_1^2s_2^3+8s_2^2s_3^3-3s_2s_3^5+8s_2s_3^4$$

$$+10s_1^2s_2^2s_3^2-12s_1^2s_2^2s_3+10s_1^2s_2s_3^3-40s_1^2s_2s_3^2+5s_1^2s_3^4-20s_1^2s_3^3-3s_1s_3^5+8s_1s_3^4$$

$$+3s_1s_2^5-6s_1s_2^4s_3-6s_1s_2^4+2s_1s_2^3s_3^2+16s_1s_2^3s_3+2s_1s_2^2s_3^3-12s_1s_2^2s_3^2+2s_1s_2s_3^4$$

$$-12s_1s_2s_3^3+3s_2^5s_3-3s_2^4s_3^2-6s_2^4s_3)f_{n+s_2}$$

$$+\frac{h^2}{(2520s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(-3s_1^5s_2-6s_1s_2s_3^4+16s_1s_2s_3^3+10s_1^2s_2^2s_3^2$$

$$+8s_1^4s_2-3s_1^4s_3^2+8s_1^4s_3+5s_1^3s_2^3+10s_1^3s_2^2s_3-20s_1^3s_2^2+2s_1^3s_2s_3^2+8s_1s_2^4+2s_1s_2^3s_3^2$$

$$-12s_1^3s_2s_3-3s_1^3s_3^3+8s_1^3s_3^2+5s_1^2s_2^4+10s_1^2s_2^3s_3-20s_1^2s_2^3-3s_1^5s_3+3s_1s_3^5-6s_1s_3^4$$

$$-40s_1^2s_2^2s_3-12s_1^2s_2s_3^2-3s_1^2s_3^4+8s_1^2s_3^3-3s_1s_2^5+2s_1s_2^4s_3-3s_2^5s_3-3s_2^4s_3^2+8s_2^4s_3$$

$$+2s_1^2s_2s_3^3-12s_1s_2^3s_3+2s_1s_2^2s_3^3-12s_1s_2^2s_3^2-3s_2^3s_3^3+8s_2^3s_3^2+5s_1^4s_2^2+2s_1^4s_2s_3$$

$$-3s_2^2s_3^4+8s_2^2s_3^3+3s_2s_3^5-6s_2s_3^4)f_{n+s_3}$$

$$+\frac{h^2}{(2520s_1-2520)(s_2-1)(s_3-1)}(3s_1^5s_2+3s_1^5s_3-5s_1^4s_2^2-10s_1s_2^4s_3+10s_1s_2^3s_3^2$$

$$-10s_1^4s_2s_3-5s_1^4s_3^2-5s_1^3s_2^3+10s_1^3s_2^2s_3+10s_1^3s_2s_3^2-5s_1^3s_3^3+10s_1s_2^2s_3^3-10s_1s_2s_3^4$$

$$-5s_1^2s_2^4+10s_1^2s_2^3s_3+30s_1^2s_2^2s_3^2+10s_1^2s_2s_3^3-5s_1^2s_3^4+3s_1s_2^5+3s_1s_3^5+3s_2^5s_3-5s_2^4s_3^2$$

$$-5s_2^3s_3^3-5s_2^2s_3^4+3s_2s_3^5)f_{n+1} \tag{5.19}$$

$$y''_{n+s_1}-\frac{(2(3s_1-s_2-s_3))}{(h^2s_1(s_1-s_2)(s_1-s_3))}y_{n+s_1}+\frac{(2(2s_1-s_3))}{(h^2s_2(s_1-s_2)(s_2-s_3))}y_{n+s_2}$$

$$-\frac{(2(2s_1-s_2))}{(h^2s_3(s_1-s_3)(s_2-s_3))}=\frac{(2s_2-4s_1+2s_3)}{(h^2s_1s_2s_3)}y_n-\frac{h^2}{(2520s_1s_2s_3)}(33s_1^6-66s_1^5s_2$$

$$-66s_1^5s_3-60s_1^5+10s_1^4s_2^2+155s_1^4s_2s_3+142s_1^4s_2+10s_1^4s_3^2+142s_1^4s_3+10s_1^3s_2^3$$

$$-35s_1^3s_2^2s_3-40s_1^3s_2^2-35s_1^3s_2s_3^2-440s_1^3s_2s_3+10s_1^3s_3^3-40s_1^3s_3^2+10s_1^2s_2^4-40s_1^2s_2^3$$

$$-35s_1^2s_2^3s_3-15s_1^2s_2^2s_3^2+260s_1^2s_2^2s_3-35s_1^2s_2s_3^3+260s_1^2s_2s_3^2+10s_1^2s_3^4-40s_1^2s_3^3$$

$$-6s_1s_2^5+5s_1s_2^4s_3+16s_1s_2^4+25s_1s_2^3s_3^2-20s_1s_2^3s_3+25s_1s_2^2s_3^3-160s_1s_2^2s_3^2-6s_1s_3^5$$

$$-20s_1s_2s_3^3+16s_1s_3^4+3s_2^5s_3-5s_2^4s_3^2-8s_2^4s_3-5s_2^3s_3^3+20s_2^3s_3^2-5s_2^2s_3^4+20s_2^2s_3^3$$

$$+3s_2s_3^5+5s_1s_2s_3^4-8s_2s_3^4)f_n$$

$$-\frac{h^2}{(2520s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(-75s_1^6+114s_1^5s_2+114s_1^5s_3+108s_1^5-6s_1^4s_2^2$$

$$-189s_1^4s_2s_3-180s_1^4s_2-6s_1^4s_3^2-180s_1^4s_3-6s_1^3s_2^3+13s_1^3s_2^2s_3+16s_1^3s_2^2+13s_1^3s_2s_3^2$$

$$+344s_1^3s_2s_3-6s_1^3s_3^3+16s_1^3s_3^2-6s_1^2s_2^4+13s_1^2s_2^3s_3+16s_1^2s_2^3+5s_1^2s_2^2s_3^2-48s_1^2s_2^2s_3$$

$$+13s_1^2s_2s_3^3-48s_1^2s_2s_3^2-6s_1^2s_3^4+16s_1^2s_3^3-6s_1s_2^5+13s_1s_2^4s_3+16s_1s_2^4+5s_1s_2^3s_3^2$$

$$-48s_1s_2^3s_3+5s_1s_2^2s_3^3-20s_1s_2^2s_3^2+13s_1s_2s_3^4-48s_1s_2s_3^3-6s_1s_3^5+16s_1s_3^4+3s_2^5s_3$$
$$-5s_2^4s_3^2-8s_2^4s_3-5s_2^3s_3^3+20s_2^3s_3^2-5s_2^2s_3^4+20s_2^2s_3^3+3s_2s_3^5-8s_2s_3^4)f_{n+s_1}$$
$$+\frac{h^2}{(2520s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(33s_1^6-6s_1^5s_2-66s_1^5s_3-6s_1^4s_2^2+13s_1^4s_2s_3$$
$$-60s_1^5+16s_1^4s_2+10s_1^4s_3^2+142s_1^4s_3-6s_1^3s_2^3+13s_1^3s_2^2s_3+16s_1^3s_2^2+5s_1^3s_2s_3^2$$
$$-48s_1^3s_2s_3+10s_1^3s_3^3-40s_1^3s_3^2-6s_1^2s_2^4+13s_1^2s_2^3s_3+16s_1^2s_2^3+5s_1^2s_2^2s_3^2-48s_1^2s_2^2s_3$$
$$+5s_1^2s_2s_3^3-20s_1^2s_2s_3^2+10s_1^2s_3^4-40s_1^2s_3^3+6s_1s_2^5-3s_1s_2^4s_3-12s_1s_2^4-11s_1s_2^3s_3^2$$
$$+8s_1s_2^3s_3-11s_1s_2^2s_3^3+36s_1s_2^2s_3^2-11s_1s_2s_3^4+36s_1s_2s_3^3-6s_1s_3^5+16s_1s_3^4-3s_2^5s_3$$
$$+3s_2^4s_3^2+6s_2^4s_3+3s_2^3s_3^3-8s_2^3s_3^2+3s_2^2s_3^4-8s_2^2s_3^3+3s_2s_3^5-8s_2s_3^4)f_{n+s_2}$$
$$-\frac{h^2}{(2520s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(33s_1^6-66s_1^5s_2-6s_1^5s_3+10s_1^4s_2^2+13s_1^4s_2s_3$$
$$+142s_1^4s_2-6s_1^4s_3^2+16s_1^4s_3+10s_1^3s_2^3+5s_1^3s_2^2s_3-40s_1^3s_2^2+13s_1^3s_2s_3^2-48s_1^3s_2s_3$$
$$-6s_1^3s_3^3+16s_1^3s_3^2+10s_1^2s_2^4+5s_1^2s_2^3s_3-40s_1^2s_2^3+5s_1^2s_2^2s_3^2-20s_1^2s_2^2s_3+13s_1^2s_2s_3^3$$
$$-48s_1^2s_2s_3^2-6s_1^2s_3^4+16s_1^2s_3^3-6s_1s_2^5-11s_1s_2^4s_3+16s_1s_2^4-11s_1s_2^3s_3^2+36s_1s_2^3s_3$$
$$-60s_1^5-11s_1s_2^2s_3^3+36s_1s_2^2s_3^2-3s_1s_2s_3^4+8s_1s_2s_3^3+6s_1s_3^5-12s_1s_3^4+3s_2^5s_3+3s_2^4s_3^2$$
$$-8s_2^4s_3+3s_2^3s_3^3-8s_2^3s_3^2+3s_2^2s_3^4-8s_2^2s_3^3-3s_2s_3^5+6s_2s_3^4)f_{n+s_3}$$
$$+\frac{h^2}{(2520s_1-2520)(s_2-1)(s_3-1)}(33s_1^6-66s_1^5s_2-66s_1^5s_3+10s_1^4s_2^2+155s_1^4s_2s_3$$
$$+10s_1^4s_3^2+10s_1^3s_2^3-35s_1^3s_2^2s_3-35s_1^3s_2s_3^2+10s_1^3s_3^3+10s_1^2s_2^4-35s_1^2s_2^3s_3-15s_1^2s_2^2s_3^2$$
$$-35s_1^2s_2s_3^3+10s_1^2s_3^4-6s_1s_2^5+5s_1s_2^4s_3+25s_1s_2^3s_3^2+25s_1s_2^2s_3^3+5s_1s_2s_3^4-6s_1s_3^5$$
$$+3s_2^5s_3-5s_2^4s_3^2-5s_2^3s_3^3-5s_2^2s_3^4+3s_2s_3^5)f_{n+1}$$

(5.20)

$$y''_{n+s_2} - \frac{(2(2s_2-s_3))}{(h^2 s_1(s_1-s_2)(s_1-s_3))}y_{n+s_1} - \frac{(2(s_1-3s_2+s_3))}{(h^2 s_2(s_1-s_2)(s_2-s_3))}y_{n+s_2}$$

$$+\frac{(2(s_1-2s_2))}{(h^2 s_3(s_1-s_3)(s_2-s_3))}y_{n+s_3} = \frac{(2s_1-4s_2+2s_3)}{(h^2 s_1 s_2 s_3)}y_n$$

$$+\frac{h^2}{(2520 s_1 s_2 s_3)}(6s_1^5 s_2 - 3s_1^5 s_3 - 10s_1^4 s_2^2 - 5s_1^4 s_2 s_3 - 16s_1^4 s_2 + 5s_1^4 s_3^2 + 8s_1^4 s_3$$

$$-10s_1^3 s_2^3 + 35s_1^3 s_2^2 s_3 + 40s_1^3 s_2^2 - 25s_1^3 s_2 s_3^2 + 20s_1^3 s_2 s_3 + 5s_1^3 s_3^3 - 20s_1^3 s_3^2 - 10s_1^2 s_2^4$$

$$+35s_1^2 s_2^3 s_3 + 40s_1^2 s_2^3 + 15s_1^2 s_2^2 s_3^2 - 260s_1^2 s_2^2 s_3 - 25s_1^2 s_2 s_3^3 + 160s_1^2 s_2 s_3^2 + 5s_1^2 s_3^4$$

$$-20s_1^2 s_3^3 + 66s_1 s_2^5 - 155s_1 s_2^4 s_3 - 142s_1 s_2^4 + 35s_1 s_2^3 s_3^2 + 440s_1 s_2^3 s_3 + 35s_1 s_2^2 s_3^3$$

$$-260s_1 s_2^2 s_3^2 - 5s_1 s_2 s_3^4 + 20s_1 s_2 s_3^3 - 3s_1 s_3^5 + 8s_1 s_3^4 - 33s_2^6 + 66s_2^5 s_3 + 60s_2^5 - 10s_2^4 s_3^2$$

$$-142s_2^4 s_3 - 10s_2^3 s_3^3 + 40s_2^3 s_3^2 - 10s_2^2 s_3^4 + 40s_2^2 s_3^3 + 6s_2 s_3^5 - 16s_2 s_3^4)$$

$$-\frac{h^2}{(2520 s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(6s_1^5 s_2 - 3s_1^5 s_3 - 6s_1^4 s_2^2 - 3s_1^4 s_2 s_3 - 12s_1^4 s_2$$

$$+3s_1^4 s_3^2 + 6s_1^4 s_3 - 6s_1^3 s_2^3 + 13s_1^3 s_2^2 s_3 + 16s_1^3 s_2^2 - 11s_1^3 s_2 s_3^2 + 8s_1^3 s_2 s_3 + 3s_1^3 s_3^3 - 8s_1^3 s_3^2$$

$$-6s_1^2 s_2^4 + 13s_1^2 s_2^3 s_3 + 16s_1^2 s_2^3 + 5s_1^2 s_2^2 s_3^2 - 48s_1^2 s_2^2 s_3 - 11s_1^2 s_2 s_3^3 + 36s_1^2 s_2 s_3^2 + 3s_1^2 s_3^4$$

$$-8s_1^2 s_3^3 - 6s_1 s_2^5 + 13s_1 s_2^4 s_3 + 16s_1 s_2^4 + 5s_1 s_2^3 s_3^2 - 48s_1 s_2^3 s_3 + 5s_1 s_2^2 s_3^3 - 20s_1 s_2^2 s_3^2$$

$$-11s_1 s_2 s_3^4 + 36s_1 s_2 s_3^3 + 3s_1 s_3^5 - 8s_1 s_3^4 + 33s_2^6 - 66s_2^5 s_3 - 60s_2^5 + 10s_2^4 s_3^2 + 142s_2^4 s_3$$

$$+10s_2^3 s_3^3 - 40s_2^3 s_3^2 + 10s_2^2 s_3^4 - 40s_2^2 s_3^3 - 6s_2 s_3^5 + 16s_2 s_3^4)f_{n+s_1}$$

$$+\frac{h^2}{(2520 s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(-6s_1^5 s_2 + 3s_1^5 s_3 - 6s_1^4 s_2^2 + 13s_1^4 s_2 s_3 + 16s_1^4 s_2$$

$$-5s_1^4 s_3^2 - 8s_1^4 s_3 - 6s_1^3 s_2^3 + 13s_1^3 s_2^2 s_3 + 16s_1^3 s_2^2 + 5s_1^3 s_2 s_3^2 - 48s_1^3 s_2 s_3 - 5s_1^3 s_3^3 + 20s_1^3 s_3^2$$

$$-6s_1^2 s_2^4 + 13s_1^2 s_2^3 s_3 + 16s_1^2 s_2^3 + 5s_1^2 s_2^2 s_3^2 - 48s_1^2 s_2^2 s_3 + 5s_1^2 s_2 s_3^3 - 20s_1^2 s_2 s_3^2 - 5s_1^2 s_3^4$$

$$+20s_1^2 s_3^3 + 114s_1 s_2^5 - 189s_1 s_2^4 s_3 - 180s_1 s_2^4 + 13s_1 s_2^3 s_3^2 + 344s_1 s_2^3 s_3 + 13s_1 s_2^2 s_3^3$$

$$-48s_1 s_2^2 s_3^2 + 13s_1 s_2 s_3^4 - 48s_1 s_2 s_3^3 + 3s_1 s_3^5 - 8s_1 s_3^4 - 75s_2^6 + 114s_2^5 s_3 + 108s_2^5$$

$$-6s_2^4 s_3^2 - 180s_2^4 s_3 - 6s_2^3 s_3^3 + 16s_2^3 s_3^2 - 6s_2^2 s_3^4 + 16s_2^2 s_3^3 - 6s_2 s_3^5 + 16s_2 s_3^4)f_{n+s_2}$$

$$-\frac{h^2}{(2520 s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(-6s_1^5 s_2 + 36s_1^3 s_2 s_3 - 11s_1^4 s_2 s_3 - 11s_1^3 s_2 s_3^2$$

$$+3s_1^3 s_3^3 + 10s_1^4 s_2^2 + 3s_1^5 s_3 + 16s_1^4 s_2 + 3s_1^4 s_3^2 - 8s_1^4 s_3 + 10s_1^3 s_2^3 + 5s_1^3 s_2^2 s_3 - 40s_1^3 s_2^2$$

$$
\begin{aligned}
&+13s_1s_2^2s_3^3-8s_1^3s_3^2+10s_1^2s_2^4+5s_1^2s_2^3s_3-40s_1^2s_2^3+5s_1^2s_2^2s_3^2-20s_1^2s_2^2s_3-11s_1^2s_2s_3^3\\
&+36s_1^2s_2s_3^2+3s_1^2s_3^4-8s_1^2s_3^3-66s_1s_2^5+13s_1s_2^4s_3+142s_1s_2^4+13s_1s_2^3s_3^2-48s_1s_2^3s_3\\
&-48s_1s_2^2s_3^2-3s_1s_2s_3^4+8s_1s_2s_3^3-3s_1s_3^5+6s_1s_3^4+33s_2^6-6s_2^5s_3-60s_2^5-6s_2^4s_3^2\\
&+16s_2^4s_3-6s_2^3s_3^3+16s_2^3s_3^2-6s_2^2s_3^4+16s_2^2s_3^3+6s_2s_3^5-12s_2s_3^4)f_{n+s_3}\\
&+\frac{h^2}{(2520s_1-2520)(s_2-1)(s_3-1)}(-6s_1^5s_2+3s_1^5s_3+10s_1^4s_2^2+5s_1^4s_2s_3-5s_1^4s_3^2\\
&+10s_1^3s_2^3-35s_1^3s_2^2s_3+25s_1^3s_2s_3^2-5s_1^3s_3^3+10s_1^2s_2^4-35s_1^2s_2^3s_3-15s_1^2s_2^2s_3^2+10s_2^2s_3^4\\
&-5s_1^2s_3^4-66s_1s_2^5+155s_1s_2^4s_3-35s_1s_2^3s_3^2-35s_1s_2^2s_3^3+5s_1s_2s_3^4+3s_1s_3^5+33s_2^6\\
&+25s_1^2s_2s_3^3-66s_2^5s_3+10s_2^4s_3^2+10s_2^3s_3^3-6s_2s_3^5)f_{n+1} \qquad (5.21)
\end{aligned}
$$

$$
\begin{aligned}
&y''_{n+s_3}+\frac{(2(s_2-2s_3))}{(h^2s_1(s_1-s_2)(s_1-s_3))}\,y_{n+s_1}-\frac{(2(s_1-2s_3))}{(h^2s_2(s_1-s_2)\,(s_2-s_3))}\,y_{n+s_2}\\
&+\frac{(2(s_1+s_2-3s_3))}{(h^2\ s_3(s_1-s_3)(s_2-s_3))}\,y_{n+s_3}=\frac{(2s_1+2s_2-4s_3)}{(h^2\ s_1s_2s_3)}\,y_n\\
&-\frac{h^2}{(2520s_1s_2s_3)}(3s_1^5s_2-6s_1^5s_3-5s_1^4s_2^2+5s_1^4s_2s_3-8s_1^4s_2+10s_1^4s_3^2+16s_1^4s_3-5s_1^3s_2^3\\
&+25s_1^3s_2^2s_3+20s_1^3s_2^2-35s_1^3s_2s_3^2-20s_1^3s_2s_3+10s_1^3s_3^3-40s_1^3s_3^2-5s_1^2s_2^4+25s_1^2s_2^3s_3\\
&+20s_1^2s_2^3-15s_1^2s_2^2s_3^2-160s_1^2s_2^2s_3-35s_1^2s_2s_3^3+260s_1^2s_2s_3^2+10s_1^2s_3^4-40s_1^2s_3^3+33s_3^6\\
&+3s_1s_2^5+5s_1s_2^4s_3-8s_1s_2^4-35s_1s_2^3s_3^2-20s_1s_2^3s_3-35s_1s_2^2s_3^3+260s_1s_2^2s_3^2-40s_2^3s_3^2\\
&+155s_1s_2s_3^4-440s_1s_2s_3^3-66s_1s_3^5+142s_1s_3^4-6s_2^5s_3+10s_2^4s_3^2+16s_2^4s_3+10s_2^3s_3^3\\
&+10s_2^2s_3^4-40s_2^2s_3^3-66s_2s_3^5+142s_2s_3^4-60s_3^5)f_n\\
&-\frac{h^2}{(2520s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(-3s_1^5s_2+6s_1^5s_3+3s_1^4s_2^2-3s_1^4s_2s_3+6s_1^4s_2\\
&-6s_1^4s_3^2-12s_1^4s_3+3s_1^3s_2^3-11s_1^3s_2^2s_3-8s_1^3s_2^2+13s_1^3s_2s_3^2+8s_1^3s_2s_3-6s_1^3s_3^3+16s_1^3s_3^2\\
&+3s_1^2s_2^4-11s_1^2s_2^3s_3-8s_1^2s_2^3+5s_1^2s_2^2s_3^2+36s_1^2s_2^2s_3+13s_1^2s_2s_3^3-48s_1^2s_2s_3^2-6s_1^2s_3^4\\
&+16s_1^2s_3^3+3s_1s_2^5-11s_1s_2^4s_3-8s_1s_2^4+5s_1s_2^3s_3^2+36s_1s_2^3s_3+5s_1s_2^2s_3^3-20s_1s_2^2s_3^2\\
&+13s_1s_2s_3^4-48s_1s_2s_3^3-6s_1s_3^5+16s_1s_3^4-6s_2^5s_3+10s_2^4s_3^2+16s_2^4s_3+10s_2^3s_3^3+33s_3^6\\
&-40s_2^3s_3^2+10s_2^2s_3^4-40s_2^2s_3^3-66s_2s_3^5+142s_2s_3^4-60s_3^5)f_{n+s_1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{h^2}{(2520s_2(s_1-s_2)(s_2-s_3)(s_2-1))}\,(3s_1^5s_2-6s_1^5s_3+3s_1^4s_2^2-11s_1^4s_2s_3-8s_1^4s_2\\
&+10s_1^4s_3^2+16s_1^4s_3+3s_1^3s_2^3-11s_1^3s_2^2s_3-8s_1^3s_2^2+5s_1^3s_2s_3^2+36s_1^3s_2s_3+10s_1^3s_3^3\\
&-40s_1^3s_3^2+3s_1^2s_2^4-11s_1^2s_2^3s_3-8s_1^2s_2^3+5s_1^2s_2^2s_3^2+36s_1^2s_2^2s_3+5s_1^2s_2s_3^3-20s_1^2s_2s_3^2\\
&+10s_1^2s_3^4-40s_1^2s_3^3-3s_1s_2^5-3s_1s_2^4s_3+6s_1s_2^4+13s_1s_2^3s_3^2+8s_1s_2^3s_3+13s_1s_2^2s_3^3\\
&-48s_1s_2^2s_3^2+13s_1s_2s_3^4-48s_1s_2s_3^3-66s_1s_3^5+142s_1s_3^4+6s_2^5s_3-6s_2^4s_3^2-12s_2^4s_3\\
&-6s_2^3s_3^3+16s_2^3s_3^2-6s_2^2s_3^4+16s_2^2s_3^3-6s_2s_3^5+16s_2s_3^4+33s_3^6-60s_3^5)f_{n+s_2}\\
&-\frac{h^2}{(2520s_3(s_1-s_3)(s_2-s_3)(s_3-1))}\,(3s_1^5s_2-6s_1^5s_3-5s_1^4s_2^2+13s_1^4s_2s_3-8s_1^4s_2\\
&-6s_1^4s_3^2+16s_1^4s_3-5s_1^3s_2^3+5s_1^3s_2^2s_3+20s_1^3s_2^2+13s_1^3s_2s_3^2-48s_1^3s_2s_3-6s_1^3s_3^3-75s_3^6\\
&+16s_1^3s_3^2-5s_1^2s_2^4+5s_1^2s_2^3s_3+20s_1^2s_2^3+5s_1^2s_2^2s_3^2-20s_1^2s_2^2s_3+13s_1^2s_2s_3^3-48s_1^2s_2s_3^2\\
&-6s_1^2s_3^4+16s_1^2s_3^3+3s_1s_2^5+13s_1s_2^4s_3-8s_1s_2^4+13s_1s_2^3s_3^2-48s_1s_2^3s_3+13s_1s_2^2s_3^3\\
&-48s_1s_2^2s_3^2-189s_1s_2s_3^4+344s_1s_2s_3^3+114s_1s_3^5-180s_1s_3^4-6s_2^5s_3-6s_2^4s_3^2+16s_2^4s_3\\
&-6s_2^3s_3^3+16s_2^3s_3^2-6s_2^2s_3^4+16s_2^2s_3^3+114s_2s_3^5-180s_2s_3^4+108s_3^5)f_{n+s_3}\\
&+\frac{h^2}{((2520s_1-2520)(s_2-1)(s_3-1))}\,(3s_1^5s_2-6s_1^5s_3-5s_1^4s_2^2+5s_1^4s_2s_3+10s_1^4s_3^2\\
&-5s_1^3s_2^3+25s_1^3s_2^2s_3-35s_1^3s_2s_3^2+10s_1^3s_3^3-5s_1^2s_2^4+25s_1^2s_2^3s_3-15s_1^2s_2^2s_3^2-35s_1^2s_2s_3^3\\
&+10s_1^2s_3^4+3s_1s_2^5+5s_1s_2^4s_3-35s_1s_2^3s_3^2-35s_1s_2^2s_3^3+155s_1s_2s_3^4-66s_1s_3^5-6s_2^5s_3\\
&+10s_2^4s_3^2+10s_2^3s_3^3+10s_2^2s_3^4-66s_2s_3^5+33s_3^6)f_{n+1} \qquad (5.22)
\end{aligned}
$$

$$
\begin{aligned}
&y''_{n+1} + \frac{(2\,(s_2+s_3-3))}{(h^2\,s_1\,(s_1-s_2)\,(s_1-s_3))}\,y_{n+s_1} - \frac{(2\,(s_1+s_3-3))}{(h^2\,s_2\,(s_1-s_2)\,(s_2-s_3))}\,y_{n+s_2} \\
&+ \frac{(2(s_1+s_2-3))}{(h^2 s_3(s_1-s_3)(s_2-s_3))} y_{n+s_3} = \frac{(2s_1+2s_2+2s_3-6)}{(h^2 s_1 s_2 s_3)} y_n \\
&-\frac{(h^2}{(2520 s_1 s_2 s_3)}\,(3s_1^5 s_2 + 3s_1^5 s_3 - 9s_1^5 - 5s_1^4 s_2^2 - 10 s_1^4 s_2 s_3 + 7s_1^4 s_2 - 5s_1^4 s_3^2 + 7s_1^4 s_3 \\
&+24 s_1^4 - 5s_1^3 s_2^3 + 10 s_1^3 s_2^2 s_3 + 35 s_1^3 s_2^2 + 10 s_1^3 s_2 s_3^2 - 5s_1^3 s_2 s_3 - 60 s_1^3 s_2 - 5 s_1^3 s_3^3 - 9 s_3^5 \\
&+35 s_1^3 s_3^2 - 60 s_1^3 s_3 - 5 s_1^2 s_2^4 + 10 s_1^2 s_2^3 s_3 + 35 s_1^2 s_2^3 + 30 s_1^2 s_2^2 s_3^2 - 145 s_1^2 s_2^2 s_3 - 60 s_1^2 s_2^2 \\
&+10 s_1^2 s_2 s_3^3 - 145 s_1^2 s_2 s_3^2 + 360 s_1^2 s_2 s_3 - 5 s_1^2 s_3^4 + 35 s_1^2 s_3^3 - 60 s_1^2 s_3^2 + 3 s_1 s_2^5 - 60 s_1 s_2^3 \\
&-10 s_1 s_2^4 s_3 + 10 s_1 s_2^3 s_3^2 + 10 s_1 s_2^2 s_3^3 - 145 s_1 s_2^2 s_3^2 + 360 s_1 s_2^2 s_3 + 360 s_1 s_2 s_3^2 + 7 s_1 s_2^4 \\
&-5 s_1 s_2^3 s_3 - 10 s_1 s_2 s_3^4 - 5 s_1 s_2 s_3^3 - 840 s_1 s_2 s_3 + 210 s_1 s_2 + 3 s_1 s_3^5 + 7 s_1 s_3^4 + 35 s_2^3 s_3^2 \\
&-60 s_1 s_3^3 + 210 s_1 s_3 - 84 s_1 + 3 s_2^5 s_3 - 9 s_2^5 - 5 s_2^4 s_3^2 + 7 s_2^4 s_3 + 24 s_2^4 - 5 s_2^3 s_3^3 + 24 s_3^4 \\
&-60 s_2^3 s_3 - 5 s_2^2 s_3^4 + 35 s_2^2 s_3^3 - 60 s_2^2 s_3^2 + 3 s_2 s_3^5 + 7 s_2 s_3^4 - 60 s_2 s_3^3 + 210 s_2 s_3 - 84 s_2 \\
&-84 s_3 + 42) f_n + \frac{h^2}{(2520 s_1 (s_1-s_2)(s_1-s_3)(s_1-1))}(3 s_1^5 s_2 + 3 s_1^5 s_3 - 9 s_1^5 - 3 s_1^4 s_2^2 \\
&-6 s_1^4 s_2 s_3 + 3 s_1^4 s_2 - 3 s_1^4 s_3^2 + 3 s_1^4 s_3 + 18 s_1^4 - 3 s_1^3 s_2^3 + 2 s_1^3 s_2^2 s_3 + 17 s_1^3 s_2^2 + 10 s_1^2 s_2^2 s_3^2 \\
&+s_1^3 s_2 s_3 - 24 s_1^3 s_2 - 3 s_1^3 s_3^3 + 17 s_1^3 s_3^2 - 24 s_1^3 s_3 - 3 s_1^2 s_2^4 + 2 s_1^2 s_2^3 s_3 + 17 s_1^2 s_2^3 + 2 s_1^3 s_2 s_3^2 \\
&-27 s_1^2 s_2^2 s_3 - 24 s_1^2 s_2^2 + 2 s_1^2 s_2 s_3^3 - 27 s_1^2 s_2 s_3^2 + 60 s_1^2 s_2 s_3 - 3 s_1^2 s_3^4 + 17 s_1^2 s_3^3 - 24 s_1^2 s_3^2 \\
&-3 s_1 s_2^5 + 2 s_1 s_2^4 s_3 + 17 s_1 s_2^4 + 10 s_1 s_2^3 s_3^2 - 27 s_1 s_2^3 s_3 - 24 s_1 s_2^3 + 10 s_1 s_2^2 s_3^3 - 55 s_1 s_2^2 s_3^2 \\
&+60 s_1 s_2^2 s_3 + 2 s_1 s_2 s_3^4 - 27 s_1 s_2 s_3^3 + 60 s_1 s_2 s_3^2 - 3 s_1 s_3^5 + 17 s_1 s_3^4 - 24 s_1 s_3^3 - 3 s_2^5 s_3 \\
&+9 s_2^5 + 5 s_2^4 s_3^2 - 7 s_2^4 s_3 - 24 s_2^4 + 5 s_2^3 s_3^3 - 35 s_2^3 s_3^2 + 60 s_2^3 s_3 + 5 s_2^2 s_3^4 - 35 s_2^2 s_3^3 + 60 s_2^2 s_3^2 \\
&-3 s_2 s_3^5 - 7 s_2 s_3^4 + 60 s_2 s_3^3 - 210 s_2 s_3 + 84 s_2 + 9 s_3^5 - 24 s_3^4 + 84 s_3 - 42) f_{n+s_1} \\
&-\frac{h^2}{(2520 s_2 (s_1-s_2)(s_2-s_3)(s_2-1))}\,(-3 s_1^5 s_2 - 3 s_1^5 s_3 + 9 s_1^5 - 3 s_1^4 s_2^2 + 2 s_1^4 s_2 s_3 \\
&+17 s_1^4 s_2 + 5 s_1^4 s_3^2 - 7 s_1^4 s_3 - 24 s_1^4 - 3 s_1^3 s_2^3 + 2 s_1^3 s_2^2 s_3 + 17 s_1^3 s_2^2 + 10 s_1^3 s_2 s_3^2 - 27 s_1^3 s_2 s_3 \\
&-24 s_1^3 s_2 + 5 s_1^3 s_3^3 - 35 s_1^3 s_3^2 + 60 s_1^3 s_3 - 3 s_1^2 s_2^4 + 2 s_1^2 s_2^3 s_3 + 17 s_1^2 s_2^3 + 10 s_1^2 s_2^2 s_3^2 + 5 s_1^2 s_3^4 \\
&-27 s_1^2 s_2^2 s_3 - 24 s_1^2 s_2^2 + 10 s_1^2 s_2 s_3^3 - 55 s_1^2 s_2 s_3^2 + 60 s_1^2 s_2 s_3 - 35 s_1^2 s_3^3 - 3 s_2^2 s_3^4 + 17 s_2^2 s_3^3 \\
&+60 s_1^2 s_3^2 + 3 s_1 s_2^5 - 6 s_1 s_2^4 s_3 + 3 s_1 s_2^4 + 2 s_1 s_2^3 s_3^2 + s_1 s_2^3 s_3 - 24 s_1 s_2^3 + 2 s_1 s_2^2 s_3^3 + 3 s_2^4 s_3 \\
&-27 s_1 s_2^2 s_3^2 + 60 s_1 s_2^2 s_3 + 2 s_1 s_2 s_3^4 - 27 s_1 s_2 s_3^3 + 60 s_1 s_2 s_3^2 - 3 s_1 s_3^5 - 7 s_1 s_3^4 + 60 s_1 s_3^3
\end{aligned}
$$

$$-210s_1s_3+84s_1+3s_2^5s_3-9s_2^5-3s_2^4s_3^2+18s_2^4-3s_2^3s_3^3+17s_2^3s_3^2-24s_2^3s_3-24s_2s_3^3$$
$$-24s_2^2s_3^2-3s_2s_3^5+17s_2s_3^4+9s_3^5-24s_3^4+84s_3-42)f_{n+s_2}$$
$$+\frac{h^2}{(2520s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(-3s_1^5s_2-3s_1^5s_3+9s_1^5+5s_1^4s_2^2+2s_1^4s_2s_3$$
$$-7s_1^4s_2-3s_1^4s_3^2+17s_1^4s_3-24s_1^4+5s_1^3s_2^3+10s_1^3s_2^2s_3-35s_1^3s_2^2+2s_1^3s_2s_3^2-27s_1^3s_2s_3$$
$$+60s_1^3s_2-3s_1^3s_3^3+17s_1^3s_3^2-24s_1^3s_3+5s_1^2s_2^4+10s_1^2s_2^3s_3-35s_1^2s_2^3+10s_1^2s_2^2s_3^2+18s_3^4$$
$$-55s_1^2s_2^2s_3+60s_1^2s_2^2+2s_1^2s_2s_3^3-27s_1^2s_2s_3^2+60s_1^2s_2s_3-3s_1^2s_3^4+17s_1^2s_3^3-24s_1^2s_3^2$$
$$+2s_1s_2^4s_3-7s_1s_2^4+2s_1s_2^3s_3^2-27s_1s_2^3s_3+60s_1s_2^3+2s_1s_2^2s_3^3-27s_1s_2^2s_3^2-9s_3^5-24s_2^4$$
$$-6s_1s_2s_3^4+s_1s_2s_3^3+60s_1s_2s_3^2-210s_1s_2+3s_1s_3^5+3s_1s_3^4-24s_1s_3^3-3s_2^5s_3-3s_1s_2^5$$
$$+84s_1+60s_1s_2^2s_3+9s_2^5-3s_2^4s_3^2+17s_2^4s_3-3s_2^3s_3^3+17s_2^3s_3^2-24s_2^3s_3-3s_2^2s_3^4+84s_2$$
$$+17s_2^2s_3^3-24s_2^2s_3^2+3s_2s_3^5+3s_2s_3^4-24s_2s_3^3-42)f_{n+s_3}$$
$$+\frac{h^2}{(2520s_1-2520)(s_2-1)(s_3-1)}(3s_1^5s_2+3s_1^5s_3-5s_1^4s_2^2-10s_1^4s_2s_3+126s_2$$
$$+15s_1^4s_2-5s_1^4s_3^2+15s_1^4s_3-5s_1^3s_2^3+10s_1^3s_2^2s_3+15s_1^3s_2^2+10s_1^3s_2s_3^2-45s_1^3s_2s_3$$
$$-5s_1^3s_3^3+15s_1^3s_3^2-5s_1^2s_2^4+10s_1^2s_2^3s_3+15s_1^2s_2^3+30s_1^2s_2^2s_3^2-45s_1^2s_2^2s_3+10s_1^2s_2s_3^3$$
$$-45s_1^2s_2s_3^2-5s_1^2s_3^4+15s_1^2s_3^3+3s_1s_2^5-10s_1s_2^4s_3+15s_1s_2^4+10s_1s_2^3s_3^2-45s_1s_2^3s_3$$
$$+10s_1s_2^2s_3^3-45s_1s_2^2s_3^2-10s_1s_2s_3^4-45s_1s_2s_3^3+420s_1s_2s_3-210s_1s_2+3s_1s_3^5-9s_1^5$$
$$-210s_1s_3+126s_1+3s_2^5s_3-9s_2^5-5s_2^4s_3^2+15s_2^4s_3-5s_2^3s_3^3+15s_2^3s_3^2-5s_2^2s_3^4-9s_3^5$$
$$+15s_1s_3^4+15s_2^2s_3^3+3s_2s_3^5+15s_2s_3^4-210s_2s_3+126s_3-84)f_{n+1} \tag{5.23}$$

