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ONE STEP HYBRID BLOCK METHODS WITH GENERALISED OFF-STEP POINTS FOR SOLVING DIRECTLY HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS.

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DOCTOR OF PHILOSOPHY
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2016
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Abstrak


Kata kunci: Interpolasi, kolokasi, kaedah blok hibrid satu langkah, penyeleasaian langsung masalah nilai awal peringkat tinggi, titik pinggir-langkah teritlak.
Abstract

Real life problems particularly in sciences and engineering can be expressed in differential equations in order to analyse and understand the physical phenomena. These differential equations involve rates of change of one or more independent variables. Initial value problems of higher order ordinary differential equations are conventionally solved by first converting them into their equivalent systems of first order ordinary differential equations. Appropriate existing numerical methods will then be employed to solve the resulting equations. However, this approach will enlarge the number of equations. Consequently, the computational complexity will increase and thus may jeopardise the accuracy of the solution. In order to overcome these setbacks, direct methods were employed. Nevertheless, most of these methods approximate numerical solutions at one point at a time. Therefore, block methods were then introduced with the aim of approximating numerical solutions at many points simultaneously. Subsequently, hybrid block methods were introduced to overcome the zero-stability barrier occurred in the block methods. However, the existing one step hybrid block methods only focus on the specific off-step point(s). Hence, this study proposed new one step hybrid block methods with generalised off-step point(s) for solving higher order ordinary differential equations. In developing these methods, a power series was used as an approximate solution to the problems of ordinary differential equations of order $\gamma$. The power series was interpolated at $\gamma$ points while its highest derivative was collocated at all points in the selected interval. The properties of the new methods such as order, error constant, zero-stability, consistency, convergence and region of absolute stability were also investigated. Several initial value problems of higher order ordinary differential equations were then solved using the new developed methods. The numerical results revealed that the new methods produced more accurate solutions than the existing methods when solving the same problems. Hence, the new methods are viable alternatives for solving initial value problems of higher order ordinary differential equations directly.

Keywords: Interpolation, collocation, one step hybrid block method, direct solution, higher order initial value problems, generalised off-step point(s).
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Ra’ft Abdelrahim

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CHAPTER ONE
INTRODUCTION

1.1 Background of the Study

Mathematicians develop mathematical models to help them understanding the physical phenomena in real life problems. These models frequently lead to equations involving some derivatives of an unknown function of single or several variables, which are called differential equations. Differential equations have vast application in many fields such as engineering, medicine, economics, operation research, psychology and anthropology.

There are two types of differential equation namely Ordinary Differential Equation (ODE) and Partial Differential Equation (PDE). ODE is a differential equation that has single independent variable, while PDE is differential equation with two or more variables (Omar & Suleiman, 1999). The general form of ODE on the interval \([a, b]\) is denoted as

\[
y^{(\gamma)} = f(x, y, y', y'', \ldots, y^{(\gamma-1)}).
\]

In order to solve the equation (1.1), the conditions stated below need to be imposed.

\[
y(a) = \eta_0, \quad y'(a) = \eta_1, \ldots, y^{(\gamma-1)}(a) = \eta_{\gamma-1} \quad (1.2)
\]

Equation (1.1) and equation (1.2) are called initial value problem (IVP). If there is another condition at the different value of \(x\) such as \(b\), then it is called boundary value problem (BVP) (Lambert, 1973).
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REFERENCES


