A FAMILY OF CLASSES IN NESTED CHAIN ABACUS AND RELATED GENERATING FUNCTIONS

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Abstrak

Model abakus telah digunakan secara meluas untuk mewakili pemetakan bagi sebarang integer positif. Walau bagaimanapun, tiada kajian yang telah dilakukan untuk membangunkan manik abakus terkait dalam perwakilan bergraf bagi objek diskrit. Untuk mengatasi masalah keterkaitan, kajian ini tertumpu kepada pencirian n-objek terkait yang dikenali sebagai n-omino terkait, seterusnya menjana abakus rantai tersarang. Selanjutnya, sifat konsep teori bagi abakus rantai tersarang dibangunkan. Di samping itu, tiga jenis penjelmaan berbeza yang penting dalam pembinaan famili kelas turut dihasilkan. Fungsi penjana turut dirumuskan berdasarkan kelas ini dengan menggunakan pengangkaan objek kombinatorik (ECO). Dalam kaedah ECO, setiap objek diperoleh daripada objek yang lebih kecil dengan membuat pengembangan setempat. Pengembangan setempat ini dihuraikan dengan cara yang mudah melalui petua turutan. Kemudian petua turutan boleh diterjemahkan menjadi persamaan fungsian untuk fungsi penjana. Kesimpulannya, kajian ini berjaya menghasilkan perwakilan bergraf baru bagi abakus rantai tersarang yang dapat diaplikasikan dalam grid terhingga penjubinan.

Abstract

Abacus model has been employed widely to represent partitions for any positive integer. However, no study has been carried out to develop connected beads of abacus in graphical representation for discrete objects. To resolve this connectedness problem this study is oriented in characterising *n* - connected objects knows as *n* connected ominoes, which then generate nested chain abacus. Furthermore, the theoretical conceptual properties for the nested chain abacus are being formulated. Along the construction, three different types of transformation are being created that are essential in building a family of classes. To enhance further, based on theses classes, generating functions are also being formulated by employing enumeration of combinatorial objects (ECO). In ECO method, each object is obtained from smaller object by making some local expansions. These local expansions are described in a simple way by a succession rule which can be translated into a function equation for the generating function. In summary, this stud has succeeded in producing novel graphical representation of nested chain abacus, which can be applied in tiling finite grid.

Keywords: abacus, partition, n-connected omnioes, generating function

.

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In the name of Allah, the Most Gracious and the Most Merciful. Thank you Allah for it is from Your will and blessings that this research is able to be completed.

I would like to thank Professor Dr. Haslinda Ibrahim for her guidance in completing this thesis, along with Associate Professor Dr. Nazihah Ahmad who have provided clear and precise guidance on how to complete this thesis.

Lastly, I would also like to thank my parents and my son Anmar, as well as all my family members.

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List of Symbols

$\mu^{(e,r)}$	Connected Partition with e Columns and r Rows
$\mathfrak{N}$	Nested Chin Abacus
SR	Set-Row
SC	Set-Column
P_{ρ}^{Rec}	Sequence of Rectangular Nested Chain Abacus
P_{ρ}^{Rec-h}	Sequence of Rectangle Path Nested Chain Abacus
$SNC2$	Single Nested Chain Abacus Transformation
SNC	Singular Nested Chain Abacus Transformation
MNC	Multiple nested Chain Abacus
$\mathfrak{D}_{single}$	Singular Transformation Class
$\mathfrak{D}_{outer}$	Single Transformation Class, if $i = 1$
$\mathfrak{D}_{inner-i}$	Single Transformation Class, if $i > 1$
$\mathfrak{D}_{inner}$	Multi Transformation Classes
b_i	Number of Beads Positions in Chain i
$b_i^{'}$	Number of Empty Bead Positions in Chain i
$\mathfrak{I}^2$	Classes of Nested Chain Abacus with two Columns
$\mathfrak{N}_c$	Class of Nested Chin Abacus w.r.t columns
$\mathfrak{N}_r$	Class of Nested Chin Abacus w.r.t rows

Declaration Associated With This Thesis

Journal

1. Mohommed, E. F., Ahmad, N., Ibrahim, H. (2016). Intersection of Main James Abacus Diagram for the Outer Chain Movement with Length [1, 0, 0...]. Journal of Telecommunication, Electronic and Computer Engineering (JTEC), 8(8), 51-56 (scopus).
2. Mohommed, E. F., Ahmad, N., Ibrahim, H., Mahmood, A. S. (2016). Nested Chain Movement of length 1 of Beta Number in James Abacus Diagram. Global Journal of Pure and Applied Mathematics, 12(4), 2953-2969 (scopus).
3. Mohommed, E. F., Ibrahim, H. Ahmad, N. (2017). Enumeration of n-connected objects inscribed in an abacus. Journal of Algebra, Number Theory Applications, (scopus).
4. Mohommed, E. F., Ahmad, N., Ibrahim, H. (2017). Enumeration Class of Polyominoes Defined by Two Column. IOSR Journal of Mathematics (IOSR-JM), 13(1), 44-49 .

proceeding

1. Mohommed, E. F., Ibrahim, H., Mahmood, A. S., Ahmad, N. (2015, December). Embedding chain movement in James diagram for partitioning beta number. In AIP Conference Proceedings (Vol. 1691, No. 1, p. 040019). AIP Publishing.
2. Mohommed, E. F., Ibrahim, H., Ahmad, N., Mahmood, A. (2016, August). Embedding the outer chain movement for main partition of β-number with length [1, 0, 0,]. In AIP Conference Proceedings (Vol. 1761, No. 1, p. 020076). AIP Publishing.

CHAPTER ONE
INTRODUCTION

1.1 Introduction

The theory of partition is a fundamental area of number theory, it is concerning the representation of integer as sum of other integers. The theory of partition has been applied in many different areas such as combinatorics, statistical and particle physic. The partitions can be graphically represented with diagrams such as Ferrers diagram and Young diagram. Agraphical representation of partition is important in the partition theory because it can design and facilitate a visual structure of any shape in the form of discrete object. Henceforth, this thesis focuses on the use of graphical illustration of partition to develop a new design structure of connected ominoes. The beauty of this construction is further extended to be used in tiling fnite grid.

1.2 Graphical Representation of Partition

Diagrams are used to represent a partition of any positive integer. Since 1800s, the famous diagrams are the Ferrers diagram and the Young diagram (Benjamin & Quinn, 2003; Hardy & Wright, 1979). On the other hand, a James diagram or known as *e*-abacus uses a β-number to represent a sequence of non-decreasing integer numbers (Gyoja et al., 2010). Next, the concept of partition and graphical representation of the partition are reviewed.

Definition 1.2.1. *(Andrews, 1998) A partition of a positive integer, t, is a finite non-increasing sequence of non-negative integers* $(\mu_1, \mu_2, \ldots, \mu_n)$ *such that* $\sum_{i=1}^{n} \mu_i = t$ *and n is the number of parts of any partition.*

Example 1.2.2. $(5, 3, 3, 2, 1), (5, 5, 2, 2), (6, 4, 2, 1, 1), \cdots$ *are partitions of* $t = 14$. *If* $\mu = (5, 3, 3, 2, 1)$, *then* $n = 5$.

Normally, for repeating parts in a partition of the integers number exponent is used. So, $(5, 3, 3, 2, 1)$ can be written as $(5, 3^2, 2, 1)$.

Partitions can be represented graphically in several ways, such as: Ferrers diagram, Young diagram and abacus.

1. **Ferrers Diagram**: named after a 19^{th} century British mathematician, Ferrers (Carroll, 1867). This diagram represents a partition as patterns of dots, with the m^{th} rows having the same number of dots as the m^{th} terms in the partition. The graphical representation of the partition $(5, 3, 3, 2, 1)$ is shown in Figure1.1.

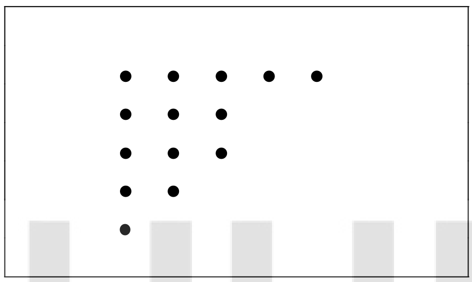


Figure 1.1. Partition $\mu = (5, 3, 3, 2, 1)$ in Ferrers diagram

2. **Young Diagram** An alternative visual representation of an integer partition, named after a 20^{th} century British mathematician, Alfred Young (Young, 1934). Rather than representing a partition with dots, as in a Ferrers diagram, a Young diagram uses boxes or squares to represent graphically the partition. An example for the partition $(5, 3, 3, 2, 1)$ is shown in Figure 1.2.

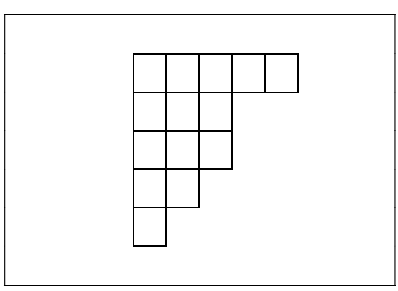


Figure 1.2. Partition $\mu = (5, 3, 3, 2, 1)$ in Young diagram

A Young diagram is also known the Ferrers diagram, particularly when representing using dots (Stanton & White, 1986; Abramovich, 2012). Thus, these two diagrams are always known as the Ferrers-Young diagram or as the Young-Ferrers diagram. Young diagram plays an important role in the drafting of the first step of many types of algebras particularly significant role in classifying the block of q-schur algebra and specific composition of positive integer numbers in which the sum is a non negative integer called t. Furthermore, Young diagram can represent the conjugate of the partition in a simple way. The conjugate of a partition is the partition that one obtains by merely changing the rows with columns of diagram (De Hoyos, 1990). Consider Example 1.2.1, the conjugate of the partition of $\mu = (5,3,3,2,1)$ is $\mu^* = (5,4,3,1,1)$, which is illustrated by the following Figure 1.3.

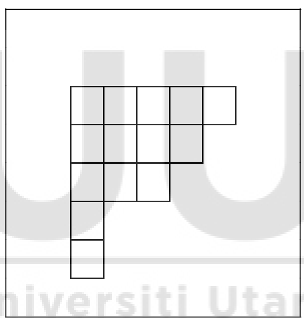


Figure 1.3. The conjugate of the partition of $\mu = (5,3,3,2,1)$

In addition, Young diagram provides a shortcut to find hook length μ_{kj} of column k and row j from the diagram where

μ_{kj} = the number of the boxes that are in row j to the right of μ_{kj} + the number of the boxes in column k below μ_{kj} + 1.

While the hook length formal is

$$\mu_{kj} = \mu_j - k + \mu_k^* - j + 1$$

where μ^* is a conjugate of partition μ and

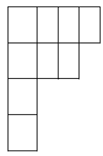
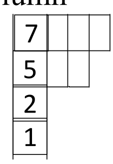
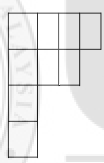
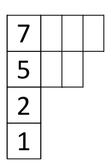
$$\mu_k^* = \{i : \mu_i > k\}$$

for $1 \leqslant k \leqslant b$ (Mathas, 1999).

Consider Example 1.2.1 Table 1.1 illustrates how to find the hook length μ_{kj} of first column using the Young diagram and the hook length formal $\mu_{11} = 7$,.

Table 1.1

Hook length μ_{kj} *of first column*

Partition	Young diagram	Formal
Without zero (4,3,1,1)	 The hook length of first column  $\mu_{k1} = \{\mu_{11}, \mu_{21}, \mu_{31}, \mu_{41}\}$ = {7,5,2,1}.	1-μ^* = (4,2,2,1). 2-$\mu_{k1} = \{\mu_{11}, \mu_{21}, \mu_{31}, \mu_{41}\}$ = {7,5,2,1}, where $\mu_{11} = \mu_1 - 1 + \mu_1^* - 1 + 1 = 7$, $\mu_{21} = \mu_2 - 1 + \mu_1^* - 2 + 1 = 5$, $\mu_{31} = \mu_3 - 1 + \mu_1^* - 3 + 1 = 2$, $\mu_{41} = \mu_4 - 1 + \mu^*{}^{1} - 4 + 1 = 1$.
With zero (4,3,1,1,0)	 The hook length of first column  $\mu_{k1} = \{\mu_{11}, \mu_{21}, \mu_{31}, \mu_{41}\}$ ={7,5,2,1}.	1-μ^* = (5,3,2,2,1). 2-$\mu_{k1} = \{\mu_{11}, \mu_{21}, \mu_{31}, \mu_{41}, \mu_{51}\}$ ={8,6,3,2,0}, where $\mu_{11} = \mu_1 - 1 + \mu_1^* - 1 + 1 = 8$, $\mu_{21} = \mu_2 - 1 + \mu_1^* - 2 + 1 = 6$, $\mu_{31} = \mu_3 - 1 + \mu_1^* - 3 + 1 = 3$, $\mu_{41} = \mu_4 - 1 + \mu_1^* - 4 + 1 = 2$, $\mu_{51} = \mu_5 - 1 + \mu_1^* - 5 + 1 = 0$.

Based on the Table 1.1 the Young diagram can not be used to represent the conjugate of the partition with one or more zeros then, it is necessary to extend represents partition idea to the case where partition any positive number has some zero parts. A new model called James abacus was constructed by using the theory of abaci (James, 1978).

1.2.1 James Abacus

At first, James depended on Young diagram to obtain abacus diagrm. To understand how James derived abacus diagram, the concept of rim shall be explained.

Definition 1.2.3. *(Fulton, 1997) A rim hook is a connected series of boxes of a Young diagram which located on the lower-right edge of a partition which produces a valid Young diagram when removed.*

In Figure1.4, the boxes marked with stars are a rim.

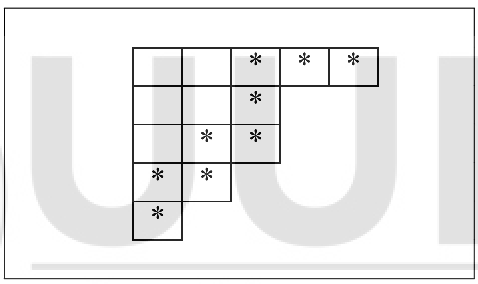


Figure 1.4. Rim location in Young diagram where $\mu = (5,3,3,2,1)$

By examining the Young diagram, James (1978) introduced a diagram with non - increasing positive integer numbers of which its sum is a positive integer number. In the beginning James employed the rim hook idea for the construction of abacus diagram. The representation of partition μ in the James abacus can be obtained by considering the partition μ in a Young diagram and starting from the south-west corner with '-' along its rim hook, towards the north-east corner. As the rim goes up pot 'o' and goes right, '-' is placed (James, 1987).

Thus, using the 'o' for bead position and '-' for empty bead position, Figure 1.4 can be transferred as seen in Figure 1.5. In order to find the James diagram of the partition

$(5, 3, 3, 2, 1)$, a rim hook in the Young diagram needs to be found as observed in Figure 1.5.

		o	-	-
		o		
	o	-		
o	-			
-				

Figure 1.5. Transformation of the rim location to bead and empty bead positions in the Young diagram where μ = (5,3,3,2,1)

Then, by taking rim hook from the bottom to the up and right of the corner, a sequence of bead and empty bead positions is obtained. After that, 'o' must be put at the end of the sequence as depicted as follows:

- o - o - o o - - o .

James introduced an additional property when there exists *e*, where *e* is the number of the abacus columns. Initially, James consider *e* as a prime number and then Fayers assume *e* as any positive integer number greater than or equal to 2 (Fayers, 2007). For instance, if we arrange the above sequence into two columns, it is represented as shown below.

-	o
-	o
-	o
o	-
-	o

Figure 1.6. James diagram when $\mu = (5, 3, 3, 2, 1)$, $e = 2$

If the Figure 1.6 is rearranged in 13 columns, then the diagram will consist of one row and is represented by

- o - o - o o - - o .

Later, James found a quick way to construct his abacus using a set of decreasing numbers called Beta numbers (β-number). Therefore, a set for a new abacus is defined in next section.

1.2.2 Beta Number

James found another way to represent abacus diagram by placing a bead at position of β-number.

Definition 1.2.4. *(Littlewood, 1951) A set of non-increasing of positive integers* $\{\beta_1, \beta_2, \beta_3, \cdots, \beta_b\}$ *is called* β *-number such that* $\beta_i = \mu_i + b - i$ *where* $1 \leq i \leq b$ *and b is greater than or equal to the number of the parts of* μ.

Example 1.2.5. *Consider* $\mu = (5, 3, 3, 2, 1)$ *in Example 1.2.2 then,* β *- numbers of* μ *are given as follows.*

$\beta_1 = \mu_1 + b - 1 = 5 + 5 - 1 = 9,$

$\beta_2 = \mu_2 + b - 2 = 3 + 5 - 2 = 6,$

$\beta_3 = \mu_3 + b - 3 = 3 + 5 - 3 = 5,$

$\beta_4 = \mu_4 + b - 4 = 2 + 5 - 4 = 3,$

$\beta_5 = \mu_5 + b - 5 = 1 + 5 - 5 = 1,$

then β *-number is* $\{9, 6, 5, 3, 1\}$.

Different β *-number generates different abacus.*

There is a suitable way of arranging β-numbers to each partition as an abacus with e columns such that e � 2. Therefore, the definition of abacus will be given as a preliminary step towards the James abacus.

An abacus refers to a counting frame that has been in use since centuries before the modern numeral system was adopted as a calculating tool. Aabacus are usually built as a bamboo frame with beads sliding on wires. It manually enhances the calculation consisting of up-down movable beads on sticks, as demonstrated Figure 1.7.

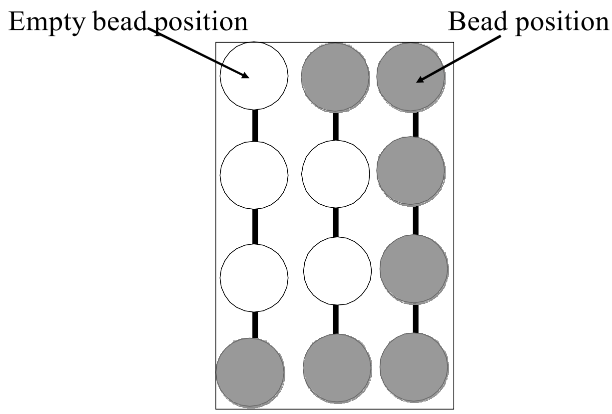


Figure 1.7. Abacus

We can associate to each partition an abacus diagram, this consist of columns which equal to e, numbered from left to right $0, 1, 2, \cdots e-1$ and positions on the abacus are numbered from right to left, working from top row to down, starting with 0, as shown in Figure 1.8.

0	1	$\cdots$	$e-1$
e	$e+1$	$\cdots$	$2e-1$
$2e$	$2e+1$	$\cdots$	$3e-1$
$3e$	$3e+1$	$\cdots$	$4e-1$
.	.	.	.

Figure 1.8. Abacus diagram

Figure1.9 gives an example of a James abacus when $e = 3$ and $e = 4$.

0	1	2
3	4	5
6	7	8
.	.	.

0	1	2	3
4	5	6	7
8	9	10	11
.	.	.	.

Figure 1.9. Abacus diagram for $e = 3$ and $e = 4$

The next step is to place the beads on the abacus in the corresponding positions where any β-number will represent a bead. Consider Example 1.2.5 where β-numbers are

$\{9, 6, 5, 3, 1\}$ so the position 1, 3, 5, 6, 9 are bead positions. Any bead can be represented as (o), while another empty bead position can be represented as (-).

-	o
-	o
-	o
o	-
-	o

a

-	o	-
o	-	o
o	-	-
o	-	-

b

Figure 1.10. James abacus for partitioned $\mu = (5, 3^2, 2, 1)$ when (a) $e = 2$ and (b) $e = 3$

The total elements of any non-increasing sequential of positive integers is a positive integer that does not change when a zero or any number of zeros is added to the sequence, but the β-number of any sequence will change when a zero or more is added. Thus, every partition can be represented by finite in a James abacus by adding zeros in the partition (James et al., 2006). Consider Example 1.2.5, then

- If $\mu = (5, 3, 3, 2, 1)$ then μ is a partition of 14 of $b_1 = n = 5$ where β- number $= \{9, 6, 5, 3, 1\}$.
- If $\mu = (5, 3, 3, 2, 1, 0)$ then μ is a partition of 14 of $b_2 = b + 1 = 6$ where β - number $= \{10, 7, 6, 4, 2, 0\}$.
- If $\mu = (5, 3, 3, 2, 1, 0, 0)$ then μ is a partition of 14 of $b_3 = b + 2 = 7$ where β - number $= \{11, 8, 7, 5, 3, 1, 0\}$.

The idea of a James abacus was based on the idea of an abacus which can easily distinguish the change of the places of the beads' position to which they belong and also the possibility of changing the order of the columns of the abacus. This feature has attracted many researchers to apply some of the beads' move.

1.2.3 James Abacus Development

About 20 years after James abacus was introduced, (James et al., 2006) established the abacus by adding one empty column to the James abacus called ($e+1$)-abacus. It was proven in a theorem related to the determination of the decomposition numbers for any e where e is a prime positive integer greater than or equal to 2. James and Mathas investigated several relationships between the original and the new partition. Fayers (2007) contracted a similar theorem but different weight of partition by adding full column to the James abacus. Furthermore, a procedure was described to remove a column from the James abacus to find a new abacus.

The parallelism idea of the previous theorem was used by considering of a finit composition of weight of the partition (Fayers, 2009). Fayers (2010) expands the use of the James abacus to compute the e-regularisation of a partition by implementating another movement. This movement is considered to be more complicated than the previous movement. A bead moving from b_k to b_{k-e} and a bead from S_{k-e} to b_k, $k=1, \dots, c$, where positions b_{k-e} and S_k are empty beads while positions b_{k-e} and S_k are beads as long as the volume $1, \dots, c$ is in order. The literature on James abacus shows the construction of a variety of diagrams based on the movement of the bead positions of James abacus, such as a work by Wildon (2008) how discovered a new way to find the conjugate of any partition by reflecting James abacus in its leading diagonal.

Afterwards Loehr (2010) constructed diagram W^* via move a set of beads which called scan movement. In this movement the beads will be moving one step to the right, then either a bead collides with another bead or no bead collision occurs. In case no bead collision occurs Abacus W^* used to proof the three pieri rules. Scan movement was the development in the case the bead jumps k positions to the right, which used to prove the combinatorial definition of Schur polynomials equivalent the algebraic definition of Schur polynomials (Loehr, 2011). Another movement of the bead to give a combinatorial proof of a plethstic generalization of the Murnaghan-Nakayama rule called single-step bead move. In this movement, all bead positions will be changing locations from position β to position $\beta - e$ above it in the same column such that $\beta - e \leqslant 0$ and $\beta - e$ is empty bead position (Wildon, 2014).

Tingley (2008) place all beads on the horizontal axis and then moves one bead exactly steps to the right in the corresponding row of beads, possibly jumping over other beads. Then, he put the beads into groups and rotating each group 90 degrees anticlockwise.

Meanwhile, Mahmood (2011) defined the main diagrams from a James abacus which give a new direction of study on James abacus. The work of Mahmood has been an eye opener for researchers to find main diagrams by applying movement in β- number in several translations in the James abacus beads to establish a new abacus.

The first of the translation is Upside Down. In this translation, the diagram is reflected in the line x and denoted as A'. The order of the James abacus rows will change as follows:

Rows numbered j in the original diagram become rows numbered $(m - j + 1)$ in new abacus diagram while rows numbered $(m - j + 1)$ in the original abacus diagram become rows numbered j in new abacus diagram. If r is odd, then rows numbered $\frac{r+1}{2}$ in the original abacus diagram become rows numbered $\frac{r+1}{2}$ in the new abacus diagram (Mahmood & Ali, 2013a) where r is the number of the abacus diagram rows and m=1,2,...,r such that

$$m = \begin{cases} 1,2,\cdots\frac{r}{2} & if \quad r \text{ is even,} \\ 1,2,\cdots\frac{r-1}{2} & if \quad r \text{ is odd.} \end{cases}$$

0	1	$\cdots$	$e-1$
e	$e+1$	$\cdots$	$2e-1$
$2e$	$2e+1$	$\cdots$	$3e-1$
$\vdots$	$\vdots$	$\cdots$	$\vdots$
me	$me+1$	$\cdots$	$re-1$

James diagram

me	$me+1$	$\cdots$	$re-1$
$\vdots$	$\vdots$	$\cdots$	$\vdots$
$2e$	$2e+1$	$\cdots$	$3e-1$
e	$e+1$	$\cdots$	$2e-1$
0	1	$\cdots$	$e-1$

Diagram A^I

Figure 1.11. Upside down

The second translation is right side-left. In this translation, the abacus diagram will be reflected in the line y and denoted as A^{II}. The order of the column in the original abacus diagram will changed as follows:

Column numbered r in the original abacus diagram becomes column numbered $(e-r+1)$ in new abacus diagram, where $e \geq 2$ and $r = 1, 2, .., (e-1)$ (Mahmood & Ali, 2013c).

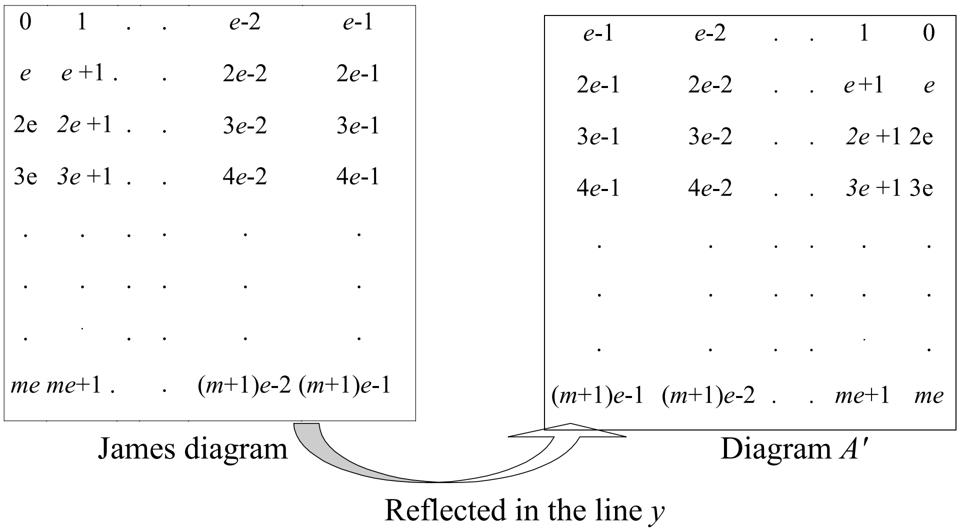


Figure 1.12. Right side-left

The third translation involves the direct rotation of β- number by considering different degrees, namely 90^o, 180^o and 270^o on the original James abacus. For 90^o, the new abacus diagram is denoted by A^{90}, for 180^o the new abacus diagram is denoted as A^{180} and for 270^o, the new abacus diagram is denoted as A^{270}, as follows:

- Diagram A^{90} is achieved by rotating the James abacus 90 degrees anticlockwise. In this case, the positions in the column h and row k in the James abacus become the positions in row $(r-h+1)$ and column k in diagram A^{90} where $h = 1, 2, \cdots, e$ and $k = 1, 2, \cdots, r$.
- Diagram A^{180} is achieved by rotating the James abacus anticlockwise 180 degrees. In this case the positions in row y and column w in the James abacus become positions in row $r - y + 1$ and column $(e - w + 1)$ in the Diagram A^{180} where $w = 1, 2, .., e$ and $y = 1, 2, .., r$.
- Diagram A^{270} is achieved by rotating the James abacus anticlockwise 270 degrees. In this case, the positions in row y and column w in original abacus

diagram become positions in row y and column $(e-w+1)$ in the Diagram A^{270} where $w=1,2,..,e$ and $y=1,2,..,r$.

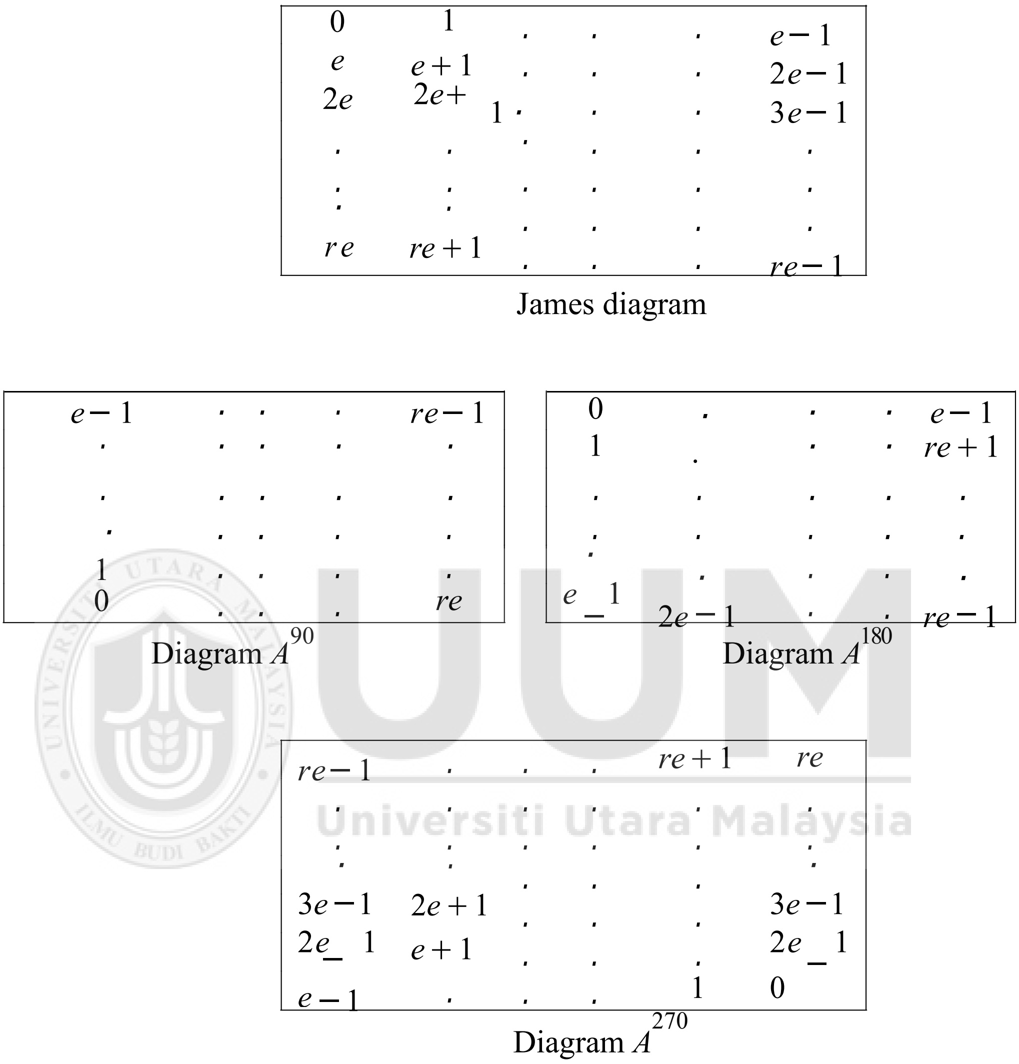


Figure 1.13. Direct rotation

Fourthly, the upside down and direct rotation translations are found in three abacus diagrams A^1, A^2 and A^3 this translation is composed of upside down and direct rotation applications of three different degrees namely, $90^o, 180^o, 270^o$ as follows:

- Diagram A^1, is achieved by applying the 4^{th} translation in 90^o anticlockwise, in which the position in row p and column L in the original abacus diagram become row L and column p in abacus diagram A^1 where $p = 1, 2, , e$ and $L = 1, 2, \cdots, r$.

- Diagram A^2 is achieved by applying the 4^{th} translation in 180^o anticlockwise; in this case, column number p in the James abacus is converted into column number $e - p + 1$ in abacus diagram A^2 where $p = 1, 2, \cdots, e$ and $L = 1, 2, \cdots, r$.
- Diagram A^3 is achieved by applying the 4^{th} translation of 270^o anticlockwise. In this case, all positions located in row number s and column number c in the original abacus diagram are converted into column numbered $(e - s + 1)$ and row $(r - c + 1)$ in abacus diagram A^3 where $s = 1, 2, .., r$ and $c = 1, 2, , e$ (Mahmood & Ali, 2013b).

The fifth movement will be exchanged with the rows by a fixed value y, respectively, to find diagram A_{x_y}. This moment is applied if $y = 1, 2, 3, \ldots$ (Sami, 2014).

King (2014) found a new diagram by applying Brauer algebra on James abacus along with consideration of the Temperley-Lieb algebra. A summary of the James abacus development is presented in Table 1.2

Table 1.2

A summary of the James abacus digram development by applying several movement

	Author-year	Movement	Name of diagram	Advantage
1	James et al-2006	Add empty column	e+1-abacus	Determination of decompositio number for any e in weight three.
2	Fayers-2007	Add full column	e+k-abacus	The decomposition number for an e in weight four are determined.
3	Wildon-2008	Reflecting in its leading diagonal	Conjugate diagram	The conjugate of any partition was found.
4	Fayers-2009	Removing a column	k- abacus	Determined the decompositio numbers for any e with infinit composition of weight.
5	Wildon-2014	single-step		Give combinatorial proof of plethystic generalization of th MurnaghanNakayama rule.
6	(Mahmood, 2013)	Upside down	Diagram A^{I}	The James diagram of b_1 was proved and this played a major rol to find all guides partition.
7	Mahmood& Ali-2013	Right side Left	Diagram A^{II}	Proof James diagram of b_1 and thi played a vital role to finding all guides partition.
8	Mahmood& Ali-2013	Direct rotation in line x by 90^o, 180^o, 270^o	Diagram A^1, A^2, A^3	James diagram of b_1 was prove and advantageous in finding a guides partition.
9	Mahmood& Ali-2013	Directrotation in line y by 90^o, 180^o, 270^o	Diagram A^{90}, A^{180}, A^{270}	James diagram of b_1 played a majo role to find all guides partition.
10	Mahmood& Ali-2014	Change rows	Diagram A_x	Help the managers to program th distribution of workers during a limited time.
11	King-2014	Brauer algebra		Find the number of permutation.

1.2.4 Advantages of James Abacus

The advantages of James diagram compared to a Young and Ferrers diagrams are as follows:

- It gives the quickest way of finding the first column hook lengths. In addition, every partition can be represented by infinite abacus diagrams (James, 1978).
- Useful to draw *e*-core of a partition. Furthermore, can be displayed to find addable and removable beads position (Fayers, 2007).
- Can be used to classify blocks of *I*wahori-Hecke algebras(James et al., 2006).
- Provide a quicker method of regularising a partition (Fayers, 2010).
- Every partition can be represented by infinite diagrams, *e* of this diagram are different (Mohammad, 2008).
- Represent conjugate to partition with zero.
- Played a major role in facilitating the understanding of many the concepts such as:
  - *e*- regular of any partition.
  - *e*- restricted of any partition.
  - *e*-hook of any partition.

The James abacus diagram is used widely in partition theory, graph theory and combinatorial to represent discrete objects which can be counted or be classified. In addition, it allows us to introduce different representations of a partition by employing different transformations in the abacus diagram. In this work, we will construct a new abacus with connected beads depending on James' abacus idea to represent discrete objects known as *n*-connected ominoes.

1.3 *n*-Connected Ominoes

An *n*-connected ominoes is a plane figure consisting of *n* ominoes connected from edge to edge. The *n*-connected ominoes have been utilized as a part of mainstream riddles since no less than 1907, and the identification of pentominoes (5-omenoes) is dated to antiquity (Surhone et al., 2010). Many results with the bits of 1 to 6 ominoes were initially distributed in Fairy Chess Review between the years 1937 to 1954 under the name of "dissection issues" (Golomb, 1954; Aval et al., 2014). Golomb (1954), proposed a connection of *n* squares adjacent edge to edge with a connected internal as polyominoes as well as a unit square as ominoes. In advancing in this field, Klarner (1966) defines a connectivity of finite number of unit square devoid of cut point as *n*-ominoes. This thesis considers the *n*-connected ominoes designed for finite connection of the unit square adjacent edge to edge with connected internal or with internal holes. The *n*-connected ominoes is an object of many combinatorial problems.

The two basic categories of these problems with *n*-connected ominoes are plane-tiling and enumeration. The first problem concerns, which figures can tile a plane, or rectangle, or parts of on plane (Beauquier & Nivat, 1990; Beauquier et al., 1995). The second category contains the problems on how to enumerate all *n*-connected ominoes or how many figures can *n* unit squares form. It is still an open question, but mathematicians tried to restrict on more special classes and so they were able to answer the question partially. The *n*-connected ominoes constitute one of the most popular subject in mathematical with long history starting of the 19th century. It have been studied for a long time in combinatorics, but they have also drawn the attention of physicists and chemists. The former in particular established a relationship with *n*-connected ominoes by defining equivalent objects named animals (Rechnitzer, 2001), obtained by taking the canter of the cells of a *n*-connected ominoes. These models allowed to simplify the description of phenomena like phase transitions (Barcucci et al., 2005) or percolation (Barequet et al., 2016), Ising model (Cipra, 1987) and polymer model (Gao & Wang, 2014).

1.3.1 The Representation of *n*-Connected Ominoes

The *n*-connected ominoes are classified according to the number of ominoes they have up to $n = 12$ (Golomb, 1954). In Table 1.3, represents the number of *n*-connected ominoes in each family (Redelmeier, 1981).

Table 1.3

Family of n-connected ominoes and the numbers in each family for $n = 1, 2, \ldots, 12$

n	Family	Number in family
1	monomino	1
2	domino	2
3	tromino	6
4	tetromino	19
5	pentomino	63
6	hexomino	216
7	heptomino	760
8	octomino	2725
9	nonomino	9910
10	decomino	36446
11	undecomino	135268
12	dodecomino	505861

However, for each $n > 1$ there exists many different shapes of *n*-connected ominoes that are associated with it. For example, for $n = 4$, the 4-connected ominoes has 4 different shapes as shown in Figure 1.14.

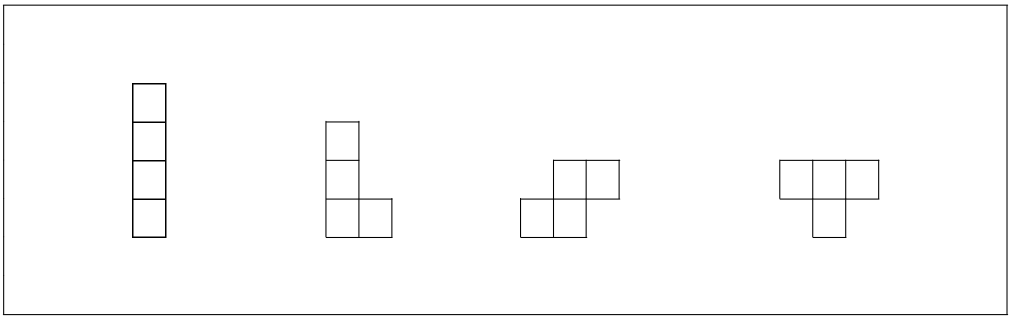


Figure 1.14. Family of tetromino (4-connected ominoes)

Table 1.3 displays the number of n-connected ominoes in each family. Since there are at least two different shapes in each family of n-connected ominoes for $n > 1$, another form of representation has been established in order to give a more specific expression to any form of n-connected ominoes by associating some shapes of the n-connected ominoes with letters as shown in Figure 1.15 (Berlekamp et al., 2003).

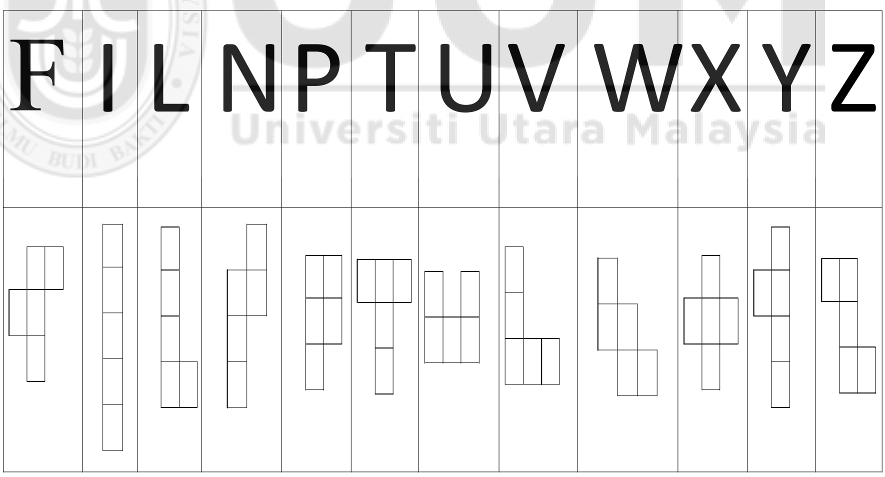


Figure 1.15. A 5-connected square in different shapes

Figure 1.15 shows 5-connected ominoes in different shapes, each represented as a letter of the English alphabet. However, not all shapes of the n-connected ominoes can be associated with letters. In Figure 1.16, the 11-connected ominoes cannot be represented as a letter.

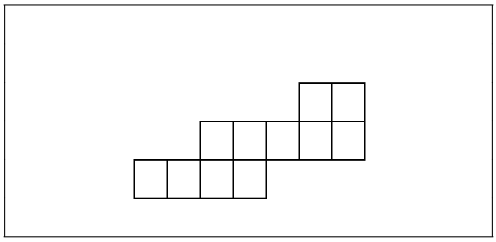


Figure 1.16. An 11-connected ominoes

A new form of representation describing *n*-connected ominoes using contour words has also been established (Berstel, 1985). A contour word is a finite set of ordered alphabets where each alphabet indicates a direction of either right-ward, left-ward, up-ward or down-ward. The contour word is used to describe a walk along the outline of the *n*-connected ominoes. The 11-connected ominoes in Figure 1.17 is represented by the contour word with four direction where x right, x^{I} left, y up and y^{I} down.

$$xxxxy\ xxxyyx\ x^{I}y^{I}x^{I}x^{I}\ x^{I}y^{I}x^{I}x^{I}$$

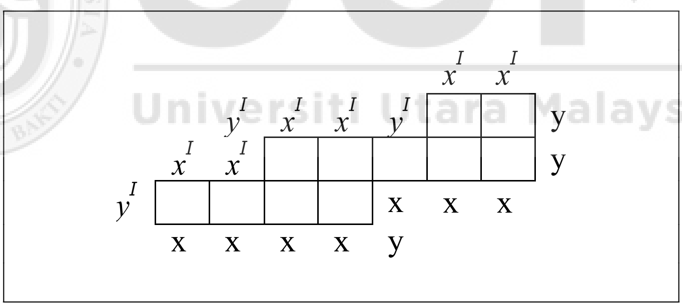


Figure 1.17. An 11-connected ominoes

However, there exists many contour words that can represent one *n*-connected ominoes. The 11-connected ominoes in Figure 1.17 can also be represented by

$$xxxx\ yxxx\ yyxx^{I}\ y^{I}x^{I}x^{I}\ x^{I}y^{I}x^{I}x^{I}\ xxxxyxx\ xyyxx^{I}$$

and by

$$y^{I}x^{I}x^{I}\ x^{I}y^{I}x^{I}x^{I}\ xxxxyxx\ xyyxx^{I}\ xxxx\ yxxx\ yyxx^{I}.$$

Therefore, there is no one to one mapping between the sets of n-connected ominoes and their contour words because there exist many contour words that can represent one shape of n-connected ominoes. Furthermore, since every n-connected ominoes has to be represented by at least one alphabet, the contour word will become very long if n is large. For example, the contour word for 35-connected ominoes in Figure 1.18 is represented by

$$xyxyxyxyxyxyx^{I}yx^{I}yx^{I}yx^{I}y^{I}x^{I}y^{I}x^{I}y^{I}x^{I}y^{I}x^{I}y^{I}x^{I}y^{I}xy^{I}xy^{I}y^{I}.$$

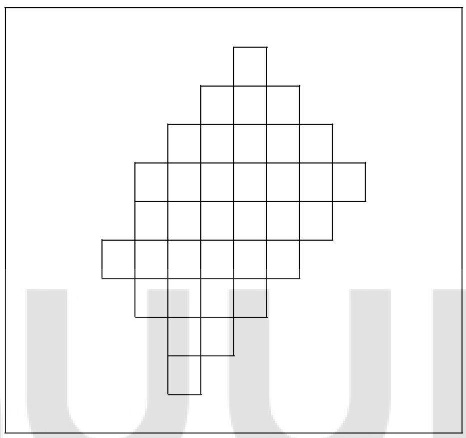


Figure 1.18. A 35-connected ominoes

The contour word has provided a better representation for each n-connected ominoes. However, we cannot use a contour word to represent the n-connected ominoes that contains at least one hole such as the 26-connected ominoes shown in Figure 1.19.

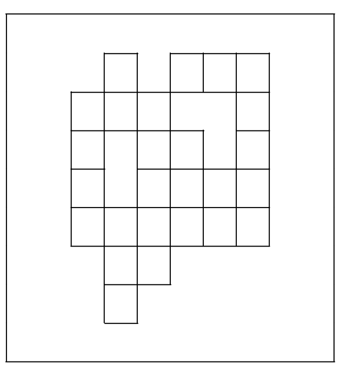


Figure 1.19. A 26-connected ominoes with two hole

The geometric properties which used to represent the *n*-connected ominoes are called column-convex if every intersection of the *n*-connected ominoes is connected with a vertical line is , and are called row-convex if every intersection of the *n*-connected ominoes with a horizontal line is connected. The *n*-connected ominoes are called convex if they are both column-convex and row-convex (Bender, 1974; Barcucci et al., 1997; Del Lungo et al., 2004; Guttmann & Enting, 1988). In addition, the *n*-connected ominoes is a directed if there exists a cell, called the root or source, from which all other cells can be reach by a path (Barcucci et al., 1996, 2005; Castiglione & Restivo, 2003; Duchi et al., 2008).

Further, Chow & Ruskey (2009) used Gray codes to represent a family of column-convex as shown in Figure 1.20.