Equation (5.12) is evaluated at all points in the selected interval. This produces the following schemes

$$
\begin{aligned}
y_n''' &- \frac{6}{(h^3 s_1(s_1-s_2)(s_1-s_3))}\, y_{n+s_1} + \frac{6}{(h^3 s_2(s_1-s_2)(s_2-s_3))}\, y_{n+s_2} \\
&- \frac{6}{(h^3 s_3(s_1-s_3)(s_2-s_3))}\, y_{n+s_3} = -\frac{6}{(h^3 s_1 s_2 s_3)}\, y_n \\
&+\frac{h}{(840 s_1 s_2 s_3)}(3s_1^5 - 5s_1^4 s_2 - 5s_1^4 s_3 - 8s_1^4 - 5s_1^3 s_2^2 + 15 s_1^3 s_2 s_3 + 20 s_2^2 s_3^2 - 5 s_2 s_3^4 \\
&+20 s_1^3 s_2 - 5s_1^3 s_3^2 + 20 s_1^3 s_3 - 5 s_1^2 s_2^3 + 15 s_1^2 s_2^2 s_3 + 20 s_1^2 s_2^2 + 15 s_1^2 s_2 s_3^2 - 120 s_1^2 s_2 s_3 \\
&-5 s_1^2 s_3^3 + 20 s_1^2 s_3^2 - 5 s_1 s_2^4 + 15 s_1 s_2^3 s_3 + 20 s_1 s_2^3 + 15 s_1 s_2^2 s_3^2 - 120 s_1 s_2^2 s_3 - 5 s_2^2 s_3^3 \\
&-120 s_1 s_2 s_3^2 - 5 s_1 s_3^4 + 20 s_1 s_3^3 + 3 s_2^5 - 5 s_2^4 s_3 - 8 s_2^4 - 5 s_2^3 s_3^2 + 20 s_2^3 s_3 + 15 s_1 s_2 s_3^3 \\
&+20 s_2 s_3^3 + 3 s_3^5 - 8 s_3^4) f_n \\
&-\frac{h}{(840 s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(3s_1^5 - 3s_1^4 s_2 - 3 s_1^4 s_3 - 6 s_1^4 - 3 s_1^3 s_2^2 + 5 s_1^3 s_2 s_3 \\
&+8 s_1^3 s_2 - 3 s_1^3 s_3^2 + 8 s_1^3 s_3 - 3 s_1^2 s_2^3 + 5 s_1^2 s_2^2 s_3 + 8 s_1^2 s_2^2 + 5 s_1^2 s_2 s_3^2 - 20 s_1^2 s_2 s_3 - 20 s_2^2 s_3^2 \\
&-3 s_1^2 s_3^3 + 8 s_1^2 s_3^2 - 3 s_1 s_2^4 + 5 s_1 s_2^3 s_3 + 8 s_1 s_2^3 + 5 s_1 s_2^2 s_3^2 - 20 s_1 s_2^2 s_3 + 5 s_1 s_2 s_3^3 + 5 s_2 s_3^4 \\
&-20 s_1 s_2 s_3^2 - 3 s_1 s_3^4 + 8 s_1 s_3^3 - 3 s_2^5 + 5 s_2^4 s_3 + 8 s_2^4 + 5 s_2^3 s_3^2 - 20 s_2^3 s_3 + 5 s_2^2 s_3^3 - 20 s_2 s_3^3 \\
&-3 s_3^5 + 8 s_3^4) f_{n+s_1} \\
&+\frac{h}{840 s_2(s_1-s_2)(s_2-s_3)(s_2-1)}(8 s_3^4 - 3 s_1^5 - 3 s_1^4 s_2 + 5 s_1^4 s_3 - 3 s_1^3 s_2^2 + 5 s_1^3 s_2 s_3 \\
&+8 s_1^3 s_2 + 5 s_1^3 s_3^2 - 20 s_1^3 s_3 - 3 s_1^2 s_2^3 + 5 s_1^2 s_2^2 s_3 + 8 s_1^2 s_2^2 + 5 s_1^2 s_2 s_3^2 - 20 s_1^2 s_2 s_3 + 8 s_2^2 s_3^2 \\
&+5 s_1^2 s_3^3 - 20 s_1^2 s_3^2 - 3 s_1 s_2^4 + 5 s_1 s_2^3 s_3 + 8 s_1 s_2^3 + 5 s_1 s_2^2 s_3^2 - 20 s_1 s_2^2 s_3 + 5 s_1 s_2 s_3^3 \\
&-20 s_1 s_2 s_3^2 + 5 s_1 s_3^4 - 20 s_1 s_3^3 + 3 s_2^5 - 3 s_2^4 s_3 - 6 s_2^4 - 3 s_2^3 s_3^2 + 8 s_2^3 s_3 - 3 s_2^2 s_3^3 + 8 s_2 s_3^3 \\
&+8 s_1^4 - 3 s_3^5 - 3 s_2 s_3^4) f_{n+s_2} \\
&-\frac{h}{840 s_3 (s_1-s_3)(s_2-s_3)(s_3-1)}(5 s_1^4 s_2 - 3 s_1^5 - 3 s_1^4 s_3 + 5 s_1^3 s_2^2 + 5 s_1^3 s_2 s_3 + 8 s_1^4 \\
&-20 s_1^3 s_2 - 3 s_1^3 s_3^2 + 8 s_1^3 s_3 + 5 s_1^2 s_2^3 + 5 s_1^2 s_2^2 s_3 - 20 s_1^2 s_2^2 + 5 s_1^2 s_2 s_3^2 - 20 s_1^2 s_2 s_3 - 3 s_1^2 s_3^3 \\
&+8 s_1^2 s_3^2 + 5 s_1 s_2^4 + 5 s_1 s_2^3 s_3 - 20 s_1 s_2^3 + 5 s_1 s_2^2 s_3^2 - 20 s_1 s_2^2 s_3 + 5 s_1 s_2 s_3^3 - 20 s_1 s_2 s_3^2 \\
&-3 s_1 s_3^4 + 8 s_1 s_3^3 - 3 s_2^5 - 3 s_2^4 s_3 + 8 s_2^4 - 3 s_2^3 s_3^2 + 8 s_2^3 s_3 - 3 s_2^2 s_3^3 + 8 s_2^2 s_3^2 - 3 s_2 s_3^4 \\
&+8 s_2 s_3^3 + 3 s_3^5 - 6 s_3^4) f_{n+s_3} \\
&+\frac{h}{840(s_1-1)(s_2-1)(s_3-1)}(5 s_1^4 s_2 - 3 s_1^5 + 5 s_1^4 s_3 - 3 s_3^5 + 5 s_1^3 s_3^2 + 5 s_1^3 s_2^2 - 3 s_2^5
\end{aligned}
$$

$$
\begin{aligned}
&+5s_1^2s_2^3-15s_1^2s_2^2s_3-15s_1^2s_2s_3^2+5s_1^2s_3^3+5s_1s_2^4-15s_1s_2^3s_3-15s_1s_2^2s_3^2-15s_1s_2s_3^3\\
&-15s_1^3s_2s_3+5s_1s_3^4+5s_2^4s_3+5s_2^3s_3^2+5s_2^2s_3^3+5s_2s_3^4)f_{n+1} \qquad (5.24)
\end{aligned}
$$

$$
\begin{aligned}
&y'''_{n+s_1} - \frac{6}{(h^3s_1(s_1-s_2)(s_1-s_3))}y_{n+s_1} + \frac{6}{(h^3s_2(s_1-s_2)(s_2-s_3))}y_{n+s_2}\\
&-\frac{6}{(h^3s_3(s_1-s_3)(s_2-s_3))}y_{n+s_3} = -\frac{6}{(h^3s_1s_2s_3)}y_n-\frac{h}{(840s_1s_2s_3)}(39s_1^5-65s_1^4s_2\\
&-65s_1^4s_3-62s_1^4+5s_1^3s_2^2+125s_1^3s_2s_3+120s_1^3s_2+5s_1^3s_3^2+120s_1^3s_3+5s_1^2s_2^3+8s_3^4\\
&-15s_1^2s_2^2s_3-20s_1^2s_2^2-15s_1^2s_2s_3^2-300s_1^2s_2s_3+5s_1^2s_3^3-20s_1^2s_3^2+5s_1s_2^4-15s_1s_2^3s_3\\
&-20s_1s_2^3-15s_1s_2^2s_3^2+120s_1s_2^2s_3-15s_1s_2s_3^3+120s_1s_2s_3^2+5s_1s_3^4-20s_1s_3^3-3s_2^5\\
&+5s_2^4s_3+8s_2^4+5s_2^3s_3^2-20s_2^3s_3+5s_2^2s_3^3-20s_2^2s_3^2+5s_2s_3^4-20s_2s_3^3-3s_3^5)f_n\\
&-\frac{h}{(840s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(-165s_1^5+207s_1^4s_2+207s_1^4s_3+204s_1^4+8s_3^4\\
&-275s_1^3s_2s_3-272s_1^3s_2-3s_1^3s_3^2-272s_1^3s_3-3s_1^2s_2^3+5s_1^2s_2^2s_3+8s_1^2s_2^2+5s_1^2s_2s_3^2\\
&+400s_1^2s_2s_3-3s_1^2s_3^3+8s_1^2s_3^2-3s_1s_2^4+5s_1s_2^3s_3+8s_1s_2^3+5s_1s_2^2s_3^2-20s_1s_2^2s_3\\
&+5s_1s_2s_3^3-20s_1s_2s_3^2-3s_1s_3^4+8s_1s_3^3-3s_2^5+5s_2^4s_3+8s_2^4+5s_2^3s_3^2-20s_2^3s_3-3s_3^5\\
&+5s_2^2s_3^3-20s_2^2s_3^2+5s_2s_3^4-20s_2s_3^3-3s_1^3s_2^2)f_{n+s_1}\\
&+\frac{h}{(840s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(39s_1^5-3s_1^4s_2-65s_1^4s_3-62s_1^4-3s_1^3s_2^2+8s_3^4\\
&+5s_1^3s_2s_3+8s_1^3s_2+5s_1^3s_3^2+120s_1^3s_3-3s_1^2s_2^3+5s_1^2s_2^2s_3+8s_1^2s_2^2+5s_1^2s_2s_3^2+8s_2s_3^3\\
&-20s_1^2s_2s_3+5s_1^2s_3^3-20s_1^2s_3^2-3s_1s_2^4+5s_1s_2^3s_3+8s_1s_2^3+5s_1s_2^2s_3^2-20s_1s_2^2s_3\\
&+5s_1s_2s_3^3-20s_1s_2s_3^2+5s_1s_3^4-20s_1s_3^3+3s_2^5-3s_2^4s_3-6s_2^4-3s_2^3s_3^2+8s_2^3s_3-3s_3^5\\
&-3s_2^2s_3^3+8s_2^2s_3^2-3s_2s_3^4)f_{n+s_2}\\
&-\frac{h}{(840s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(39s_1^5-65s_1^4s_2-3s_1^4s_3+5s_1^3s_2^2+5s_1s_2s_3^3\\
&-62s_1^4+5s_1^3s_2s_3+120s_1^3s_2-3s_1^3s_3^2+8s_1^3s_3+5s_1^2s_2^3+5s_1^2s_2^2s_3-20s_1^2s_2^2+5s_1^2s_2s_3^2\\
&-20s_1^2s_2s_3-3s_1^2s_3^3+8s_1^2s_3^2+5s_1s_2^4+5s_1s_2^3s_3-20s_1s_2^3+5s_1s_2^2s_3^2-20s_1s_2^2s_3\\
&-20s_1s_2s_3^2-3s_1s_3^4+8s_1s_3^3-3s_2^5-3s_2^4s_3+8s_2^4-3s_2^3s_3^2+8s_2^3s_3-3s_2^2s_3^3+8s_2^2s_3^2\\
&-3s_2s_3^4+8s_2s_3^3+3s_3^5-6s_3^4)f_{n+s_3}
\end{aligned}
$$

$$+\frac{h}{((840s_1-840)(s_2-1)(s_3-1))}(39s_1^5-65s_1^4s_2-65s_1^4s_3+5s_1^3s_2^2+125s_1^3s_2s_3$$
$$+5s_1^3s_3^2+5s_1^2s_2^3-15s_1^2s_2^2s_3-15s_1^2s_2s_3^2+5s_1^2s_3^3+5s_1s_2^4-15s_1s_2^3s_3-15s_1s_2^2s_3^2$$
$$-15s_1s_2s_3^3+5s_1s_3^4-3s_2^5+5s_2^4s_3+5s_2^3s_3^2+5s_2^2s_3^3+5s_2s_3^4-3s_3^5)f_{n+1}$$

(5.25)

$$y'''_{n+s_2}-\frac{6}{(h^3s_1(s_1-s_2)(s_1-s_3))}y_{n+s_1}+\frac{6}{(h^3s_2(s_1-s_2)(s_2-s_3))}y_{n+s_2}$$
$$-\frac{6}{(h^3s_3(s_1-s_3)(s_2-s_3))}y_{n+s_3}=\frac{6}{(h^3s_1s_2s_3)}y_n+\frac{h}{(840s_1s_2s_3)}(3s_1^5-5s_1^4s_2$$
$$-5s_1^4s_3-8s_1^4-5s_1^3s_2^2+15s_1^3s_2s_3+20s_1^3s_2-5s_1^3s_3^2+20s_1^3s_3-5s_1^2s_2^3+15s_1^2s_2^2s_3$$
$$+20s_1^2s_2^2+15s_1^2s_2s_3^2-120s_1^2s_2s_3-5s_1^2s_3^3+20s_1^2s_3^2+65s_1s_2^4-125s_1s_2^3s_3-120s_1s_2^3$$
$$+15s_1s_2^2s_3^2+300s_1s_2^2s_3+15s_1s_2s_3^3-120s_1s_2s_3^2-5s_1s_3^4+20s_1s_3^3-39s_2^5+65s_2^4s_3$$
$$+62s_2^4-5s_2^3s_3^2-120s_2^3s_3-5s_2^2s_3^3+20s_2^2s_3^2-5s_2s_3^4+20s_2s_3^3+3s_3^5-8s_3^4)f_n$$
$$-\frac{h}{(840s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(3s_1^5-3s_1^4s_2-3s_1^4s_3-6s_1^4-3s_1^3s_2^2+5s_1^3s_2s_3$$
$$+8s_1^3s_2-3s_1^3s_3^2+8s_1^3s_3-3s_1^2s_2^3+5s_1^2s_2^2s_3+8s_1^2s_2^2+5s_1^2s_2s_3^2-20s_1^2s_2s_3-3s_1^2s_3^3$$
$$+8s_1^2s_3^2-3s_1s_2^4+5s_1s_2^3s_3+8s_1s_2^3+5s_1s_2^2s_3^2-20s_1s_2^2s_3+5s_1s_2s_3^3-20s_1s_2s_3^2$$
$$-3s_1s_3^4+8s_1s_3^3+39s_2^5-65s_2^4s_3-62s_2^4+5s_2^3s_3^2+120s_2^3s_3+5s_2^2s_3^3-20s_2^2s_3^2$$
$$+5s_2s_3^4-20s_2s_3^3-3s_3^5+8s_3^4)f_{n+s_1}+\frac{h}{(840s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(-3s_1^5$$
$$-3s_1^4s_2+5s_1^4s_3+8s_1^4-3s_1^3s_2^2+5s_1^3s_2s_3+8s_1^3s_2+5s_1^3s_3^2-20s_1^3s_3-3s_3^5-3s_1^2s_2^3$$
$$+5s_1^2s_2^2s_3+8s_1^2s_2^2+5s_1^2s_2s_3^2-20s_1^2s_2s_3+5s_1^2s_3^3-20s_1^2s_3^2+207s_1s_2^4-275s_1s_2^3s_3$$
$$-272s_1s_2^3+5s_1s_2^2s_3^2+400s_1s_2^2s_3+5s_1s_2s_3^3-20s_1s_2s_3^2+5s_1s_3^4-20s_1s_3^3-165s_2^5$$
$$+207s_2^4s_3+204s_2^4-3s_2^3s_3^2-272s_2^3s_3-3s_2^2s_3^3+8s_2^2s_3^2-3s_2s_3^4+8s_2s_3^3+8s_3^4)f_{n+s_2}$$
$$-\frac{h}{(840s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(-3s_1^5+5s_1^4s_2-3s_1^4s_3+8s_1^4+5s_1^3s_2^2+5s_1^3s_2s_3$$
$$-20s_1^3s_2-3s_1^3s_3^2+8s_1^3s_3+5s_1^2s_2^3+5s_1^2s_2^2s_3-20s_1^2s_2^2+5s_1^2s_2s_3^2-20s_1^2s_2s_3-3s_1^2s_3^3$$
$$+8s_1^2s_3^2-65s_1s_2^4+5s_1s_2^3s_3+120s_1s_2^3+5s_1s_2^2s_3^2-20s_1s_2^2s_3+5s_1s_2s_3^3-20s_1s_2s_3^2$$

$$-3s_1s_3^4+8s_1s_3^3+39s_2^5-3s_2^4s_3-62s_2^4-3s_2^3s_3^2+8s_2^3s_3-3s_2^2s_3^3+8s_2^2s_3^2-3s_2s_3^4$$

$$+8s_2s_3^3+3s_3^5-6s_3^4)\,f_{n+s_3}+\frac{h}{(840s_1-840)(s_2-1)(s_3-1)}(-3s_1^5+5s_1^4s_2$$

$$+5s_1^4s_3+5s_1^3s_2^2-15s_1^3s_2s_3+5s_1^3s_3^2+5s_1^2s_2^3-15s_1^2s_2^2s_3-15s_1^2s_2s_3^2+5s_1^2s_3^3+5s_2^2s_3^3$$

$$-65s_1s_2^4+125s_1s_2^3s_3-15s_1s_2^2s_3^2-15s_1s_2s_3^3+5s_1s_3^4+39s_2^5-65s_2^4s_3+5s_2^3s_3^2$$

$$+5s_2s_3^4-3s_3^5)f_{n+1} \tag{5.26}$$

$$
\begin{aligned}
&y'''_{n+s_3} - \frac{6}{(h^3 s_1 (s_1-s_2)(s_1-s_3))} y_{n+s_1} + \frac{6}{(h^3 s_2 (s_1-s_2)(s_2-s_3))} y_{n+s_2} \\
&-\frac{6}{(h^3 s_3(s_1-s_3)(s_2-s_3))} y_{n+s_3} = \frac{-6}{(h^3 s_1 s_2 s_3)} y_n + \frac{h}{(840 s_1 s_2 s_3)}(3s_1^5 - 5s_1^4 s_2 - 39 s_3^5 \\
&-5s_1^4 s_3 - 8s_1^4 - 5s_1^3 s_2^2 + 15 s_1^3 s_2 s_3 + 20 s_1^3 s_2 - 5s_1^3 s_3^2 + 20 s_1^3 s_3 - 5s_1^2 s_2^3 + 15 s_1^2 s_2^2 s_3 \\
&+20 s_1^2 s_2^2 + 15 s_1^2 s_2 s_3^2 - 120 s_1^2 s_2 s_3 - 5 s_1^2 s_3^3 + 20 s_1^2 s_3^2 - 5 s_1 s_2^4 + 15 s_1 s_2^3 s_3 + 20 s_1 s_2^3 \\
&+15 s_1 s_2^2 s_3^2 - 120 s_1 s_2^2 s_3 - 125 s_1 s_2 s_3^3 + 300 s_1 s_2 s_3^2 + 65 s_1 s_3^4 - 120 s_1 s_3^3 + 3 s_2^5 - 5 s_2^4 s_3 \\
&-8 s_2^4 - 5 s_2^3 s_3^2 + 20 s_2^3 s_3 - 5 s_2^2 s_3^3 + 20 s_2^2 s_3^2 + 65 s_2 s_3^4 - 120 s_2 s_3^3 + 62 s_3^4) f_n \\
&-\frac{h}{(840 s_1 (s_1-s_2)(s_1-s_3)(s_1-1))}(3s_1^5 - 3s_1^4 s_2 - 3 s_1^4 s_3 - 6 s_1^4 - 3 s_1^3 s_2^2 + 5 s_1^3 s_2 s_3 \\
&+8 s_1^3 s_2 - 3 s_1^3 s_3^2 + 8 s_1^3 s_3 - 3 s_1^2 s_2^3 + 5 s_1^2 s_2^2 s_3 + 8 s_1^2 s_2^2 + 5 s_1^2 s_2 s_3^2 - 20 s_1^2 s_2 s_3 - 3 s_1^2 s_3^3 \\
&+8 s_1^2 s_3^2 - 3 s_1 s_2^4 + 5 s_1 s_2^3 s_3 + 8 s_1 s_2^3 + 5 s_1 s_2^2 s_3^2 - 20 s_1 s_2^2 s_3 + 5 s_1 s_2 s_3^3 - 20 s_1 s_2 s_3^2 \\
&-3 s_1 s_3^4 + 8 s_1 s_3^3 - 3 s_2^5 + 5 s_2^4 s_3 + 8 s_2^4 + 5 s_2^3 s_3^2 - 20 s_2^3 s_3 + 5 s_2^2 s_3^3 - 20 s_2^2 s_3^2 - 65 s_2 s_3^4 \\
&+120 s_2 s_3^3 + 39 s_3^5 - 62 s_3^4) f_{n+s_1} + \frac{h}{(840 s_2 (s_1-s_2)(s_2-s_3)(s_2-1))}(-3s_1^5 - 3 s_1^4 s_2 \\
&+5 s_1^4 s_3 + 8 s_1^4 - 3 s_1^3 s_2^2 + 5 s_1^3 s_2 s_3 + 8 s_1^3 s_2 + 5 s_1^3 s_3^2 - 20 s_1^3 s_3 - 3 s_1^2 s_2^3 + 5 s_1^2 s_2^2 s_3 + 8 s_2 s_3^3 \\
&+8 s_1^2 s_2^2 + 5 s_1^2 s_2 s_3^2 - 20 s_1^2 s_2 s_3 + 5 s_1^2 s_3^3 - 20 s_1^2 s_3^2 - 3 s_1 s_2^4 + 5 s_1 s_2^3 s_3 + 8 s_1 s_2^3 + 5 s_1 s_2^2 s_3^2 \\
&-20 s_1 s_2^2 s_3 + 5 s_1 s_2 s_3^3 - 20 s_1 s_2 s_3^2 - 65 s_1 s_3^4 + 120 s_1 s_3^3 + 3 s_2^5 - 3 s_2^4 s_3 - 6 s_2^4 - 3 s_2^3 s_3^2 \\
&+8 s_2^3 s_3 - 3 s_2^2 s_3^3 + 8 s_2^2 s_3^2 - 3 s_2 s_3^4 + 39 s_3^5 - 62 s_3^4) f_{n+s_2} \\
&-\frac{h}{(840 s_3 (s_1-s_3)(s_2-s_3)(s_3-1))}(-3s_1^5 + 5 s_1^4 s_2 - 3 s_1^4 s_3 + 8 s_1^4 + 5 s_1^3 s_2^2 + 5 s_1^3 s_2 s_3 \\
&-20 s_1^3 s_2 - 3 s_1^3 s_3^2 + 8 s_1^3 s_3 + 5 s_1^2 s_2^3 + 5 s_1^2 s_2^2 s_3 - 20 s_1^2 s_2^2 + 5 s_1^2 s_2 s_3^2 - 20 s_1^2 s_2 s_3 - 3 s_1^2 s_3^3 \\
&+8 s_1^2 s_3^2 + 5 s_1 s_2^4 + 5 s_1 s_2^3 s_3 - 20 s_1 s_2^3 + 5 s_1 s_2^2 s_3^2 - 20 s_1 s_2^2 s_3 - 275 s_1 s_2 s_3^3 + 400 s_1 s_2 s_3^2 \\
&+207 s_1 s_3^4 - 272 s_1 s_3^3 - 3 s_2^5 - 3 s_2^4 s_3 + 8 s_2^4 - 3 s_2^3 s_3^2 + 8 s_2^3 s_3 - 3 s_2^2 s_3^3 + 8 s_2^2 s_3^2 + 204 s_3^4 \\
&+207 s_2 s_3^4 - 272 s_2 s_3^3 - 165 s_3^5) f_{n+s_3} \\
&+\frac{h}{(840 s_1 - 840)(s_2-1)(s_3-1)}(-3s_1^5 + 5 s_1^4 s_2 + 5 s_1^4 s_3 + 5 s_1^3 s_2^2 - 65 s_2 s_3^4 - 3 s_2^5 \\
&-15 s_1^3 s_2 s_3 + 5 s_1^3 s_3^2 + 5 s_1^2 s_2^3 - 15 s_1^2 s_2^2 s_3 - 15 s_1^2 s_2 s_3^2 + 5 s_1^2 s_3^3 + 5 s_1 s_2^4 - 15 s_1 s_2^3 s_3 \\
&-15 s_1 s_2^2 s_3^2 + 125 s_1 s_2 s_3^3 - 65 s_1 s_3^4 + 5 s_2^4 s_3 + 5 s_2^3 s_3^2 + 5 s_2^2 s_3^3 + 39 s_3^5) f_{n+1} \qquad (5.27)
\end{aligned}
$$

$$y'''_{n+1} - \frac{6}{(h^3 s_1(s_1-s_2)(s_1-s_3))} y_{n+s_1} + \frac{6}{(h^3 s_2(s_1-s_2)(s_2-s_3))} y_{n+s_2}$$
$$-\frac{6}{(h^3 s_3(s_1-s_3)(s_2-s_3))} y_{n+s_3} = -\frac{6}{(h^3 s_1 s_2 s_3)} y_n + \frac{h}{(840 s_1 s_2 s_3)} (3s_1^5 - 5s_1^4 s_2$$
$$-5s_1^4 s_3 - 8s_1^4 - 5s_1^3 s_2^2 + 15s_1^3 s_2 s_3 + 20s_1^3 s_2 - 5s_1^3 s_3^2 + 20s_1^3 s_3 - 5s_1^2 s_2^3 - 5s_1 s_3^4$$
$$+15s_1^2 s_2^2 s_3 + 20s_1^2 s_2^2 + 15s_1^2 s_2 s_3^2 - 120s_1^2 s_2 s_3 - 5s_1^2 s_3^3 + 20s_1^2 s_3^2 - 5s_1 s_2^4 + 15s_1 s_2^3 s_3$$
$$+20s_1 s_2^3 + 15s_1 s_2^2 s_3^2 - 120s_1 s_2^2 s_3 + 15s_1 s_2 s_3^3 - 120s_1 s_2 s_3^2 + 420s_1 s_2 s_3 - 140s_1 s_2$$
$$+20s_1 s_3^3 - 140s_1 s_3 + 70s_1 + 3s_2^5 - 5s_2^4 s_3 - 8s_2^4 - 5s_2^3 s_3^2 + 20s_2^3 s_3 - 5s_2^2 s_3^3 - 8s_3^4$$
$$-140s_2 s_3 + 20s_2^2 s_3^2 - 5s_2 s_3^4 + 20s_2 s_3^3 + 70s_2 + 3s_3^5 + 70s_3 - 42) f_n$$
$$-\frac{h}{(840 s_1(s_1-s_2)(s_1-s_3)(s_1-1))} (3s_1^5 - 3s_1^4 s_2 - 3s_1^4 s_3 - 6s_1^4 - 3s_1^3 s_2^2 + 5s_1^3 s_2 s_3$$
$$+8s_1^3 s_2 - 3s_1^3 s_3^2 + 8s_1^3 s_3 - 3s_1^2 s_2^3 + 5s_1^2 s_2^2 s_3 + 8s_1^2 s_2^2 + 5s_1^2 s_2 s_3^2 - 20s_1^2 s_2 s_3 + 140s_2 s_3$$
$$-3s_1^2 s_3^3 + 8s_1^2 s_3^2 - 3s_1 s_2^4 + 5s_1 s_2^3 s_3 + 8s_1 s_2^3 + 5s_1 s_2^2 s_3^2 - 20s_1 s_2^2 s_3 + 5s_1 s_2 s_3^3 + 5s_2 s_3^4$$
$$-20s_1 s_2 s_3^2 - 3s_1 s_3^4 + 8s_1 s_3^3 - 3s_2^5 + 5s_2^4 s_3 + 8s_2^4 + 5s_2^3 s_3^2 - 20s_2^3 s_3 + 5s_2^2 s_3^3 - 20s_2^2 s_3^2$$
$$-20s_2 s_3^3 - 70s_2 - 3s_3^5 + 8s_3^4 - 70s_3 + 42) f_{n+s_1}$$
$$+\frac{h}{(840 s_2(s_1-s_2)(s_2-s_3)(s_2-1))} (-3s_1^5 - 3s_1^4 s_2 + 5s_1^4 s_3 + 8s_1^4 - 3s_1^3 s_2^2 + 5s_1^3 s_2 s_3$$
$$+8s_1^3 s_2 + 5s_1^3 s_3^2 - 20s_1^3 s_3 - 3s_1^2 s_2^3 + 5s_1^2 s_2^2 s_3 + 8s_1^2 s_2^2 + 5s_1^2 s_2 s_3^2 - 20s_1^2 s_2 s_3 - 3s_3^5$$
$$+5s_1^2 s_3^3 - 20s_1^2 s_3^2 - 3s_1 s_2^4 + 5s_1 s_2^3 s_3 + 8s_1 s_2^3 + 5s_1 s_2^2 s_3^2 - 20s_1 s_2^2 s_3 + 5s_1 s_2 s_3^3 + 5s_1 s_3^4$$
$$-20s_1 s_2 s_3^2 - 20s_1 s_3^3 + 140s_1 s_3 - 70s_1 + 3s_2^5 - 3s_2^4 s_3 - 6s_2^4 - 3s_2^3 s_3^2 + 8s_2^3 s_3 - 3s_2^2 s_3^3$$
$$+8s_2^2 s_3^2 - 3s_2 s_3^4 + 8s_2 s_3^3 + 8s_3^4 - 70s_3 + 42) f_{n+s_2}$$
$$-\frac{h}{(840 s_3(s_1-s_3)(s_2-s_3)(s_3-1))} (-3s_1^5 + 5s_1^4 s_2 - 3s_1^4 s_3 + 8s_1^4 + 5s_1^3 s_2^2 + 5s_1^3 s_2 s_3$$
$$-20s_1^3 s_2 - 3s_1^3 s_3^2 + 8s_1^3 s_3 + 5s_1^2 s_2^3 + 5s_1^2 s_2^2 s_3 - 20s_1^2 s_2^2 + 5s_1^2 s_2 s_3^2 - 20s_1^2 s_2 s_3 - 3s_1^2 s_3^3$$
$$+8s_1^2 s_3^2 + 5s_1 s_2^4 + 5s_1 s_2^3 s_3 - 20s_1 s_2^3 + 5s_1 s_2^2 s_3^2 - 20s_1 s_2^2 s_3 + 5s_1 s_2 s_3^3 - 20s_1 s_2 s_3^2$$
$$+140s_1 s_2 - 3s_1 s_3^4 + 8s_1 s_3^3 - 70s_1 - 3s_2^5 - 3s_2^4 s_3 + 8s_2^4 - 3s_2^3 s_3^2 + 8s_2^3 s_3 - 3s_2^2 s_3^3$$
$$+8s_2^2 s_3^2 - 3s_2 s_3^4 + 8s_2 s_3^3 - 70s_2 + 3s_3^5 - 6s_3^4 + 42) f_{n+s_3}$$
$$+\frac{h}{(840s_1-840)(s_2-1)(s_3-1)} (-3s_1^5 + 5s_1^4 s_2 + 5s_1^4 s_3 + 5s_1^3 s_2^2 - 15s_1^3 s_2 s_3 + 210s_2$$

$$+5s_1^3s_3^2+5s_1^2s_2^3-15s_1^2s_2^2s_3-15s_1^2s_2s_3^2+5s_1^2s_3^3+5s_1s_2^4-15s_1s_2^3s_3-15s_1s_2^2s_3^2-3s_3^5$$
$$-15s_1s_2s_3^3+420s_1s_2s_3-280s_1s_2+5s_1s_3^4-280s_1s_3+210s_1-3s_2^5+5s_2^4s_3+5s_2^3s_3^2$$
$$+5s_2^2s_3^3+5s_2s_3^4-280s_2s_3+210s_3-168)f_{n+1} \tag{5.28}$$

Combining (5.13), (5.14), (5.19) and (5.24) of discrete schemes to form a block

$$\begin{aligned} A^{[3]_4}Y_m^{[3]_4} &= B_1^{[3]_4}R_1^{[3]_4}+B_2^{[3]_4}R_2^{[3]_4}+B_3^{[3]_4}R_3^{[3]_4}+B_4^{[3]_4}R_4^{[3]_4} \\ &+ h^4D^{[3]_4}R_5^{[3]_4}+h^4E^{[3]_4}R_6^{[3]_4} \end{aligned} \tag{5.29}$$

where

$$A^{[3]_4}=\begin{pmatrix} \frac{-((s_2-1)(s_3-1))}{(s_1(s_1-s_2)(s_1-s_3))} & \frac{((s_1-1)(s_3-1))}{(s_2(s_1-s_2)(s_2-s_3))} & \frac{-((s_1-1)(s_2-1))}{(s_3(s_1-s_3)(s_2-s_3))} & 1 \\ \frac{-(s_2s_3)}{(hs_1(s_1-s_2)(s_1-s_3))} & \frac{(s_1s_3)}{(hs_2(s_1-s_2)(s_2-s_3))} & \frac{-(s_1s_2)}{(hs_3(s_1-s_3)(s_2-s_3))} & 0 \\ \frac{(2s_2+2s_3)}{(h^2s_1(s_1-s_2)(s_1-s_3))} & \frac{-(2s_1+2s_3)}{(h^2s_2(s_1-s_2)(s_2-s_3))} & \frac{(2s_1+2s_2)}{(h^2s_3(s_1-s_3)(s_2-s_3))} & 0 \\ \frac{-6}{(h^3s_1(s_1-s_2)(s_1-s_3))} & \frac{6}{(h^3s_2(s_1-s_2)(s_2-s_3))} & \frac{-6}{(h^3s_3(s_1-s_3)(s_2-s_3))} & 0 \end{pmatrix}$$

$$Y_m^{[3]_4}=\begin{pmatrix} y_{n+s_1} \\ y_{n+s_2} \\ y_{n+s_3} \\ y_{n+1} \end{pmatrix},\ B_1^{[3]_4}=\begin{pmatrix} 0 & 0 & 0 & \frac{((s_1-1)(s_2-1)(s_3-1))}{(s_1s_2s_3)} \\ 0 & 0 & 0 & \frac{-(s_1s_2+s_1s_3+s_2s_3)}{(hs_1s_2s_3)} \\ 0 & 0 & 0 & \frac{(2(s_1+s_2+s_3))}{(h^2s_1s_2s_3)} \\ 0 & 0 & 0 & \frac{-6}{(h^3s_1s_2s_3)} \end{pmatrix},$$

$$R_1^{[3]_4}=\begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix},\ B_2^{[3]_4}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ R_2^{[3]_4}=\begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix},$$

$$B_3^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; R_3^{[3]_4} = \begin{pmatrix} y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_{n} \end{pmatrix}$$

$$B_4^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; R_4^{[3]_4} = \begin{pmatrix} y'''_{n-3} \\ y'''_{n-2} \\ y'''_{n-1} \\ y'''_{n} \end{pmatrix},$$

$$D^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & D_{14}^{[3]_4} \\ 0 & 0 & 0 & D_{24}^{[3]_4} \\ 0 & 0 & 0 & D_{34}^{[3]_4} \\ 0 & 0 & 0 & D_{44}^{[3]_4} \end{pmatrix}, \; R_5^{[3]_4} = \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix},$$

$$E^{[3]_4} = \begin{pmatrix} E_{11}^{[3]_4} & E_{12}^{[3]_4} & E_{13}^{[3]_4} & E_{14}^{[3]_4} \\ E_{21}^{[3]_4} & E_{22}^{[3]_4} & E_{23}^{[3]_4} & E_{24}^{[3]_4} \\ E_{31}^{[3]_4} & E_{32}^{[3]_4} & E_{33}^{[3]_4} & E_{34}^{[3]_4} \\ E_{41}^{[3]_4} & E_{42}^{[3]_4} & E_{43}^{[3]_4} & E_{44}^{[3]_4} \end{pmatrix} \text{ and } R_6^{[3]_4} = \begin{pmatrix} f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix}$$

The elements of $D^{[3]_4}$ and $E^{[3]_4}$ are given by

$$D_{14}^{[3]_4} = \frac{-(s_1-1)(s_2-1)(s_3-1)}{(5040 s_1 s_2 s_3)}(5s_1^3 s_2 + 5s_1^3 s_3 + 5s_1 - 15s_1 s_2^2 s_3 + 5s_3^2 + 5s_3$$

$$-3s_1^4 + 5s_1^3 + 5s_1^2 s_2^2 - 15s_1^2 s_2 s_3 - 15s_1^2 s_2 + 5s_1^2 s_3^2 - 15s_1^2 s_3 + 5s_1^2 + 5s_1 s_2^3 - 3s_2^4$$

$$+5s_2^3 - 15s_1 s_2^2 - 15s_1 s_2 s_3^2 - 15s_1 s_2 + 5s_1 s_3^3 - 15s_1 s_3^2 - 15s_1 s_3 + 5s_3^3 - 3s_3^4 + 5s_2$$

$$+105 s_1 s_2 s_3 + 5s_2^3 s_3 + 5s_2^2 s_3^2 - 15s_2^2 s_3 + 5s_2^2 + 5s_2 s_3^3 - 15s_2 s_3^2 - 15s_2 s_3 - 3)$$

$$D_{24}^{[3]_4} = \frac{1}{5040h}(3s_1^4 - 5s_1^3 s_2 + 15s_1 s_2 s_3^2 - 120s_1 s_2 s_3 - 5s_1 s_3^3 + 20s_1 s_3^2 + 3s_2^4 - 5s_2^3 s_3$$

$$-8s_1^3 - 5s_1^2 s_2^2 + 15s_1^2 s_2 s_3 + 20s_1^2 s_2 - 5s_1^2 s_3^2 + 20s_1^2 s_3 - 5s_1 s_2^3 + 15s_1 s_2^2 s_3 + 20s_1 s_2^2$$

$$-8s_2^3 - 5s_1^3 s_3 - 5s_2^2 s_3^2 + 3s_3^4 + 20s_2^2 s_3 - 5s_2 s_3^3 + 20s_2 s_3^2 - 8s_3^3)$$

$$D_{34}^{[3]_4} = \frac{-1}{(2520h^2 s_1 s_2 s_3)}(3s_1^5 s_2 + 3s_1^5 s_3 - 5s_1^4 s_2^2 - 10s_1^4 s_2 s_3 - 8s_1^4 s_2 - 5s_1^4 s_3^2 + 20s_1^3 s_3^2$$

$$-8s_1^4 s_3 - 5s_1^3 s_2^3 + 10s_1^3 s_2^2 s_3 + 20s_1^3 s_2^2 + 10s_1^3 s_2 s_3^2 + 40s_1^3 s_2 s_3 - 5s_1^3 s_3^3 - 5s_1^2 s_2^4 - 5s_2^2 s_3^4$$

$$+10s_1^2 s_2 s_3^3 - 100s_1^2 s_2 s_3^2 - 5s_1^2 s_3^4 + 20s_1^2 s_3^3 + 3s_1 s_2^5 - 10s_1 s_2^4 s_3 - 8s_1 s_2^4 + 10s_1^2 s_2^3 s_3$$

$$+10s_1 s_2^3 s_3^2 + 40s_1 s_2^3 s_3 + 10s_1 s_2^2 s_3^3 - 100s_1 s_2^2 s_3^2 - 10s_1 s_2 s_3^4 + 40s_1 s_2 s_3^3 + 20s_1^2 s_2^3$$

$$+3s_1 s_3^5 - 8s_1 s_3^4 + 3s_2^5 s_3 - 5s_2^4 s_3^2 - 8s_2^4 s_3 - 5s_2^3 s_3^3 + 20s_2^3 s_3^2 + 30s_1^2 s_2^2 s_3^2 - 100s_1^2 s_2^2 s_3$$

$$+20s_2^2 s_3^3 + 3s_2 s_3^5 - 8s_2 s_3^4)$$

$$D_{44}^{[3]_4} = \frac{1}{(840h^3 s_1 s_2 s_3)}(3s_1^5 - 5s_1^4 s_2 - 5s_1^4 s_3 - 8s_1^4 - 5s_1^3 s_2^2 + 15s_1^3 s_2 s_3 + 20s_2^2 s_3^2$$

$$+20s_1^3 s_2 - 5s_1^3 s_3^2 + 20s_1^3 s_3 - 5s_1^2 s_2^3 + 15s_1^2 s_2^2 s_3 + 20s_1^2 s_2^2 + 15s_1^2 s_2 s_3^2 - 120s_1^2 s_2 s_3$$

$$-5s_1^2 s_3^3 + 20s_1^2 s_3^2 - 5s_1 s_2^4 + 15s_1 s_2^3 s_3 + 20s_1 s_2^3 + 15s_1 s_2^2 s_3^2 - 120s_1 s_2^2 s_3 + 15s_1 s_2 s_3^3$$

$$-5s_2 s_3^4 - 120s_1 s_2 s_3^2 - 5s_1 s_3^4 + 20s_1 s_3^3 - 5s_2^4 s_3 - 8s_2^4 - 5s_2^3 s_3^2 + 20s_2^3 s_3 - 5s_2^2 s_3^3$$

$$+20s_2 s_3^3 + 3s_3^5 - 8s_3^4 + 3s_2^5)$$

$$E_{11}^{[3]_4} = \frac{-(s_2-1)(s_3-1)}{5040 s_1 (s_1-s_2)(s_1-s_3)}(3s_1^4 - 3s_1^3 s_2 - 3s_1^3 s_3 - 3s_1^3 - 3s_1^2 s_2^2 + 5s_1^2 s_2 s_3$$

$$+5s_1^2 s_2 - 3s_1^2 s_3^2 + 5s_1^2 s_3 - 3s_1^2 - 3s_1 s_2^3 + 5s_1 s_2^2 s_3 + 5s_1 s_2^2 + 5s_1 s_2 s_3^2 - 15s_1 s_2 s_3$$

$$-3s_1 s_3^3 + 5s_1 s_3^2 + 5s_1 s_3 - 3s_1 - 3s_2^4 + 5s_2^3 s_3 + 5s_2^3 + 5s_2^2 s_3^2 - 15s_2^2 s_3 + 5s_2^2 + 5s_3^2$$

$$+5s_1 s_2 + 5s_2 s_3^3 - 15s_2 s_3^2 - 15s_2 s_3 + 5s_2 - 3s_3^4 + 5s_3^3 + 5s_3^2 + 5s_3 - 3)$$

$$E_{12}^{[3]_4} = \frac{(s_1-1)(s_3-1)}{5040s_2(s_1-s_2)(s_2-s_3)}(5s_1^3 - 3s_1^4 - 3s_1^3s_2 + 5s_1^3s_3 - 3s_1^2s_2^2 + 5s_1^2s_2s_3$$

