ROBUST LINEAR DISCRIMINANT ANALYSIS USING MOM-*Qn* AND WMOM-*Qn* ESTIMATORS: COORDINATE-WISE APPROACH

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Abstrak

Kaedah analisis diskriminan linear (RLDA) teguh menjadi pilihan yang lebih baik untuk masalah pengklasifikasi berbanding dengan analisis diskriminan linear (LDA) klasik disebabkan kemampuan kaedah tersebut dalam mengatasi isu titik terpencil. LDA klasik bergantung kepada penganggar lokasi dan skala yang biasa iaitu min sampel dan kovarians matriks. Sensitiviti penganggar ini ke arah data terpencil akan menjejaskan proses pengelasan. Untuk mengurangkan isu ini, penganggar teguh lokasi dan kovarians dicadangkan. Sehubungan itu, dalam kajian ini, dua RLDA untuk pengelasan dua kumpulan telah diubah suai menggunakan dua penganggar lokasi yang amat teguh yang dinamakan Penganggar-M satu langkah terubahsuai (MOM) dan Penganggar-M satu langkah terubahsuai terwinsor (WMOM). Satu penganggar skala yang amat teguh, *Qn*, disepadukan dalam kriteria pemangkasan MOM dan WMOM, menghasilkan dua RLDA yang baharu yang masing-masing dikenali sebagai $RLDA_{MQ}$ dan $RLDA_{WMQ}$. Dalam pengiraan RLDA yang baharu, min biasa digantikan dengan MOM-*Qn* dan WMOM-*Qn*. Prestasi kaedah RLDA baharu diuji ke atas data simulasi begitu juga data sebenar, dan seterusnya dibandingkan dengan LDA klasik. Bagi data simulasi, beberapa pemboleh ubah telah dimanipulasi untuk mewujudkan pelbagai keadaan yang sering berlaku dalam kehidupan sebenar. Pembolehubah tersebut ialah kehomogenan kovarians (sama dan tidak sama), saiz sampel (seimbang dan tidak seimbang), dimensi pembolehubah, dan peratus pencemaran. Secara umumnya, keputusan menunjukkan bahawa prestasi RLDA baharu adalah lebih baik daripada LDA klasik dari segi purata ralat kesilapan pengelasan, walaupun RLDA yang baharu mempunyai kelemahan iaitu memerlukan lebih banyak masa pengiraan. $RLDA_{MQ}$ memberi hasil yang terbaik pada saiz sampel seimbang manakala $RLDA_{WMQ}$ lebih baik dari yang lainnya pada keadaan saiz sampel tidak seimbang. Apabila data kewangan yang sebenar dipertimbangkan, $RLDA_{MQ}$ menunjukkan keupayaan dalam menangani data terpencil dengan ralat kesilapan pengelasan yang paling kecil. Sebagai penutup, kajian ini telah mencapai objektif utama iaitu untuk memperkenalkan RLDA baharu untuk mengklasifikasi data multi pembolehubah dua kumpulan dengan kehadiran titik terpencil.

Kata kunci: Ralat kesilapan pengelasan, Penganggar-M satu langkah terubahsuai, Data terpencil, Analisis diskriminan linear teguh, Terwinsor.

Abstract

Robust linear discriminant analysis (RLDA) methods are becoming the better choice for classification problems as compared to the classical linear discriminant analysis (LDA) due to their ability in circumventing outliers issue. Classical LDA relies on the usual location and scale estimators which are the sample mean and covariance matrix. The sensitivity of these estimators towards outliers will jeopardize the classification process. To alleviate the issue, robust estimators of location and covariance are proposed. Thus, in this study, two RLDA for two groups classification were modified using two highly robust location estimators namely Modified One-Step M-estimator (MOM) and Winsorized Modified One-Step M-estimator (WMOM). Integrated with a highly robust scale estimator, *Qn*, in the trimming criteria of MOM and WMOM, two new RLDA were developed known as $\mathrm{RLDA}_{\mathrm{MQ}}$ and $\mathrm{RLDA}_{\mathrm{WMQ}}$ respectively. In the computation of the new RLDA, the usual mean is replaced by MOM-*Qn* and WMOM-*Qn* accordingly. The performance of the new RLDA were tested on simulated as well as real data and then compared against the classical LDA. For simulated data, several variables were manipulated to create various conditions that always occur in real life. The variables were homogeneity of covariance (equal and unequal), samples (balanced and unbalanced), dimension of variables, and the percentage of contamination. In general, the results show that the performance of the new RLDA are more favorable than the classical LDA in terms of average misclassification error for contaminated data, although the new RLDA have the shortcoming of requiring more computational time. $\mathrm{RLDA}_{\mathrm{MQ}}$ works best under balanced sample sizes while $\mathrm{RLDA}_{\mathrm{WMQ}}$ surpasses the others under unbalanced sample sizes. When real financial data were considered, $\mathrm{RLDA}_{\mathrm{MQ}}$ shows capability in handling outliers with lowest misclassification error. As a conclusion, this research has achieved its primary objective which is to develop new RLDA for two groups classification of multivariate data in the presence of outliers.