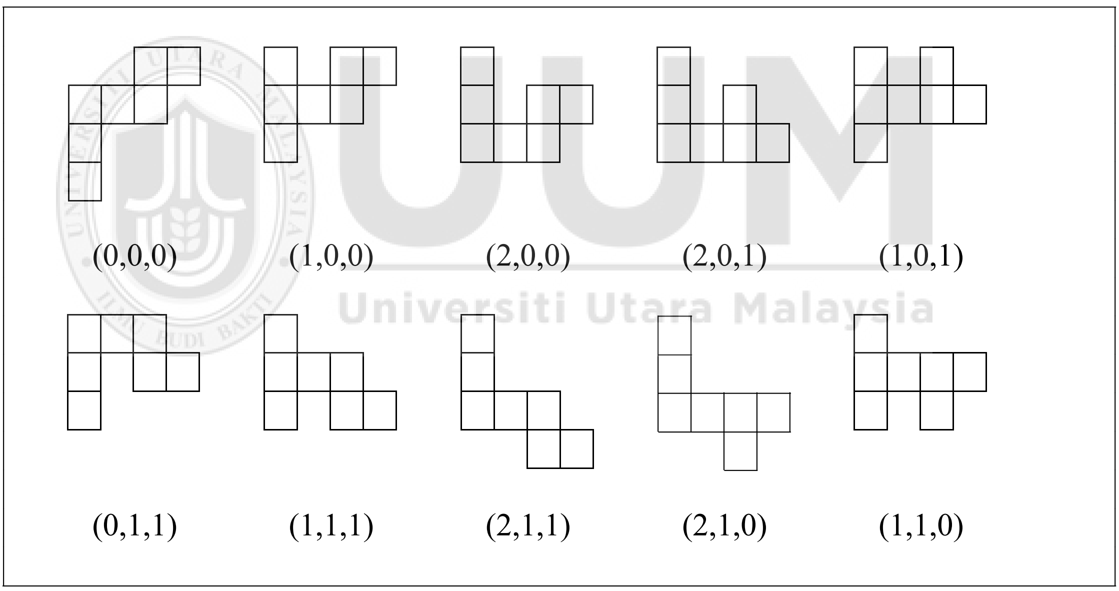


Figure 1.20. Twelve [3,1,2,1]-ominoes and their Gray codes

The concept of partition as a mathematical expression for *n*-connected omenoies has been used in a Young diagram (Rechnitzer, 2001). However, it has only been used for a special type for *n*-connected omenoies. The Young diagram refers to the arrangement of *n*-connected omenoies in left justified rows with lengths in non-increasing order as shown in Figure 1.21 for the partition $\mu = (5, 4, 3^2, 1)$. Previous studies on connected ominoes have managed to give

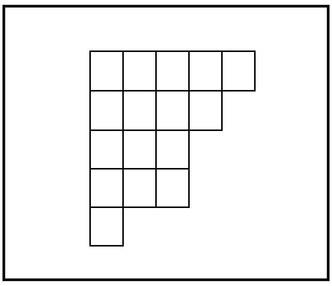


Figure 1.21. Young diagram of partition $\mu = (5, 4, 3^2, 1)$

Previous studies on different shapes of connected ominoes have managed to give representation to some classes of n-connected ominoes. Some studies have even managed to give representation to specific class of n-connected ominoes. However, the representation for each and every shape of n-connected ominoes remains unknown.

1.4 Research Motivation

James abacus was used as a graphical representation for any partition of positive integers. It was able to represent partitions in simple ways and can illustrate the finding of addable and removable bead positions. The idea of the abacus has been used widely to solve several problems which cannot be solved using Young or Ferrers diagrams such as:

- To give a graphical representation in cases that do not need arrangements for a non-increasing order.
- To represent the partition in cases μ has some zero parts.
- To display to find addable and removable bead position.

However, Beta numbers in James abacus are not necessary to be connected since it may have empty columns and rows. Moreover, adding one or more columns (rows) will make each partition have many representations. Henceforth, this work focuses on the development of n-connected ominoes. Hence to the best of our knowledge, no one used n-connected omines to give a mathematical model of an abacus. Thus, this study amalgamated the concept of abacus and n-connected ominoes to fill in the gap of the aforementioned scenario in the James abacus model. The newly proposed diagram can then hopefully answer the following questions:

1. How will newly proposed abacus represent n-connected ominoes?
2. What are the classes for newly proposed abacus diagrams based on the formulated properties?
3. How will the structure in newly proposed abacus be employed to construct generating functions?
4. How will the newly proposed abacus be used in related applications?

Thus, an extension of interest to determines n-connected ominoes in James' abacus diagram is deemed significant to visualize the idea.

1.5 Research Objectives

The main objective of this study is to develop a new abacus to represent the n-connected ominoes, which will be called nested chain abacus, and consider the problem of constructing and enumerating a family of classes of n-connected ominoes. In order to accomplish the main objective, the following sub-objectives must be fulfilled:

1. To establish and prove a conceptual framework for the new abacus, including definitions and theories.
2. To construct a new algorithm for the nested chain abacus.
3. To formulate and prove a conceptual framework for nested chain abacus transformation, including definitions and theories.
4. To construct a family of classes associated with transformations in (3) and partition properties.
5. To develop a generating function of n-connected ominoes as represented in (4).
6. To apply the newly constructed abacus in tiling field.

1.6 Scope of Study

This study focuses on developing a mathematical model of an abacus and using the structure of the new developments to propose three different types of transformation. In addition, a families of classes for the nested chain abacus are also being proposed. Subsequently, formulas for generating function of classes of nested chain abacus are developed.

1.7 Thesis Outline

This thesis contains seven chapters, the contribution of this thesis is presented in Chapter Two. We beginning with Chapter One which provides the overview of this study including the introduction, research background, research motivation, research of study, scope of study and thesis outline.

Next, in Chapter Two a new algorithm to represent abacus, called nested chain abacus using n-connected ominoes is constructed. First, we established a graphical form of n-connected ominoes. Then, we used the new development to propose representation of n-connected ominoes. In addition, we developed the properties of the nested chain abacus, such as how the beads are connected and the formation of the chain structures. The design structures of nested chain abacus are also represented. Finally, two different types of sequences related to the different design structures of the nested chain abacus were developed.

In Chapter Three we focus on the constructing an algorithm for the nested chain abacus transformation which is fundamental for enumerating classes of the nested chain abacus that will be presented in Chapter Four. Therefore, three different types of transformations in the chains are formulated in rectangle chain, path chain and in singleton chain. This is followed by the development of three types of nested chain abacus transformation: single nested chain abacus transformation, stratum nested chain abacus transformation and multi nested chain abacus transformation.

Chapter Four presents types of nested chain abacus defined with respect to chains and to formulation new classes. In addition the chain concept and two families of sequences presented in Chapter Two will be used to obtain the generating function.

The family of classes of nested chain abacus are developed in Chapter Five by using two methods, namely *e*-core and spinal design. In this chapter, we develop a formula to obtain generating functions.

In Chapter Six, we consider the problem of tiling in finite region using the nested chain abacus. Two algorithms for tiling in a finite region using two classes of nested chain abacus with respect to columns and rows are developed.

Lastly, Chapter Seven summarizes the study and provides some suggestions for future work. The framework of this study is presented in Figure 1.22

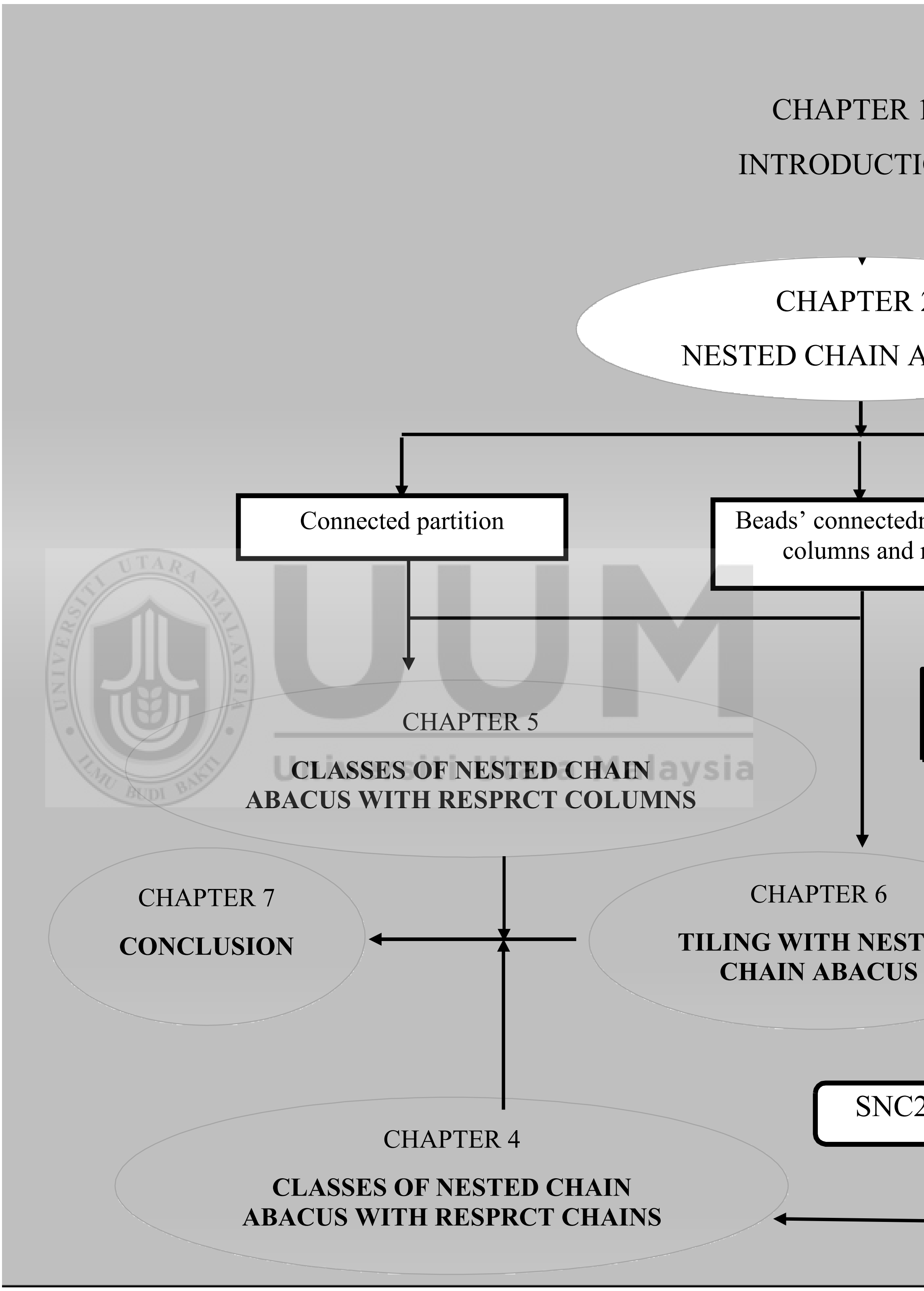


Figure 1.22. Research Framework

CHAPTER TWO
NESTED CHAIN ABACUS

2.1 Introduction

In this chapter, a new abacus called nested chain abacus is constructed using combinatorial analysis. First, we established a graphical form of *n*-connected ominoes in a minimal frame. Then, we represented a connected partition for the nested chain abacus where the concept of the partition is to provide a representation for each *n*-connected ominoes. In addition, we developed the properties of the nested chain abacus such as how the beads are connected and the formation of the chain structures. We then construct a rectangular, rectangle-path and singleton nested chain abacus as well as its various properties. Finally, we generated a family of sequences related to the different design structures of the nested chain abacus. This chapter focuses on the fundamental concept of the nested chain abacus to establish a family of classes in the following chapters.

We begin by providing some basic definitions used in Section 2.2. In Section 2.3 the algorithm of the nested chain abacus constructed is presented; then, the theoretical concept of the uniqueness of the nested chain abacus is formulated and then proven. Meanwhile, in Section 2.4 we present the definitions on the connectedness of the nested chain abacus with respect to the rows and columns. In addition, in Section 2.5, the design structure of the nested chain abacus is presented. Then, results from the design structure are developed.

2.2 Definition and Terminologies

This section provides the definitions needed to develop the nested chain abacus.

Definition 2.2.1. *An n-connected ominoes is a plane geometric figure formed by one* *or more ominoes, such that there exists a path from one ominoes to another for any pair of ominoes.*

Figure 2.1 illustrates Definition 2.2.1 for 7-connected ominoes

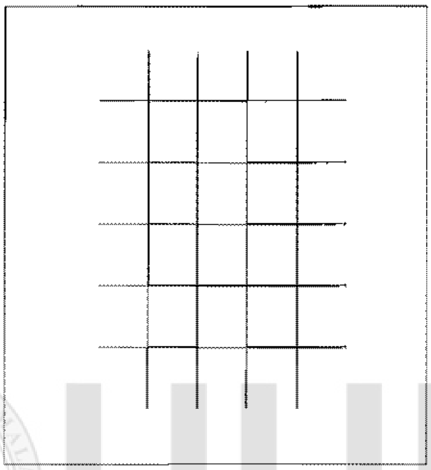


Figure 2.1. A 7-connected ominoes

For the remained of this section, we construct the definitions needed to develop the graphical form of n-connected ominoes and graphical examples are given to demonstrate the definitions. We start by defining a minimal frame of n-connected ominoes.

Definition 2.2.2. *A minimal frame is a minimal rectangle containing the n-connected ominoes itself, such that there is at least one omino in each column and each row where* $n \leqslant re$.

Consider Figure 2.1 where the minimal frame of 7-connected ominoes is as illustrated in Figure 2.2

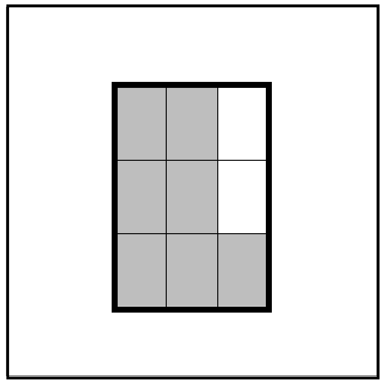


Figure 2.2. A 7-connected ominoes in a minimal frame

Definition 2.2.3. *Let n-connected ominoes in a minimal frame have e columns and r rows. The function* $f(m, j) : Z \times Z \to Z$ *such that, if location* (m, j) *in the minimal frame contains an ominoes, then*

$$w_k = f(m, j) = (m-1)e + (j-1)$$

for $k = 1, 2, , n$ *where e and r refer to the number of the rows and columns of the minimal frame respectively, for* $1 \leq j \leq e$ *and* $1 \leq m \leq r$.

Definition 2.2.4. *A nested chain abacus is an abacus with e columns, r rows and n of connected bead positions which satisfy the following conditions:*

1. *The columns are labelled from left to right as* $0, 1, \ldots, e-1$.
2. *The rows are labelled from up to down as* $0, 1, \ldots, r-1$.
3. *The connected bead locations are labelled with numbers* $0, 1, \ldots, er-1$ *across the rows from left to right beginning from the number 0 in the top-leftmost location until the number* $er-1$ *is in the bottom-rightmost location.*
4. *Each column and row has at least one bead position.*

Table 2.1 corresponding placement of position numbers on the nested chain abacus with e columns number from 0 to $e-1$ and r rows number from 0 to $r-1$.

Table 2.1

Placement of position numbers on the nested chain abacus with $e \times r$ positions

col.0	col.1	col.2	...	col. $e-1$
0	1	2	...	e-1
e	e+1	e+2	...	2e-1
.	.	.	...	.
.	.	.	...	.
.	.	.	...	.
	$e(r$-2)	$e(r$-2)+1	$e(r$-2)+2	...
$e(r$-1)-1				
$e(r$-1)	$e(r$-1)+1	$e(r$-1)+2	...	re-1

Definition 2.2.5. *A connected partition $\mu^{(e,r)}$ is a sequence of non-increasing positive integer numbers $(\mu_1, \mu_2, ..., \mu_n)$ such that μ_b represents to a connected bead positions with e columns and r rows, where $1 \leqslant b \leqslant n$.*

Repeated entries of partition $(\mu_1, \mu_2, ..., \mu_n)$ can be combined and exponents are used to represent the partition of positive integer numbers. So, $(\mu_1^{\tau_1}, \mu_2^{\tau_2}, ..., \mu_b^{\tau_b})$ is a partition such that $\sum_{b'=1}^{b} \tau_{b'} = n$. Next, we will construct an algorithm for the connectedness of bead positions in nested chain abacus.

2.3 Nested Chain Abacus

This section provides a graphical form of nested chain abacus and a unique connected partition $\mu^{(e,r)}$ for any n-connected ominoes.

We begin by discussing the graphical form of n-connected ominoes with respect to a minimal frame, which enables us to define ominoe positions and empty positions in terms of rows and columns of the minimal frame. Then, we redefine ominoe positions and empty ominoe positions as bead positions and empty bead positions, respectively, which enables us to apply the concept of beads on an abacus to represent the n-connected ominoes on a nested chain abacus. This is followed by the theoretical discussion on the concept of partition representation for the nested chain abacus for n-c onnected ominoes. For the following, we use $n = 7$ to explain the algorithm for nested chain abacus.

Step 1: Establishing a graphical form of *n*-connected ominoes

1. Form a graph of n-connected ominoes.
2. Identify the first column (leftmost row), last column (rightmost row), first row (topmost column) and last row (bottommost column) with at least one ominoe as a minimal frame. We numbered the columns from the leftmost, working from left to right 1,2,...,e and numbered the rows from topmost to bottommost 1,2,. ,r in the minimal frame where r and e number of rows and columns respectively.

Consider Figure 2.1 for the 7-connected ominoes in a minimal frame with 3 rows and 3 columns as shown in Figure 2.2.

Step 2: Creating a direction:

In this step, we created a direction to obtain a nested chain abacus with respect to the minimal frame.

Identify the first omino which located in the top-leftmost, from left to right, working down from the top-leftmost ominoe to the bottom in the minimal frame.

Consider the Figure 2.3 of 7-connected ominoes, beginning from bead position A which located in row 1 and column 1, then bead position B in row 1 and column 2. The third, fourth,.... positions for the minimal frame for 7-connected ominoes are substituted with the remaining positions C, D, E, F and G. Subsequently, we can also observe that there are two empty positions: The first empty position is in row 1 and column 3, and the second in row 2 and column 3.

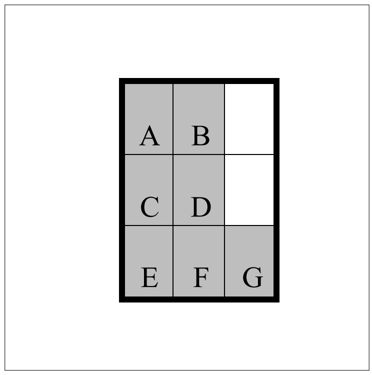


Figure 2.3. Direction of 7-connected ominoes

Step 3: Creating connected bead positions

In this step, we followed Definition 2.2.3 to create bead positions on the nested chain abacus. According to Step 2, we begin at the top-leftmost ominoe of the minimal frame and the rest of the ominoes.

Consider 7-connected ominoes in Figure 2.3, for finding w_k for $k = 1$, in which we inspected the location in row 1 and column 1. Since this location contains omino A, we calculate w_1 by applying the function f where $m = 1$ and $j = 1$. Then,

$$w_1 = f(1,1) = (1-1)3 + (1-1) = 0. \tag{2.1}$$

After that, we increment k by 1 so that the next application of the function f would yield w_2 where

$$w_2 = f(1,2) = (1-1)3 + (2-1) = 1. \tag{2.2}$$

The inspection process is continued in the same manner and subsequently, function f would only be applied accordingly to obtain $w_3 = 3$, $w_4 = 4$, $w_5 = 6$, $w_6 = 7$ and $w_7 = 8$. Consider Definition 2.2.4 in which the positions of the 7 beads in the nested chain abacus is as shown in Figure 2.4.

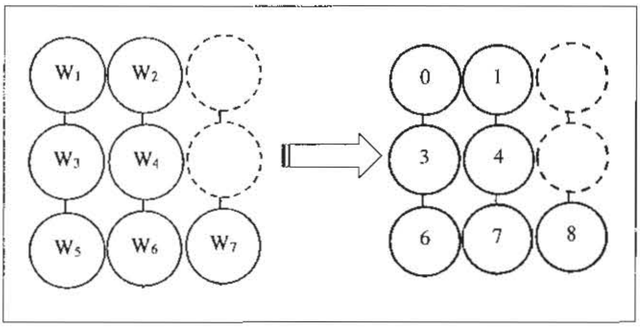


Figure 2.4. Nested chain abacus with 7-connected beads

Step 4: Constructing a connected partition of the nested chain abacus

Using the w_k's obtained from Step 3, we produce a partition called connected partition which represents the nested chain abacus with n beads, e columns and r rows for $k = 1, 2, ..., n$. The transformation process of the w_k's into connected partition $\mu^{(e,r)}$ is as follows:

$\mu_n = w_1$, $\mu_{n-g} = w_{g+1} - w_g + \mu_{n-g+1} - 1$ where $1 \leqslant g \leqslant n-1$.

Then, $\mu^{(e,r)} = (\mu_1^{t_1}, \mu_2^{t_2}, ..., \mu_b^{t_b})$ is a connected partition with e columns and r rows where $\mu_1 \geqslant \mu_2 \geqslant ... \geqslant \mu_b$ and $\sum_{b'=1}^{b} t_{b'} = n$.

Consider Figure 2.4 where from Steps 1, 2 and 3 we found that

$\{w_1, w_2, w_3, w_4, w_5, w_6, w_7\} = \{0, 1, 3, 4, 6, 7, 8\}$,

we find connected partition $\mu^{(3,3)}$ is as follows

$\mu_7 = w_1 = 0$,

$\mu_6 = w_2 - w_1 + \mu_7 - 1 = 0$,

$\mu_5 = w_3 - w_2 + \mu_6 - 1 = 1$,

$\mu_4 = w_4 - w_3 + \mu_5 - 1 = 1$,

$\mu_3 = w_5 - w_4 + \mu_4 - 1 = 2$,

$\mu_2 = w_6 - w_5 + \mu_3 - 1 = 2$,

$\mu_1 = w_7 - w_6 + \mu_2 - 1 = 2$.

Then, the connected partition representing to 7-connected ominoes in Figure 2.1 is

$\mu^{(3,3)}$=(2, 2, 2, 1, 1, 0, 0). Repeated entries can be combined and exponents are used to represent partition of positive integer numbers. So, $\mu^{(3,3)} = (2^3, 1^2, 0^2)$.

Based on the algorithm discussed earlier, we present a unique expression for n-connected omenoes as shown in Theorem 2.3.3. To prove it, we need Proposition 2.3.1 and Lemma 2.3.2.

Proposition 2.3.1. *Let $f : Z \times Z \to Z$ be defined by $f(m, j) = e(m-1) + (j-1)$. Function f is an injective function where e and r are positive integers for $1 \leqslant m \leqslant r$ and $1 \leqslant j \leqslant e$.*

Proof. Let (m, j) and (m', j') be ominoes positions where m, j, m' and j' are positive integers for $1 \leqslant m' \leqslant r$ and $1 \leqslant j' \leqslant e$. Suppose that $f(m, j) = f(m', j')$.
Case one: If $m = m'$, $f(m, j) = f(m', j')$ then $e(m-1) + (j-1) = e(m'-1) + (j'-1)$. Since $m = m'$ then, $(j-1) = (j'-1)$. Thus, $j = j'$.
Case two: If $j = j'$, $f(m, j) = f(m', j')$ then $e(m-1) + (j-1) = e(m'-1) + (j'-1)$. Since $j = j'$, then, $e(m-1) = e(m'-1)$. Thus, $m = m'$.
Case three: If $m \neq m'$ and $j \neq j'$, then $e = \dfrac{j' - j}{m - m'} < j'$ or j. This contradiction because $1 \leqslant j \leqslant e$ and $1 \leqslant j' \leqslant e$. Therefore, $(m, j) = (m', j')$. □

Lemma 2.3.2. *For every nested chain abacus $\mathfrak{N}$ with set S of beads, e columns and r rows, there exists a unique connected partition representing $\mathfrak{N}$ where $S = \{w_1, w_2, ..., w_n\}$.*

Proof. Suppose that $\mu^{(e,r)}$ and $\lambda^{(e,r)}$ are two connected partitions representing $\mathfrak{N}$ with set S, e columns and r rows. Based on nested chain abacus algorithm Step 4, for each set of n-connected beads $\{w_1, w_2, ..., w_n\}$, then

$\mu_n = w_1$ and $\mu_{n-g} = w_{g+1} - w_g + \mu_{n-g+1} - 1$ where $1 \leqslant g \leqslant n-1$.

Since $\lambda^{(e,r)} = (\lambda_1, \lambda_2, ..., \lambda_n)$ is a connected partition with e columns and r rows representing $\mathfrak{N}$, then,

$\lambda_n = w_1$ and $\lambda_{n-g} = w_{g+1} - w_g + \lambda_{n-g+1} - 1$ where $1 \leqslant g \leqslant n-1$.

Such that,

$$
\begin{aligned}
w_1 &= \mu_n = \lambda_n, \\
\mu_{n-1} &= w_2 - w_1 + \mu_n - 1 = w_2 - w_1 + \lambda_n - 1 = \lambda_{n-1}, \\
\mu_{n-2} &= w_3 - w_2 + \mu_{n-1} - 1 = w_3 - w_2 + \lambda_{n-1} - 1 = \lambda_{n-1}, \\
&. \\
&. \\
&. \\
\mu_1 &= w_n - w_{n-1} + \mu_2 - 1 = w_n - w_{n-1} + \lambda_2 - 1 = \lambda_1.
\end{aligned}
$$

Hence, $\mu_b = \lambda_b$, and subsequently $\mu^{(e,r)}$ and $\lambda^{(e,r)}$ are connected partitions representing the same nested chain abacus with e columns, r rows and set S where $1 \leqslant b \leqslant n$. Hence, $\mathfrak{N}$ is an associator with exactly one connected partition. □

Theorem 2.3.3. *For any form of n-connected ominoes, there exists a unique connected partition, $\mu^{(e,r)}$, representing it with r rows and e columns.*

Proof. On the contrary, suppose that $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are two nested chain abacus with e columns, r rows and n beads representing one form of n-connected ominoes. Based on the algorithm of representing nested chain abacus in Step 2, there exist $f : Z \times Z \to Z$ such that

$$f(m, j) = e(m-1) + (j-1)$$

where $e \geqslant j$, $e > 0$ and $mj \leqslant er$. Since $\mathfrak{N}_1$ represents a n-connected ominoes with respect to the minimal frame with e columns and r rows so

$$f(m, j) = w_k.$$

Meanwhile, $\mathfrak{N}_2$ represents a n-connected ominoes with respect to minimal frame with e columns and r rows. Then

$$f(m,j) = w'_k$$

where (m,j) is a location in the minimal frame containing an ominoe, $1 \leqslant m \leqslant r$ and $1 \leqslant j \leqslant e$ for k = 1, 2,..., n. Based on Proposition 2.3.1 $w_k = w'_k$.

Thus, any form of n-connected ominoes is represented by exactly one nested chain abacus $\mathfrak{N}$. Based on Lemma 2.3.2, there exists a unique connected partition represented by $\mathfrak{N}$. Thus, there exists a unique connected partition that represents n-connected ominoes. □

Previous researches, have shown that the four shapes of 4-connected ominoes in Figure 2.5(a) are known as tetrominos. Based on the algorithm discussed earlier, we can associate each of the four shapes with a unique connected partition as shown in Figure 2.5.

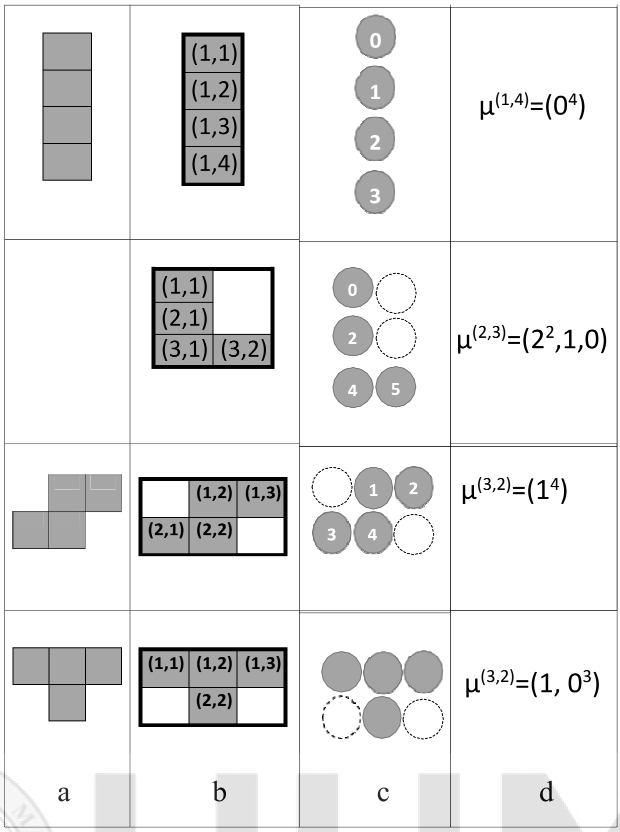


Figure 2.5. Representation of the 4 shapes of family of tetromino (a) A 4-connected ominoes (b) A 4-connected ominoes w.r.t minimal frame (c) Nested chain abacus (d) Connected partition

The next remark demonstrates the development of the nested chain abacus from a connected partition.

Remark 2.3.4. *Let* $\mu^{(e,r)}=(\mu_1, \mu_2, \ldots, \mu_n)$ *be a connected partition of a nested chain abacus with e columns and r rows. Then, we can develop the nested chain abacus as follows:*

$$w_1 = \mu_n$$

, $w_g = \mu_{n-g+1} - \mu_{n-g+2} + w_{g-1} + 1$. Example 2.3.5 and Figure 2.6 are given to illustrate Remark 2.3.4.

Example 2.3.5. *Let the connected partition be* $\mu^{(4,4)} = (6, 4, 4, 3, 3, 3, 3, 1)$. *The 8-connected ominoes which correspond to a nested chain abacus with 8 beads, 4 columns* *and 4 rows will be as follows.*

$w_1 = \mu_8 = 1,$

$w_2 = \mu_{8-2+1} - \mu_{8-2+2} + w_1 + 1 = \mu_7 - \mu_8 + w_1 + 1 = 4,$

$w_3 = \mu_{8-3+1} - \mu_{8-3+2} + w_2 + 1 = \mu_6 - \mu_7 + w_2 + 1 = 5,$

$w_4 = \mu_{8-4+1} - \mu_{8-4+2} + w_3 + 1 = \mu_5 - \mu_6 + w_3 + 1 = 6,$

$w_5 = \mu_{8-5+1} - \mu_{8-5+2} + w_4 + 1 = \mu_4 - \mu_5 + w_4 + 1 = 7,$

$w_6 = \mu_{8-6+1} - \mu_{8-6+2} + w_5 + 1 = \mu_3 - \mu_4 + w_5 + 1 = 9,$

$w_7 = \mu_{8-7+1} - \mu_{8-7+2} + w_6 + 1 = \mu_2 - \mu_3 + w_6 + 1 = 10,$

$w_8 = \mu_{8-8+1} - \mu_{8-8+2} + w_7 + 1 = \mu_1 - \mu_2 + w_7 + 1 = 13.$

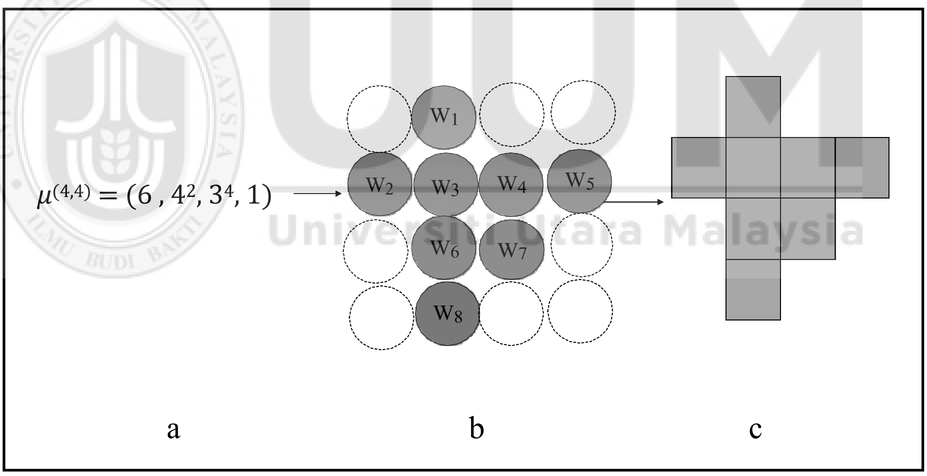


Figure 2.6. (a) Connected partition with 4 columns and 4 rows (b) Nested chain abacus (c) 8-connected omenoes

Next, we will develop the properties of the connected beads of the nested chain abacus.

2.4 The Connectedness of Beads in the Nested Chain Abacus

In Definition 2.4.1, we describe the connectedness of two bead positions w_δ and $w_{\delta'}$ in nested chain abacus such that w_δ and $w_{\delta'}$ are adjacent.

Definition 2.4.1. *Let w_δ and $w_{\delta'}$ be two bead positions in the nested chain abacus with e columns and r rows. w_δ and $w_{\delta'}$ are adjacent if one of the following conditions are satisfied:*

- $|w_\delta - w_{\delta'}| = e$ *if w_δ and $w_{\delta'}$ are located in one column.*
- $|w_\delta - w_{\delta'}| = 1$ *if w_δ and $w_{\delta'}$ are located in one row.*

Based on Definition 2.4.1 and considering Figure 2.4, since w_1 - w_2 = $|0-1| = 1$, then, w_1 and w_2 are adjacent. In addition, $w_1 - w_3$ = $|0-3| = 3 = e$, thus, w_1 and w_3 are also adjacent.

Definition 2.4.1 describes the connectedness of any adjacent bead positions. To provide a general meaning for the connectedness of 3 or more bead positions, we define the connectedness of beads with respect to rows and columns.

2.4.1 Connectedness of Beads with Respect to the Rows in Nested Chain Abacus

In this section, we will discuss the connectedness of beads in the same row.

Definition 2.4.2. *A sequence of bead positions, called a set-row of connected beads is denoted by SR if the bead positions are adjacent and belong to the same row.*

Lemma 2.4.3. *Let $A = \{w_1, w_2, ..., w_q\}$ be a set-row. Then, $w_p - w_{p-1} = 1$, where $2 \leqslant p \leqslant q$ and $2 \leqslant q \leqslant n$.*

Proof. Suppose w_p and w_{p-1} belong to A, then, based on Definition 2.4.2, the bead positions are adjacent and belong to the same row. Thus, based on Definition 2.4.1, $w_p - w_{p-1} = 1, \forall\, w_p, w_{p-1} \in SR$. □

Based on Lemma 2.4.3 and Definition 2.4.2, the bead positions located in the same row are connected if they belong to one set-row, as shown in the following example.

Example 2.4.4. *Let* $\{w_1, w_2, \ldots, w_{15}\}$ *be a set of bead positions in a nested chain abacus which are represented by connected partition* $\mu^{(6,3)} = (2^3, 1^{10}, 0^3)$ *as shown in Figure 2.7.*

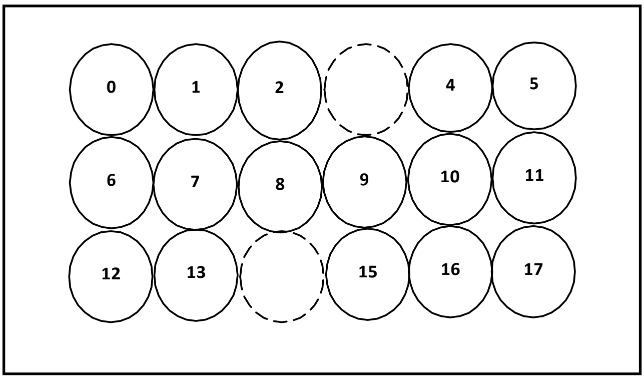


Figure 2.7. Nested chain abacus of 16-connected beads with 6 columns and 3 rows

Based on the Figure 2.7 $w_1 = 0$, $w_2 = 1$, $w_3 = 2$,$w_4 = 4$, $w_5 = 5$, $w_6 = 6$, $w_7 = 7$, $w_8 = 8$, $w_9 = 9$, $w_{10} = 10$, $w_{11} = 11$, $w_{12} = 12$, $w_{13} = 13$, $w_{14} = 15$, $w_{15} = 16$ and $w_{16} = 17$. From Lemma 2.4.3, the sequences of the first set-rows of the 16-connected beads is given as follows:

$SR_1 = \{w_1, w_2, w_3\} = \{0, 1, 2\}$, so w_1, w_2 and w_3 are connected.

For the rest of this section, we discuss the connectedness of any two bead positions located in different rows.

Let us suppose that w_δ and w_{δ^I} be are two bead positions belonging to SR_α and SR_{α^I}, respectively, such that SR_α and SR_{α^I} are two set-rows located in row α and α^I.

First, we define the connectedness between w_δ and w_{δ^I} if m and m^I are consecutive numbers.

Definition 2.4.5. *Let* SR_α *and* SR_{α^I} *be set-rows of connected beads located in rows* m *and* m^I*, respectively, in a nested chain abacus such that* $m = m^I + 1$*. Then,* SR_α *is* *connected with SR_{α^I} if at least one of the beads in SR_α is adjacent to a bead in SR_{α^I}.*

Lemma 2.4.6. *Two set-rows belonging to rows m and $m+1$ are connected if $\exists\ w_a \in SR_\alpha$ and $\exists\ w_b \in SR_{\alpha^I}$ such that $|w_a - w_b| = e$.*

Proof. The two set-rows belong to two consecutive rows and so by Definition 2.4.5 w_a and w_b are adjacent. Based on Definition 2.4.1, we obtain

$$|w_a - w_b| = e$$

where $w_a \in SR_\alpha$ and $w_b \in SR_{\alpha^I}$ □

Based on Lemma 2.4.6, w_δ and w_{δ^I} are connected if SR_α and SR_{α^I} are connected. Consider Example 2.4.4, and based on Lemma 2.4.6 SR_1 and SR_3 are connected because $|0-6| = 6 = e$ where $SR_1 = \{0, 1, 2\}$ and $SR_3 = \{6, 7, 8, 9, 10, 11\}$. Table 2.2 shows set-rows sequences that are connected, and those not connected, in the nested chain abacus of 16-connected beads in Figure 2.7 where Yes-connected and No-not connected.

Table 2.2

Connectedness sequence of set-columns in the 16-connected beads

	SR_1	SR_2	SR_3	SR_4	SR_5
SR_1	Yes	Yes	No	No	No
SR_2	No	Yes	Yes	No	No
SR_3	No	Yes	Yes	Yes	Yes
SR_4	No	No	Yes	Yes	No
SR_5	No	No	Yes	No	Yes

Next, we define the connectedness between w_δ and w_{δ^I} if m and m^I are not consecutive numbers.

Definition 2.4.7. *Let SR_α and $SR_{\alpha'}$ be two set-rows of connected beads located in rows m and m', respectively, in the nested chain abacus such that $m > m' + 1$. Then, SR_α is connected with $SR_{\alpha'}$ if there exists $SR_{k_1}, SR_{k_2}, \ldots, SR_{k_z}$ which satisfy the following conditions:*

1. *SR_α is connected with SR_{k_1}*
2. *SR_{k_z} is connected with $SR_{\alpha'}$,*

where $SR_{k_{z'}}$ set-rows of connected beads and $SR_{k_{z'}}$ is connected with $SR_{k_{z'+1}}$ for $1 \leqslant z' \leqslant z-1$ and $k_1, k_2, \ldots, k_z$ are consecutive numbers.

Recall Example 2.4.4, and based on Definition 2.4.7 SR_1 and SR_4 are connected because there exists SR_3 such that SR_1 connected with SR_3 and SR_3 connected with SR_4 as shown in Figure 2.8.

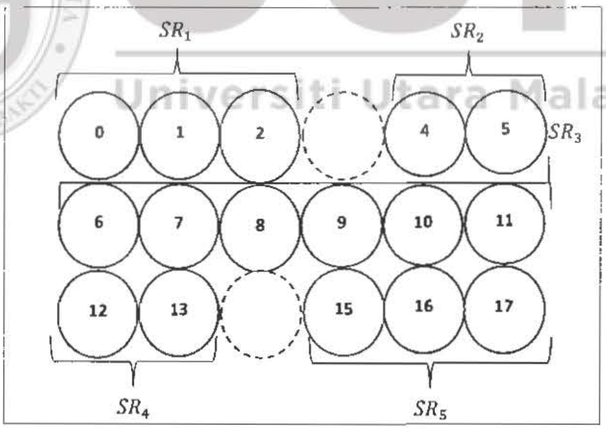


Figure 2.8. Nested chain abacus of 16-connected beads with 6 columns and 3 rows

Next, we will focus on connectedness of the bead positions with respect to columns

2.4.2 Connectedness of Beads with Respect to the Columns in Nested Chain Abacus

In this section, we introduce definitions of the of connectedness of bead positions with respect to columns as illustrated by some examples. We begin by introducing the definition of the connected beads located in the same column.

Definition 2.4.8. *A sequence $\{w_{\lambda_1}, w_{\lambda_2}, ..., w_{\lambda_b}\}$ of bead positions in the nested chain abacus with e columns and r rows is called set-column of connected beads and denoted by SC if they belong to the same column and every two consecutive elements in the sequence are adjacent.*

Lemma 2.4.9. *Let the sequence of set-columns be $w_{\lambda_1} < w_{\lambda_2} < ... < w_{\lambda_b}$ then, we have $w_{\lambda_2} - w_{\lambda_1} = ... = w_{\lambda_{b-1}} - w_{\lambda_b} = e$ where $1 \leqslant b \leqslant n$.*

Proof. Since the sequence $w_{\lambda_1} < w_{\lambda_2} < ... < w_{\lambda_b}$ is belonging to one column and from Definition 2.4.8, every two consecutive elements are adjacent, therefore, from Definition 2.4.1, $w_{\lambda_2} - w_{\lambda_1} = ... = w_{\lambda_{b-1}} - w_{\lambda_b} = e$ where $1 \leqslant b \leqslant n$. □

Based on Definition 2.4.8, the bead positions located in same column are connected if they belong to one set-column. Based on Example 2.4.4, we explain Definition 2.4.8 as follows:

$SC_1 = \{w_1, w_6, w_{12}\} = \{0, 6, 12\}$, since 6 - 0 = 12 - 6 = 6, then, w_1, w_6 and w_{12} are connected.

Similarly, the beads are connected in the SC_2, SC_3, SC_4, SC_5 and SC_6 where $SC_2 = \{w_2, w_7, w_{13}\}$, $SC_3 = \{w_3, w_8\}$, $SC_4 = \{w_9, w_{15}\}$, $SC_5 = \{w_4, w_{10}, w_{16}\}$, $SC_6 = \{w_5, w_{11}, w_{16}\}$ as shown in Figure 2.9.

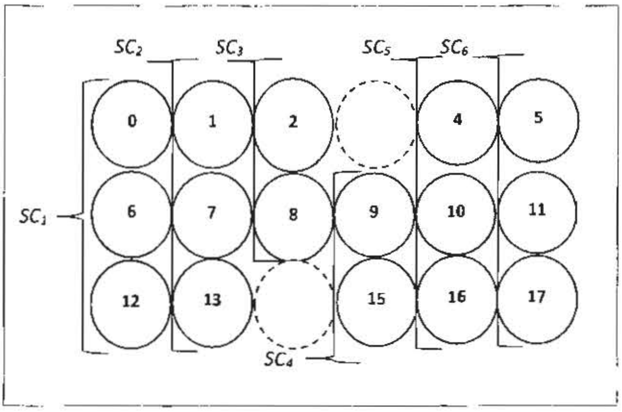


Figure 2.9. Nested chain abacus of 16-connected beads with 6 columns and 3 rows

In the rest of this section, we discuss the connectedness of any two bead positions located in different columns. Let us suppose that w_δ and $w_{\delta'}$ are two bead positions belonging to SC_ξ and $SC_{\xi'}$, respectively, such that SC_ξ and $SC_{\xi'}$ are two set-columns located in column j and j' respectively. First, we define the connectedness between w_δ and $w_{\delta'}$ if j and j' are consecutive numbers.

Definition 2.4.10. *Let SC_ξ and $SC_{\xi'}$ be set-columns of connected beads located in columns j and j', respectively, in the nested chain abacus such that $j = j' + 1$. Then, SC_ξ is connected with $SC_{\xi'}$ if at least one of the beads in SC_ξ is adjacent to a bead in $SC_{\xi'}$.*

Lemma 2.4.11. *Let $SC_{\xi'}$ and $SC_{\xi'}$ be set-columns of connected beads located in columns j and $j+1$, respectively, in the nested chain abacus. Then, $SC_{\xi'}$, $SC_{\xi'}$ are connected if $|w_a - w_b| = 1$ where $\exists w_a \in SC_\xi$ and $\exists w_b \in SC_{\xi'}$.*

Proof. The two set-columns belong to two consecutive columns and from Definition 2.4.10, $\exists w_a \in SC_\xi$ and $\exists w_b \in SC_{\xi'}$ are adjacent; therefore, from Definition 2.4.2, we get $|w_a - w_b| = 1$. □

Based on Definition 2.4.10, then w_δ and $w_{\delta'}$ are connected if SC_ξ and $SC_{\xi'}$ are connected. We explain Definition 2.4.10 based on Example 2.4.4 and Figure 2.9.

Compare the element in SC_1 and SC_2: Since $\exists\ 7 \in SC_1$ and $6 \in SC_2$ such that $|7-6| = 1$, then, SC_1 is connected with SC_2.

Table 2.3 shows sequences that are connected-Yes, and not connected-No in the nested chain abacus in Figure 2.7.

Table 2.3

Sequence of set-columns in the 16-connected beads

	SC_1	SC_2	SC_3	SC_4	SC_5	SC_6
SC_1	Yes	Yes	No	No	No	No
SC_2	Yes	Yes	Yes	No	No	No
SC_3	No	Yes	Yes	Yes	No	No
SC_4	No	No	Yes	Yes	Yes	No
SC_5	No	No	No	Yes	Yes	Yes
SC_6	No	No	No	No	Yes	Yes

Next, we define the connectedness between w_δ and $w_{\delta'}$ if j and j' are not consecutive numbers.

Definition 2.4.12. *Let SC_ξ and $SC_{\xi'}$ be two set-columns of connected beads located in columns j and j', respectively, in the nested chain abacus such that $j > j'+1$. SC_ξ is connected with $SC_{\xi'}$ if there exists $SC_{k_1}, SC_{k_2}, \ldots, SC_{k_z}$ which satisfy the following conditions:*

1. *SC_ξ is connected with SC_{k_1}*
2. *SC_{k_z} is connected with $SC_{\xi'}$,*

where $SC_{k_{z'}}$ set-columns of connected beads and $SC_{k_{z'}}$ is directly linked to $SC_{k_{z'+1}}$ for $1 \leqslant z' \leqslant z-1$ and $k_1, k_2, \ldots, k_z - 1$ are consecutive numbers.

Consider Figure 2.9 and Table 2.3, where from Definition 2.4.12 there exists connectedness between SC_1 and SC_4 because there exist set-columns SC_2, SC_3 such that

SC_1 is connected with SC_2, SC_2 with SC_3 and SC_3 with SC_4 respectively. In the next definition, we discuss the connectedness of w_δ and $w_{\delta'}$ if one is located in a set-column and another is located in a set-row.