$$+5s_1^2s_3^2 - 15s_1^2s_3 + 5s_1^2 - 3s_1s_2^3 + 5s_1s_2^2s_3 + 5s_1s_2^2 + 5s_1s_2s_3^2 - 15s_1s_2s_3 + 5s_1s_2$$

$$+5s_1s_3^3 - 15s_1s_3^2 - 15s_1s_3 + 5s_1 + 3s_2^4 - 3s_2^3s_3 - 3s_2^3 - 3s_2^2s_3^2 + 5s_2^2s_3 - 3s_2^2 + 5s_3$$

$$+5s_1^2s_2 - 3s_2s_3^3 + 5s_2s_3^2 + 5s_2s_3 - 3s_2 - 3s_3^4 + 5s_3^3 + 5s_3^2 - 3)$$

$$E_{13}^{[3]_4} = -\frac{(s_1-1)(s_2-1)}{(5040s_3(s_1-s_3)(s_2-s_3))}(-3s_1^4 + 5s_1^3s_2 - 3s_1^3s_3 + 5s_1^3 + 5s_1^2s_2^2 + 5s_1^2s_2s_3$$

$$-15s_1^2s_2 - 3s_1^2s_3^2 + 5s_1^2s_3 + 5s_1^2 + 5s_1s_2^3 + 5s_1s_2^2s_3 - 15s_1s_2^2 + 5s_1s_2s_3^2 - 15s_1s_2s_3$$

$$-15s_1s_2 - 3s_1s_3^3 + 5s_1s_3^2 + 5s_1s_3 + 5s_1 - 3s_2^4 - 3s_2^3s_3 + 5s_2^3 - 3s_2^2s_3^2 + 5s_2^2s_3 + 5s_2^2$$

$$-3s_2s_3^3 + 5s_2s_3^2 - 3s_3 + 5s_2s_3 + 5s_2 + 3s_3^4 - 3s_3^3 - 3s_3^2 - 3)$$

$$E_{14}^{[3]_4} = \frac{1}{5040}(-3s_1^4 + 5s_1^3s_2 + 5s_1^3s_3 - 3s_1^3 + 5s_1^2s_2^2 - 15s_1^2s_2s_3 + 5s_1^2s_2 + 5s_1^2s_3^2$$

$$-3s_1^2 + 5s_1s_2^3 - 15s_1s_2^2s_3 + 5s_1s_2^2 - 15s_1s_2s_3^2 - 15s_1s_2s_3 + 5s_1s_2 + 5s_1s_3^3 + 5s_1s_3^2$$

$$+5s_1s_3 - 3s_1 - 3s_2^4 + 5s_2^3s_3 - 3s_2^3 + 5s_2^2s_3^2 + 5s_2^2s_3 - 3s_2^2 + 5s_2s_3^3 + 5s_2s_3^2 + 5s_2s_3$$

$$+5s_1^2s_3 - 3s_2 - 3s_3^4 - 3s_3^3 - 3s_3^2 - 3s_3 + 3)$$

$$E_{21}^{[3]_4} = -\frac{s_2s_3}{5040h(s_1-s_2)(s_1-s_3)(s_1-1)}(-3s_1^3s_2 - 3s_1^3s_3 - 3s_1^2s_2^2 + 8s_3^3 + 5s_2^2s_3^2$$

$$+5s_1^2s_2s_3 + 8s_1^2s_2 - 3s_1^2s_3^2 + 8s_1^2s_3 - 3s_1s_2^3 + 5s_1s_2^2s_3 + 8s_1s_2^2 + 5s_1s_2s_3^2 - 20s_1s_2s_3$$

$$3s_1^4 - 6s_1^3 - 3s_1s_3^3 + 8s_1s_3^2 - 3s_2^4 + 5s_2^3s_3 + 8s_2^3 - 20s_2^2s_3 + 5s_2s_3^3 - 20s_2s_3^2 - 3s_3^4)$$

$$E_{22}^{[3]_4} = \frac{s_1s_3}{5040h(s_1-s_2)(s_2-s_3)(s_2-1)}(5s_1^3s_3 - 3s_1^3s_2 - 3s_1^2s_2^2 - 3s_3^4 + 5s_1^2s_2s_3$$

$$+8s_1^3 + 8s_1^2s_2 + 5s_1^2s_3^2 - 20s_1^2s_3 - 3s_1s_2^3 + 5s_1s_2^2s_3 + 8s_1s_2^2 + 5s_1s_2s_3^2 - 20s_1s_2s_3$$

$$-3s_1^4 + 5s_1s_3^3 - 20s_1s_3^2 + 3s_2^4 - 3s_2^3s_3 - 6s_2^3 - 3s_2^2s_3^2 + 8s_2^2s_3 - 3s_2s_3^3 + 8s_2s_3^2 + 8s_3^3)$$

$$E_{23}^{[3]_4} = -\frac{s_1s_2}{5040h(s_1-s_3)(s_2-s_3)(s_3-1)}(3s_3^4 + 5s_1^3s_2 - 3s_1^3s_3 + 5s_1^2s_2^2 + 5s_1^2s_2s_3$$

$$-20s_1^2s_2 - 3s_1^2s_3^2 + 8s_1^2s_3 + 5s_1s_2^3 + 5s_1s_2^2s_3 - 20s_1s_2^2 + 5s_1s_2s_3^2 - 20s_1s_2s_3 - 3s_1s_3^3$$

$$+8s_1^3 + 8s_1s_3^2 - 3s_2^4 - 3s_2^3s_3 + 8s_2^3 - 3s_2^2s_3^2 + 8s_2^2s_3 - 3s_2s_3^3 + 8s_2s_3^2 - 6s_3^3 - 3s_1^4)$$

$$E_{24}^{[3]_4} = \frac{s_1s_2s_3}{5040h(s_1-1)(s_2-1)(s_3-1)}(5s_2^2s_3^2 5s_2^3s_3 - 3s_1^4 + 5s_1^3s_2 + 5s_1^3s_3 + 5s_1^2s_2^2$$

$$-15s_1^2s_2s_3 + 5s_1^2s_3^2 + 5s_1s_2^3 - 15s_1s_2^2s_3 - 15s_1s_2s_3^2 + 5s_1s_3^3 - 3s_2^4 + 5s_2s_3^3 - 3s_3^4)$$

$$E_{31}^{[3]_4} = \frac{1}{(2520h^2\, s_1(s_1-s_2)(s_1-s_3)\,(s_1-1))}\,(3s_1^5s_2+3s_1^5s_3-3s_1^4s_2^2-6s_1^4s_2s_3$$
$$-6s_1^4s_2-3s_1^3s_2^3+2s_1^3s_2^2s_3+8s_1^3s_2^2+2s_1^3s_2s_3^2+16s_1^3s_2s_3-3s_1^3s_3^3+8s_1^3s_3^2-3s_1^4s_3^2$$
$$-3s_1^2s_2^4+2s_1^2s_2^3s_3+8s_1^2s_2^3+10s_1^2s_2^2s_3^2-12s_1^2s_2^2s_3+2s_1^2s_2s_3^3-12s_1^2s_2s_3^2-6s_1^4s_3$$
$$-3s_1^2s_3^4+8s_1^2s_3^3-3s_1s_2^5+2s_1s_2^4s_3+8s_1s_2^4+10s_1s_2^3s_3^2-12s_1s_2^3s_3+5s_2^4s_3^2$$
$$+10s_1s_2^2s_3^3-40s_1s_2^2s_3^2+2s_1s_2s_3^4-12s_1s_2s_3^3-3s_1s_3^5+8s_1s_3^4-3s_2^5s_3+8s_2^4s_3$$
$$+5s_2^3s_3^3-20s_2^3s_3^2+5s_2^2s_3^4-20s_2^2s_3^3-3s_2s_3^5+8s_2s_3^4)$$
$$E_{32}^{[3]_4} = -\frac{1}{(2520h^2s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(-3s_1^5s_2-3s_1^5s_3-3s_1^4s_2^2+2s_1^4s_2s_3$$
$$+8s_1^4s_2+5s_1^4s_3^2+8s_1^4s_3-3s_1^3s_2^3+2s_1^3s_2^2s_3+8s_1^3s_2^2+10s_1^3s_2s_3^2+2s_1s_2s_3^4+3s_1s_2^5$$
$$-12s_1^3s_2s_3+5s_1^3s_3^3-20s_1^3s_3^2-3s_1^2s_2^4+2s_1^2s_2^3s_3+8s_1^2s_2^3-3s_1s_3^5+8s_1s_3^4+3s_2^5s_3$$
$$+10s_1^2s_2^2s_3^2-12s_1^2s_2^2s_3+10s_1^2s_2s_3^3-40s_1^2s_2s_3^2+5s_1^2s_3^4-20s_1^2s_3^3-3s_2^4s_3^2-6s_2^4s_3$$
$$-6s_1s_2^4s_3-6s_1s_2^4+2s_1s_2^3s_3^2+16s_1s_2^3s_3+2s_1s_2^2s_3^3-12s_1s_2^2s_3^2-12s_1s_2s_3^3-3s_2^3s_3^3$$
$$+8s_2^3s_3^2-3s_2^2s_3^4+8s_2^2s_3^3-3s_2s_3^5+8s_2s_3^4)$$
$$E_{33}^{[3]_4} = \frac{1}{(2520h^2s_3(s_1-s_3)(s_2-s_3)(s_3-1))}\,(-3s_1^5s_2-3s_1^5s_3+5s_1^4s_2^2+2s_1^4s_2s_3$$
$$+8s_1^4s_2-3s_1^4s_3^2+8s_1^4s_3+5s_1^3s_2^3+10s_1^3s_2^2s_3-20s_1^3s_2^2+2s_1^3s_2s_3^2+3s_1s_3^5-6s_1s_3^4$$
$$-12s_1^3s_2s_3-3s_1^3s_3^3+8s_1^3s_3^2+5s_1^2s_2^4+10s_1^2s_2^3s_3-20s_1^2s_2^3+10s_1^2s_2^2s_3^2+16s_1s_2s_3^3$$
$$+8s_2^4s_3+2s_1^2s_2s_3^3-12s_1^2s_2s_3^2-3s_1^2s_3^4+8s_1^2s_3^3-3s_1s_2^5+2s_1s_2^4s_3-3s_2^3s_3^3+8s_2^3s_3^2$$
$$+8s_1s_2^4+2s_1s_2^3s_3^2-12s_1s_2^3s_3+2s_1s_2^2s_3^3-12s_1s_2^2s_3^2-6s_1s_2s_3^4-3s_2^5s_3-40s_1^2s_2^2s_3$$
$$-3s_2^4s_3^2-3s_2^2s_3^4+8s_2^2s_3^3+3s_2s_3^5-6s_2s_3^4)$$
$$E_{34}^{[3]_4} = \frac{1}{h^2(2520s_1-2520)(s_2-1)(s_3-1)}(3s_1^5s_2+3s_1^5s_3-5s_1^4s_2^2+3s_1s_3^5+3s_2^5s_3$$
$$-10s_1^4s_2s_3-5s_1^4s_3^2-5s_1^3s_2^3+10s_1^3s_2^2s_3+10s_1^3s_2s_3^2-5s_1^3s_3^3-10s_1s_2^4s_3+10s_1s_2^3s_3^2$$
$$-5s_1^2s_2^4+10s_1^2s_2^3s_3+30s_1^2s_2^2s_3^2+10s_1^2s_2s_3^3-5s_1^2s_3^4+3s_1s_2^5+10s_1s_2^2s_3^3-10s_1s_2s_3^4$$
$$-5s_2^4s_3^2-5s_2^3s_3^3-5s_2^2s_3^4+3s_2s_3^5)$$

$$E_{41}^{[3]_4} = \frac{-1}{840h^3 s_1(s_1-s_2)(s_1-s_3)(s_1-1)}(3s_1^5 - 3s_1^4s_2 - 3s_1^4s_3 - 6s_1^4 - 3s_1^3s_2^2 - 3s_3^5$$
$$+8s_1^3s_2 - 3s_1^3s_3^2 + 8s_1^3s_3 - 3s_1^2s_2^3 + 5s_1^2s_2^2s_3 + 8s_1^2s_2^2 + 5s_1^2s_2s_3^2 - 20s_1^2s_2s_3 - 20s_2^2s_3^2$$
$$-3s_1^2s_3^3 + 8s_1^2s_3^2 - 3s_1s_2^4 + 5s_1s_2^3s_3 + 8s_1s_2^3 + 5s_1s_2^2s_3^2 - 20s_1s_2^2s_3 + 5s_1s_2s_3^3 + 5s_2s_3^4$$
$$-20s_1s_2s_3^2 - 3s_1s_3^4 + 8s_1s_3^3 - 3s_2^5 + 5s_2^4s_3 + 8s_2^4 + 5s_2^3s_3^2 - 20s_2^3s_3 + 5s_2^2s_3^3 - 20s_2s_3^3$$
$$+5s_1^3s_2s_3 + 8s_3^4)$$
$$E_{42}^{[3]_4} = \frac{1}{840h^3 s_2(s_1-s_2)(s_2-s_3)(s_2-1)}(8s_3^4 - 3s_1^5 - 3s_1^4s_2 + 5s_1^4s_3 - 3s_1^3s_2^2 - 3s_3^5$$
$$+8s_1^3s_2 + 5s_1^3s_3^2 - 20s_1^3s_3 - 3s_1^2s_2^3 + 5s_1^2s_2^2s_3 + 8s_1^2s_2^2 + 5s_1^2s_2s_3^2 - 20s_1^2s_2s_3 + 8s_2^2s_3^2$$
$$+5s_1^2s_3^3 - 20s_1^2s_3^2 - 3s_1s_2^4 + 5s_1s_2^3s_3 + 8s_1s_2^3 + 5s_1s_2^2s_3^2 - 20s_1s_2^2s_3 + 5s_1s_2s_3^3$$
$$-20s_1s_2s_3^2 + 5s_1s_3^4 - 20s_1s_3^3 + 3s_2^5 - 3s_2^4s_3 - 6s_2^4 - 3s_2^3s_3^2 + 8s_2^3s_3 - 3s_2^2s_3^3 + 8s_2s_3^3$$
$$+5s_1^3s_2s_3 + 8s_1^4 - 3s_2s_3^4)$$
$$E_{43}^{[3]_4} = \frac{-1}{840h^3 s_3(s_1-s_3)(s_2-s_3)(s_3-1)}(5s_1^4s_2 - 3s_1^5 - 3s_1^4s_3 + 5s_1^3s_2^2 + 5s_1^3s_2s_3$$
$$-20s_1^3s_2 - 3s_1^3s_3^2 + 8s_1^3s_3 + 5s_1^2s_2^3 + 5s_1^2s_2^2s_3 - 20s_1^2s_2^2 + 5s_1^2s_2s_3^2 - 20s_1^2s_2s_3 - 3s_1^2s_3^3$$
$$+8s_1^2s_3^2 + 5s_1s_2^4 + 5s_1s_2^3s_3 - 20s_1s_2^3 + 5s_1s_2^2s_3^2 - 20s_1s_2^2s_3 + 5s_1s_2s_3^3 - 20s_1s_2s_3^2$$
$$-3s_1s_3^4 + 8s_1s_3^3 - 3s_2^5 - 3s_2^4s_3 + 8s_2^4 - 3s_2^3s_3^2 + 8s_2^3s_3 - 3s_2^2s_3^3 + 8s_2^2s_3^2 - 3s_2s_3^4$$
$$+8s_2s_3^3 + 3s_3^5 - 6s_3^4 + 8s_1^4)$$
$$E_{44}^{[3]_4} = \frac{1}{840h^3(s_1-1)(s_2-1)(s_3-1)}(5s_1^4s_2 - 3s_1^5 + 5s_1^4s_3 - 3s_3^5 + 5s_1^3s_3^2 + 5s_1^3s_2^2$$
$$+5s_1^2s_2^3 - 15s_1^2s_2^2s_3 - 15s_1^2s_2s_3^2 + 5s_1^2s_3^3 + 5s_1s_2^4 - 15s_1s_2^3s_3 - 15s_1s_2^2s_3^2 - 15s_1s_2s_3^3$$
$$-15s_1^3s_2s_3 + 5s_1s_3^4 + 5s_2^4s_3 + 5s_2^3s_3^2 + 5s_2^2s_3^3 + 5s_2s_3^4 - 3s_2^5)$$

Multiplying Equation (5.29) by inverse of $A^{[3]_4}$ gives a hybrid block method of the form

$$
\begin{aligned}
I^{[3]_4} Y_m^{[3]_4} &= \bar{B}_1^{[3]_4} R_1^{[3]_4} + \bar{B}_2^{[3]_4} R_2^{[3]_4} + \bar{B}_3^{[3]_4} R_3^{[3]_4} + \bar{B}_4^{[3]_4} R_4^{[3]_4} \\
&+ h^4 \bar{D}^{[3]_4} R_5^{[3]_4} + h^4 \bar{E}^{[3]_4} R_6^{[3]_4} \qquad (5.30)
\end{aligned}
$$

where

$$
I^{[3]_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \bar{B}_1^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
\bar{B}_2^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & s_1 h \\ 0 & 0 & 0 & s_2 h \\ 0 & 0 & 0 & s_3 h \\ 0 & 0 & 0 & h \end{pmatrix}, \bar{B}_3^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \frac{s_1^2}{2} h \\ 0 & 0 & 0 & \frac{s_2^2}{2} h \\ 0 & 0 & 0 & \frac{s_3^2}{2} h \\ 0 & 0 & 0 & \frac{1}{2} h \end{pmatrix}
$$

$$
\bar{B}_4^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \frac{s_1^3}{6} h \\ 0 & 0 & 0 & \frac{s_2^3}{6} h \\ 0 & 0 & 0 & \frac{s_3^3}{6} h \\ 0 & 0 & 0 & \frac{1}{6} h \end{pmatrix}, \bar{D}^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \bar{D}_{14}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{24}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{34}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{44}^{[3]_4} \end{pmatrix}
$$

$$
\bar{E}^{[3]_4} = \begin{pmatrix} \bar{E}_{11}^{[3]_4} & \bar{E}_{12}^{[3]_4} & \bar{E}_{13}^{[3]_4} & \bar{E}_{14}^{[3]_4} \\ \bar{E}_{21}^{[3]_4} & \bar{E}_{22}^{[3]_4} & \bar{E}_{23}^{[3]_4} & \bar{E}_{24}^{[3]_4} \\ \bar{E}_{31}^{[3]_4} & \bar{E}_{32}^{[3]_4} & \bar{E}_{33}^{[3]_4} & \bar{E}_{34}^{[3]_4} \\ \bar{E}_{41}^{[3]_4} & \bar{E}_{42}^{[3]_4} & \bar{E}_{43}^{[3]_4} & \bar{E}_{44}^{[3]_4} \end{pmatrix}
$$

and the non -zero terms of $\bar{D}^{[3]_4}$ and $\bar{E}^{[3]_4}$ are given by

$$\bar{D}^{[3]_4}_{14} = -\frac{(s_1^4(28s_1s_2+28s_1s_3-168s_2s_3-8s_1^2s_2-8s_1^2s_3-8s_1^2+3s_1^3+28s_1s_2s_3))}{(5040s_2s_3)}$$
$$\bar{D}^{[3]_4}_{24} = \frac{(s_2^4(168s_1s_3-28s_1s_2-28s_2s_3+8s_1s_2^2+8s_2^2s_3+8s_2^2-3s_2^3-28s_1s_2s_3))}{(5040s_1s_3)}$$
$$\bar{D}^{[3]_4}_{34} = -\frac{(s_3^4(28s_1s_3-168s_1s_2+28s_2s_3-8s_1s_3^2-8s_2s_3^2-8s_3^2+3s_3^3+28s_1s_2s_3))}{(5040s_1s_2)}$$
$$\bar{D}^{[3]_4}_{44} = \frac{((8s_1+8s_2+8s_3-28s_1s_2-28s_1s_3-28s_2s_3+168s_1s_2s_3-3))}{(5040s_1s_2s_3)}$$
$$\bar{E}^{[3]_4}_{11} = \frac{(s_1^4(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))}$$
$$\bar{E}^{[3]_4}_{12} = \frac{(s_1^6(28s_3-8s_1-8s_1s_3+3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}$$
$$\bar{E}^{[3]_4}_{13} = -\frac{(s_1^6(28s_2-8s_1-8s_1s_2+3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\bar{E}^{[3]_4}_{14} = \frac{(s_1^6(28s_2s_3-8s_1s_3-8s_1s_2+3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))}$$
$$\bar{E}^{[3]_4}_{21} = -\frac{(s_2^6(28s_3-8s_2-8s_2s_3+3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$
$$\bar{E}^{[3]_4}_{22} = \frac{(s_2^4(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))}$$
$$\bar{E}^{[3]_4}_{23} = \frac{(s_2^6(8s_2-28s_1+8s_1s_2-3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\bar{E}^{[3]_4}_{24} = -\frac{(s_2^6(8s_1s_2-28s_1s_3+8s_2s_3-3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))}$$
$$\bar{E}^{[3]_4}_{31} = \frac{(s_3^6(8s_3-28s_2+8s_2s_3-3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$
$$\bar{E}^{[3]_4}_{32} = -\frac{(h^4s_3^6(8s_3-28s_1+8s_1s_3-3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}$$
$$\bar{E}^{[3]_4}_{33} = \frac{(s_3^4(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\bar{E}^{[3]_4}_{34} = \frac{(s_3^6(28s_1s_2-8s_1s_3-8s_2s_3+3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))}$$
$$\bar{E}^{[3]_4}_{41} = -\frac{((28s_2s_3-8s_3-8s_2+3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$
$$\bar{E}^{[3]_4}_{42} = \frac{((28s_1s_3-8s_3-8s_1+3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}$$
$$\bar{E}^{[3]_4}_{43} = -\frac{((28s_1s_2-8s_2-8s_1+3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\bar{E}^{[3]_4}_{44} = \frac{((6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3))}{(5040(s_3-1)(s_2-1)(s_1-1))}$$

Equation (5.30) can also be written as

$$
\begin{aligned}
y_{n+s_1} &= y_n + s_1 h y_n^{'} + \frac{s_1^2 h^2}{2} y_n^{''} + \frac{s_1^3 h^3}{6} y_n^{'''} \\
&- \frac{(h^4 s_1^4 (28 s_1 s_2 + 28 s_1 s_3 - 168 s_2 s_3 - 8 s_1^2 s_2 - 8 s_1^2 s_3 - 8 s_1^2 + 3 s_1^3 + 28 s_1 s_2 s_3))}{(5040 s_2 s_3)} f_n \\
&+ \frac{(h^4 s_1^4 (14 s_1 s_2 + 14 s_1 s_3 - 42 s_2 s_3 - 6 s_1^2 s_2 - 6 s_1^2 s_3 - 6 s_1^2 + 3 s_1^3 + 14 s_1 s_2 s_3))}{(5040 (s_1 - 1)(s_1 - s_3)(s_1 - s_2))} f_{n+s_1} \\
&+ \frac{(h^4 s_1^6 (28 s_3 - 8 s_1 - 8 s_1 s_3 + 3 s_1^2))}{(5040 s_2 (s_2 - 1)(s_2 - s_3)(s_1 - s_2))} f_{n+s_2} \\
&- \frac{(h^4 s_1^6 (28 s_2 - 8 s_1 - 8 s_1 s_2 + 3 s_1^2))}{(5040 s_3 (s_3 - 1)(s_2 - s_3)(s_1 - s_3))} f_{n+s_3} \\
&+ \frac{(h^4 s_1^6 (28 s_2 s_3 - 8 s_1 s_3 - 8 s_1 s_2 + 3 s_1^2))}{(5040 (s_3 - 1)(s_2 - 1)(s_1 - 1))} f_{n+1} \qquad (5.31)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+s_2} &= y_n + s_2 h y_n^{'} + \frac{s_2^2 h^2}{2} y_n^{''} + \frac{s_2^3 h^3}{6} y_n^{'''} \\
&+ \frac{(h^4 s_2^4 (168 s_1 s_3 - 28 s_1 s_2 - 28 s_2 s_3 + 8 s_1 s_2^2 + 8 s_2^2 s_3 + 8 s_2^2 - 3 s_2^3 - 28 s_1 s_2 s_3))}{(5040 s_1 s_3)} f_n \\
&- \frac{(h^4 s_2^6 (28 s_3 - 8 s_2 - 8 s_2 s_3 + 3 s_2^2))}{(5040 s_1 (s_1 - 1)(s_1 - s_3)(s_1 - s_2))} f_{n+s_1} \\
&+ \frac{(h^4 s_2^4 (42 s_1 s_3 - 14 s_1 s_2 - 14 s_2 s_3 + 6 s_1 s_2^2 + 6 s_2^2 s_3 + 6 s_2^2 - 3 s_2^3 - 14 s_1 s_2 s_3))}{(5040 (s_2 - 1)(s_2 - s_3)(s_1 - s_2))} f_{n+s_2} \\
&+ \frac{(h^4 s_2^6 (8 s_2 - 28 s_1 + 8 s_1 s_2 - 3 s_2^2))}{(5040 s_3 (s_3 - 1)(s_2 - s_3)(s_1 - s_3))} f_{n+s_3} \\
&- \frac{(h^4 s_2^6 (8 s_1 s_2 - 28 s_1 s_3 + 8 s_2 s_3 - 3 s_2^2))}{(5040 (s_3 - 1)(s_1 - 1)(s_2 - 1))} f_{n+1} \qquad (5.32)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+s_3} = {} & y_n + s_3 h y_n^{'} + \frac{s_3^2 h^2}{2} y_n^{''} + \frac{s_3^3 h^3}{6} y_n^{'''} \\
& - \frac{(h^4 s_3^4 (28 s_1 s_3 - 168 s_1 s_2 + 28 s_2 s_3 - 8 s_1 s_3^2 - 8 s_2 s_3^2 - 8 s_3^2 + 3 s_3^3 + 28 s_1 s_2 s_3))}{(5040 s_1 s_2)} f_n \\
& + \frac{(h^4 s_3^6 (8 s_3 - 28 s_2 + 8 s_2 s_3 - 3 s_3^2))}{(5040 s_1 (s_1 - 1)(s_1 - s_3)(s_1 - s_2))} f_{n+s_1} \\
& - \frac{(h^4 s_3^6 (8 s_3 - 28 s_1 + 8 s_1 s_3 - 3 s_3^2))}{(5040 s_2 (s_2 - 1)(s_2 - s_3)(s_1 - s_2))} f_{n+s_2} \\
& + \frac{(h^4 s_3^4 (14 s_1 s_3 - 42 s_1 s_2 + 14 s_2 s_3 - 6 s_1 s_3^2 - 6 s_2 s_3^2 - 6 s_3^2 + 3 s_3^3 + 14 s_1 s_2 s_3))}{(5040 (s_3 - 1)(s_2 - s_3)(s_1 - s_3))} f_{n+s_3} \\
& + \frac{(h^4 s_3^6 (28 s_1 s_2 - 8 s_1 s_3 - 8 s_2 s_3 + 3 s_3^2))}{(5040 (s_2 - 1)(s_1 - 1)(s_3 - 1))} f_{n+1}
\end{aligned}
\tag{5.33}
$$

$$
\begin{aligned}
y_{n+1} = {} & y_n + h y_n^{'} + \frac{h^2}{2} y_n^{''} + \frac{h^3}{6} y_n^{'''} \\
& + \frac{(h^4 (8 s_1 + 8 s_2 + 8 s_3 - 28 s_1 s_2 - 28 s_1 s_3 - 28 s_2 s_3 + 168 s_1 s_2 s_3 - 3))}{(5040 s_1 s_2 s_3)} f_n \\
& - \frac{(h^4 (28 s_2 s_3 - 8 s_3 - 8 s_2 + 3))}{(5040 s_1 (s_1 - 1)(s_1 - s_3)(s_1 - s_2))} f_{n+s_1} \\
& + \frac{(h^4 (28 s_1 s_3 - 8 s_3 - 8 s_1 + 3))}{(5040 s_2 (s_2 - 1)(s_2 - s_3)(s_1 - s_2))} f_{n+s_2} \\
& - \frac{(h^4 (28 s_1 s_2 - 8 s_2 - 8 s_1 + 3))}{(5040 s_3 (s_3 - 1)(s_2 - s_3)(s_1 - s_3))} f_{n+s_3} \\
& + \frac{(h^4 (6 s_1 + 6 s_2 + 6 s_3 - 14 s_1 s_2 - 14 s_1 s_3 - 14 s_2 s_3 + 42 s_1 s_2 s_3 - 3))}{(5040 (s_3 - 1)(s_2 - 1)(s_1 - 1))} f_{n+1}
\end{aligned}
\tag{5.34}
$$

Substituting Equations (5.31), (5.32) and (5.33) into (5.15) –(5.18) gives the first derivative of the block as below

$$
\begin{aligned}
y'_{n+s_1} &= y'_n + s_1 h y''_n + \frac{s_1^2 h^2}{2} y'''_n \\
&- \frac{(h^3 s_1^3(21s_1s_2 + 21s_1s_3 - 105s_2s_3 - 7s_1^2s_2 - 7s_1^2s_3 - 7s_1^2 + 3s_1^3 + 21s_1s_2s_3))}{(840s_2s_3)} f_n \\
&+ \frac{(h^3 s_1^3(14s_1s_2 + 14s_1s_3 - 35s_2s_3 - 7s_1^2s_2 - 7s_1^2s_3 - 7s_1^2 + 4s_1^3 + 14s_1s_2s_3))}{(840(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{(h^3 s_1^5(21s_3 - 7s_1 - 7s_1s_3 + 3s_1^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&- \frac{(h^3 s_1^5(21s_2 - 7s_1 - 7s_1s_2 + 3s_1^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{(h^3 s_1^5(21s_2s_3 - 7s_1s_3 - 7s_1s_2 + 3s_1^2))}{(840(s_3-1)(s_2-1)(s_1-1))} f_{n+1}
\end{aligned}
\tag{5.35}
$$

$$
\begin{aligned}
y'_{n+s_2} &= y'_n + s_2 h y''_n + \frac{s_2^2 h^2}{2} y'''_n \\
&+ \frac{(h^3 s_2^3(105s_1s_3 - 21s_1s_2 - 21s_2s_3 + 7s_1s_2^2 + 7s_2^2s_3 + 7s_2^2 - 3s_2^3 - 21s_1s_2s_3))}{(840s_1s_3)} f_n \\
&- \frac{(h^3 s_2^5(21s_3 - 7s_2 - 7s_2s_3 + 3s_2^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{(h^3 s_2^3(35s_1s_3 - 14s_1s_2 - 14s_2s_3 + 7s_1s_2^2 + 7s_2^2s_3 + 7s_2^2 - 4s_2^3 - 14s_1s_2s_3))}{(840(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{(h^3 s_2^5(7s_2 - 21s_1 + 7s_1s_2 - 3s_2^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&- \frac{(h^3 s_2^5(7s_1s_2 - 21s_1s_3 + 7s_2s_3 - 3s_2^2))}{(840(s_3-1)(s_1-1)(s_2-1))} f_{n+1}
\end{aligned}
\tag{5.36}
$$

$$
\begin{aligned}
y'_{n+s_3} &= y'_n + s_3 h y''_n + \frac{s_3^2 h^2}{2} y'''_n \\
&- \frac{(h^3 s_3^3(21s_1s_3 - 105s_1s_2 + 21s_2s_3 - 7s_1s_3^2 - 7s_2s_3^2 - 7s_3^2 + 3s_3^3 + 21s_1s_2s_3))}{(840s_1s_2)} f_n \\
&+ \frac{(h^3 s_3^5(7s_3 - 21s_2 + 7s_2s_3 - 3s_3^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&- \frac{(h^3 s_3^5(7s_3 - 21s_1 + 7s_1s_3 - 3s_3^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{(h^3 s_3^3(14s_1s_3 - 35s_1s_2 + 14s_2s_3 - 7s_1s_3^2 - 7s_2s_3^2 - 7s_3^2 + 4s_3^3 + 14s_1s_2s_3))}{(840(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{(h^3 s_3^5(21s_1s_2 - 7s_1s_3 - 7s_2s_3 + 3s_3^2))}{(840(s_2-1)(s_1-1)(s_3-1))} f_{n+1}
\end{aligned}
\tag{5.37}
$$

$$
\begin{aligned}
y'_{n+1} &= y'_n + h y''_n + \frac{h^2}{2} y'''_n \\
&+ \frac{(h^3(7s_1 + 7s_2 + 7s_3 - 21s_1s_2 - 21s_1s_3 - 21s_2s_3 + 105s_1s_2s_3 - 3))}{(840s_1s_2s_3)} f_n \\
&- \frac{(h^3(21s_2s_3 - 7s_3 - 7s_2 + 3))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{(h^3(21s_1s_3 - 7s_3 - 7s_1 + 3))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&- \frac{(h^3(21s_1s_2 - 7s_2 - 7s_1 + 3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{(h^3(7s_1 + 7s_2 + 7s_3 - 14s_1s_2 - 14s_1s_3 - 14s_2s_3 + 35s_1s_2s_3 - 4))}{(840(s_3-1)(s_2-1)(s_1-1))} f_{n+1}
\end{aligned}
\tag{5.38}
$$

In order to find the properties of the first derivative method shown above, (5.35) to (5.38) are written in a block form

$$
\acute{Y}_m^{[3]_4} \;=\; \acute{B}_2^{[3]_4} R_2^{[3]_4} + \acute{B}_3^{[3]_4} R_3^{[3]_4} + \acute{B}_4^{[3]_4} R_4^{[3]_4} + h^3 \acute{D}^{[3]_4} R_5^{[3]_4} + h^3 \acute{E}^{[3]_4} R_6^{[3]_4} \tag{5.39}
$$

where

$$\acute{Y}_m^{[3]_4} = \begin{pmatrix} y'_{n+s_1} \\ y'_{n+s_2} \\ y'_{n+s_3} \\ y'_{n+1} \end{pmatrix}, \ \acute{B}_2^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\acute{B}_3^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & s_1h \\ 0 & 0 & 0 & s_2h \\ 0 & 0 & 0 & s_3h \\ 0 & 0 & 0 & h \end{pmatrix}, \ \acute{B}_4^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \frac{s_1^2}{2}h \\ 0 & 0 & 0 & \frac{s_2^2}{2}h \\ 0 & 0 & 0 & \frac{s_3^2}{2}h \\ 0 & 0 & 0 & \frac{1}{2}h \end{pmatrix},$$

$$\acute{D}^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \acute{D}_{14}^{[3]_4} \\ 0 & 0 & 0 & \acute{D}_{24}^{[3]_4} \\ 0 & 0 & 0 & \acute{D}_{34}^{[3]_4} \\ 0 & 0 & 0 & \acute{D}_{44}^{[3]_4} \end{pmatrix}, \ \acute{E}^{[3]_4} = \begin{pmatrix} \acute{E}_{11}^{[3]_4} & \acute{E}_{12}^{[3]} & \acute{E}_{13}^{[3]_4} & \acute{E}_{14}^{[3]_4} \\ \acute{E}_{21}^{[3]_4} & \acute{E}_{22}^{[3]_4} & \acute{E}_{23}^{[3]_4} & \acute{E}_{24}^{[3]_4} \\ \acute{E}_{31}^{[3]_4} & \acute{E}_{32}^{[3]_4} & \acute{E}_{33}^{[3]_4} & \acute{E}_{34}^{[3]_4} \\ \acute{E}_{41}^{[3]_4} & \acute{E}_{42}^{[3]_4} & \acute{E}_{43}^{[3]_4} & \acute{E}_{44}^{[3]_4} \end{pmatrix}$$

and the non-zero terms of $\acute{D}^{[3]_4}$ and $\acute{E}^{[3]_4}$ are given by

$$\acute{D}_{14}^{[3]_4} = -\frac{(s_1^3(21s_1s_2 + 21s_1s_3 - 105s_2s_3 - 7s_1^2s_2 - 7s_1^2s_3 - 7s_1^2 + 3s_1^3 + 21s_1s_2s_3))}{(840s_2s_3)}$$

$$\acute{D}_{24}^{[3]_4} = \frac{(s_2^3(105s_1s_3 - 21s_1s_2 - 21s_2s_3 + 7s_1s_2^2 + 7s_2^2s_3 + 7s_2^2 - 3s_2^3 - 21s_1s_2s_3))}{(840s_1s_3)}$$

$$\acute{D}_{34}^{[3]_4} = \frac{(s_3^3(21s_1s_3 - 105s_1s_2 + 21s_2s_3 - 7s_1s_3^2 - 7s_2s_3^2 - 7s_3^2 + 3s_3^3 + 21s_1s_2s_3))}{(840s_1s_2)}$$

$$\acute{D}_{44}^{[3]_4} = \frac{((7s_1 + 7s_2 + 7s_3 - 21s_1s_2 - 21s_1s_3 - 21s_2s_3 + 105s_1s_2s_3 - 3))}{(840s_1s_2s_3)}$$

$$\acute{E}_{11}^{[3]_4} = \frac{(s_1^3(14s_1s_2 + 14s_1s_3 - 35s_2s_3 - 7s_1^2s_2 - 7s_1^2s_3 - 7s_1^2 + 4s_1^3 + 14s_1s_2s_3))}{(840(s_1 - 1)(s_1 - s_3)(s_1 - s_2))}$$

$$\acute{E}_{12}^{[3]_4} = \frac{(s_1^5(21s_3 - 7s_1 - 7s_1s_3 + 3s_1^2))}{(840s_2(s_2 - 1)(s_2 - s_3)(s_1 - s_2))}$$

$$\acute{E}_{13}^{[3]_4} = -\frac{(s_1^5(21s_2 - 7s_1 - 7s_1s_2 + 3s_1^2))}{(840s_3(s_3 - 1)(s_2 - s_3)(s_1 - s_3))}$$

$$\acute{E}_{14}^{[3]_4} = \frac{(s_1^5(21s_2s_3 - 7s_1s_3 - 7s_1s_2 + 3s_1^2))}{(840(s_3 - 1)(s_2 - 1)(s_1 - 1))}$$

$$\acute{E}_{21}^{[3]_4} = -\frac{(s_2^5(21s_3 - 7s_2 - 7s_2s_3 + 3s_2^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$

$$\acute{E}_{22}^{[3]_4} = \frac{(s_2^3(35s_1s_3 - 14s_1s_2 - 14s_2s_3 + 7s_1s_2^2 + 7s_2^2s_3 + 7s_2^2 - 4s_2^3 - 14s_1s_2s_3))}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}$$

$$\acute{E}_{23}^{[3]_4} = \frac{(s_2^5(7s_2 - 21s_1 + 7s_1s_2 - 3s_2^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}$$

$$\acute{E}_{24}^{[3]_4} = -\frac{(s_2^5(7s_1s_2 - 21s_1s_3 + 7s_2s_3 - 3s_2^2))}{(840(s_3-1)(s_1-1)(s_2-1))}$$

$$\acute{E}_{31}^{[3]_4} = \frac{(s_3^5(7s_3 - 21s_2 + 7s_2s_3 - 3s_3^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$

$$\acute{E}_{32}^{[3]_4} = -\frac{(s_3^5(7s_3 - 21s_1 + 7s_1s_3 - 3s_3^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}$$

$$\acute{E}_{33}^{[3]_4} = \frac{(s_3^3(14s_1s_3 - 35s_1s_2 + 14s_2s_3 - 7s_1s_3^2 - 7s_2s_3^2 - 7s_3^2 + 4s_3^3 + 14s_1s_2s_3))}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}$$

$$\acute{E}_{34}^{[3]_4} = \frac{(s_3^5(21s_1s_2 - 7s_1s_3 - 7s_2s_3 + 3s_3^2))}{(840(s_2-1)(s_1-1)(s_3-1))}$$

$$\acute{E}_{41}^{[3]_4} = -\frac{((21s_2s_3 - 7s_3 - 7s_2 + 3))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$

$$\acute{E}_{42}^{[3]_4} = \frac{((21s_1s_3 - 7s_3 - 7s_1 + 3))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}$$

$$\acute{E}_{43}^{[3]_4} = -\frac{((21s_1s_2 - 7s_2 - 7s_1 + 3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}$$

$$\acute{E}_{44}^{[3]_4} = \frac{((7s_1 + 7s_2 + 7s_3 - 14s_1s_2 - 14s_1s_3 - 14s_2s_3 + 35s_1s_2s_3 - 4))}{(840(s_3-1)(s_2-1)(s_1-1))}$$