Keywords: Misclassification Error, Modified One-Step M-Estimator, Outliers, Robust linear discriminant analysis, Winsorized.

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List of Abbreviations

MOM	Modified One-step M-estimator
WMOM	Winosrized Modified One-step M-estimator
CA	Classical Approach
Q_n	A scale estimator
CV	Cross- Validation
LDA	Linear Discriminant Analysis
MOM-Q_n	Modified One-Step M-Estimator with Q_n
WMOM-Q_n	Winsorized Modified One-Step M-Estimator with Q_n
$RLDA_{MQ}$	RLDA with MOM-Q_n
$RLDA_{WMQ}$	RLDA with WMOM-Q_n
QDA	Quadratic Discriminant Analysis
LR	Logistic Regression
RDA	Regularized Discriminant Analysis
MVE	Minimum Volume Ellipsoid
MCD	Minimum Covariant Determinant
MAD	Mean Absolute Deviation
PCA	Principal Component Analysis
RLDA	Robust Linear Discriminant Analysis
KPCA	Kernel Principal Component Analysis
CKFD	Complete Kernel Fisher Discriminant
KFD	Kernel Fisher Discriminant
LLDA	Locally Linear Discriminant Analysis

MODA	Multimodal Oriented Discriminant Analysis
$\mathrm{MAD_n}$	Median Absolute Deviation
S_n	A scale estimator
T_n	A scale estimator
LSE	Least-Squares Estimation
MSE	Mean Squared Error
AER	Apparent Error Rates

CHAPTER ONE
INTRODUCTION

1.1 Overview

Statistical classification techniques are basically of two types; cluster analysis and discriminant analysis. In cluster analysis, the rule to classify and the independent variables that describe the classification of the object are known but the category of the object is not known. Whereas, in discriminant analysis the object groups and several training examples of objects that have been grouped are known and the model of classification is also given. Discriminant analysis is one of the methods that give more information to the structure of multivariate data; which are data arising from variables greater than one (Fidler & Leonardis, 2003). The construction of a discriminant procedure comes from a training sample used for classifying every member of the sample. One of the primary objectives of discriminant analysis is to make inference about the unknown class membership of a new observation.

As stated in Chen and Muirhead (1994), distributional assumptions on the observation which involves the measurement of groups separately and the examination of the properties of the intended algorithms are the major root of statistical considerations in discriminant analysis. These rationales form the two stages of separation and allocation of the discriminant analysis. The separation stage is aimed to obtain functions known as discriminant functions which can conveniently make a separation of the groups, while the allocation stage involves assigning an unclassified object to one of the given groups using discriminant functions. On the other hand, the most crucial stage is the separation stage where the outcomes on the discriminant analysis are determined (Yan & Dai, 2011).

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Appendix A

Program Calculates the Value of the Robust Scale Estimator *Qn*

```
function Result=Qn(X)
[s1 s2]=size(X);
dist=zeros(s1,s2);
count=0;
for i=1:s1
    for j=1:s1
        if i<j
            count=count+1;
            dist(count,s2)=abs(X(i,s2)-X(j,s2));
        end
    end
end
sortdist=sort(dist);
h=floor(s1/2)+1;
k=nchoosek(h,2);
Result=sortdist(k,s2)*2.2219;
```

Appendix B

Programs for Calculates Modified One-Step M-Estimator $RLDA_{MQ}$ and Winsorized Modified One-Step M-Estimator $RLDA_{WMQ}$ Sample with the scale estimator Qn