Definition 2.4.13. *Let SC_ξ and $SR_{\xi'}$ be a set-column and set-row, respectively, in the nested chain abacus. Then the set-row and set-column are connected if at least one bead from SC_ξ and another bead from $SR_{\xi'}$ are adjacent.*

Lemma 2.4.14. *If SC_ξ and $SR_{\xi'}$ are set-column and set-row, respectively, in the nested chain abacus such that $w_\delta \in SC_\xi$ and $w_{\delta'} \in SR_{\xi'}$ are adjacent, then*

$$|w_\delta - w_{\delta'}| = \{1, e\} \text{ and } w_\delta \neq w_{\delta'}.$$

Proof. Based on Definition 2.4.2, if two beads are adjacent, then $|w_\delta - w_{\delta'}| = \{1, e\}$ for $w_\delta \in SC_\xi$ and $w_{\delta'} \in SR_{\xi'}$ respectively. □

From Definitions 2.4.5, 2.4.7, 2.4.10 and 2.4.12, the connectedness of bead positions with respect to the sequence of bead positions is provide by the following Theorem 2.4.15.

Theorem 2.4.15. *Bead positions w_a and w_b, in the nested chain abacus are connected if there exists a sequence of bead positions $\{w_a, w_{a_1}, w_{a_2}, \ldots, w_b\}$ such that $\{w_{a_1} - w_a, w_{a_2} - w_{a_1}, \ldots\} \in \{1, e\}$.*

Proof. Two beads w_a and w_b in the nested chain abacus are connected if the set-columns, or set-rows, or both in which they belong to are connected. Then, $S(R/C)_a$ and $S(R/C)_b$ must be connected and thus there exists at least a sequence of set-row, or set-column, or both i.e. the sequence,

$$\{S(R/C)_{a_1}, S(R/C)_{a_2}, \ldots, S(R/C)_{a_k}\}$$

such that $S(R/C)_{a_1}$ is connected with $S(R/C)_a$ and $S(R/C)_{a_k}$ is connected with $S(R/C)_b$. Beginning from w_a, we have a sequence of connected beads to the adjacent beads, $|w_a - w_b| = 1$. connecting $S(R/C)_a$ and $S(R/C)_{a_1}$ since $S(R/C)_a$ is a connected set-row or set-column. From the adjacent beads we can follow the same procedure for each elements in the sequence

$$\{S(R/C)_{a_1}, S(R/C)_{a_2}, \ldots, S(R/C)_{a_k}\}$$

and finally reach w_b with the sequence

$$\{w_{a_1} - w_a, w_{a_2} - w_{a_1}, \ldots\} \in \{1, e\}$$

where $w_a \in S(R/C)_a$ and $w_b \in S(R/C)_b$. □

Definition 2.4.16. *A partition, μ, for a nested chain abacus with n beads, e columns and r rows is a connected partition $\mu^{(e,r)}$ if every pair of beads is connected.*

We attempt to view the abacus for n-connected beads in terms of nested chains. The following section describes the design structure of the constructed nested chain abacus which is fundamental for developing chain move transformation that will be presented in chapter three.

2.5 Design Structure of Nested Chain Abacus

In this sections we will use matrix form to the abacus to give the detail description of the design structure of the nested chain abacus as shown in next lemma.

Lemma 2.5.1. *Suppose that* $\{w_1, w_2, \dots, w_n\}$ *is a set of bead positions given by a connected partition* $\mu^{(e,r)} = (\mu_1, \mu_2, \dots, \mu_n)$*. Then, every location in the nested chain abacus with e columns and r rows can be converted to an element in a matrix* $A_{r\times e}$ *by*

$$me + j \Rightarrow a_{(m+1)(j+1)}$$

for $0 \leqslant m \leqslant r-1$ *and* $0 \leqslant j \leqslant e-1$.

Proof. In nested chain abacus, the bead positions in column j and row m are numbered as $(me+j)$. The row numbers are from 0 to $r-1$ and column numbers are from 0 to $e-1$ while every matrix $A_{r\times e}$ consists of r rows from 1 to r and e columns from 1 to e; so any position $me+j$ in the nested chain abacus is an element $a_{(m+1)(j+1)}$ in the matrix $(r\times e)$. Then

$$me + j \Rightarrow a_{(m+1)(j+1)}$$

for $1 \leqslant m \leqslant r-1$ and $1 \leqslant j \leqslant e-1$. □

The general conversion of the nested chain abacus with e columns and r rows into matrix form is illustrated in Figure 2.10.

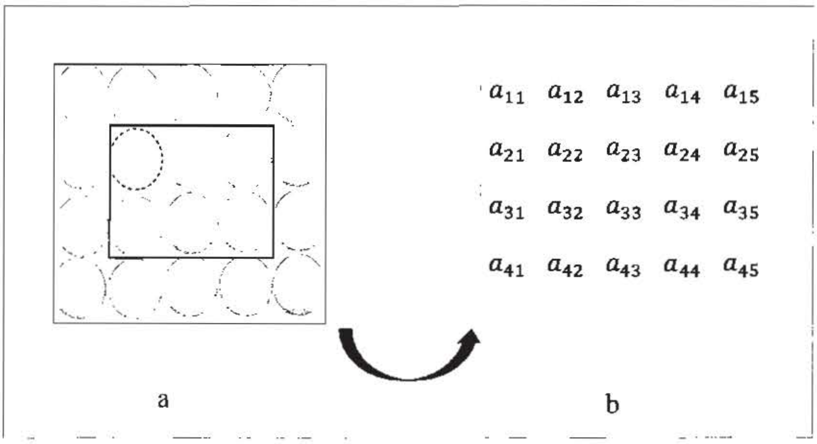


Figure 2.10. Conversion of the nested chain abacus in to matrix

Remark 2.5.2. *In this work, nested chain abacus positions is depicted as a union of nested disjoint chains of bead and empty bead positions.*

A chain starting from the i^{th} column will start at the i^{th} row because rows 1 to $(i-1)$ will be covered by the chains starting from rows 1, 2,....,$(i-1)$. After the starting point we will come down along the same column so the row numbers will be changing and will come down till the i^{th} row from the end so that it will be the $(r-i+1)^{th}$ row from the beginning. Now in the i^{th} row we go to the $(e-i+1)^{th}$ column and then we should cover the chain by coming down till the $(r-i+1)^{th}$, so two vertical columns have been covered. Now from the starting point in the i^{th} row we should cover up the chain on the right of it on that row till the $(e-i+1)^{th}$ column; so then the row will remain fixed and the columns will vary till we reach $(e-i+1)^{th}$ column in the i^{th} row and cover the chains in the $(r-i+1)^{th}$ row by keeping the row fixed and varying the columns from i to $(e-i+1)$. Each chain covers two rows and two columns the i^{th} column and the $(e-i+1)^{th}$ column and for the chain the i^{th} column and the $(e-i+1)^{th}$ column becomes the same and thus we get; $i=e-i+1$, so the chain will be in column. In another hand, the i^{th} rows and the $(r-i+1)^{th}$ rows becomes the same and thus we get; $i=r-i+1$, so the chain will be in row. In addition, we can get i^{th} column and the $(e-i+1)^{th}$ column becomes the same and thus we get $i=e-i+1$ and, i^{th} row *and the $(r-i+1)^{th}$ row becomes the same and thus we get $i = r-i+1$.*

A chain has rectangular form if the chain derived by two columns and two rows, while a chain has path form if the chain derived by a column or a row. This is basically how it is done;

1. $a_{ii} \to a_{(r-i+1)i}$ *and then* $a_{i(e-i+1)} \to a_{(r-i+1)(e-i+1)}$ *and then to cover the rest;* $a_{ii} \to a_{i(e-i+1)}$ *and* $a_{(r-i+1)i} \to a_{(r-i+1)(e-i+1)}$. *In this case we have two chooses:*
   - $[(r-i+1)e+i] - [ie+i] > [ie+(e-i+1)] - [ie+i]$.
   - $[(r-i+1)e+i] - [ie+i] < [ie+(e-i+1)] - [ie+i]$.
2. $[(r-i+1)e+i] - [ie+i] = 0$ *or* $[ie+(e-i+1)] - [ie+i] = 0$.
3. $[(r-i+1)e+i] - [ie+i] = 0$ *and* $[ie+(e-i+1)] - [ie+i] = 0$.

Based on three types of chains there are three design structures of the nested chain abacus: rectangular, rectangle-path and singleton nested chain abacus we begin by discussing the construction of rectangular nested chain abacus.

2.5.1 Rectangular Nested Chain Abacus

The rectangular nested chain abacus consists of rectangular chains. Definition 2.5.3 clarify the constructing of rectangular chain in nested chain abacus

Definition 2.5.3. *Let there $r \times e$ matrix A represents bead positions and empty bead positions in the nested chain abacus with e columns, r rows and c chains. Then,*

1. *A vertical rectangular chain is an arrangement of the bead positions and empty bead positions in a vertical rectangular format in the nested chain abacus such that* $[(r-i+1)e+i] - [ie+i] > [ie+(e-i+1)] - [ie+i]$ *and the element chain*

$$\{a_{mi}, a_{m(e-i+1)}, a_{ij}, a_{(r-i+1)j} \mid i \leqslant m \leqslant (r-i+1),\ i \leqslant j \leqslant (e-i+1)\}$$

where e is even number and $e \leqslant r$ *for* $1 \leqslant i \leqslant c$.

2. *A horizontal rectangular chain is an arrangement of the bead positions and empty bead positions in a horizontal rectangular format in the nested chain abacus such that* $[(r-i+1)e+i]-[ie+i] < [ie+(e-i+1)]-[ie+i]$ *and the elements in chain*

$$\{a_{mi}, a_{m(e-i+1)}, a_{ij}, a_{(r-i+1)j} \mid i \leqslant m \leqslant (r-i+1),\ i \leqslant j \leqslant (e-i+1)\}$$

where r is even number and $r < e$ *for* $1 \leqslant i \leqslant c$.

Based on Definition 2.5.3 there are two designs structure of nested chains abacus. Next we define the first design structure of the nested chain abacus.

Definition 2.5.4. *Vertical rectangular nested chain abacus is a nested chain abacus with e columns,r rows and c vertical rectangular chains where* $e \leqslant r$ *and e even.*

Example 2.5.5 provide the illustration of the vertical rectangular design structure.

Example 2.5.5. *Let* $\mu^{(4,6)} = (8^2, 6^3, 5, 4^3, 3^2, 1^5)$ *be a connected partition for a nested chain abacus with 4 columns and 6 rows that represents in Figure 2.11(a). Based on Definition 2.5.3 the nested chain abacus created from two vertical rectangular chains as illustrated in Figure 2.11(b) and Figure 2.11(c).*

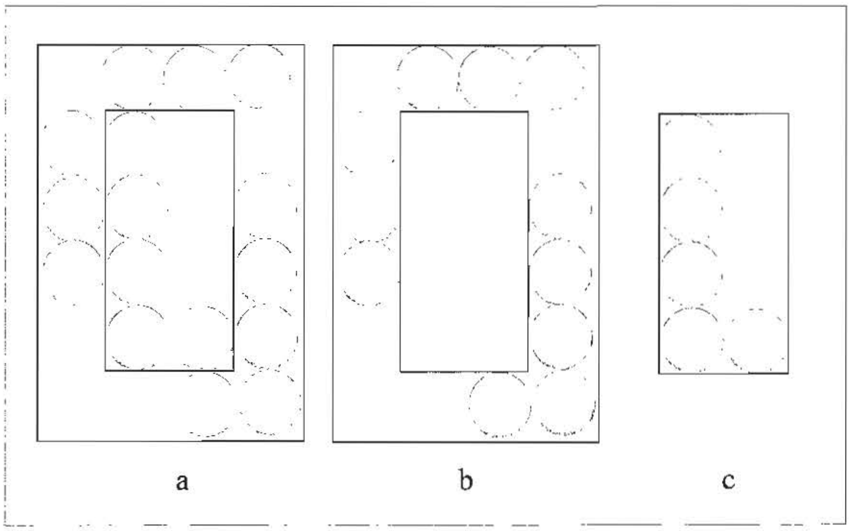


Figure 2.11. (a) Nested chain abacus of $\mu^{(4,6)} = (8^2, 6^3, 5, 4^3, 3^2, 1^5)$ where $c = 2$ (b) Outer vertical rectangular chain (c) Inner vertical rectangular chain

From Figure 2.11, we observe that yhe nested chain abacus construct from vertical rectangular chains where

chain 1 = $\{a_{m1}, a_{m6}, a_{1j}, a_{6j} : 1 \leqslant m \leqslant 6 : 1 \leqslant j \leqslant 4\}$ and

chain 2 = $\{a_{m2}, a_{m3} : 2 \leqslant m \leqslant 5\}$.

Based on Definition 2.5.3(2) The horizontal rectangular nested chain abacus structure as given by the next definition

Definition 2.5.6. *Horizontal rectangular nested chain abacus is a nested chain abacus with c horizontal rectangular chains, e columns and r rows where r < e and r even.*

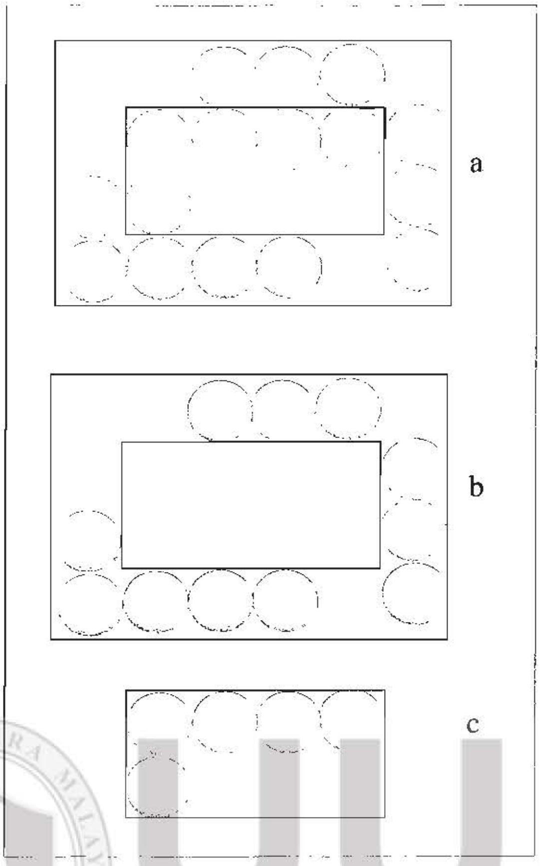


Figure 2.12. (a) Nested chain abacus of $\mu^{(6,4)}=(8,5,7^5,4^7,2^3)$ where $c=2$ (b) Outer horizontal rectangular chain (c) Inner horizontal rectangular chain

From Figure 2.12, we observe that nested chain abacus construct from two horizontal rectangular chains where

chain 1 = $\left\{a_{m1}, a_{m6}, a_{1j}, a_{4j} : 1 \leqslant m \leqslant 4, 1 \leqslant j \leqslant 6\right\}$ and

chain 2 = $\left\{a_{2j}, a_{3j} : 2 \leqslant j \leqslant 5\right\}$.

Lemma 2.5.7. *Let nested chain abacus be rectangular design structure then*

(i) $i < e-i+1$ *and* $i < r-i+1$.

(ii) $1 \leqslant i \leqslant c$ *and c is a positive integer.*

(iii) Every chain is derived from two columns columns i and $e-i+1$ *and two rows :* *i and* $r-i+1$.

(iv) *The last chain is derived from two consecutive columns if nested chain abacus be vertical rectangular design structure and the last chain is derived from two consecutive rows if nested chain abacus be horizontal rectangular structure.*

Proof.

(i) The i^{th} column is where the rectangular design structure starts and in column $(e-i+1)$ the rectangular design structure ends and thus $i < e-i+1$. Similarity, $i < r-i+1$

(ii) Let i denotes the value from 1 to the i^{th} column (respectively the i^{th} row) where the vertical rectangular design structure ends, based on Remark 2.5.2 the maximum number of chains formed will be the last value of i taken to form the chains and thus we have $1 \leq i \leq c$ and c is the number of chains.

(iii) As we discussed the formation of the structures of the chains at the end of the proof of Definition 2.5.3 and see we have that every chain is derived from two columns: columns i and $e-i+1$ and two rows i and $r-i+1$.

(iv) Since e is an even number and each vertical rectangular nested chain covers two columns so the last chain is derived from two consecutive columns.

□

The number of chains in vertical and horizontal rectangular nested chain abacus are determined by Lemma 2.5.8.

Lemma 2.5.8. *The number of chains in nested chain abacus* N *is*

(i) $\frac{e}{2}$ *if* N *is vertical rectangular nested chain abacus.*

(ii) $\frac{r}{2}$ *if* N *is horizontal rectangular nested chain abacus.*

where e and r are the number of columns and rows respectively.

Proof.

(i) Based on Lemma 2.5.7(3), every vertical rectangular chain is derived from two columns, which are columns i and $e-i+1$. Since the last chain is derived from two consecutive columns, the difference between these two consecutive column numbers is

$$(e-i+1)-i=1.$$

Thus,

$$i=\frac{e}{2}.$$

(ii) Similar to proof (i) Lemma 2.5.8. □

Referring to Example 2.5.5 where $\mu^{(4,6)}=(8^2,6^3,5,4^3,3^2,1^5)$ and $e=4$ then by Lemma 2.5.8 we have two chains as has been shown in Figure 2.11.

The next theorem shows that the number of positions in any vertical (respectively horizontal) rectangular chain is $2r+2e-4(2i-1)$.

Theorem 2.5.9. *The number of bead and empty bead positions in each rectangular chains i in nested chain abacus is*

$$2r+2e-4(2i-1)$$

where e and r are the number of columns and rows respectively.

Proof. Since each chain i form a rectangle, based on Lemma 2.5.7 the length,

$$(r-i+1)-i=r-2i+1.$$

Based on Lemma 2.5.7 the width of chain i is

$$(e - i + 1) - i = e - 2i + 1.$$

Therefore, the perimeter of chain i is given by

$$2[(r - 2i + 1) + (e - 2i + 1)] = 2r + 2e - 4(2i - 1).$$ □

Example 2.5.10. *Let* $\mu^{(4,5)} = (8^2, 6, 3, 2, 1^4)$ *be connected partition. Based on Lemma 2.5.8, the nested chain abacus has two chains. By Theorem 2.5.9 the first chain (chain 1) has 14 positions while the second chain (chain 2) has 6 positions.*

Next theorem is a result of Definition 2.5.3, Lemma 2.5.8 and Theorem 2.5.9

Theorem 2.5.11. *Let* N *be nested chain abacus with e columns and r rows and c chain then,* $< V_i > = < V_1, V_2, V_3, \cdots, V_c >$ *is a arithmetic sequence with common difference of successive is* -8.

Proof. Let V_{i+1} and V_i represent the number of positions in chain $i + 1$ and chain i respectively where $i = 1, 2, \cdots, c$ and $c = \frac{e}{2}$ if the nested chain abacus is a vertical nested chain abacus and $c = \frac{r}{2}$ if the nested chain abacus is a horizontal nested chain abacus by Theorem 2.5.8. Thus,

$$[2r + 2e - 4(2(i + 1) - 1)] - [2r + 2e - 4(2i - 1)] = -8.$$ □

2.5.2 Rectangle-Path Nested Chain Abacus

The rectangle-path nested chain abacus consists of rectangular chains and one path chain. Definition 2.5.12 clarify the constructing of rectangle-path chain in nested chain abacus

Definition 2.5.12. *Let the matrix $A_{r\times e}$ represent bead positions and empty bead positions in the nested chain abacus with e columns and r rows. Then*

- *Vertical-path chain is an arrangement of bead and empty bead positions in column $\frac{e+1}{2}$ in the nested chain abacus such that $[(r-i+1)e+i]-[ie+i]=0$ and the elements in*

$$chain\left(\frac{e+1}{2}\right)=\left\{a_{m(\frac{e+1}{2})}:\frac{e-1}{2}\leqslant m\leqslant\frac{2r-e+3}{2}\right\}$$

  where $e<r$, e is odd and c is a positive integer.

- *Horizontal-path chain is an arrangement of bead and empty bead positions in row $\frac{r+1}{2}$ in the nested chain abacus such that $[ie+(e-i+1)]-[ie+i]=0$ and the elements in*

$$chain\left(\frac{r+1}{2}\right)=\left\{a_{(\frac{r+1}{2})j}:\frac{r-1}{2}\leqslant m\leqslant\frac{2e-r+3}{2}\right\}$$

  where $r<e$, r is odd and c is a positive integer.

Based on Definition 2.5.12 there are two designs structure of the rectangle-path nested chains abacus.

Definition 2.5.13. *The vertical rectangle-path nested chains abacus is a nested chain abacus with e columns, r rows, c vertical rectangular chains and one vertical-path chain where $e<r$ and e odd.*

Example 2.5.14 provides the illustration of this design structure.

Example 2.5.14. *Let $\mu^{(3,7)}=(4^8,3,1^8)$ be a connected partition with $e=3$ where the corresponding nested chain abacus that represents the 17-connected beads is as in*

Figure 2.13(a). It follows that the nested chain abacus has one vertical rectangular chain and one vertical path chain as illustrated in Figure 2.13(b) and Figure 2.13(c).

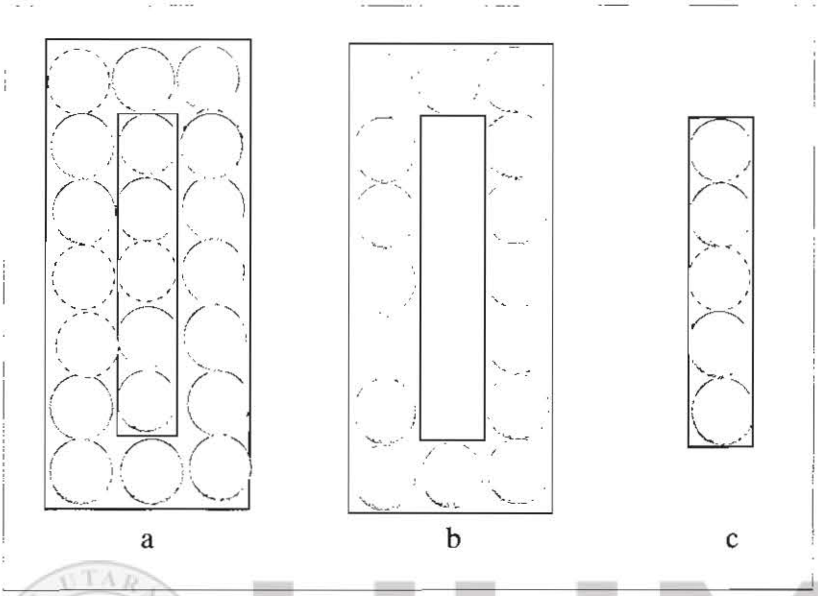


Figure 2.13. (a) The nested chain abacus where $c = 2$ (b) Outer vertical chain and (c) Vertical path chain

From Figure 2.13, we observe that if $e = 3$ the vertical rectangle-path nested chain abacus has one vertical rectangular chain and one vertical path chain where

chain 1 = $\{a_{m1}, a_{m3}, a_{1j}, a_{7j} : 1 \leqslant m \leqslant 7, 1 \leqslant j \leqslant 3\}$,

chain 2 = $\{a_{m2} : 2 \leqslant m \leqslant 6\}$.

Notice that chain 2 is a vertical path chain.

In the next definition we proposed the horizontal rectangle-path nested chain abacus

Definition 2.5.15. *The horizontal rectangle-path nested chain abacus is a nested chain abacus with e columns, r rows, $c-1$ horizontal rectangular chains and one horizontal path chain where $r < e$ and r odd.*

The number of chains constructed in the vertical (respectively horizontal) rectangle-path nested chain abacus determined by the following lemma.

Lemma 2.5.16. *The number of chains in nested chain abacus $\mathfrak{N}$ is*

(i) $\dfrac{e+1}{2}$ *if $\mathfrak{N}$ is vertical rectangle-path nested chain abacus.*

(ii) $\dfrac{r+1}{2}$ *if $\mathfrak{N}$ is horizontal rectangle-path nested chain abacus.*

Proof.

(i) Given that e is odd, then $e = 2v+1$ where v is a positive integer. By Lemma 2.5.7(3) every rectangle chain is derived from two columns then there are

$$v = \frac{e-1}{2}$$

vertical rectangular chains and one vertical path chain. Hence, the vertical rectangular nested chains abacus has $\dfrac{e+1}{2}$ chains derived from one column.

(ii) Given that r is odd, then $r = 2v+1$ where v is a positive integer. By Lemma 2.5.7(3) the rectangular chain is derived from two rows so the nested chain abacus with $2v+1$ rows have v rectangular chains. Then, $v = \dfrac{r-1}{2}$ rectangle chain. Thus, the horizontal rectangular chain has

$$\frac{r-1}{2} + 1 = \frac{r+1}{2} \text{ chains.}$$ □

The number of positions either in the vertical path chain or in the horizontal path chain constructed in the horizontal rectangle-path nested chain abacus or vertical rectangle-path nested chain abacus respectively is determined by the following theorem.

Theorem 2.5.17. *The number of positions in the path chain in the rectangle-path nested chain abacus with e columns and r rows is*

$$re - (c-1)(2r+2e) - \sum_{i=1}^{c-1} 4(2i-1).$$

Proof. Let k be the number of positions in the path chain and i the chain number. By Theorem 2.5.9 every rectangular chain has

$$2e+2r-4(2i-1)$$

positions. For the nested chain abacus with $c-1$ rectangular chains and one path chain, the number of positions in the $c-1$ chains is

$$(c-1)2e+2r-\sum_{i=1}^{c-1}4(2i-1).$$

Since the number of positions in the nested chain abacus is er, then,

$$er=(c-1)(2e+2r)-\sum_{i=1}^{c-1}4(2i-1)+k.$$

Thus,

$$k=er-(c-1)(2e-2r)+\sum_{i=1}^{c-1}4(2i-1),$$

where k is the number of positions in chain c. □

In the next section we proposed the last structure in nested chain abacus

2.5.3 Singleton Nested Chain Abacus

The singleton nested chain abacus consists of rectangular chains and a singleton chain. Definition 2.5.18 clarify the constructing of singleton chain in nested chain abacus.

Definition 2.5.18. *A singleton chain is a position $a_{m(\frac{e+1}{2})}$ located in column $\frac{e+1}{2}$ and row $\frac{e+1}{2}$ such that $[(r-i+1)e+i]-[ie+i]=[ie+(e-i+1)]-[ie+i]=0$.*

Beset on Definition 2.5.3 and Definition 2.5.18, we developed a singleton nested chain abacus.

Definition 2.5.19. *The singleton rectangular nested chain abacus is a nested chain abacus with $c-1$ rectangular chains and one singleton chain where $e=r$ and e odd.*

Example 2.5.20 provide the illustration of the design structure for singleton nested chain abacus.

Example 2.5.20. *Let $\mu^{(3,3)}=(2^2,1^4)$ be a connected partition where the singleton nested chain abacus is as in Figure 2.14(a). Based on Definition 2.5.3 and Definition 2.5.18 the nested chain abacus created from one rectangular chains and one singleton chain as illustrated in Figure 2.14(b) and Figure2.14(c).*

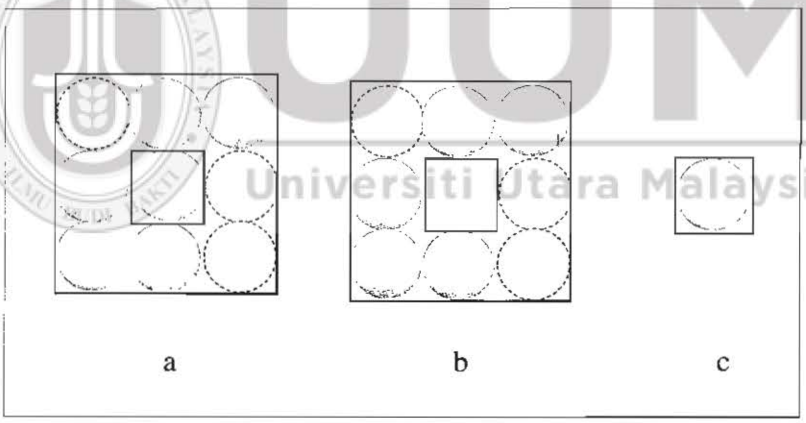


Figure 2.14. (a) Nested chain abacus where $c=2$, (b) Outer chain and (c) Singleton chain

From Figure 2.14, we observe that nested chain abacus construct from rectangular chains and singleton where

Chain 1 = $\{a_{m1}, a_{m3}, a_{1j}, a_{3j} : 1 \leqslant m \leqslant 3, 1 < j < 3\}$,

Chain 2 = $\{a_{22}\}$

Notice that chain 2 is a singleton chain.

The number of chains in the singleton nested chain abacus is determined in the next theorem.

Lemma 2.5.21. *The number of chains in the singleton nested chain abacus is* $\frac{e+1}{2}$ *where e is the number of columns and r is the number of rows.*

Proof. The proof is similar to the proof of Lemma 2.5.16. □

From the previous structures with respect to columns we can establish the e-nested chain abacus sequence as shows in Remark 2.5.22

Remark 2.5.22. *Let* $e \leqslant r$ *and* F_e *be a function* $F_e : \mathbb{N} \rightarrow \mathbb{N}$ *such that*

$$F_e = \begin{cases} \dfrac{e}{2} & if \quad e \text{ is even,} \\ \dfrac{e+1}{2} & if \quad e \text{ is odd.} \end{cases}$$

Table 2.4 gives the number of chains for different values of e.

Table 2.4

The number of chains for different values of $r = 1, 2, ..., 12$ *and* $e = 1, 2, ..., 8$

F_e r	1	2	3	4	5	6	7	8	9	10	11	12
F_1	1	1	1	1	1	1	1	1	1	1	1	1
F_2	1	1	1	1	1	1	1	1	1	1	1	1
F_3			2	2	2	2	2	2	2	2	2	2
F_4				2	2	2	2	2	2	2	2	2
F_5					3	3	3	3	3	3	3	3
F_6						3	3	3	3	3	3	3
F_7							4	4	4	4	4	4
F_8								4	4	4	4	4

Based on Table 2.4 and Remark 2.5.22 we can summarize the values as presented below

e	1	2	3	4	5	6	7	8	9	10
F_e	1	1	2	2	3	3	4	4	5	5

The e-nested chain abacus sequence is a series of number $\{F_e\}_{e=2}^{\infty}$ based on Remark 2.5.22 where

$$F_e + F_{e+1} = F_{2e+1} = F_{2e+2}$$

with $F_2 = 1$ and $F_3 = 2$. The e-nested chain abacus sequence is $1,2,2,3,3,...$ for $e = 2,3,4,5,6,....$ Similarly, from the previous structures with respect to rows, we can establish the r-nested chain abacus sequence as shows in Remark 2.5.23

Remark 2.5.23. *Let $r < e$ and F_r be a function $F_r : \mathbb{N} \rightarrow \mathbb{N}$ such that*

$$F_r = \begin{cases} \dfrac{r}{2} & if \quad r \text{ is even number,} \\ \dfrac{r+1}{2} & if \quad r \text{ is odd number.} \end{cases}$$

Similar, the r-nested chain abacus sequence is a series of number $\{F_r\}_{r=2}^{\infty}$ get by Remark 2.5.23

$$F_r + F_{r+1} = F_{2r+1} = F_{2r+2}$$

with $F_1 = 1$ and $F_2 = 2$. The r-nested chain abacus sequence is $1,2,2,3,3,...$ for $r = 2,3,4,5,6,....$ The result follows from Lemma 2.5.7, Lemma 2.5.8 Theorem 2.5.9, Theorem 2.5.11, 2.5.16 and Theorem 2.5.17 , two sequences P_ρ^{Rec-P} and P_ρ^{Rec} for rectangle path nested chain abacus and rectangular nested chain abacus respectively can be obtained from the number of positions in each chain. In Theorem 2.5.24 we developed the first sequence with rectangle-path nested chain abacus.

Theorem 2.5.24. *Let $\mathfrak{N}$ be the rectangle-path nested chain abacus with e column, r rows and c chains.*

(i) *If $e < r$ then,*

$$P_{\rho}^{Rec-P} = \begin{cases} r-e+1 & if \quad \rho = 1, \\ 2P_{1}^{Rec-P}+6 & if \quad \rho = 2, \\ P_{\rho-1}^{Rec-P}+8 & if \quad \rho \geqslant 3. \end{cases}$$

(ii) *If $r < e$ then,*

$$P_{\rho}^{Rec-P} = \begin{cases} e-r+1 & if \quad \rho = 1, \\ 2P_{1}^{Rec-P}+6 & if \quad \rho = 2, \\ P_{\rho-1}^{Rec-P}+8 & if \quad \rho \geqslant 3. \end{cases}$$

where P_{ρ}^{Rec-P} be the number of positions in chain i and $\rho = c-i+1$ for $1 \leqslant i \leqslant c$.

Proof.

(i) Based on Theorem 2.5.17, the number of position in chain c is

$$re - c(2r+2e) + \sum_{i=1}^{c} 4(2i-1).$$

Since $c = \frac{e+1}{2}$, then,

$$\begin{aligned} k &= re - \frac{e-1}{2}(2r+2e) + 8\sum_{i=1}^{c-1} i - \sum_{i=1}^{c-1} 4 \\ &= re - e^2 - e - er + r + 8\sum_{i=1}^{c-1} i + 4\frac{e-1}{2}. \end{aligned}$$

Since $\sum_{i=1}^{c-1} i = \frac{(c-1)c}{2}$, then,

$$k = e - r + 1.$$

Thus,

$$P_1^{Rec-P} = r - e + 1.$$

Since chain $c-1$ is a rectangular chain then by Lemma 2.5.16 and Theorem 2.5.9 the number of positions in chain $c-1$ is

$$2r - 2e + 8 = 2(r - e + 1) + 6 = 2P_1^{Rec-P} + 6.$$

Based on Theorem 2.5.11 the different between two rectangular chains in $\mathfrak{N}$ is 8 then

$$\begin{aligned} P_{c-2}^{Rec-P} - P_{c-1}^{Rec-P} &= P_{c-3}^{Rec-P} - P_{c-2}^{Rec-P} &= \cdots = 8 \\ P_3^{Rec-P} - P_2^{Rec-P} &= P_4^{Rec-P} - P_3^{Rec-P} &= \cdots = 8. \end{aligned}$$

Hence

$$4P_\rho^{Rec-P} = P_{\rho-1}^{Rec-P} + 8.4$$

(ii) Follow directly by proof 1 of Theorem 2.5.24. □

Next, we developed the second sequence with rectangular nested chain.

Theorem 2.5.25. *Let $\mathfrak{N}$ be the rectangular nested chain abacus with e column, r rows and c chains.*

(i) If $e < r$ then,

$$P_\rho^{Rec} = \begin{cases} 2r-2e+1 & if \quad \rho = 1, \\ \\ P_{\rho-1}^{Rec}+8 & if \quad \rho \geqslant 2. \end{cases}$$

(ii) If $r < e$ then,

$$P_\rho^{Rec} = \begin{cases} 2e-2r+1 & if \quad \rho = 1, \\ \\ P_{\rho-1}^{Rec}+8 & if \quad \rho \geqslant 2. \end{cases}$$

where P_ρ^{Rec} be the number of positions in chain i and $\rho = c-i+1$ for $1 \leqslant i \leqslant c$.

Proof.

(i) Since $\mathfrak{N}$ is a rectangular nested chain abacus then, by Lemma 2.5.7 chain c derived by two consecutive columns. Based on Theorem 2.5.9 and Lemme 2.5.8, the number of position in chain i is

$$2e+2r-4(2i-1),$$

and $c = \frac{e}{2}$, respectively. Thus,

$$2e+2r-4(2\frac{e}{2}-1)$$

and

$$P_1^{Rec} = 2r-2e+1.$$

By Theorem 2.5.11 there is arithmetical sequence for the number of positions in the chains with common difference of succession equal to (-8). So,

$P_\rho = P_{\rho-1}^{Rec}+8$ where $\rho \geqslant 2$.

(ii) See Proof (i) Theorem 2.5.25. □

2.6 Conclusion

In this chapter, a new combinatorial interpretation called nested chain abacus for *n*-connected ominoes is presented. Furthermore, a new partition for every *n*-connected ominoes is developed. Then, we formulate and prove the uniqueness of nested chain abacus. In addition, we formulate the definition of connected bead positions in the nested chain abacus that will be fundamental in constructing a design structure for the nested chain abacus. Then, two different types of sequences were developed and proved. Based on the constructed design structure of the nested chain abacus, three transformations will be developed in Chapter Three.

Bellow is the summary of Chapter Two.

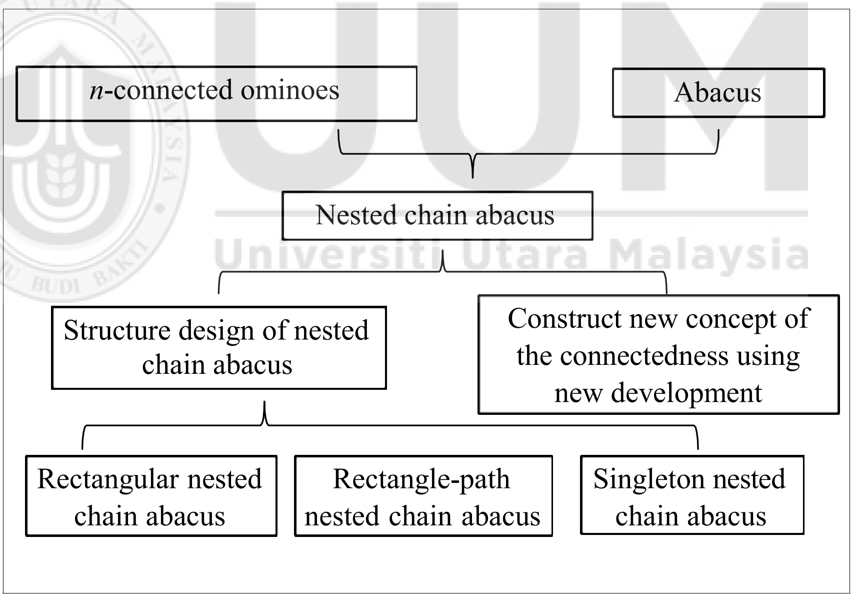


Figure 2.15. The structure of nested chain abacus.

CHAPTER THREE
NESTED CHAIN ABACUS TRANSFORMATION

3.1 Introduction

In this chapter, an algorithm for the nested chain abacus transformation is constructed, which is fundamental for constructing classes of the nested chain abacus that will be presented in chapter 4. First, related terminologies are formulated. Then, different types of transformation in the chains are formulated in rectangle chain, path chain and singleton chain. This is followed by the development of three different types of nested chain abacus transformation: Single nested chain abacus transformation with $e = 2$ (SNC2-Transformation), stratum nested chain abacus transformation with $e > 2$ (SNC-Transformation) and multiple nested chain abacus transformation (MNC-Transformation).

This chapter begins with the introduction. The necessary definitions and terminologies are defined in Section 3.2. The transformation in the chains with the three types are formulated in Section 3.3. Then, the nested chain abacus transformation algorithms are constructed based on chain transformation in Section 3.4.

3.2 Definition and Terminologies

Some terminologies that are needed in the nested chain abacus transformation constructed.

Definition 3.2.1. *Chain transformation* (Ch) *is a moving transformation when*

$$Ch:(m-1)e+(j-1)\longrightarrow\begin{cases}(m-1)e+(j+d-1), & d\in\mathbb{Z},\\(m+k-1)e+(j-1), & k\in\mathbb{Z},\end{cases}$$

which is in anticlockwise direction for all bead positions in chain i in the nested chain abacus with e columns and r rows where $0 \leqslant m \leqslant r-1$ *and* $0 \leqslant j \leqslant e-1$.

Remark 3.2.2. *In the chain transformation the beads will move by a specific distance. A bead located in the chain i and in the column i will move to the downwards if c is a positive integer. While, a bead located in the row* $r-i+1$ *will move to the rightwards if d is a positive integer. Meanwhile, the bead located in the column* $e-i+1$ *will move to the upwards if c is a negative integer. Finally, the bead located in row i will move to the leftwards if d is a positive integer, as show in Figure 3.1.*

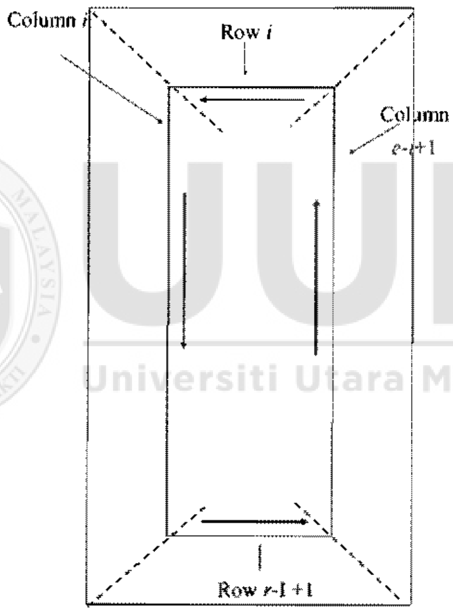


Figure 3.1. Chain transformation' direction

Remark 3.2.3. *Let* a_{mj} *be an element of a matrix* $A_{r\times e}$ *with e columns and r rows which represents the bead positions and empty bead positions in the nested chain abacus* $\mathfrak{N}$ *with c chains. Based on Definition3.2.1 and Lemma 2.5.1, then,*

$$Ch : a_{mj} \longrightarrow \begin{cases} a_{m(j+d)}, & d \in \mathbb{Z}, \\ a_{(m+k)j}, & k \in \mathbb{Z}, \end{cases}$$

which is in anticlockwise direction for all bead positions in chain i in the nested chain abacus with e columns and r rows where $1 \leqslant m \leqslant r$ *and* $1 \leqslant j \leqslant e$.

In our transformation we select one position in the chain as the initial point. When this point is moved rotationally anticlockwise to a new location, all other positions in the rectangle chain will move rotationally to a new location accordingly.

Definition 3.2.4. *Nested chain abacus transformation is a chain transformation in one or more chain in the nested chain abacus.*

Next, we construct the chain transformation of the bead positions inside the chains. Lemma 3.2.5 provides the basic concept for bead position movements.

Lemma 3.2.5. *Let* a_{mj} *be an element of a matrix* $A_{r\times e}$ *which represents the bead positions and empty bead positions in the nested chain abacus* $\mathfrak{N}$ *with e columns, r rows and c chains. Then,*

(i) *the bead positions in the rectangle chain i are located in columns* $\{i, e-i+1\}$ *and rows* $\{i, r-i+1\}$.

(ii) *the direction of the positions are such that*

- a_{ii} *through* $a_{(r-i+1)i}$ *is downwards.*
- $a_{(r-i+1)i}$ *through* $a_{(r-i+1)(e-i+1)}$ *is rightwards.*
- $a_{(r-i+1)(e-i+1)}$ *through* $a_{i(e-i+1)}$ *is upwards.*
- $a_{i(e-i+1)}$ *through* a_{ii} *is leftwards.*

Proof.

(i) Based on Definition 2.5.3 the elements of the rectangle chain are
$\left\{a_{mi}\, a_{m(e-i+1)}\, a_{ij}\, a_{(r-i+1)j} : i \leqslant m \leqslant (r-i+1) i \leqslant j \leqslant (e-i+1)\right\}$.

Thus, the bead positions of the rectangle chain i are located in columns $\{i, e-i+1\}$ and rows $\{i, r-i+1\}$.

(ii) Since the chain transformation is in anticlockwise direction for all bead positions in chain i in the nested chain abacus and based on Definition 3.2.1, Remark 3.2.2 and Remark 3.2.3 then,

- $Ch(a_{ii}) \longrightarrow a_{(i+k)i}, \ldots, Ch(a_{(r-i)i}) \longrightarrow a_{(r-i+1)i}$, and the positions will move downward.
- $Ch(a_{(r-i+1)i}) \longrightarrow a_{(r-i+1)(i+d)}, \ldots, Ch(a_{(r-i+1)(e-i)}) \longrightarrow a_{(r-i+1)(e-i+1)}$, and the positions will move rightwards.
- $Ch(a_{(r-i+1)(e-i+1)}) \longrightarrow a_{(r-i+1+k)(e-i+1)}, \ldots, Ch(a_{(i+1)(e-i+1)}) \longrightarrow a_{(i)(e-i+1)}$ and the positions will move upwards.
- $Ch(a_{i(e-i+1)}) \longrightarrow a_{i(e-i+1+d)}, \ldots, Ch(a_{i(i-1)}) \longrightarrow a_{ii}$ and the position will move leftwards.

Where d and k are positive integer. □

Definition 3.2.6. *The full chain is a chain in the nested chain abacus with e columns and r rows such that all positions are beads.*

Since all the positions are beads the form of chain will not change. The full chain with 2 columns and 6 rows as shown in Figure 3.2 (a) the chain transformation in full chain as in Figure 3.2 (b) and Figure 3.2 (c).

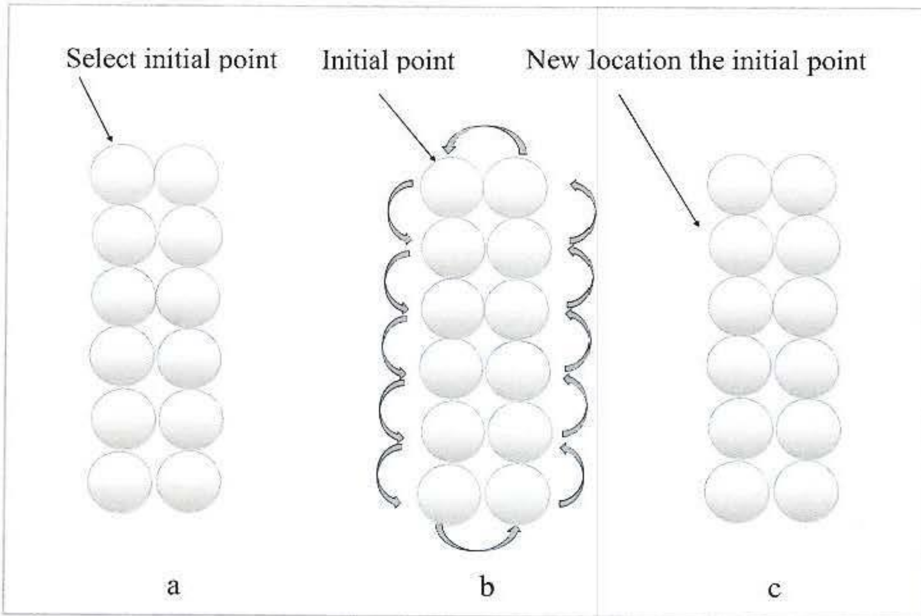


Figure 3.2. (a) Full chain, (b) Arrow indicating one step full chain transformation and (c) The new location of the initial point after transformation

3.3 Transformation in Chains

There are three cases of transformation based on three types of chains: transformation in rectangle chain, transformation in path chain and transformation in singleton chain. Each transformation is design details in the following subsections.