Substituting equations (5.31), (5.32) and (5.33) into (5.20) –(5.23) yields the second derivative of the block as below

$$\begin{aligned}
y''_{n+s_1} &= y''_n + hs_1y'''_n \\
&- \frac{(h^2s_1^2(5s_1s_2 + 5s_1s_3 - 20s_2s_3 - 2s_1^2s_2 - 2s_1^2s_3 - 2s_1^2 + s_1^3 + 5s_1s_2s_3))}{(60s_2s_3)}f_n \\
&+ \frac{(h^2s_1^2(5s_1s_2 + 5s_1s_3 - 10s_2s_3 - 3s_1^2s_2 - 3s_1^2s_3 - 3s_1^2 + 2s_1^3 + 5s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&+ \frac{(h^2s_1^4(5s_3 - 2s_1 - 2s_1s_3 + s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&- \frac{(h^2s_1^4(5s_2 - 2s_1 - 2s_1s_2 + s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \\
&+ \frac{(h^2s_1^4(5s_2s_3 - 2s_1s_3 - 2s_1s_2 + s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}f_{n+1}
\end{aligned} \tag{5.40}$$

$$\begin{aligned}
y''_{n+s_2} &= y''_n + s_2 h y'''_n \\
&+ \frac{(h^2 s_2^2(20s_1s_3 - 5s_1s_2 - 5s_2s_3 + 2s_1s_2^2 + 2s_2^2s_3 + 2s_2^2 - s_2^3 - 5s_1s_2s_3))}{(60s_1s_3)} f_n \\
&- \frac{(h^2 s_2^4(5s_3 - 2s_2 - 2s_2s_3 + s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{(h^2 s_2^2(10s_1s_3 - 5s_1s_2 - 5s_2s_3 + 3s_1s_2^2 + 3s_2^2s_3 + 3s_2^2 - 2s_2^3 - 5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{(h^2 s_2^4(2s_2 - 5s_1 + 2s_1s_2 - s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&- \frac{(h^2 s_2^4(2s_1s_2 - 5s_1s_3 + 2s_2s_3 - s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))} f_{n+1}
\end{aligned} \tag{5.41}$$

$$\begin{aligned}
y''_{n+s_3} &= y''_n + s_3 h y'''_n \\
&- \frac{(h^2 s_3^2(5s_1s_3 - 20s_1s_2 + 5s_2s_3 - 2s_1s_3^2 - 2s_2s_3^2 - 2s_3^2 + s_3^3 + 5s_1s_2s_3))}{(60s_1s_2)} f_n \\
&+ \frac{(h^2 s_3^4(2s_3 - 5s_2 + 2s_2s_3 - s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&- \frac{(h^2 s_3^4(2s_3 - 5s_1 + 2s_1s_3 - s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{(h^2 s_3^2(5s_1s_3 - 10s_1s_2 + 5s_2s_3 - 3s_1s_3^2 - 3s_2s_3^2 - 3s_3^2 + 2s_3^3 + 5s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{(h^2 s_3^4(5s_1s_2 - 2s_1s_3 - 2s_2s_3 + s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))} f_{n+1}
\end{aligned} \tag{5.42}$$

$$\begin{aligned}
y''_{n+1} &= y''_n + h y'''_n \\
&+ \frac{(h^2(2s_1 + 2s_2 + 2s_3 - 5s_1s_2 - 5s_1s_3 - 5s_2s_3 + 20s_1s_2s_3 - 1))}{(60s_1s_2s_3)} f_n \\
&- \frac{(h^2(5s_2s_3 - 2s_3 - 2s_2 + 1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{(h^2(5s_1s_3 - 2s_3 - 2s_1 + 1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&- \frac{(h^2(5s_1s_2 - 2s_2 - 2s_1 + 1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{(h^2(3s_1 + 3s_2 + 3s_3 - 5s_1s_2 - 5s_1s_3 - 5s_2s_3 + 10s_1s_2s_3 - 2))}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1}
\end{aligned} \tag{5.43}$$

Block of second derivative can be represented as below

$$\acute{\acute{Y}}_m^{[3]_4} = \acute{\acute{B}}_3^{[3]_4} R_3^{[3]_4} + \acute{\acute{B}}_4^{[3]_4} R_4^{[3]_4} + h^2 \acute{\acute{D}}^{[3]_4} R_5^{[3]_4} + h^2 \acute{\acute{E}}^{[3]_4} R_6^{[3]_4} \tag{5.44}$$

where

$$\acute{\acute{Y}}_m^{[3]_4} = \begin{pmatrix} y''_{n+s_1} \\ y''_{n+s_2} \\ y''_{n+s_3} \\ y''_{n+1} \end{pmatrix}, \acute{\acute{B}}_3^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \acute{\acute{B}}_4^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & s_1 h \\ 0 & 0 & 0 & s_2 h \\ 0 & 0 & 0 & s_3 h \\ 0 & 0 & 0 & h \end{pmatrix}$$

$$\acute{\acute{D}}^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & \acute{\acute{D}}_{14}^{[3]_4} \\ 0 & 0 & 0 & \acute{\acute{D}}_{24}^{[3]_4} \\ 0 & 0 & 0 & \acute{\acute{D}}_{34}^{[3]_4} \\ 0 & 0 & 0 & \acute{\acute{D}}_{44}^{[3]_4} \end{pmatrix}, \acute{\acute{E}}^{[3]_4} = \begin{pmatrix} \acute{\acute{E}}_{11}^{[3]_4} & \acute{\acute{E}}_{12}^{[3]} & \acute{\acute{E}}_{13}^{[3]_4} & \acute{\acute{E}}_{14}^{[3]_4} \\ \acute{\acute{E}}_{21}^{[3]_4} & \acute{\acute{E}}_{22}^{[3]_4} & \acute{\acute{E}}_{23}^{[3]_4} & \acute{\acute{E}}_{24}^{[3]_4} \\ \acute{\acute{E}}_{31}^{[3]_4} & \acute{\acute{E}}_{32}^{[3]_4} & \acute{\acute{E}}_{33}^{[3]_4} & \acute{\acute{E}}_{34}^{[3]_4} \\ \acute{\acute{E}}_{41}^{[3]_4} & \acute{\acute{E}}_{42}^{[3]_4} & \acute{\acute{E}}_{43}^{[3]_4} & \acute{\acute{E}}_{44}^{[3]_4} \end{pmatrix},$$

and the non-zero terms of $\acute{\acute{D}}^{[3]_2}$ and $\acute{\acute{E}}^{[3]_4}$ are given by

$$\acute{\acute{D}}_{11}^{[3]_4} = -\frac{(s_1^2(5s_1s_2 + 5s_1s_3 - 20s_2s_3 - 2s_1^2s_2 - 2s_1^2s_3 - 2s_1^2 + s_1^3 + 5s_1s_2s_3))}{(60s_2s_3)}$$
$$\acute{\acute{D}}_{12}^{[3]_4} = \frac{(s_2^2(20s_1s_3 - 5s_1s_2 - 5s_2s_3 + 2s_1s_2^2 + 2s_2^2s_3 + 2s_2^2 - s_2^3 - 5s_1s_2s_3))}{(60s_1s_3)}$$
$$\acute{\acute{D}}_{13}^{[3]_4} = -\frac{(s_3^2(5s_1s_3 - 20s_1s_2 + 5s_2s_3 - 2s_1s_3^2 - 2s_2s_3^2 - 2s_3^2 + s_3^3 + 5s_1s_2s_3))}{(60s_1s_2)}$$
$$\acute{\acute{D}}_{14}^{[3]_4} = \frac{((2s_1 + 2s_2 + 2s_3 - 5s_1s_2 - 5s_1s_3 - 5s_2s_3 + 20s_1s_2s_3 - 1))}{(60s_1s_2s_3)}$$
$$\acute{\acute{E}}_{11}^{[3]_4} = \frac{(s_1^2(5s_1s_2 + 5s_1s_3 - 10s_2s_3 - 3s_1^2s_2 - 3s_1^2s_3 - 3s_1^2 + 2s_1^3 + 5s_1s_2s_3))}{(60(s_1 - 1)(s_1 - s_3)(s_1 - s_2))}$$
$$\acute{\acute{E}}_{12}^{[3]_4} = \frac{(s_1^4(5s_3 - 2s_1 - 2s_1s_3 + s_1^2))}{(60s_2(s_2 - 1)(s_2 - s_3)(s_1 - s_2))}$$
$$\acute{\acute{E}}_{13}^{[3]_4} = -\frac{(s_1^4(5s_2 - 2s_1 - 2s_1s_2 + s_1^2))}{(60s_3(s_3 - 1)(s_2 - s_3)(s_1 - s_3))}$$

$$\acute{E}_{14}^{[3]_4} = \frac{(s_1^4(5s_2s_3 - 2s_1s_3 - 2s_1s_2 + s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}$$
$$\acute{E}_{21}^{[3]_4} = -\frac{(s_2^4(5s_3 - 2s_2 - 2s_2s_3 + s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$
$$\acute{E}_{22}^{[3]_4} = \frac{(s_2^2(10s_1s_3 - 5s_1s_2 - 5s_2s_3 + 3s_1s_2^2 + 3s_2^2s_3 + 3s_2^2 - 2s_2^3 - 5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}$$
$$\acute{E}_{23}^{[3]_4} = \frac{(s_2^4(2s_2 - 5s_1 + 2s_1s_2 - s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\acute{E}_{24}^{[3]_4} = -\frac{(s_2^4(2s_1s_2 - 5s_1s_3 + 2s_2s_3 - s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}$$
$$\acute{E}_{31}^{[3]_4} = \frac{(s_3^4(2s_3 - 5s_2 + 2s_2s_3 - s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$
$$\acute{E}_{32}^{[3]_4} = -\frac{(s_3^4(2s_3 - 5s_1 + 2s_1s_3 - s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}$$
$$\acute{E}_{33}^{[3]_4} = \frac{(s_3^2(5s_1s_3 - 10s_1s_2 + 5s_2s_3 - 3s_1s_3^2 - 3s_2s_3^2 - 3s_3^2 + 2s_3^3 + 5s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\acute{E}_{34}^{[3]_4} = \frac{(s_3^4(5s_1s_2 - 2s_1s_3 - 2s_2s_3 + s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}$$
$$\acute{E}_{41}^{[3]_4} = -\frac{((5s_2s_3 - 2s_3 - 2s_2 + 1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$
$$\acute{E}_{42}^{[3]_4} = \frac{((5s_1s_3 - 2s_3 - 2s_1 + 1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}$$
$$\acute{E}_{43}^{[3]_4} = -\frac{((5s_1s_2 - 2s_2 - 2s_1 + 1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\acute{E}_{44}^{[3]_4} = \frac{((3s_1 + 3s_2 + 3s_3 - 5s_1s_2 - 5s_1s_3 - 5s_2s_3 + 10s_1s_2s_3 - 2))}{(60(s_3-1)(s_2-1)(s_1-1))}$$

Substituting equations (5.31), (5.32) and (5.33) into (5.25) –(5.28) leads to the third derivative of the block as following

$$
\begin{aligned}
y'''_{n+s_1} &= y'''_n \\
&-\frac{(hs_1(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3))}{(60s_2s_3)}f_n \\
&+\frac{hs_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{60(s_1-1)(s_1-s_3)(s_1-s_2)}f_{n+s_1} \\
&+\frac{(hs_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&-\frac{(hs_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \\
&+\frac{(hs_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}f_{n+1} \qquad (5.45)
\end{aligned}
$$

$$
\begin{aligned}
y'''_{n+s_2} &= y'''_n \\
&+\frac{(hs_2(30s_1s_3-10s_1s_2-10s_2s_3+5s_1s_2^2+5s_2^2s_3+5s_2^2-3s_2^3-10s_1s_2s_3))}{(60s_1s_3)}f_n \\
&-\frac{(hs_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&+\frac{hs_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3)}{60(s_2-1)(s_2-s_3)(s_1-s_2)}f_{n+s_2} \\
&+\frac{(hs_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \\
&-\frac{(hs_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}f_{n+1} \qquad (5.46)
\end{aligned}
$$

$$
\begin{aligned}
y'''_{n+s_3} &= y'''_n \\
&-\frac{hs_3(10s_1s_3-30s_1s_2+10s_2s_3-5s_1s_3^2-5s_2s_3^2-5s_3^2+3s_3^3+10s_1s_2s_3)}{(60s_1s_2)}f_n \\
&+\frac{(hs_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&-\frac{(hs_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&+\frac{hs_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3)}{60(s_3-1)(s_2-s_3)(s_1-s_3)}f_{n+s_3} \\
&+\frac{(hs_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}f_{n+1} \qquad (5.47)
\end{aligned}
$$

$$
\begin{aligned}
y'''_{n+1} &= y'''_n \\
&+\frac{(h(5s_1+5s_2+5s_3-10s_1s_2-10s_1s_3-10s_2s_3+30s_1s_2s_3-3))}{(60s_1s_2s_3)}f_n \\
&-\frac{(h(10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&+\frac{(h(10s_1s_3-5s_3-5s_1+3))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&+\frac{(h(15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12))}{(60(s_3-1)(s_2-1)(s_1-1))}f_{n+1} \\
&-\frac{(h(10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \qquad (5.48)
\end{aligned}
$$

Block of third derivative can be represented in block form as below

$$
\acute{\dddot{Y}}_m^{[3]_4} = \acute{\dddot{B}}_4^{[3]_4} R_4^{[3]_4} + h\acute{\dddot{D}}^{[3]_4} R_5^{[3]_4} + h\acute{\dddot{E}}^{[3]_4} R_6^{[3]_4} \qquad (5.49)
$$

where

$$
\acute{\dddot{Y}}_m^{[3]_4} = \begin{pmatrix} y'''_{n+s_1} \\ y'''_{n+s_2} \\ y'''_{n+s_3} \\ y'''_{n+1} \end{pmatrix}, \; \acute{\dddot{B}}_4^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
\acute{\dddot{D}}^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & \acute{\dddot{D}}^{[3]_4}_{14} \\ 0 & 0 & 0 & \acute{\dddot{D}}^{[3]_4}_{24} \\ 0 & 0 & 0 & \acute{\dddot{D}}^{[3]_4}_{34} \\ 0 & 0 & 0 & \acute{\dddot{D}}^{[3]_4}_{44} \end{pmatrix}, \ \acute{\dddot{E}}^{[3]_4} = \begin{pmatrix} \acute{\dddot{E}}^{[3]_4}_{11} & \acute{\dddot{E}}^{[3]}_{12} & \acute{\dddot{E}}^{[3]_4}_{13} & \acute{\dddot{E}}^{[3]_4}_{14} \\ \acute{\dddot{E}}^{[3]_4}_{21} & \acute{\dddot{E}}^{[3]_4}_{22} & \acute{\dddot{E}}^{[3]_4}_{23} & \acute{\dddot{E}}^{[3]_4}_{24} \\ \acute{\dddot{E}}^{[3]_4}_{31} & \acute{\dddot{E}}^{[3]_4}_{32} & \acute{\dddot{E}}^{[3]_4}_{33} & \acute{\dddot{E}}^{[3]_4}_{34} \\ \acute{\dddot{E}}^{[3]_4}_{41} & \acute{\dddot{E}}^{[3]_4}_{42} & \acute{\dddot{E}}^{[3]_4}_{43} & \acute{\dddot{E}}^{[3]_4}_{44} \end{pmatrix},
$$

and the non-zero terms of $\acute{\dddot{D}}^{[3]_2}$ and $\acute{\dddot{E}}^{[3]_4}$ are given by

$$
\begin{aligned}
\acute{\dddot{D}}^{[3]_4}_{14} &= -\frac{(s_1(10s_1s_2 + 10s_1s_3 - 30s_2s_3 - 5s_1^2s_2 - 5s_1^2s_3 - 5s_1^2 + 3s_1^3 + 10s_1s_2s_3))}{(60s_2s_3)} \\
\acute{\dddot{D}}^{[3]_4}_{24} &= \frac{(s_2(30s_1s_3 - 10s_1s_2 - 10s_2s_3 + 5s_1s_2^2 + 5s_2^2s_3 + 5s_2^2 - 3s_2^3 - 10s_1s_2s_3))}{(60s_1s_3)} \\
\acute{\dddot{D}}^{[3]_4}_{34} &= -\frac{(s_3(10s_1s_3 - 30s_1s_2 + 10s_2s_3 - 5s_1s_3^2 - 5s_2s_3^2 - 5s_3^2 + 3s_3^3 + 10s_1s_2s_3))}{(60s_1s_2)} \\
\acute{\dddot{D}}^{[3]_4}_{44} &= \frac{((5s_1 + 5s_2 + 5s_3 - 10s_1s_2 - 10s_1s_3 - 10s_2s_3 + 30s_1s_2s_3 - 3))}{(60s_1s_2s_3)} \\
\acute{\dddot{E}}^{[3]_4}_{11} &= \frac{(s_1(20s_1s_2 + 20s_1s_3 - 30s_2s_3 - 15s_1^2s_2 - 15s_1^2s_3 - 15s_1^2 + 12s_1^3 + 20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))} \\
\acute{\dddot{E}}^{[3]_4}_{12} &= \frac{(s_1^3(10s_3 - 5s_1 - 5s_1s_3 + 3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \\
\acute{\dddot{E}}^{[3]_4}_{13} &= -\frac{(s_1^3(10s_2 - 5s_1 - 5s_1s_2 + 3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \\
\acute{\dddot{E}}^{[3]_4}_{14} &= \frac{(s_1^3(10s_2s_3 - 5s_1s_3 - 5s_1s_2 + 3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))} \\
\acute{\dddot{E}}^{[3]_4}_{21} &= -\frac{(s_2^4(5s_3 - 2s_2 - 2s_2s_3 + s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \\
\acute{\dddot{E}}^{[3]_4}_{22} &= \frac{(s_2(30s_1s_3 - 20s_1s_2 - 20s_2s_3 + 15s_1s_2^2 + 15s_2^2s_3 + 15s_2^2 - 12s_2^3 - 20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))} \\
\acute{\dddot{E}}^{[3]_4}_{23} &= \frac{(s_2^3(5s_2 - 10s_1 + 5s_1s_2 - 3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \\
\acute{\dddot{E}}^{[3]_4}_{24} &= -\frac{(s_2^3(5s_1s_2 - 10s_1s_3 + 5s_2s_3 - 3s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))} \\
\acute{\dddot{E}}^{[3]_4}_{31} &= \frac{(s_3^3(5s_3 - 10s_2 + 5s_2s_3 - 3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \\
\acute{\dddot{E}}^{[3]_4}_{32} &= -\frac{(s_3^3(5s_3 - 10s_1 + 5s_1s_3 - 3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}
\end{aligned}
$$

$$\acute{\acute{\acute{E}}}_{33}^{[3]_4} = \frac{(s_3(20s_1s_3 - 30s_1s_2 + 20s_2s_3 - 15s_1s_3^2 - 15s_2s_3^2 - 15s_3^2 + 12s_3^3 + 20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\acute{\acute{\acute{E}}}_{34}^{[3]_4} = \frac{(s_3^3(10s_1s_2 - 5s_1s_3 - 5s_2s_3 + 3s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}$$
$$\acute{\acute{\acute{E}}}_{41}^{[3]_4} = -\frac{((10s_2s_3 - 5s_3 - 5s_2 + 3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$
$$\acute{\acute{\acute{E}}}_{42}^{[3]_4} = \frac{((10s_1s_3 - 5s_3 - 5s_1 + 3))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}$$
$$\acute{\acute{\acute{E}}}_{43}^{[3]_4} = -\frac{((10s_1s_2 - 5s_2 - 5s_1 + 3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\acute{\acute{\acute{E}}}_{44}^{[3]_4} = \frac{((15s_1 + 15s_2 + 15s_3 - 20s_1s_2 - 20s_1s_3 - 20s_2s_3 + 30s_1s_2s_3 - 12))}{(60(s_3-1)(s_2-1)(s_1-1))}$$

5.1.1 Establishing of the Properties of One Step Hybrid Block Method with Generalised Three Off-Step Points for Fourth Order ODEs

In this section of this chapter, numerical properties which includes order, zero stability, consistent, convergence, error constant, region of absolute stability are established.

5.1.1.1 Order of One Step Hybrid Block Method with Generalised Three Off-Step for Fourth Order ODEs

In finding the order of the block (5.30), Definition (3.1.2) is applied. Expanding y and f-function in Taylor series, that gives

$$
\begin{bmatrix}
\begin{array}{c}
\sum_{j=0}^{\infty}\frac{(s_1)^j h^j}{j!}y_n^j - y_n - s_1 h y_n^{'} - \frac{s_1^2h^2}{2}y_n^{''} - \frac{s_1^3h^3}{6}y_n^{'''} \\
+\frac{(h^4s_1^4(28s_1s_2+28s_1s_3-168s_2s_3-8s_1^2s_2-8s_1^2s_3-8s_1^2+3s_1^3+28s_1s_2s_3))}{(5040s_2s_3)}y_n^{iv} \\
-\frac{(s_1^4(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_1^6(28s_3-8s_1-8s_1s_3+3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+4}}{j!}y_n^{j+4} \\
+\frac{(s_1^6(28s_2-8s_1-8s_1s_2+3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_1^6(28s_2s_3-8s_1s_3-8s_1s_2+3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+4}}{j!}y_n^{j+4}
\end{array} \\
\begin{array}{c}
\sum_{j=0}^{\infty}\frac{(s_2)^j h^j}{j!}y_n^j - y_n - s_2 h y_n^{'} - \frac{s_2^2h^2}{2}y_n^{''} - \frac{s_2^3h^3}{6}y_n^{'''} \\
-\frac{(h^4s_2^4(168s_1s_3-28s_1s_2-28s_2s_3+8s_1s_2^2+8s_2^2s_3+8s_2^2-3s_2^3-28s_1s_2s_3))}{(5040s_1s_3)}y_n^{iv} \\
+\frac{(s_2^6(28s_3-8s_2-8s_2s_3+3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_2^4(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_2^6(8s_2-28s_1+8s_1s_2-3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+4}}{j!}y_n^{j+4} \\
+\frac{(s_2^6(8s_1s_2-28s_1s_3+8s_2s_3-3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))}\sum_{j=0}^{\infty}\frac{h^{j+4}}{j!}y_n^{j+4}
\end{array} \\
\begin{array}{c}
\sum_{j=0}^{\infty}\frac{(s_3)^j h^j}{j!}y_n^j - y_n - s_3 h y_n^{'} - \frac{s_3^2h^2}{2}y_n^{''} - \frac{s_3^3h^3}{6}y_n^{'''} \\
+\frac{(h^4s_3^4(28s_1s_3-168s_1s_2+28s_2s_3-8s_1s_3^2-8s_2s_3^2-8s_3^2+3s_3^3+28s_1s_2s_3))}{(5040s_1s_2)}y_n^{iv} \\
-\frac{(s_3^6(8s_3-28s_2+8s_2s_3-3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+4}}{j!}y_n^{j+4} \\
+\frac{(h^4s_3^6(8s_3-28s_1+8s_1s_3-3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_3^4(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_3^6(28s_1s_2-8s_1s_3-8s_2s_3+3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))}\sum_{j=0}^{\infty}\frac{h^{j+4}}{j!}y_n^{j+4}
\end{array} \\
\begin{array}{c}
\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j - y_n - h y_n^{'} - \frac{h^2}{2}y_n^{''} - \frac{h^3}{6}y_n^{'''} \\
-\frac{(h^4(8s_1+8s_2+8s_3-28s_1s_2-28s_1s_3-28s_2s_3+168s_1s_2s_3-3))}{(5040s_1s_2s_3)}y_n^{iv} \\
+\frac{((28s_2s_3-8s_3-8s_2+3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+4}}{j!}y_n^{j+4} \\
-\frac{((28s_1s_3-8s_3-8s_1+3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+4}}{j!}y_n^{j+4} \\
+\frac{((28s_1s_2-8s_2-8s_1+3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+4}}{j!}y_n^{j+4} \\
-\frac{((6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3))}{(5040(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+4}}{j!}y_n^{j+4}
\end{array}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Comparing the coefficients of h^j and y^j . This gives

$$\bar{C}_0 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} s_1 - s_1 \\ s_2 - s_2 \\ s_3 - s_3 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} \frac{s_1^2}{2} - \frac{s_1^2}{2} \\ \frac{s_2^2}{2} - \frac{s_2^2}{2} \\ \frac{s_3^2}{2} - \frac{s_3^2}{2} \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} \frac{s_1^3}{6} - \frac{s_1^3}{6} \\ \frac{s_2^3}{6} - \frac{s_2^3}{6} \\ \frac{s_3^3}{6} - \frac{s_3^3}{6} \\ \frac{1}{6} - \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\bar{C}_4 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^4}{4!} + \frac{(s_1^4(28s_1s_2+28s_1s_3-168s_2s_3-8s_1^2s_2-8s_1^2s_3-8s_1^2+3s_1^3+28s_1s_2s_3))}{(5040s_2s_3)} \\
-\frac{(s_1^4(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{(s_1^6(28s_3-8s_1-8s_1s_3+3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{(s_1^6(28s_2-8s_1-8s_1s_2+3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{(s_1^6(28s_2s_3-8s_1s_3-8s_1s_2+3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_2)^4}{4!} - \frac{(s_2^4(168s_1s_3-28s_1s_2-28s_2s_3+8s_1s_2^2+8s_2^2s_3+8s_2^2-3s_2^3-28s_1s_2s_3))}{(5040s_1s_3)} \\
+\frac{(s_2^6(28s_3-8s_2-8s_2s_3+3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{(s_2^4(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{(s_2^6(8s_2-28s_1+8s_1s_2-3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
+\frac{(s_2^6(8s_1s_2-28s_1s_3+8s_2s_3-3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))}\frac{1}{0!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_3)^4}{4!} + \frac{(s_3^4(28s_1s_3-168s_1s_2+28s_2s_3-8s_1s_3^2-8s_2s_3^2-8s_3^2+3s_3^3+28s_1s_2s_3))}{(5040s_1s_2)} \\
-\frac{(s_3^6(8s_3-28s_2+8s_2s_3-3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
+\frac{(h^4s_3^6(8s_3-28s_1+8s_1s_3-3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{(s_3^4(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{(s_3^6(28s_1s_2-8s_1s_3-8s_2s_3+3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))}\frac{1}{0!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{1}{4!} - \frac{((8s_1+8s_2+8s_3-28s_1s_2-28s_1s_3-28s_2s_3+168s_1s_2s_3-3))}{(5040s_1s_2s_3)} \\
+\frac{((28s_2s_3-8s_3-8s_2+3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{((28s_1s_3-8s_3-8s_1+3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{((28s_1s_2-8s_2-8s_1+3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{((6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\[2em] 0 \\[2em] 0 \\[2em] 0 \end{bmatrix}
$$

$$
\bar{C}_5 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^5}{5!} - \frac{(s_1^4(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{(s_1^6(28s_3-8s_1-8s_1s_3+3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+\frac{(s_1^6(28s_2-8s_1-8s_1s_2+3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
-\frac{(s_1^6(28s_2s_3-8s_1s_3-8s_1s_2+3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_2)^5}{5!} + \frac{(s_2^6(28s_3-8s_2-8s_2s_3+3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{(s_2^4(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
-\frac{(s_2^6(8s_2-28s_1+8s_1s_2-3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
+\frac{(s_2^6(8s_1s_2-28s_1s_3+8s_2s_3-3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))}\frac{1}{1!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_3)^5}{5!} - \frac{(s_3^6(8s_3-28s_2+8s_2s_3-3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
+\frac{(s_3^6(8s_3-28s_1+8s_1s_3-3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
-\frac{(s_3^4(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
-\frac{(s_3^6(28s_1s_2-8s_1s_3-8s_2s_3+3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))}\frac{1}{1!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{1}{5!} + \frac{((28s_2s_3-8s_3-8s_2+3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{((28s_1s_3-8s_3-8s_1+3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+\frac{((28s_1s_2-8s_2-8s_1+3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
-\frac{((6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_6 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^6}{6!} - \frac{(s_1^4(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{(s_1^6(28s_3-8s_1-8s_1s_3+3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+\frac{(s_1^6(28s_2-8s_1-8s_1s_2+3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
-\frac{(s_1^6(28s_2s_3-8s_1s_3-8s_1s_2+3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_2)^6}{6!} + \frac{(s_2^6(28s_3-8s_2-8s_2s_3+3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{(s_2^4(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
-\frac{(s_2^6(8s_2-28s_1+8s_1s_2-3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
+\frac{(s_2^6(8s_1s_2-28s_1s_3+8s_2s_3-3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))}\frac{1}{2!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_3)^6}{6!} - \frac{(s_3^6(8s_3-28s_2+8s_2s_3-3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
+\frac{(s_3^6(8s_3-28s_1+8s_1s_3-3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
-\frac{(s_3^4(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
-\frac{(s_3^6(28s_1s_2-8s_1s_3-8s_2s_3+3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))}\frac{1}{2!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{1}{6!} + \frac{((28s_2s_3-8s_3-8s_2+3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{((28s_1s_3-8s_3-8s_1+3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+\frac{((28s_1s_2-8s_2-8s_1+3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
-\frac{((6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\[2em] 0 \\[2em] 0 \\[2em] 0 \end{bmatrix}
$$

$$
\bar{C}_7 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^7}{7!} - \frac{(s_1^4(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
-\frac{(s_1^6(28s_3-8s_1-8s_1s_3+3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
+\frac{(s_1^6(28s_2-8s_1-8s_1s_2+3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\
-\frac{(s_1^6(28s_2s_3-8s_1s_3-8s_1s_2+3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!}
\end{array} \\[4ex]
\begin{array}{c}
\frac{(s_2)^7}{7!} + \frac{(s_2^6(28s_3-8s_2-8s_2s_3+3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
-\frac{(s_2^4(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
-\frac{(s_2^6(8s_2-28s_1+8s_1s_2-3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\
+\frac{(s_2^6(8s_1s_2-28s_1s_3+8s_2s_3-3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))}\frac{1}{3!}
\end{array} \\[4ex]
\begin{array}{c}
\frac{(s_3)^7}{7!} - \frac{(s_3^6(8s_3-28s_2+8s_2s_3-3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
+\frac{(s_3^6(8s_3-28s_1+8s_1s_3-3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
-\frac{(s_3^4(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\
-\frac{(s_3^6(28s_1s_2-8s_1s_3-8s_2s_3+3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))}\frac{1}{3!}
\end{array} \\[4ex]
\begin{array}{c}
\frac{1}{7!} + \frac{((28s_2s_3-8s_3-8s_2+3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
-\frac{((28s_1s_3-8s_3-8s_1+3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
+\frac{((28s_1s_2-8s_2-8s_1+3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\
-\frac{((6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!}
\end{array}
\end{bmatrix}
= \begin{bmatrix} 0 \\[4ex] 0 \\[4ex] 0 \\[4ex] 0 \end{bmatrix}
$$

$$
\bar{C}_8 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^8}{8!} - \frac{(s_1^4(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{(s_1^6(28s_3-8s_1-8s_1s_3+3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+\frac{(s_1^6(28s_2-8s_1-8s_1s_2+3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
-\frac{(s_1^6(28s_2s_3-8s_1s_3-8s_1s_2+3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!}
\end{array} \\[3em]
\begin{array}{c}
\frac{(s_2)^8}{8!} + \frac{(s_2^6(28s_3-8s_2-8s_2s_3+3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{(s_2^4(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
-\frac{(s_2^6(8s_2-28s_1+8s_1s_2-3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
+\frac{(s_2^6(8s_1s_2-28s_1s_3+8s_2s_3-3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))}\frac{1}{4!}
\end{array} \\[3em]
\begin{array}{c}
\frac{(s_3)^8}{8!} - \frac{(s_3^6(8s_3-28s_2+8s_2s_3-3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
+\frac{(s_3^6(8s_3-28s_1+8s_1s_3-3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
-\frac{(s_3^4(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
-\frac{(s_3^6(28s_1s_2-8s_1s_3-8s_2s_3+3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))}\frac{1}{4!}
\end{array} \\[3em]
\begin{array}{c}
\frac{1}{8!} + \frac{((28s_2s_3-8s_3-8s_2+3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{((28s_1s_3-8s_3-8s_1+3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+\frac{((28s_1s_2-8s_2-8s_1+3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
-\frac{((6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!}
\end{array}
\end{bmatrix} = \begin{bmatrix} 0 \\[3em] 0 \\[3em] 0 \\[3em] 0 \end{bmatrix}
$$

$$
\bar{C}_9 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^9}{9!} - \frac{(s_1^4(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
-\frac{(s_1^6(28s_3-8s_1-8s_1s_3+3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
+\frac{(s_1^6(28s_2-8s_1-8s_1s_2+3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
-\frac{(s_1^6(28s_2s_3-8s_1s_3-8s_1s_2+3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!}
\end{array} \\ \\
\begin{array}{c}
\frac{(s_2)^9}{9!} + \frac{(s_2^6(28s_3-8s_2-8s_2s_3+3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
-\frac{(s_2^4(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
-\frac{(s_2^6(8s_2-28s_1+8s_1s_2-3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
+\frac{(s_2^6(8s_1s_2-28s_1s_3+8s_2s_3-3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))}\frac{1}{5!}
\end{array} \\ \\
\begin{array}{c}
\frac{(s_3)^9}{9!} - \frac{(s_3^6(8s_3-28s_2+8s_2s_3-3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
+\frac{(s_3^6(8s_3-28s_1+8s_1s_3-3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
-\frac{(s_3^4(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
-\frac{(s_3^6(28s_1s_2-8s_1s_3-8s_2s_3+3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))}\frac{1}{5!}
\end{array} \\ \\
\begin{array}{c}
\frac{1}{9!} + \frac{((28s_2s_3-8s_3-8s_2+3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
-\frac{((28s_1s_3-8s_3-8s_1+3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
+\frac{((28s_1s_2-8s_2-8s_1+3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
-\frac{((6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3))}{(5040(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!}
\end{array}
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Therefore, the order of the block (5.30) is $[5,5,5,5]^T$ with general vector error constants

$$
\bar{C}_9 = \begin{bmatrix}
-\frac{(s_1^6(24s_1s_2+24s_1s_3-84s_2s_3-9s_1^2s_2-9s_1^2s_3-9s_1^2+4s_1^3+24s_1s_2s_3))}{1814400} \\
\frac{(s_2^6(84s_1s_3-24s_1s_2-24s_2s_3+9s_1s_2^2+9s_2^2s_3+9s_2^2-4s_2^3-24s_1s_2s_3))}{1814400} \\
-\frac{(s_3^6(24s_1s_3-84s_1s_2+24s_2s_3-9s_1s_3^2-9s_2s_3^2-9s_3^2+4s_3^3+24s_1s_2s_3))}{1814400} \\
\frac{(9s_1+9s_2+9s_3-24s_1s_2-24s_1s_3-24s_2s_3+84s_1s_2s_3-4)}{1814400}
\end{bmatrix}
$$

which is true for all $s_1, s_2, s_3 \in (0,1) \setminus \{s_2 = \frac{9s_1^2s_3+9s_1^2-4s_1^3-24s_1s_3}{24s_1-84s_3-9s_1^2+24s_1s_3}\} \cup \{s_1 = \frac{24s_2s_3-9s_2^2s_3-9s_2^2+4s_2^3}{84s_3-24s_2+9s_2^2-24s_2s_3}\}$ $\cup \{s_2 = \frac{-24s_1s_3+9s_1s_3^2+9s_3^2-4s_3^3}{-84s_2+24s_3-9s_3^2+24s_1s_3}\} \cup \{s_1 = \frac{-9s_2-9s_3+24s_2s_3+4}{9-24s_2-24s_3+84s_2s_3}\}$.

