ONE STEP HYBRID BLOCK METHODS WITH GENERALISED OFF-STEP POINTS FOR SOLVING DIRECTLY HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS.

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Abstrak

Permasalahan kehidupan nyata terutamanya dalam sains dan kejuruteraan boleh diungkapkan dalam persamaan pembeza untuk tujuan menganalisis dan memahami fenomena fizikal. Persamaan pembeza ini melibatkan kadar perubahan satu atau lebih pembolehubah tak bersandar. Masalah nilai awal persamaan pembeza biasa peringkat tinggi diselesaikan secara konvensional dengan menukarkan persamaan tersebut ke sistem persamaan pembeza biasa peringkat pertama yang setara terlebih dahulu. Kaedah berangka bersesuaian yang sedia ada kemudiannya digunakan untuk menyelesai persamaan yang terhasil. Walau bagaimanapun, pendekatan ini akan menambah bilangan persamaan. Akibatnya, kekompleksan pengiraan akan bertambah dan ianya boleh menjejaskan kejituan penyelesaian. Bagi mengatasi kelemahan ini, kaedah langsung digunakan. Namun, kebanyakan kaedah ini menganggar penyelesaian berangka pada satu titik pada satu masa. Oleh itu, beberapa kaedah blok diperkenalkan bertujuan untuk menganggar penyelesaian berangka pada beberapa titik serentak. Seterusnya, kaedah blok hibrid diperkenalkan bagi mengatasi sawar kestabilan-sifar yang berlaku dalam kaedah blok. Walau bagaimanapun, kaedah blok hibrid satu langkah sedia ada hanya tertumpu kepada titik pinggir-langkah yang spesifik. Oleh yang demikian, kajian ini mencadangkan beberapa kaedah blok hibrid satu langkah dengan titik pinggir-langkah teritlak bagi menyelesaikan persamaan pembeza biasa peringkat tinggi secara langsung. Dalam pembangunan kaedah ini, siri kuasa telah digunakan sebagai penyelesaian hampir kepada permasalahan persamaan pembeza biasa peringkat γ. Siri kuasa diinterpolasi pada γ titik sementara terbitannya yang tertinggi dikolokasi pada semua titik dalam selang terpilih. Sifat bagi kaedah baharu seperti peringkat, pemalar ralat, kestabilan-sifar, ketekalan, penumpuan dan rantau kestabilan mutlak juga turut dikaji. Beberapa masalah nilai awal persamaan pembeza biasa peringkat tinggi kemudiannya diselesaikan dengan menggunakan kaedah baharu yang telah dibangunkan. Keputusan berangka mendedahkan kaedah baharu menghasilkan penyelesaian yang lebih jitu berbanding dengan kaedah yang sedia ada apabila menyelesaikan masalah yang sama. Oleh itu, kaedah baharu adalah alternatif berdaya saing dalam menyelesaikan masalah nilai awal persamaan pembeza biasa peringkat tinggi secara langsung.

Kata kunci: Interpolasi, kolokasi, kaedah blok hibrid satu langkah, penyelesaian langsung masalah nilai awal peringkat tinggi, titik pinggir-langkah teritlak.

Abstract

Real life problems particularly in sciences and engineering can be expressed in differential equations in order to analyse and understand the physical phenomena. These differential equations involve rates of change of one or more independent variables. Initial value problems of higher order ordinary differential equations are conventionally solved by first converting them into their equivalent systems of first order ordinary differential equations. Appropriate existing numerical methods will then be employed to solve the resulting equations. However, this approach will enlarge the number of equations. Consequently, the computational complexity will increase and thus may jeopardise the accuracy of the solution. In order to overcome these setbacks, direct methods were employed. Nevertheless, most of these methods approximate numerical solutions at one point at a time. Therefore, block methods were then introduced with the aim of approximating numerical solutions at many points simultaneously. Subsequently, hybrid block methods were introduced to overcome the zero-stability barrier occurred in the block methods. However, the existing one step hybrid block methods only focus on the specific off-step point(s). Hence, this study proposed new one step hybrid block methods with generalised off-step point(s) for solving higher order ordinary differential equations. In developing these methods, a power series was used as an approximate solution to the problems of ordinary differential equations of order γ. The power series was interpolated at γ points while its highest derivative was collocated at all points in the selected interval. The properties of the new methods such as order, error constant, zero-stability, consistency, convergence and region of absolute stability were also investigated. Several initial value problems of higher order ordinary differential equations were then solved using the new developed methods. The numerical results revealed that the new methods produced more accurate solutions than the existing methods when solving the same problems. Hence, the new methods are viable alternatives for solving initial value problems of higher order ordinary differential equations directly.

Keywords: Interpolation, collocation, one step hybrid block method, direct solution, higher order initial value problems, generalised off-step point(s).

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CHAPTER ONE
INTRODUCTION

1.1 Background of the Study

Mathematicians develop mathematical models to help them understanding the physical phenomena in real life problems. These models frequently lead to equations involving some derivatives of an unknown function of single or several variables, which are called differential equations. Differential equations have vast application in many fields such as engineering, medicine, economics, operation research, psychology and anthropology.

There are two types of differential equation namely Ordinary Differential Equation (ODE) and Partial Differential Equation (PDE). ODE is a differential equation that has single independent variable, while PDE is differential equation with two or more variables (Omar & Suleiman, 1999). The general form of ODE on the interval $[a,b]$ is denoted as

$$y^{\gamma} = f(x, y, y', y'', \ldots, y^{\gamma-1}). \tag{1.1}$$

In order to solve the equation (1.1), the conditions stated below need to be imposed.

$$y(a) = \eta_0, \quad y'(a) = \eta_1, \ldots, y^{\gamma-1}(a) = \eta_{\gamma-1} \tag{1.2}$$

Equation (1.1) and equation (1.2) are called initial value problem(IVP). If there is another condition at the different value of x such as b, then it is called boundary value problem(BVP) (Lambert, 1973).

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