3.3.1 Transformation in Rectangle Chain

The transformation in rectangle chain is a chain transformation (Ch) employs in rectangle chain. First, we construct the transformation (Ch) if the beads skip one position anticlockwise in rectangle chain $\{b,k\} = \pm 1$ (see Definition 3.2.1)

Lemma 3.3.1. *Let a_{mj} be an element in matrix $A_{r\times e}$ which represents bead and empty bead positions in rectangle chain with e columns and r rows. Then, the transformation chain of a_{mj} is*

$$Ch(a_{mj}) \longrightarrow \begin{cases} a_{(m-1)j} & if \quad i+1 \leqslant m \leqslant (r-i+1),\ j=e-i+1, \\ a_{(m+1)j} & if \quad i \leqslant m \leqslant (r-i),\ j=i, \\ a_{m(j-1)} & if \quad m=i,\ i+1<j \leqslant e-i+1, \\ a_{m(j+1)} & if \quad m=(r-i+1),\ i \leqslant j < e-i, \end{cases}$$

where $1 \leqslant i \leqslant c$.

Proof. By Lemma 3.2.5(1) the position in the rectangle chain i location in two columns $\{i, e-i+1\}$ and two rows $\{r, r-i+1\}$. Based on Definition 3.2.1 and $\{a,b\} = \pm 1$ then the bead position in column i will skip one position downward so

$$Ch(a_{mj}) \longrightarrow a_{(m+1)j}$$

where $1 \leqslant m < r$. Since the direction of the positions movement anticlockwise then the positions $a_{(r-i+1)j}$ will skip to $a_{(r-i+1)(j+1)}$ one position where $i \leqslant j < e-i+1$. Since the direction of the positions move anticlockwise then the bead position location a_{mj} on the column $e-i+1$ will skip to $a_{(m-1)j}$ one position where $1 < m \leqslant r$. If $a_{mj} \in \{a_{mj} | m=i, i \leqslant j < e-i+1\}$ then

$$a_{mj} \longrightarrow a_{m(j-1)}.$$

□

Example 3.3.2 illustrates the transformation in rectangle chains.

Example 3.3.2. *Let* $\mu^{(4,7)} = (10, 8^6, 5, 2, 0^6)$ *be connected partition represents to* $\mathfrak{N}$ *for 15-connected beads as shown in Figure 3.4 (a). Based Definition 2.5.3 and Lemma 2.5.8 the nested chain abacus consists of two rectangle chains.*

First, for detail explanation we will show single movement for the Example 3.3.2. The bead position 19 which location in row 5 and column 4 (a_{54}) will be moved to the position 15 (a_{44}). The bead position 21 which location in row 6 and column 2 (a_{62}) will be moved to the position 22 (a_{63}) as shown in Figure 3.3.

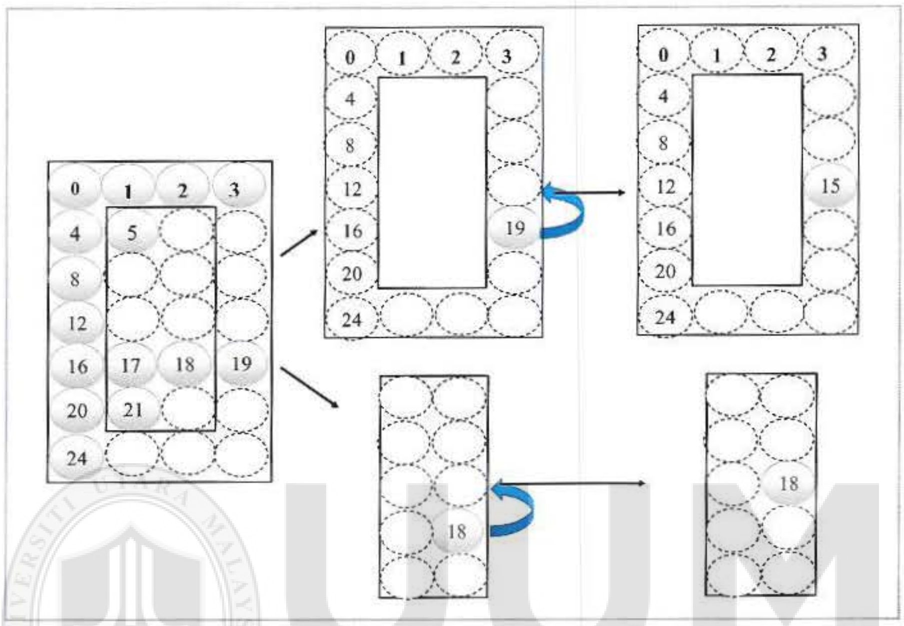


Figure 3.3. Single movement for $\mu^{(4,7)} = (10, 8^6, 5, 2, 0^6)$

The transformation in rectangle chain clarify in Figure 3.4 (b) and Figure 3.4 (c) where $\{b, k\} = \pm 1$.

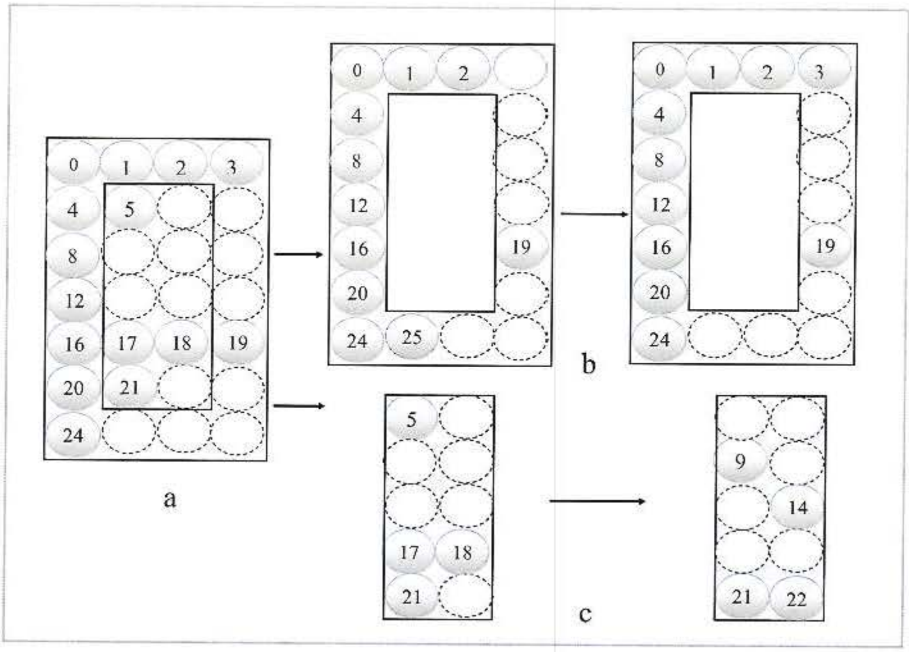


Figure 3.4. (a) Nested chain abacus of $\mu^{(4,7)} = (10, 8^6, 5, 2, 0^6)$ with 2 rectangle chains, (b) Rectangle chain transformation applied to the outer rectangle chains and (c) Rectangle chain transformation applied to the inner rectangle chain

The algorithm constructed to generate classes of nested chain abacus with c chains depended on the chain transformation. The fundamentals of the transformation in rectangle chains are constructed in Lemma 3.3.1 where each beads in chain i skip one position where $1 \leqslant i \leqslant c$.

In the following theorem, we establish the maximal x number of transformation (Ch^x) in rectangle chain.

Theorem 3.3.3. *Let $\mathfrak{N}$ be a nested chain abacus with e columns, r rows and c chains. Then, the maximal number of transformation in rectangle chain is*

$$2e + 2r - 8i + 3$$

where $1 \leqslant i \leqslant c$.

Proof. Let a_{mj} be an initial position in rectangle chain i for the nested chain abacus with e columns, r rows and c chains. Based on Lemma 3.2.5, a_{mj} can be moved downwards, upwards, rightwards or leftwards depending on m and j where $1 \leqslant m \leqslant r$ and $1 \leqslant j \leqslant e$. Based on Definition 2.5.3 and Lemma 3.2.5, if $j = i$ then the initial position will be moved downwards along the column i until location $a_{(r-i+1)i}$ after skipping

$$r - i + 1 - m$$

positions. Then, the position $a_{(r-i+1)i}$ will move from left to the right where the last location of the initial position is $a_{(r-i+1)(e-i+1)}$ after skipping

$$e - 2i + 1$$

positions. Since $a_{(r-i+1)(e-i+1)}$ in column $e - i + 1$ then the position $a_{(r-i+1)(e-i+1)}$ will move up and skip

$$r - 2i + 1$$

positions to get to the position $a_{i(e-i+1)}$. Furthermore, the initial position will move from right to the left and there are skip

$$e - 2i + 1$$

positions until it reach location a_{ii}. Finally, the initial positions a_{ii} skip $m - i$ positions to return back. Thus the initial position a_{mj} will skip

$$2e + 2r - 4$$

positions to move and to return to its original position. The same applies if the initial position is $a_{(r-i+1)j}$ or $a_{m(e-i+1)}$ or a_{ij}. Hence,

$$2e + 2r - 8i + 3$$

is the maximum of chain transformation for each position in the rectangle chain. □

The following corollary describes the number of possible transformations for a positions in outer chain.

Corollary 3.3.4. *Let $\mathfrak{N}$ be the nested chain abacus with e columns, r rows and c chains. Then, the maximum number of transformation in the outer chain is*

$$2e+2r-5.$$

Proof. From Theorem 3.3.3, the maximal number of transformation in a rectangle chain is

$$2e+2r-8i+3.$$

Since the outer chain is chain 1 in the nested chain abacus then the maximal number of transformation in the outer chain is

$$2e+2r-5.$$

□

Not: Notation $a_{mj} \xrightarrow{y}$ means a_{mj} will skip y positions

We formulated Theorem 3.3.5 to find the transformation in rectangle chain (Ch^r) if the beads skips x positions in four cases depending on the location of the beads in the rectangle chain where $1 \leqslant x \leqslant 2e+2r-8i+3$. First, we will describe the location of the bead positions in each case which are T_1, T_2, T_3 and T_4 where

$T_1 = \{a_{mi} | i \leqslant m < r-i+1\}$,

$T_2 = \{a_{(r-i+1)j} | i \leqslant j < e-i+1\}$,

$T_3 = \{a_{m(e-i+1)} | i \leqslant m < r-i+1\}$ and

$T_4 = \{a_{ij} | i \leqslant j \leqslant e - i + 1\}$.

Figures 3.5 until 3.8 provide an indication to determine the location of the positions in the sets T_1, T_2, T_3 and T_4.

$$
\begin{array}{llllll}
a_{ii} & a_{i(i+1)} & a_{i(i+2)} & a_{i(i+3)} & \cdots & a_{i(e-i+1)} \\
a_{(i+1)i} & . & . & . & \cdots & a_{(i+1)(e-i+1)} \\
a_{(i+2)i} & . & . & . & \cdots & a_{(i+2)(e-i+1)} \\
a_{(i+3)i} & . & . & . & \cdots & a_{(i+3)(e-i+1)} \\
. & . & . & . & \cdots & . \\
. & . & . & . & \cdots & . \\
. & . & . & . & \cdots & . \\
a_{(r-i)i} & . & . & . & \cdots & a_{(r-i)(e-i+1)} \\
a_{(r-i+1)i} & a_{(r-i+1)(i+1)} & a_{(r-i+1)(i+2)} & a_{(r-i+1)(i+3)} & \cdots & a_{(r-i+1)(e-i+1)}
\end{array}
$$

Figure 3.5. Elements for T_1

$$
\begin{array}{llllll}
a_{ii} & a_{i(i+1)} & a_{i(i+2)} & a_{i(i+3)} & \cdots & a_{i(e-i+1)} \\
a_{(i+1)i} & . & . & . & \cdots & a_{(i+1)(e-i+1)} \\
a_{(i+2)i} & . & . & . & \cdots & a_{(i+2)(e-i+1)} \\
a_{(i+3)i} & . & . & . & \cdots & a_{(i+3)(e-i+1)} \\
. & . & . & . & \cdots & . \\
. & . & . & . & \cdots & . \\
. & . & . & . & \cdots & . \\
a_{(r-i)i} & . & . & . & \cdots & a_{(r-i)(e-i+1)} \\
a_{(r-i+1)i} & a_{(r-i+1)(i+1)} & a_{(r-i+1)(i+2)} & a_{(r-i+1)(i+3)} & \cdots & a_{(r-i+1)(e-i+1)}
\end{array}
$$

Figure 3.6. Elements for T_2

$$
\begin{array}{llllll}
a_{ii} & a_{i(i+1)} & a_{i(i+2)} & a_{i(i+3)} & \cdots & a_{(i+1)(e-i+1)} \\
a_{(i+2)i} & . & . & . & \cdots & a_{(i+2)(e-i+1)} \\
a_{(i+3)i} & . & . & . & \cdots & a_{(i+3)(e-i+1)} \\
. & . & . & . & \cdots & . \\
. & . & . & . & \cdots & . \\
. & . & . & . & \cdots & . \\
a_{(r-i)i} & . & . & . & \cdots & a_{(r-i)(e-i+1)} \\
a_{(r-i+1)i} & a_{(r-i+1)(i+1)} & a_{(r-i+1)(i+2)} & a_{(r-i+1)(i+3)} & \cdots & a_{(r-i+1)(e-i+1)}
\end{array}
$$

Figure 3.7. Elements for T_3

$$
\begin{array}{cccccc}
a_{ii} & a_{i(i+1)} & a_{i(i+2)} & a_{i(i+3)} & \cdots & a_{i(e-i+1)} \\
a_{(i+1)i} & . & . & . & \cdots & a_{(i+1)(e-i+1)} \\
a_{(i+2)i} & . & . & . & \cdots & a_{(i+2)(e-i+1)} \\
a_{(i+3)i} & . & . & . & \cdots & a_{(i+3)(e-i+1)} \\
. & . & . & . & \cdots & . \\
. & . & . & . & \cdots & . \\
. & . & . & . & \cdots & . \\
a_{(r-i)i} & . & . & . & \cdots & a_{(r-i)(e-i+1)} \\
a_{(r-i+1)i} & a_{(r-i+1)(i+1)} & a_{(r-i+1)(i+2)} & a_{(r-i+1)(i+3)} & \cdots & a_{(r-i+1)(e-i+1)}
\end{array}
$$

Figure 3.8. Elements for T_4

Theorem 3.3.5. *Let a_{mj} be an element in a rectangle chain in the nested chain abacus $\mathfrak{N}$ with e columns and r rows represented by matrix $A_{r\times e}$. Then, the transformation of a_{mj} is*

Case one

$$
a_{mi} \longrightarrow \begin{cases}
a_{(m+x)i} & if \quad i \le m+x < r-i+1, \\
a_{(r-i+1)(x-r+2i+m-1)} & if \quad r-i+1 \le m+x < r+e-3i+2, \\
a_{(2r-x-m+e-4i+3)(e-i+1))} & if \quad e+r-3i+2 \le m+x < 2r+e-5i+3, \\
a_{i(2e+2r-m-x-6i+4)} & if \quad 2r+e-5i+3 \leqslant m+x < 2r+2e-7i+4, \\
a_{(x-2r-2e+m+8i-4)i} & if \quad 2r+2e-7i+4 \le m+x < 3r+2e-9i+5,
\end{cases}
$$

if $a_{mj} \in T_1$ where $T_1 = \{a_{mj} | j = i, i \le m < -i+1\}$.

Case two

$$
a_{nm} \longrightarrow \begin{cases}
a_{m(j+x)} & if \quad (j+x) \le e-i+1, \\
a_{(r+e-2i-x-j+2)(e-i+1)} & if \quad e-i+1 < j+x \le e+r-3i+2, \\
a_{i(2e+r-4i-x-j+3)} & if \quad e+r-3i+2 < j+x \le 2e+r-5i+3, \\
a_{(x-2e-r+6i+j-3)i} & if \quad 2e+r-5i+3 < j+x \le 2e+2r-7i+4, \\
a_{f(x-2e-2r+8i+j-4)} & if \quad 2e+2r-7i+4 < n+x \le 3e+2r-9i+5,
\end{cases}
$$

if $a_{mj} \in T_2$ *where* $T_2 = \{a_{mj} | m = r-i+1, i \leqslant j < e-i+1\}$,

Case three

$$a_{mj} \longrightarrow \begin{cases} a_{(m-x)(e-i+1)} & if \quad m-x \geqslant i, \\ a_{i(m+e-2i-x+1)} & if \quad 3i-e-1 \geqslant m-x \geqslant i, \\ a_{(x+4i-m-e-1)i} & if \quad 5i-e-r-2 \geqslant m-x \geqslant 3i-e-1, \\ a_{(r-i+1)(x-m-e-r+6i-2)} & if \quad 7i-2e-r-3 \geqslant m-x \geqslant 5i-e-r-2, \\ a_{(2r+2e-8i-x+4+m)(e-i+1)} & if \quad 9i-2e-2r-4 \geqslant m-x \geqslant 7i-2e-r-3, \end{cases}$$

if $a_{mj} \in T_3$ *where* $T_3 = \{a_{mj} | j = e-i+1, i < m \leqslant r-i+1\}$,

Case four:

$$a_{mj} \longrightarrow \begin{cases} a_{i(j-x)} & if \quad j-x \geqslant i, \\ a_{(x-j+2i)i} & if \quad i > j-x \geq 3i-r-1, \\ a_{(r-i+1)(x-j-r+4i-1)} & if \quad 3i-r-1 > j-x \geqslant 5i-e-r-2, \\ a_{(j-x+2r+e-6i+3)(e-i+1)} & if \quad 5i-e-r-2 > n-x \geq 7i-e-2r-3, \\ a_{i(j-x+2r+2e-8i+4)} & if \quad 7i-e-2r-3 > n-x \geq 9i-2e-2r-4, \end{cases}$$

if $a_{mj} \in T_4$ *where* $T_4 = \{a_{mj} | i \leqslant j \leqslant e-i+1, m = i\}$, *for* $1 \leqslant i < c$ *and*

$1 \leqslant x \leqslant 2e+2r-8i+3$.

Proof.

Case one:

(i) If $m+x \leqslant r-i+1$.

Based on Lemma 3.2.5(2) a_{mi} will move downwards and skip x positions. Since

$x \leqslant r-i+1-m$ then $Ch(a_{mi}) \in T_1$ and

$$Ch : a_{mi} \longrightarrow a_{(m+x)i}.$$

(ii) If $r-i+1 < m+x \leqslant r+e-3i+2$.

Based on Definition 2.5.3 and Lemma 3.2.5 a_{mi} will move downwards from a_{mi} through $a_{(r-i+1)i}$. Since $x > r-i+1-m$ and $a_{mi} \in T_1$ then, a_{mi} will skip

$$r-i-m+1$$

positions and then move rightwards to a new location after skipping

$$x-r+i+m-1$$

positions (See Lemma 3.2.5). Thus

$$Ch : a_{mi} \longrightarrow a_{(r-i+1)(x-r+2i+m-1)}.$$

(iii) If $e+r-3i+2 < m+x \leqslant 2r+e-5i+3$.

Based on Definition 2.5.3 and Lemma 3.2.5 a_{mi} will move downwards to $a_{(r-i+1)i}$. Since $x > (r-i+1-m)+(e-2i+1)$ then the new location is $\in T_3$. Thus, a_{mi} will move rightwards until $a_{(r-i+1)(e-i+1)}$ and then upward. Based on Lemma 3.3.1,

$$\begin{aligned} Ch : a_{mi} &\xrightarrow{r-i-m+1} a_{(r-i+1)i}, \\ Ch : a_{(r-i+1)i} &\xrightarrow{e-2i+1} a_{(r-i+1)(e-i+1)}, \end{aligned}$$

and

$$Ch : a_{(r-i+1)(e-i+1)} \longrightarrow a_{i(e+2r-m-x-4i+3)}.$$

Thus,

$$Ch : a_{mi} \longrightarrow a_{i(e+2r-m-x-4i+3)}.$$

(iv) If $2r+e-5i+3 < m+x \leqslant 2r+2e-7i+4$.

Since

$$x > (r-i+1-m)+(e-2i+1)+(r-2i+1)$$

and Based on Definition 2.5.3 and Lemma 3.2.5, then, the new location $\in T_4$. Based on Lemma 3.3.1,

$$Ch : a_{mi} \xrightarrow{r-i-m+1} a_{(r-i+1)i};$$
$$Ch : a_{(r-i+1)i} \xrightarrow{e-2i+1} a_{(r-i+1)(e-i+1))}.$$

Then,

$$Ch : a_{(r-i+1)(e-i+1))} \xrightarrow{(r-2i+1)} a_{i(e+2r-m-5i+3)};$$

and

$$Ch : a_{i(e+2r-m-5i+3)} \xrightarrow{x-2r-e+5i+m-3} a_{(2r+2e-x-6i-m+4)i}.$$

Thus,

$$Ch : a_{mi} \longrightarrow a_{(2r+2e-x-6i-m+4)i}.$$

(v) If $2r+2e-7i+4 \leq m+x < 3r+2e-9i+5$.

Since $x > (r-i+1-m)+(e-2i+1)+(r-2i+1)+(e-2i+1)$ and based on Definition 2.5.3 and Lemma 3.2.5, the new location is $\in T_5$. Thus a_{mi} will be moved downwards to $a_{(r-i+1)i}$ and from $a_{(r-i+1)i}$ through $a_{(r-i+1)(e-i+1)}$ rightwards and then upwards until $a_{i(e-i+1)}$. From $a_{i(e-i+1)}$ through a_{ii} the bead positions will move leftwards, and at a_{ii} the bead position will move downward. Based on Lemma 3.3.1,

$$Ch : a_{mi} \xrightarrow{r-i-m+1} a_{(r-i+1)i},$$

and,

$$a_{(r-i+1)i} \xrightarrow{e-2i+1} a_{(r-i+1)(e-i+1))}.$$

Thus

$$Ch : a_{(r-i+1)(e-i+1))} \xrightarrow{r-2i+1} a_{i(e+2r-m-5i+3)},$$
$$Ch : a_{i(e+2r-m-5i-3)} \xrightarrow{e-2i+1} a_{(x-2(r-e)+m+7i-3)i},$$
$$Ch : a_{(x-2r-2e+m+7i-3)i} \xrightarrow{i} a_{(x-2r-2e+m+8i-4)i}.$$

Hence

$$Ch : a_{mi} \longrightarrow a_{(x-2r-2e+m+8i-4)i}.$$

□

The proof for case two, three and four are similar as case one.

Corollary 3.3.6. *Let a_{mj} be an element in a rectangle chain in the nested chain abacus $\mathfrak{N}$ with 2 columns and r rows represented by matrix $A_{r\times 2}$. Then, the x of Ch^x transformation of a_{mj} is*

$$a_{mj} \longrightarrow \begin{cases} a_{(m+x)1} & if \quad j=1, m+x \leqslant r, \\ a_{(m-x)2} & if \quad j=2, m-x \geqslant 1, \\ a_{(x-m+1)1} & if \quad j=2, 1-r \leqslant m-x < 1, \\ a_{(2r-x+m)2} & if \quad j=2, 1-2r \leqslant m-x \leqslant -r, \\ a_{(2r-x-m+1)2} & if \quad j=1, r < m+x \leqslant 2r, \\ a_{(x-2r+m)1} & if \quad j=1, 2r < m+x \leqslant x-2r-1, \end{cases}$$

where $1 \leqslant x \leqslant 2e+2r-8i+3$ for $1 \leqslant m \leqslant r$ and $j = 1, 2e$.

Proof. It follows immediately from the proof in Theorem 3.3.5. □

Example 3.3.7 illustrates a chain transformation in the nested chain abacus where $e = 2$.

Example 3.3.7. *Let* $(3^7, 2, 1)^{(2,6)}$ *be a connected partition representing nested chain abacus* $\mathfrak{N}$ *for 9-connected beads as shown in Table 3.1 and Figure 3.9. For example position 1 or* (a_{12}) *move by Ch to 0 or* (a_{11}).

Table 3.1

Original and new location of bead positions where $e = 2$.

a_{m2}	$\longrightarrow$	$Ch(a_{m2})$
1 (a_{12})	$\longrightarrow$	0 (a_{11})
3 (a_{22})	$\longrightarrow$	1 (a_{12})
5 (a_{32})	$\longrightarrow$	3 (a_{22})
6 (a_{41})	$\longrightarrow$	8 (a_{51})
7 (a_{42})	$\longrightarrow$	5 (a_{32})
8 (a_{51})	$\longrightarrow$	10 (a_{61})
9 (a_{52})	$\longrightarrow$	7 (a_{42})
10 (a_{61})	$\longrightarrow$	11 (a_{62})
11 (a_{62})	$\longrightarrow$	9 (a_{52})

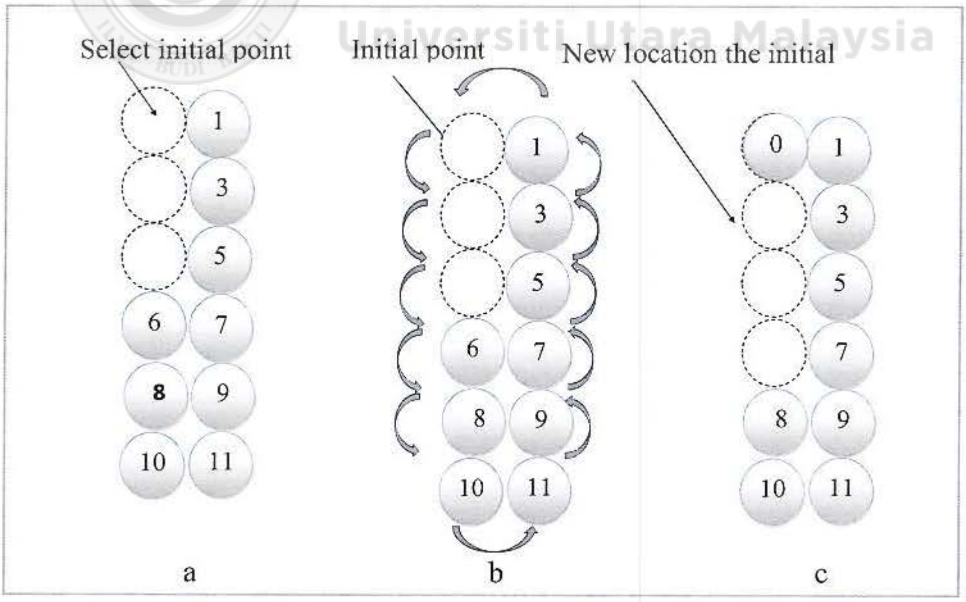


Figure 3.9. (a) Selected initial point in a rectangle chain with 9 bead positions and 3 empty bead positions, (b) Arrows indicating rectangle chain transformation and (c) The result of chain transformation

3.3.2 Transformation in Path Chain

Transformation in Path Chain is a chain transformation where $d = 0$. Based on Chapter two there are two designs of path chain. Thus there are two types of transformation in path chain; First, we construct the transformation in vertical path chain (Ch) if the beads skip one position anticlockwise as shown in Lemma 3.3.8.

Lemma 3.3.8. *Let a_{mj} be an element in a vertical-path chain in the nested chain abacus $\mathfrak{N}$ with e columns, r rows and c chains represented by matrix $A_{r\times e}$. Then, (Ch) transformation of a_{mj} is*

$$a_{mj} \longrightarrow \begin{cases} a_{(m+1)j} & if \quad \frac{e+1}{2} \leqslant m < \frac{2r-e+1}{2}, j = \frac{e+1}{2}, \\ a_{\left(\frac{e+1}{2}\right)\left(\frac{e+1}{2}\right)} & if \quad m = \frac{2r-e+1}{2}, j = \frac{e+1}{2}. \end{cases}$$

Proof. Based on Definition 2.5.12, T_c is a set of positions of the vertical-path chain c where $T_c = \{a_{cc}, a_{(c+1)c}, \ldots, a_{(r-c+1)c}\}$ and

$$a_{cc} = a_{\left(\frac{e+1}{2}\right)\left(\frac{e+1}{2}\right)}$$

such that

$$a_{(r-c+1)c} = a_{\left(\frac{2r-e+1}{2}\right)\left(\frac{e+1}{2}\right)}.$$

Since a_{mj} is a bead position in the path chain and the bead movement is anticlockwise then, bead positions $\{a_{cc}, \ldots, a_{(r-c)c}\}$ will be skip one position downwards and

$$a_{mj} \longrightarrow a_{(m+1)j}.$$

Thus,

$$a_{\left(\frac{e+1}{2}\right)\left(\frac{e+1}{2}\right)} \longrightarrow a_{\left(\frac{2r-e+1}{2}\right)\left(\frac{e+1}{2}\right)}$$

where $\frac{e+1}{2} \leqslant m < \frac{2r-e+1}{2}$ and $j = \frac{e+1}{2}$. □

Figure3.10 illustrates the Lemma 3.3.8.

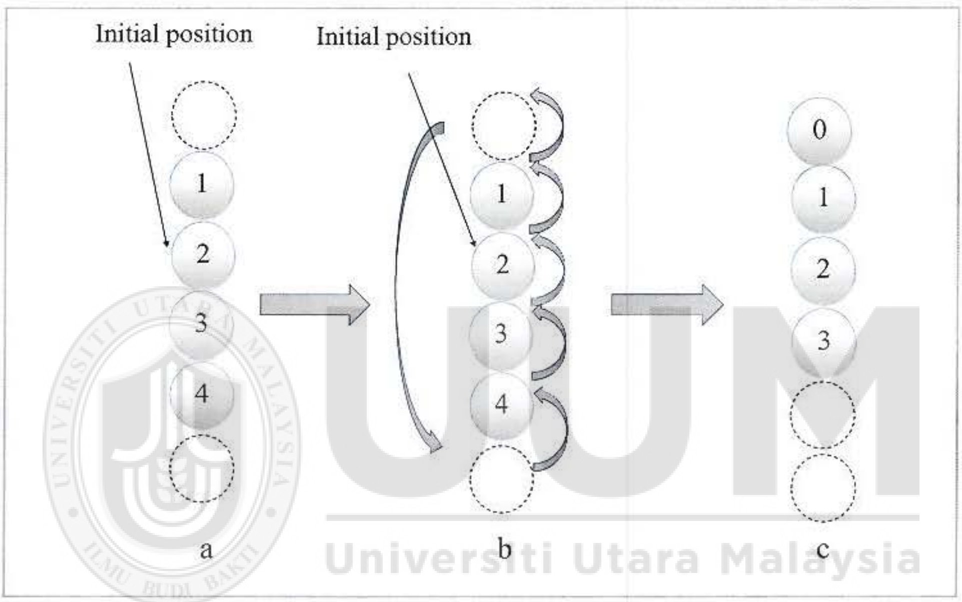


Figure 3.10. (a) Selected initial point in a path chain with 4 bead positions, (b) Arrows indicating x-steps path chain transformation for $x = 1$ and (c) The result of chain transformation for $x = 1$

The fundamentals of the transformation in the vertical-path chain is constructed in Lemma 3.3.8 where each bead in path chain skips one position anticlockwise.

In Theorem 3.3.9 we determine the maximal number of transformations in the vertical-path chain.

Theorem 3.3.9. *Let a_{mj} be an element in the vertical-path chain in the nested chain abacus $\mathfrak{N}$ with e columns and r rows. Then, the maximal number of the vertical-path transformations (Ch) is*

$$r - e + 1.$$

Proof. Let a_{mj} be a position in the vertical-path chain for the nested chain abacus with e columns, r rows and c chains. Based on Lemma 3.2.5, a_{mj} can be moved downwards and upwards where

$$\frac{e+1}{2} \leqslant m \leqslant \frac{2r-e+1}{2},\ j=c.$$

Therefore

$$a_{\left(\frac{e+1}{2}\right)\left(\frac{e+1}{2}\right)} \longrightarrow a_{\left(\frac{e+3}{2}\right)\left(\frac{e+1}{2}\right)} \longrightarrow \cdots \longrightarrow a_{\left(\frac{2r-e+1}{2}\right)}$$

and

$$a_{\left(\frac{2r-e+1}{2}\right)\left(\frac{e+1}{2}\right)} \longrightarrow a_{\left(\frac{e+1}{2}\right)\left(\frac{e+1}{2}\right)}.$$

Thus a_{mj} will skip

$$\frac{2r-e+1}{2} - \frac{e+1}{2} + 1 = r-e+1$$

positions to return to its original location. □

In Theorem 3.3.10 we observe that all transformations are in the vertical-path chain (Ch^x) if the beads skips x positions.

Theorem 3.3.10. *Let a_{mj} be an element in the vertical-path chain in the nested chain abacus $\mathfrak{N}$ with e columns, r rows and c chains represented by the matrix $A_{r\times e}$. Then, transformation Ch^x*

$$a_{mj} \longrightarrow \begin{cases} a_{(m+x)j} & if \quad m+x \leqslant \dfrac{2r-e+1}{2}, j=\dfrac{e+1}{2}, \\ & \\ a_{\left(\frac{2x-2r+2e+2m-2}{2}\right)\left(\frac{e+1}{2}\right)} & if \quad m+x > \dfrac{2r-e+1}{2}, j=\dfrac{e+1}{2}, \end{cases}$$

where $1 \leqslant x \leqslant r-e+1$.

Proof. Based on Definition 2.5.12, T_c is a set of the positions of vertical-path in chain c for $T_c = \{a_{cc}, a_{(c+1)c}, \ldots, a_{(r-c+1)c}\}$ and $c = \dfrac{e+1}{2}$ where x refers to the number of positions that the bead positions will skip to get to the new location. Based on Lemma 3.3.8 the direction of bead position movement is downwards, upwards and downwards. If $m+x \in T_c$ then the bead positions will move downwards and skip x positions. Then

$$a_{(mj)} \xrightarrow{\frac{2r-e-m+1}{2}} a_{\left(\frac{2r-e+1}{2}\right)} \xrightarrow{1} a_{\left(\frac{e+1}{2}\right)\left(\frac{e+1}{2}\right)}$$

and

$$a_{\left(\frac{e+1}{2}\right)\left(\frac{e+1}{2}\right)} \xrightarrow{\left(\frac{2x-2r+e+2m-3}{2}\right)} a_{\left(\frac{2x-2r+2e+2m-2}{2}\right)\left(\frac{e+1}{2}\right)}.$$

Thus,

$$a_{(mj)} \xrightarrow{\left(\frac{2x-2r+e+2m-3}{2}\right)} a_{\left(\frac{2x-2r+2e+2m-2}{2}\right)\left(\frac{e+1}{2}\right)}$$

where $m+x \notin T_c$. □

Next, we construct the horizontal-path chain transformation in the chain. Firstly, the chain transformation is formulated when the position skip for one time.

Lemma 3.3.11. *Let a_{mj} be an element in the horizontal-path chain in the nested chain abacus $\mathfrak{N}$ with e columns, r rows and c chains represented by matrix $A_{r\times e}$. Then, transformation Ch:*

$$a_{mj} \longrightarrow \begin{cases} a_{m(j+1)} & if \quad \frac{r+1}{2} \leqslant j < \frac{2e-r+1}{2}, m = \frac{r+1}{2}, \\ a_{(\frac{r+1}{2})(\frac{r+1}{2})} & if \quad m = \frac{r+1}{2}, j = \frac{2e-r+1}{2}. \end{cases}$$

Proof. Suppose T_c is a set of the positions of chain c, Since the nested chain abacus is horizontal rectangle-path structure and based on Definition 2.5.12, then

$$T_c = \{a_{cc}, a_{c(c+1)}, \ldots, a_{c(e-c+1)}\}.$$

Based on Lemma 2.5.16(2) then

$$a_{cc} = a_{\left(\frac{r+1}{2}\right)\left(\frac{r+1}{2}\right)},$$

and

$$a_{c(e-c+1)} = a_{\left(\frac{r+1}{2}\right)\left(\frac{2e-r+1}{2}\right)}$$

where $c = \frac{r+1}{2}$. Since a_{mj} is a bead position in the path chain and the bead movement is anticlockwise, then bead positions

$$\{a_{c(c+1)}, \ldots, a_{c(e-c+1)}\}$$

will skip one position rightwards and

$$a_{mj} \longrightarrow a_{m(j+1)}, \; a_{cc} \longrightarrow a_{c(e-c+1)}$$

where $\frac{r+1}{2} \leqslant j < \frac{2e-r+1}{2}$ and $m = c$. □

In the following theorem we will determine the number of possible transformations in the horizontal-path chain.

Theorem 3.3.12. *Let a_{mj} be an element in the horizontal-path chain in the nested chain abacus $\mathfrak{N}$ with e columns, r rows and c chains represented by matrix $A_{r\times e}$. Then, the total number of movement of each position in the path-chain is*

$$e-r+1.$$

Proof. Let a_{mj} be a position in the horizontal-path for nested chain abacus with e columns, r rows and c chains. From Definition 2.5.12 and Lemma 3.2.5, a_{mj} can be moved rightwards and leftwards depending on m. Therefore

$$a_{\left(\frac{r+1}{2}\right)\left(\frac{r+1}{2}\right)} \longrightarrow a_{\left(\frac{r+1}{2}\right)\left(\frac{r+3}{2}\right)} \longrightarrow \cdots \longrightarrow a_{\left(\frac{r+1}{2}\right)\left(\frac{2e-r+1}{2}\right)}$$

Then

$$a_{\left(\frac{r+1}{2}\right)\left(\frac{2e-r+1}{2}\right)} \longrightarrow a_{\left(\frac{r+1}{2}\right)\left(\frac{r+1}{2}\right)}$$

where $\frac{e+1}{2} \leqslant m \leqslant \frac{2e-r+1}{2}$ and $j=c$. Thus a_{mj} will be move

$$\frac{2e-r+1}{2}-\frac{r+1}{2}+1=e-r+1$$

steps to return to its original location. □

Next, we develop transformation Ch^x where the beads will skip x positions.

Theorem 3.3.13. *Let a_{mj} be an element in matrix $A_{r\times e}$ which represents the bead and empty bead positions in the horizontal-rectangle path chain with e columns and r rows. Then, the chain transformation Ch^x is:*

$$a_{mj} \longrightarrow \begin{cases} a_{m(j+x)} & if \quad \frac{r+1}{2} \leqslant j < \frac{2e-r+1}{2}, m = \frac{r+1}{2}, \\ a_{\left(\frac{2x-2e+2r+2m-2}{2}\right)\left(\frac{r+1}{2}\right)} & if \quad m = \frac{r+1}{2}, j = \frac{2e-r+1}{2}, \end{cases}$$

where $1 \leqslant x \leqslant e-r+1$.

Proof. It follows immediately from Theorem 3.3.10 and Lemma 3.3.11 □

3.3.3 Transformation in Singleton Chain

Now we construct a transformation in singleton chain.

Lemma 3.3.14. *Let a_{mj} be an element in matrix $A_{r\times e}$ which represents the bead positions and empty bead positions in the singleton nested chain abacus $\mathfrak{N}$ with r rows, e columns and c chains such that the bead position a_{mj} is located in singleton-chain. Then, transformation $Ch : a_{mj} \longrightarrow a_{mj}$ where $m = \frac{r+1}{2}, n = \frac{r+1}{2}$*

Proof. Based on Definition 2.5.18, the singleton chain consists of one position located in column $\frac{e+1}{2}$ and row $\frac{e+1}{2}$ then $a_{mj} \longrightarrow a_{mj}$. □

Figure 3.11 illustrates the rectangle chain transformation and singleton chain transformation.

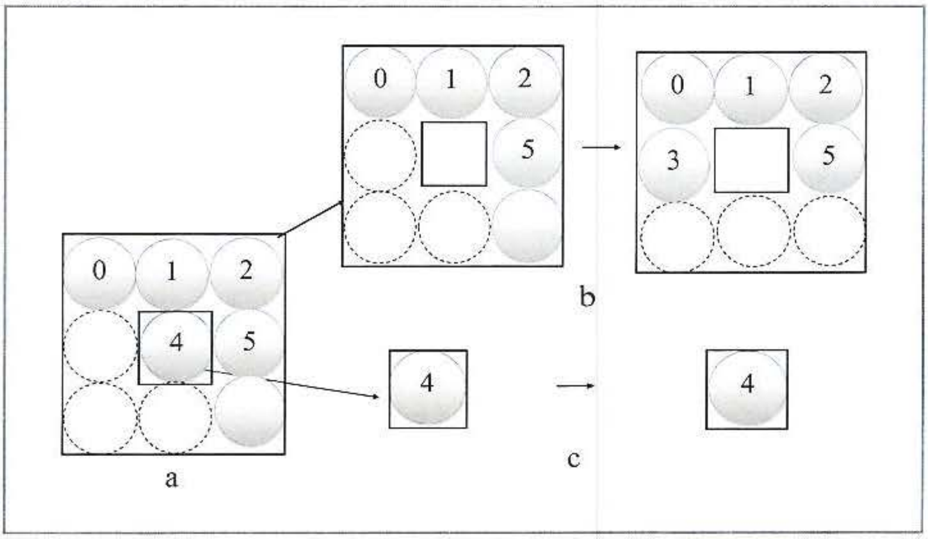


Figure 3.11. (a) Nested chain abacus with one rectangle chain and one singleton chain, (b) Rectangle chain transformation applied to the outer rectangle chain and (c) Singleton chain transformation applied as the singleton chain

In the next section, a new algorithm called nested chain abacus transformation is constructed.

3.4 Nested Chain Abacus Transformation Algorithm

This is followed by the development of three different types of nested chain abacus transformation which are single nested chain abacus transformation with $e = 2$ (SNC2-Transformation), stratum nested chain abacus transformation with $e > 2$ (SNC-Transformation) and multiple chain transformation (MNC-Transformation).

3.4.1 SNC2-Transformation

A transformation of a nested chain abacus with a chain is called SNC2-Transformation construction. We use $n = 7$ to explain the algorithm for SNC2-Transformation

Step 1: Convert the nested chain abacus with n connected beads and one chain into $A_{r\times 2}$.

Consider Figure 3.12(a) for a nested chain abacus with one chain and 7-beads represented by connected partition $\mu^{(2,5)} = (3^5, 2, 1)$. The nested chain abacus is then converted in to $A_{5\times 2}$ as shown in Figure 3.12(b).

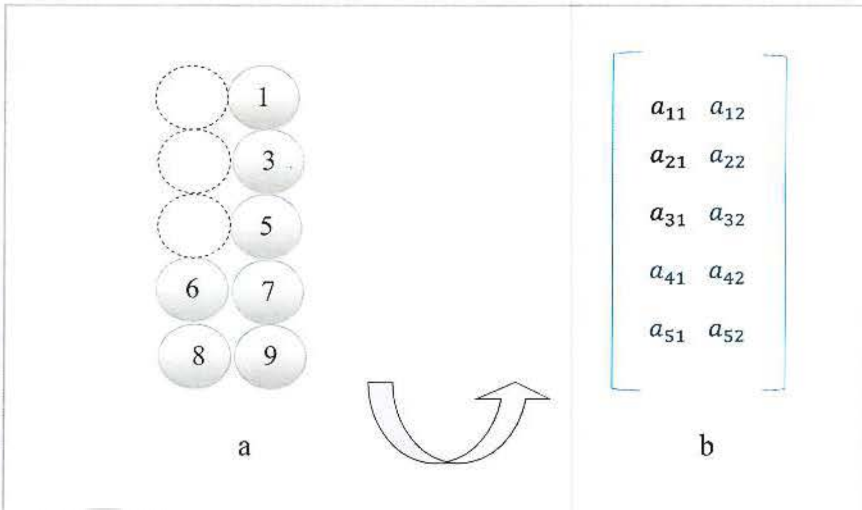


Figure 3.12. Nested chain abacus converted into matrix

Step 2: Select a_{mj} as an initial point where a_{mj} is an element in the $r \times e$ matrix. Consider Figure 3.12(b) in which a_{41} is selected as an initial point.

Step 3: Generate different of nested chain abacus $\mathfrak{N}$ with one chain by employing Ch^x with 2 columns (see Corollary 3.3.6), based on Theorem 3.3.3 with $e = 2$, $1 \leqslant x \leqslant 2r$. Consider Figure 3.12(b) of 7 connected beads where we employ the transformation in rectangle chain Ch^x, where $x = 9$. Based on Corollary 3.3.6, Table 3.2 show the new location of beads. For example a_{41} move by Ch to a_{51}, by Ch^2 to a_{52}, by Ch^3 to a_{42}, by Ch^4 to a_{32}, by Ch^5 to a_{22}, by Ch^6 to a_{12}, by Ch^7 to a_{11}, by Ch^8 to a_{21} and by Ch^9 to a_{31}, similar for the rest.

Table 3.2

New location of the positions in the nested chain abacus by SNC2-Transformation

a_{mj}	$\longrightarrow$	Ch	Ch^2	Ch^3	Ch^4	Ch^5	Ch^6	Ch^7	Ch^8	Ch^9
a_{41}	$\longrightarrow$	a_{51}	a_{52}	a_{42}	a_{32}	a_{22}	a_{12}	a_{11}	a_{21}	a_{31}
a_{51}	$\longrightarrow$	a_{52}	a_{42}	a_{32}	a_{22}	a_{12}	a_{11}	a_{21}	a_{31}	a_{41}
a_{12}	$\longrightarrow$	a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{52}	a_{42}	a_{32}	a_{22}
a_{22}	$\longrightarrow$	a_{12}	a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{52}	a_{42}	a_{32}
a_{32}	$\longrightarrow$	a_{22}	a_{12}	a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{52}	a_{42}
a_{42}	$\longrightarrow$	a_{32}	a_{22}	a_{12}	a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{52}
a_{52}	$\longrightarrow$	a_{42}	a_{32}	a_{22}	a_{12}	a_{11}	a_{21}	a_{31}	a_{31}	a_{51}

Figure below illustrates the result of the SNC2-Transformation algorithm.

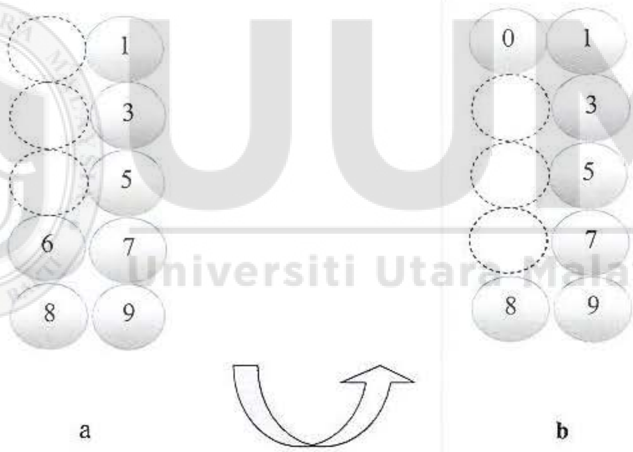


Figure 3.13. Nested chain abacus with one chain in (a) and the result of applying Ch^1 in (b)

3.4.2 SNC-Transformation

A transformation of a nested chain abacus with c chains is called SNC-Transformation construction. In SNC-Transformation, we employ the chain transformation in only one chain. We use $n = 19$ to explain the algorithm for SNC-Transformation employed in chain 2.

Step 1: Convert the nested chain abacus into $A_{r \times e}$.

Consider Figure 3.14(a) for a nested chain abacus with two chains and 19 beads represented by connected partition $\mu^{(5,4)} = (1^{13}, 0^{6})$. The nested chain abacus is converted into $A_{4\times5}$ as shown in Figure 3.14(b).