The same strategy as mention earlier is also employed in order to find the order of first derivative block (5.39). Expanding y' and f- function Taylor series gives

$$
\begin{bmatrix}
\begin{array}{c}
\sum_{j=0}^{\infty} \frac{(s_1)^j h^j}{j!} y_n^{j+1} - y_n' - s_1 h y_n'' - \frac{s_1^2 h^2}{2} y_n''' \\
+\frac{(h^3 s_1^3(21s_1s_2+21s_1s_3-105s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+3s_1^3+21s_1s_2s_3))}{(840s_2s_3)} y_n^{iv} \\
-\frac{(s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3))}{(840(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+3}}{j!} y_n^{j+4} \\
-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+3}}{j!} y_n^{j+4} \\
+\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+3}}{j!} y_n^{j+4} \\
-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2))}{(840(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+4}
\end{array} \\
\begin{array}{c}
\sum_{j=0}^{\infty} \frac{(s_2)^j h^j}{j!} y_n^{j+1} - y_n' - s_2 h y_n'' - \frac{s_2^2 h^2}{2} y_n''' \\
-\frac{(h^3 s_2^3(105s_1s_3-21s_1s_2-21s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-3s_2^3-21s_1s_2s_3))}{(840s_1s_3)} y_n^{iv} \\
+\frac{(s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+3}}{j!} y_n^{j+4} \\
-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3))}{(840(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+3}}{j!} y_n^{j+4} \\
-\frac{(s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+3}}{j!} y_n^{j+4} \\
+\frac{(s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2))}{(840(s_3-1)(s_1-1)(s_2-1))} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+4}
\end{array} \\
\begin{array}{c}
\sum_{j=0}^{\infty} \frac{(s_3)^j h^j}{j!} y_n^{j+1} - y_n' - s_3 h y_n'' - \frac{s_3^2 h^2}{2} y_n''' \\
+\frac{(h^3 s_3^3(21s_1s_3-105s_1s_2+21s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+3s_3^3+21s_1s_2s_3))}{(840s_1s_2)} y_n^{iv} \\
-\frac{(s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+3}}{j!} y_n^{j+4} \\
+\frac{(s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+3}}{j!} y_n^{j+4} \\
-\frac{(s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3))}{(840(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+3}}{j!} y_n^{j+4} \\
-\frac{(s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2))}{(840(s_2-1)(s_1-1)(s_3-1))} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+4}
\end{array} \\
\begin{array}{c}
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{j+1} - y_n' - h y_n'' - \frac{h^2}{2} y_n''' \\
-\frac{(h^3(7s_1+7s_2+7s_3-21s_1s_2-21s_1s_3-21s_2s_3+105s_1s_2s_3-3))}{(840s_1s_2s_3)} y_n^{iv} \\
+\frac{((21s_2s_3-7s_3-7s_2+3))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+3}}{j!} y_n^{j+4} \\
-\frac{((21s_1s_3-7s_3-7s_1+3))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+3}}{j!} y_n^{j+4} \\
+\frac{((21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+3}}{j!} y_n^{j+4} \\
-\frac{((7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4))}{(840(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+4}
\end{array}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Comparing the coefficients of h^j and y^j yields

$$\bar{C}_0 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} s_1-s_1 \\ s_2-s_2 \\ s_3-s_3 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} \frac{s_1^2}{2}-\frac{s_1^2}{2} \\ \frac{s_2^2}{2}-\frac{s_2^2}{2} \\ \frac{s_3^2}{2}-\frac{s_3^2}{2} \\ \frac{1}{2}-\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_4 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^3}{3!} + \frac{(s_1^3(21s_1s_2+21s_1s_3-105s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+3s_1^3+21s_1s_2s_3))}{(840s_2s_3)} \\
-\frac{(s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3))}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!}
\end{array} \\[2em]
\begin{array}{c}
\frac{(s_2)^3}{3!} - \frac{(s_2^3(105s_1s_3-21s_1s_2-21s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-3s_2^3-21s_1s_2s_3))}{(840s_1s_3)} \\
+\frac{(s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3))}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{(s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
+\frac{(s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2))}{(840(s_3-1)(s_1-1)(s_2-1))}\frac{1}{0!}
\end{array} \\[2em]
\begin{array}{c}
\frac{(s_3)^3}{3!} + \frac{(s_3^3(21s_1s_3-105s_1s_2+21s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+3s_3^3+21s_1s_2s_3))}{(840s_1s_2)} \\
-\frac{(s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
+\frac{(s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{(s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3))}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{(s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2))}{(840(s_2-1)(s_1-1)(s_3-1))}\frac{1}{0!}
\end{array} \\[2em]
\begin{array}{c}
\frac{1}{3!} - \frac{((7s_1+7s_2+7s_3-21s_1s_2-21s_1s_3-21s_2s_3+105s_1s_2s_3-3))}{(840s_1s_2s_3)} \\
+\frac{((21s_2s_3-7s_3-7s_2+3))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{((21s_1s_3-7s_3-7s_1+3))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{((21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{((7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!}
\end{array}
\end{bmatrix} = \begin{bmatrix} 0 \\[2em] 0 \\[2em] 0 \\[2em] 0 \end{bmatrix}$$

$$
\bar{C}_5 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^4}{4!} - \frac{(s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3))}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}
\end{gathered} \\ \\
\begin{gathered}
\frac{(s_2)^4}{4!} + \frac{(s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3))}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
-\frac{(s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
+\frac{(s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2))}{(840(s_3-1)(s_1-1)(s_2-1))}\frac{1}{1!}
\end{gathered} \\ \\
\begin{gathered}
\frac{(s_3)^4}{4!} - \frac{(s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
+\frac{(s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
-\frac{(s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3))}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
-\frac{(s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2))}{(840(s_2-1)(s_1-1)(s_3-1))}\frac{1}{1!}
\end{gathered} \\ \\
\begin{gathered}
\frac{1}{4!} + \frac{((21s_2s_3-7s_3-7s_2+3))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{((21s_1s_3-7s_3-7s_1+3))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+\frac{((21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
-\frac{((7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{bmatrix}
$$

$$
\bar{C}_6 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^5}{5!} - \frac{(s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3))}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
- \frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+ \frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
- \frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{array} \\[2em]
\begin{array}{c}
\frac{(s_2)^5}{5!} + \frac{(s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
- \frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3))}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
- \frac{(s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
+ \frac{(s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2))}{(840(s_3-1)(s_1-1)(s_2-1))}\frac{1}{2!}
\end{array} \\[2em]
\begin{array}{c}
\frac{(s_3)^5}{5!} - \frac{(s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
+ \frac{(s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
- \frac{(s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3))}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
- \frac{(s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2))}{(840(s_2-1)(s_1-1)(s_3-1))}\frac{1}{2!}
\end{array} \\[2em]
\begin{array}{c}
\frac{1}{5!} + \frac{((21s_2s_3-7s_3-7s_2+3))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
- \frac{((21s_1s_3-7s_3-7s_1+3))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+ \frac{((21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
- \frac{((7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{array}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_7 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^6}{6!} - \frac{(s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3))}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
+\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\
-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!}
\end{array} \\
\\
\begin{array}{c}
\frac{(s_2)^6}{6!} + \frac{(s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3))}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
-\frac{(s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\
+\frac{(s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2))}{(840(s_3-1)(s_1-1)(s_2-1))}\frac{1}{3!}
\end{array} \\
\\
\begin{array}{c}
\frac{(s_3)^3}{3!} - \frac{(s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
+\frac{(s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
-\frac{(s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3))}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\
-\frac{(s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2))}{(840(s_2-1)(s_1-1)(s_3-1))}\frac{1}{3!}
\end{array} \\
\\
\begin{array}{c}
\frac{1}{6!} + \frac{((21s_2s_3-7s_3-7s_2+3))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\
-\frac{((21s_1s_3-7s_3-7s_1+3))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\
+\frac{((21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\
-\frac{((7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!}
\end{array}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_8 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^7}{7!} - \frac{(s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3))}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!}
\end{gathered} \\[4ex]
\begin{gathered}
\frac{(s_2)^7}{7!} + \frac{(s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3))}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
-\frac{(s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
+\frac{(s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2))}{(840(s_3-1)(s_1-1)(s_2-1))}\frac{1}{4!}
\end{gathered} \\[4ex]
\begin{gathered}
\frac{(s_3)^7}{7!} - \frac{(s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
+\frac{(s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
-\frac{(s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3))}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
-\frac{(s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2))}{(840(s_2-1)(s_1-1)(s_3-1))}\frac{1}{4!}
\end{gathered} \\[4ex]
\begin{gathered}
\frac{1}{7!} + \frac{((21s_2s_3-7s_3-7s_2+3))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{((21s_1s_3-7s_3-7s_1+3))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+\frac{((21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
-\frac{((7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_9 = \begin{bmatrix}
\begin{aligned}
&\frac{(s_1)^8}{8!} - \frac{(s_1^3(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3))}{(840(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!}\\
&-\frac{(s_1^5(21s_3-7s_1-7s_1s_3+3s_1^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!}\\
&+\frac{(s_1^5(21s_2-7s_1-7s_1s_2+3s_1^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!}\\
&-\frac{(s_1^5(21s_2s_3-7s_1s_3-7s_1s_2+3s_1^2))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!}
\end{aligned}\\[2em]
\begin{aligned}
&\frac{(s_2)^8}{8!} + \frac{(s_2^5(21s_3-7s_2-7s_2s_3+3s_2^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!}\\
&-\frac{(s_2^3(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3))}{(840(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!}\\
&-\frac{(s_2^5(7s_2-21s_1+7s_1s_2-3s_2^2))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!}\\
&+\frac{(s_2^5(7s_1s_2-21s_1s_3+7s_2s_3-3s_2^2))}{(840(s_3-1)(s_1-1)(s_2-1))}\frac{1}{5!}
\end{aligned}\\[2em]
\begin{aligned}
&\frac{(s_3)^8}{8!} - \frac{(s_3^5(7s_3-21s_2+7s_2s_3-3s_3^2))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!}\\
&+\frac{(s_3^5(7s_3-21s_1+7s_1s_3-3s_3^2))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!}\\
&-\frac{(s_3^3(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3))}{(840(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!}\\
&-\frac{(s_3^5(21s_1s_2-7s_1s_3-7s_2s_3+3s_3^2))}{(840(s_2-1)(s_1-1)(s_3-1))}\frac{1}{5!}
\end{aligned}\\[2em]
\begin{aligned}
&\frac{1}{8!} + \frac{((21s_2s_3-7s_3-7s_2+3))}{(840s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!}\\
&-\frac{((21s_1s_3-7s_3-7s_1+3))}{(840s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!}\\
&+\frac{((21s_1s_2-7s_2-7s_1+3))}{(840s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!}\\
&-\frac{((7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4))}{(840(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!}
\end{aligned}
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Hence, the order of the first derivative block method is $[5,5,5,5]^T$

for all $s_1, s_2, s_3 \in (0,1) \setminus \{s_2 = \frac{14s_1s_3+6s_1^2s_3+6s_1^2-3s_1^3}{14s_1-42s_3-6s_1^2+14s_2s_3}\} \cup \{s_1 = \frac{14s_2s_3-6s_2^2s_3-6s_2^2+3s_2^3}{42s_3-14s_2+6s_2^2-14s_2s_3}\}$
$\cup \{s_2 = \frac{-14s_1s_3+6s_1s_3^2+6s_3^2-3s_3^3}{-42s_1+14s_3-6s_3^2+14s_1s_3}\} \cup \{s_1 = \frac{-6s_2-6s_3+14s_2s_3+3}{6-14s_2-14s_3+42s_2s_3}\}\}$.

with general error constants vector

$$
\bar{C}_9 = \begin{bmatrix}
-\frac{(s_1^5(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{201600} \\
\frac{(s_2^5(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{201600} \\
-\frac{(s_3^5(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{201600} \\
\frac{(6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3)}{201600}
\end{bmatrix}
$$

In finding the order of second derivative block (5.44), Expanding y'' and f- function Taylor series, that gives

$$
\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{(s_1)^j h^j}{j!}y_n^{j+2}-y_n''-hs_1y_n''' \\
+\frac{(h^2s_1^2(5s_1s_2+5s_1s_3-20s_2s_3-2s_1^2s_2-2s_1^2s_3-2s_1^2+s_1^3+5s_1s_2s_3))}{(60s_2s_3)}y_n^{iv} \\
-\frac{(s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+2}}{j!}y_n^{j+4} \\
-\frac{(s_1^4(5s_3-2s_1-2s_1s_3+s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+2}}{j!}y_n^{j+4} \\
+\frac{(s_1^4(5s_2-2s_1-2s_1s_2+s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+2}}{j!}y_n^{j+4} \\
-\frac{(s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+4} \\
\sum_{j=0}^{\infty}\frac{(s_2)^j h^j}{j!}y_n^{j+2}-y_n''-s_2hy_n''' \\
-\frac{(h^2s_2^2(20s_1s_3-5s_1s_2-5s_2s_3+2s_1s_2^2+2s_2^2s_3+2s_2^2-s_2^3-5s_1s_2s_3))}{(60s_1s_3)}y_n^{iv} \\
+\frac{(s_2^4(5s_3-2s_2-2s_2s_3+s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+2}}{j!}y_n^{j+4} \\
-\frac{(s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+2}}{j!}y_n^{j+4} \\
-\frac{(s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+2}}{j!}y_n^{j+4} \\
+\frac{(s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+4} \\
\sum_{j=0}^{\infty}\frac{(s_3)^j h^j}{j!}y_n^{j+2}-y_n''-s_3hy_n''' \\
+\frac{(h^2s_3^2(5s_1s_3-20s_1s_2+5s_2s_3-2s_1s_3^2-2s_2s_3^2-2s_3^2+s_3^3+5s_1s_2s_3))}{(60s_1s_2)}y_n^{iv} \\
-\frac{(s_3^4(2s_3-5s_2+2s_2s_3-s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+2}}{j!}y_n^{j+4} \\
+\frac{(s_3^4(2s_3-5s_1+2s_1s_3-s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+2}}{j!}y_n^{j+4} \\
-\frac{(s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+2}}{j!}y_n^{j+4} \\
-\frac{(s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+4} \\
\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{j+2}-y_n''-hy_n''' \\
-\frac{(h^2(2s_1+2s_2+2s_3-5s_1s_2-5s_1s_3-5s_2s_3+20s_1s_2s_3-1))}{(60s_1s_2s_3)}y_n^{iv} \\
+\frac{((5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^j h^{j+2}}{j!}y_n^{j+4} \\
-\frac{((5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^j h^{j+2}}{j!}y_n^{j+4} \\
+\frac{((5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^j h^{j+2}}{j!}y_n^{j+4} \\
-\frac{((3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2))}{(60(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+4}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Comparing the coefficients of h^j and y^j produces

$$\bar{C}_0 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} s_1-s_1 \\ s_2-s_2 \\ s_3-s_3 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\bar{C}_4 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^2}{2} + \frac{(s_1^2(5s_1s_2+5s_1s_3-20s_2s_3-2s_1^2s_2-2s_1^2s_3-2s_1^2+s_1^3+5s_1s_2s_3))}{(60s_2s_3)} \\
-\frac{(s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{(s_1^4(5s_3-2s_1-2s_1s_3+s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{(s_1^4(5s_2-2s_1-2s_1s_2+s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{(s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!}
\end{gathered} \\[3ex]
\begin{gathered}
\frac{(s_2)^2}{2!} - -\frac{(s_2^2(20s_1s_3-5s_1s_2-5s_2s_3+2s_1s_2^2+2s_2^2s_3+2s_2^2-s_2^3-5s_1s_2s_3))}{(60s_1s_3)} \\
+\frac{(s_2^4(5s_3-2s_2-2s_2s_3+s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{(s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{(s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
+\frac{(s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{0!}
\end{gathered} \\[3ex]
\begin{gathered}
\frac{(s_3)^2}{2!} + \frac{(s_3^2(5s_1s_3-20s_1s_2+5s_2s_3-2s_1s_3^2-2s_2s_3^2-2s_3^2+s_3^3+5s_1s_2s_3))}{(60s_1s_2)} \\
-\frac{(s_3^4(2s_3-5s_2+2s_2s_3-s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
+\frac{(s_3^4(2s_3-5s_1+2s_1s_3-s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{(s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{(s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{0!}
\end{gathered} \\[3ex]
\begin{gathered}
\frac{1}{2!} - \frac{(2s_1+2s_2+2s_3-5s_1s_2-5s_1s_3-5s_2s_3+20s_1s_2s_3-1)}{(60s_1s_2s_3)} \\
+\frac{((5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{((5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{((5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{((3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!}
\end{gathered}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_5 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^3}{3} - \frac{(s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
- \frac{(s_1^4(5s_3-2s_1-2s_1s_3+s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+ \frac{(s_1^4(5s_2-2s_1-2s_1s_2+s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
- \frac{(s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_2)^3}{3!} + \frac{(s_2^4(5s_3-2s_2-2s_2s_3+s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
- \frac{(s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
- \frac{(s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
+ \frac{(s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{1!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_3)^3}{3!} - \frac{(s_3^4(2s_3-5s_2+2s_2s_3-s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
+ \frac{(s_3^4(2s_3-5s_1+2s_1s_3-s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
- \frac{(s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
- \frac{(s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{1!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{1}{3!} + \frac{((5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
- \frac{((5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+ \frac{((5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
- \frac{((3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}
\end{gathered}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_6 = \begin{bmatrix}
\begin{array}{c}
\frac{(s_1)^4}{4} - \frac{(s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{(s_1^4(5s_3-2s_1-2s_1s_3+s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+\frac{(s_1^4(5s_2-2s_1-2s_1s_2+s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
-\frac{(s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{array} \\ \\
\begin{array}{c}
\frac{(s_2)^4}{4!} + \frac{(s_2^4(5s_3-2s_2-2s_2s_3+s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{(s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
-\frac{(s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
+\frac{(s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{2!}
\end{array} \\ \\
\begin{array}{c}
\frac{(s_3)^4}{4!} - \frac{(s_3^4(2s_3-5s_2+2s_2s_3-s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
+\frac{(s_3^4(2s_3-5s_1+2s_1s_3-s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
-\frac{(s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
-\frac{(s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{2!}
\end{array} \\ \\
\begin{array}{c}
\frac{1}{4!} + \frac{((5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\
-\frac{((5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\
+\frac{((5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\
-\frac{((3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!}
\end{array}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$\bar{C}_7 = \begin{bmatrix} \begin{array}{c} \frac{(s_1)^5}{5} - \frac{(s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{(s_1^4(5s_3-2s_1-2s_1s_3+s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ +\frac{(s_1^4(5s_2-2s_1-2s_1s_2+s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ -\frac{(s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \end{array} \\ \\ \begin{array}{c} \frac{(s_2)^5}{5!} + \frac{(s_2^4(5s_3-2s_2-2s_2s_3+s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{(s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ -\frac{(s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ +\frac{(s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{3!} \end{array} \\ \\ \begin{array}{c} \frac{(s_3)^5}{5!} - \frac{(s_3^4(2s_3-5s_2+2s_2s_3-s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ +\frac{(s_3^4(2s_3-5s_1+2s_1s_3-s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ -\frac{(s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ -\frac{(s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{3!} \end{array} \\ \\ \begin{array}{c} \frac{1}{5!} + \frac{((5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{((5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ +\frac{((5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ -\frac{((3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \end{array} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\bar{C}_8 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^6}{6} - \frac{(s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
- \frac{(s_1^4(5s_3-2s_1-2s_1s_3+s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+ \frac{(s_1^4(5s_2-2s_1-2s_1s_2+s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
- \frac{(s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_2)^6}{6!} + \frac{(s_2^4(5s_3-2s_2-2s_2s_3+s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
- \frac{(s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
- \frac{(s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
+ \frac{(s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{4!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{(s_3)^6}{6!} - \frac{(s_3^4(2s_3-5s_2+2s_2s_3-s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
+ \frac{(s_3^4(2s_3-5s_1+2s_1s_3-s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
- \frac{(s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
- \frac{(s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{4!}
\end{gathered} \\[2em]
\begin{gathered}
\frac{1}{6!} + \frac{((5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
- \frac{((5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+ \frac{((5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
- \frac{((3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\[2em] 0 \\[2em] 0 \\[2em] 0 \end{bmatrix}
$$

$$
\bar{C}_9 = \begin{bmatrix}
\frac{(s_1)^7}{7} - \frac{(s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
-\frac{(s_1^4(5s_3-2s_1-2s_1s_3+s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
+\frac{(s_1^4(5s_2-2s_1-2s_1s_2+s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
-\frac{(s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!} \\
\\
\frac{(s_2)^7}{7!} + \frac{(s_2^4(5s_3-2s_2-2s_2s_3+s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
-\frac{(s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
-\frac{(s_2^4(2s_2-5s_1+2s_1s_2-s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
+\frac{(s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{5!} \\
\\
\frac{(s_3)^7}{7!} - \frac{(s_3^4(2s_3-5s_2+2s_2s_3-s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
+\frac{(s_3^4(2s_3-5s_1+2s_1s_3-s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
-\frac{(s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
-\frac{(s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{5!} \\
\\
\frac{1}{7!} + \frac{((5s_2s_3-2s_3-2s_2+1))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\
-\frac{((5s_1s_3-2s_3-2s_1+1))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\
+\frac{((5s_1s_2-2s_2-2s_1+1))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\
-\frac{((3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Therefore, the order of second derivative method (5.44) is $[5,5,5,5]^T$ with general error vector of constants

$$
\bar{C}_9 = \begin{bmatrix}
-\frac{(s_1^4(14s_1s_2+14s_1s_3-35s_2s_3-7s_1^2s_2-7s_1^2s_3-7s_1^2+4s_1^3+14s_1s_2s_3))}{50400} \\
\frac{(s_2^4(35s_1s_3-14s_1s_2-14s_2s_3+7s_1s_2^2+7s_2^2s_3+7s_2^2-4s_2^3-14s_1s_2s_3))}{50400} \\
-\frac{(s_3^4(14s_1s_3-35s_1s_2+14s_2s_3-7s_1s_3^2-7s_2s_3^2-7s_3^2+4s_3^3+14s_1s_2s_3))}{50400} \\
\frac{(7s_1+7s_2+7s_3-14s_1s_2-14s_1s_3-14s_2s_3+35s_1s_2s_3-4)}{50400}
\end{bmatrix}
$$

This is true for all $s_1, s_2, s_3 \in (0,1) \setminus \{s_2 = \frac{-14s_1s_3+7s_1^2s_3+7s_1^2-4s_1^3}{14s_1-35s_3-7s_1^2+14s_1s_3}\} \cup \{s_1 = \frac{14s_2s_3-7s_2^2s_3-7s_2^2+4s_2^3}{35s_3-14s_2+7s_2^2-14s_2s_3}\}$ $\cup \{s_2 = \frac{-14s_1s_3+7s_1s_3^2+7s_3^2-4s_3^3}{-35s_1+14s_3-7s_3^2+14s_1s_3}\} \cup \{s_1 = \frac{-7s_2-7s_3+14s_2s_3+4}{7-14s_2-14s_3+35s_2s_3}\}$. In order to find the order of third derivative (5.49), strategy in section (5.1.1.1) is applied. Similarly, y''' and f-function are expanded in Taylor series. This is illustrated below.

$$\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{(s_1)^jh^j}{j!}y_n^{j+3}-y_n'''+\frac{(hs_1(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3))}{(60s_2s_3)}y_n^{iv} \\
-\frac{(s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+1}}{j!}y_n^{j+4} \\
-\frac{(s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+1}}{j!}y_n^{j+4} \\
+\frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+1}}{j!}y_n^{j+4} \\
-\frac{(s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+4} \\
\\
\sum_{j=0}^{\infty}\frac{(s_2)^jh^j}{j!}y_n^{j+3}-y_n'''-\frac{(hs_2(30s_1s_3-10s_1s_2-10s_2s_3+5s_1s_2^2+5s_2^2s_3+5s_2^2-3s_2^3-10s_1s_2s_3))}{(60s_1s_3)}y_n^{iv} \\
+\frac{(s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+1}}{j!}y_n^{j+4} \\
-\frac{(s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+1}}{j!}y_n^{j+4} \\
-\frac{(s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+1}}{j!}y_n^{j+4} \\
+\frac{(s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+4} \\
\\
\sum_{j=0}^{\infty}\frac{(s_3)^jh^j}{j!}y_n^{j+3}-y_n'''+\frac{(hs_3(10s_1s_3-30s_1s_2+10s_2s_3-5s_1s_3^2-5s_2s_3^2-5s_3^2+3s_3^3+10s_1s_2s_3))}{(60s_1s_2)}y_n^{iv} \\
-\frac{(s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+1}}{j!}y_n^{j+4} \\
+\frac{(s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+1}}{j!}y_n^{j+4} \\
-\frac{(s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+1}}{j!}y_n^{j+4} \\
-\frac{(s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+4} \\
\\
\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{j+3}-y_n'''-\frac{(h(5s_1+5s_2+5s_3-10s_1s_2-10s_1s_3-10s_2s_3+30s_1s_2s_3-3))}{(60s_1s_2s_3)}y_n^{iv} \\
+\frac{((10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+1}}{j!}y_n^{j+4} \\
-\frac{((10s_1s_3-5s_3-5s_1+3))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+1}}{j!}y_n^{j+4} \\
+\frac{((10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+1}}{j!}y_n^{j+4} \\
-\frac{((15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12))}{(60(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+4}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Comparing the coefficients of h^j and y^j produces

$$\bar{C}_0 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\bar{C}_4 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^1}{1!} - +\frac{(s_1(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3))}{(60s_2s_3)} \\
-\frac{(s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0h^{j+1}}{0!} \\
-\frac{(s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{(s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!}
\end{gathered} \\
\\
\begin{gathered}
\frac{(s_2)^1}{1!} - -\frac{(s_2(30s_1s_3-10s_1s_2-10s_2s_3+5s_1s_2^2+5s_2^2s_3+5s_2^2-3s_2^3-10s_1s_2s_3))}{(60s_1s_3)} \\
+\frac{(s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{(s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{(s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
+\frac{(s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{0!}
\end{gathered} \\
\\
\begin{gathered}
\frac{(s_3)^1}{1!} - +\frac{(s_3(10s_1s_3-30s_1s_2+10s_2s_3-5s_1s_3^2-5s_2s_3^2-5s_3^2+3s_3^3+10s_1s_2s_3))}{(60s_1s_2)} \\
-\frac{(s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
+\frac{(s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
-\frac{(s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{(s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{0!}
\end{gathered} \\
\\
\begin{gathered}
\frac{1}{1!} - \frac{(5s_1+5s_2+5s_3-10s_1s_2-10s_1s_3-10s_2s_3+30s_1s_2s_3-3)}{(60s_1s_2s_3)} \\
+\frac{((10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^0}{0!} \\
-\frac{((10s_1s_3-5s_3-5s_1+3))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^0}{0!} \\
+\frac{((10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^0}{0!} \\
-\frac{((15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{0!}
\end{gathered}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{C}_5 = \begin{bmatrix}
\begin{gathered}
\frac{(s_1)^2}{2!} - \frac{(s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{(s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+\frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
-\frac{(s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}
\end{gathered} \\
\\
\begin{gathered}
\frac{(s_2)^2}{2!} + \frac{(s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{(s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
-\frac{(s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
+\frac{(s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{1!}
\end{gathered} \\
\\
\begin{gathered}
\frac{(s_3)^2}{2!} - \frac{(s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
+\frac{(s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
-\frac{(s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
-\frac{(s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{1!}
\end{gathered} \\
\\
\begin{gathered}
\frac{1}{2!} + \frac{((10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^1}{1!} \\
-\frac{((10s_1s_3-5s_3-5s_1+3))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^1}{1!} \\
+\frac{((10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^1}{1!} \\
-\frac{((15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{1!}
\end{gathered}
\end{bmatrix}
= \begin{bmatrix} 0 \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{bmatrix}
$$

$$\bar{C}_6 = \begin{bmatrix} \frac{(s_1)^3}{3!} - \frac{(s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\ -\frac{(s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\ +\frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\ -\frac{(s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!} \\ \\ \frac{(s_2)^3}{3!} + \frac{(s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\ -\frac{(s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\ -\frac{(s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\ +\frac{(s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{2!} \\ \\ \frac{(s_3)^3}{3!} - \frac{(s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\ +\frac{(s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\ -\frac{(s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\ -\frac{(s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{2!} \\ \\ \frac{1}{3!} + \frac{((10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^2}{2!} \\ -\frac{((10s_1s_3-5s_3-5s_1+3))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^2}{2!} \\ +\frac{((10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^2}{2!} \\ -\frac{((15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{2!} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}_7 = \begin{bmatrix} \begin{gathered} \frac{(s_1)^4}{4!} - \frac{(s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{(s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ +\frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ -\frac{(s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \end{gathered} \\ \\ \begin{gathered} \frac{(s_2)^4}{4!} + \frac{(s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{(s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ -\frac{(s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ +\frac{(s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{3!} \end{gathered} \\ \\ \begin{gathered} \frac{(s_3)^4}{4!} - \frac{(s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ +\frac{(s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ -\frac{(s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ -\frac{(s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{3!} \end{gathered} \\ \\ \begin{gathered} \frac{1}{4!} + \frac{((10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^3}{3!} \\ -\frac{((10s_1s_3-5s_3-5s_1+3))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^3}{3!} \\ +\frac{((10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^3}{3!} \\ -\frac{((15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{3!} \end{gathered} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\bar{C}_8 = \begin{bmatrix}
\frac{(s_1)^5}{5!} - \frac{(s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{(s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+\frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
-\frac{(s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!} \\
\\
\frac{(s_2)^5}{5!} + \frac{(s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{(s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
-\frac{(s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
+\frac{(s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{4!} \\
\\
\frac{(s_3)^5}{5!} - \frac{(s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
+\frac{(s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
-\frac{(s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
-\frac{(s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{4!} \\
\\
\frac{1}{5!} + \frac{((10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^4}{4!} \\
-\frac{((10s_1s_3-5s_3-5s_1+3))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^4}{4!} \\
+\frac{((10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^4}{4!} \\
-\frac{((15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{4!}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$\bar{C}_9 = \begin{bmatrix} \frac{(s_1)^6}{6!} - \frac{(s_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3))}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ - \frac{(s_1^3(10s_3-5s_1-5s_1s_3+3s_1^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ + \frac{(s_1^3(10s_2-5s_1-5s_1s_2+3s_1^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\ - \frac{(s_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \\ \\ \frac{(s_2)^6}{6!} + \frac{(s_2^3(10s_3-5s_2-5s_2s_3+3s_2^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ - \frac{(s_2(30s_1s_3-20s_1s_2-20s_2s_3+15s_1s_2^2+15s_2^2s_3+15s_2^2-12s_2^3-20s_1s_2s_3))}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ - \frac{(s_2^3(5s_2-10s_1+5s_1s_2-3s_2^2))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\ + \frac{(s_2^3(5s_1s_2-10s_1s_3+5s_2s_3-3s_2^2))}{(60(s_3-1)(s_1-1)(s_2-1))}\frac{1}{5!} \\ \\ \frac{(s_3)^6}{6!} - \frac{(s_3^3(5s_3-10s_2+5s_2s_3-3s_3^2))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ + \frac{(s_3^3(5s_3-10s_1+5s_1s_3-3s_3^2))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ - \frac{(s_3(20s_1s_3-30s_1s_2+20s_2s_3-15s_1s_3^2-15s_2s_3^2-15s_3^2+12s_3^3+20s_1s_2s_3))}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\ - \frac{(s_3^3(10s_1s_2-5s_1s_3-5s_2s_3+3s_3^2))}{(60(s_2-1)(s_1-1)(s_3-1))}\frac{1}{5!} \\ \\ \frac{1}{6!} + \frac{((10s_2s_3-5s_3-5s_2+3))}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\frac{(s_1)^5}{5!} \\ - \frac{((10s_1s_3-5s_3-5s_1+3))}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\frac{(s_2)^5}{5!} \\ + \frac{((10s_1s_2-5s_2-5s_1+3))}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\frac{(s_3)^5}{5!} \\ - \frac{((15s_1+15s_2+15s_3-20s_1s_2-20s_1s_3-20s_2s_3+30s_1s_2s_3-12))}{(60(s_3-1)(s_2-1)(s_1-1))}\frac{1}{5!} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, the order of third derivative block method(5.49) is $[5,5,5,5]^T$ with general vector of error constants

$$\bar{C}_9 = \begin{bmatrix} -\frac{(s_1^3(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3))}{7200} \\ \frac{(s_2^3(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3))}{7200} \\ -\frac{(s_3^3(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3))}{7200} \\ \frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{7200} \end{bmatrix}$$

and this is true for all $s_1, s_2, s_3 \in (0,1) \setminus \{s_2 = \frac{-5s_1s_3+3s_1^2s_3+3s_1^2-2s_1^3}{5s_1-10s_3-3s_1^2+5s_1s_3}\} \cup \{s_1 = \frac{5s_2s_3-3s_2^2s_3-3s_2^2+2s_2^3}{10s_3-5s_2+3s_2^2-5s_2s_3}\}$ $\cup \{s_2 = \frac{-5s_1s_3+3s_1s_3^2+3s_3^2-2s_3^3}{-10s_1+5s_3-3s_3^2+5s_1s_3}\} \cup \{s_1 = \frac{-3s_2-3s_3+5s_2s_3+2}{3-5s_2-5s_3+10s_2s_3}\}$.

5.1.1.2 Zero Stability of One Step Hybrid Block Method with Generalised Three Off-Step Points for Fourth Order ODEs

In finding the zero-stability of the block (5.30), we only consider the first characteristic function according to Definition (3.1.3), that is

$$\begin{aligned}
\Pi(z) &= |z\,I^{[3]_4} - \bar{B}_1^{[3]_4}| \\
&= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^3(z-1) = 0
\end{aligned}$$

whose solution is $z = 0,0,0,1$. The first characteristic polynomial of the first derivative block method (5.39) according to Definition (3.1.3) is given by

$$\begin{aligned}
\Pi(z) &= |z\,I^{[3]_4} - \acute{B}_2^{[3]_4}| \\
&= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^3(z-1) = 0
\end{aligned}$$

whose solution is $z = 0,0,0,1$.

The characteristic polynomial of second derivative(5.44) is

$$
\begin{aligned}
\Pi(z) &= |z\, I^{[3]_4} - \acute{B}_3^{[3]_4}| \\
&= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^3(z-1) = 0
\end{aligned}
$$

which implies $z = 0,0,0,1$. The same strategy as earlier mention is also used to prove zero stability of third derivative block (5.49), that is

$$
\begin{aligned}
\Pi(z) &= |z\, I^{[3]_4} - \acute{\acute{\acute{B}}}_3^{[3]_4}| \\
&= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= z^3(z-1)
\end{aligned}
$$

This implies $z = 0,0,0,1$. Hence, the conditions in Definition (3.1.3) are satisfied. Therefore, the block method(5.30) and its derivatives (5.39), (5.44) and (5.49) are zero stable.

5.1.1.3 Consistency and Convergent of One Step Hybrid Block Method with Generalised Three Off-Step Points for Fourth Order ODEs

Consistency and convergent of the method (5.30) and its derivatives (5.39),(5.44) and (5.49) are proved by Definition (3.1.4) and Theorem (3.1)

5.1.1.4 Region of Absolute Stability of One Step Hybrid Block Method with Generalised Three Off-Step Points for Fourth Order ODEs

Applying (3.29) for one step hybrid block with generalised three off-step points (5.30), it gives

$$\bar{h}(\theta,h)=\frac{I^{[3]_4}Y_m^{[3]_4}(\theta)-\bar{B}_1^{[3]_4}R_1^{[3]_4}(\theta)}{[\bar{D}^{[3]_4}Y_{R_5^{[3]_4}}(\theta)+\bar{E}^{[3]_4}Y_{R_6^{[3]_4}}(\theta)]} \tag{5.50}$$

where

$$I^{[3]_4}Y_m^{[3]_4}(\theta)=\begin{bmatrix} e^{is_1\theta} & 0 & 0 & 0 \\ 0 & e^{is_2\theta} & 0 & 0 \\ 0 & 0 & e^{is_3\theta} & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{bmatrix}$$

$$\bar{B}_1^{[3]_4}R_1^{[3]_4}(\theta)=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{D}^{[3]_4}Y_{R_5^{[3]_4}}(\theta)=\begin{bmatrix} 0 & 0 & 0 & \bar{D}_{14}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{24}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{34}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{44}^{[3]_4} \end{bmatrix}$$

$$\bar{E}^{[3]_4}Y_{R_6^{[3]_4}}(\theta)=\begin{bmatrix} \bar{E}_{11}^{[3]_4}e^{is_1\theta} & \bar{E}_{12}^{[3]}e^{is_2\theta} & \bar{E}_{13}^{[3]_4}e^{is_3\theta} & \bar{E}_{14}^{[3]_4}e^{i\theta} \\ \bar{E}_{21}^{[3]_4}e^{is_1\theta} & \bar{E}_{22}^{[3]_4}e^{is_2\theta} & \bar{E}_{23}^{[3]_4}e^{is_3\theta} & \bar{E}_{24}^{[3]_4}e^{i\theta} \\ \bar{E}_{31}^{[3]_4}e^{is_1\theta} & \bar{E}_{32}^{[3]_4}e^{is_2\theta} & \bar{E}_{33}^{[3]_4}e^{is_3\theta} & \bar{E}_{34}^{[3]_4}e^{i\theta} \\ \bar{E}_{41}^{[3]_4}e^{is_1\theta} & \bar{E}_{42}^{[3]_4}e^{is_2\theta} & \bar{E}_{43}^{[3]_4}e^{is_3\theta} & \bar{E}_{44}^{[3]_4}e^{i\theta} \end{bmatrix}$$

simplifying the above matrix and finding its determinant, we get

$$\bar{h}(\theta,h)=\frac{60963840000(e^{i\theta}-1)}{(s_1^3s_2^3s_3^3(20s_1+20s_2+20s_3-10s_1s_2-10s_1s_3-10s_2s_3+4s_1s_2s_3+s_1s_2s_3e^{i\theta}-35))}$$

Expanding the above equation trigonometrically and equating the imaginary part to zero, the equation of stability region for one step hybrid block method with three generalised off-step points for fourth order ODE is obtained as below

$$\bar{h}(\theta,h)=\frac{60963840000(\cos(\theta)-1)}{(s_1^3s_2^3s_3^3(20s_1+20s_2+20s_3-10s_1s_2-10s_1s_3-10s_2s_3+4s_1s_2s_3+s_1s_2s_3\cos(\theta)-35))} \tag{5.51}$$

5.2 Numerical Results for Solving Fourth Order ODEs

This section considers specific numerical method for hybrid block method with generalised three off-step points for fourth ODEs.

Substituting $s_1=\frac{1}{4}$, $s_2=\frac{1}{2}$, $s_3=\frac{3}{4}$ into equations (5.31)-(5.34), (5.35)-(5.38), (5.40)-(5.43)and (5.45)-(5.48), the following block of one step with three hybrid points and its derivatives are obtained

$$
\begin{aligned}
y_{n+\frac{1}{4}} &= y_n + \frac{h}{4}y_n^{'} + \frac{h^2}{32}y_n^{''} + \frac{h^3}{384}y_n^{'''} + \frac{3373h^4}{30965760}f_n - \frac{131h^4}{30965760}f_{n+1} \\
&+ \frac{139h^4}{1548288}f_{n+\frac{1}{4}} - \frac{283h^4}{5160960}f_{n+\frac{1}{2}} + \frac{179h^4}{7741440}f_{n+\frac{3}{4}} \\
y_{n+\frac{1}{2}} &= y_n + \frac{h}{2}y_n^{'} + \frac{h^3}{48}y_n^{'''} + \frac{h^2}{8}y_n^{''} + \frac{37h^4}{30240}f_n - \frac{h^4}{15120}f_{n+1} \\
&+ \frac{59h^4}{30240}f_{n+\frac{1}{4}} - \frac{h^4}{1152}f_{n+\frac{1}{2}} + \frac{11h^4}{30240}f_{n+\frac{3}{4}} \\
y_{n+\frac{3}{4}} &= y_n + \frac{3h}{4}y_n^{'} + \frac{9h^2}{32}y_n^{''} + \frac{9h^3}{128}y_n^{'''} + \frac{5319h^4}{1146880}f_n - \frac{297h^4}{1146880}f_{n+1} \\
&+ \frac{2889h^4}{286720}f_{n+\frac{1}{4}} - \frac{1539h^4}{573440}f_{n+\frac{1}{2}} + \frac{81h^4}{57344}f_{n+\frac{3}{4}} \\
y_{n+1} &= y_n + hy_n^{'} + \frac{h^2}{2}y_n^{''} + \frac{h^3}{6}y_n^{'''} + \frac{11h^4}{945}f_n - \frac{h^4}{1512}f_{n+1} \\
&+ \frac{4h^4}{135}f_{n+\frac{1}{4}} - \frac{h^4}{315}f_{n+\frac{1}{2}} + \frac{4h^4}{945}f_{n+\frac{3}{4}}
\end{aligned}
\tag{5.52}
$$

$$
\begin{aligned}
y^{'}_{n+\frac{1}{4}} &= +y_n^{'} + \frac{h}{4}y_n^{'''} + \frac{h^2}{32}y_n^{'''} + \frac{113h^3}{71680}f_n - \frac{h^3}{645120}47f_{n+1} + \frac{107h^3}{64512}f_{n+\frac{1}{4}} \\
&- \frac{103h^3}{107520}f_{n+\frac{1}{2}} + \frac{43h^3}{107520}f_{n+\frac{3}{4}} \\
y^{'}_{n+\frac{1}{2}} &= y_n^{'} + \frac{h}{2}y_n^{''} + \frac{h^2}{8}y_n^{'''} + \frac{331h^3}{40320}f_n - \frac{19h^3}{40320}f_{n+1} + \frac{83h^3}{5040}f_{n+\frac{1}{4}} \\
&- \frac{h^3}{168}f_{n+\frac{1}{2}} + \frac{13h^3}{5040}f_{n+\frac{3}{4}} \\
y^{'}_{n+\frac{3}{4}} &= y_n^{'} + \frac{3}{4}y_n^{''} + \frac{9h^2}{32}y_n^{'''} + \frac{1431h^3}{71680}f_n - \frac{81h^3}{71680}f_{n+1} + \frac{1863h^3}{35840}f_{n+\frac{1}{4}} \\
&- \frac{243h^3}{35840}f_{n+\frac{1}{2}} + \frac{45h^3}{7168}f_{n+\frac{3}{4}} \\
y^{'}_{n+1} &= y_n^{'} + hy_n^{''} + \frac{h^2}{2}y_n^{'''} + \frac{31h^3}{840}f_n - \frac{h^3}{504}f_{n+1} + \frac{34h^3}{315}f_{n+\frac{1}{4}} \\
&+ \frac{h^3}{210}f_{n+\frac{1}{2}} + \frac{2h^3}{105}f_{n+\frac{3}{4}}
\end{aligned}
\tag{5.53}
$$

$$
\begin{aligned}
y''_{n+\frac{1}{4}} &= y''_n + \frac{h}{4}y'''_n + \frac{367h^2}{23040}f_n - \frac{7h^2}{7680}f_{n+1} + \frac{3h^2}{128}f_{n+\frac{1}{4}} \\
&-\frac{47h^2}{3840}f_{n+\frac{1}{2}} + \frac{29h^2}{5760}f_{n+\frac{3}{4}} \\
y''_{n+\frac{1}{2}} &= y''_n + \frac{h}{2}y'''_n + \frac{53h^2}{1440}f_n - \frac{h^2}{480}f_{n+1} + \frac{h^2}{10}f_{n+\frac{1}{4}} \\
&-\frac{h^2}{48}f_{n+\frac{1}{2}} + \frac{h^2}{90}f_{n+\frac{3}{4}} \\
y''_{n+\frac{3}{4}} &= y''_n + \frac{3}{4}y'''_n + \frac{147h^2}{2560}f_n - \frac{9h^2}{2560}f_{n+1} + \frac{117h^2}{640}f_{n+s_1} \\
&+\frac{27h^2}{1280}f_{n+\frac{1}{2}} + \frac{3h^2}{128}f_{n+\frac{3}{4}} \\
y''_{n+1} &= y''_n + hy'''_n + \frac{7h^2}{90}f_n + \frac{4h^2}{15}f_{n+\frac{1}{4}} + \frac{h^2}{15}f_{n+\frac{1}{2}} \\
&+\frac{4h^2}{45}f_{n+\frac{3}{4}}
\end{aligned}
\tag{5.54}
$$

$$
\begin{aligned}
y'''_{n+\frac{1}{4}} &= y'''_n + \frac{251h}{2880}f_n - \frac{19h}{2880}f_{n+1} + \frac{323h}{1440}f_{n+\frac{1}{4}} - \frac{11h}{120}f_{n+\frac{1}{2}} \\
&+\frac{53h}{1440}f_{n+\frac{3}{4}} \\
y'''_{n+\frac{1}{2}} &= y'''_n + \frac{29h}{360}f_n - \frac{h}{360}f_{n+1} + \frac{h}{90}31f_{n+s_1} + \frac{h}{15}f_{n+\frac{1}{2}} \\
&+\frac{h}{90}f_{n+\frac{3}{4}} \\
y'''_{n+\frac{3}{4}} &= y'''_n + \frac{27h}{320}f_n - \frac{3h}{320}f_{n+1} + \frac{51h}{160}f_{n+\frac{1}{4}} + \frac{9h}{40}f_{n+\frac{1}{2}} \\
&+\frac{21h}{160}f_{n+\frac{3}{4}} \\
y'''_{n+1} &= y'''_n + \frac{7h}{90}f_n + \frac{7h}{90}f_{n+1} + \frac{16h}{45}f_{n+\frac{1}{4}} + \frac{2h}{15}f_{n+\frac{1}{2}} \\
&+\frac{16h}{45}f_{n+\frac{3}{4}}
\end{aligned}
\tag{5.55}
$$

Based on the approach used in section (5.1.1.1), the block method and its derivative above are of order $[5,5,5,5]^T$, $[5,5,5,5]^T$, $[5,5,5,5]^T$ and $[5,5,5,6]^T$ with error constant $[3.088509e^{-9}, 4.736414e^{-8}, 1.885210e^{-7}, -2.066566e^{-6}]^T$, $[5.260346e^{-8}, 3.390842e^{-7}, 8.276531e^{-7}, 1.550099e^{-6}]^T$, $[6.478930e^{-7}, 1.550099e^{-6}, 2.452305e^{-6}, 3.100198e^{-6}]^T$, $[4.577637e^{-6}, 2.712674e^{-6}, 4.577637e^{-6}, -5.1670e^{-7}]^T$. In order to find the region of absolute stability of (5.52), $s_1 = \frac{1}{4}$,$s_2 = \frac{1}{2}$,$s_3 = \frac{3}{4}$ is substituted into Equation (5.51), this gives

$$\bar{h}(\theta,h) = \frac{(2367600721920000(\cos(\theta)-1))}{(3\cos(\theta)-368)} \tag{5.56}$$

Equation (5.56) is evaluated at intervals of 30° , this produces tabulated results below.