1- Program calculates the $RLDA_{MQ}$

```
function Result=MOM_Qn_sample(Y)
[S1 S2]=size(Y);
if S2>1
   disp('error Only vectors not coulumns or Matrices');
return;
end

Med=median(Y);
QN= Qn(Y);
const = 2.24;
Low=-const*QN;
High=const*QN;
k=0;
for i=1:S1,
   if ((Y(i) - Med) >= Low) && ((Y(i) - Med) <= High)
      k= k+1;
   end
end
X = zeros(k,S2);
k=1;
for i=1:S1
   if ((Y(i) - Med) >= Low) && ((Y(i) - Med) <= High)
      X(i) = Y(i);
      k= k+1;
   else
      X(i)=nan;
   end
end

Result=X;
```

2- Program calculates the $RLDA_{WMQ}$

```
function Result=WQn_sample(Y)
[S1 S2]=size(Y);
if S2>1
   disp('error Only vectors not coulumns or Matrices');
return;
end

Med=median(Y);
QN= Qn(Y);
const = 2.24;
Low=-const*QN;
```

```
High=const*QN;
k=0;
for i=1:S1,
    if ((Y(i) - Med) >= Low) && ((Y(i) - Med) <= High)
        k= k+1;
    end
end
X = zeros(k,S2);
k=1;
for i=1:S1
    if ((Y(i) - Med) >= Low) && ((Y(i) - Med) <= High)
        X(i) = Y(i);
        k= k+1;
    end
end
Max = max(X);
Min = min(X);
for i=1:S1
if ((Y(i) - Med) < Low)
    X(i) = Min;
elseif((Y(i) - Med) > High)
    X(i) = Max;
end
end
Result=X;
```

Appendix C

Programs for Simulation Study

1- Programs for Simulation $RLDA_{MQ}$

```
function result = simulation_MOM_Qn
clear all;
start_time = cputime;

N1=2000;
N2=2000;
n1=20;
n2=20;
p1=2;
err = 0.4;
R=2000;

miscl = zeros(R,1);

for r=1:R

    seed1 = 12954+r;
    randn('seed',seed1);

    G1=randn(N1,p1);
    G2=1+2*randn(N2,p1);

    V1 = repmat(1:1, [N1 1]);
    V2 = repmat(2:2, [N2 1]);

    test_data=[G1 V1
              G2 V2];

    [n,p] = size(test_data);

    seed = 3984+r;
    randn('seed',seed);

    X1=[randn((1-err)*n1,p1)
        3+randn(err*n1,p1)];
    X2=[1+2*randn((1-err)*n2,p1)
        -2+2*(randn(err*n2,p1))];

    MS_Qn1 = zeros(n1,p1);
    MS_Qn2 = zeros(n2,p1);
    Qn_X1=zeros(1,p1);
    Qn_X2=zeros(1,p1);

 for i=1:p1
   MS_Qn1(1:n1,i) = MOM_Qn_sample(X1(1:n1,i));
```

```
  MS_Qn2(1:n2,i) = MOM_Qn_sample(X2(1:n2,i));
 end

   dim = p-1;
   a = log (n2/n1);

  for i=1:p1
   Qn_X1(i) = Qn(X1(1:n1,i));
   Qn_X2(i) = Qn(X2(1:n2,i));
  end

   Product_Qn_X1=Qn_X1'*Qn_X1;
   Product_Qn_X2=Qn_X2'*Qn_X2;

   mu1  = nanmean(MS_Qn1); mu2  = nanmean(MS_Qn2);
   cov1 = corr(X1,'type','Spearman').*Product_Qn_X1;
   cov2 = corr(X2,'type','Spearman').*Product_Qn_X2;

   sigma  = ((n1-1)*cov1+(n2-1)*cov2)/(n1+n2-2);
   linear   = (mu1-mu2)/sigma;
   constant = 1/2*linear*(mu1+mu2)';
   scores   = linear*test_data(1:n,1:dim)' - constant ;

   group  = (scores < a) + 1;
   miscl(r) = mean(group ~= test_data(:,p)');
end

end_time = cputime;

result.average_MOM_Qn_miscl =mean(miscl);
result.std_dev_MOM_Qn_miscl =std(miscl);
result.exec_time = end_time-start_time;
```