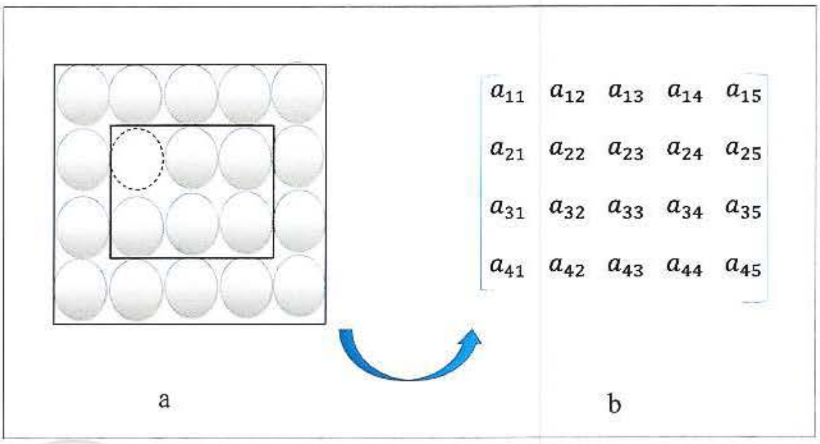


Figure 3.14. Nested chain abacus in (a) and convert into matrix in (b)

Step 2: Select the chain i and then a_{mj} as an initial point where $i \leqslant j \leqslant e-i+1$ and $i \leqslant m \leqslant r-i+1$.

Consider Figure 3.14(b) where $i = 2$ and a_{32} are selected as number of chain and an initial point respectively.

Step 3: Generate different of nested chain abacus $\mathfrak{N}$ with c chain by employing

- Theorem 3.3.5 to find the transformation Ch^x in chain i if chain i is rectangle chain, based on Theorem 3.3.3, $1 \leqslant x \leqslant 2e+2r-8i+3$ or,
- Theorem 3.3.10 to find the transformation Ch^x in chain c if chain c is vertical-path chain, based on Theorem 3.3.9, $1 \leqslant x \leqslant r-e+1$ or,
- Theorem 3.3.13 to find the transformation Ch^x in chain c if chain c is horizontal-path chain, based on Theorem 3.3.12, $1 \leqslant x \leqslant e-r+1$ or,
- Lemma 3.3.14 to find the transformation Ch^x in chain c if chain c is singleton chain.

Consider Figure 3.14(b) of 19 connected beads in which we employ transformation in rectangle chain Ch^x where $x = 5$. Table 3.3 show the original and new location of beads.

Table 3.3

New location of the positions in the nested chain abacus by SNC-Transformation

a_{mj}	$\longrightarrow$	Ch	Ch^2	Ch^3	Ch^4	Ch^5
a_{32}	$\longrightarrow$	a_{33}	a_{34}	a_{24}	a_{23}	a_{22}
a_{33}	$\longrightarrow$	a_{34}	a_{24}	a_{23}	a_{22}	a_{32}
a_{34}	$\longrightarrow$	a_{24}	a_{23}	a_{22}	a_{32}	a_{33}
a_{24}	$\longrightarrow$	a_{23}	a_{22}	a_{32}	a_{33}	a_{34}
a_{23}	$\longrightarrow$	a_{22}	a_{32}	a_{33}	a_{34}	a_{24}

Make SNC-Transformation Ch^3 in chain 2 as illustrated in Figure 3.15.

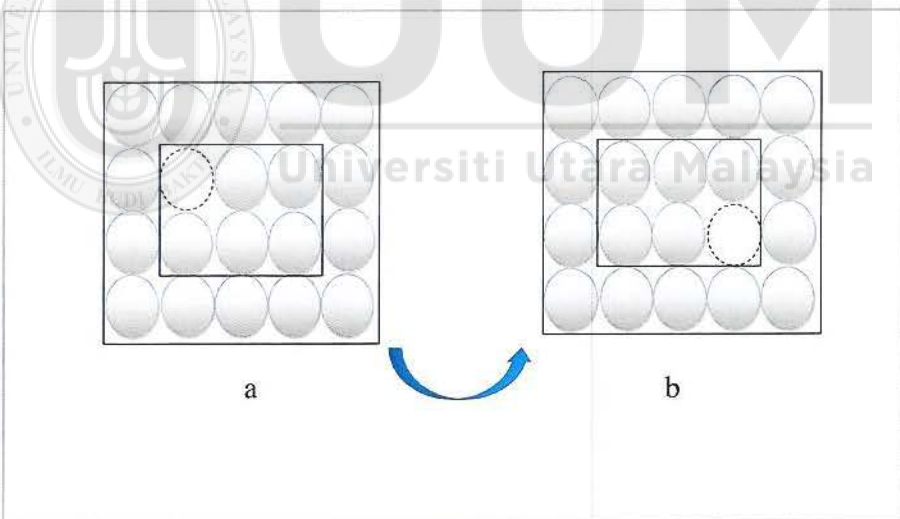


Figure 3.15. (a) Nested chain abacus of $\mu^{(5,4)}$ (b) Nested chain abacus of $\mu^{+3(5,4)}$

3.4.3 MNC-Transformation

A transformation of a nested chain abacus with c chains called MNC-Transformation construction. In MNC-Transformation we employ transformation in all chains. We use $n = 10$ to explain the algorithm for MNC-Transformation for chain 1, chain 2 and chain 3.

Step 1: Converted the nested chain abacus into $A_{r\times e}$.

Consider Figure 3.16 (a) for a nested chain abacus with two chains and 10 beads represented by connected partition $\mu^{(3,4)} = (2,1^5,0^4)$. The nested chain abacus is then converted into $A_{4\times 3}$ as shown in Figure 3.16 (b).

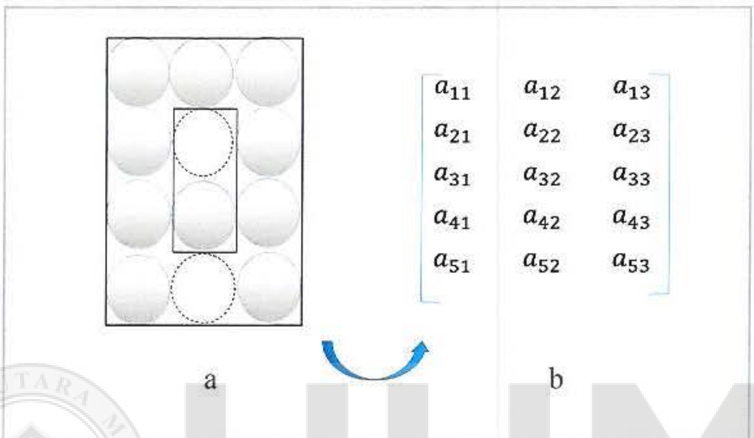


Figure 3.16. Nested chain abacus converted into matrix

Step 2: Select c element in c chains as an initial point where:

Consider Figure 3.16(b) in which 2 initial points a_{11} and a_{32} are selected, where a_{11} and a_{32} are elements in the 4×3 matrix.

Step 3: Generate different of nested chain abacus $\mathfrak{N}$ with c chain by employing

- Theorem 3.3.5 to find the transformation Ch^x in all chains if $\mathfrak{N}$ is rectangular nested chain abacus, based on Theorem 3.3.3, $1 \leqslant x \leqslant 2e+2r-8i+3$ or,
- Theorem 3.3.5 to find the transformation Ch^x in chain i where $1 \leqslant i < c$, based on Theorem 3.3.3 $1 \leqslant x \leqslant 2e+2r-8i+3$. In addition, employ Theorem 3.3.10 to find the transformation Ch^x in chain c, based on Theorem 3.3.9 $1 \leqslant x \leqslant r-e+1$, if $\mathfrak{N}$ is vertical rectangle-path nested chain abacus or,
- Theorem 3.3.5 to find the transformation Ch^x in chain i where $1 \leqslant i < c$, based on Theorem 3.3.3 $1 \leqslant x \leqslant 2e+2r-8i+3$. In addition, employ Theorem 3.3.13 to find the transformation Ch^x in chain c, based on Theorem 3.3.12

$1 \leqslant x \leqslant e-r+1$, if $\mathfrak{N}$ is horizontal-path rectangle nested chain abacus or,

- Theorem 3.3.5 to find the transformation Ch^x in chain i where $1 \leqslant i < c$, based on Theorem 3.3.3 $1 \leqslant x \leqslant 2e+2r-8i+3$. In addition, employ Theorem 3.3.14 to find the transformation Ch^x in chain c if $\mathfrak{N}$ is singleton nested chain abacus.

Consider Figure 3.16(b) of 10 connected beads where we employ transformation Ch^9 chain 1 and transformation Ch^1 in chain 2, as described in Theorem 3.3.5 and Theorem 3.3.10 respectively. Table 3.4 and Table 3.5 show the the original and new location of beads is as shown in .

Table 3.4

New location of the positions in the nested chain abacus by MNC-Transformation

a_{mj}		Ch	Ch^2	Ch^3	Ch^4	Ch^5	Ch^6	Ch^7	Ch^8	Ch^9
a_{11}	$\longrightarrow$	a_{11}	a_{21}	a_{21}	a_{31}	a_{31}	a_{41}	a_{41}	a_{42}	a_{42}
a_{21}	$\longrightarrow$	a_{21}	a_{31}	a_{31}	a_{41}	a_{41}	a_{42}	a_{42}	a_{43}	a_{43}
a_{31}	$\longrightarrow$	a_{31}	a_{41}	a_{41}	a_{42}	a_{42}	a_{43}	a_{43}	a_{33}	a_{33}
a_{41}	$\longrightarrow$	a_{41}	a_{42}	a_{42}	a_{43}	a_{43}	a_{33}	a_{33}	a_{23}	a_{23}
a_{43}	$\longrightarrow$	a_{43}	a_{33}	a_{33}	a_{23}	a_{23}	a_{13}	a_{13}	a_{12}	a_{12}
a_{33}	$\longrightarrow$	a_{33}	a_{22}	a_{23}	a_{13}	a_{13}	a_{12}	a_{12}	a_{11}	a_{11}
a_{23}	$\longrightarrow$	a_{23}	a_{13}	a_{13}	a_{12}	a_{12}	a_{11}	a_{11}	a_{21}	a_{21}
a_{13}	$\longrightarrow$	a_{13}	a_{12}	a_{12}	a_{11}	a_{11}	a_{21}	a_{21}	a_{31}	a_{31}
a_{12}	$\longrightarrow$	a_{12}	a_{11}	a_{11}	a_{21}	a_{21}	a_{31}	a_{31}	a_{41}	a_{41}
a_{32}	$\longrightarrow$	a_{22}	a_{32}	a_{22}	a_{32}	a_{22}	a_{32}	a_{22}	a_{32}	a_{22}

Table 3.5

New location of the positions in the nested chain abacus by MNC-Transformation

Ch^{10}	Ch^{11}	Ch^{12}	Ch^{13}	Ch^{14}	Ch^{15}	Ch^{16}	Ch^{17}	Ch^{18}	Ch^{19}
a_{43}	a_{43}	a_{33}	a_{33}	a_{23}	a_{23}	a_{13}	a_{13}	a_{12}	a_{12}
a_{33}	a_{33}	a_{23}	a_{23}	a_{13}	a_{13}	a_{12}	a_{12}	a_{11}	a_{11}
a_{23}	a_{23}	a_{13}	a_{13}	a_{12}	a_{12}	a_{11}	a_{11}	a_{21}	a_{21}
a_{13}	a_{13}	a_{12}	a_{12}	a_{11}	a_{11}	a_{21}	a_{21}	a_{33}	a_{23}
a_{11}	a_{11}	a_{21}	a_{21}	a_{31}	a_{31}	a_{41}	a_{41}	a_{42}	a_{42}
a_{21}	a_{21}	a_{31}	a_{31}	a_{41}	a_{41}	a_{42}	a_{42}	a_{43}	a_{43}
a_{31}	a_{31}	a_{41}	a_{41}	a_{42}	a_{42}	a_{43}	a_{43}	a_{33}	a_{33}
a_{41}	a_{41}	a_{42}	a_{42}	a_{43}	a_{43}	a_{33}	a_{33}	a_{23}	a_{23}
a_{42}	a_{42}	a_{43}	a_{43}	a_{33}	a_{33}	a_{23}	a_{23}	a_{13}	a_{13}
a_{32}	a_{22}	a_{32}	a_{22}	a_{32}	a_{22}	a_{32}	a_{22}	a_{32}	a_{22}

Make MNC-Transformation Ch^9 in nested chain abacus with 10-beads as illustrated in Figure 3.17.

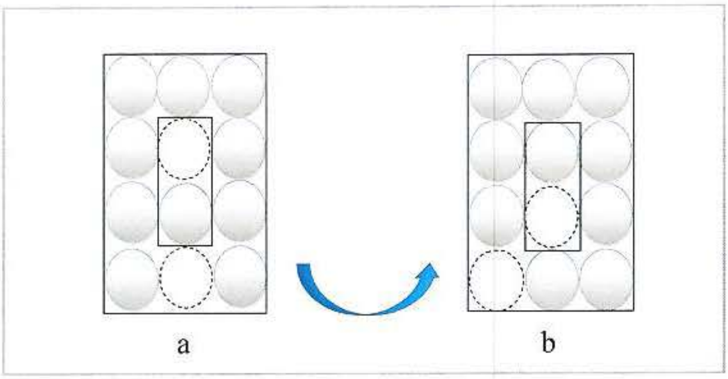


Figure 3.17. (a) Nested chain abacus and (b) Nested chain abacus after employ Ch^9

3.5 Conclusion

This chapter presented and formulated three types of transformations based on these transformations, algorithms for SNC2-Transformation, SNC-Transformation and MNC-Transformation are developed as presented in Figure 3.18.

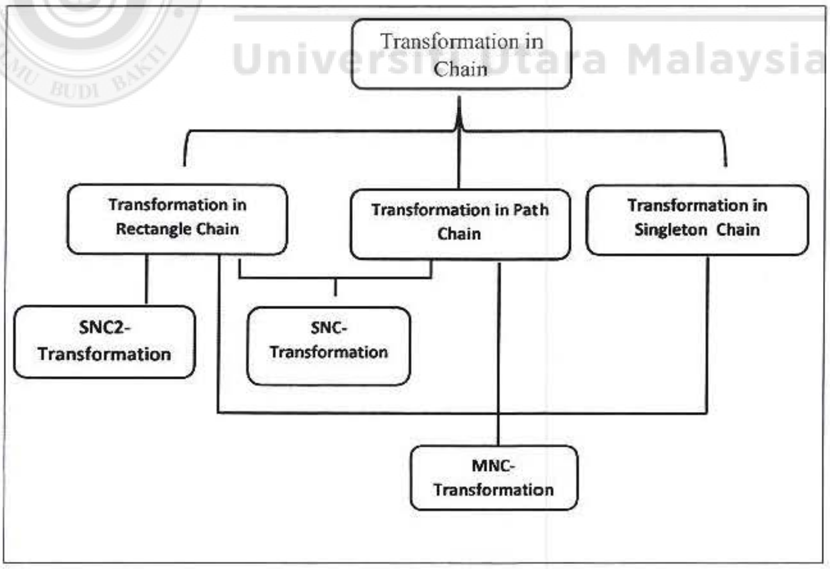


Figure 3.18. Transformations in nested chain abacus

These transformations are needed to formulate a family of classes for the nested chain abacus that will be developed in chapter 4.

CHAPTER FOUR

CLASSES OF NESTED CHAIN ABACUS WITH RESPECT TO CHAINS

4.1 Introduction

In this chapter we employ a sets of transformations to construct classes of nested chain abacus. In addition, a new method is presented to obtain the generating functions by building the nested chain abacus from the lower order with respect to chains, where order is simply the number of chains.

This chapter begins with the introduction. The necessary definitions for classes in nested chain abacus are defined in Section 4.2. Then in Section 4.3, classes of nested chain abacus based on SNC2-Transformation is developed. Meanwhile in Section 4.4, classes of nested chain abacus based on SNC-Transformation are presented. In addition in Section 4.5, classes of nested chain abacus based on MNC-Transformation are presented. Finally, a new method to calculate generating functions is developed.

4.2 Definitions for classes in nested chain abacus

We provide some basic definitions that are needed for the rest of the chapter.

Definition 4.2.1. *(Martínez & Molinero, 2001) A class is a countable family of mathematical objects with respect to some characterizations such as geometrical constraints or combinatorial properties.*

Definition 4.2.2. *(Goulden & Jackson, 2004) Let* (a_n) *be a sequence of numbers where* $n = 0,1,2,\ldots.$ *The ordinary form of the generating function to* (a_n) *is* $\sum_n (a_n)x^n$.

Definition 4.2.3. *(Apostol, 2013) A point lattice is a regularly spaced array of points, in which often the array is called a grid.*

Remark 4.2.4. *Since the plan lattice points are the vertices of unit squares, then nested chain abacus $\mathfrak{N}$ is inscribed in the grid with e' columns and r' rows such that each position in the nested chain abacus is a point lattice.*

Figure 4.1 Illustrates the lattice nested chain abacus in the grid.

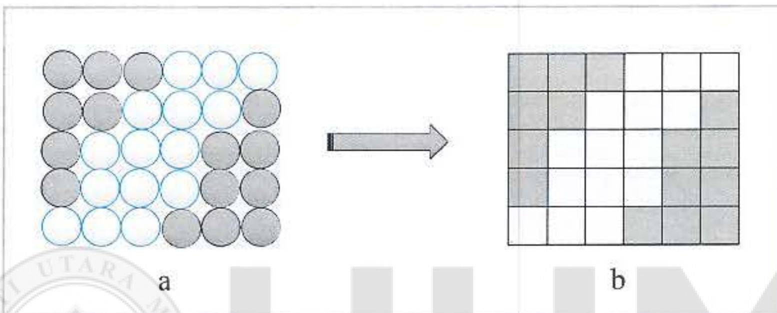


Figure 4.1. (a) Nested chain abacus for 15-connected beads and (b) The nested chain abacus for 15-connected beads embedded in square lattice

Now, we define two cases of nested chain abacus with four definitions based on the beads and empty beads locations with respect to chains.

Case one: Nested chain abacus with $c-1$ full chains.

Definition 4.2.5. *$\mathfrak{D}_{singl}$ is a nested chain abacus with one chain which has at most a sequence of connected beads, that is, there exists $\{a_1, a_2,...,a_k : |a_{k'} - a_{k'+1}| \in \{1,e\}\}$ where $a_{k'}$ is a positive integer corresponding to connected bead positions and $1 \leqslant k' < k$ for $1 \leqslant k \leqslant 2r$.*

Based on Remark 4.2.4 Figure 4.2 (a) is $\mathfrak{D}_{singl}$ nested chain abacus representing connected partition is $\mu^{(2,5)} = (2^4, 1, 0^3)$ while Figure 4.2 (b) is a case where the connected partition $\mu^{(2,5)} = (2^4, 1^3, 0)$ nested chain abacus not $\mathfrak{D}_{singl}$.

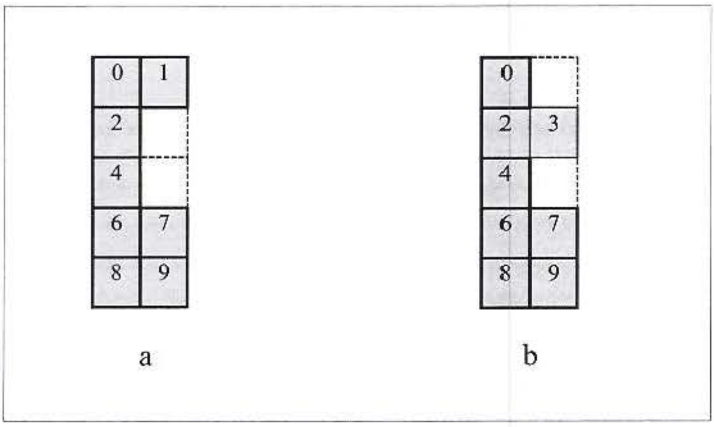


Figure 4.2. (a) A $\mathfrak{D}_{\text{singl}}$ nested chain abacus and (b) nested chain abacus but not $\mathfrak{D}_{\text{singl}}$

Definition 4.2.6. *$\mathfrak{D}_{\text{outer}}$ is a vertical (respectively, vertical-path) rectangle nested chain abacus with c chains, $c > 1$ and all inner chains are full chains and empty bead positions b' is in the outer chain for the following:*

(i) $\mathfrak{D}_{\text{outer}}$ is $\mathfrak{D}_{\text{outer}-1}$ if the outer chain has g_p sets of connected bead positions.

(ii) $\mathfrak{D}_{\text{outer}}$ is $\mathfrak{D}_{\text{outer}-2}$ if $1 \leqslant b' < e$ and the empty bead positions b' are consecutive positions.

(iii) $\mathfrak{D}_{\text{outer}}$ is $\mathfrak{D}_{\text{outer}-3}$ if $e \leqslant b' < e+r-3$ and the empty bead positions b' are consecutive positions.

Definition 4.2.7. *$\mathfrak{D}'_{\text{outer}}$ is a horizontal (respectively, horizontal path) rectangle nested chain abacus with c chains, $c > 1$ and all inner chains are full chains and empty bead positions b' located in the outer chain for the following;*

(i) $\mathfrak{D}'_{\text{outer}}$ is $\mathfrak{D}'_{\text{outer}-1}$ if the outer chain has g_p sets of connected bead positions.

(ii) $\mathfrak{D}'_{\text{outer}}$ is $\mathfrak{D}'_{\text{outer}-2}$ if $1 \leqslant b' < r$ and empty bead positions b' are consecutive positions.

(iii) $\mathfrak{D}'_{\text{outer}}$ is $\mathfrak{D}'_{\text{outer}-3}$ if $r \leqslant b' < r+e-3$ and the empty bead positions b' are consecutive positions.

The number of elements in set g_p is denoted by $\#g_p$ such that $\#g_p > 1$ where $g_p = \{a_1, a_2, ..., a_k\}$, $|a_{k'+1} - a_{k'}| \in \{1, e\}$ and $1 \leqslant k' \leqslant k-1$ for $1 \leqslant \sum_{p'=1}^{p} \#g_p \leqslant 2r+2e-4$ and $1 \leqslant k \leqslant 2e+2r-4$

Figure 4.3(a), (b) and (c) illustrate $\mathfrak{D}_{\text{outer}-1}$, $\mathfrak{D}_{\text{outer}-2}$ and $\mathfrak{D}_{\text{outer}-3}$ while (d) is not $\mathfrak{D}_{\text{outer}}$.

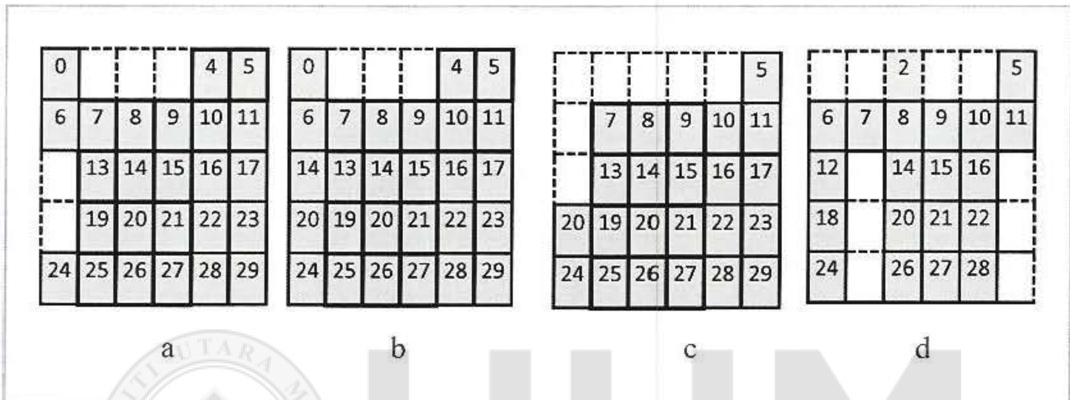


Figure 4.3. (a) A $\mathfrak{D}_{\text{outer}-1}$, (b) $\mathfrak{D}_{\text{outer}-2}$, (c) $\mathfrak{D}_{\text{outer}-3}$ and (d) not $\mathfrak{D}_{\text{outer}}$

Definition 4.2.8. *$D_{inner-i}$ is a nested chain abacus with c chains and all chains are a full chain except chain i with empty bead positions b_i' where $1 < i \leqslant c$.*

Case two: The empty bead positions are located in at least two chains.

Definition 4.2.9.

1. *$\mathfrak{D}_{\text{inner}}$ is a vertical (respectively horizontal) rectangle nested chain abacus which satisfies the following conditions:*

   (i) $b_{i+1}+1 \geqslant b_i'$ *if* $1 \leqslant b_{i+1} < e-2i$

   (respectively, $1 \leqslant b_{i+1} < r-2i$*)*

   (ii) $b_{i+1}+3 \geqslant b_i'$ *if* $e-2i \leqslant b_{i+1} < r+e-6i$

   (respectively, $r-2i \leqslant b_{i+1} < r+e-6i$ *)*

   (iii) $b_{i+1}+5 \geqslant b_i'$ *if* $r+e-6i \leqslant b_{i+1} < 2e+r-8i$

   (respectively, $r+e-6i \leqslant b_{i+1} < 2r+e-2i$*)*

(iv) $b_{i+1}+7 \geqslant b_i^{'}$ if $b_{i+1} \geqslant 2e+r-8i$

(respectively, $b_{i+1} \geqslant 2r+e-8i$)

such that b_i and $b_i^{'}$ are bead and empty bead positions, respectively, in chain i where $1 \leqslant i \leqslant c$.

2. $\mathfrak{D}_{\text{inner}}^{'}$ *is a vertical (respectively, horizontal)-path rectangle nested chain abacus with c chains and satisfies the following conditions:*

   (i) $b_{i+1}+1 \geqslant b_i^{'}$ if $1 \leqslant b_{i+1} < e-2i$

   (respectively, $(r-2i)$)

   (ii) $b_{i+1}+3 \geqslant b_i^{'}$ if $e-2i$

   (respectively $(r-2i)) \leqslant b_{i+1} < r+e-6i$

   (iii) $b_{i+1}+5 \geqslant b_i^{'}$ if $r+e-6i \leqslant b_{i+1} < 2e+r-8i$

   (respectively, $(2r+e-2i)$)

   (iv) $b_{i+1}+7 \geqslant b_i^{'}$ if $b_{i+1} \geqslant 2e+r-8i$

   (respectively, $(2r+e-8i)$)

   (v) *Satisfy one of the following conditions:*

   - *Chain* $\frac{e+1}{2}$ *is full columns*
   - $b_{\frac{e-1}{2}}^{'} \leqslant 4$

   such that b_i and $b_i^{'}$ are bead and empty bead positions, respectively, in chain i where $1 \leqslant i \leqslant c$.

Based on Definitions 4.2.5, 4.2.6, 4.2.8 and 4.2.9 and SNC2-Transformation, SNC-Transformation and MNC-Transformation respectively, we will now define the classes of nested chain abacus.

Note that, all formulas formulated in the remaining of this chapter have been verified with computer programs according to Appendix E.

4.3 Single Transformation Class

A new class of nested chain abacus with a chain is generated by employing SNC2-Transformation in $\mathfrak{D}_{\mathfrak{singl}}$.

Note, $\mathfrak{D}_{\mathfrak{singl}}$ with e columns and r rows are denoted by $\mathfrak{D}_{\mathfrak{singl}}^{(2,r)}$.

The next lemma provides the method of generating $\mathfrak{D}_{\mathfrak{singl}}^{(2,r)}$ in fixed n (number of beads).

In addition, the $\mathfrak{D}_{\mathfrak{singl}}^{(2,r)}$ elements are enumerated.

Lemma 4.3.1. *Let b' be the number of empty bead positions. For fixed b', the number of $\mathfrak{D}_{\mathfrak{singl}}^{(e,r)}$ generated by employing SNC2-Transformation is $2r$.*

Proof. Let a_{mj} be an initial position in $\mathfrak{D}_{\mathfrak{singl}}^{(2,r)}$. Based on Lemma 3.2.5, a_{mj} can be moved downwards, upwards, rightwards or leftwards depending on m and j where $1 \leqslant m \leqslant r$ and $1 \leqslant j \leqslant 2$. Based on Definition 2.5.3 and Lemma 3.2.5, if $j = 1$ then the initial position will be moved downwards along the column 1 until location a_{r1} after skipping $r - m$ locations. Then, the position of location a_{r1} will move from left to the right from a_{r1} to a_{r2} after skipping one location. Afterwards, the initial position will move up and skip $r - 1$ locations to arrive at location a_{12}. Furthermore, the initial position will move from right to left and skip one location. Finally, a_{11} will skip $m - 1$ locations downwards to return back. Thus, the initial position a_{mj} will skip

$$r - m + 1 + r - 1 + m - 1 = 2r - 1$$

locations to move and return to its original place. The same applies if the initial position is a_{m2}. Since each move generates a new nested chain abacus, then the class of $\mathfrak{D}_{\mathfrak{singl}}$ generated by SNC2-Transformation consists of $2r$ elements. □

Example 4.3.2 illustrates Lemma 4.3.1.

Example 4.3.2. *Let* $\mu^{(4,5)} = (2^4, 1, 0^3)$ *be a connected partition where* $b' = 2$. *Based on Lemma 4.3.1, the number of* $\mathfrak{D}_{\mathfrak{singl}}$ *with 8 beads is 10 as shown in Figure 4.4.*

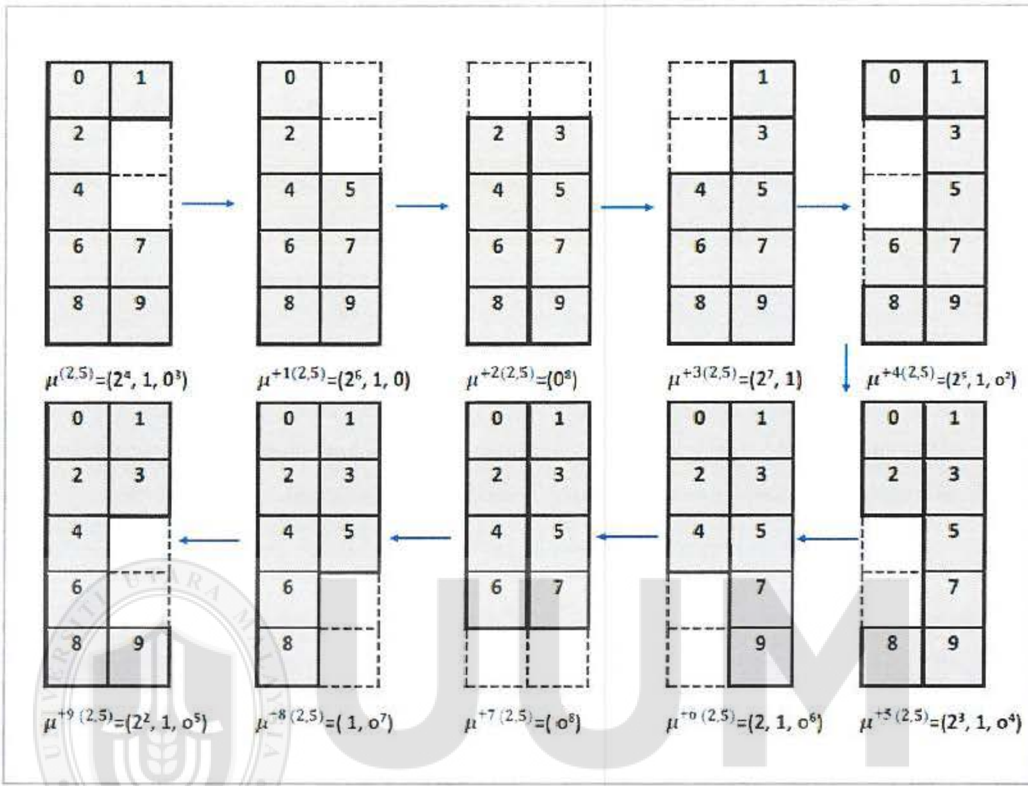


Figure 4.4. Nested chain abacus of class $\mathfrak{D}_{\mathfrak{singl}}$

Figure 4.4 illustrates the generation of $\mathfrak{D}_{\mathfrak{singl}}$ nested chain abacus class by employing SNC2-Transformation where $\mu^{(2,5)} = (2^4, 1, 0^3)$ and then

$$
\begin{aligned}
Ch(\mu^{(2,5)}) &= \mu^{+1(2,5)} &&= (2^6,1,0),\\
Ch(\mu^{+1(2,5)}) &= \mu^{+2(2,5)} &&= (0^8),\\
Ch^1(\mu^{+2(2,5)}) &= \mu^{+3(2,5)} &&= (2^7,1),\\
Ch^1(\mu^{+3(2,5)}) &= \mu^{+4(2,5)} &&= (2^5,1,0^2),\\
Ch^1(\mu^{+4(2,5)}) &= \mu^{+5(2,5)} &&= (2^3,1,0^4),\\
Ch^1(\mu^{+5(2,5)}) &= \mu^{+6(2,5)} &&= (2,1,0^6),\\
Ch^1(\mu^{+6(2,5)}) &= \mu^{+7(2,5)} &&= (0^8),\\
Ch^1(\mu^{+7(2,5)}) &= \mu^{+8(2,5)} &&= (1,0^7),\\
Ch^1(\mu^{+8(2,5)}) &= \mu^{+9(2,5)} &&= (2^2,1,0^5).
\end{aligned}
$$

Theorem 4.3.3 is a generalization of Lemma 4.3.1.

Theorem 4.3.3. *The number of* $\mathfrak{D}^{(c,c)}_{singl}$ *generated by employing SNC2-Transformation is*

$$\sum_{r\geqslant 2}(r-1)2r+1.$$

Proof. Since $\mathfrak{N}$ is $\mathfrak{D}_{singl}$ then there is b consecutive bead positions and the number of beads in $\mathfrak{D}^{(2,r)}_{singl}$ are $\{r+1, r+2, ..., 2r\}$. Thus, there are

$$2r-(r+1)$$

of $\mathfrak{D}^{(2,r)}_{singl}$ nested chain abacus. Based on Lemma 4.3.1, for a fixed number of b and r, there exist $2r$ of $\mathfrak{D}^{(2,r)}_{singl}$. Thus, there exist

$$(r-1)2r$$

of $\mathfrak{D}_{singl}^{(2,r)}$ with fixed r. In addition, there is a $\mathfrak{D}_{singl}^{(2,r)}$ with full chain. Hence, there exist

$$(r-1)2r+1$$

of $\mathfrak{D}_{singl}^{(2,r)}$ with fixed r where $1 \leqslant b \leqslant n$. Since $r = 2,3,4,\ldots$ then, the number of $\mathfrak{D}_{singl}^{(r,r)}$ generated by employing SNC2-Transformation is

$$\sum_{r\geqslant 2}(r-1)2r+1. \qquad \square$$

Next, we will find the generating function of $\mathfrak{D}_{singl}^{(2,r)}$ class.

Theorem 4.3.4. *For $e=2$, The generating function for the numbers $\mathfrak{D}_{singl}^{(2,r)}$ has the following rational form:*

$$2x^2\frac{(y+1)y}{(1-y)^3} - 2\frac{(x^2y)}{(y-1)^2} + x^2\frac{1}{1-y} - 1.$$

Proof. Based on Theorem 4.3.3, for $e=2$ the number of nested chain abacus in class $\mathfrak{D}_{singl}^{(2,r)}$ generated by employing SNC2-Transformation is

$$(r-1)2r+1.$$

Based on the ordinary form of the generating function

$$\begin{aligned}
\sum_{e=2,r\geqslant 0} a_{e,r}x^e y^r &= \sum_{e=2,r\geqslant 2}((r-1)2r+1)x^2y^r \\
&= 2\sum_{r\geqslant 2,e=2} r^2x^2y^r - 2\sum_{r\geqslant 2,e=2} rx^2y^r + \sum_{r\geqslant 2,e=2} x^2y^r \\
&= 2x^2\frac{(y+1)y}{(1-y)^3} - 2\frac{(x^2y)}{(y-1)^2} + x^2\frac{1}{1-y} - 1. \qquad \square
\end{aligned}$$

4.4 Stratum Transformation Class

Stratum transformation class is class of nested chain abacus generated by employing a SNC-Transformation in $\mathfrak{D}_{\text{outer}}$, $\mathfrak{D}_{\text{outer}}^{'}$ and $\mathfrak{D}_{\text{inner}-i}$. The $\mathfrak{D}_{\text{outer}}$ and $\mathfrak{D}_{\text{inner}}$ nested chain abacus with e columns and r rows are denoted by $\mathfrak{D}_{\text{outer}}^{(e,r)}$, $\mathfrak{D}_{\text{outer}}^{'(e,r)}$ and $\mathfrak{D}_{\text{inner}-i}^{(e,r)}$.

Case one: Nested chain abacus is $\mathfrak{D}_{\text{outer}-2}^{(e,r)}$ and $b^{'} < e$.

Lemma 4.4.1. *The number of $\mathfrak{D}_{\text{outer}-2}^{(e,r)}$ (respectively, $\mathfrak{D}_{\text{outer}-2}^{'(e,r)}$) generated by employing SNC-Transformation is*

$$\sum_{r=1}^{\infty}\sum_{e=1}^{r-1}(2e+2r-4)(e-1)(respectively, \sum_{e=1}^{\infty}\sum_{r=1}^{e-1}(2e+2r-4)(r-1)).$$

Proof. Since the nested chain abacus is $\mathfrak{D}_{\text{outer}-2}^{(e,r)}$ (respectively, $\mathfrak{D}_{\text{outer}-2}^{'(e,r)}$), then the inner chains are full. For fixed $b^{'}$, based on SNC-Transformation and Corollary 3.3.4 where $i = 1$ the different shapes of $\mathfrak{D}_{\text{outer}-2}^{(e,r)}$ (respectively, $\mathfrak{D}_{\text{outer}-2}^{'(e,r)}$) that can be obtained by employing rectangle chain transformation on outer chain is

$$2e+2r-4.$$

Since the empty bead positions are consecutive and

$$1 \leqslant b^{'} < e(respectively, 1 \leqslant b^{'} < r),$$

then there exist

$$e-1$$

of $\mathfrak{D}_{\text{outer}-2}^{(e,r)}$. Thus, there exist

$$(e-1)(2e+2r-4)\,\mathfrak{D}_{\text{outer}-2}^{(e,r)}\left(respectively, (r-1)(2e+2r-4)\,\mathfrak{D}_{\text{outer}-2}^{'(e,r)}\right).$$

Since $e < r$ and $r > 1$, then,

$$\sum_{r=1}^{\infty}\sum_{e=1}^{r-1}(2e+2r-4)(e-1)(respectively, \sum_{e=1}^{\infty}\sum_{r=1}^{e-1}(2e+2r-4)(r-1)).$$

Example 4.4.2. *Let $\mu^{(4,3)} = (3^{16}, 0)$ be a connected partition of $\mathfrak{D}^{(4,5)}_{\text{outer}-2}$ with 3 empty bead positions. Based on Lemma 4.4.1 the number of nested chain abacus generated by employing SNC-Transformation on $\mathfrak{D}^{(e,r)}_{\text{outer}-2}$ of 17-connected bead positions is represented by $\mu^{(4,3)} = (3^{16}, 0)$.* □

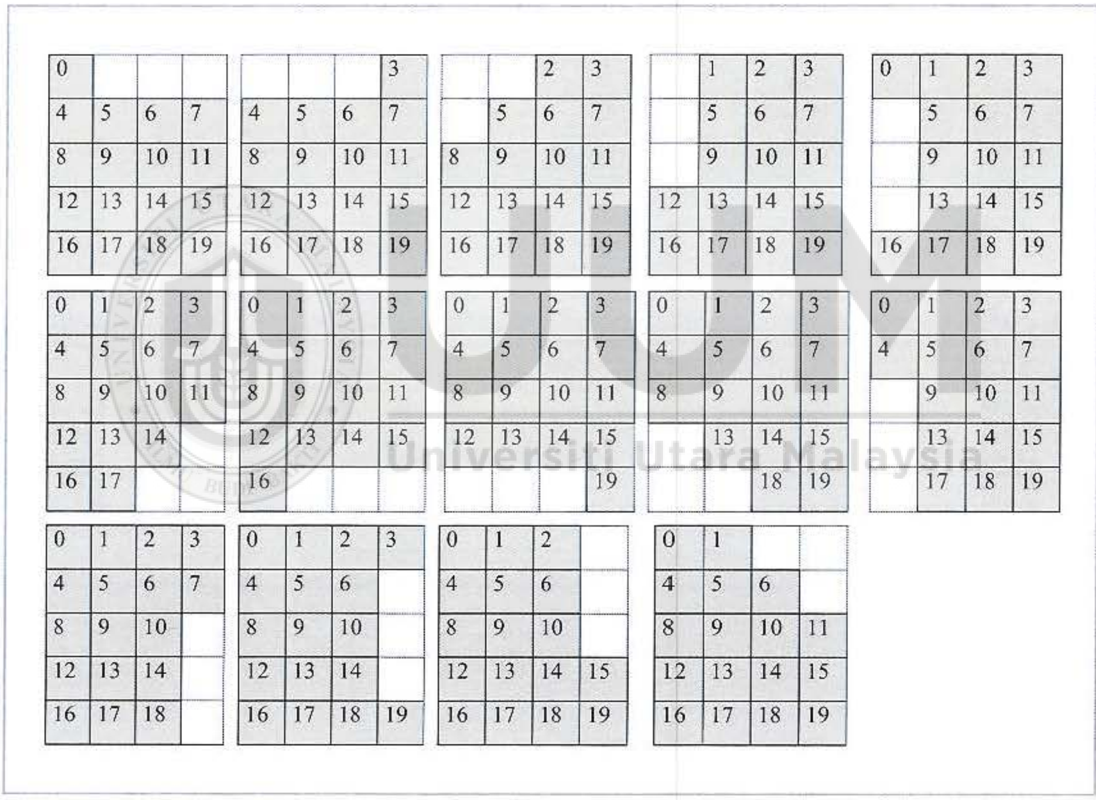


Figure 4.5. Fourteen of $\mathfrak{D}^{(4,5)}_{\text{outer}-2}$ for 17-connected beads where $b' = 3$

Based on SNC-Transformation, the following lemma provides the way to generate $\mathfrak{D}^{(e,r)}_{\text{outer}-3}$ ($\mathfrak{D}'^{(e,r)}_{\text{outer}-3}$) class. In addition, the $\mathfrak{D}^{(e,r)}_{\text{outer}-3}$ ($\mathfrak{D}'^{(e,r)}_{\text{outer}-3}$) elements are enumerated. **Case two:** If nested chain abacus is $\mathfrak{D}^{(e,r)}_{\text{outer}-3}$ and $b' = e = r$.

Lemma 4.4.3. *Let $b^{'}$ be the number of empty bead positions. For fixed $b^{'} = e = r$, the number of nested chain abacus generated by employing SNC-Transformation in $\mathfrak{D}^{(\mathfrak{e},\mathfrak{r})}_{\mathfrak{outer}-3}$ is:*

(i) Two of $\mathfrak{D}^{(\mathfrak{e},\mathfrak{r}-1)}_{\mathfrak{outer}-2}$

(ii) Two of $\mathfrak{D}^{(\mathfrak{e}-1,\mathfrak{r})}_{\mathfrak{outer}-2}$

(iii) $4e-8$ of $\mathfrak{D}^{(\mathfrak{e},\mathfrak{r})}_{\mathfrak{outer}-3}$

with $b^{'}$ empty bead positions where $e \geqslant 2$ and $r \geqslant 2$.

Proof. Let S be a set of empty bead positions in $\mathfrak{D}_{\mathfrak{outer}-3}$, then,

$S = \{a_{mj} : k \leqslant m \leqslant k+b^{'}-1, j \in \{1,e\}\}$, or

$S = \{a_{mj} : k^{'} \leqslant j \leqslant k^{'}+b^{'}-1, m \in \{1,r\}\}$, or

$S = \{a_{mj} : r-k_1 \leqslant m \leqslant r, j = 1\} \cup \{a_{mj} : 1 \leqslant j \leqslant k_2, m = r\}$, or

$S = \{a_{mj} : e-k_1 \leqslant j \leqslant e, m = r\} \cup \{a_{mj} : r-k_2 \leqslant m \leqslant r, j = e\}$, or

$S = \{a_{mj} : e-k_1 \leqslant j \leqslant e, m = 1\} \cup \{a_{mj} : 1 \leqslant m \leqslant k_2, j = e\}$, or

$S = \{a_{mj} : 1 \leqslant j \leqslant k_1, m = 1\} \cup \{a_{mj} : 1 \leqslant m \leqslant k_2, j = 1\}$.

Since class $\mathfrak{D}^{(\mathfrak{e},\mathfrak{r})}_{\mathfrak{outer}-3}$ is generated by employing SNC-Transformation and $b^{'} = e = r$ then $\exists\{x_1, x_2\}$ such that

$Ch^{x_1}(S) = \{a_{mj} : 1 \leqslant j \leqslant e,\ m = 1\}$ and $Ch^{x_2}(S) = \{a_{mj} : 1 \leqslant J \leqslant e,\ m = r\}$ where $1 \leqslant x_1 \leqslant 2r+2e-5$ and $1 \leqslant x_2 \leqslant 2r+2e-5$ for $1 \leqslant k \leqslant e-b^{'}+1$, $1 \leqslant k^{'} \leqslant e-b^{'}+1$ and $k_1+k_2 = b^{'}$. Then, there are two $\mathfrak{D}^{(\mathfrak{e},\mathfrak{r}-1)}_{\mathfrak{outer}-2}$. In addition, $\exists\{x_3, x_4\}$ such that $Ch^{x_3}(S) = \{a_{mj} : 1 \leqslant m \leqslant r, j = 1\}$ and $Ch^{x_4}(S) = \{a_{mn} : 1 \leqslant m \leqslant r, n = e\}$ where $1 \leqslant x_3 \leqslant 2r+2e-5$ and $1 \leqslant x_4 \leqslant 2r+2e-5$. Then, there are two $\mathfrak{D}^{(\mathfrak{e}-1,\mathfrak{r})}_{\mathfrak{outer}-2}$. Based on Corollary 3.3.4, the maximal number of nested chain abacus generated by SNC-Transformation is

$$2e+2r-5$$

in addition the original one; therefore, the number of nested chain abacus in $\mathfrak{D}^{(\mathfrak{e},\mathfrak{r})}_{\mathfrak{outer}-3}$

is

$$2e + 2r - 4$$

such that two of them are $\mathfrak{D}^{(e,r-1)}_{outer-2}$ ($\mathfrak{D}'^{(e,r-1)}_{outer-2}$) with full chains and another two are $\mathfrak{D}^{(e-1,r)}_{outer-2}$ ($\mathfrak{D}'^{(e-1,r)}_{outer-2}$). Hence, the number of nested chain abacus in $\mathfrak{D}_{outer-3}$ with e columns and r rows is

$$2e + 2r - 8.$$

Since $e = r$, then, the number of nested chain abacus in $\mathfrak{D}_{outer-3}$ with e columns and r rows is

$$4e - 8.$$

□

Consider Lemma 4.4.3 where the number of $\mathfrak{D}^{(e,r)}_{outer-3}$ represents the connected partition $\mu^{(3,3)} = (3^5, 2)$. There are four $\mathfrak{D}^{(3,3)}_{outer-3}$ and two $\mathfrak{D}^{(2,3)}_{outer-2}$ as well as two $\mathfrak{D}^{(3,2)}_{outer-2}$ as shown in Figure 4.6.