θ	0	30°	60°	90°	120°	150°	180°
$\bar{h}(\theta,h)$	0	$8.6808e^{11}$	$3.2300e^{12}$	$6.4337e^{12}$	$9.6114e^{12}$	$1.1921e^{13}$	$1.2763e^{13}$

Hence, the interval of absolute stability is $(0, 1.2763e^{13})$ as demonstrated in Figure 5.2 as a region in polar coordinate system.

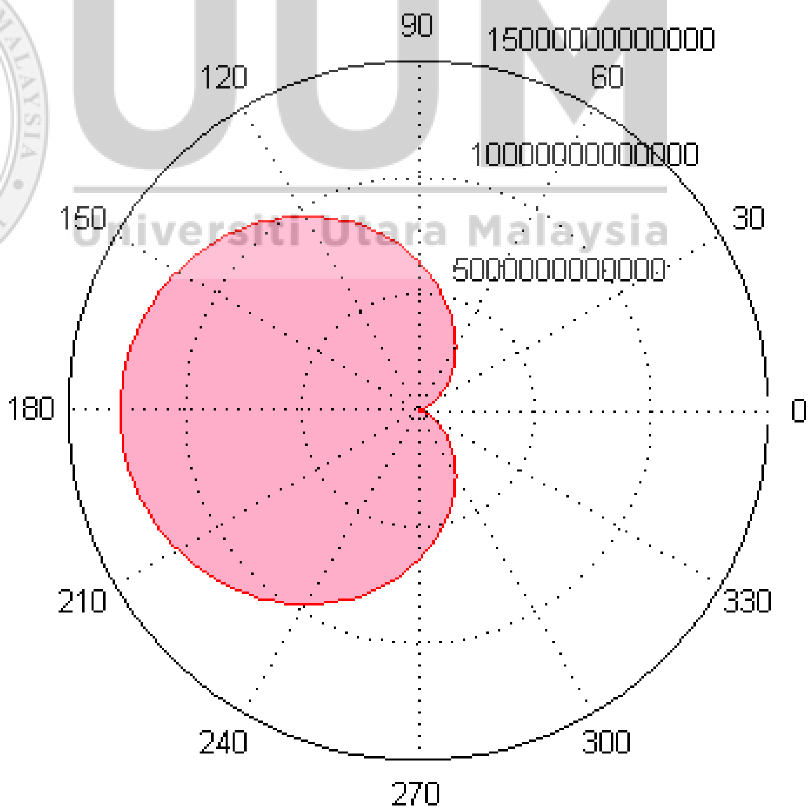

Figure 5.2. Region stability of one step hybrid block method with three off-step points $s_1 = \frac{1}{4}$, $s_2 = \frac{1}{2}$ and $s_3 = \frac{3}{4}$ for fourth order ODEs.

In order to test the accuracy of our methods, the following fourth order ODEs are considered. We solved the same problems the existing methods solved in order to compare our results in terms of error.

Problem 17: $y^{iv} = x,\ y(0) = 0,\ y^{'}(0) = 1,\ y^{''}(0) = y^{'''}(0) = 0,\ h = \dfrac{1}{10}$

Exact solution: $y(x) = \dfrac{x^2}{120} + x \qquad 0 \le x \le 1$

Source: *Kayode et al.*, 2014

Problem 18: $y^{iv} - (y^{'})^2 + yy^{''} + 4x^2 - e^x(1 - 4x + x^2) = 0, \quad y(0) = 1,$

$y^{'}(0) = 1,\ y^{''}(0) = 1,\ h = \dfrac{1}{100}$

Exact solution: $y(x) = x^2 + e^x \qquad 0 \le x \le 1$

Source: *Olabode and Omole*, 2015

Problem 19: $y^{iv} + y^{''} = 0,\ y(0) = 0,\ y^{'}(0) = \dfrac{-1.1}{72 - 50\pi},\ y^{''}(0) = \dfrac{1}{144 - 100\pi},$

$y^{'''}(0) = \dfrac{1.2}{144 - 100\pi},\ h = \dfrac{1}{320}$

Exact solution: $y(x) = \dfrac{1 - x - \cos x - 1.2\sin x}{144 - 100\pi} \qquad 0 \le x \le 1$

Source: *Kuboye and Omar*, 2015

Problem 20: $y^{iv} - 4y^{''} = 0,\ y(0) = 1,\ y^{'}(0) = 3, \quad y^{''}(0) = 3,$

$y^{'''}(0) = 16,\ h = \dfrac{1}{320}$

Exact solution: $y(x) = \dfrac{1 - x - \cos x - 1.2\sin x}{144 - 100\pi} \qquad 0 \le x \le 1$

Source: *Awoyemi et al.*, 2015

Table 5.1

Comparison of the New Method with One Step Hybrid Block Method (Kayode et al. , 2014) and Six Step Block Method (Olabode, 2009) for Solving Problem 17 where $h = \frac{1}{10}$

x		$s_1 = \frac{1}{4}, s_2 = \frac{1}{2}, s_3 = \frac{3}{4}, P = 5$	Olabode(2009), $P = 6$	Kayode et al.(2014), $P = 8$
0.1	Exact solution	0.100000083333333340	0.100000083	0.100000083333334000
	Computed solution	0.100000083333333340	0.100000084	0.10000008333351720
	Error	$0.000000e^{0}$	$1.66666667e^{-10}$	$1.832e^{-13}$
0.2	Exact solution	0.200002666666666690	0.200002667	0.200002666666666900
	Computed Solution	0.200002666666666660	0.200002667	0.20000266667150250
	Error	$2.775558e^{-17}$	$3.33333305e^{-10}$	$4.835e^{-12}$
0.3	Exact solution	0.300020250000000040	0.30002025	0.300020250000000004
	Computed solution	0.300020249999999990	0.300020251	0.30002025000721480
	Error	$5.551115e^{-17}$	$5.99999994e^{-10}$	$7.214e^{-12}$
0.4	Exact solution	0.400085333333333350	0.400085333	0.400008533333333333
	Computed solution	0.400085333333333400	0.400085334	0.40000853340160457
	Error	$5.551115e^{-17}$	$7.66666675e^{-10}$	$6.832e^{-11}$
0.5	Exact Solution	0.500260416666666650	0.500260417	0.500260416666666665
	Computed solution	0.500260416666666870	0.500260418	0.50026041674083458
	Error	$2.220446e^{-16}$	$9.33333300e^{-10}$	$7.416e^{-11}$
0.6	Exact Solution	0.600647999999999960	0.600648	0.600648000000000007
	Computed solution	0.600648000000000290	0.600648001	0.60064800002714565
	Error	$3.330669e^{-16}$	$1.10000009e^{-9}$	$2.714e^{-11}$
0.7	Exact solution	0.701400583333333330	0.701400583	0.701400583333333344
	Computed solution	0.701400583333333330	0.701400585	0.701400583333333660
	Error	$3.330669e^{-16}$	$1.27166666e^{-9}$	$2.815e^{-10}$
0.8	Exact solution	0.802730666666666590	0.802730667	0.802730666666666670
	Computed solution	0.802730666666667040	0.802730668	0.80273066700848838
	Error	$4.440892e^{-16}$	$1.45333334e^{-9}$	$3.412e^{-10}$
0.9	Exact solution	0.904920749999999940	0.90492075	0.904920750000000005
	Computed Solution	0.904920750000000270	0.904920752	0.90492075019356814
	Error	$3.330669e^{-16}$	$1.64999991e^{-9}$	$1.936e^{-10}$
1.0	Exact solution	1.008333333333333300	1.008333333	1.008333333333333000
	Computed solution	1.008333333333333700	1.008333335	1.00833333361984509
	Error	$4.440892e^{-16}$	$1.87666660e^{-9}$	$2.865e^{-10}$

Table 5.2

Comparison of the new method with One and Two Hybrid Block Method (Olabode and Omole, 2015) for Solving Problem 18 where $h = \frac{1}{320}$

x		$s_1 = \frac{1}{4}, s_2 = \frac{1}{3}, s_3 = \frac{2}{3}, P = 5$	Olabode and Omole (2015), $P = 5$	Olabode and Omole (2015), $P = 6$
0.0031250	Exact solution	1.003139653527739000	1.003139653527739149	1.003139653527739149
	Computed solution	1.003139653527739300	1.003139653526590265	1.003139653526590265
	Error	$2.220446e^{-16}$	$1.148884e^{-12}$	$1.148884e^{-12}$
0.0062500	Exact solution	1.006308634503762000	1.006308634503762010	1.006308634503762010
	Computed Solution	1.006308634503762200	1.006308634484910542	1.006308634484910542
	Error	$2.220446e^{-16}$	$1.8851468e^{-11}$	$1.8851468e^{-11}$
0.0093750	Exact solution	1.009506973589070900	1.009506973589071086	1.009506973589071086
	Computed solution	1.009506973589071200	1.009506973491318106	1.009506973491318106
	Error	$2.220446e^{-16}$	$9.7752980e^{-11}$	$9.7752980e^{-11}$
0.0125000	Exact solution	1.012734701540634500	1.012734701540634377	1.012734701540634377
	Computed solution	1.012734701540634300	1.012734701224875248	1.012734701224875248
	Error	$2.220446e^{-16}$	$3.15759129e^{-10}$	$3.15759129e^{-10}$
0.0156250	Exact Solution	1.015991849211685900	1.015991849211685747	1.015991849211685747
	Computed solution	1.015991849211680800	1.015991848424806972	1.015991848424806972
	Error	$2.220446e^{-16}$	$7.86878775e^{-10}$	$1.15463e^{-10}$
0.0187500	Exact Solution	1.019278447552026500	1.019278447552026225	1.019278447552026225
	Computed solution	1.019278447552020500	1.019278445888007693	1.019278445888007693
	Error	$2.220446e^{-16}$	$1.66401853e^{-9}$	$1.66401853e^{-9}$
0.0218750	Exact solution	1.022594527608326400	1.022594527608326245	1.022594527608326245
	Computed solution	1.022594527608326400	1.022594524466586058	1.022594524466586058
	Error	0.000000000	$3.14174019e^{-9}$	$3.14174019e^{-9}$
0.0250000	Exact solution	1.025940120524429000	1.025940120524428841	1.025940120524428841
	Computed solution	1.025940120524429200	1.025940115065457591	1.025940115065457591
	Error	$2.220446e^{-16}$	$5.45897125e^{-9}$	$5.45897125e^{-9}$
0.0281250	Exact solution	1.029315257541653800	1.029315257541653783	1.029315257541653783
	Computed Solution	1.029315257541654000	1.029315248639974225	1.029315248639974225
	Error	$2.220446e^{-16}$	$8.90167956e^{-9}$	$8.90167956e^{-9}$
0.0312500	Exact solution	1.032719969999102800	1.032719969999102671	1.032719969999102671
	Computed solution	1.032719969999102800	1.032719956193600508	1.032719956193600508
	Error	0.000000000	$1.38055022e^{-8}$	$1.38055022e^{-8}$

Table 5.3

Comparison of the New Method with Six Step Block Method (Kuboye and Omar, 2015) for Solving Problem 19 where $h = \frac{1}{100}$

x		$s_1 = \frac{1}{10}, s_2 = \frac{1}{4}, s_3 = \frac{1}{2}, P = 5$	Kuboye and Omar (2015), $P = 7$
0.01	Exact solution	0.000128995622844037	0.000128995622844037
	Computed solution	0.000128995622844037	0.000128995622844037
	Error	$5.421011e^{-20}$	$5.421011e^{-20}$
0.02	Exact solution	0.000257396543210136	0.000257396543210136
	Computed Solution	0.000257396543210136	0.000257396543210136
	Error	$5.421011e^{0}$	$5.421011e^{-20}$
0.03	Exact solution	0.000385195797911474	0.000385195797911474
	Computed solution	0.000385195797911474	0.000385195797911474
	Error	$2.710505e^{-19}$	$2.710505e^{-19}$
0.04	Exact solution	0.000512386483927295	0.000512386483927295
	Computed solution	0.000512386483927295	0.000512386483927295
	Error	$1.084202e^{-19}$	$1.084202e^{-19}$
0.05	Exact Solution	0.000638961759093202	0.000638961759093202
	Computed solution	0.000638961759093201	0.000638961759093201
	Error	$4.336809e^{-19}$	$3.252607e^{-19}$
0.06	Exact Solution	0.000764914842785370	0.000764914842785370
	Computed solution	0.000764914842785370	0.000764914842785370
	Error	$5.421011e^{-19}$	$3.252607e^{-19}$
0.07	Exact solution	0.000890239016598606	0.000890239016598605
	Computed solution	0.000890239016598605	0.000890239016598605
	Error	$6.505213e^{-19}$	$000000e^{00}$
0.08	Exact solution	0.001014927625018177	0.001014927625018177
	Computed solution	0.001014927625018176	0.001014927625018175
	Error	$8.673617e^{-19}$	$1.734723e^{-18}$
0.09	Exact solution	0.001138974076085363	0.001138974076085363
	Computed Solution	0.001138974076085362	0.001138974076085358
	Error	$4.336809e^{-19}$	$4.336809e^{-18}$
0.10	Exact solution	0.001262371842056641	0.001262371842056641
	Computed solution	0.001262371842056640	0.001262371842056632
	Error	$1.517883e^{-18}$	$8.456777e^{-18}$

Table 5.4

Comparison of the New Method with Five Step Predictor-Corrector Method (Kayode, 2008b) and Five Step Block Method (Kayode, 2008a) for Solving Problem 19 where $h = \frac{1}{320}$

x		$s_1 = \frac{1}{4}, s_2 = \frac{1}{3}, s_3 = \frac{2}{3}, P = 5$	Kayode(2008b), $P = 6$	Kayode(2008a), $P = 6$
0.1031250	Exact solution	0.001300799589367158	0.0013007993	0.00130079934027
	Computed solution	0.001300799589367158	0.0013007993	0.00130079934027
	Error	0.00000000	$0.48355417e^{-16}$	$0.498732999343e^{-15}$
0.2062500	Exact solution	0.002531773700195635	0.0025317732	0.00253177321538
	Computed Solution	0.002531773700195640	0.0025317732	0.00253177321538
	Error	$4.770490e^{-18}$	$0.13933299E^{-15}$	$0.676542155631e^{-15}$
0.3062500	Exact solution	0.003652478978884993	0.0036524788	0.00365247827946
	Computed solution	0.003652478978884998	0.0036524783	0.00365247827947
	Error	$4.770490e^{-18}$	$0.66893539e^{-15}$	$0.313507900196e^{-14}$
0.4062500	Exact solution	0.004695953231804849	0.0046959523	0.00469595233257
	Computed solution	0.004695953231804853	0.0046959523	0.00469595233258
	Error	$4.336809e^{-16}$	$0.20129384e^{-14}$	$0.943602834758e^{-14}$
0.5062500	Exact Solution	0.005657642360593461	0.0056576413	0.00565764127720
	Computed solution	0.005657642360593461	0.0056576413	0.00565764127722
	Error	0.00000000	$0.46736053e^{-14}$	$0.221168569570e^{-13}$
0.6031250	Exact Solution	0.006507754608034526	0.0065077534	0.00650775336185
	Computed solution	0.006507754608034500	0.0065077534	0.00650775336190
	Error	$2.602085e^{-17}$	$0.91874598e^{-14}$	$0.433793626020e^{-13}$
0.7031250	Exact solution	0.007298314767638524	0.0072983134	0.00729831337007
	Computed solution	0.007298314767638463	0.0072983134	0.00729831337015
	Error	$6.071532e^{-17}$	$0.16069038e^{-13}$	$0.778708694749e^{-13}$
0.8031250	Exact solution	0.007998520222728983	0.0079985187	0.00799851869108
	Computed solution	0.007998520222728879	0.0079985187	0.00799851869121
	Error	$1.040834e^{-16}$	$0.25407974e^{-13}$	$0.128634949914e^{-12}$
0.9031250	Exact solution	0.008607246703302495	0.0086072451	0.00860724505508
	Computed Solution	0.008607246703302337	0.0086072451	0.00860724505528
	Error	$1.578598e^{-16}$	$0.38108926e^{-13}$	$0.199271155132e^{-12}$
1.0031250	Exact solution	0.009124283967030095	0.0091242822	0.00912428221980
	Computed solution	0.009124283967029866	0.0091242822	0.00912428222010
	Error	$2.289835e^{-15}$	$0.54051538e^{-1}$	$0.293232452209e^{-12}$

Table 5.5

Comparison of the New Method with Six Step Multistep Method (Awoyemi et al., 2015) for Solving Problem 20 where $h = \frac{1}{320}$

x		$s_1=\frac{1}{4}, s_2=\frac{2}{4}, s_3=\frac{3}{4}, P=5$	$s_1=\frac{1}{4}, s_2=\frac{1}{3}, s_3=\frac{3}{4}, P=5$	Awoyemi et al.(2015), $p=6$
0.0031250	Exact solution	1.009375081380367200	1.009375081380367200	$0.100937508152e^{1}$
	Computed solution	1.009375081380367200	1.009375081380367000	$0.1009375082e^{1}$
	Error	0.00000000	$2.22044e^{-16}$	000000000
0.0062500	Exact solution	1.018750651046753200	1.018750651046753200	$0.101875065133e^{1}$
	Computed Solution	1.018750651046752600	1.018750651046753000	$0.1018750651e^{1}$
	Error	$6.661338e^{-16}$	$2.22044e^{16}$	$0000000e^{00}$
0.0093750	Exact solution	1.028127197304249400	1.028127197304249400	$0.102812719772e^{1}$
	Computed solution	1.028127197304248700	1.028127197304249000	$0.1028127198e^{1}$
	Error	$6.661338e^{-16}$	$4.440892e^{-16}$	$2.22045e^{-16}$
0.0125000	Exact solution	1.037505208496096300	1.037505208496096300	$0.103750520906e^{1}$
	Computed solution	1.037505208496095600	1.037505208496095800	$0.1037505209e^{1}$
	Error	$6.661338e^{-16}$	$4.440892e^{-16}$	$2.44249e^{-15}$
0.0156250	Exact Solution	1.046885173022758400	1.046885173022758400	$0.104688517372e^{1}$
	Computed solution	1.046885173022757500	1.046885173022757700	$0.1046885174e^{1}$
	Error	$8.881784e^{-16}$	$6.661338e^{-16}$	$1.15463e^{-14}$
0.0187500	Exact Solution	1.056267579361003000	1.056267579361003000	$0.105626758020e^{1}$
	Computed solution	1.056267579361001700	1.056267579361001900	$0.1056267580e^{1}$
	Error	$1.332268e^{-15}$	$1.110223e^{-15}$	$3.30846e^{-14}$
0.0218750	Exact solution	1.065652916082981100	1.065652916082981100	$0.106565291706e^{1}$
	Computed solution	1.065652916082978000	1.065652916082978400	$0.1065652917e^{1}$
	Error	$3.108624e^{-15}$	$2.664535e^{-15}$	$7.28306e^{-14}$
0.0250000	Exact solution	1.075041671875310000	-1.075041671875310000	$0.107504167299e^{1}$
	Computed solution	1.075041671875305300	1.075041671875305500	$0.1075041673e^{1}$
	Error	$4.662937e^{-15}$	$4.440892e^{-15}$	$1.37002e^{-13}$
0.0281250	Exact solution	1.084434335558167600	-1.084434335558167600	$0.108443433682e^{1}$
	Computed Solution	1.084434335558160500	1.084434335558160700	$0.1084434337e^{1}$
	Error	$7.105427e^{-15}$	$6.883383e^{-15}$	$2.30926e^{-13}$
0.0312500	Exact solution	1.093831396104383300	1.093831396104383300	$0.109383139751e^{1}$
	Computed solution	1.093831396104372600	1.093831396104372900	$0.1093831398e^{1}$
	Error	$1.065814e^{-14}$	$1.043610e^{-14}$	$3.60822e^{-13}$

5.3 Comments on the Results

The benefit of using the one-step hybrid block method for solving IVPs of fourth order ODEs is obvious. The new method displays its superiority by producing less error if compared to the present methods as shown in Tables 5.1-5.5. Based on the numerical results, it can be concluded that the one-step hybrid block method would be a better choice for solving IVPs of fourth order ODEs directly.

5.4 Conclusion

We have developed one-step hybrid block method with generalised three off-step points for the solution of fourth order initial value problems. The properties of the developed block method which include: zero stability, order, consistency and convergence are established. The existing one step hybrid block method only focus on the specific off step points. The new method, however, are capable of generalising the off step points and also able to solve fourth order IVPs directly. Thus, these method are more robust and flexible. Besides having good numerical property the method is also claim superior to the existing method in term of error.

CHAPTER SIX
CONCLUSION AND AREA OF FURTHER RESEARCH

6.1 Conclusion

This study has successfully developed a new class of one step hybrid block methods with generalized off-step points for solving higher order ordinary differential equations directly. There are several advantages of the new methods if compared to the existing ones. These methods are capable of solving initial value problems of higher order ODEs directly. Therefore, the new methods do not require converting initial value problems of higher order ODEs to their equivalent system of first order ODEs. As a result, the number of equations will not be increased and this can avoid evaluating more functions which may lead to computational burden, lots of human effort and complexity in writing the computer program.

The second advantage of the new hybrid block methods is that the zero-stability barrier occurs in multistep block methods can be overcome. It was discovered that the multi-step block methods are subjected to zero-stability barrier which states that the highest order linear multistep method is $k+2$ when the step length k is even and $k+1$ when k is odd. In order to overcome this barrier, hybrid methods which use information at off-step points were proposed. Hybrid block methods take the advantages of hybrid and block methods that is numerical approximation not only can be computed at more than one point at the same time, but zero-stability barrier can also be avoided.

The existing one step hybrid block methods only focus on the specific off-step points. The new developed methods, however, are capable of generalising the off-step points. Thus, these methods are more robust and flexible. The derivation of these methods includes one step block methods with q generalised off-step points where $q = 1, 2$ and 3 for solving second order ODEs, one step block methods with q generalised off-step points where $q = 2$ and 3 for third order ODEs and one step block methods with three generalised off-step points for the solution of fourth order ODEs. Interpolation and collocation technique is adopted in developing these methods where power series approximate solution is used as interpolation polynomials while its highest derivative is used as a collocation equation. Different strategy of interpolation was considered based on the order of differential equation while collocation points were made at all points in developing the methods. The properties of the new developed block methods which include: zero-stability, order, error constant, consistency, convergence and region of absolute stability are established.

It was observed that the order of one step block method with q hybrid points where $q = 1, 2$ and 3 for second order ODEs is between three and five. Similarly, the order of one step block method with q hybrid points where $q = 2$ and 3 for third order ODEs is between four and five while the order of one step block method with three hybrid points for fourth order ODEs is five. This implies that the block methods are consistent because the order is greater than one. The developed block methods are convergent since they are zero-stable and consistent. It is noted that the more off-step points considered on the interval, the higher the order of the method will be. Furthermore, it is also found that the more hybrid points, the larger the interval of absolute stability of the developed block method for second, third and fourth order ODEs. The application of the developed methods was then made to several linear and non-linear initial value problems of higher order ordinary differential equations. The generated numerical results claim superiority over the existing methods in terms of error.

6.2 Areas for Further Research

In this research work the derivation of one step block with q hybrid points where q is the number of generalised off-step points for second, third and fourth order ODEs.

Further researchers can extend the work by developing a generalised hybrid block method that will cater for any step-length k for solving n-th order initial value problems of ODEs directly using interpolation and collocation strategy. These methods can also be extended to solving boundary value problems of higher order ODEs directly.

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APPENDIX A
MATLAB CODE OF THE NEW METHOD WITH GENERALISED ONE OFF-STEP POINT FOR SOLVING SECOND ORDER ODE

Matlab code of new method with one off-step point for solving problem 4 of second order ODE clear

clc

% y' is represented by z

syms $x0\ y0\ z0\ f0\ x\ y\ z$

%$f(x,y,y') = y'$

disp ('x – value exact – solution computed – solution error')

tic

$x0 = 0;\ y0 = 0;\ z0 = -1;\ h = (0.1);$

$s = 2/5;$

$for\ j = 0 : h : 1;$

$f0 = z0;$

$x(1) = x0 + (s * h);$

$y(1) = (z0 * h^3 * s^3)/6 + (z0 * h^2 * s^2)/2 + z0 * h * s + y0;$

$z(1) = +(z0 * h^2 * s^2)/2 + z0 * h * s + z0; f(1) = z(1);$

$x(2) = x0 + (1 * h);$

$y(2) = +(z0 * h^3)/6 + (z0 * h^2)/2 + z0 * h + y0; z(2) = (z0 * h^4)/24 + (z0 * h^3)/6 + (z0 * h^2)/2 + z0 * h + z0;$

$f(2) = z(2);$

$yp1 = y0 + z0 * h * s - (f0 * h^2 * s^2 * (s-4))/12 + (f(2) * h^2 * s^4)/(12 * (s-1)) + (f(1) * h^2 * s^2 * (s-2))/(12 * (s-1))$

$yp2 = y0 + z0 * h + (f0 * h^2 * (4 * s - 1))/(12 * s) - (f(1) * h^2)/(12 * s * (s-1)) + (f(2) * h^2 * (2 * s - 1))/(12 * (s-1))$

$yp3 = z0 - (f0*h*s*(s-3))/6 + (f(1)*h*s*(2*s-3))/(6*(s-1)) + (f(2)*h*s^3)/(6*(s-1))$

$yp4 = z0 + (f0*h*(3*s-1))/(6*s) + (f(2)*h*(3*s-2))/(6*(s-1)) - (f(1)*h)/(6*s*(s-1))$

$fr1 = yp1;$

$fr2 = yp2;$

$yr1 = y0 + z0*h*s - (f0*h^2*s^2*(s-4))/12 + (fr2*h^2*s^4)/(12*(s-1)) + (fr1*h^2*s^2*(s-2))/(12*(s-1))$

$m1 = toc;$

$err1 = abs((1-exp(x(1))) - yr1); fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(1), (1-exp(x(1))), yr1, err4))$

$yr2 = y0 + z0*h + (f0*h^2*(4*s-1))/(12*s) - (fr1*h^2)/(12*s*(s-1)) + (fr2*h^2*(2*s-1))/(12*(s-1));$

$m2 = toc;$

$err2 = abs((1-exp(x(2))) - yr2);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(2), (1-exp(x(2))), yr2, err4))$

$x0 = x(2);\ y0 = yr2;\ z0 = yp2;$

end

APPENDIX B
MATLAB CODE OF THE NEW METHOD WITH GENERALISED TWO OFF-STEP POINT FOR SOLVING SECOND ORDER ODE

Matlab code of new method with two off-step point for solving problem 4 of second order ODE clear

clc

$\%y^{'}$ is represented by z

syms $x0$ $y0$ $z0$ $f0$ x y z

$\%f(x,y,y^{'}) = y^{'}$

disp ('x – value exact – solution computed – solution error')

tic

$x0 = 0;$ $y0 = 0$ $;z0 = -1;$ $h = (0.1);$

for $j = 0 : h : 1;$

$s = 1/5;$

$r = 3/5;$

$f0 = z0;$

$x(1) = x0 + (s*h);$

$y(1) = (z0*h^4*s^4)/24 + (z0*h^3*s^3)/6 + (z0*h^2*s^2)/2 + z0*h*s + y0;$

$z(1) = (z0*h^3*s^3)/6 + (z0*h^2*s^2)/2 + z0*h*s + z0;$

$f(1) = z(1);$

$x(2) = x0 + (r*h);$

$y(2) = (z0*h^4*r^4)/24 + (z0*h^3*r^3)/6 + (z0*h^2*r^2)/2 + z0*h*r + y0;$

$z(2) = (z0*h^3*r^3)/6 + (z0*h^2*r^2)/2 + z0*h*r + z0;$

$f(2) = z(2);$

$x(3) = x0 + (1*h);$

$y(3) = (z0*h^4)/24 + (z0*h^3)/6 + (z0*h^2)/2 + z0*h + y0;$

$z(3) = (z0*h^3)/6 + (z0*h^2)/2 + z0*h + z0;$

$f(3) = z(3);$

$yp1 = y0 + z0*h*s + (f(1)*h^2*s^2*(5*s - 10*r + 5*r*s - 3*s^2))/(60*(s-1)*(r-s)) + (f(3)*h^2*s^4*(5*r - 2*s))/(60*(r-1)*(s-1)) - (f(2)*h^2*s^4*(2*s-5))/(60*r*(r-1)*(r-s)) - (f0*h^2*s^2*(5*s-20*r+5*r*s-2*s^2))/(60*r)$

$yp2 = y0 + z0*h*r + (f(2)*h^2*r^2*(10*s - 5*r - 5*r*s + 3*r^2))/(60*(r-1)*(r-s)) - (f(3)*h^2*r^4*(2*r - 5*s))/(60*(s-1)*(r-1)) + (f(1)*h^2*r^4*(2*r-5))/(60*s*(s-1)*(r-s)) + (f0*h^2*r^2*(20*s-5*r-5*r*s+2*r^2))/(60*s)$

$yp3 = y0 + z0*h + (f(2)*h^2*(5*s-2))/(60*r*(r-1)*(r-s)) - (f(1)*h^2*(5*r-2))/(60*s*(s-1)*(r-s)) + (f(3)*h^2*(10*r*s - 5*s - 5*r + 3))/(60*(s-1)*(r-1)) + (f0*h^2*(20*r*s - 5*s - 5*r + 2))/(60*r*s)$

$yp4 = z0 + (f(3)*h*s^3*(2*r-s))/(12*(s-1)*(r-1)) - (f0*h*s*(2*s - 6*r + 2*r*s - s^2))/(12*r) - (f(2)*h*s^3*(s-2))/(12*r*(r-1)*(r-s)) + (f(1)*h*s*(4*s - 6*r + 4*r*s - 3*s^2))/(12*(s-1)*(r-s))$

$yp5 = z0 - (f(3)*h*r^3*(r-2*s))/(12*(s-1)*(r-1) + (f0*h*r*(6*s - 2*r - 2*r*s + r^2))/(12*s) + (f(2)*h*r*(6*s - 4*r - 4*r*s + 3*r^2))/(12*(r-1)*(r-s)) + (f(1)*h*r^3*(r-2))/(12*s*(s-1)*(r-s))$

$yp6 = z0 + (f(2)*h*(2*s-1))/(12*r*(r-1)*(r-s)) - (f(1)*h*(2*r-1))/(12*s*(s-1)*(r-s)) + (f(3)*h*(6*r*s - 4*s - 4*r + 3))/(12*(s-1)*(r-1)) + (f0*h*(6*r*s - 2*s - 2*r + 1))/(12*r*s)$

$fr1 = yp4;$

$fr2 = yp5;$

$fr3 = yp6;$

$yr1 = y0+z0*h*s-(f0*h^2*s^2*(5*s-20*r+5*r*s-2*s^2))/(60*r)+(fr1*h^2*s^2*(5*s-10*r+5*r*s-3*s^2))/(60*(r-s)*(s-1))-(fr2*h^2*s^4*(2*s-5))/(60*r*(r-s)*(r-1))+(fr3*h^2*s^4*(5*r-2*s))/(60*(r-1)*(s-1));$

$m1 = toc;$

$err1 = abs((1-exp(x(1)))-yr1);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(1), (1-exp(x(1))), yr1, err1))$

$yr2 = y0+z0*h*r+(f0*h^2*r^2*(20*s-5*r-5*r*s+2*r^2))/(60*s)+(fr1*h^2*r^4*(2*r-5))/(60*s*(r-s)*(s-1))+(fr2*h^2*r^2*(10*s-5*r-5*r*s+3*r^2))/(60*(r-s)*(r-1))-(fr3*h^2*r^4*(2*r-5*s))/(60*(r-1)*(s-1));$

$m2 = toc;$

$err2 = abs((1-exp(x(2)))-yr2);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(2), (1-exp(x(2))), yr2, err2))$

$yr3 = y0+z0*h+(f0*h^2*(20*r*s-5*s-5*r+2))/(60*r*s)-(fr1*h^2*(5*r-2))/(60*s*(s-1)*(r-s))+(fr2*h^2*(5*s-2))/(60*r*(r-1)*(r-s))+(fr3*h^2*(10*r*s-5*s-5*r+3))/(60*(s-1)*(r-1));$

$m3 = toc;$

$err3 = abs((1-exp(x(3)))-yr3);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(3), (1-exp(x(3))), yr3, err3))$

$x0 = x(3);\ y0 = yr3;\ z0 = yp6;$

end

APPENDIX C
MATLAB CODE OF THE NEW METHOD WITH GENERALISED THREE OFF-STEP POINT FOR SOLVING SECOND ORDER ODE

Matlab code of new method with three off-step point for solving problem 1 of second order ODE

clear

clc

$\%y'$ is represented by z

syms $x0\ y0\ z0\ f0\ x\ y\ z$

$\%f(x,y,y') = x(y')^2$

$x0 = 0;\ y0 = 1;\ z0 = 1/2;\ h = 1/320;$

disp ('x – value exact – solution computed – solution error')

tic

$for\ j = 0 : h : 1;$

$s1 = 1/5;$

$s2 = 2/5;$

$s3 = 3/5;$

$x(1) = x0 + (s1 * h);$

$y(1) = y0 + (s1 * h) * z0 + ((s1 * h)^2/2) * (x0 * z0^2) + ((s1 * h)^3/6) * (z0^2 + 2 * x0^2 * z0^3) + ((s1 * h)^4/24) * (6 * x0 * z0^3 + 6 * x0^3 * z0^4) + ((s1 * h)^5/120) * (36 * x0^2 * z0^4 + 24 * x0^4 * z0^5 + 6 * z0^3);$

$z(1) = z0 + (s1 * h) * (x0 * z0^2) + ((s1 * h)^2/2) * (z0^2 + 2 * x0^2 * z0^3) + ((s1 * h)^3/6) * (6 * x0 * z0^3 + 6 * x0^3 * z0^4) + ((s1 * h)^4/24) * (36 * x0^2 * z0^4 + 24 * x0^4 * z0^5 + 6 * z0^3);$

$f(1) = x(1) * z(1)^2;$

$x(2) = x0 + (s2 * h);$

$y(2) = y0 + (s2*h)*z0 + ((s2*h)^2/2)*(x0*z0^2) + ((s2*h)^3/6)*(z0^2 + 2*x0^2*z0^3) + ((s2*h)^4/24)*(6*x0*z0^3 + 6*x0^3*z0^4); + ((s2*h)^5/120)*(36*x0^2*z0^4 + 24*x0^4*z0^5 + 6*z0^3);$

$z(2) = z0 + (s2*h)*(x0*z0^2) + ((s2*h)^2/2)*(z0^2 + 2*x0^2*z0^3) + ((s2*h)^3/6)*(6*x0*z0^3 + 6*x0^3*z0^4) + ((s2*h)^4/24)*(36*x0^2*z0^4 + 24*x0^4*z0^5 + 6*z0^3);$

$f(2) = x(2)*z(2)^2;$

$x(3) = x0 + (s3*h);$

$y(3) = y0 + (s3*h)*z0 + ((s3*h)^2/2)*(x0*z0^2) + ((s3*h)^3/6)*(z0^2 + 2*x0^2*z0^3) + ((s3*h)^4/24)*(6*x0*z0^3 + 6*x0^3*z0^4) + ((s3*h)^5/120)*(36*x0^2*z0^4 + 24*x0^4*z0^5 + 6*z0^3);$

$z(3) = z0 + (s3*h)*(x0*z0^2) + ((s3*h)^2/2)*(z0^2 + 2*x0^2*z0^3) + ((s3*h)^3/6)*(6*x0*z0^3 + 6*x0^3*z0^4) + ((s3*h)^4/24)*(36*x0^2*z0^4 + 24*x0^4*z0^5 + 6*z0^3);$

$f(3) = x(3)*z(3)^2;$

$x(4) = x0 + (1*h);$

$y(4) = y0 + (1*h)*z0 + ((1*h)^2/2)*(x0*z0^2) + ((1*h)^3/6)*(z0^2 + 2*x0^2*z0^3) + ((1*h)^4/24)*(6*x0*z0^3 + 6*x0^3*z0^4) + ((1*h)^5/120)*(36*x0^2*z0^4 + 24*x0^4*z0^5 + 6*z0^3);$

$z(4) = z0 + (1*h)*(x0*z0^2) + ((1*h)^2/2)*(z0^2 + 2*x0^2*z0^3) + ((1*h)^3/6)*(6*x0*z0^3 + 6*x0^3*z0^4) + ((1*h)^4/24)*(36*x0^2*z0^4 + 24*x0^4*z0^5 + 6*z0^3);$

$f(4) = x(4)*z(4)^2;$

$yp1 = y0 + z0*h*s1 - (f(3)*h^2*s1^4*(5*s2 - 2*s1 - 2*s1*s2 + s1^2))/(60*s3*(s3-1)*(s2-s3)*(s1-s3)) + (f(2)*h^2*s1^4*(5*s3 - 2*s1 - 2*s1*s3 + s1^2))/(60*s2*(s2-1)*(s2-s3)*(s1-s2)) - (f(1)*h^2*s1^2*(5*s1*s2 + 5*s1*s3 - 10*s2*s3 - 3*s1^2*s2 - 3*s1^2*s3 - 3*s1^2 + 2*s1^3 + 5*s1*s2*s3))/(60*(s1-1)*(s1-s3)*(s1-s2) + (f(4)*h^2*s1^4*(5*s2*s3 - 2*s1*s3 - 2*s1*s2 + s1^2))/(60*(s3-1)*(s2-1)*(s1-1)) - (f0*h^2*s1^2*(5*s1*s2 + 5*s1*s3 - 20*s2*s3 - 2*s1^2*s2 - 2*s1^2*s3 - 2*s1^2 + s1^3 + 5*s1*s2*s3))/(60*s2*s3)$

$yp2 = y0 + z0*h*s2 + (f(3)*h^2*s2^4*(2*s2-5*s1+2*s1*s2-s2^2))/(60*s3*(s3-1)*(s2-s3)*(s1-s3)) - (f(4)*h^2*s2^4*(2*s1*s2-5*s1*s3+2*s2*s3-s2^2))/(60*(s3-1)*(s1-1)*(s2-1)) - (f(1)*h^2*s2^4*(5*s3-2*s2-2*s2*s3+s2^2))/(60*s1*(s1-1)*(s1-s3)*(s1-s2)) + (f0*h^2*s2^2*(20*s1*s3-5*s1*s2-5*s2*s3+2*s1*s2^2+2*s2^2*s3+2*s2^2-s2^3-5*s1*s2*s3))/(60*s1*s3) + (f(2)*h^2*s2^2*(10*s1*s3-5*s1*s2-5*s2*s3+3*s1*s2^2+3*s2^2*s3+3*s2^2-2*s2^3-5*s1*s2*s3))/(60*(s2-1)*(s2-s3)*(s1-s2))$