2- Programs for Simulation $RLDA_{WMQ}$

```
function result = simulation_WMOM_Qn
clear all;
start_time = cputime;

N1=2000;
N2=2000;
n1=50;
n2=20;
p1=2;
err = 0.4;
R=2000;

miscl = zeros(R,1);

for r=1:R
```

```
    seed1 = 12954+r;
    randn('seed',seed1);

    G1=randn(N1,p1);
    G2=1+2*randn(N2,p1);

    V1 = repmat(1:1, [N1 1]);
    V2 = repmat(2:2, [N2 1]);

    test_data=[G1 V1
               G2 V2];

    [n,p] = size(test_data);

    seed = 3984+r;
    randn('seed',seed);

    X1=[randn((1-err)*n1,p1)
        3+randn(err*n1,p1)];
    X2=[1+2*randn((1-err)*n2,p1)
        -2+2*(randn(err*n2,p1))];

    WG1 = zeros(n1,p1);
    WG2 = zeros(n2,p1);

for i=1:p1
   WG1(1:n1,i) = WQn_sample(X1(1:n1,i));
   WG2(1:n2,i) = WQn_sample(X2(1:n2,i));
end

    dim = p-1;
    a = log (n2/n1);

    mu1  = mean(WG1); mu2  = mean(WG2);
    cov1 = cov(WG1); cov2 = cov(WG2);

    sigma  = ((n1-1)*cov1+(n2-1)*cov2)/(n1+n2-2);
    linear   = (mu1-mu2)/sigma;
    constant = 1/2*linear*(mu1+mu2)';
    scores   = linear*test_data(1:n,1:dim)' - constant ;

    group  = (scores < a) + 1;
    miscl(r) = mean(group ~= test_data(:,p)');
end

end_time = cputime;

result.average_WMOM_Qn_miscl =mean(miscl);
result.std_dev_WMOM_Qn_miscl =std(miscl);
result.exec_time = end_time-start_time;
```

Appendix D

Programs for Real Data

1- Programs for Real Data $RLDA_{MQ}$

```
[n,p] = size(datafull);
   [N,P] = size(datafull);

   dim = p-1;
   Dim = P-1;

   X1 = datafull(datafull(:,p)==1,1:dim);
   X2 = datafull(datafull(:,p)==2,1:dim);
   n1 = size(X1,1);
   n2 = size(X2,1);
   a = log (n2/n1);
   MS_Qn1 = zeros(n1,dim);
   MS_Qn2 = zeros(n2,dim);
   Qn_X1=zeros(1,dim);
   Qn_X2=zeros(1,dim);

 for i=1:dim
   MS_Qn1(1:n1,i) = MOM_Qn_sample(X1(1:n1,i));
   MS_Qn2(1:n2,i) = MOM_Qn_sample(X2(1:n2,i));
 end

  for i=1:dim
   Qn_X1(i) = Qn(X1(1:n1,i));
   Qn_X2(i) = Qn(X2(1:n2,i));
  end

   Product_Qn_X1=Qn_X1'*Qn_X1;
   Product_Qn_X2=Qn_X2'*Qn_X2;

   mu1  = nanmean(MS_Qn1); mu2  = nanmean(MS_Qn2);
   cov1 = corr(X1,'type','Spearman').*Product_Qn_X1;
   cov2 = corr(X2,'type','Spearman').*Product_Qn_X2;

   sigma  = ((n1-1)*cov1+(n2-1)*cov2)/(n1+n2-2);
   linear   = (mu1-mu2)/(sigma);
   constant = 0.5*linear*(mu1+mu2)';
   scores   = linear*datafull(1:N,1:Dim)' - constant ;
   group  = (scores < a) + 1;
   miscl = mean(group ~= datafull(:,P)');
```

2- Programs for Real Data $RLDA_{WMQ}$

```
[n,p] = size(datafull);
   [N,P] = size(datafull);

   dim = p-1;
   Dim = P-1;

   X1 = data27(data27(:,p)==1,1:dim);
   X2 = data27(data27(:,p)==2,1:dim);
   n1 = size(X1,1);
```

```
    n2 = size(X2,1);
    a = log (n2/n1);
    WG1 = zeros(n1,dim);
    WG2 = zeros(n2,dim);

for i=1:dim
   WG1(1:n1,i) = WQn_sample(X1(1:n1,i));
   WG2(1:n2,i) = WQn_sample(X2(1:n2,i));
end

    mu1  = mean(WG1); mu2  = mean(WG2);
    cov1 = cov(WG1);  cov2 = cov(WG2);

    sigma  = ((n1-1)*cov1+(n2-1)*cov2)/(n1+n2-2);
    linear   = (mu1-mu2)/(sigma);
    constant = 0.5*linear*(mu1+mu2)';
    scores   = linear*datafull(1:N,1:Dim)' - constant ;
    group  = (scores < a) + 1;
    miscl = mean(group ~= datafull(:,P)');
```