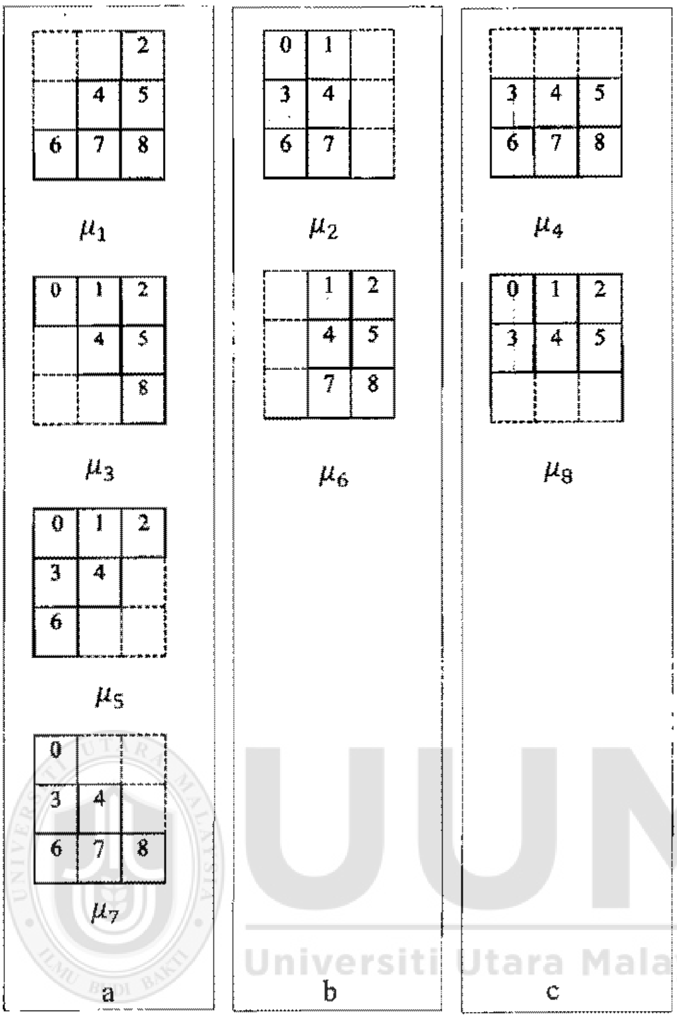


Figure 4.6. Eight nested chain abacus, (a) four $\mathfrak{D}^{(3,3)}_{outer-3}$, (b) two $\mathfrak{D}^{(2,3)}_{outer-2}$ and (c) two $\mathfrak{D}^{(3,2)}_{outer-2}$

In the next theorem we will found a generating function of $\mathfrak{D}^{(e,e)}_{outer-3}$ class

Theorem 4.4.4. *Let b' be the number of empty bead positions. The generating function for the number of $\mathfrak{D}^{(e,e)}_{outer-3}$ has the following ordinary form:*

$$\frac{4}{(1-x)^3}$$

where $b' = r = e$.

Proof. Based on Lemma 4.4.3, the number of nested chain abacus in $\mathfrak{D}^{(e,e)}_{outer-3}$ is $4e-8$. Since $e > 2$, then, $4, 8, 12, \ldots = 4\{1,2,3,\ldots\}$. The ordinary generating function for the infinite sequence $\{1,2,3,\ldots\}$ is the power series: $1+3x^2+5x^3+\ldots$ where $\{1,3,5,\ldots\}$ whose term n is the binomial coefficient $4\binom{n+2}{2}\cdot$ Since

$$\sum_{n=1}^{\infty}\binom{n+2}{2}x^n = \binom{3}{2}x+\binom{4}{2}x^2+\binom{5}{2}x^3+\cdots$$

Then, the ordinary generating function is

$$\sum_{n=1}^{\infty}4\binom{n+2}{2}x^n = \frac{4}{(1-x)^3}.$$

□

Case three: If nested chain abacus is $\mathfrak{D}^{(e,r)}_{outer-3}$ and $e \leqslant b' \leqslant e+r-3$.

Lemma 4.4.5. *For fixed $e \leqslant b' \leqslant e+r-3$, the number of nested chain abacus generated by employing SNC-Transformation in $\mathfrak{D}^{(e,r)}_{outer-3}$ is*

(i) $2(b'-e+1)\ \mathfrak{D}^{(e,r-1)}_{outer-3}$ *if* $b' < r$

(ii) $2(b'-e+1)\ \mathfrak{D}^{(e,r-1)}_{outer-3}$ *and* $2(b'-r+1)\ \mathfrak{D}^{(e-1,r)}_{outer-3}$ *if* $b' > r$

(iii) $(2e+2r-4)$ - $2(b'-e+1)$ - $2(b'-r+1)\ \mathfrak{D}^{(e-1,r)}_{outer-3}$

where b' be the number of empty bead positions.

Proof.

(i) Suppose that S is set of empty bead positions. Since $b' < r$ then $\exists\{x_1, x_2\}$ such that

$$Ch^{x_1}(S) = \{a_{11}, a_{12}, \ldots, a_{1e}, a_{2e}, \ldots, a_{(b'-e+1)e}\},$$

$$Ch^{x_1+1}(S) = \{a_{21}, a_{11}, a_{12}, \ldots, a_{1e}, a_{2e}, \ldots, a_{(b'-e)e}\}$$

.

.

.

$$Ch^{x_1+b'-e+1}(S) = \{a_{(b'-e+1)1}, a_{(b'-e)1}, \ldots, a_{11}, a_{12}, \ldots, a_{1e}\}$$

and

$$Ch^{x_2}(S) = \{a_{(r-b'+e)1}, a_{(r-b'+e+1)1}, \ldots, a_{r1}, a_{r2}, \ldots, a_{re}\}$$

.

.

.

$$Ch^{x_2+b'-e+1}(S) = \{a_{r1}, a_{r2}, \ldots, a_{re}, a_{(r-1)e}, \ldots, a_{(r-b'+e)e}$$

Thus, there are $2(b' - e + 1)\ \mathfrak{D}^{(e,\mathfrak{r}-1)}_{\text{outer}-3}$.

(ii) Suppose that S is set of empty bead positions. Since $b' < r$ then $\exists\{x_3, x_4\}$ such that

$$Ch^{x_3}(S) = \{a_{1(b'-r+1)}, a_{1(b'-r-1)}, \ldots, a_{11}, a_{21}, \ldots, a_{r1}\},$$

$$Ch^{x_3+1}(S) = \{a_{1(b'-r-1)}, a_{1(b'-r-2)}, \ldots, a_{11}, a_{21}, \ldots, a_{r1}, a_{r2}\},$$

.

.

.

$$Ch^{x_3+b'-r+1}(S) = \{a_{11}, a_{21}, \ldots, a_{r1}, a_{r2}, \ldots, a_{r(b'-r+1)}\},$$

and

$$Ch^{x_4}(S) = \{a_{r(e-b'+r)}, a_{r(e-b'+r+1)}, \ldots, a_{re}, a_{(r-1)e}, \ldots, a_{1e}\},$$
$$Ch^{x_4+1}(S) = \{a_{r(b'-r+2)}, a_{r(b'-r+3)}, \ldots, a_{re}, a_{(r-1)e}, \ldots, a_{1e}, a_{1(e-1)}\},$$

.

.

.

$$Ch^{x_4+b'-r+1}(S) = \{a_{re}, a_{(r-1)e}, \ldots, a_{1e}, a_{1(e-1)}, \ldots, a_{1(e-b'+r)}\}.$$

Thus, there are

$$2(b' - r + 1)$$

of $\mathfrak{D}^{(e-1,r)}_{outer-3}$. Since $e < r$ then and based on 1 there are

$$2(b' - e + 1)$$

of $\mathfrak{D}^{(e-1,r)}_{outer-3}$.

(iii) Based on Corollary 3.3.4, there is

$$(2e + 2r - 4) - 2(b' - e + 1) - 2(b' - r + 1)$$

of $\mathfrak{D}^{(e,r)}_{outer-3}$ where $1 \leqslant v \leqslant (r + 2e - b' + 1)$.

□

Lemma 4.4.6. *Let b' be the number of empty bead positions. The number of nested chain abacus generated by employing SNC-Transformation in $\mathfrak{D}'^{(e,r)}_{outer-3}$ is*

(i) $2(b' - r + 1)\ \mathfrak{D}^{(e-1,r)}_{outer-3}$ *if* $b' < e$

(ii) $2(b' - e + 1)\ \mathfrak{D}'^{(e,r-1)}_{outer-3}$ *and* $2(b' - r + 1)\ \mathfrak{D}'^{(e-1,r)}_{outer-3}$ *if* $b' > e$

(iii) $(2e + 2r - 4)$ - $2(b' - e + 1)$ - $2(b' - r + 1)\ \mathfrak{D}^{(e-1,r)}_{outer-3}$

where $r \leqslant b' \leqslant e + r - 3$.

Proof. See Lemma 4.4.5 □

Theorem 4.4.7. *Let $b^{'}$ be the number of empty bead positions and $e \leqslant b^{'} < e+r-3$. The number of nested chain abacus generated by employing SNC-Transformation in $\mathfrak{D}^{(e,r)}_{outer-3}$ is*

(i) $2\sum_{d=1}^{b^{'}-e+1} d\,\mathfrak{D}^{(e,r-1)}_{outer}$

(ii) $2\sum_{d^{'}=1}^{b^{'}-r+1} d^{'}\,\mathfrak{D}^{(e-1,r)}_{outer}$

(iii) $(2e+2r-4)-2\sum_{d=0}^{b^{'}-e+1} d-2\sum_{d^{'}=1}^{b^{'}-r+1} d^{'}\,\mathfrak{D}^{(e,r)}_{outer}$

where $b^{'}-e=0$ if $b^{'} \leqslant e$ and $e \geqslant 2\cdot$

Proof.

(i) We will prove the above theorem by induction.

Basic step: When $b^{'}=1$, then, the number of nested chain abacus generated by employing SNC-Transformation in $\mathfrak{D}^{(e,r)}_{outer-3}$ is $(2e+2r-4)$.

Induction step: Let $k \in Z^{+}$ and suppose $2\sum_{d=0}^{b^{'}-e+1} d$ is true for $b^{'}=k$. If we add one empty bead position in outer chain then,

$$k-e+2+\sum_{d=1}^{k-e+1} d=\sum_{d=0}^{k-e+2} d=1+2+...+(k-e+1)+(k-e+2).$$

(ii) See Proof (*i*) Theorem 4.4.7.

(iii) Based on Corollary 3.3.4, there is $2e+2r-4$ nested chain abacus generated by employing SNC-Transformation. Thus, there is

$$2e+2r-4-2\sum_{d=0}^{k-e+1} d-\sum_{d^{'}=0}^{k-r+1} d$$

of $\mathfrak{D}^{(e,r)}_{outer-3}$.

□

The next example is illustrated according to Theorem 4.4.7 if $b^{'}=3$.

Example 4.4.8. *Let* $\mu^{(3,4)} = (1,0^8)$ *be a connected partition of* $\mathfrak{D}^{(3,4)}_{\text{outer}-3}$ *with 3 consecutive empty bead positions. Based on Theorem 4.4.7, the number of nested chain abacus generated by employing SNC- Transformational in* $\mathfrak{D}^{(3,4)}_{\text{outer}-3}$ *is 8 and two* $\mathfrak{D}^{(3,3)}_{\text{outer}-2}$.

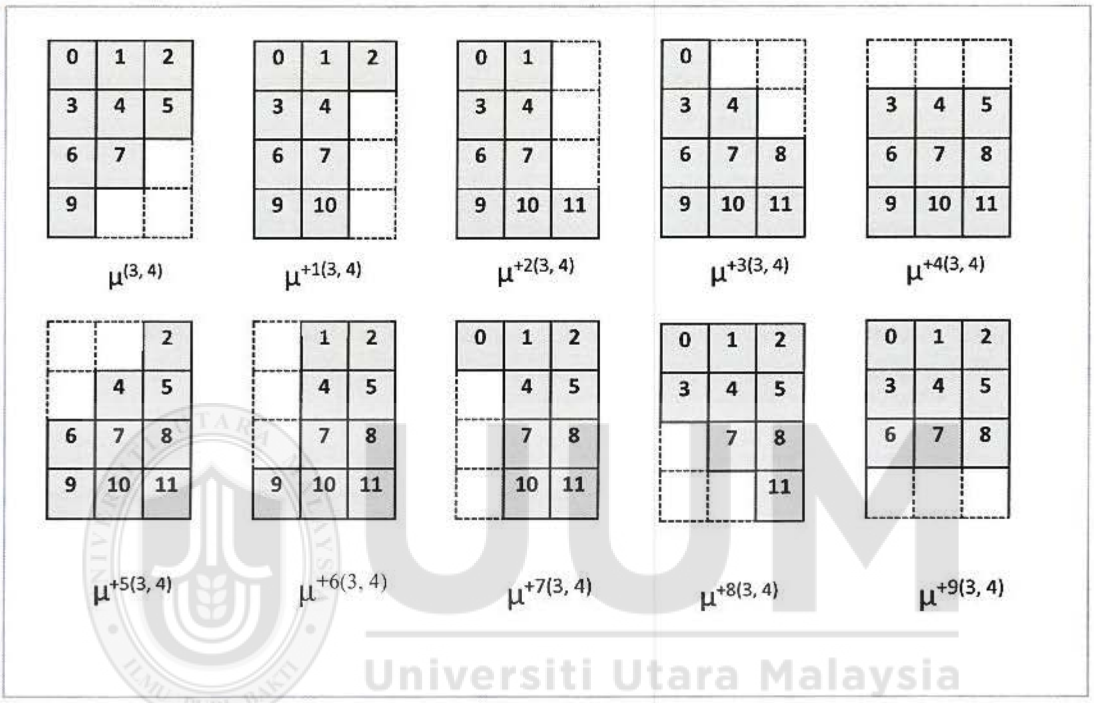


Figure 4.7. Eight $\mathfrak{D}^{(3,4)}_{\text{outer}-3}$ and two $\mathfrak{D}^{(3,3)}_{\text{outer}-2}$

The enumeration of nested chain abacus generated by employing SNC Transformation in $\mathfrak{D}_{\text{outer}-1}$ is similar to the enumeration of vested chain abacus generated by employing SNC-Transformation in $\mathfrak{D}_{\text{outer}-2}$ and $\mathfrak{D}_{\text{outer}-3}$ except if $\#g_1 = \#g_2 = \ldots = \#g_p$ and $\#\alpha_1 = \#\alpha_2 = \ldots = \#\alpha_p$ where g_p and $\alpha_{p'}$ are the sets of connected bead and empty bead positions in the outer chain, respectively, such that # denotes to the number of elements in the set as shown in Example 4.4.9 and Theorem 4.4.10, respectively.

Example 4.4.9. *Let* $\mu = (3,2^6)$ *be a connected partition of* $\mathfrak{D}_{\text{outer}-1}$ *nested chain abacus of 7-connected beads with two sets of bead positions* $\{g_1, g_2\}$ *where* $g_1 =$

$\{1,2\}$ *and* $g_2 = \{3,6,9\}$. *By using SNC-Transformation, there exists 8* $\mathfrak{D}^{(3,4)}_{\text{outer}-1}$ *and 2* $\mathfrak{D}^{(3,3)}_{\text{outer}-2}$.

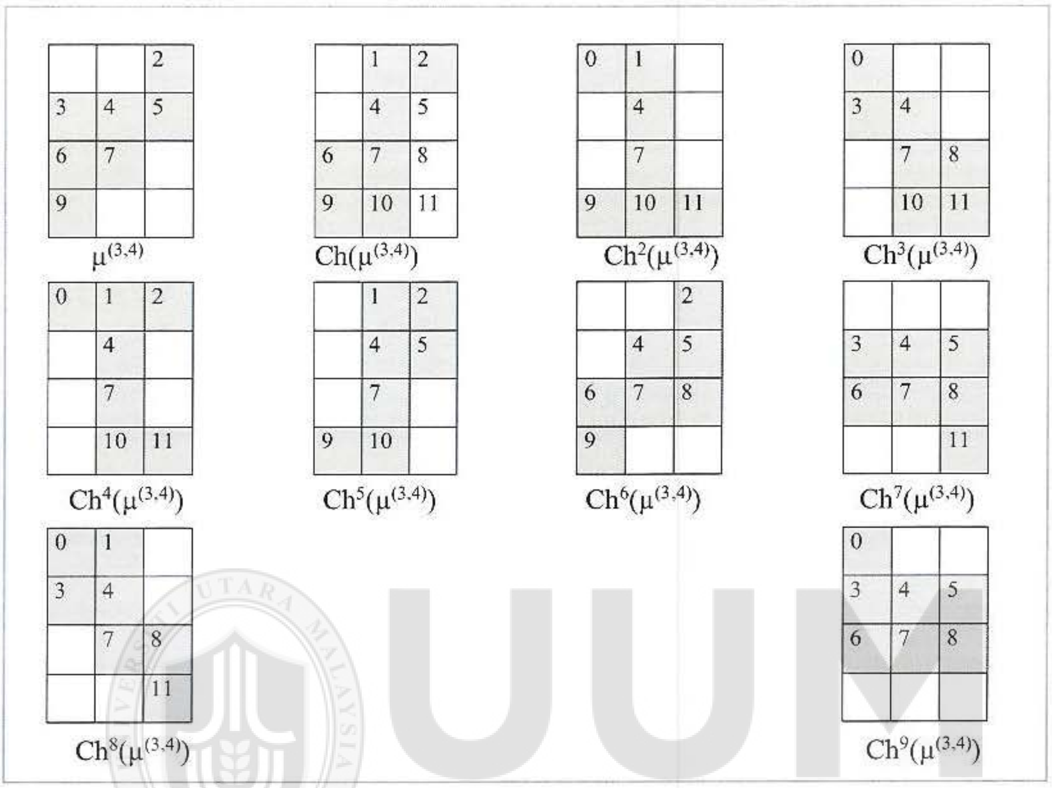


Figure 4.8. Ten nested chain abacus

Theorem 4.4.10. *Let* $\#g_t$ *and* $\#\alpha_{t'}$ *be the number of the bead and empty bead positions in the sets* g_t *and* $\alpha_{t'}$, *respectively. Then, there exists* L *of* $\mathfrak{D}_{\text{outer}-1}$ *(respectively,* $\mathfrak{D}'_{\text{outer}-1}$*), where* $L = \#g_1 + \#\alpha_1$, $\#g_1 = \#g_2 = \ldots = \#g_p$ *and* $\#\alpha_1 = \#\alpha_2 = \ldots = \#\alpha_p$ *for* $1 \leqslant t \leqslant p$ *and* $1 \leqslant t' \leqslant p'$.

Proof. Since the nested chain abacus $\mathfrak{N}$ is a $\mathfrak{D}_{\text{outer}-1}$ (respectively, $\mathfrak{D}'_{\text{outer}-1}$), then $\#g_1 = \#g_2 = \ldots = \#g_p$, $\#\alpha_1 = \#\alpha_2 = \ldots = \#\alpha_p$ and $\#g_t > 1$ where $g_t = \{d^p_1, d^p_2, \ldots, d^p_z\}$ and $1 \leqslant z' < z$ and $1 \leqslant z \leqslant p$. $\#g_1 + \alpha_1 = \#g_2 + \alpha_2 = \ldots = \#g_p + \alpha_p = L$ since $Ch^L(a_{mj}) = \underbrace{Ch(a_{mj}) \circ Ch(a_{mj}) \circ \cdots \circ Ch(a_{mj})}_{L-time}$ then $Ch^L(d^{p''}_{z'}) = d^{p''+1}_{z-z'+1}$ and $Ch_L(d^{p}_{z'}) = d^1_{z}$ where $1 \leqslant p'' < p$. Thus, $\mu^{+L'} = \mu^{+(L+L')}$, so there exist L of $\mathfrak{D}_{\text{outer}-1}$ (respectively, $\mathfrak{D}'_{\text{outer}-1}$) where $1 \leqslant L' < L$. □

Example 4.4.11 illustrates Theorem 4.4.10.

Example 4.4.11. *Let* $\mu^{(3,4)} = (4, 2^6, 0)$ *be a connected partition of* $\mathfrak{D}_{\text{outer}-1}$ *nested chain abacus of 7-connected beads with two sets of bead positions* $\{g_1, g_2\}$ *where* $g_1 = \{1, 2\}$ *and* $g_2 = \{9, 10\}$ *such that* $\alpha_1 = 2$ *and* $\alpha_2 = 2$. *Based on Theorem 4.4.10, the number of nested chain abacus generated by employing SNC-Transformation in* $\mathfrak{D}_{\text{outer}-1}$ *is 5 as shown in Figure 4.9.*

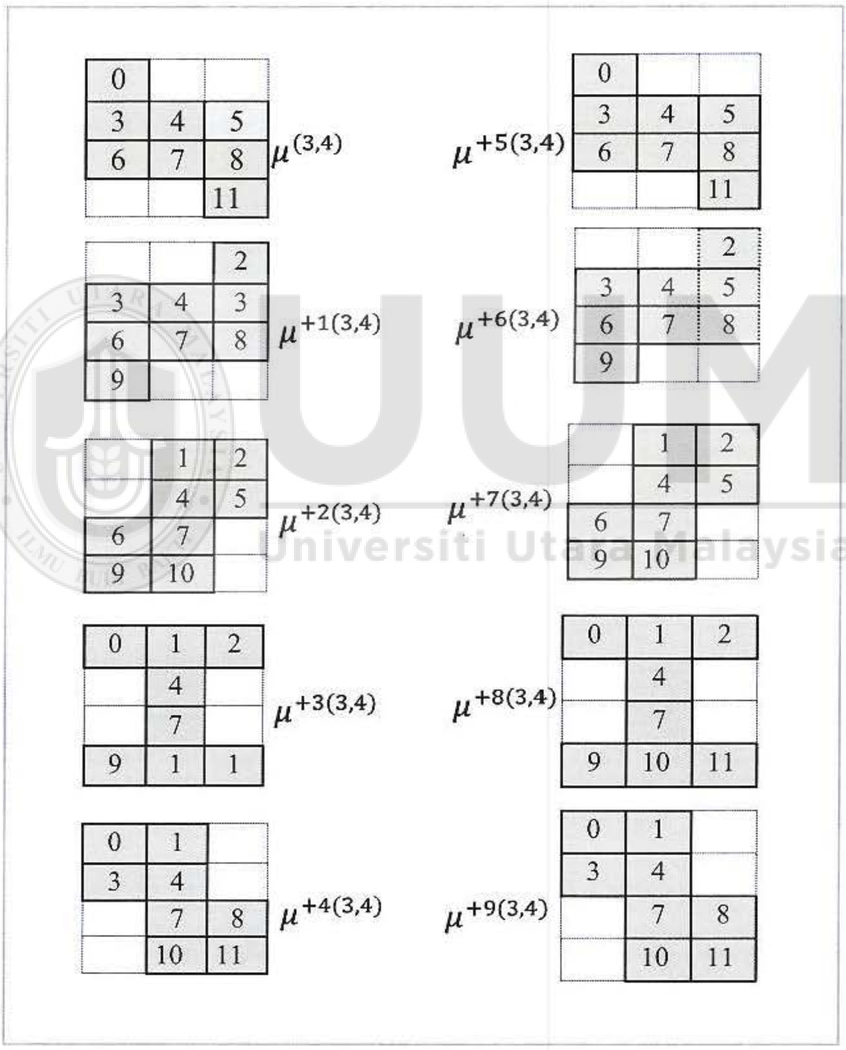


Figure 4.9. Ten $\mathfrak{D}'^{(3,4)}_{\text{outer}-1}$, for 8-connected beads

Previously, the SNC-Transformation was applied in chain 1 (outer chain). In the next theorem, the SNC-Transformation will be employed in chain i where $1 < i \leqslant c$.

Lemma 4.4.12. *Let b' be the number of empty bead positions. For fixed number of b', the number of $D^{(e,r)}_{inner-i}$ generated by employing SNC-Transformation in $D^{(e,r)}_{inner-i}$ is*

(i) $2r+2e-4(2i-1)$ *nested chain abacus if $D^{(e,r)}_{inner-i}$ is rectangular nested chain abacus where $1 < i \leqslant c$ (respectively, rectangle-path nested chain abacus where $1 < i < c$).*

(ii) $r-e+1$ *nested chain abacus if $D^{(e,r)}_{inner-i}$ is vertical rectangle-path nested chain abacus where $i = c$.*

(iii) $e-r+1$ *nested chain abacus if $D^{(e,r)}_{inner-i}$ is horizontal rectangle-path nested chain abacus where $i = c$.*

Proof.

(i) Since the SNC-Transformation algorithm, then, the chain transformation is employed in one chain (chain with empty bead positions). Based on Theorem 3.3.3, the number of nested chain abacus in class $D^{(e,r)}_{inner-i}$ is

$$2r+2e-4(i-1)$$

if chain i is rectangular design.

(ii) See proof (*i*) Lemma 4.4.12 .

(iii) Based on Theorem 3.3.10 (respectively, Theorem 3.3.13), the number of nested chain abacus generated by employing SNC-Transformation in chain c with b'_c empty bead position is

$$r-e+1(respectively, e-r+1), D^{(e,r)}_{inner-i}.$$

(iv) See Proof (*iii*) in Theorem 4.4.12.

□

Theorem 4.4.13. *Let b' be the number of empty bead positions. For fixed b', the number of $D^{(e,r)}_{inner-i}$ nested chain abacus with c chains is*

(i) $4r^2+4e^2+8er-8rk-8rk-4ek-2r-2e+2k+4k^2\ D^{(e,r)}_{inner-i}$

if $\mathfrak{N}$ is a rectangular design structure where $1 \leqslant i \leqslant c$ and $k = 4(2i-1)$.

(ii) $4r^2+4e^2+8er-8rk-8rk-4ek-2r-2e+2k+4k^2\ D^{(e,r)}_{inner-i}$

if $\mathfrak{N}$ is a rectangle-path design structure where $1 < i < c-1$.

(iii) $r^2+e^2-2er+r-e\ D^{(e,r)}_{inner-i}$

if $\mathfrak{N}$ is a vertical rectangle-path design structure where $i = c$.

(iv) $r^2+e^2-2er-r+e\ D^{(e,r)}_{inner-i}$

if $\mathfrak{N}$ is a horizontal rectangle-path design structure where $i = c$.

Proof.

(i) Since the number of nested chain abacus in class $D^{(e,r)}_{inner-i}$ is

$$2r+2e-4(2i-1)$$

if chain i is rectangle design and the beads are of consecutive positions in chain i, then, we have

$$2e+2r-4(2i-1)-1$$

with H different sequences of bead positions, such that sequence H has H beads where $H = 1, 2, \ldots, 2e+2r-4(2i-1)-1$. Thus, there are

$$(2e+2r-4(2i-1))(2e+2r-4(2i-1)-1)$$

$$= 4r^2+4e^2+8er-8rk-8rk-4ek-2r-2e+2k+4k^2r^2+e^2-2er+r-e$$

nested chain abacus in class $D^{(e,r)}_{inner-i}$ with H sets of beads where $1 \leqslant i \leqslant c$.

(ii) See Proof (*i*) in Theorem 4.4.

(iii) Since the nested chain abacus of vertical rectangle-path design structure and $c-1$ chains are full chains and chain c, which is a path chain, is not full a chain, based on Theorem 3.3.9, path chain transformation of chain c with fixed number of empty bead position will generate

$$r-e+1$$

different chains. Since the beads are consecutive positions in chain c, then, we have H_d different sequences with d bead positions such that $d = 1, 2, ..., r-e$. Thus, there are

$$(r-e+1)(r-e) = r^2+e^2-2er+r-e\, D^{(e,r)}_{inner-i}.$$

(iv) See Proof (*iii*) in Theorem 4.4.

□

In the next theorem we will found a generating function of $\mathfrak{D}^{(e,r)}_{\text{inner}-\text{i}}$ class

Theorem 4.4.14. *Let $\mathfrak{D}^{(e,r)}_{\text{inner}-\text{i}}$ be nested chain abacus with e columns and r rows. The generating function for the number $\mathfrak{D}^{(e,r)}_{\text{inner}-\text{i}}$ has the following rational form*

$$10\left(\frac{x}{(1-x)^2}\right)\left(\frac{1}{1-y}-1\right)+2\left(\frac{y}{(1-y)^2}\right)\left(\frac{1}{1-x}-1\right)-4\left(\frac{xy}{(1-x)(1-y)}\right).$$

Proof.

$$\begin{aligned}
\sum_{r\geqslant 1} x^r &= x+x^2+x^3+\ldots &&= \frac{1}{1-x}-1 \\
x\sum_{r\geqslant 1} rx^{r-1} &= 1+2x+3x^2+\ldots &&= x\frac{d}{dx}\left(\frac{1}{1-x}-1\right) \\
& &&= \frac{x}{(1-x)^2}.
\end{aligned}$$

Thus

$$x\sum_{r,e\geqslant 1} rx^{r-1}y^{e} = x\left(\frac{1}{(1-x)^2}\right)\left(\frac{1}{1-y}-1\right).$$

Similarity

$$y\sum_{r,e\geqslant 1} ey^{e-1}x^{r} = y\left(\frac{1}{(1-y)^2}\right)\left(\frac{1}{1-x}-1\right).$$

Since i is the number of chain with empty position then,

$$\sum_{r,e\geqslant 1} ix^{r}y^{e} = \frac{ixy}{(1-x)(1-y)}.$$

Based on Theorem there is $2r+2e-4(2i-1)$ $\mathfrak{D}^{(e,r)}_{\text{inner}-i}$. Thus, the the generating function for $\mathfrak{D}^{(e,r)}_{\text{inner}-i}$ is

$$\sum_{r,e\geqslant 1}(2r+2e-4(2i-1))x^{r}y^{e} = 2\left(\frac{2xy}{(1-x)^2(1-y)}\right)+2\left(\frac{xy}{(1-y)^2(1-x)}\right)$$

$$-\frac{(8i+4)xy}{(1-x)(1-y)}.$$

□

4.5 Multi Transformation Classes

Multi transformation classes are classes of nested chain abacus generated by employing a MNC-Transformation in $\mathfrak{D}_{\text{inner}}$ (respectively, $\mathfrak{D}'_{\text{inner}}$) nested chain abacus. The $\mathfrak{D}_{\text{inner}}$ and $\mathfrak{D}'_{\text{inner}}$ nested chain abacus with e columns and r rows are denoted by $\mathfrak{D}^{(e,r)}_{\text{inner}}$ (respectively, $\mathfrak{D}'^{(e,r)}_{\text{inner}}$) we would like to point out that, for any nested chain abacus $\mathfrak{N}$ in $\mathfrak{D}_{\text{inner}}$ then $Ch(\mathfrak{N})$ in $\mathfrak{D}_{\text{inner}}$.

The next lemma provides the way to generate $\mathfrak{D}^{(e,r)}_{\text{inner}}$ ($\mathfrak{D}'^{(e,r)}_{\text{inner}}$) nested chain abacus in different cases. In addition, the $\mathfrak{D}_{\text{inner}}^{(e,r)}$ ($\mathfrak{D}_{\text{inner}}^{'(e,r)}$) elements are enumerated.

Case one: If $b_1^{'} < e$ (respectively, $b_1^{'} < r$).

Lemma 4.5.1. *Let $b_1^{'}$ be the number of empty bead positions in chain 1. For fixed $b_1^{'} < e$, then, there exist*

(i) $\prod_{i=1}^{c}(2r+2e-4(2i-1))$
of $\mathfrak{D}_{\text{inner}}^{(e,r)}$ rectangle nested chain abacus.

(ii) $(r-e+1)\prod_{i=1}^{c-1}(2r+2e-4(2i-1))$
of $\mathfrak{D}_{\text{inner}}^{'(e,r)}$ vertical rectangle-path nested chain abacus.

(iii) $(e-r+1)\prod_{i=1}^{c-1}(2r+2e-4(2i-1))$
of $\mathfrak{D}_{\text{inner}}^{'(e,r)}$ is horizontal rectangle-path nested chain abacus.

generated by employing MNC-Transformation.

Proof.

(i) Based on MNC-Transformation algorithm in $\mathfrak{D}_{\text{inner}}^{(e,r)}$, $\exists Ch(i)$ in this case the chain transformation application in the chains one by one such that each chain will be move $2e+2r-4(2i-1)$ then, $Ch(i): \mathbb{Z} \to \mathbb{Z}$ such that

$$a_{mj} \to \begin{cases} a_{(m-1)(e-i+1)} & if \quad 1 < m \leqslant r, j = e-i+1, \\ a_{(m+1)i} & if \quad 1 \leqslant m < r, j = i, \\ a_{i(j-1)} & if \quad m = i, i < j \leqslant e-i+1, \\ a_{(r-i+1)(j+1))} & if \quad m = r-i+1, i \leqslant j < e-i+1, \\ a_{mj} & if \quad j \neq \{i, e-i+1\} \text{ and } m \neq \{i, r-i+1\}, \end{cases}$$

where $1 \leqslant i \leqslant c$.

Based on Theorem 3.3.3, the maximal number of nested chain abacus can be generated by employing MNC-Transformation in $\mathfrak{D}_{\text{inner}}^{(e,r)}$ is

$$2r+2e-4(2i-1)$$

for each rectangle chain. Then, there exist

$$(2r+2e-4(2\times 1-1))(2r+2e-4(2\times 2-1))...(2r+2e-4(2\times c-1))$$

of $\mathfrak{D}_{\text{inner}}^{(e,r)}$. Thus, there is

$$\prod_{i=1}^{c}(2r+2e-4(2i-1))\ \mathfrak{D}_{\text{inner}}^{(e,r)}.$$

(ii) Based on MNC-Transformation algorithm in $\mathfrak{D}_{\text{inner}}^{\prime(e,r)}$ then $\exists\, Ch(i):\mathbb{Z}\to\mathbb{Z}$

$$a_{mj}\to\begin{cases} a_{(m-1)(e-i+1)} & if \quad 1<m\leqslant r, j=e-i+1,\\ a_{(m+1)i} & if \quad 1\leqslant m<r, j=i,\\ a_{i(j-1)} & if \quad m=i, i<j\leqslant e-i+1,\\ a_{(r-i+1)(j+1))} & if \quad m=r-i+1, i\leqslant j<e-i+1,\\ a_{mj} & if \quad j\neq\{i,e-i+1\} \text{ and } m\neq\{i,r-i+1\}.\end{cases}$$

$$a_{mj}\to\begin{cases} a_{(m+1)(\frac{e+1}{2})} & if \quad j=\frac{e+1}{2}, \frac{e+1}{2}<m\leqslant\frac{r-2e+1}{2},\\ a_{(\frac{r-2e+1}{2})(\frac{e+1}{2})} & if \quad j=\frac{e+1}{2} m=\frac{e+1}{2},\\ a_{mj} & if \quad j\neq\frac{e+1}{2}, m\notin[\frac{e+1}{2},\frac{r-2e+1}{2}].\end{cases}$$

Based on Theorem 3.3.12 and Theorem 3.3.10, there exist

$$r-e+1(respectively, e-r+1)$$

different transformations for a rectangle-path chain, and each transformation will generate a nested chain abacus. Since $\mathfrak{D}_{\text{inner}}^{\prime(e,r)}$ consists of $c-1$ rectangle nested chain, and a vertical-path (respectively, horizontal-path) chain, then, there exist

$$(r-e+1)\left(\prod_{i=1}^{c-1} 2r+2e-4(2i-1)\right)\ \mathfrak{D}_{\text{inner}}{}^{(e,r)}$$

respectively,

$$(e-r+1)\left(\prod_{i=1}^{c-1} 2r+2e-4(2i-1))\right)\ \mathfrak{D}_{\text{inner}}{}'^{(e,r)}.$$

□

Case two: If $e \leqslant b_1^{'} \leqslant e+r-3$ and $b_2 < e-2$ (respectively, $r \leqslant b_1^{'} \leqslant e+r-3$, $b_2 < r-2$).

Theorem 4.5.2. *Let $b_1^{'}$ be the number of empty bead positions in chain 1. For fixed $e \leqslant b_1^{'} \leqslant e+r-3$ and $b_2 < e-2$ (respectively, $r \leqslant b_1^{'} \leqslant e+r-3$, $b_2 < r-2$). The number of $\mathfrak{D}_{\text{inner}}$ with c chains and $e \leqslant r$ (respectively, $r < e$) generated by MNC-Transformation is*

(i) $2(b_1^{'}-e+1)\prod_{i=2}^{c}(2r+2e-4(2i-1))\mathfrak{D}_{\text{inner}}^{(e,r-1)}$ *if $b^{'} < r$ (respectively, if $b^{'} < e$).*

(ii) $2(b_1^{'}-r+1)\prod_{i=2}^{c}(2r+2e-4(2i-1))\mathfrak{D}_{\text{inner}}^{(e-1,r)}$.

(iii) $2(b_1^{'}-e+1)\prod_{i=2}^{c}(2r+2e-4(2i-1))\mathfrak{D}_{\text{inner}}^{(e,r-1)}$.

(iv) $\prod_{i=1}^{c}(2r+2e-4(2i-1))-[2(b_1^{'}-e+1)\prod_{i=1}^{c-1}2r+2e-4(2i-1)+2(b_1^{'}-r+1)$
$\prod_{i=1}^{c-1}2r+2e-4(2i-1)]\mathfrak{D}_{\text{inner}}^{(e,r)}$.

Proof. Suppose $P=\{v_1,v_2,\ldots,v_{b_1^{'}}\}$ is a set of consecutive empty bead positions in chain 1 such that $|v_{h+1}-v_h| \in \{1,e\}$ and each position in P is an element in matrix $A_{r\times e}$ after converting the nested chain abacus to matrix where $1 \leqslant h < b_1^{'}$. Since $\mathfrak{D}_{\text{inner}}$ class is generated by employing MNC-Transformation which works in anticlockwise direction, then, there exist rectangle chain transformations

$Ch^{u}(P), Ch^{u+1}(P),\ldots, Ch^{u+b_1^{'}-e+1}(P)$ employed in chain 1 such that

$$\mathrm{Ch}^{u}(P) = \{ a_{1j} : 1 \leqslant j \leqslant e\} \cup \{a_{me} : 2 \leqslant m \leqslant b_1' - e + 1\},$$

$$\mathrm{Ch}^{u+1}(P) = \{a_{21}\} \cup \{a_{1j} : 1 \leqslant j \leqslant e\} \cup \{a_{me} : 2 \leqslant m \leqslant b_1' - e\},$$

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$$\mathrm{Ch}_1^{u+b_1'-e+1}(P) = \{ a_{m1} : 2 \leqslant m \leqslant b_1' - e + 1\} \cup \{a_{1j} : 1 \leqslant j \leqslant e\},$$

Thus, row 1 is an empty row in $b_1' - e + 1$ nested chain abacus where

$$1 \leqslant u \leqslant (2e + 2r - 5) - (b' - e + 1).$$

A similar case can be applied if

$$Ch^{u}(P) = \{a_{rj} : 1 \leqslant j \leqslant e\} \cup \{a_{m1} : (r-1) - (b_1' - e) < m \leqslant r - 1\}.$$

Thus, the number of $\mathfrak{D}_{\mathrm{inner}}^{(e,r-1)}$ is

$$2(b_1' - e + 1)\prod_{i=2}^{c}(2r + 2e - 4(2i - 1)).$$

Since $r < e + r$, then, there exist rectangle chain transformations

$$Ch^{v}(P), Ch^{v+1}(P), \ldots, Ch^{v+b_1'-r+1}(P)$$

employed in chain 1 such that

$$\begin{aligned}
\mathrm{Ch}^{v}(P) &= \{\, a_{m1} : 1 \leqslant m \leqslant r\} \cup \{a_{1j} : 2 \leqslant j \leqslant b_1^{'} - r\}, \\
\mathrm{Ch}^{v+1}(P) &= \{a_{2r}\} \cup \{a_{m1} : 1 \leqslant m \leqslant r\} \cup \{a_{1j} : 2 \leqslant j \leqslant b_1^{'} - r - 1\}, \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
\mathrm{Ch}^{v+b_1^{'}-r+1}(P) &= \{\, a_{m1} : 1 \leqslant m \leqslant r\} \cup \{a_{rj} : 2 \leqslant j \leqslant b_1^{'} - r\}.
\end{aligned}$$

Thus, column 1 is an empty row in $b_1^{'} - r + 1$ nested chain abacus where

$$1 \leqslant v \leqslant (2e + 2r - 5) - (b_1^{'} - r + 1).$$

Based on Lemma 4.5.1 the number of $\mathfrak{D}_{\text{inner}}^{(c-1,c)}$ is

$$2(b_1^{'} - r + 1) \prod_{i=2}^{c} (2r + 2e - 4(2i - 1)).$$

Based on Lemma 4.5.1, the number of nested chain abacus generated by employing MNC-Transformation in $\mathfrak{D}_{\text{inner}}^{(c,c)}$ is

$$\prod_{i=1}^{c} (2r + 2e - 4(2i - 1)).$$

Based on 1 and 2 as previously mention in Theorem 4.5.2, then, there is $\prod_{i=1}^{c}(2r+2e-4(2i-1)) - [2(b_1^{'} - e + 1)\prod_{i=1}^{c-1}(2r+2e-4(2i-1)) + 2(b_1^{'} - r + 1)$ $\prod_{i=1}^{c-1}(2r+2e-4(2i-1))$ of $\mathfrak{D}_{\text{inner}}^{(c,c)}$. □

Corollary 4.5.3. *Let $b_1^{'}$ be the number of empty bead positions in chain 1. For fixed $e \leqslant b_1^{'} \leqslant e + r - 3$ and $b^2 < e - 2$ (respectively, $r \leqslant b_1^{'} \leqslant e + r - 3$ and $b^2 < r - 2$), the number of nested chain abacus with c chains generated by MNC-Transformation* in $\mathfrak{D}^{'(e,r)}_{\text{inner}}$ is

(i) $2(b_1^{'} - e + 1)(r - e + 1) \prod_{i=2}^{c-1} 2r + 2e - 4(2i - 1)$,

(respectively, $2(b_1^{'} - e + 1)(e - r + 1) \prod_{i=2}^{c-1} 2r + 2e - 4(2i - 1)\ \mathfrak{D}^{'(e,r-1)}_{\text{inner}}$).

(ii) $2(b_1^{'} - r + 1)(r - e + 1) \prod_{i=2}^{c-1} 2r + 2e - 4(2i - 1)$ if $b_1^{'} \geqslant r$,

(respectively, $2(b_1^{'} - r + 1)(e - r + 1) \prod_{i=2}^{c-1} 2r + 2e - 4(2i - 1)$ if $b_1^{'} \geqslant e\ \mathfrak{D}^{'(e-1,r)}_{\text{inner}}$).

(iii) $\prod_{i=1}^{c} 2r + 2e - 4(2i - 1) - [2(b_1^{'} - e + 1)(r - e + 1) + 2(b_1^{'} - r + 1)(r - e + 1)]$
$\prod_{i=2}^{c-1} 2r + 2e - 4(2i - 1)$,

(respectively, $\prod_{i=1}^{c} 2r + 2e - 4(2i - 1) - [2(b_1^{'} - e + 1)(e - r + 1)]$
$\prod_{i=2}^{c-1} 2r + 2e - 4(2i - 1) + 2(b_1^{'} - r + 1)(e - r + 1) \prod_{i=2}^{c-1} 2r + 2e - 4(2i - 1)]\ \mathfrak{D}^{'(e,r)}_{\text{inner}}$).

Proof. See Proof of Theorem 4.5.2. □

Case three: If $e + 2 \leqslant b_1^{'} \leqslant e + r - 3$, $b_2^{'} \geqslant e - 2$ and $b_3^{'} < e - 4$.

Corollary 4.5.4. *Let $b_i^{'}$ be the number of empty bead positions in chain i. For fixed $e + 2 \leqslant b_1^{'} \leqslant e + r - 3$, $b_2^{'} \geqslant e - 2$ and $b_3^{'} < e - 4$, the number of $\mathfrak{D}^{(e,r)}_{\text{inner}}$ generated by employ MNC-Transformation is*

(i) $(b_1^{'} - e + 1)(3e + 2r - b_1^{'} - 15) \prod_{i=3}^{c} (2r + 2e - 4(2i - 1))\ \mathfrak{D}^{(e,r-1)}_{\text{inner}}$ if
$b_2^{'} < r - 2$ and $b_1^{'} < r$.

(ii) $(b_1^{'} - e + 1)(b_2^{'} - e - 1) \prod_{i=3}^{c} (2r + 2e - 4(2i - 1))\ \mathfrak{D}^{(e,r-2)}_{\text{inner}}$ if
$b_2^{'} < r - 2$ and $b_1^{'} < r$.

(iii) $\prod_{i=1}^{c} (2r + 2e - 4(2i - 1)) - [(b_1^{'} - e + 1)(3e + 2r - b_1^{'} - 15) - (b_1^{'} - e + 1)$
$(b_2^{'} - e - 1)] \prod_{i=3}^{c} (2r + 2e - 4(2i - 1))\ \mathfrak{D}^{(e,r)}_{\text{inner}}$ if
$b_2^{'} < r - 2$ and $b_1^{'} < r$.

(iv) $(b_1^{'}-r+1)(2e+3r-b_1^{'}-15)\prod_{i=3}^{c}(2r+2e-4(2i-1))\ \mathfrak{D}_{\text{inner}}^{(e-1,r)}$ *if*

$b_1^{'}\geqslant r$ *and* $b_2^{'}\geqslant r-2$.

(v) $(b_1^{'}-e+1)(b_2^{'}-e-1)\prod_{i=3}^{c}(2r+2e-4(2i-1))\ \mathfrak{D}_{\text{inner}}^{(e-2,r)}$ *if*

$b_1^{'}\geqslant r$ *and* $b_2^{'}\geqslant r-2$.

(vi) $\prod_{i=3}^{c}(2r+2e-4(2i-1))$ - $(b_1^{'}-e+1)(3e+2r-b_1^{'}-15)$- $(b_1^{'}-e+1)(b_2^{'}-e-1)$

- $(b_1^{'}-r+1)(2e+3r-b_1^{'}-15)$- $(b_1^{'}-e+1)(b_2^{'}-e-1)]\prod_{i=3}^{c}(2r+2e-4(2i-1))$

$\mathfrak{D}_{\text{inner}}^{(e,r)}$ *if*

$b_1^{'}\geqslant r$ *and* $b_2^{'}\geqslant r-2$.