$yp3 = y0 + z0*h*s3 - (f(2)*h^2*s3^4*(2*s3-5*s1+2*s1*s3-s3^2))/(60*s2*(s2-1)*(s2-s3)*(s1-s2)) + (f(4)*h^2*s3^4*(5*s1*s2-2*s1*s3-2*s2*s3+s3^2))/(60*(s2-1)*(s1-1)*(s3-1)) + (f(1)*h^2*s3^4*(2*s3-5*s2+2*s2*s3-s3^2))/(60*s1*(s1-1)*(s1-s3)*(s1-s2)) - (f0*h^2*s3^2*(5*s1*s3-20*s1*s2+5*s2*s3-2*s1*s3^2-2*s2*s3^2-2*s3^2+s3^3+5*s1*s2*s3))/(60*s1*s2) + (f(3)*h^2*s3^2*(5*s1*s3-10*s1*s2+5*s2*s3-3*s1*s3^2-3*s2*s3^2-3*s3^2+2*s3^3+5*s1*s2*s3))/(60*(s3-1)*(s2-s3)*(s1-s3))$

$yp4 = y0 + z0*h - (f(2)*h^2*(5*s1*s3-2*s3-2*s1+1))/(60*s2*(s2-1)*(s2-s3)*(s1-s2)) - (f(3)*h^2*(5*s1*s2-2*s2-2*s1+1))/(60*s3*(s3-1)*(s2-s3)*(s1-s3)) + (f0*h^2*(2*s1+2*s2+2*s3-5*s1*s2-5*s1*s3-5*s2*s3+20*s1*s2*s3-1))/(60*s1*s2*s3) + (f(4)*h^2*(3*s1+3*s2+3*s3-5*s1*s2-5*s1*s3-5*s2*s3+10*s1*s2*s3-2))/(60*(s3-1)*(s2-1)*(s1-1)) - (f(1)*h^2*(5*s2*s3-2*s3-2*s2+1))/(60*s1*(s1-1)*(s1-s3)*(s1-s2))$

$yp5 = z0 + (f(4)*h*s1^3*(10*s2*s3-5*s1*s3-5*s1*s2+3*s1^2))/(60*(s3-1)*(s2-1)*(s1-1)) + (f(1)*h*s1*(20*s1*s2+20*s1*s3-30*s2*s3-15*s1^2*s2-15*s1^2*s3-15*s1^2+12*s1^3+20*s1*s2*s3))/(60*(s1-1)*(s1-s3)*(s1-s2)) - (f0*h*s1*(10*s1*s2+10*s1*s3-30*s2*s3-5*s1^2*s2-$

$$
\begin{aligned}
&5\ast s1^2\ast s3-5\ast s1^2+3\ast s1^3+10\ast s1\ast s2\ast s3))/(60\ast s2\ast s3)+(f(2)\ast h\ast s1^3\ast(10\ast\\
&s3-5\ast s1-5\ast s1\ast s3+3\ast s1^2))/(60\ast s2\ast(s2-1)\ast(s2-s3)\ast(s1-s2))-(f(3)\ast\\
&h\ast s1^3\ast(10\ast s2-5\ast s1-5\ast s1\ast s2+3\ast s1^2))/(60\ast s3\ast(s3-1)\ast(s2-s3)\ast(s1-s3))
\end{aligned}
$$

$$
\begin{aligned}
&yp6=z0+(f(2)\ast h\ast s2\ast(30\ast s1\ast s3-20\ast s1\ast s2-20\ast s2\ast s3+15\ast s1\ast s2^2+\\
&15\ast s2^2\ast s3+15\ast s2^2-12\ast s2^3-20\ast s1\ast s2\ast s3))/(60\ast(s2-1)\ast(s2-s3)\ast(s1-\\
&s2))+(f0\ast h\ast s2\ast(30\ast s1\ast s3-10\ast s1\ast s2-10\ast s2\ast s3+5\ast s1\ast s2^2+5\ast s2^2\ast s3+\\
&5\ast s2^2-3\ast s2^3-10\ast s1\ast s2\ast s3))/(60\ast s1\ast s3)-(f(4)\ast h\ast s2^3\ast(5\ast s1\ast s2-10\ast\\
&s1\ast s3+5\ast s2\ast s3-3\ast s2^2))/(60\ast(s3-1)\ast(s2-1)\ast(s1-1))-(f(1)\ast h\ast s2^3\ast(10\ast\\
&s3-5\ast s2-5\ast s2\ast s3+3\ast s2^2))/(60\ast s1\ast(s1-1)\ast(s1-s3)\ast(s1-s2))+(f(3)\ast\\
&h\ast s2^3\ast(5\ast s2-10\ast s1+5\ast s1\ast s2-3\ast s2^2))/(60\ast s3\ast(s3-1)\ast(s2-s3)\ast(s1-s3))
\end{aligned}
$$

$$
\begin{aligned}
&yp7=z0+(f(3)\ast h\ast s3\ast(20\ast s1\ast s3-30\ast s1\ast s2+20\ast s2\ast s3-15\ast s1\ast s3^2-\\
&15\ast s2\ast s3^2-15\ast s3^2+12\ast s3^3+20\ast s1\ast s2\ast s3))/(60\ast(s3-1)\ast(s2-s3)\ast(s1-\\
&s3))-(f0\ast h\ast s3\ast(10\ast s1\ast s3-30\ast s1\ast s2+10\ast s2\ast s3-5\ast s1\ast s3^2-5\ast s2\ast s3^2-\\
&5\ast s3^2+3\ast s3^3+10\ast s1\ast s2\ast s3))/(60\ast s1\ast s2)+(f(4)\ast h\ast s3^3\ast(10\ast s1\ast s2-5\ast\\
&s1\ast s3-5\ast s2\ast s3+3\ast s3^2))/(60\ast(s3-1)\ast(s2-1)\ast(s1-1))+(f(1)\ast h\ast s3^3\ast(5\ast\\
&s3-10\ast s2+5\ast s2\ast s3-3\ast s3^2))/(60\ast s1\ast(s1-1)\ast(s1-s3)\ast(s1-s2))-(f(2)\ast\\
&h\ast s3^3\ast(5\ast s3-10\ast s1+5\ast s1\ast s3-3\ast s3^2))/(60\ast s2\ast(s2-1)\ast(s2-s3)\ast(s1-s2))
\end{aligned}
$$

$$
\begin{aligned}
&yp8=z0-(f(1)\ast h\ast(10\ast s2\ast s3-5\ast s3-5\ast s2+3))/(60\ast s1\ast(s1-1)\ast(s1-s3)\ast\\
&(s1-s2))+(f0\ast h\ast(5\ast s1+5\ast s2+5\ast s3-10\ast s1\ast s2-10\ast s1\ast s3-10\ast s2\ast s3+30\ast\\
&s1\ast s2\ast s3-3))/(60\ast s1\ast s2\ast s3)+(f(4)\ast h\ast(15\ast s1+15\ast s2+15\ast s3-20\ast s1\ast s2-\\
&20\ast s1\ast s3-20\ast s2\ast s3+30\ast s1\ast s2\ast s3-12))/(60\ast(s3-1)\ast(s2-1)\ast(s1-1))+\\
&(f(2)\ast h\ast(10\ast s1\ast s3-5\ast s3-5\ast s1+3))/(60\ast s2\ast(s2-1)\ast(s2-s3)\ast(s1-s2))-\\
&(f(3)\ast h\ast(10\ast s1\ast s2-5\ast s2-5\ast s1+3))/(60\ast s3\ast(s3-1)\ast(s2-s3)\ast(s1-s3))
\end{aligned}
$$

$fr1=x(1)\ast yp5^2;$

$fr2=x(2)\ast yp6^2;$

$fr3 = x(3)*yp7^2;$

$fr4 = x(4)*yp8^2;$

$yr1 = (y0 + z0*h*s1 - (fr3*h^2*s1^4*(5*s2 - 2*s1 - 2*s1*s2 + s1^2))/(60*s3*(s3-1)*(s2-s3)*(s1-s3)) + (fr2*h^2*s1^4*(5*s3 - 2*s1 - 2*s1*s3 + s1^2))/(60*s2*(s2-1)*(s2-s3)*(s1-s2)) - (fr1*h^2*s1^2*(5*s1*s2 + 5*s1*s3 - 10*s2*s3 - 3*s1^2*s2 - 3*s1^2*s3 - 3*s1^2 + 2*s1^3 + 5*s1*s2*s3))/(60*(s1-1)*(s1-s3)*(s1-s2) + (fr4*h^2*s1^4*(5*s2*s3 - 2*s1*s3 - 2*s1*s2 + s1^2))/(60*(s3-1)*(s2-1)*(s1-1)) - (f0*h^2*s1^2*(5*s1*s2 + 5*s1*s3 - 20*s2*s3 - 2*s1^2*s2 - 2*s1^2*s3 - 2*s1^2 + s1^3 + 5*s1*s2*s3))/(60*s2*s3)$

$err1 = abs(1 + 1/2*log((2 + x(1))/(2 - x(1))) - yr1);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(1), 1+1/2*log((2+x(1))/(2-x(1))), yr1, err1))$

$yr2 = y0 + z0*h*s2 + (fr3*h^2*s2^4*(2*s2 - 5*s1 + 2*s1*s2 - s2^2))/(60*s3*(s3-1)*(s2-s3)*(s1-s3)) - (fr4*h^2*s2^4*(2*s1*s2 - 5*s1*s3 + 2*s2*s3 - s2^2))/(60*(s3-1)*(s1-1)*(s2-1)) - (fr1*h^2*s2^4*(5*s3 - 2*s2 - 2*s2*s3 + s2^2))/(60*s1*(s1-1)*(s1-s3)*(s1-s2)) + (f0*h^2*s2^2*(20*s1*s3 - 5*s1*s2 - 5*s2*s3 + 2*s1*s2^2 + 2*s2^2*s3 + 2*s2^2 - s2^3 - 5*s1*s2*s3))/(60*s1*s3) + (fr2*h^2*s2^2*(10*s1*s3 - 5*s1*s2 - 5*s2*s3 + 3*s1*s2^2 + 3*s2^2*s3 + 3*s2^2 - 2*s2^3 - 5*s1*s2*s3))/(60*(s2-1)*(s2-s3)*(s1-s2))$

$err2 = abs(1 + 1/2*log((2 + x(2))/(2 - x(2))) - yr2);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(2), 1+1/2*log((2+x(2))/(2-x(2))), yr2, err2))$

$yr3 = y0 + z0*h*s3 - (fr2*h^2*s3^4*(2*s3 - 5*s1 + 2*s1*s3 - s3^2))/(60*s2*(s2-1)*(s2-s3)*(s1-s2)) + (fr4*h^2*s3^4*(5*s1*s2 - 2*s1*s3 - 2*s2*s3 + s3^2))/(60*(s2-1)*(s1-1)*(s3-1)) + (fr1*h^2*s3^4*(2*s3 - 5*s2 + 2*s2*s3 - s3^2))/(60*s1*(s1-1)*(s1-s3)*(s1-s2)) - (f0*h^2*s3^2*(5*s1*s3 - 20*s1*s2 + 5*s2*s3 - 2*s1*s3^2 - 2*s2*s3^2 - 2*s3^2 + s3^3 + 5*s1*s2*s3))/(60*$

$$s1*s2)+(fr3*h^2*s3^2*(5*s1*s3-10*s1*s2+5*s2*s3-3*s1*s3^2-3*s2*$$

$$s3^2-3*s3^2+2*s3^3+5*s1*s2*s3))/(60*(s3-1)*(s2-s3)*(s1-s3))$$

$$err3=abs(1+1/2*log((2+x(3))/(2-x(3)))-yr3);$$

$$fprintf('\%2.7f\ \ \%3.18f\ \ \%3.18f\ \ \%1.6\ \backslash n',x(3),1+1/2*log((2+x(3))/(2-x(3))),yr3,err3))$$

$$yr4=y0+z0*h-(fr2*h^2*(5*s1*s3-2*s3-2*s1+1))/(60*s2*(s2-1)*(s2-$$

$$s3)*(s1-s2))-(fr3*h^2*(5*s1*s2-2*s2-2*s1+1))/(60*s3*(s3-1)*(s2-$$

$$s3)*(s1-s3))+(f0*h^2*(2*s1+2*s2+2*s3-5*s1*s2-5*s1*s3-5*s2*s3+$$

$$20*s1*s2*s3-1))/(60*s1*s2*s3)+(fr4*h^2*(3*s1+3*s2+3*s3-5*s1*$$

$$s2-5*s1*s3-5*s2*s3+10*s1*s2*s3-2))/(60*(s3-1)*(s2-1)*(s1-1))-$$

$$(fr1*h^2*(5*s2*s3-2*s3-2*s2+1))/(60*s1*(s1-1)*(s1-s3)*(s1-s2))$$

$$err4=abs(1+1/2*log((2+x(4))/(2-x(4)))-yr4);$$

$$fprintf('\%2.7f\ \ \%3.18f\ \ \%3.18f\ \ \%1.6\ \backslash n',x(4),1+1/2*log((2+x(4))/(2-x(4))),yr4,err4))$$

$$x0=x(4);\ y0=yr4;\ z0=yp8;$$

end

APPENDIX D

MATLAB CODE OF THE NEW METHOD WITH GENERALISED TWO OFF-STEP POINT FOR SOLVING THIRD ORDER ODE

Matlab code of new method with two off-step point for solving problem 8 of third order ODE clear

clc

$\%y^{'}$ is represented by z

$\%y^{''}$is represented by v

syms $x0\ y0\ z0\ f0\ x\ y\ z\ v\ v0$

$\%f(x,y,y^{'},y^{''}) = -y$

$x0 = 0;\ y0 = 1\ ; z0 = -1;\ v0 = 1;\ h = (0.1);$

disp ('x – value exact – solution computed – solution error')

tic

$s = 3/10;$

$r = 17/20;$

$for j = 0 : h : 5;$

$f0 = -y0;$

$x(1) = x0 + (s*h);$

$y(1) = -(z0*h^4*s^4)/24 + (z0*h^3*s^3)/6 - (z0*h^2*s^2)/2 + z0*h*s + y0;$

$z(1) = -(z0*h^3*s^3)/6 + (z0*h^2*s^2)/2 - z0*h*s + z0;$

$v(1) = -(z0*h^2*s^2)/2 + z0*h*s - z0;$

$f(1) = -y(1);$

$x(2) = x0 + (r*h);$

$y(2) = -(z0*h^4*r^4)/24 + (z0*h^3*r^3)/6 - (z0*h^2*r^2)/2 + z0*h*r + y0;$

$z(2) = -(z0*h^3*r^3)/6 + (z0*h^2*r^2)/2 - z0*h*r + z0;$

$v(2) = -(z0*h^2*r^2)/2 + z0*h*r - z0;$

$f(2) = -y(2);$

$x(3) = x0 + (1*h);$

$y(3) = -(z0*h^4*1^4)/24 + (z0*h^3*1^3)/6 - (z0*h^2*1^2)/2 + z0*h*1 + y0;$

$z(3) = -(z0*h^3*1^3)/6 + (z0*h^2*1^2)/2 - z0*h*1 + z0;$

$v(3) = -(z0*h^2*1^2)/2 + z0*h*1 - z0;$

$f(3) = -y(3);$

$yp1 = (v0*h^2*s^2 + 2*z0*h*s + 2*y0)/2 - (f0*h^3*s^3*(3*s - 15*r + 3*r*s - s^2))/(120*r) + (h^3*s^3*(2*s - 5*r + 2*r*s - s^2))/(120*(s-1)*(r-s))*f(1) - (h^3*s^5*(s-3))/(120*r*(r-1)*(r-s))*f(2) + (h^3*s^5*(3*r-s))/(120*(r-1)*(s-1))*f(3)$

$yp2 = (v0*h^2*r^2 + 2*z0*h*r + 2*y0)/2 + (f0*h^3*r^3*(15*s - 3*r - 3*r*s + r^2))/(120*s) + (h^3*r^5*(r-3))/(120*s*(s-1)*(r-s))*f(1), +(h^3*r^3*(5*s - 2*r - 2*r*s + r^2))/(120*(r-1)*(r-s))*f(2) - (h^3*r^5*(r-3*s))/(120*(s-1)*(r-1))*f(3)$

$yp3 = (v0*h^2 + 2*z0*h + 2*y0)/2 + (f0*h^3*(15*r*s - 3*s - 3*r + 1))/(120*r*s) - (h^3*(3*r-1))/(120*s*(s-1)*(r-s))*f(1) + (h^3*(3*s-1))/(120*r*(r-1)*(r-s))*f(2) + (h^3*(5*r*s - 2*s - 2*r + 1))/(120*(s-1)*(r-1))*f(3)$

$yp4 = z0 + v0*h*s - (f0*h^2*s^2*(5*s - 20*r + 5*r*s - 2*s^2))/(60*r) + (h^2*s^2*(5*s - 10*r + 5*r*s - 3*s^2))/(60*(s-1)*(r-s))*f(1) - (h^2*s^4*(2*s-5))/(60*r*(r-1)*(r-s))*f(2) + (h^2*s^4*(5*r - 2*s))/(60*(r-1)*(s-1))*f(3)$

$yp5 = z0 + v0*h*r + (f0*h^2*r^2*(20*s - 5*r - 5*r*s + 2*r^2))/(60*s) + (h^2*r^4*(2*r-5))/(60*s*(s-1)*(r-s))*f(1) + (h^2*r^2*(10*s - 5*r - 5*r*s + 3*r^2))/(60*(r-1)*(r-s))*f(2) - (h^2*r^4*(2*r - 5*s))/(60*(s-1)*(r-1))*f(3)$

$yp6 = z0 + v0*h + (f0*h^2*(20*r*s - 5*s - 5*r + 2))/(60*r*s) - (h^2*(5*r-2))/(60*s*(s-1)*(r-s))*f(1) + (h^2*(5*s-2))/(60*r*(r-1)*(r-s))*f(2) + (h^2*(10*r*s - 5*s - 5*r + 3))/(60*(s-1)*(r-1))*f(3)$

$yp7 = v0 - (f0*h*s*(2*s - 6*r + 2*r*s - s^2))/(12*r) + (h*s*(4*s - 6*r + 4*$

$r*s-3*s^2))/(12*(s-1)*(r-s))*f(1)-(h*s^3*(s-2))/(12*r*(r-1)*(r-s))*f(2)(h*s^3*(2*r-s))/(12*(r-1)*(s-1))*f(3)$

$yp8 = v0+(f0*h*r*(6*s-2*r-2*r*s+r^2))/(12*s)+(h*r^3*(r-2))/(12*s*(s-1)*(r-s))*f(1)+(h*r*(6*s-4*r-4*r*s+3*r^2))/(12*(r-1)*(r-s))*f(2)-(h*r^3*(r-2*s))/(12*(s-1)*(r-1))*f(3)$

$yp9 = v0+(f0*h*(6*r*s-2*s-2*r+1))/(12*r*s)-(h*(2*r-1))/(12*s*(s-1)*(r-s))*f(1)+(h*(2*s-1))/(12*r*(r-1)*(r-s))*f(2)+(h*(6*r*s-4*s-4*r+3))/(12*(s-1)*(r-1))*f(3)$

$yr1 = (v0*h^2*s^2+2*z0*h*s+2*y0)/2-(f0*h^3*s^3*(3*s-15*r+3*r*s-s^2))/(120*r)+(h^3*s^3*(2*s-5*r+2*r*s-s^2))/(120*(s-1)*(r-s))*fr1-(h^3*s^5*(s-3))/(120*r*(r-1)*(r-s))*fr2+(h^3*s^5*(3*r-s))/(120*(r-1)*(s-1))*fr3$

$m1 = toc;$

$err1 = abs(exp(-x(1))-yr1);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(1), exp(-x(1)), yr1, err1))$

$yr2 = (v0*h^2*r^2+2*z0*h*r+2*y0)/2+(f0*h^3*r^3*(15*s-3*r-3*r*s+r^2))/(120*s)+(h^3*r^5*(r-3))/(120*s*(s-1)*(r-s))*fr1,+(h^3*r^3*(5*s-2*r-2*r*s+r^2))/(120*(r-1)*(r-s))*fr2-(h^3*r^5*(r-3*s))/(120*(s-1)*(r-1))*fr3$

$err2 = abs(exp(-x(2))-yr2);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(2), exp(-x(2)), yr2, err2))$

$yr3 = (v0*h^2+2*z0*h+2*y0)/2+(f0*h^3*(15*r*s-3*s-3*r+1))/(120*r*s)-(h^3*(3*r-1))/(120*s*(s-1)*(r-s))*fr1+(h^3*(3*s-1))/(120*r*(r-1)*(r-s))fr2+(h^3*(5*r*s-2*s-2*r+1))/(120*(s-1)*(r-1))*fr3$

$err3 = abs(exp(-x(3))-yr3);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(3), exp(-x(3)), yr3, err3))$

$x0 = x(3);\ z0 = yp6;\ y0 = yr3;\ v0 = yp9;$

end

APPENDIX E

MATLAB CODE OF THE NEW METHOD WITH GENERALISED THREE OFF-STEP POINT FOR SOLVING THIRD ORDER ODE

Matlab code of new method with two off-step point for solving problem 10 of third order ODE clear

clc

$\%y^{'}$ is represented by z

$\%y^{''}$is represented by v

syms $x0\ y0\ z0\ f0\ x\ y\ z\ v\ v0$

$\%f(x,y,y^{'},y^{''}) = -y^{'}$

$x0 = 0;\ y0 = 0\ ; z0 = 1;\ v0 = 2;\ h = (0.1);$

disp ('x – value exact – solution computed – solution error') tic

$for j = 0 : h : 2;$

$s1 = 1/5;$

$s2 = 3/5;$

$s3 = 4/5;$

$f0 = -z0;$

$x(1) = x0 + (s1*h); y(1) = (z0*h^5*s1^5)/120 - (v0*h^4*s1^4)/24 - (z0*h^3*s1^3)/6 + (v0*h^2*s1^2)/2 + z0*h*s1 + y0;$

$z(1) = (z0*h^4*s1^4)/24 - (v0*h^3*s1^3)/6 - (z0*h^2*s1^2)/2 + v0*h*s1 + z0;$

$v(1) = (z0*h^3*s1^3)/6 - (v0*h^2*s1^2)/2 - z0*h*s1 + v0;$

$f(1) = -z(1);$

$x(2) = x0 + (s2*h);$

$y(2) = (z0*h^5*s2^5)/120 - (v0*h^4*s2^4)/24 - (z0*h^3*s2^3)/6 + (v0*h^2*s2^2)/2 + z0*h*s2 + y0;$

$z(2) = (z0*h^4*s2^4)/24 - (v0*h^3*s2^3)/6 - (z0*h^2*s2^2)/2 + v0*h*s2 + z0;$

$v(2) = (z0*h^3*s2^3)/6 - (v0*h^2*s2^2)/2 - z0*h*s2 + v0;$

$f(2) = -z(2);$

$x(3) = x0 + (s3*h);$

$y(3) = (z0*h^5*s3^5)/120 - (v0*h^4*s3^4)/24 - (z0*h^3*s3^3)/6 + (v0*h^2*s3^2)/2 + z0*h*s3 + y0;$

$z(3) = (z0*h^4*s2^4)/24 - (v0*h^3*s3^3)/6 - (z0*h^2*s3^2)/2 + v0*h*s3 + z0;$

$v(3) = (z0*h^3*s3^3)/6 - (v0*h^2*s3^2)/2 - z0*h*s3 + v0;$

$f(3) = -z(3);$

$x(4) = x0 + (1*h);$

$y(4) = (z0*h^5*1^5)/120 - (v0*h^4*1^4)/24 - (z0*h^3*1^3)/6 + (v0*h^2*1^2)/2 + z0*h*1 + y0;$

$z(4) = (z0*h^4*1^4)/24 - (v0*h^3*1^3)/6 - (z0*h^2*1^2)/2 + v0*h*1 + z0;$

$v(4) = (z0*h^3*1^3)/6 - (v0*h^2*1^2)/2 - z0*h*1 + v0;$

$f(4) = -z(4);$

$yp1 = -(f(3)*h^3*s1^5*(21*s2 - 7*s1 - 7*s1*s2 + 3*s1^2))/(840*s3*(s3-1)*(s2-s3)*(s1-s3)) + (f(4)*h^3*s1^5*(21*s2*s3 - 7*s1*s3 - 7*s1*s2 + 3*s1^2))/(840*(s3-1)*(s2-1)*(s1-1)) + (f(2)*h^3*s1^5*(21*s3 - 7*s1 - 7*s1*s3 + 3*s1^2))/(840*s2*(s2-1)*(s2-s3)*(s1-s2)) + y0 + (f(1)*h^3*s1^3*(14*s1*s2 + 14*s1*s3 - 35*s2*s3 - 7*s1^2*s2 - 7*s1^2*s3 - 7*s1^2 + 4*s1^3 + 14*s1*$

$s2*s3))/(840*(s1-1)*(s1-s3)*(s1-s2))-(f0*h^3*s1^3*(21*s1*s2+21*s1*$
$s3-105*s2*s3-7*s1^2*s2-7*s1^2*s3-7*s1^2+3*s1^3+21*s1*s2*s3))/(840*$
$s2*s3)+(v0*h^2*s1^2)/2+z0*h*s1;$

$yp2=(f(3)*h^3*s2^5*(7*s2-21*s1+7*s1*s2-3*s2^2))/(840*s3*(s3-1)*$
$(s2-s3)*(s1-s3))+(f(2)*h^3*s2^3*(35*s1*s3-14*s1*s2-14*s2*s3+7*s1*$
$s2^2+7*s2^2*s3+7*s2^2-4*s2^3-14*s1*s2*s3))/(840*(s2-1)*(s2-s3)*(s1-$
$s2))-(f(4)*h^3*s2^5*(7*s1*s2-21*s1*s3+7*s2*s3-3*s2^2))/(840*(s3-1)*$
$(s1-1)*(s2-1))-(f(1)*h^3*s2^5*(21*s3-7*s2-7*s2*s3+3*s2^2))/(840*$
$s1*(s1-1)*(s1-s3)*(s1-s2))+y0+(f0*h^3*s2^3*(105*s1*s3-21*s1*s2-$
$21*s2*s3+7*s1*s2^2+7*s2^2*s3+7*s2^2-3*s2^3-21*s1*s2*s3))/(840*s1*$
$s3)+(v0*h^2*s2^2)/2+z0*h*s2;$

$yp3=y0-(f0*h^3*s3^3*(21*s1*s3-105*s1*s2+21*s2*s3-7*s1*s3^2-7*$
$s2*s3^2-7*s3^2+3*s3^3+21*s1*s2*s3))/(840*s1*s2)+(f(4)*h^3*s3^5*(21*$
$s1*s2-7*s1*s3-7*s2*s3+3*s3^2))/(840*(s2-1)*(s1-1)*(s3-1))+(f(1)*$
$h^3*s3^5*(7*s3-21*s2+7*s2*s3-3*s3^2))/(840*s1*(s1-1)*(s1-s3)*(s1-$
$s2))-(f(2)*h^3*s3^5*(7*s3-21*s1+7*s1*s3-3*s3^2))/(840*s2*(s2-1)*$
$(s2-s3)*(s1-s2))+(f(3)*h^3*s3^3*(14*s1*s3-35*s1*s2+14*s2*s3-7*$
$s1*s3^2-7*s2*s3^2-7*s3^2+4*s3^3+14*s1*s2*s3))/(840*(s3-1)*(s2-s3)*$
$(s1-s3))+(v0*h^2*s3^2)/2+z0*h*s3;$

$yp4=z0*h+y0+(f(4)*h^3*(7*s1+7*s2+7*s3-14*s1*s2-14*s1*s3-$
$14*s2*s3+35*s1*s2*s3-4))/(840*(s2-1)*(s1-1)*(s3-1))+(v0*h^2)/2-$
$(f(1)*h^3*(21*s2*s3-7*s3-7*s2+3))/(840*s1*(s1-1)*(s1-s3)*(s1-$
$s2))+(f(2)*h^3*(21*s1*s3-7*s3-7*s1+3))/(840*s2*(s2-1)*(s2-s3)*$
$(s1-s2))+(f0*h^3*(7*s1+7*s2+7*s3-21*s1*s2-21*s1*s3-21*s2*s3+$
$105*s1*s2*s3-3))/(840*s1*s2*s3)-(f(3)*h^3*(21*s1*s2-7*s2-7*s1+$
$3))/(840*s3*(s3-1)*(s2-s3)*(s1-s3));$

$yp5=z0+(f(4)*h^2*s1^4*(5*s2*s3-2*s1*s3-2*s1*s2+s1^2))/(60*(s3-1)*$
$(s2-1)*(s1-1))-(f0*h^2*s1^2*(5*s1*s2+5*s1*s3-20*s2*s3-2*s1^2*s2-$

$2*s1^2*s3-2*s1^2+s1^3+5*s1*s2*s3))/(60*s2*s3)-(f(3)*h^2*s1^4*(5*s2-2*s1-2*s1*s2+s1^2))/(60*s3*(s3-1)*(s2-s3)*(s1-s3))+(f(2)*h^2*s1^4*(5*s3-2*s1-2*s1*s3+s1^2))/(60*s2*(s2-1)*(s2-s3)*(s1-s2))+(f(1)*h^2*s1^2*(5*s1*s2+5*s1*s3-10*s2*s3-3*s1^2*s2-3*s1^2*s3-3*s1^2+2*s1^3+5*s1*s2*s3))/(60*(s1-1)*(s1-s3)*(s1-s2))+v0*h*s1;$

$yp6=z0-(f(4)*h^2*s2^4*(2*s1*s2-5*s1*s3+2*s2*s3-s2^2))/(60*(s3-1)*(s1-1)*(s2-1))+(f0*h^2*s2^2*(20*s1*s3-5*s1*s2-5*s2*s3+2*s1*s2^2+2*s2^2*s3+2*s2^2-s2^3-5*s1*s2*s3))/(60*s1*s3)+(f(3)*h^2*s2^4*(2*s2-5*s1+2*s1*s2-s2^2))/(60*s3*(s3-1)*(s2-s3)*(s1-s3))-(f(1)*h^2*s2^4*(5*s3-2*s2-2*s2*s3+s2^2))/(60*s1*(s1-1)*(s1-s3)*(s1-s2))+(f(2)*h^2*s2^2*(10*s1*s3-5*s1*s2-5*s2*s3+3*s1*s2^2+3*s2^2*s3+3*s2^2-2*s2^3-5*s1*s2*s3))/(60*(s2-1)*(s2-s3)*(s1-s2))+v0*h*s2;$

$yp7=z0-(f0*h^2*s3^2*(5*s1*s3-20*s1*s2+5*s2*s3-2*s1*s3^2-2*s2*s3^2-2*s3^2+s3^3+5*s1*s2*s3))/(60*s1*s2)+(f(1)*h^2*s3^4*(2*s3-5*s2+2*s2*s3-s3^2))/(60*s1*(s1-1)*(s1-s3)*(s1-s2))-(f(2)*h^2*s3^4*(2*s3-5*s1+2*s1*s3-s3^2))/(60*s2*(s2-1)*(s2-s3)*(s1-s2))+(f(4)*h^2*s3^4*(5*s1*s2-2*s1*s3-2*s2*s3+s3^2))/(60*(s3-1)*(s2-1)*(s1-1))+(f(3)*h^2*s3^2*(5*s1*s3-10*s1*s2+5*s2*s3-3*s1*s3^2-3*s2*s3^2-3*s3^2+2*s3^3+5*s1*s2*s3))/(60*(s3-1)*(s2-s3)*(s1-s3))+v0*h*s3;$

$yp8=z0+v0*h+(f(4)*h^2*(3*s1+3*s2+3*s3-5*s1*s2-5*s1*s3-5*s2*s3+10*s1*s2*s3-2))/(60*(s3-1)*(s2-1)*(s1-1))-(f(3)*h^2*(5*s1*s2-2*s2-2*s1+1))/(60*s3*(s3-1)*(s2-s3)*(s1-s3))+(f0*h^2*(2*s1+2*s2+2*s3-5*s1*s2-5*s1*s3-5*s2*s3+20*s1*s2*s3-1))/(60*s1*s2*s3)-(f(1)*h^2*(5*s2*s3-2*s3-2*s2+1))/(60*s1*(s1-1)*(s1-s3)*(s1-s2))+(f(2)*h^2*(5*s1*s3-2*s3-2*s1+1))/(60*s2*(s2-1)*(s2-s3)*(s1-s2));$

$yp9=v0+(f(2)*h*s1^3*(10*s3-5*s1-5*s1*s3+3*s1^2))/(60*s2*(s2-1)*(s2-s3)*(s1-s2))+(f(1)*h*s1*(20*s1*s2+20*s1*s3-30*s2*s3-15*s1^2*s2-15*s1^2*s3-15*s1^2+12*s1^3+20*s1*s2*s3))/(60*(s1-1)*(s1-s3)*$

$(s1-s2))-(f(3)*h*s1^3*(10*s2-5*s1-5*s1*s2+3*s1^2))/(60*s3*(s3-1)*$

$(s2-s3)*(s1-s3))-(f0*h*s1*(10*s1*s2+10*s1*s3-30*s2*s3-5*s1^2*$

$s2-5*s1^2*s3-5*s1^2+3*s1^3+10*s1*s2*s3))/(60*s2*s3)+(f(4)*h*s1^3*$

$(10*s2*s3-5*s1*s3-5*s1*s2+3*s1^2))/(60*(s3-1)*(s2-1)*(s1-1));$

$yp10=v0-(f(1)*h*s2^3*(10*s3-5*s2-5*s2*s3+3*s2^2))/(60*s1*(s1-1)*$

$(s1-s3)*(s1-s2))+(f(2)*h*s2*(30*s1*s3-20*s1*s2-20*s2*s3+15*s1*$

$s2^2+15*s2^2*s3+15*s2^2-12*s2^3-20*s1*s2*s3))/(60*(s2-1)*(s2-s3)*$

$(s1-s2))+(f(3)*h*s2^3*(5*s2-10*s1+5*s1*s2-3*s2^2))/(60*s3*(s3-1)*$

$(s2-s3)*(s1-s3))+(f0*h*s2*(30*s1*s3-10*s1*s2-10*s2*s3+5*s1*$

$s2^2+5*s2^2*s3+5*s2^2-3*s2^3-10*s1*s2*s3))/(60*s1*s3)-(f(4)*h*s2^3*$

$(5*s1*s2-10*s1*s3+5*s2*s3-3*s2^2))/(60*(s3-1)*(s2-1)*(s1-1));$

$yp11=v0+(f(4)*h*s3^3*(10*s1*s2-5*s1*s3-5*s2*s3+3*s3^2))/(60*(s3-$

$1)*(s2-1)*(s1-1))+(f(3)*h*s3*(20*s1*s3-30*s1*s2+20*s2*s3-15*$

$s1*s3^2-15*s2*s3^2-15*s3^2+12*s3^3+20*s1*s2*s3))/(60*(s3-1)*(s2-$

$s3)*(s1-s3))+(f(1)*h*s3^3*(5*s3-10*s2+5*s2*s3-3*s3^2))/(60*s1*(s1-$

$1)*(s1-s3)*(s1-s2))-(f(2)*h*s3^3*(5*s3-10*s1+5*s1*s3-3*s3^2))/(60*$

$s2*(s2-1)*(s2-s3)*(s1-s2))-(f0*h*s3*(10*s1*s3-30*s1*s2+10*s2*$

$s3-5*s1*s3^2-5*s2*s3^2-5*s3^2+3*s3^3+10*s1*s2*s3))/(60*s1*s2);$

$yp12=v0-(f(1)*h*(10*s2*s3-5*s3-5*s2+3))/(60*s1*(s1-1)*(s1-s3)*$

$(s1-s2))+(f(2)*h*(10*s1*s3-5*s3-5*s1+3))/(60*s2*(s2-1)*(s2-s3)*$

$(s1-s2))-(f(3)*h*(10*s1*s2-5*s2-5*s1+3))/(60*s3*(s3-1)*(s2-s3)*$

$(s1-s3))+(f0*h*(5*s1+5*s2+5*s3-10*s1*s2-10*s1*s3-10*s2*s3+$

$30*s1*s2*s3-3))/(60*s1*s2*s3)+(f(4)*h*(15*s1+15*s2+15*s3-20*s1*$

$s2-20*s1*s3-20*s2*s3+30*s1*s2*s3-12))/(60*(s3-1)*(s2-1)*(s1-1));$

$fr1=-yp5;$

$fr2=-yp6;$

$fr3=-yp7;$

$fr4=-yp8;$

$$yr1=-(fr3\ast h^3\ast s1^5\ast(21\ast s2-7\ast s1-7\ast s1\ast s2+3\ast s1^2))/(840\ast s3\ast(s3-1)\ast (s2-s3)\ast(s1-s3))+(fr4\ast h^3\ast s1^5\ast(21\ast s2\ast s3-7\ast s1\ast s3-7\ast s1\ast s2+3\ast s1^2))/(840\ast(s3-1)\ast(s2-1)\ast(s1-1))+(fr2\ast h^3\ast s1^5\ast(21\ast s3-7\ast s1-7\ast s1\ast s3+3\ast s1^2))/(840\ast s2\ast(s2-1)\ast(s2-s3)\ast(s1-s2))+y0+(fr1\ast h^3\ast s1^3\ast(14\ast s1\ast s2+14\ast s1\ast s3-35\ast s2\ast s3-7\ast s1^2\ast s2-7\ast s1^2\ast s3-7\ast s1^2+4\ast s1^3+14\ast s1\ast s2\ast s3))/(840\ast(s1-1)\ast(s1-s3)\ast(s1-s2))-(f0\ast h^3\ast s1^3\ast(21\ast s1\ast s2+21\ast s1\ast s3-105\ast s2\ast s3-7\ast s1^2\ast s2-7\ast s1^2\ast s3-7\ast s1^2+3\ast s1^3+21\ast s1\ast s2\ast s3))/(840\ast s2\ast s3)+(v0\ast h^2\ast s1^2)/2+z0\ast h\ast s1$$

$m1=toc;$

$err1=abs(-2\ast cos(x(1))+sin(x(1))+2-yr1);$

$fprintf('\%2.7f\%3.18f\%3.18f\%1.6e\backslash n',x(1),-2\ast cos(x(1))+sin(x(1))+2,yr1,err1)$

$$yr2=(fr3\ast h^3\ast s2^5\ast(7\ast s2-21\ast s1+7\ast s1\ast s2-3\ast s2^2))/(840\ast s3\ast(s3-1)\ast(s2-s3)\ast(s1-s3))+(fr2\ast h^3\ast s2^3\ast(35\ast s1\ast s3-14\ast s1\ast s2-14\ast s2\ast s3+7\ast s1\ast s2^2+7\ast s2^2\ast s3+7\ast s2^2-4\ast s2^3-14\ast s1\ast s2\ast s3))/(840\ast(s2-1)\ast(s2-s3)\ast(s1-s2))-(fr4\ast h^3\ast s2^5\ast(7\ast s1\ast s2-21\ast s1\ast s3+7\ast s2\ast s3-3\ast s2^2))/(840\ast(s3-1)\ast(s1-1)\ast(s2-1))-(fr1\ast h^3\ast s2^5\ast(21\ast s3-7\ast s2-7\ast s2\ast s3+3\ast s2^2))/(840\ast s1\ast(s1-1)\ast(s1-s3)\ast(s1-s2))+y0+(f0\ast h^3\ast s2^3\ast(105\ast s1\ast s3-21\ast s1\ast s2-21\ast s2\ast s3+7\ast s1\ast s2^2+7\ast s2^2\ast s3+7\ast s2^2-3\ast s2^3-21\ast s1\ast s2\ast s3))/(840\ast s1\ast s3)+(v0\ast h^2\ast s2^2)/2+z0\ast h\ast s2;$$

$m2=toc;$

$err2=abs(-2\ast cos(x(2))+sin(x(2))+2-yr2);$

$fprintf('\%2.7f\%3.18f\%3.18f\%1.6e\backslash n',x(2),-2\ast cos(x(2))+sin(x(2))+2,yr2,err2)$

$$yr3=y0-(f0\ast h^3\ast s3^3\ast(21\ast s1\ast s3-105\ast s1\ast s2+21\ast s2\ast s3-7\ast s1\ast s3^2-7\ast s2\ast s3^2-7\ast s3^2+3\ast s3^3+21\ast s1\ast s2\ast s3))/(840\ast s1\ast s2)+(fr4\ast h^3\ast s3^5\ast(21\ast s1\ast s2-7\ast s1\ast s3-7\ast s2\ast s3+3\ast s3^2))/(840\ast(s2-1)\ast(s1-1)\ast(s3-1))+(fr1\ast h^3\ast s3^5\ast(7\ast s3-21\ast s2+7\ast s2\ast s3-3\ast s3^2))/(840\ast s1\ast(s1-1)\ast(s1-s3)\ast(s1-$$