Proof. Based on Definition 4.2.9, there are sets P and K with $b_1^{'}$ and $b_2^{'}$ empty bead positions in chains 1 and 2, respectively, where $b_1^{'}\geqslant e$ and $b_2^{'}\geqslant e-2$. Suppose that rectangle chain transformation number u in chain 1 for set P is $(Ch_1^u(P))$ then,

$$\mathrm{Ch}_1^u(P) = \{\,a_{1j}:1\leqslant j\leqslant e\}\cup\{a_{me}:2\leqslant m\leqslant b_1^{'}-e+1\},$$

$$\mathrm{Ch}_1^{u+1}(P) = \{a_{21}\}\cup\{a_{1j}:1\leqslant j\leqslant e\}\cup\{a_{me}:2\leqslant m\leqslant b_1^{'}-e\},$$

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$$\mathrm{Ch}_1^{u+b_1^{'}-e-2}(P)= \{a_{2e}\}\cup\{a_{1j}:1\leqslant j\leqslant e\}\cup\{a_{m1}:2\leqslant m\leqslant b_1^{'}-e\},$$

$$\mathrm{Ch}_1^{u+b_1^{'}-e}(P) = \{\,a_{m1}:2\leqslant m\leqslant b_1^{'}-e+1\}\cup\{a_{1j}:1\leqslant j\leqslant e\}.$$

Hence, Ch_1 will produce,

- $(b_1^{'}-e+1)$ chain 1 with $\{a_{1j}:1\leqslant j\leqslant e\}$ (row 1) empty bead positions
- $(b_1^{'}-e-1)$ chain 1 with $\{a_{1j}:1\leqslant j\leqslant e\}\cup\{a_{21},a_{2e}\}$ empty bead positions.

Similarly, Ch_1 will produce,

- $(b_1^{'}-e+1)$ chain 1 with $\{a_{rj}:1\leqslant j\leqslant e\}$ (row r) empty bead positions

- $(b_1^{'} - e - 1)$ chain 1 with $\{a_{rj} : 1 \leqslant j \leqslant e\} \cup \{a_{(r-1)1}, a_{(r-1)e}\}$ empty bead positions.

Based on Corollary 3.3.4 the number of transformation in chain 1 is $2e + 2r - 5$, then,

$$1 \leqslant u \leqslant (2e + 2r - 5) - (b_1^{'} - e + 1).$$

Since the MNC-Transformation employ in all chains and $b_2^{'} \geqslant e - 2$, then,
$\exists\, Ch_2^{v}(K)$ such that

$$\begin{aligned}
\mathrm{Ch}_2^{v}(P) &= \{\, a_{2j} : 2 \leqslant j \leqslant e - 1\} \cup \{a_{m(e-1)} : 2 \leqslant m \leqslant b_2^{'} - e + 4\},\\
\mathrm{Ch}_2^{v+1}(K) &= \{a_{32}\} \cup \{a_{2j} : 2 \leqslant j \leqslant e - 1\} \cup \{a_{m(e-1)} : 2 \leqslant m \leqslant b_2^{'} - e + 3\},
\end{aligned}$$

$\vdots$

$$\mathrm{Ch}_2^{v+b_2^{'}-e+3}(K) = \{\, a_{m2} : 3 \leqslant m \leqslant b_2^{'} - e + 4\} \cup \{a_{2j} : 2 \leqslant j \leqslant e - 1\},$$

Hence, Ch_2 will produce $(b_2^{'} - e + 3)$ chain 2 with $\{a_{2j} : 2 \leqslant j \leqslant e - 1\}$ empty bead positions.
Similarly, Ch_2 will produce $(b_1^{'} - e + 3)$ of chain 2 with $\{a_{(r-1)j} : 2 \leqslant j \leqslant e - 1\}$ empty bead positions. Based on Theorem 3.3.3 the number of transformation in chain two is $2e + 2r - 13$, then, Ch_2 will produce

$$(2e + 2r - 12) - (b_2^{'} - e + 3) = 3e + 2r - b_2^{'} - 15$$

of chain 2 with $\{a_{(r-1)j} : 2 \leqslant j \leqslant e - 1\}$ not empty bead positions where

$$1 \leqslant v \leqslant 3e + 2r - 16 - b_2^{'}.$$

Since MNC-Transformation will employ in all chins then, the number of $\mathfrak{D}_{\text{inner}}^{(e,r-1)}$ is

$$(b_1' - e + 1)(3e + 2r - b_1' - 15)\prod_{i=3}^{c}(2r + 2e - 4(2i-1))$$

and the number of $\mathfrak{D}_{\text{inner}}^{(e,r-2)}$ is

$$(b_1' - e + 1)(b_2' - e + 3)\prod_{i=3}^{c}(2r + 2e - 4(2i-1)).$$

Based on Lemma 4.5.1(1) MNC-Transformation generated

$$\prod_{i=1}^{c} 2r + 2e - 4(2i-1)$$

nested chain abacus. Thus there is
$\prod_{i=1}^{c}(2r+2e-4(2i-1)) - (b_1' - e + 1)(3e + 2r - b_1' - 15) - (b_1' - e + 1)(b_2' - e + 3)$
$\prod_{i=3}^{c} 2r + 2e - 4(2i-1)\ \mathfrak{D}_{\text{inner}}^{(e,r)}$.
Similarly, we can prove that, the number of $\mathfrak{D}_{\text{inner}}^{(e-1,r)}$ if $b_2' < r - 2$ is

$$(b_1' - r + 1)(2e + 3r - b_1' - 15)\prod_{i=3}^{c}(2r + 2e - 4(2i-1))$$

with $e - 1$ columns and r rows if $b_1' \geqslant r$ and $b_2' \geqslant r - 2$

$$(b_1' - e + 1)(b_2' - e + 3)\prod_{i=3}^{c}(2r + 2e - 4(2i-1))$$

with $e - 2$ columns and r rows if $b_1' \geqslant r$ and $b_2' \geqslant r - 2$. Thus, the number of $\mathfrak{D}_{\text{inner}}^{(e,r)}$ is
$\prod_{i=1}^{c}(2r+2e-4(2i-1)) - (b_1' - e + 1)(3e + 2r - b_1' - 15)$ - $(b_1' - e + 1)(b_2' - e + 3)$ -
$(b_1' - r + 1)(2e + 3r - b_1' - 15)$ - $(b_1' - e + 1)(b_2' - e + 3)\prod_{i=3}^{c}(2r + 2e - 4(2i-1))$. □

Corollary 4.5.5. *Let b_i' be the number of empty bead positions in chain i. For fixed $e + 2 \leqslant b_1' \leqslant e + r - 3$, $b_2' \geqslant e - 2$ and $b_3' < e - 4$, then, the number of $\mathfrak{D}'^{(e,r)}_{\text{inner}}$ generated* *by employing MNC-Transformation is*

(i) $(b_1' - e + 1)(3e + 2r - b_1' - 15)(r - e + 1)\prod_{i=4}^{c}(2r + 2e - 4(2i - 1))\ \mathfrak{D}'^{(e,r-1)}_{\text{inner}}$ *if* $b_1' < r$ *and* $b_2' < r - 2$.

(ii) $(b_1' - e + 1)(b_2' - e - 1)(r - e + 1)\prod_{i=4}^{c}(2r + 2e - 4(2i - 1))\ \mathfrak{D}'^{(e,r-2)}_{\text{inner}}$ *if* $b_1' < r$ *and* $b_2' < r - 2$.

(iii) $(b_1' - r + 1)(2e + 3r - b_1' - 15)(r - e + 1)\prod_{i=4}^{c}(2r + 2e - 4(2i - 1))\ \mathfrak{D}'^{(e-1,r)}_{\text{inner}}$ *if* $b_1' \geqslant r$ *and* $b_2' \geqslant r - 2$.

(iv) $(b_1' - e + 1)(b_2' - e - 1)(r - e + 1)\prod_{i=4}^{c}(2r + 2e - 4(2i - 1))\ \mathfrak{D}'^{(e-2,r)}_{\text{inner}}$ *if* $b_1' \geqslant r$ *and* $b_2' \geqslant r - 2$.

(v) $(r - e + 1)\prod_{i=2}^{c}(2r + 2e - 4(2i - 1))$- $(b_1' - e + 1)(3e + 2r - b_1' - 15)(r - e + 1)$ - $(b_1' - e + 1)(b_2' - e - 1)(r - e + 1)$ - $(b_1' - r + 1)(2e + 3r - b_1' - 15)$ $(r - e + 1)$ - $(b_1' - e + 1)(b_2' - e - 1)(r - e + 1)\prod_{i=4}^{c}(2r + 2e - 4(2i - 1))\ \mathfrak{D}^{(e,r)}_{\text{inner}}$.

Proof. It follows immediately from Theorem 4.5.4. □

Case four: If $e + 2 \leqslant b_1' \leqslant e + r - 3$, $b_i' \geqslant e - 2(i - 1)$ where $2 \leqslant i \leqslant c$.

Theorem 4.5.6. *Let* $\mathfrak{D}^{(e,r)}_{\text{inner}}$ *be nested chain abacus with c chains and* b_i' *the number of empty bead positions in chain i such that* $e \leqslant b_1' \leqslant e + r - 3$. *Then, the number of nested chain abacus generated by employing MNC-Transformation in* $\mathfrak{D}^{(e,r)}_{\text{inner}}$ *is*

(i) $\prod_{i=1}^{k-1}\Big(b_i' - (e - 2(i - 1)) - (2k - 2(i - 1)) + 1\Big)(2e + 2r - 4(2(k + 1) - 1))$ $-\Big(b_{k+1}' - (e - 2k) + 1\Big)\prod_{i=k+2}^{c}(2e + 2r - 4(2i - 1))\mathfrak{D}^{(e,r-\ell)}_{\text{inner}}$ *if* $b_{k+1}' < e - 2k$.

(ii) $\prod_{i=1}^{k-1}\Big(b_i' - (r - 2(i - 1)) - (2k - 2(i - 1)) + 1\Big)(2e + 2r - 4(2(k + 1) - 1))$ $-\Big(b_{k+1}' - (r - 2k) + 1\Big)\prod_{i=k+2}^{c}(2e + 2r - 4(2i - 1))$ *with* $\mathfrak{D}^{(e-\ell,r)}_{\text{inner}}$ *if* $b_{k+1}' < r - 2k$.

(iii) $\prod_{i=1}^{c}(2e+2r-4(2i-1))$

$$- \prod_{i=1}^{k-1}\left(b_i^{'}-(e-2(i-1))+(2k-2(i-1))+1\right)(2e+2r-4(2(k+1)-1))$$
$$\left[\left(b_{k+1}^{'}-(e-2k)+1\right)\prod_{i=k+2}^{c}(2e+2r-4(2i-1))\right.$$
$$\left.+\left(b_{k+1}^{'}-(r-2k)+1\right)\prod_{i=k+2}^{c}(2e+2r-4(2i-1))\right] \text{ of } \mathfrak{D}_{\text{inner}}^{(\mathfrak{r},\mathfrak{r})} \text{ where } k \geqslant 2.$$

Proof.

(i) We will proof the above theorem by induction.

Basic step:

When $k=2$. Based on Theorem 4.5.2 then, the number of nested chain abacus is

$$(b_1^{'}-e+1)\prod_{i=1}^{c-1}(2e+2r-4(2i-1).$$

Induction step:

We suppose its true for $i=i^{'}$, we proof 1 is true if $i^{'}=i+1$, since we can add chain in the outer such that $e \leqslant b^{'} \leqslant e+r-3$, then,

$$(b_1^{'}-(e+2)+1)\left(\prod_{i=2}^{k}\left(b_i^{'}-(e-2(i-1))-(2k-2(i-1))+1\right)\right)$$
$$-\left(2(k+2)-1)\right)-(b_{k+2}^{'}-(e-2k+2)+1\Big)\prod_{i=k+3}^{c}(2e+2r-4(2i-1)).$$

Then

$$\prod_{i=1}^{k^{'}}\left(b_i^{'}-(e-2(i-1))-(2k-2(i-1))+1\right)$$
$$-\left(2(k+2)-1)\right)-(b_{k+2}^{'}-(e-2k+2)+1\Big)\prod_{i=k+3}^{c}(2e+2r-4(2i-1)).$$

Where $e^{'}=e+1$, $r^{'}=r+2$ and $k^{'}=k+1$.

(ii) See proof (i) Theorem 4.5.6.

(iii) Based on Lemma 4.5.1(1) MNC-Transformation generated

$$\prod_{i=1}^{c}2e+2r-4(2i-1)]$$

nested chain abacus with r columns and r rows. Based on Theorem 4.5.6$(i)(ii)$, there is

$$\prod_{i=1}^{c}(2e+2r-4(2i-1))$$
$$-\prod_{i=1}^{k-1}\left(b_i'-(e-2(i-1))+(2k-2(i-1))+1\right)(2e+2r-4(2(k+1)-1))$$
$$\left[\left(b_{k+1}'-(e-2k)+1\right)\prod_{i=k+2}^{c}(2e+2r-4(2i-1))\right.$$
$$\left.+\left(b_{k+1}'-(r-2k)+1\right)\prod_{i=k+2}^{c}(2e+2r-4(2i-1))\right] \text{ of } \mathfrak{D}_{\text{inner}}^{(e,r)} \text{ where } k\geqslant 2. \qquad \square$$

Corollary 4.5.7. *Let $\mathfrak{D}_{\text{inner}}^{'(e,r)}$ be vertical-path rectangle nested chain abacus with c chains and b_i' empty bead positions in chain i such that $e \leqslant b_1' \leqslant e+r-3$. Then, the number of nested chain abacus generated by employing MNC-Transformation in $\mathfrak{D}_{\text{inner}}^{'(e,r)}$ is*

(i) $\prod_{i=1}^{k-1}\left(b_i'-(e-2(i-1))-(2k-2(i-1))+1\right)[2e+2r-4(2(k+1)-1)]$
$-\left(b_{k+1}'-(e-2k)+1\right)(r-e+1)\prod_{i=k+2}^{c-1}(2e+2r-4(2i-1))\ \mathfrak{D}_{\text{inner}}^{'(e,r-\ell)}$ *if* $b_{k+1}'<e-2k.$

(ii) $\prod_{i=1}^{k-1}\left(b_i'-(r-2(i-1))-(2k-2(i-1))+1\right)(2e+2r-4(2(k+1)-1))$
$-\left(b_{k+1}'-(r-2k)+1\right)(r-e+1)\prod_{i=k+2}^{c-1}(2e+2r-4(2i-1))$ *with* $\mathfrak{D}_{\text{inner}}^{'(e-\ell,r)}$ *if* $b_{k+1}'<r-2k.$

(iii) $\prod_{i=1}^{c}(2e+2r-4(2i-1))-1-2\ \mathfrak{D}_{\text{inner}}^{'(e,r)}$ *where* $k\geqslant 2.$

Proof. This follows immediately from 3.3.9 and Theorem 4.5.6 $\square$

Corollary 4.5.8. *Let $\mathfrak{D}_{\text{inner}}^{'(e,r)}$ be horizontal-path rectangle nested chain abacus with c chains and b_i' the number of empty bead positions in chain i. Then, the number of nested chain abacus generated by employing MNC-Transformation in $\mathfrak{D}_{\text{inner}}^{'(e,r)}$ is*

(i) $\prod_{i=1}^{k-1}\left(b_i'-(e-2(i-1))-(2k-2(i-1))+1\right)(2e+2r-4(2(k+1)-1))$
$-\left(b_{k+1}'-(e-2k)+1\right)(e-r+1)\prod_{i=k+2}^{c-1}(2e+2r-4(2i-1))\ \mathfrak{D}_{\text{inner}}^{'(e,r-\ell)}$ *if*

$b'_{k+1} < e - 2k.$

(ii) $\prod_{i=1}^{k-1}\left(b'_i - (r - 2(i-1))\right) - (2k - 2(i-1) + 1)(2e + 2r - 4(2(k+1) - 1))$
$- \left(b'_{k+1} - (r - 2k) + 1\right)(e - r + 1)\prod_{i=k+2}^{c-1}(2e + 2r - 4(2i-1))$ *with* $\mathfrak{D}'^{(c-k,r)}_{\text{inner}}$ *if*
$b'_{k+1} < r - 2k.$

(iii) $\prod_{i=1}^{c}(2e + 2r - 4(2i-1)) - 1 - 2,\ \mathfrak{D}'^{(c,r)}_{\text{inner}}$ *if* $k \geqslant 2,$

where $e \leqslant b'_1 \leqslant e + r - 3.$

Proof. This follows immediately from Theorem 3.3.12 and Theorem 4.5.6 □

The generating function of nested chain abacus of connected beads according to various parameters was studied (Redelmeier, 1981; Goupil et al., 2013; James, 1987). In next theorem we used the (Goupil et al., 2010) methodology for establishing a recurrence relation for which we can deduce the number of nested chain abacus having c chain.

Theorem 4.5.9. *Let* $\mathfrak{D}'^{(c,r)}_{\text{inner}}$ *be nested chain abacus with e columns and r rows. The number of* $\mathfrak{D}'^{(c,r)}_{\text{inner}}$ *having* b_i *bead positions and* b'_i *empty beads satisfies the following recurrence relation*

$$\prod_{i=1}^{c}\frac{b_i + b'_i}{1 - b_i - b'_i}$$

Proof. A nested chain abacus having b_i beads and b'_i empty bead positions is obtained by joining i chains where $1 < i < c$ and $b'_i < b_{i+1}$. There is

$$\binom{b_i + b'_i}{b'_i}$$

of chain i. Based on Lemma 4.5.1 the number of $\mathfrak{D}_{\text{inner}}^{\prime(\mathfrak{c},\mathfrak{r})}$ generated by employ MNC-transformation depended on beads and empty bead positions is

$$\prod_{i=1}^{c}(b_i + b_i^{'})$$

Based on the ordinary form the generating function of $\mathfrak{D}_{\text{inner}}^{\prime(\mathfrak{c},\mathfrak{r})}$ is

$$\sum a_{n,m} x^n y^m.$$

Thus the generating function of $\mathfrak{D}_{\text{inner}}^{\prime(\mathfrak{c},\mathfrak{r})}$ by adding chain with beads and empty bead positions and employing MNC-transformation is

$$\sum (b_i + b_i^{'}) x^i y^i.$$

Based on (Barequet et al., 2016)

$$\sum (b_i + b_i^{'}) x^i y^i = \frac{1}{1 - b_i - b_i^{'}}.$$

Hence,

$$\sum \prod_{i=1}^{c} b_i + b_i^{'} x^i y^i = \prod_{i=1}^{c} \frac{1}{1 - b_i - b_i^{'}}. \qquad \square$$

Based on Theorem 2.5.24 and Theorem 2.5.24, in the next section we will develop the generated function with respect to chains to count the number of nested chain abacus with n-connected beads.

4.6 Generating Function with Respect to Chains

This section employs the design structure of nested chain abacus based on Theorem 2.5.24 and Theorem 2.5.25 to construct a succession rule. Furthermore, based on this rule, generating function will be developed.

4.6.1 Succession Rule

A succession rule, Ω. is a system $((a);\mathscr{P})$, consisting of an axiom (a) and a set, $\mathscr{P}$, of productions or rewriting OF rules defined on a set of labels $M \subseteq N^{+}$.

$$\Omega = \begin{cases} a \\ k \rightsquigarrow (c_1(k))(c_2(k))(c_3(k))...(c_t(k)), \end{cases} \tag{4.1}$$

where $a \in M$ is a constant and the c_i are functions $M \to M$ (Ferrari et al., 2003). One of the main properties of a succession rule is the consistency principle, i.e. each label (k) must produce exactly k elements. A succession rule induces, and is suitably represented by, a generating tree whose root is labelled by the axiom (a), and a node labelled (k) produces at the next level k sons labelled by $(c_1(k),...,c_k(k))$ respectively (which in turn will produce $(c_1(k),...,c_k(k))$ sons, etc.). The succession rule produces a sequence, $\{f_n\}_n$, of positive integers, where f_n is the number of nodes at level n of the generating tree and its generating function is denoted by $f_\Omega = \sum_{n \geqslant 0} f_n x^n$ (Bacchelli et al., 2010).

We will construct our succession rule starting from a single chain, which will grow step by step by adding C_i beads, where C_i is the number of positions in chain i as shown in Figure 4.10 which illustrates the number of nested chain abacus in levels 1 and 2.

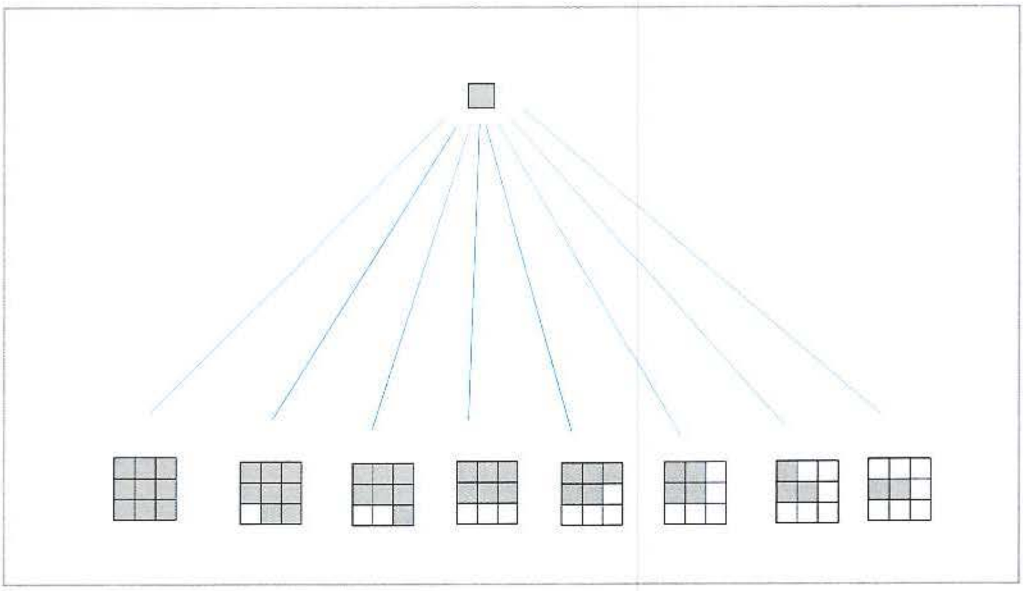

Figure 4.10. First levels of the generating tree of Ω if the first chain consists of one position

Based on the Figure 4.10, we constructed the generating tree where L_n corresponds to the number of nested chain abacus at level n for $n \geqslant 0$.

Lemma 4.6.1. *The number of nested chain abacus in the generating tree is done by adding one chain at level N where $1 \leqslant N \leqslant n$ is*

$$\begin{cases} L_0 = 1 \\ L_1 = P_2 \\ L_2 = P_2(P_2+8) \\ L_3 = P_2(P_2+8)(P_2+16) \\ L_4 = P_2(P_2+8)(P_2+16)(P_2+24) \\ \qquad\vdots \\ L_n = \prod_{k=0}^{n-1}(P_2+8k) \end{cases} \tag{4.2}$$

where L_n is the number of nested chain abacus in level n and $n \geqslant 0$.

Proof. Since we are starting from single chain, then, there is a nested chain abacus in L_0. Next, we will add full chain with P_2 beads where the number of the beads in the second chain depends on the structure of the nested chain abacus (see Theorems 2.5.24 and 2.5.25). Thus, there are P_2 of the nested chain abacus in L_1. Continue to add full chain with P_3 beads where $P_3 = P_2 + 8$ (see Theorem 2.5.25). Thus, there are

$$P_2(P_2 + 8)$$

nested chain abacus. Hence, there are

$$\prod_{k=0}^{n-1}(P_2 + 8k)$$

nested chain abacus at level n. □

Figure 4.11 illustrates to Lemma 4.6.1.

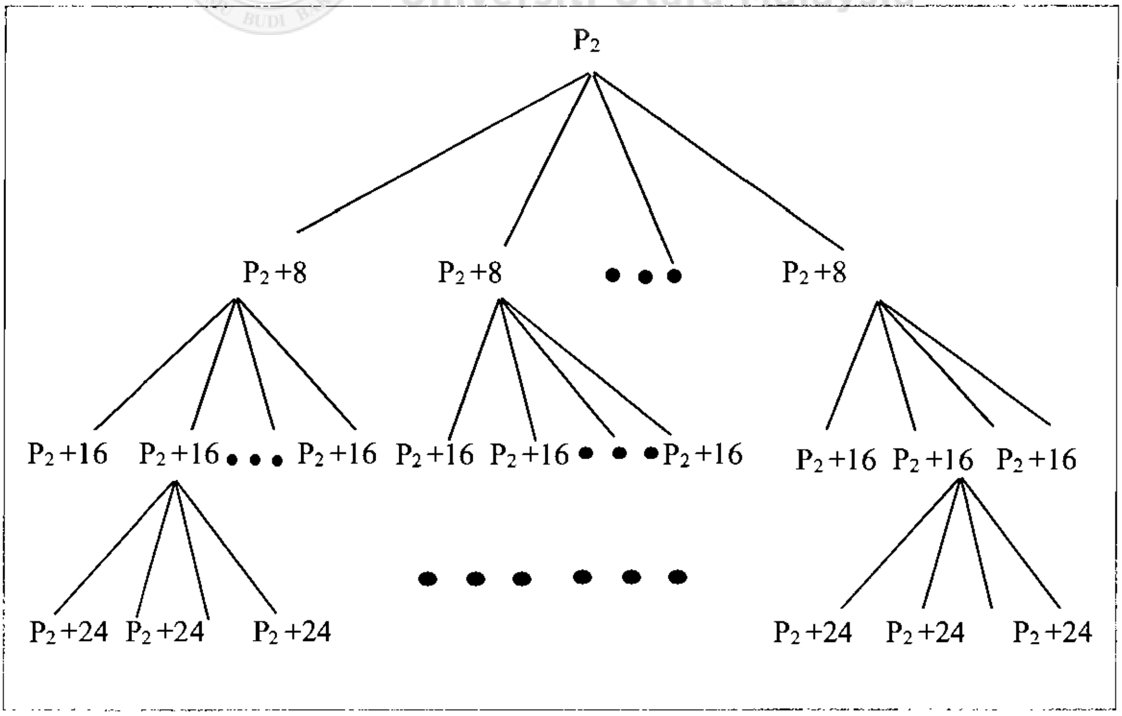

Figure 4.11. Number of nested chain abacus at level N where $N \geqslant 0$

The formal of the generating tree can be sketched using the following succession rule

$$\Omega = \begin{cases} P_2, \\ P_2 + 8k \quad [P_2 + 8(k+1)]^{P_2+8k}. \end{cases} \tag{4.3}$$

4.6.2 Generating Function

A succession rule, Ω, defines a sequence of positive integers f_n, $n \geqslant 0$, f_n being the number of the nodes at level n in the generating tree. In succession equation (4.3), all elements are changed following to $(P_2 + 8k)$. Thus,

$$f_\Omega(x) = \sum_{n \geqslant 0} f_n x^n$$

$$f_\Omega(x,y) = \sum_{n \geqslant 0, k \geqslant 1} f_{n,k}\, x^n y^k. \tag{4.4}$$

Using this succession rule (where power notation denotes the repeating number of levels), since $f_{n,k} = 0$, $k \neq P_2 + 8n$ $(n \geqslant 0)$. Since the starting number of the tree is P_2, the first term is $x^0 y^{P_2}$ we can transfer (4.4) as shown below

$$
\begin{aligned}
f_\Omega(x,y) &= \sum_{n\geqslant 0,k\geqslant 0} f_{n,P_2+8k} \\
&= x^0 y^{P_2} + \sum_{n\geqslant 1,k\geqslant 0} f_{n,P_2+8k}\, x^n y^{P_2+8k} \\
&= y^{P_2} + x \sum_{n\geqslant 0,k\geqslant 0} f_{n,P_2+8k}\, x^n \left(\overbrace{y^{P_2+8(k+1)} + \cdots + y^{P_2+8(k+1)}}^{P_2+8k} \right) \\
&= y^{P_2} + x \sum_{n\geqslant 0,k\geqslant 0} f_{n,P_2+8k}\, x^n (P_2+8k)(y^{P_2+8(k+1)}) \\
&= y^{P_2} + \sum_{n\geqslant 0,k\geqslant 0} f_{n,P_2+8k}\, x^n (P_2+8k).y^{P_2+8(k+1)} \\
&= y^{P_2} + xy^9 \sum_{n\geqslant 0,k\geqslant 0} f_{n,P_2+8k}\, x^n (P_2+8k).y^{P_2+8k-1} \\
&= y^{P_2} + xy^9 \frac{\partial}{\partial y}\left(\sum_{n\geqslant 0,k\geqslant 0} f_\Omega(x,y) \right) \\
&= y^{P_2} + xy^9 \frac{\Omega(x,y)}{\partial y}.
\end{aligned}
\tag{4.5}
$$

Thus,

$$
\frac{\partial f_\Omega(x,y)}{\partial y} - \frac{1}{xy^9} f_\Omega(x,y) = -\frac{y^{P_2}-9}{x}. \tag{4.6}
$$

The solution of $\frac{dy}{dx} + q(x)y = Q(x)$ is $y = e^{-\int q(x)dx}(c + \int Q(x)e^{\int q(x)dx}dx)$.

Let $q(y) = -\frac{1}{xy^9}, Q(y) = -\frac{y^{P_2}-9}{x}$, then,

$$f_\Omega(x,y) = e^{\int -\frac{1}{xy^9}dy}\left(c + \int -\frac{y^{P_2}-9}{x} e^{\int \frac{1}{xy^9}dy}dy\right)$$

$$= e^{\frac{1}{8xy^8}}\left(c - \frac{\int y^{P_2-9} e^{-\frac{1}{8xy^8}}dy}{x}\right). \tag{4.7}$$

$$\lim_{x \to +0} f_\Omega(x,1) = \lim_{x \to +0} e^{\frac{1}{8x}}\left(c - \frac{\int e^{-\frac{1}{8x}}dy}{x}\right)$$

$$= \lim_{x \to +0} e^{\frac{1}{8x}}\left(c - \lim_{x \to +0} \frac{e^{-\frac{1}{8x}}}{x}\right)$$

$$= e^{\infty} \times (c - 0).$$

$$= 0. \tag{4.8}$$

Let $z = \frac{1}{x}, x \to +0, z \to +\infty$,

$$\lim_{x \to +0} \frac{e^{-\frac{1}{8x}}}{x} = \lim_{z \to +\infty} ze^{-z}$$

$$= \lim_{z \to +\infty} \frac{z}{e^z}$$

$$= \lim_{z \to +\infty} \frac{(z)'}{(e^z)'}$$

$$= \lim_{z \to +\infty} \frac{1}{e^z}$$

$$= 0. \tag{4.9}$$

Thus,

$$\lim_{x\to+0} f_\Omega(x,1) = \lim_{x\to+0} e^{\frac{1}{8x}}\left(c - \lim_{x\to+0}\frac{e^{-\frac{1}{8x}}}{x}\right)$$

$$= \lim_{x\to+0} e^{\frac{1}{8x}}(c-0)$$

$$= \mathrm{e}^{\infty}\times(c-0) = 0. \tag{4.10}$$

So from this, c=0 and

$$f_\Omega(x,y) = \mathrm{e}^{\frac{1}{8xy^8}}\left(c - \frac{\int y^{P_2-9}e^{-\frac{1}{8xy^8}}dy}{x}\right)$$

$$= -\mathrm{e}^{\frac{1}{8xy^8}}\left(\frac{\int y^{P_2-9}e^{-\frac{1}{8xy^8}}dy}{x}\right).$$

From this equation, we can find the generating functions such as $f_\Omega(x) = f_\Omega(x,1)$.

Example 4.6.2. *Suppose that $P_1 = 6$, $P_2 = 14$ and $P_3 = P_2 + 8 = 22, \ldots$. By employing programming code in Appendix A. Thus, the generating function is*

$f(x) = 1 + 14x + 308x^2 + 6776x^3 + O(x)$.

4.7 Conclusion

The crux of this chapter is to generate classes of nested chain abacus based on three types of transformations: SNC2-Transformation, SNC-Transformation and

MNC-Transformation. Furthermore, generating function with respect to chain is proposed.

Figure 4.12 presents the classes from three different types of transformation.

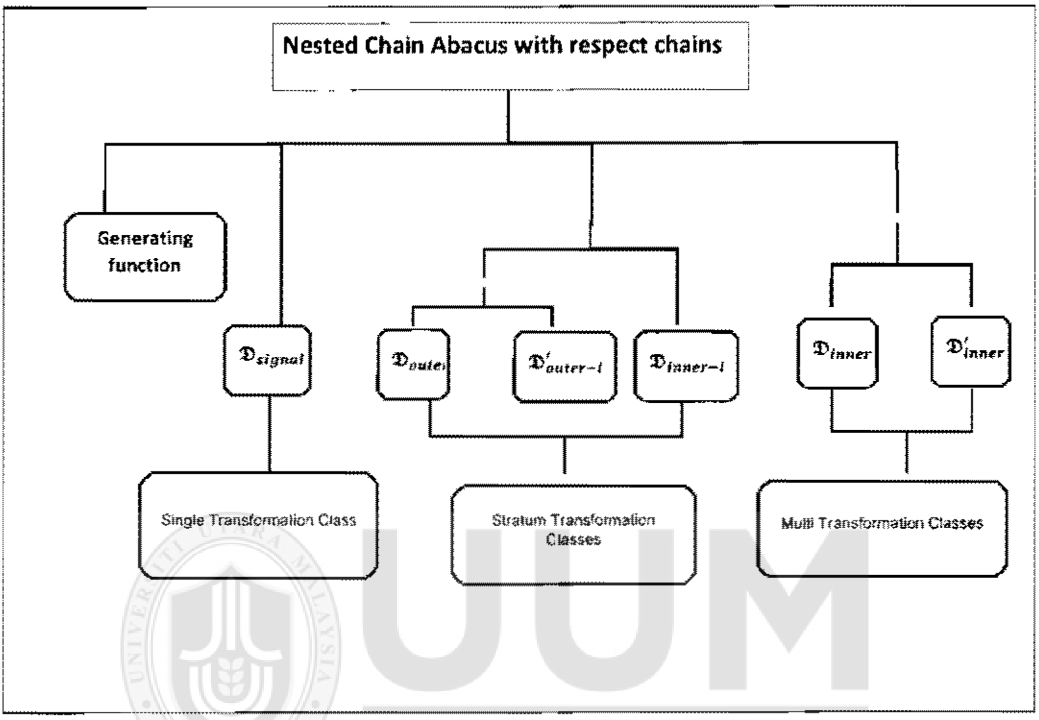

Figure 4.12. Nested chain abacus with respect to chains

In chapter 5, we will provide other classes of nested chain abacus with respect to columns using the connectedness and use the property of *e*-core.

CHAPTER FIVE

CLASSES OF NESTED CHAIN ABACUS WITH RESPECT TO THE COLUMNS

5.1 Introduction

In this chapter, we develop classes of nested chain abacus using two methods, namely e-core and spinal design. These two methods focus on the case with columns. Then, generating function are formulated using enumeration of combinatorial object (ECO) method.

This chapter begins with some definitions and related result required for this chapter in Section 5.2. In section 5.3, the e-convex class developments is presented. Then, the generating function of the class of e-convex is developed in Section 5.4. Moreover, in Section 5.5, the spinal design developments with one class of nested chain abacus which is the $\mathfrak{N}^2$.

5.2 Definition and Related result

This section provides definitions and related results necessary in building a class of nested chain abacus.

Definition 5.2.1. *(Fayers, 2008) An e-core partition is representation to an abacus configuration by sliding all of the beads on each column to their highest possible positions. The partition which corresponds to this new abacus configuration is the ecore partition.*

The following theorems are hold in (Andrews, 1998).

Theorem 5.2.2. *The number of partitions of the integer, n, with distinct parts is the coefficient of* x^n *in* $(1-x^2)(1-x^3)\ldots(1-x^n)$.

Theorem 5.2.3. *The number of partitions of the integer, n, with parts are* $\leqslant k$

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\dots(1-x^k)}.$$

Theorem 5.2.4. *The number of partitions of the integer, n, with exactly k parts is*

$$\frac{x^k}{(1-x)(1-x^2)(1-x^3)\dots(1-x^k)}.$$

Next, we will define the ECO method and explain how it is developed.

Enumeration of Combinatorial Objects (ECO) is a method for the enumeration of a class of combinatorial objects (Duchi, 2003; Barcucci et al., 2005).

Proposition 5.2.5. *Barcucci et al. (1999) Let ϑ be an operator of $\mathcal{O}$. If ϑ satisfies the following conditions:*

(i) *for each $O' \in \mathcal{O}_{n+1}$, there exists $O \in \mathcal{O}_n$ such that $O' \in \vartheta(O)$.*

(ii) *for each $O, O' \in O_n$ such that $O \neq O'$, then $\vartheta(O) \bigcap \vartheta(O') = \varnothing$.*

This method has been successfully applied to the enumeration of various classes of mathematical objects such as walks, permutations, and n-connected ominoes. Further details, theorems, proofs, and definitions can be found in (Pergola, 1999). The recursive construction determined by ϑ can be described through a generating tree, i.e. a rooted tree whose vertices are objects of O. The objects having the same value of the parameter, p, are at the same level, and the sons of an object are the objects produced through ϑ. A formal system for the description of the generating tree is the succession rule (Castiglione et al., 2005).

Next, we will define the class of nested chain abacus namely e-convex.

5.3 *e*-convex

In this section we develop *e*-convex class of nested chain abacus based on the connectedness in chapter two and combining the two notations of the *e*-core and convexity.

Definition 5.3.1. *A nested chain abacus is called column convex if each column has exactly one set-column sequence of connected beads.*

Definition 5.3.2. *e-convex class is a nested chain abacus with exactly one set-column sequence of connected bead positions* $S_j = \{a_1, a_2, a_3, \ldots, a_{C_j}\}$ *in column j, such that* $a_1 = j-1$ *and* $a_2 - a_1 = a_3 - a_2 = \ldots = a_{C_j} - a_{C_j - 1} = e$ *where* C_j *is the number of beads in column j for* $0 \leqslant j \leqslant e-1$.

In the next two examples, the case of the first example is the connected partition $\mu^{(4,5)}$ which is an *e*-convex while the case of the second example is the connected partition $\mu^{(4,5)}$ which is not *e*-convex.

Example 5.3.3. *Let* $\mu^{(4,5)} = (9,6,3,2,1^3,0^4)$ *be a connected partition of the nested chain abacus with* $e=4$ *and* $r=5$. *Based on Definition 2.4.8,* $SC_1 = \{0\}, SC_2 = \{1,5,9\}$, $SC_3 = \{2,6\}$ *and* $SC_4 = \{3,7,11,15,19\}$. *From Definition 5.3.2, Figure 5.1(a) is an e-convex nested chain abacus.*

Example 5.3.4. *Let* $\mu^{(4,5)} = (6^2,4,2^2,1^8,0)$ *be a connected partition of the nested chain abacus with* $e=4$ *and* $r=5$. *Since* $1 \notin SC_2$ *and based on Definition 5.3.2, Figure 5.1(b) is not e-convex nested chain abacus.*

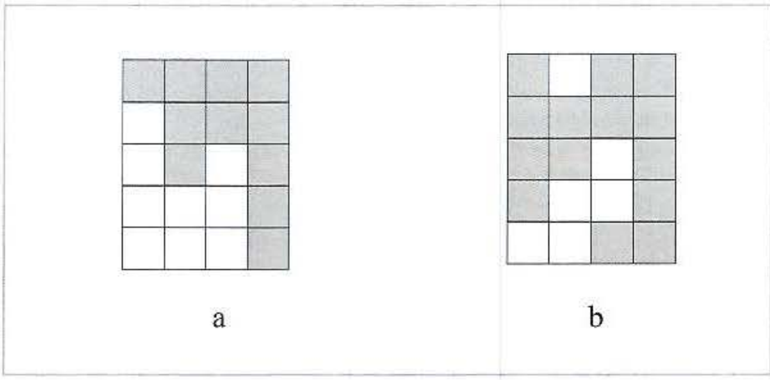

Figure 5.1. (a) Nested chain abacus of e-core connected partition and (b) Nested chain abacus of connected partition

For the rest of this section we enumerating the e-core class which consider one of the important properties for the discrete objects (objects that can be count and classified). We will find:

1. The number of e-convex class in closed form.
2. Generating function of e-convex using ECO construction.

Case one: The number of e-convex produce from a partition of n into e parts.

The next lemma provides the solution to the problem of enumerating the e-convex class, with f columns having the same number of beads and fixed $\{e, r\}$.

Lemma 5.3.5. *The number of e-convex with one partition of n into e columns and r rows is*

$$\frac{e!}{f!}$$

where f is the number of columns that have the same number of beads.

Proof. Suppose $\mathfrak{N}$ is a nested chain abacus with e columns and r rows, based on

Definition 5.3.2, the n beads will be distributed into e columns with the two following conditions:

(i) No column can be left empty.

(ii) f of e columns have the same number of beads and the remaining columns have a different number of beads, where $0 \leqslant f \leqslant e$.

Since the n beads is distributed into e columns, then there are $e!$ ways to order the e columns where each way will produce a nested chain abacus of e-convex. Thus, there is $e!$ nested chain abacus of e-convex. Since f of the e columns have the same number of beads, then, we have $f!$ same nested chain abacus. Thus, we have overcounted $f!$ nested chain abacus of $e!$. Thus, there are

$$\frac{e!}{f!}$$

different ways to distribute n beads into e columns. □

In the following lemma, we enumerated class e-convex as having er positions and fixed number of beads in g sets of columns such that each set has columns with the same number of beads.

Lemma 5.3.6. *Let nested chain abacus be e-convex with e columns and r rows, such that there are g sets of columns and that each set has columns with the same number of beads. Then, the number of e-convex with the same partition of n into e columns and r rows is*

$$\frac{e!}{\sum_{v=1}^{g} K_v!}$$

where

$$\sum_{v=1}^{g} K_v! \leqslant e.$$

Proof. Now, we induct on g. For $g = 1$, the n beads will be distributed into e with K_1 columns that have the same number of beads. Based on Lemma 5.3.5 the number of

e-convex with one partition of n into e columns have one set that have same beads is

$$\frac{e!}{K_1!}.$$

For the inductive step, consider adding K_{g+1} columns with the same number of beads to e-convex. Thus, we will overcount $K_{g+1}!$ from $e!$. Hence, there are

$$\frac{e!}{K_{g+1}\sum_{v=1}^{g} K_v!} = \frac{e!}{\sum_{v=1}^{g+1} K_v!}$$

of e-convex. □

In Theorem 5.3.7, we enumerate the class of e-convex as having fixed n bead positions while the number of beads in the rows has be extendable and has to be minimized.

Theorem 5.3.7. *For fixed e, let $\mathfrak{N}$ be e-convex with e columns and r rows such that there are g sets of columns with equal number of beads $\{K_1, K_2, ..., K_g\}$. Then, there exist*

$$\sum_{K_1=2}^{e} \sum_{K_2=2}^{e-K_1} \sum_{K_3=2}^{e-K_1-K_2} \cdots \sum_{K_g=2}^{e-K_1-K_2-\ldots-K_{g-1}} \frac{e!}{\sum_{v=1}^{g} K_v!}$$

of e-convex for $\sum_{g'=1}^{g} K_{g'} \leqslant e$ and $\sum_{g'=1}^{g} K_{g'} b_{g'} = n$ where $b_{g'}$ is the number of bead positions in $K_{g'}$ columns.

Proof. Based on Lemma 5.3.6, for fixed e and g sets of columns with same number of beads there are

$$\frac{e!}{\sum_{v=1}^{g} K_v!}$$

of e-convex. Since the number of rows can be changed such that each row has at least one bead, then, the number of beads in the columns can be extendable and has be to minimized. Thus, the number of sets with the same beads will be changed, subsequently changing the number of beads $b_{g'}$ in set g'. Hence, the number of e-convex is

$$\sum_{K_1=2}^{e}\sum_{K_2=2}^{e-K_1}\sum_{K_3=2}^{e-K_1-K_2}\cdots\sum_{K_g=2}^{e-K_1-K_2-\ldots-K_{g-1}}\frac{e!}{\sum_{v=1}^{g}K_v!}$$

where e is fixed number. □

Case two: The number of nested chain abacus of e-convex produced with different partitions of n.

The previous discussion focused on the enumeration of class e-convex within the same e columns. We can extend the enumeration problem in a case where the number of columns are expendable and has to be minimized.

Theorem 5.3.8. *For fixed n and e, the number of nested chain abacus of e-convex with different partitions of n is*

$$J\frac{e!}{\sum_{v=1}^{g}K_v!}$$

where J is the coefficient of x^n *in* $\dfrac{x^e}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^e)}$ *for* $n>1$.

Proof. Based on Theorem 5.2.4, the number of partitions of n beads into e columns is the coefficient of x^n in

$$\frac{x^e}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^e)}$$

such that each partition is associate with a e-convex. Based on Lemma 5.3.6, the number of e-convex produced from a partition of n beads into e columns is

$$\sum_{v=1}^{g} K_v! \leqslant e.$$

Hence, the number of nested chain abacus of e-convex is

$$J\frac{e!}{\sum_{v=1}^{g} K_v!}$$

where J is the coefficient of x^n in $\frac{x^e}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^e)}$ for $n > 1$. □

Theorem 5.3.9. *For fixed n, the number of nested chain abacus of e-convex with different partitions is*

$$F\frac{e!}{\sum_{v=1}^{g} K_v!}$$

where F is the coefficient of x^n in $\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^e)}$ for $n > 1$.

Proof. Based on Theorem 5.2.3, the number of partitions of n beads in different columns is the coefficient of x^n in

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^e)}$$

such that each partition is associated with a e-convex. Based on Lemma 5.3.6, the number of e-convex produced from a partition of n beads into e columns is $\sum_{v=1}^{g} K_v! \leqslant e$. Hence, the number of e-convex is

$$F\frac{e!}{\sum_{v=1}^{g} K_v!}$$

where F is the coefficient of x^n in $\frac{1}{(1-x)(1-x^2)(1-x^3)\dots(1-x^e)}$ for $n > 1$. $\square$

Theorem 5.3.10. *For any n, r and e there exist $\frac{(\alpha_1+e^{'})!}{\alpha_1!e^{'}!}-1$ of e-convex with $e^{'}$ columns and α_1 rows where $1 \leqslant \alpha_1 \leqslant r$, $1 \leqslant e^{'} \leqslant e$ and $1 \leqslant r^{'} \leqslant e$ such that $n^{'} = \alpha_1 + e^{'}$ for n, e and r positive integers.*

Proof. We will prove through induction that $\frac{(\alpha_1+e^{'})!}{\alpha_1!e^{'}!}$ is true for all $1 \leqslant e^{'} \leqslant e$.

Basic step: When $e^{'} = 1$

$$\frac{(\alpha_1+1)!}{\alpha_1!1!}-1 = \frac{\alpha_1 1(\alpha_1+1)}{\alpha_1!1!}-1 = \alpha_1,$$

so there are α_1 of e-convex with one column and α_1 rows.