$s2)) - (fr2*h^3*s3^5*(7*s3 - 21*s1 + 7*s1*s3 - 3*s3^2))/(840*s2*(s2-1)*$

$(s2-s3)*(s1-s2)) + (fr3*h^3*s3^3*(14*s1*s3 - 35*s1*s2 + 14*s2*s3 - 7*$

$s1*s3^2 - 7*s2*s3^2 - 7*s3^2 + 4*s3^3 + 14*s1*s2*s3))/(840*(s3-1)*(s2-s3)*$

$(s1-s3)) + (v0*h^2*s3^2)/2 + z0*h*s3;$

$m3 = toc;$

$err3 = abs(-2*cos(x(3)) + sin(x(3)) + 2 - yr3);$

$fprintf('\%2.7f\%3.18f\%3.18f\%1.6e\backslash n', x(3), -2*cos(x(3)) + sin(x(3)) + 2, yr3, err3)$

$yr4 = z0*h + y0 + (fr4*h^3*(7*s1 + 7*s2 + 7*s3 - 14*s1*s2 - 14*s1*s3 - 14*$

$s2*s3 + 35*s1*s2*s3 - 4))/(840*(s2-1)*(s1-1)*(s3-1)) + (v0*h^2)/2 -$

$(fr1*h^3*(21*s2*s3 - 7*s3 - 7*s2 + 3))/(840*s1*(s1-1)*(s1-s3)*(s1-$

$s2)) + (fr2*h^3*(21*s1*s3 - 7*s3 - 7*s1 + 3))/(840*s2*(s2-1)*(s2-s3)*$

$(s1-s2)) + (f0*h^3*(7*s1 + 7*s2 + 7*s3 - 21*s1*s2 - 21*s1*s3 - 21*s2*s3 +$

$105*s1*s2*s3 - 3))/(840*s1*s2*s3) - (fr3*h^3*(21*s1*s2 - 7*s2 - 7*s1 +$

$3))/(840*s3*(s3-1)*(s2-s3)*(s1-s3));$

$m4 = toc;$

$err4 = abs(-2*cos(x(4)) + sin(x(4)) + 2 - yr4);$

$fprintf('\%2.7f\%3.18f\%3.18f\%1.6e\backslash n', x(4), -2*cos(x(4)) + sin(x(4)) + 2, yr4, err4)$

$x0 = x(4); z0 = yp8; y0 = yr4; v0 = yp12;$

end

APPENDIX F
MATLAB CODE OF THE NEW METHOD WITH GENERALISED THREE OFF-STEP POINT FOR SOLVING FOURTH ORDER ODE

Matlab code of new method with two off-step point for solving problem 18 of fourth order ODE

clear

clc

$\%y'$ is represented by z

$\%y''$ is represented by v

$\%y'''$ is represented by w

syms $x0\ y0\ z0\ f0\ x\ y\ z\ v\ v0\ w\ w0$

$\%f(x,y,y',y'',y''') = (y')^2 - yy'' - 4x^2 + e^x(1-4x+x^2)$

disp ('x – value exact – solution computed – solution error')

tic;

$x0 = 0;\ y0 = 1;\ z0 = 1;\ v0 = 3;\ w0 = 1;\ h = 1/320;$

$for\ j = 0 : h : 1;$

$f0 = (z0)^2 - y0*w0 - 4*(x0)^2 + exp(x0)*(1-4*x0+(x0)^2);$

$s1 = 1/4;$

$s2 = 1/3;$

$s3 = 2/3;$

$x(1) = x0 + (s1*h);$

$y(1) = y0 + (h^4*s1^4*(exp(x0)*(x0^2-4*x0+1) - w0*y0 - 4*x0^2 + z0^2))/24 + (h^2*s1^2*v0)/2 + (h^3*s1^3*w0)/6 - (h^5*s1^5*(8*x0 - exp(x0)*(x0^2-4*x0+1) - 2*v0*z0 + w0*z0 - exp(x0)*(2*x0-4) + y0*(exp(x0)*(x0^2-4*x0+1) - w0*y0 - 4*x0^2 + z0^2)))/120 + h*s1*z0 + (h^6*s1^6*(2*exp(x0) - z0*(2*exp(x0)*$

$(x0^2-4*x0+1)-2*w0*y0-8*x0^2+2*z0^2)+exp(x0)*(x0^2-4*x0+1)-v0*$
$w0+2*w0*z0+y0*(8*x0-exp(x0)*(x0^2-4*x0+1)-2*v0*z0+w0*z0-$
$exp(x0)*(2*x0-4)+y0*(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*x0^2+z0^2))+$
$2*exp(x0)*(2*x0-4)+2*v0^2-8))/720;$

$z(1)=z0+(h^3*s1^3*(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*x0^2+z0^2))/6+$
$(h^2*s1^2*w0)/2-(h^4*s1^4*(8*x0-exp(x0)*(x0^2-4*x0+1)-2*v0*z0+w0*$
$z0-exp(x0)*(2*x0-4)+y0*(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*x0^2+$
$z0^2)))/24+h*s1*v0+(h^5*s1^5*(2*exp(x0)-z0*(2*exp(x0)*(x0^2-4*x0+$
$1)-2*w0*y0-8*x0^2+2*z0^2)+exp(x0)*(x0^2-4*x0+1)-v0*w0+2*w0*$
$z0+y0*(8*x0-exp(x0)*(x0^2-4*x0+1)-2*v0*z0+w0*z0-exp(x0)*(2*$
$x0-4)+y0*(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*x0^2+z0^2))+2*exp(x0)*$
$(2*x0-4)+2*v0^2-8))/120;$

$v(1)=v0+(h^2*s1^2*(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*x0^2+z0^2))/2-(h^3*$
$s1^3*(8*x0-exp(x0)*(x0^2-4*x0+1)-2*v0*z0+w0*z0-exp(x0)*(2*x0-$
$4)+y0*(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*x0^2+z0^2)))/6+h*s1*w0+$
$(h^4*s1^4*(2*exp(x0)-z0*(2*exp(x0)*(x0^2-4*x0+1)-2*w0*y0-8*x0^2+$
$2*z0^2)+exp(x0)*(x0^2-4*x0+1)-v0*w0+2*w0*z0+y0*(8*x0-exp(x0)*$
$(x0^2-4*x0+1)-2*v0*z0+w0*z0-exp(x0)*(2*x0-4)+y0*(exp(x0)*(x0^2-$
$4*x0+1)-w0*y0-4*x0^2+z0^2))+2*exp(x0)*(2*x0-4)+2*v0^2-8))/24;$

$w(1)=w0+h*s1*(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*x0^2+z0^2)-(h^2*s1^2*$
$(8*x0-exp(x0)*(x0^2-4*x0+1)-2*v0*z0+w0*z0-exp(x0)*(2*x0-4)+y0*$
$(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*x0^2+z0^2)))/2+(h^3*s1^3*(2*exp(x0)-$
$z0*(2*exp(x0)*(x0^2-4*x0+1)-2*w0*y0-8*x0^2+2*z0^2)+exp(x0)*(x0^2-$
$4*x0+1)-v0*w0+2*w0*z0+y0*(8*x0-exp(x0)*(x0^2-4*x0+1)-2*v0*$
$z0+w0*z0-exp(x0)*(2*x0-4)+y0*(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*$
$x0^2+z0^2))+2*exp(x0)*(2*x0-4)+2*v0^2-8))/6;$

$f(1)=(z(1))^2-y(1)*w(1)-4*(x(1))^2+exp(x(1))*(1-4*x(1)+(x(1))^2);$

$x(2)=x0+(s2*h);$

$y(2) = y0 + (h^4 \ast s2^4 \ast (exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - w0 \ast y0 - 4 \ast x0^2 + z0^2))/24 +$
$(h^2 \ast s2^2 \ast v0)/2 + (h^3 \ast s2^3 \ast w0)/6 - (h^5 \ast s2^5 \ast (8 \ast x0 - exp(x0) \ast (x0^2 - 4 \ast x0 + 1) -$
$2 \ast v0 \ast z0 + w0 \ast z0 - exp(x0) \ast (2 \ast x0 - 4) + y0 \ast (exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - w0 \ast$
$y0 - 4 \ast x0^2 + z0^2)))/120 + h \ast s2 \ast z0 + (h^6 \ast s2^6 \ast (2 \ast exp(x0) - z0 \ast (2 \ast exp(x0) \ast$
$(x0^2 - 4 \ast x0 + 1) - 2 \ast w0 \ast y0 - 8 \ast x0^2 + 2 \ast z0^2) + exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - v0 \ast$
$w0 + 2 \ast w0 \ast z0 + y0 \ast (8 \ast x0 - exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - 2 \ast v0 \ast z0 + w0 \ast z0 -$
$exp(x0) \ast (2 \ast x0 - 4) + y0 \ast (exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - w0 \ast y0 - 4 \ast x0^2 + z0^2)) +$
$2 \ast exp(x0) \ast (2 \ast x0 - 4) + 2 \ast v0^2 - 8))/720;$
$z(2) = z0 + (h^3 \ast s2^3 \ast (exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - w0 \ast y0 - 4 \ast x0^2 + z0^2))/6 +$
$(h^2 \ast s2^2 \ast w0)/2 - (h^4 \ast s2^4 \ast (8 \ast x0 - exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - 2 \ast v0 \ast z0 + w0 \ast$
$z0 - exp(x0) \ast (2 \ast x0 - 4) + y0 \ast (exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - w0 \ast y0 - 4 \ast x0^2 +$
$z0^2)))/24 + h \ast s2 \ast v0 + (h^5 \ast s2^5 \ast (2 \ast exp(x0) - z0 \ast (2 \ast exp(x0) \ast (x0^2 - 4 \ast x0 +$
$1) - 2 \ast w0 \ast y0 - 8 \ast x0^2 + 2 \ast z0^2) + exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - v0 \ast w0 + 2 \ast w0 \ast$
$z0 + y0 \ast (8 \ast x0 - exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - 2 \ast v0 \ast z0 + w0 \ast z0 - exp(x0) \ast (2 \ast$
$x0 - 4) + y0 \ast (exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - w0 \ast y0 - 4 \ast x0^2 + z0^2)) + 2 \ast exp(x0) \ast$
$(2 \ast x0 - 4) + 2 \ast v0^2 - 8))/120;$
$v(2) = v0 + (h^2 \ast s2^2 \ast (exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - w0 \ast y0 - 4 \ast x0^2 + z0^2))/2 - (h^3 \ast$
$s2^3 \ast (8 \ast x0 - exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - 2 \ast v0 \ast z0 + w0 \ast z0 - exp(x0) \ast (2 \ast x0 -$
$4) + y0 \ast (exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - w0 \ast y0 - 4 \ast x0^2 + z0^2)))/6 + h \ast s2 \ast w0 +$
$(h^4 \ast s2^4 \ast (2 \ast exp(x0) - z0 \ast (2 \ast exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - 2 \ast w0 \ast y0 - 8 \ast x0^2 +$
$2 \ast z0^2) + exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - v0 \ast w0 + 2 \ast w0 \ast z0 + y0 \ast (8 \ast x0 - exp(x0) \ast$
$(x0^2 - 4 \ast x0 + 1) - 2 \ast v0 \ast z0 + w0 \ast z0 - exp(x0) \ast (2 \ast x0 - 4) + y0 \ast (exp(x0) \ast (x0^2 -$
$4 \ast x0 + 1) - w0 \ast y0 - 4 \ast x0^2 + z0^2)) + 2 \ast exp(x0) \ast (2 \ast x0 - 4) + 2 \ast v0^2 - 8))/24;$
$w(2) = w0 + h \ast s2 \ast (exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - w0 \ast y0 - 4 \ast x0^2 + z0^2) - (h^2 \ast s2^2 \ast$
$(8 \ast x0 - exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - 2 \ast v0 \ast z0 + w0 \ast z0 - exp(x0) \ast (2 \ast x0 - 4) + y0 \ast$
$(exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - w0 \ast y0 - 4 \ast x0^2 + z0^2)))/2 + (h^3 \ast s2^3 \ast (2 \ast exp(x0) -$
$z0 \ast (2 \ast exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - 2 \ast w0 \ast y0 - 8 \ast x0^2 + 2 \ast z0^2) + exp(x0) \ast (x0^2 -$
$4 \ast x0 + 1) - v0 \ast w0 + 2 \ast w0 \ast z0 + y0 \ast (8 \ast x0 - exp(x0) \ast (x0^2 - 4 \ast x0 + 1) - 2 \ast v0 \ast$

$z0 + w0*z0 - exp(x0)*(2*x0-4) + y0*(exp(x0)*(x0^2-4*x0+1) - w0*y0 - 4*x0^2 + z0^2)) + 2*exp(x0)*(2*x0-4) + 2*v0^2 - 8))/6;$

$f(2) = (z(2))^2 - y(2)*w(2) - 4*(x(2))^2 + exp(x(2))*(1-4*x(2)+(x(2))^2);$

$x(3) = x0 + (s3*h);$

$y(3) = y0 + (h^4*s3^4*(exp(x0)*(x0^2-4*x0+1) - w0*y0 - 4*x0^2 + z0^2))/24 + (h^2*s3^2*v0)/2 + (h^3*s3^3*w0)/6 - (h^5*s3^5*(8*x0 - exp(x0)*(x0^2-4*x0+1) - 2*v0*z0 + w0*z0 - exp(x0)*(2*x0-4) + y0*(exp(x0)*(x0^2-4*x0+1) - w0*y0 - 4*x0^2 + z0^2)))/120 + h*s3*z0 + (h^6*s3^6*(2*exp(x0) - z0*(2*exp(x0)*(x0^2-4*x0+1) - 2*w0*y0 - 8*x0^2 + 2*z0^2) + exp(x0)*(x0^2-4*x0+1) - v0*w0 + 2*w0*z0 + y0*(8*x0 - exp(x0)*(x0^2-4*x0+1) - 2*v0*z0 + w0*z0 - exp(x0)*(2*x0-4) + y0*(exp(x0)*(x0^2-4*x0+1) - w0*y0 - 4*x0^2 + z0^2)) + 2*exp(x0)*(2*x0-4) + 2*v0^2 - 8))/720;$

$z(3) = z0 + (h^3*s3^3*(exp(x0)*(x0^2-4*x0+1) - w0*y0 - 4*x0^2 + z0^2))/6 + (h^2*s3^2*w0)/2 - (h^4*s3^4*(8*x0 - exp(x0)*(x0^2-4*x0+1) - 2*v0*z0 + w0*z0 - exp(x0)*(2*x0-4) + y0*(exp(x0)*(x0^2-4*x0+1) - w0*y0 - 4*x0^2 + z0^2)))/24 + h*s3*v0 + (h^5*s3^5*(2*exp(x0) - z0*(2*exp(x0)*(x0^2-4*x0+1) - 2*w0*y0 - 8*x0^2 + 2*z0^2) + exp(x0)*(x0^2-4*x0+1) - v0*w0 + 2*w0*z0 + y0*(8*x0 - exp(x0)*(x0^2-4*x0+1) - 2*v0*z0 + w0*z0 - exp(x0)*(2*x0-4) + y0*(exp(x0)*(x0^2-4*x0+1) - w0*y0 - 4*x0^2 + z0^2)) + 2*exp(x0)*(2*x0-4) + 2*v0^2 - 8))/120;$

$v(3) = v0 + (h^2*s3^2*(exp(x0)*(x0^2-4*x0+1) - w0*y0 - 4*x0^2 + z0^2))/2 - (h^3*s3^3*(8*x0 - exp(x0)*(x0^2-4*x0+1) - 2*v0*z0 + w0*z0 - exp(x0)*(2*x0-4) + y0*(exp(x0)*(x0^2-4*x0+1) - w0*y0 - 4*x0^2 + z0^2)))/6 + h*s3*w0 + (h^4*s3^4*(2*exp(x0) - z0*(2*exp(x0)*(x0^2-4*x0+1) - 2*w0*y0 - 8*x0^2 + 2*z0^2) + exp(x0)*(x0^2-4*x0+1) - v0*w0 + 2*w0*z0 + y0*(8*x0 - exp(x0)*(x0^2-4*x0+1) - 2*v0*z0 + w0*z0 - exp(x0)*(2*x0-4) + y0*(exp(x0)*(x0^2 -$

$4*x0+1)-w0*y0-4*x0^2+z0^2))+2*exp(x0)*(2*x0-4)+2*v0^2-8))/24;$

$w(3)=w0+h*s3*(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*x0^2+z0^2)-(h^2*s3^2*(8*x0-exp(x0)*(x0^2-4*x0+1)-2*v0*z0+w0*z0-exp(x0)*(2*x0-4)+y0*(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*x0^2+z0^2)))/2+(h^3*s3^3*(2*exp(x0)-z0*(2*exp(x0)*(x0^2-4*x0+1)-2*w0*y0-8*x0^2+2*z0^2)+exp(x0)*(x0^2-4*x0+1)-v0*w0+2*w0*z0+y0*(8*x0-exp(x0)*(x0^2-4*x0+1)-2*v0*z0+w0*z0-exp(x0)*(2*x0-4)+y0*(exp(x0)*(x0^2-4*x0+1)-w0*y0-4*x0^2+z0^2))+2*exp(x0)*(2*x0-4)+2*v0^2-8))/6;$

$f(3)=(z(3))^2-y(3)*w(3)-4*(x(3))^2+exp(x(3))*(1-4*x(3)+(x(3))^2);$

$x(4)=x0+h;$

$y(4)=y0+(h^2*1^3*v0)/2+(h^3*1^4*w0)/6+(h^5*1^5*(2*v0*z0-8*x0-w0*z0+exp(x0)*(2*x0-4)+y0*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2)+exp(x0)*(x0^2-4*x0+1)))/120+h*1*z0+(h^6*1^7*(2*exp(x0)-v0*w0+2*w0*z0+2*exp(x0)*(2*x0-4)-y0*(2*v0*z0-8*x0-w0*z0+exp(x0)*(2*x0-4)+y0*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2)+exp(x0)*(x0^2-4*x0+1))+z0*(2*w0*y0-2*exp(x0)*(x0^2-4*x0+1)+8*x0^2-2*z0^2)+exp(x0)*(x0^2-4*x0+1)+2*v0^2-8))/720-(h^4*1^5*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2))/24;$

$z(4)=z0+(h^2*1^3*w0)/2+(h^4*1^4*(2*v0*z0-8*x0-w0*z0+exp(x0)*(2*x0-4)+y0*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2)+exp(x0)*(x0^2-4*x0+1)))/24+h*1*v0+(h^5*1^6*(2*exp(x0)-v0*w0+2*w0*z0+2*exp(x0)*(2*x0-4)-y0*(2*v0*z0-8*x0-w0*z0+exp(x0)*(2*x0-4)+y0*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2)+exp(x0)*(x0^2-4*x0+1))+z0*(2*w0*y0-2*exp(x0)*(x0^2-4*x0+1)+8*x0^2-2*z0^2)+exp(x0)*(x0^2-4*x0+1)+2*v0^2-8))/120-(h^3*1^4*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2))/6;$

$v(4)=v0+(h^3*1^3*(2*v0*z0-8*x0-w0*z0+exp(x0)*(2*x0-4)+y0*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2)+exp(x0)*(x0^2-4*x0+1)))/6+h*1*w0+(h^4*1^5*(2*exp(x0)-v0*w0+2*w0*z0+2*exp(x0)*(2*x0-4)-$

$y0*(2*v0*z0-8*x0-w0*z0+exp(x0)*(2*x0-4)+y0*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2)+exp(x0)*(x0^2-4*x0+1))+z0*(2*w0*y0-2*exp(x0)*(x0^2-4*x0+1)+8*x0^2-2*z0^2)+exp(x0)*(x0^2-4*x0+1)+2*v0^2-8))/24-(h^2*1^3*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2))/2;$

$w(4)=w0-h*1*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2)+(h^2*1^2*(2*v0*z0-8*x0-w0*z0+exp(x0)*(2*x0-4)+y0*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2)+exp(x0)*(x0^2-4*x0+1)))/2+(h^3*1^4*(2*exp(x0)-v0*w0+2*w0*z0+2*exp(x0)*(2*x0-4)-y0*(2*v0*z0-8*x0-w0*z0+exp(x0)*(2*x0-4)+y0*(w0*y0-exp(x0)*(x0^2-4*x0+1)+4*x0^2-z0^2)+exp(x0)*(x0^2-4*x0+1))+z0*(2*w0*y0-2*exp(x0)*(x0^2-4*x0+1)+8*x0^2-2*z0^2)+exp(x0)*(x0^2-4*x0+1)+2*v0^2-8))/6;$

$f(4)=(z(4))^2-y(4)*w(4)-4*(x(4))^2+exp(x(4))*(1-4*x(4)+(x(4))^2);$

$yp1=(w0*h^3*s1^3+3*v0*h^2*s1^2+6*z0*h*s1+6*y0)/6-(f0*h^4*s1^4*(28*s1*s2+28*s1*s3-168*s2*s3-8*s1^2*s2-8*s1^2*s3-8*s1^2+3*s1^3+28*s1*s2*s3))/(5040*s2*s3)+(h^4*s1^4*(14*s1*s2+14*s1*s3-42*s2*s3-6*s1^2*s2-6*s1^2*s3-6*s1^2+3*s1^3+14*s1*s2*s3))/(5040*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h^4*s1^6*(28*s3-8*s1-8*s1*s3+3*s1^2))/(5040*s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)-(h^4*s1^6*(28*s2-8*s1-8*s1*s2+3*s1^2))/(5040*s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h^4*s1^6*(28*s2*s3-8*s1*s3-8*s1*s2+3*s1^2))/(5040*(s3-1)*(s2-1)*(s1-1))*f(4)$

$yp2=(w0*h^3*s2^3+3*v0*h^2*s2^2+6*z0*h*s2+6*y0)/6+(f0*h^4*s2^4*(168*s1*s3-28*s1*s2-28*s2*s3+8*s1*s2^2+8*s2^2*s3+8*s2^2-3*s2^3-28*s1*s2*s3))/(5040*s1*s3)-(h^4*s2^6*(28*s3-8*s2-8*s2*s3+3*s2^2))/(5040*s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h^4*s2^4*(42*s1*s3-14*s1*s2-14*s2*s3+6*s1*s2^2+6*s2^2*s3+6*s2^2-3*s2^3-14*s1*s2*s3))/(5040*(s2-1)*(s2-s3)*(s1-s2))*f(2)+(h^4*s2^6*(8*s2-28*s1+8*s1*s2-3*s2^2))/(5040*s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)-(h^4*s2^6*(8*s1*s2-28*s1*s3+8*s2*s3-3*s2^2))/(5040*(s3-1)*(s1-1)*(s2-1))*f(4)$

$yp3 = (w0*h^3*s3^3+3*v0*h^2*s3^2+6*z0*h*s3+6*y0)/6-(f0*h^4*s3^4*(28*s1*s3-168*s1*s2+28*s2*s3-8*s1*s3^2-8*s2*s3^2-8*s3^2+3*s3^3+28*s1*s2*s3))/(5040*s1*s2)+(h^4*s3^6*(8*s3-28*s2+8*s2*s3-3*s3^2))/(5040*s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)-(h^4*s3^6*(8*s3-28*s1+8*s1*s3-3*s3^2))/(5040*s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)+(h^4*s3^4*(14*s1*s3-42*s1*s2+14*s2*s3-6*s1*s3^2-6*s2*s3^2-6*s3^2+3*s3^3+14*s1*s2*s3))/(5040*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h^4*s3^6*(28*s1*s2-8*s1*s3-8*s2*s3+3*s3^2))/(5040*(s2-1)*(s1-1)*(s3-1))*f(4)$

$yp4 = (w0*h^3+3*v0*h^2+6*z0*h+6*y0)/6+(f0*h^4*(8*s1+8*s2+8*s3-28*s1*s2-28*s1*s3-28*s2*s3+168*s1*s2*s3-3))/(5040*s1*s2*s3)-(h^4*(28*s2*s3-8*s3-8*s2+3))/(5040*s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h^4*(28*s1*s3-8*s3-8*s1+3))/(5040*s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)-(h^4*(28*s1*s2-8*s2-8*s1+3))/(5040*s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h^4*(6*s1+6*s2+6*s3-14*s1*s2-14*s1*s3-14*s2*s3+42*s1*s2*s3-3))/(5040*(s3-1)*(s2-1)*(s1-1))*f(4)$

$yp5 = (w0*h^2*s1^2+2*v0*h*s1+2*z0)/2-(f0*h^3*s1^3*(21*s1*s2+21*s1*s3-105*s2*s3-7*s1^2*s2-7*s1^2*s3-7*s1^2+3*s1^3+21*s1*s2*s3))/(840*s2*s3)+(h^3*s1^3*(14*s1*s2+14*s1*s3-35*s2*s3-7*s1^2*s2-7*s1^2*s3-7*s1^2+4*s1^3+14*s1*s2*s3))/(840*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h^3*s1^5*(21*s3-7*s1-7*s1*s3+3*s1^2))/(840*s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)-(h^3*s1^5*(21*s2-7*s1-7*s1*s2+3*s1^2))/(840*s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h^3*s1^5*(21*s2*s3-7*s1*s3-7*s1*s2+3*s1^2))/(840*(s3-1)*(s2-1)*(s1-1))*f(4)$

$yp6 = (w0*h^2*s2^2+2*v0*h*s2+2*z0)/2+(f0*h^3*s2^3*(105*s1*s3-21*s1*s2-21*s2*s3+7*s1*s2^2+7*s2^2*s3+7*s2^2-3*s2^3-21*s1*s2*s3))/(840*s1*s3)-(h^3*s2^5*(21*s3-7*s2-7*s2*s3+3*s2^2))/(840*s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h^3*s2^3*(35*s1*s3-14*s1*s2-14*s2*s3+7*s1*s2^2+7*s2^2*s3+7*s2^2-4*s2^3-14*s1*s2*s3))/(840*(s2-1)*(s2-s3)*$

$(s1-s2))*f(2)+(h^3*s2^5*(7*s2-21*s1+7*s1*s2-3*s2^2))/(840*s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)-(h^3*s2^5*(7*s1*s2-21*s1*s3+7*s2*s3-3*s2^2))/(840*(s3-1)*(s1-1)*(s2-1))*f(4)$

$yp7=(w0*h^2*s3^2+2*v0*h*s3+2*z0)/2-(f0*h^3*s3^3*(21*s1*s3-105*s1*s2+21*s2*s3-7*s1*s3^2-7*s2*s3^2-7*s3^2+3*s3^3+21*s1*s2*s3))/(840*s1*s2)+(h^3*s3^5*(7*s3-21*s2+7*s2*s3-3*s3^2))/(840*s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)-(h^3*s3^5*(7*s3-21*s1+7*s1*s3-3*s3^2))/(840*s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)+(h^3*s3^3*(14*s1*s3-35*s1*s2+14*s2*s3-7*s1*s3^2-7*s2*s3^2-7*s3^2+4*s3^3+14*s1*s2*s3))/(840*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h^3*s3^5*(21*s1*s2-7*s1*s3-7*s2*s3+3*s3^2))/(840*(s2-1)*(s1-1)*(s3-1))*f(4)$

$yp8=(w0*h^2+2*v0*h+2*z0)/2+(f0*h^3*(7*s1+7*s2+7*s3-21*s1*s2-21*s1*s3-21*s2*s3+105*s1*s2*s3-3))/(840*s1*s2*s3)-(h^3*(21*s2*s3-7*s3-7*s2+3))/(840*s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h^3*(21*s1*s3-7*s3-7*s1+3))/(840*s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)-(h^3*(21*s1*s2-7*s2-7*s1+3))/(840*s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h^3*(7*s1+7*s2+7*s3-14*s1*s2-14*s1*s3-14*s2*s3+35*s1*s2*s3-4))/(840*(s3-1)*(s2-1)*(s1-1))*f(4)$

$yp9=v0+w0*h*s1-(f0*h^2*s1^2*(5*s1*s2+5*s1*s3-20*s2*s3-2*s1^2*s2-2*s1^2*s3-2*s1^2+s1^3+5*s1*s2*s3))/(60*s2*s3)+(h^2*s1^2*(5*s1*s2+5*s1*s3-10*s2*s3-3*s1^2*s2-3*s1^2*s3-3*s1^2+2*s1^3+5*s1*s2*s3))/(60*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h^2*s1^4*(5*s3-2*s1-2*s1*s3+s1^2))/(60*s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)-(h^2*s1^4*(5*s2-2*s1-2*s1*s2+s1^2))/(60*s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h^2*s1^4*(5*s2*s3-2*s1*s3-2*s1*s2+s1^2))/(60*(s3-1)*(s2-1)*(s1-1))*f(4)$

$yp10=v0+w0*h*s2+(f0*h^2*s2^2*(20*s1*s3-5*s1*s2-5*s2*s3+2*s1*s2^2+2*s2^2*s3+2*s2^2-s2^3-5*s1*s2*s3))/(60*s1*s3)-(h^2*s2^4*(5*s3-2*s2-2*s2*s3+s2^2))/(60*s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h^2*s2^2*$

$$(10*s1*s3-5*s1*s2-5*s2*s3+3*s1*s2^2+3*s2^2*s3+3*s2^2-2*s2^3-5*$$

$$s1*s2*s3))/(60*(s2-1)*(s2-s3)*(s1-s2))*f(2)+(h^2*s2^4*(2*s2-5*s1+$$

$$2*s1*s2-s2^2))/(60*s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)-(h^2*s2^4*(2*s1*$$

$$s2-5*s1*s3+2*s2*s3-s2^2))/(60*(s3-1)*(s1-1)*(s2-1))*f(4)$$

$$yp11=v0+w0*h*s3-(f0*h^2*s3^2*(5*s1*s3-20*s1*s2+5*s2*s3-2*s1*$$

$$s3^2-2*s2*s3^2-2*s3^2+s3^3+5*s1*s2*s3))/(60*s1*s2)+(h^2*s3^4*(2*s3-$$

$$5*s2+2*s2*s3-s3^2))/(60*s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)-(h^2*s3^4*$$

$$(2*s3-5*s1+2*s1*s3-s3^2))/(60*s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)+$$

$$(h^2*s3^2*(5*s1*s3-10*s1*s2+5*s2*s3-3*s1*s3^2-3*s2*s3^2-3*s3^2+$$

$$2*s3^3+5*s1*s2*s3))/(60*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h^2*s3^4*(5*$$

$$s1*s2-2*s1*s3-2*s2*s3+s3^2))/(60*(s2-1)*(s1-1)*(s3-1))*f(4)$$

$$yp12=v0+w0*h+(f0*h^2*(2*s1+2*s2+2*s3-5*s1*s2-5*s1*s3-5*s2*$$

$$s3+20*s1*s2*s3-1))/(60*s1*s2*s3)-(h^2*(5*s2*s3-2*s3-2*s2+1))/(60*$$

$$s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h^2*(5*s1*s3-2*s3-2*s1+1))/(60*$$

$$s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)-(h^2*(5*s1*s2-2*s2-2*s1+1))/(60*$$

$$s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h^2*(3*s1+3*s2+3*s3-5*s1*s2-$$

$$5*s1*s3-5*s2*s3+10*s1*s2*s3-2))/(60*(s3-1)*(s2-1)*(s1-1))*f(4)$$

$$yp13=w0-(f0*h*s1*(10*s1*s2+10*s1*s3-30*s2*s3-5*s1^2*s2-5*$$

$$s1^2*s3-5*s1^2+3*s1^3+10*s1*s2*s3))/(60*s2*s3)+(h*s1*(20*s1*s2+20*$$

$$s1*s3-30*s2*s3-15*s1^2*s2-15*s1^2*s3-15*s1^2+12*s1^3+20*s1*s2*$$

$$s3))/(60*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h*s1^3*(10*s3-5*s1-5*s1*$$

$$s3+3*s1^2))/(60*s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)-(h*s1^3*(10*s2-5*$$

$$s1-5*s1*s2+3*s1^2))/(60*s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h*s1^3*$$

$$(10*s2*s3-5*s1*s3-5*s1*s2+3*s1^2))/(60*(s3-1)*(s2-1)*(s1-1))*f(4)$$

$$yp14=w0+(f0*h*s2*(30*s1*s3-10*s1*s2-10*s2*s3+5*s1*s2^2+5*s2^2*$$

$$s3+5*s2^2-3*s2^3-10*s1*s2*s3))/(60*s1*s3)-(h*s2^3*(10*s3-5*s2-5*$$

$$s2*s3+3*s2^2))/(60*s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h*s2*(30*s1*$$

$$s3-20*s1*s2-20*s2*s3+15*s1*s2^2+15*s2^2*s3+15*s2^2-12*s2^3-20*$$

$s1*s2*s3))/(60*(s2-1)*(s2-s3)*(s1-s2))*f(2)+(h*s2^3*(5*s2-10*s1+5*s1*s2-3*s2^2))/(60*s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)-(h*s2^3*(5*s1*s2-10*s1*s3+5*s2*s3-3*s2^2))/(60*(s3-1)*(s1-1)*(s2-1))*f(4)$

$yp15 = w0-(f0*h*s3*(10*s1*s3-30*s1*s2+10*s2*s3-5*s1*s3^2-5*s2*s3^2-5*s3^2+3*s3^3+10*s1*s2*s3))/(60*s1*s2)+(h*s3^3*(5*s3-10*s2+5*s2*s3-3*s3^2))/(60*s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)-(h*s3^3*(5*s3-10*s1+5*s1*s3-3*s3^2))/(60*s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)+(h*s3*(20*s1*s3-30*s1*s2+20*s2*s3-15*s1*s3^2-15*s2*s3^2-15*s3^2+12*s3^3+20*s1*s2*s3))/(60*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h*s3^3*(10*s1*s2-5*s1*s3-5*s2*s3+3*s3^2))/(60*(s2-1)*(s1-1)*(s3-1))*f(4)$

$yp16 = w0+(f0*h*(5*s1+5*s2+5*s3-10*s1*s2-10*s1*s3-10*s2*s3+30*s1*s2*s3-3))/(60*s1*s2*s3)-(h*(10*s2*s3-5*s3-5*s2+3))/(60*s1*(s1-1)*(s1-s3)*(s1-s2))*f(1)+(h*(10*s1*s3-5*s3-5*s1+3))/(60*s2*(s2-1)*(s2-s3)*(s1-s2))*f(2)-(h*(10*s1*s2-5*s2-5*s1+3))/(60*s3*(s3-1)*(s2-s3)*(s1-s3))*f(3)+(h*(15*s1+15*s2+15*s3-20*s1*s2-20*s1*s3-20*s2*s3+30*s1*s2*s3-12))/(60*(s3-1)*(s2-1)*(s1-1))*f(4)$

$fr1 = (yp5)^2-yp1*yp13-4*(x(1))^2+exp(x(1))*(1-4*x(1)+(x(1))^2);$

$fr2 = (yp6)^2-yp2*yp14-4*(x(2))^2+exp(x(2))*(1-4*x(2)+(x(2))^2);$

$fr3 = (yp7)^2-yp3*yp15-4*(x(3))^2+exp(x(3))*(1-4*x(3)+(x(3))^2);$

$fr4 = (yp8)^2-yp4*yp16-4*(x(4))^2+exp(x(4))*(1-4*x(4)+(x(4))^2);$

$yr1 = (w0*h^3*s1^3+3*v0*h^2*s1^2+6*z0*h*s1+6*y0)/6-(f0*h^4*s1^4*(28*s1*s2+28*s1*s3-168*s2*s3-8*s1^2*s2-8*s1^2*s3-8*s1^2+3*s1^3+28*s1*s2*s3))/(5040*s2*s3)+(h^4*s1^4*(14*s1*s2+14*s1*s3-42*s2*s3-6*s1^2*s2-6*s1^2*s3-6*s1^2+3*s1^3+14*s1*s2*s3))/(5040*(s1-1)*(s1-s3)*(s1-s2))*fr1+(h^4*s1^6*(28*s3-8*s1-8*s1*s3+3*s1^2))/(5040*s2*(s2-1)*(s2-s3)*(s1-s2))*fr2-(h^4*s1^6*(28*s2-8*s1-8*s1*s2+3*s1^2))/(5040*s3*(s3-1)*(s2-s3)*(s1-s3))*fr3+(h^4*s1^6*(28*s2*s3-8*s1*s3-8*s1*s2+3*s1^2))/(5040*(s3-1)*(s2-1)*(s1-1))*fr4$

$err1 = abs(((x(1)^2) + exp(x(1))) - yr1);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(1), ((x(1)^2) + exp(x(1))) + 2, yr1, err1))$

$yr2 = (w0*h^3*s2^3 + 3*v0*h^2*s2^2 + 6*z0*h*s2 + 6*y0)/6 + (f0*h^4*s2^4*(168* s1*s3 - 28*s1*s2 - 28*s2*s3 + 8*s1*s2^2 + 8*s2^2*s3 + 8*s2^2 - 3*s2^3 - 28*s1* s2*s3))/(5040*s1*s3) - (h^4*s2^6*(28*s3 - 8*s2 - 8*s2*s3 + 3*s2^2))/(5040* s1*(s1-1)*(s1-s3)*(s1-s2))*fr1 + (h^4*s2^4*(42*s1*s3 - 14*s1*s2 - 14* s2*s3 + 6*s1*s2^2 + 6*s2^2*s3 + 6*s2^2 - 3*s2^3 - 14*s1*s2*s3))/(5040*(s2-1)* (s2-s3)*(s1-s2))*fr2 + (h^4*s2^6*(8*s2 - 28*s1 + 8*s1*s2 - 3*s2^2))/(5040* s3*(s3-1)*(s2-s3)*(s1-s3))*fr3 - (h^4*s2^6*(8*s1*s2 - 28*s1*s3 + 8*s2* s3 - 3*s2^2))/(5040*(s3-1)*(s1-1)*(s2-1))*fr4$

$err1 = abs(((x(2)^2) + exp(x(2))) - yr2);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(2), ((x(2)^2) + exp(x(2))) + 2, yr2, err2))$

$yr3 = (w0*h^3*s3^3 + 3*v0*h^2*s3^2 + 6*z0*h*s3 + 6*y0)/6 - (f0*h^4*s3^4*(28* s1*s3 - 168*s1*s2 + 28*s2*s3 - 8*s1*s3^2 - 8*s2*s3^2 - 8*s3^2 + 3*s3^3 + 28*s1* s2*s3))/(5040*s1*s2) + (h^4*s3^6*(8*s3 - 28*s2 + 8*s2*s3 - 3*s3^2))/(5040* s1*(s1-1)*(s1-s3)*(s1-s2))*fr1 - (h^4*s3^6*(8*s3 - 28*s1 + 8*s1*s3 - 3*s3^2))/(5040*s2*(s2-1)*(s2-s3)*(s1-s2))*fr2 + (h^4*s3^4*(14*s1*s3 - 42*s1*s2 + 14*s2*s3 - 6*s1*s3^2 - 6*s2*s3^2 - 6*s3^2 + 3*s3^3 + 14*s1*s2* s3))/(5040*(s3-1)*(s2-s3)*(s1-s3))*fr3 + (h^4*s3^6*(28*s1*s2 - 8*s1* s3 - 8*s2*s3 + 3*s3^2))/(5040*(s2-1)*(s1-1)*(s3-1))*fr4$

$err3 = abs(((x(3)^2) + exp(x(3))) - yr3);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(3), ((x(3)^2) + exp(x(3))), yr3, err3))$

$yr4 = (w0*h^3 + 3*v0*h^2 + 6*z0*h + 6*y0)/6 + (f0*h^4*(8*s1 + 8*s2 + 8*s3 - 28*s1*s2 - 28*s1*s3 - 28*s2*s3 + 168*s1*s2*s3 - 3))/(5040*s1*s2*s3) - (h^4*(28*s2*s3 - 8*s3 - 8*s2 + 3))/(5040*s1*(s1-1)*(s1-s3)*(s1-s2))* fr1 + (h^4*(28*s1*s3 - 8*s3 - 8*s1 + 3))/(5040*s2*(s2-1)*(s2-s3)*(s1- s2))*fr2 - (h^4*(28*s1*s2 - 8*s2 - 8*s1 + 3))/(5040*s3*(s3-1)*(s2-s3)* (s1-s3))*fr3 + (h^4*(6*s1 + 6*s2 + 6*s3 - 14*s1*s2 - 14*s1*s3 - 14*s2*$

$s3+42*s1*s2*s3-3))/(5040*(s3-1)*(s2-1)*(s1-1))*fr4$

$err4 = abs(((x(4)^2)+exp(x(4)))-yr4);$

$fprintf('\%2.7f\ \%3.18f\ \%3.18f\ \%1.6\ \backslash n', x(4), ((x(4)^2)+exp(x(4))), yr4, err4))$

$x0 = x(4);\ y0 = yr4;\ z0 = yp8;\ v0 = yp12;\ w0 = yp16;$

end