Induction step: For given $e^{''}$, suppose that $\frac{(\alpha_1+e^{'})!}{\alpha_1!e^{'}!}-1$ is true for $e^{''} = e^{'}$. Then, from the induction hypothesis, we have

$$\frac{(\alpha_1+e^{''})!(\alpha_1+e^{''}+1)}{\alpha_1!e^{''}!(e^{''}+1)}-1.$$

Thus,

$$\frac{(\alpha_1+e^{''}+1)!}{\alpha_1!(e^{''}+1)!}-1.$$

From the principle of induction, $\frac{(\alpha_1+e^{'})!}{\alpha_1!e^{'}!}-1$ is true for all $1 \leqslant e^{''} \leqslant e$. $\square$

In previous work, we obtained the number of e-convex class of nested chain abacus with a fixed number of beads, n. Since the nested chain abacus of connected beads has application in physics, especially in the movement of fluids where such movement causes the increase the number of connected beads. Next, we examine the increment of the number of connected beads by adding columns. A new nested chain abacus that satisfies the definition of e-convex by adding a sequence of set-column (SC) of connected bead positions (adding column with connected bead positions) is obtained through Fayers's method.

Fayers (2007) suggested a simple way to insert one column or more in the abacus by putting the number of beads in consideration, such that the last bead position in this column does not exceed the position of the last bead, w_n, but with an empty bead position in between. Otherwise, if this bead position exceeds the last bead positions without having an empty bead position in between, this case will be enumerated by using Fayers' work.

Given a partition, λ, and a positive integer, k, we construct a new partition, λ^{+k}, as follows.

Step one: Take $b \geqslant$ the parts of λ, where b is the number of bead positions and construct the abacus display for λ.

Step two: Add a column to the abacus immediately to the left of column d where $b+k = ce+d$ with $0 \leqslant d \leqslant e-1$.

Step three: Place c beads on this new column at the top c positions where $k \geqslant 0$ that is the positions labelled $d, d+e+1, \ldots$ and $d+(c-1)(e+1)$.

Then, the new abacus represents partition λ^{+k}. Since the c beads in the new columns will be placed in the top c positions, then the nested chain abacus of λ^{+k} is a e-convex and e-convex which is a configuration of connected partition $\lambda^{+k(e',r')}$.

For $\lambda^{(2,4)} = (3,2,1,0^2)$, we can construct cases of $\lambda^{+0(3,4)}, \lambda^{+1(3,4)}, \ldots, \lambda^{+5(3,5)}$ for class e-convex as shown in the next figure.

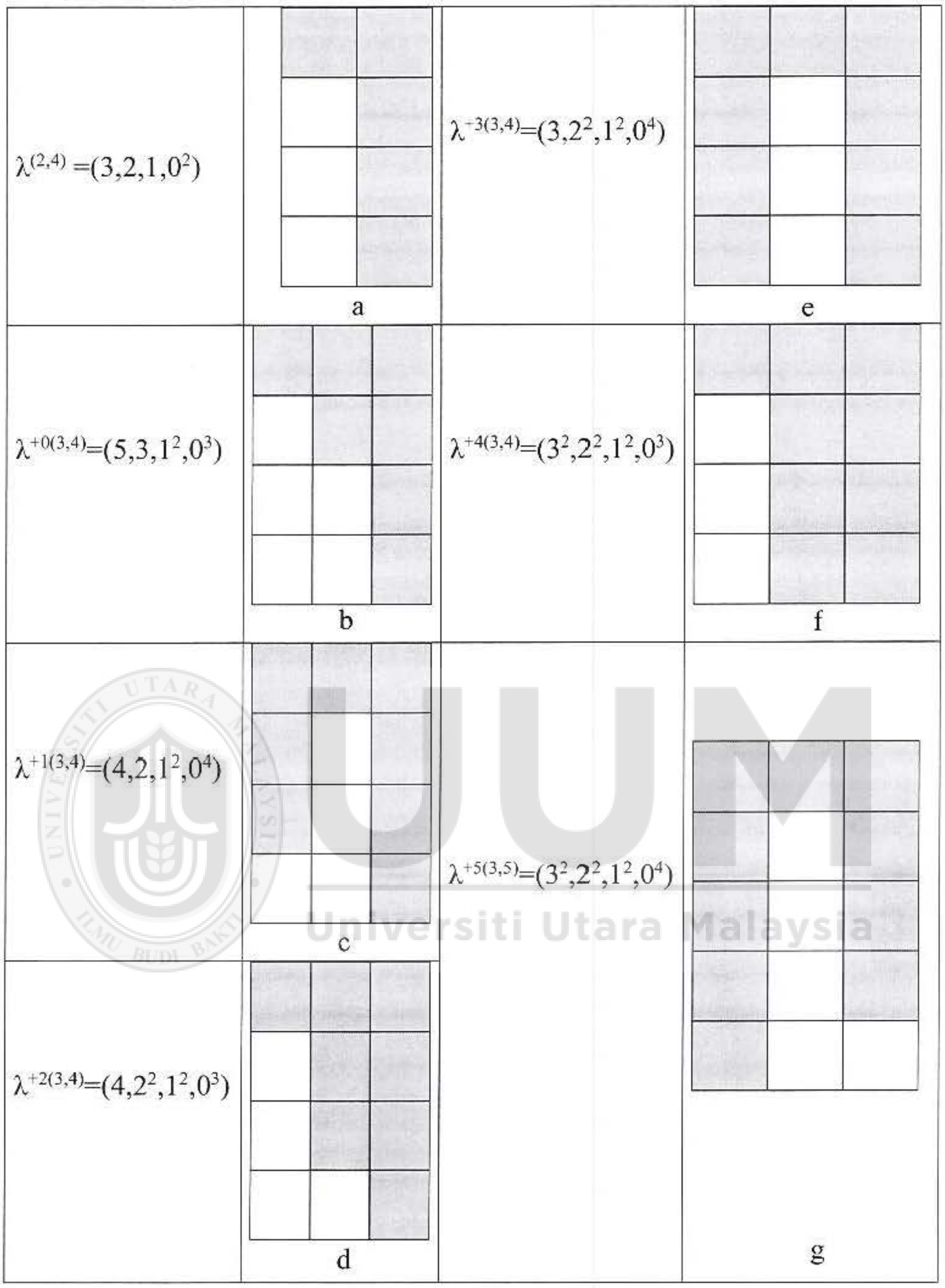

Figure 5.2. (a) A *e*-convex with 2 columns and 4 rows, (b) *e*-convex of $k = 0$, (c) *e*-convex of $k = 1$, (d) *e*-convex of $k = 2$, (e) *e*-convex of $k = 3$, (f) *e*-convex of $k = 4$ and (g) *e*-convex of $k = 5$

Theorem 5.3.11. *Let $\mathfrak{N}$ be e-convex and let $\mathfrak{N}^{+k}$ as a nested chain abacus for $(n+c)$-connected beads with e columns be e-convex such that $n+k = ce+d$ where $k \geqslant 0$ and $1 \leqslant d \leqslant e-1$. Then, there exist*

$$\begin{cases} e-d^{'}+e\rho+1-\sum_{v=1}^{r-c_1+1} r_v & if \quad C_e = r, C_e \neq c_v, \\ e-d^{'}+e\rho+2-\sum_{v=1}^{r-c_1+1} r_v & if \quad C_e = r, C_e = c_v, \\ 2e-d^{'}+e\rho-j+1-\sum_{k=1}^{e\rho+e-d^{'}} r_k & if \quad C_j = r, C_e \neq c_v, \\ 2e-d^{'}+e\rho-j-\sum_{k=1}^{e\rho+e-d^{'}} r_k & if \quad C_j = r, C_e = c_v, \end{cases}$$

of e-convex where $c_1 = \dfrac{n-d^{'}}{e}$, C_e is the number of bead positions in column e, $\rho = r - c_1$ and r_v is the number of columns with $c_v = \left(\dfrac{n+k-d}{e}\right)$ beads for $1 \leqslant j \leqslant e-1$ and $v > 1$.

Proof. Let $\mu^{(e,r)}$ be a connected partition of a e-convex with r rows and e columns. Then $\mu^{(e,r)(+k)}$ is a connected partition of e-convex after adding one column with c_v beads immediately to the left of d, where $n+k = ce+d$ and $0 \leqslant c \leqslant r+1$. Firstly, we add c_1 in column $d^{'}$ such that $c_1 = \dfrac{b-d}{e}$, and since $0 \leqslant d \leqslant e-1$ we can add $e-d^{'}$ columns with c_1 beads. Secondly, we can add e columns with c_1+1 beads. The latter is repeated $c = r$ resulting in $e-d^{'}+e\rho$ of e-convex. Then, we observe the following cases.

Case one: $C_e = r$, $C_e \neq c_v$.

The number of beads in consideration is such that the last bead position in these columns do not exceed the position of w_n but with an empty bead position in between them. Otherwise, if this bead position exceeds w_n without having an empty bead positions in between them, then there is a e-convex with $e+1$ and $C_1 = r+1$.

Since there are r_v columns of the nested chain with $\frac{n+k-d}{e}$ beads and there is a column, d, which also has $\frac{n+k-d}{e}$ beads then there are r_v+1 of the same forms or shapes of nested chain abacus. This means that we overcounted

$$\sum_{v=1}^{r-c_1+1} r_v - 1$$

of e-convex. Hence, there are

$$e-d^{'}+e\rho+1-\sum_{v=1}^{r-c_1+1} r_v$$

of e-convex.

Case two: $C_e = r, C_e = c_v$.

The number of beads under consideration is such that the last bead position in these columns do not exceed the position of w_n but with an empty bead position in between them. Otherwise, if this bead position exceeds w_n without having an empty bead positions in between them, then there is a e-convex with $e+1$ and $C_1 = r+1$. Since there are r_v columns of the nested chain with $\frac{n+k-d}{e}$ beads and there is a column d which also has $\frac{n+k-d}{e}$ beads then there are r_v+1 of the same forms or shapes of nested chain abacus and $C_e = C_v$ for $1 \leqslant d \leqslant e-1$. Thus, we have overcounted

$$\left(\sum_{v=1}^{r-c_1+1} r_v\right) - 2$$

of e-convex. Hence, there are

$$e-d^{'}+e\rho+2-\sum_{v=1}^{r-c_1+1} r_v$$

of e-convex.

Case three: $C_j = r, C_e \neq c_v$.

The number of beads under consideration is such that the last bead position in these columns do not exceed the position of w_n but with an empty bead position in between them. Otherwise, if this bead position exceeds w_n without having an empty bead positions in between them, then we have overcounted

$$\left(\sum_{k=1}^{ep+e-d'} r_k\right) + j + 1 - e$$

of e-convex where w_n is located in column j. Thus, there are

$$2e - d' + ep - j - 1 - \sum_{k=1}^{ep+e-d'} r_k$$

of e-convex.

Case four: $C_j = r, C_e = c_v$.

Since $C_e \neq r$ and the number of beads under consideration is such that the last bead position in these columns do not exceed the position of w_n but with an empty bead position in between them. Otherwise, if this bead position exceeds w_n without having an empty bead positions in between them, then we have overcounted

$$\left(\sum_{k=1}^{ep+e-d'} r_k\right) + j + 1 - e$$

of e-convex where w_n is located in column j. Thus, there are

$$2e - d' + ep - j - 1 - \sum_{k=1}^{ep+e-d'} r_k$$

of e-convex. □

5.4 ECO Method for the *e*-Convex Class

In this section we employ the ECO method to enumerate the *e*-convex class of nested chain abacus.

In this section, we define an ECO operation for the recursive construction of the set of the generation of *e*-convex to perform local expansions. Based on the five design structures of nested chain abacus, we partition the set of *e*-convex into five subsets denoted by e^V-convex, e^H-convex, e^{PV}-convex and e^{PH}-convex:

- e^V-convex subclass is a vertical rectangular nested chain abacus (see Figure 5.3(a)).
- e^{PV}-convex subclass is a vertical rectangle-path nested chain abacus (see Figure 5.3(b)).
- e^H-convex subclass is a horizontal rectangular nested chain abacus (see Figure 5.3(d)).
- e^{PH}-convex subclass is a horizontal rectangle-path nested chain abacus (see Figure 5.3(c)).
- e^S-convex subclass is a singleton nested chain abacus (see Figure 5.3(f)).

Figure 5.3 illustrates the 5 disjointed subsets of *e*-convex.

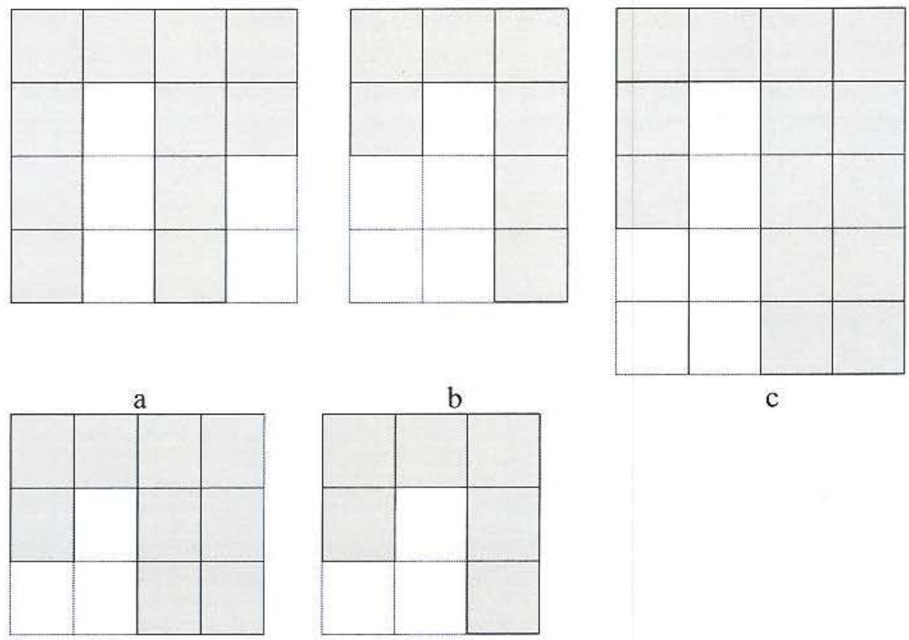

Figure 5.3. (a) A e^V-convex, (b) e^{PV}-convex, (c) e^{PH}-convex, (d) e^H-convex and (f) e^S-convex

We now define an ECO operator called ϑ which forms the following local expansions:

- For any $L = 1, 2, ..., r$ the operator ϑ glues a column with L beads to the rightmost and leftmost column of nested chain abacus; this can be done in $2r$ ways. Therefore ϑ produces r of e-convex class.
- One e-convex by adding an entire row.

Thus, the operator ϑ produced $2r+1$ of e-convex which have $e+r+1$ semi-perimeter. Moreover, the operator performs some other transformations on e-convex of classes e^V-convex, e^{PV}-convex, e^{PH}-convex, e^H-convex and e^S-convex, according to the respective classes:

1. If the nested chain abacus belongs to the class e^S-convex, then the operator ϑ produces

- $2r$ of e^{PH}-convex with $e+1$ columns and r rows by adding h cells in the rightmost and leftmost of e^{S}-convex where $1 \leqslant h \leqslant r$.
- One of e^{PV}-convex with e columns and $r+1$ rows by adding one row in the topmost of e^{S}-convex with e columns and r rows.

2. If $\mathfrak{N}$ belongs to the class e^{PH}-convex, then the operator ϑ produces
   - $2r$ of e^{PH} by adding h cells in the rightmost and leftmost of e^{PH}-convex where $1 \leqslant h \leqslant r$.
   - One of
     - e^{v}-convex by adding one row in the topmost of e^{PH}-convex if $e = r+1$, or
     - e^{H}-convex by adding one row in the topmost of e^{PH}-convex if $e > r+1$.
3. If $\mathfrak{N}$ belongs to the class e^{PV}-convex, then the operator ϑ produces
   - $2r$ of e^{V}-convex by adding h cells in the rightmost and leftmost of e^{PV}-convex where $1 \leqslant h \leqslant r$.
   - One of e^{PV}-convex by adding one row in the topmost of e^{PH}-convex.
4. If $\mathfrak{N}$ belongs to the class e^{H}-convex, then the operator ϑ produces
   - $2r$ of e^{H}-convex by adding h cells in the rightmost and leftmost of e^{H}-convex where $1 \leqslant h \leqslant r$.
   - One of
     - e^{s}-convex by adding one row in the topmost of e^{H}-convex if $r = e-1$ and e is odd, or
     - e^{PH}-convex by adding one row in the topmost of e^{H}-convex if $e > r-1$.
5. If $\mathfrak{N}$ belong to the class e^{V}-convex, then the operator ϑ produces
   - $2r$ of
     - e^{S}-convex by adding column in the rightmost and leftmost of e^{V}-convex with h beads if $e = r-1$.

- e^{PV}-convex by adding column in the rightmost and leftmost of e^{V}-convex with h beads if $e > r-1$ where $1 \leqslant h \leqslant r$.

- e^{V}-convex by adding one row with e beads in the topmost of e^{v}-convex.

Since

1. for any $\mathfrak{N} \in e$-convex, all the nested chain in $\vartheta(e$-convex) also e-convex.
2. all the produced e-convex have semi-perimeter $n+1$.

The operator, ϑ, satisfies the conditions 1 and 2 of the ECO method as show in Figure 5.4 and Figure 5.5.

Figure 5.4 illustrates ECO method applied in classes e^{S}-convex and e^{PH}-convex.

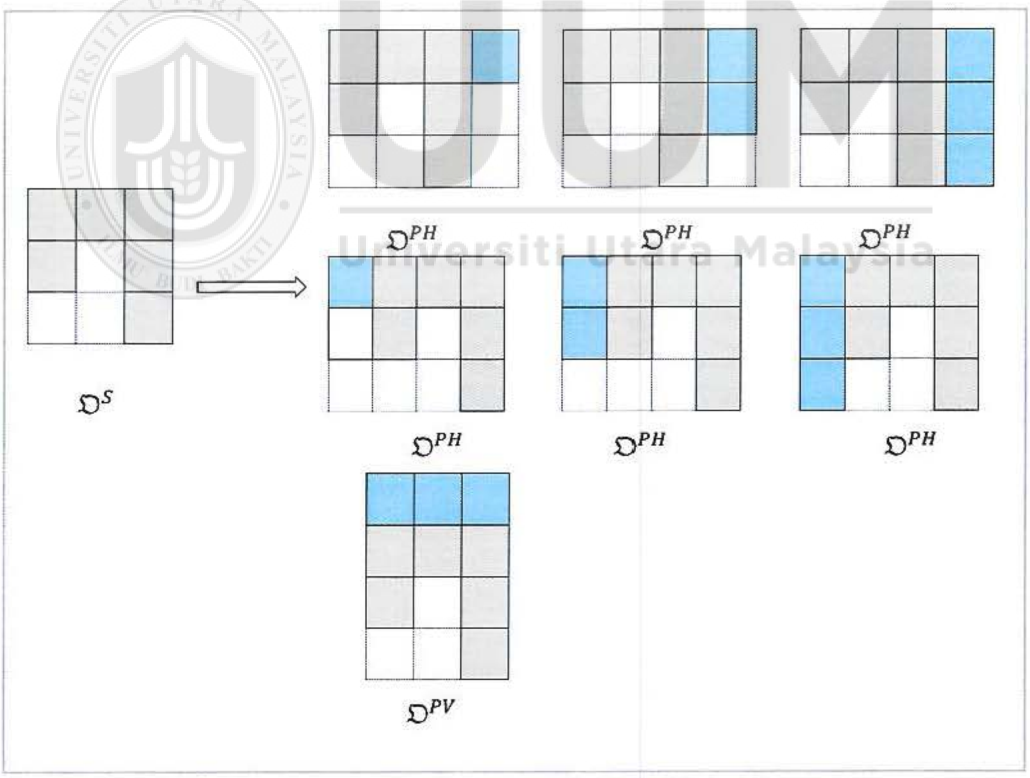

Figure 5.4. The ϑ operator applied to $\mathfrak{N}$ nested chain abacus in class e^{S}-convex

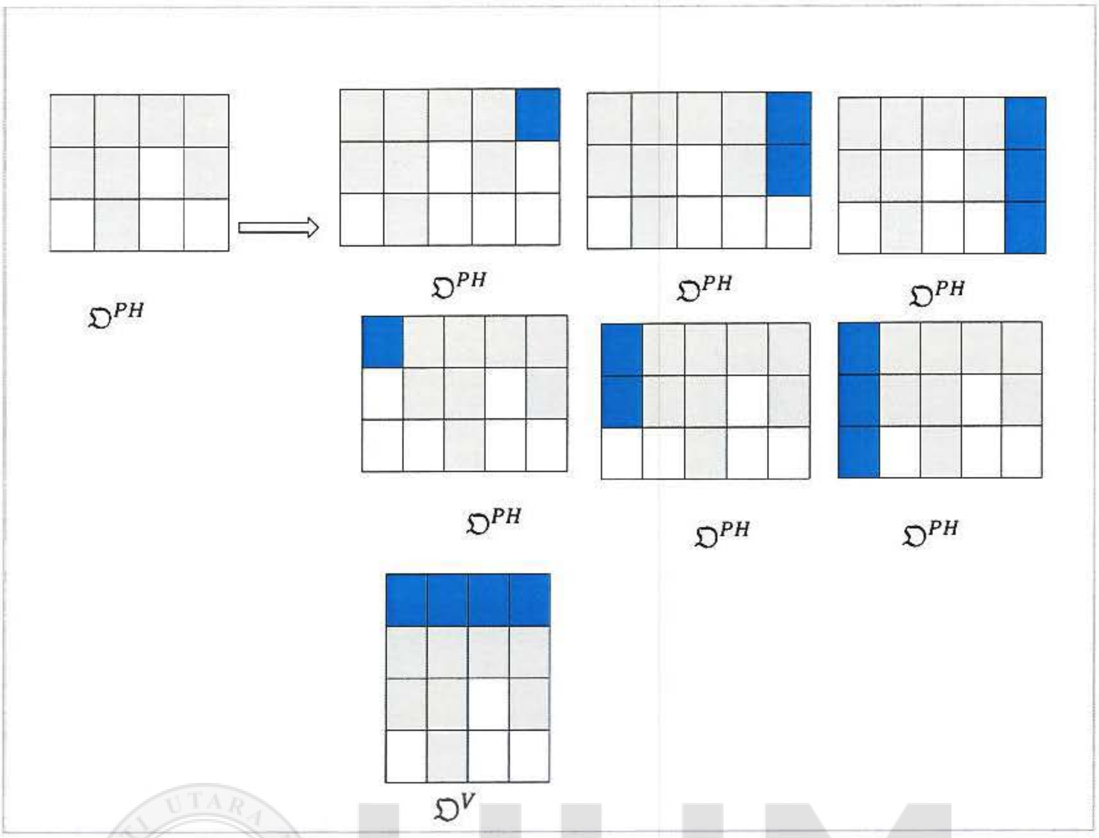

Figure 5.5. ϑ operator applied to e^{PH}-convex nested chain abacus

The next step consists of translating the previous construction into a set of equations whose solution is the generating function for e-convex. To achieve this purpose, we must introduce the concept of succession rule.

5.4.1 The Succession Rule Associated with ϑ

Translating the construction of the operator ϑ onto the framework of succession rule means to label with k, $k \in N^{+}$ for each nested chain abacus that produces exactly k beads, and then represent the performance of the operator with a set of productions. Actually, it is easy to recognize that each nested chain abacus of class e^{V}-convex, e^{PV}-convex, e^{PH}-convex, e^{H}-convex and e^{S}-convex has label $2r+1$ where r is the number of rows. Previously, we had sketched the performance of the ECO operator on a generic nested chain abacus. Let us take as an example the nested chain abacus in Figure 5.4 producing seven nested chain abacus and each of the new nested chain abacus will preduce seven nested chain abacus except for the last which preduce (9) nested chain abacus. The root of the tree is (3), which is the label of the nested chain abacus with one bead position.

Generically, the succession rule with the ECO method can be presented as

$$\Omega : \begin{cases} 3 \\ 2r+1 \quad (2r+1)^{2r}(2(r+1)+1),\ r>0 \end{cases}$$

where the power notation is used to express repetitions, that is $(2r+1)^{2r}$, which stands for $\underbrace{(2r+1),\cdots,(2r+1)}_{2r\,times}$.

Figure 5.6 illustrates the first three levels of the generating tree of Ω.

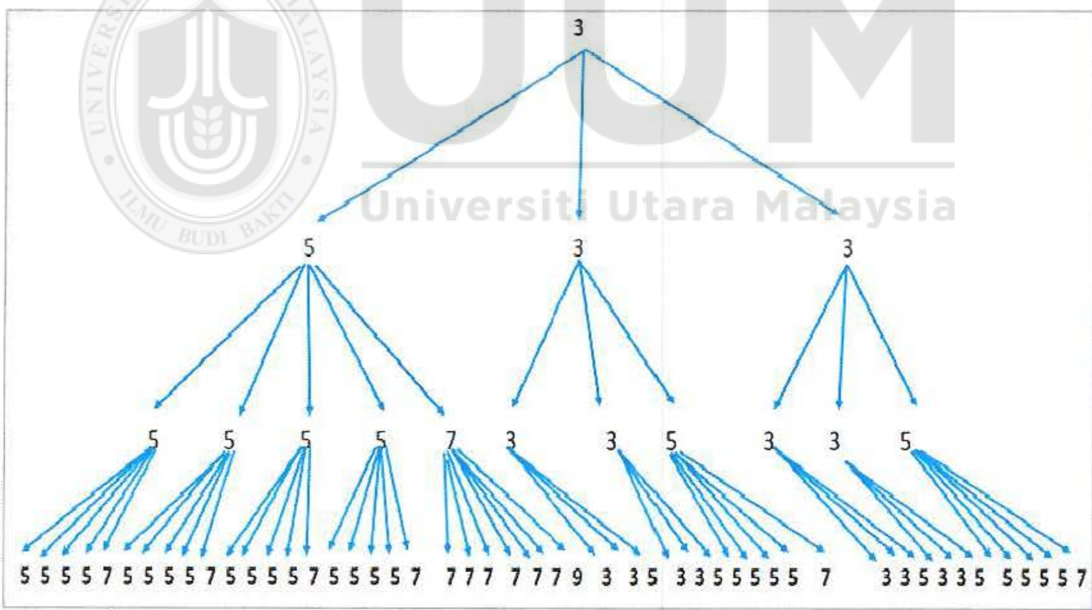

Figure 5.6. Generating tree of Ω

5.4.2 The Generating Function in Level N

Based on the result of Ω, we obtain the local expansions of the class e-convex with e columns and r rows according to the growth of the number of bead positions by adding h bead positions on the leftmost, and rightmost and e bead positions on the up-most where $1 \leqslant h \leqslant r$. Based on the operator, ϑ, each e-convex will produce $2r+1$ then,

Level 0: The number of e-convex with r rows is $2r+1$ such that $2r$ of them with r rows and 1 with $r+1$ rows

Level 1: The number of e-convex with r rows is

$$(2(r+1)+1)+2r(2r+1).$$

Level 2: The number of e-convex with r rows is

$$2(r+1)(2(r+1)+1)+(2(r+2)+1)+2\left[2r(2r+1)+(2(r+1)+1)\right].$$

Level 3: The number of e-convex with r rows is

$(2(r+1))^2(2(r+1)+1)+2(r+1)(2(r+2)+1)+2(r+2)(2(r+2)+1)$

$+(2(r+3)+1)+2[(2r)^2(2r+1)+2r(2(r+1)+1)+(2r)^2(2(r+1)+1)]$

$+(2(r+2)+1)]$

$=(2(r+1))^2(2(r+1)+1)+(2(r+1)+2(r+2))(2(r+2)+1)+(2(r+3)+1)$

$+2\left[(2r)^2(2r+1)+(2r+(2r)^2)(2(r+1)+1)+(2(r+2)+1)\right].$

Since we begin with $r=1$, then, the number of class e-convex computing is as follows:

Level 0: The number of e-convex is 3.

Level 1: The number of e-convex is

$$1(5)+2(3).$$

Level 2: The number of e-convex is

$$4(5)+1(7)+2[2(3)+1(5)].$$

Level 3: The number of e-convex is

$$4^2(5)+(4+6)7+1(9)+2[2^2(3)+(2+4)(5)+7].$$

Level 4: The number of e-convex is

$4^3(5)+(4^2+4\times6+6^2)7+(4+6+8)9+11+$

$2\left[2^2(3)+(2^2+4\times2+4^2)(5)+(2+4+6)7+1\times9\right].$

Level 5: The number of e-convex is

$4^4(5)+\left[4^3+4^2\times6+4\times6^2+6^3\right]7+$

$\left[4^2+4^1\times8^1+4^2\times6+6\times8+8^2+6^29+4+6+8+10\right]11+13.$

We have the formula for $L=0,\,1,\,2,\,3$ and we want to generalize it for level n ($L=n$). We divide the general formula of the number of class of e-convex into two parts and generalize them by analysing each part within it. Finally combine the generalizations for both parts to arrive at the generalized formula for $L=n$.

Part 1:

The first term contains one term and the second term contains two terms, and thus, the k^{th} term contains k terms. The first term had 2, then, second term had 2 and 4, then the third term had 2, 4 and 6 so the k^{th} term contains 2, 4, 6,... till ($2k$). The constant term multiplied with each term starting from 3 and increases by 2 for the next term.

So, the k^{th} term of the formula for the second part is;

K^{th} term 2^{nd} part equal to

$$2\left(\sum_{i_j=0}^{n-k}2^{i_1}\times4^{i_2}\times6^{i_3}\times\ldots\times(2k)^{i_k}\right)(3+2(k-1)),$$

such that

$$\sum_{j=1}^{k}i_j=nk$$

and for $n\geqslant3$.

Part 2:

The first term contains one term and the second term contains two terms, and thus, the kth term contains k terms. The first term had 4, then the second term had 4 and 6, then the third term had 4, 6 and 8 so the k^{th} term contains 4, 6, 8,... till $(2k+2)$. The constant term multiplied with each term starts from 5 and increases by 2 for the next term.

So the k^{th} term of the formula for the first part is

K^{th} term 1^{st} part is

$$\sum_{i_j=0}^{n-k}\left(4^{i_1}\times 6^{i_2}\times 8^{i_3}\times\ldots\times(2k+2)^{i_k}\right)\right)\left(5+2(k-1)\right),$$

such that

$$\sum_{j=1}^{k} i_j = n-k$$

and for $n \geqslant 3$.

So, we can take the k^{th} term of the whole formula as

Kth term is

$$\sum_{i_j=0}^{n-k}\left(4^{i_1}\times 6^{i_2}\times 8^{i_3}\times\ldots\times(2k+2)^{i_k}\right)\left(5+2(k-1)\right)+$$

$$2\left(\sum_{i_j=0}^{n-k}\left(2^{i_1}\times 4^{i_2}\times 6^{i_3}\times\ldots\times(2k)^{i_k}\right)\left(3+2(k-1)\right)\right),$$

such that

$$\sum_{j=1}^{k} i_j = nk$$

and for $n \geqslant 3$.

So, the formula for $L=N$ in general is

$$\sum_{k=1}^{n}\sum_{i_j=0}^{n-k}\left(4^{i_1}\times 6^{i_2}\times 8^{i_3}\times\ldots\times(2k+2)^{i_k}\right)\left(5+2(k-1)\right)$$

$$+2\sum_{k=1}^{n}\sum_{i_j=0}^{n-k} 2^{i_1}\times 4^{i_2}\times 6^{i_3}\times\ldots\times(2k)^{i_k})(3+2(k-1)),$$

such that

$$\sum_{j=1}^{k} i_j = nk$$

and for $n \geqslant 3$.

5.4.3 The Generating Function of a Succession Rule

In order to find generating function. First, we have found the function $f_\Omega(x,y)$ and substituted $y=1$ to find the generating function $f_\Omega(x)$. The given succession rule is:

$$\Omega:\begin{cases} 3 \\ 2r+1 \quad (2r+1)^{2r}(2(r+1)+1),\ r>0 \end{cases}$$

where power notation denotes the repeat number of level. Then

$$\begin{aligned} f_\Omega(x) &= \sum_{n\geqslant 0} f_n x^n, \\ f_\Omega(x,y) &= \sum_{n\geqslant 0, k\geqslant 1} f_{n,k}\, x^n y^k \\ f_\Omega(x,y) &= \sum_{n\geqslant 0, k\geqslant 1} f_{n,2k+1}\, x^n y^{2k+1}. \end{aligned}$$

From the succession rule Ω, since $f_{n,2k}=0$, $f_{n,2k}=0$, where $k=0,1,\ldots,$ we deduce that

$$\begin{aligned} f_\Omega(x,y) &= \sum_{n\geqslant 0, k\geqslant 1} f_{n,2k+1}\, x^n y^{2k+1} \\ &= x^0 y^3 + \sum_{n\geqslant 1, k\geqslant 1} f_{n,2k+1}\, x^n y^{2k+1} \\ &= y^3 + x\sum_{n\geqslant 0, k\geqslant 1} f_{n,2k+1}\, x^n (\overbrace{y^{2k+1}+y^{2k+1}+\cdots+y^{2k+1}}^{2k} + y^{2(k+1)+1}) \\ &= y^3 + x\sum_{n\geqslant 0, k\geqslant 1} f_{n,2k+1}\, x^n (2k.y^{2k+1}+y^{2k+3}). \end{aligned}$$

Since the starting number of the tree is 3, so the first term is x^0y^3.

$$
\begin{aligned}
f_\Omega(x,y) &= \sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, x^n y^{2k+1} \\
&= y^3 + x \sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, x^n (2k.y^{2k+1} + y^{2k+3}) \\
&= y^3 + x \sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, x^n y^{2k+3} + x \sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, x^n 2k.y^{2k+1} \\
&= y^3 + xy^2 \sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, x^n y^{2k+1} + x \sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, x^n 2k.y^{2k+1} \\
&= y^3 + xy^2 f_\Omega(x,y) + x \sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, 2k.x^n y^{2k+1}.
\end{aligned}
$$

We can transfer the third term as follows:

Form differentiability per each term of power series,

$$
\begin{aligned}
& x \sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, 2kx^n y^{2k+1} \\
= & x \sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, (2k+1-1).x^n y^{2k+1} \\
= & yx \sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, (2k+1)x^n y^{2k} - x \sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, x^n y^{2k+1} \\
= & xy \frac{\partial}{\partial y} \Big(\sum_{n\geqslant 0,k\geqslant 1} f_{n,2k+1}\, x^n y^{2k+1} \Big) - x f_\Omega(x,y) \\
= & xy \frac{\partial f_\Omega(x,y)}{\partial y} - x f_\Omega(x,y).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f_\Omega(x,y) &= y^3 + xy^2 f_\Omega(x,y) + xy \frac{\partial f_\Omega(x,y)}{\partial y} - x f_\Omega(x,y) \\
f_\Omega(x,y)(1 + x - xy^2) &= y^3 + xy \frac{\partial f_\Omega(x,y)}{\partial y} \\
\frac{\partial f_\Omega(x,y)}{\partial y} - f_\Omega(x,y) \Big(\frac{1 + x - xy^2}{xy} \Big) &= \frac{y^2}{x}.
\end{aligned}
$$

The solution of $\dfrac{dy}{dx} + p(x)y = Q(x)$ is $y = e^{-\int p(x)dx} \left(c + \int Q(x) e^{\int p(x)dx} dx \right)$

where $p(y) = -\dfrac{1 + x - xy^2}{xy}$, $Q(y) = \dfrac{y^2}{x}$, where we can see that x is constant.

Thus,

$$
\begin{aligned}
f_{\Omega}(x,y) &= e^{\int \frac{1+x-xy^2}{xy} dy} \left(c + \int \frac{y^2}{x} e^{\int \frac{1+x-xy^2}{xy} dy} dy \right) \\
f_{\Omega}(x,y) &= y^{\frac{1+x}{x}} e^{-\frac{y^2}{2}} \frac{1}{x} \int y^{1-\frac{1}{x}} e^{-\frac{y^2}{2}} dy.
\end{aligned}
$$

By employing programming code in Appendix F, the generating function is

$$f(x) = 1 + 3x + 11x^2 + 49x^3 + O(x^3).$$

The final equation is the expansion of the generating function which is

$$f_1 = 1, f_2 = 3, f_3 = 11, f_4 = 49, \ldots$$

where f_n denotes the number of nodes at level n of the generating tree and from the definition of succession rule, f_{n+1} denotes the sum of the numbers at level n.

Next, we construct a class of nested chain abacus with respect to columns.

5.5 Spinal Design Approach

Based on chapter Two, Section 2.4.2 we construct a class of nested chain abacus with two columns knows as $\mathfrak{N}^2$. In the rest of this section we discuss the construct of class in Definition 5.5.1 and then one of the important properties for the discrete objects (objects can be count and classified).

Definition 5.5.1. *A nested chain abacus is knows as $\mathfrak{N}^2$ if*

1. *consists of tow columns*
2. *column 1 (respectively, column 2) with at most two SC sequences of adjacent beads.*
3. *column 2 (respectively, column 1) with at least one SC sequence of adjacent beads.*

4. *all SC sequences are connected.*

Figure 5.7 illustrate the construction the the nested chain abacus of $\mathfrak{N}^2$ for 4-connected beads. 5.7.

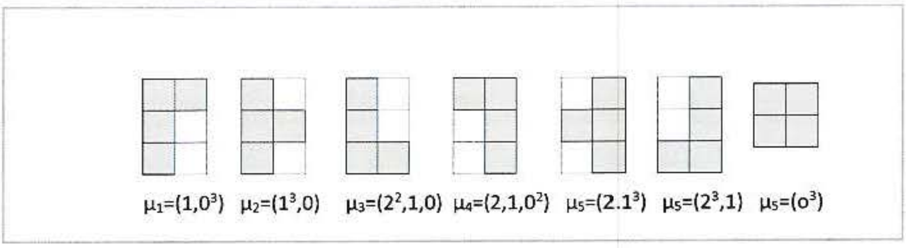

Figure 5.7. The 7 distinct forms of $\mathfrak{N}^2$ for 4-connected squares

Consider Figure 5.7 there are 7 distinct forms of $\mathfrak{N}^2$, the number of nested chain abacus of $\mathfrak{N}^2$ with one *SC* in Theorem 5.5.2 where $\binom{a}{b} = 0$ if $b > a$

Theorem 5.5.2. *Let $\mathfrak{N}^2$ be nested chain abacus with n beads and two columns such that column 1 (respectively, column 2) with one SC sequence of adjacent beads and r beads. Then, the number of $\mathfrak{N}^2$ is*

$$2\sum_{k=1}^{\delta}\binom{r}{k} + 4\left[1 + \sum_{s=1}^{k-1}\sum_{k=1}^{\delta}\binom{r-1}{s}\right] + 2\sum_{k=4}^{\delta} k - 4 + 1$$

where k is the number of beads in column 2 (respectively, column 1) and $k \leqslant r$.

Proof. Since $k \leqslant r$ then

$$\delta = \begin{cases} \dfrac{n}{2} & \text{if} \quad n \text{ is even number,} \\ \dfrac{n-1}{2} & \text{if} \quad n \text{ is odd number.} \end{cases}$$

Case 1 to prove $2\sum_{s=1}^{k-1}\binom{r}{k}$.

In this case we have bijection from column 1 (respectively, column 2) with r beads to column 2 (respectively, column 1) with k beads. Since $k \leqslant r$ then, k beads will connected with r beads in $\binom{r}{k}$ different ways. Since $1 \leqslant k \leqslant \delta$ then, there is $\sum_{k=1}^{\delta}\binom{r}{k}$

different ways to connected column 1 with column 2. Based on Definition 5.5.1 the number of $\mathfrak{N}^2$ is $\sum_{k=1}^{\delta}\binom{r}{k}$. Now, if column 2 with r and based on previous result the number of $\mathfrak{N}^2$ is $2\sum_{k=1}^{\delta}\binom{r}{k}$.

Case 2 to prove $4\left[1+\sum_{s=1}^{k-1}\sum_{k=1}^{\delta}\binom{r-1}{s}\right]$.

In this case the r beads in column 1 will connected with S beads in column 2 and $k-S$ beads connected with columns 1 by bead $q+1$ or $r+1$ as shown in next figure,

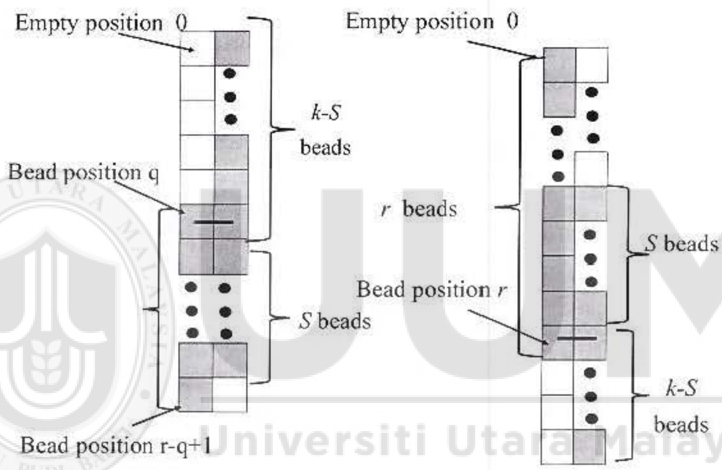

Figure 5.8

where $1 \leqslant S \leqslant k-1$. First, if $\{q+1, q-1, q-3, ...\}$ are beads positions as shown in above figure,

- $S=0$ then, there is one way to connected column 1 with column 2. Thus there is a nested chain abacus.
- $S=1$, Since $k-s$ beads connected with column 1 by a bead then, there is $\binom{r-1}{1}$ way to connected column 1 with column 2.
- $S=2$, Since $k-s$ beads connected with column 1 by a bead then, there is $\binom{r-1}{2}$ way to connected column 1 with column 2.

.

.

Thus there is $\binom{r-1}{s}$ way to connected column 1 with column 2. Since $1 \leqslant S \leqslant k-1$ and $1 \leqslant k \leqslant \delta$ there is

$$1+\sum_{s=1}^{k-1}\sum_{k=1}^{\delta}\binom{r-1}{s}$$

ways to connected column 1 with column 2. Based on Definition 5.5.3 the is

$$1+\sum_{s=1}^{k-1}\sum_{k=1}^{\delta}\binom{r-1}{s}$$

of nested chain abacus of $\mathfrak{N}^2$. Second, if $\{r+1, r+3, r+5, \ldots\}$ are beads positions as shown in above figure, then there is

$$1+\sum_{s=1}^{k-1}\sum_{k=1}^{\delta}\binom{r-1}{s}$$

ways to connected column 1 with column 2. Similarity , if the r beads in column 2. Thus the number of nested chain abacus of $\mathfrak{N}^2$ is

$$4\left[1+\sum_{s=1}^{k-1}\sum_{k=1}^{\delta}\binom{r-1}{s}\right].$$

Case 3 to prove $2\sum_{k=4}^{\delta} k-4+1$.

In this case column 2 with two set-sequences $\{SC_1, SC_2\}$ satisfies the following conditions:

(i) SC_1 and SC_2 have at least two beads.

(ii) SC_1 connected only with the first beads in column 1 and SC_2 connected only with the last beads in column 1.

Such that,

1. If $k=4$, based on the previous two condition there is only one $\mathfrak{N}^2$ in this case.
2. If $k=5$, based on previous two condition
   - SC_1 has two beads and SC_2 has three beads.
   - SC_1 has three beads and SC_2 has two beads.

   .

   .

   .
3. If column 2 with k beads, based on previous two condition
   - SC_1 has 2 beads and SC_2 has $k-2$ beads.
   - SC_1 has 3 beads and SC_2 has $k-3$ beads.

   .

   .

   .
   - SC_{k-2} has 2 beads and SC_2 has 2 beads.

Thus there is $k-3$ nested chain abacus, based on Definition 5.5.3 there is $k-3$ of $\mathfrak{N}^2$. Since $1 \leqslant k \leqslant \delta$ then, the number of $\mathfrak{N}^2$ is

$$\sum_{k=4}^{\delta} k-3.$$

Based on Cases 1,2,3 the number of $\mathfrak{N}^2$ is

$$2\sum_{k=1}^{\delta}\binom{r}{k}+4\left[1+\sum_{s=1}^{k-1}\sum_{k=1}^{\delta}\binom{r-1}{s}\right]+2\sum_{k=4}^{\delta}k-4+1.$$

□

Theorem 5.5.3. *Let $\mathfrak{N}^2$ be nested chain abacus with n beads and two columns such that column 1 (respectively, column 2) with two SC sequence of adjacent beads and r*

beads. Then, the number of $\mathfrak{N}^2$ *is*

$$1 + 2\sum_{M=0}^{a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=1}^{2}\sum_{k=3}^{\delta}\binom{r-h-2}{M} + 2\sum_{M=0}^{a}\sum_{N=0}^{k-g-a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=3}^{r-1}\sum_{k=3}^{\delta}\binom{h-2}{N}\binom{r-h-2}{M}.$$

Proof. We will prove the above theorem by induction.

Basic step:

When $r = 1$, since $k \leqslant r$, then, $k = 1$. Thus, the number of $\mathfrak{N}^2$ is 1. Induction step: We suppose is true for $r = r'$. Then,

$$1 + 2\sum_{M=0}^{a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=1}^{2}\sum_{k=3}^{\delta}\binom{r'-h-2}{M} + 2\sum_{M=0}^{a}\sum_{a=0}^{\delta+1-g}\sum_{g=3}^{k}\sum_{h=1}^{2}\binom{r'-h-1}{M}$$
$$+ 2\sum_{M=0}^{a}\sum_{N=0}^{k-g-a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=3}^{r'-1}\sum_{k=3}^{\delta}\binom{h-2}{N}\binom{r'-h-2}{M}$$
$$+ 2\sum_{M=0}^{a}\sum_{N=0}^{\delta+1-g-a}\sum_{a=0}^{\delta+1-g}\sum_{g=3}^{\delta+1}\binom{h-2}{N}\binom{r'-h-1}{M}.$$

Then,

$$1 + 2\sum_{M=0}^{a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=1}^{2}\sum_{k=3}^{\delta+1}\binom{r'+1-h-2}{M}$$
$$+ 2\sum_{M=0}^{a}\sum_{N=0}^{k-g-a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=3}^{r'-1}\sum_{k=3}^{\delta}\binom{h-2}{N}\binom{r'-h-1}{M}.$$

Then,

$$1 + 2\sum_{M=0}^{a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=1}^{2}\sum_{k=3}^{\delta+1}\binom{r'-h-1}{M}$$
$$+ 2\sum_{M=0}^{a}\sum_{N=0}^{k-g-a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=3}^{r'}\sum_{k=3}^{\delta}\binom{h-2}{N}\binom{r'-h-1}{M}.$$

□

5.6 Conclusion

This chapter is devoted to generate classes of nested chain abacus by using using two methods, namely e-core and spinal design. Further, we determine generating function of the e-convex class with respect to the semi-perimeter, as application of the ECO method. In addition, property of the classes are proposed.

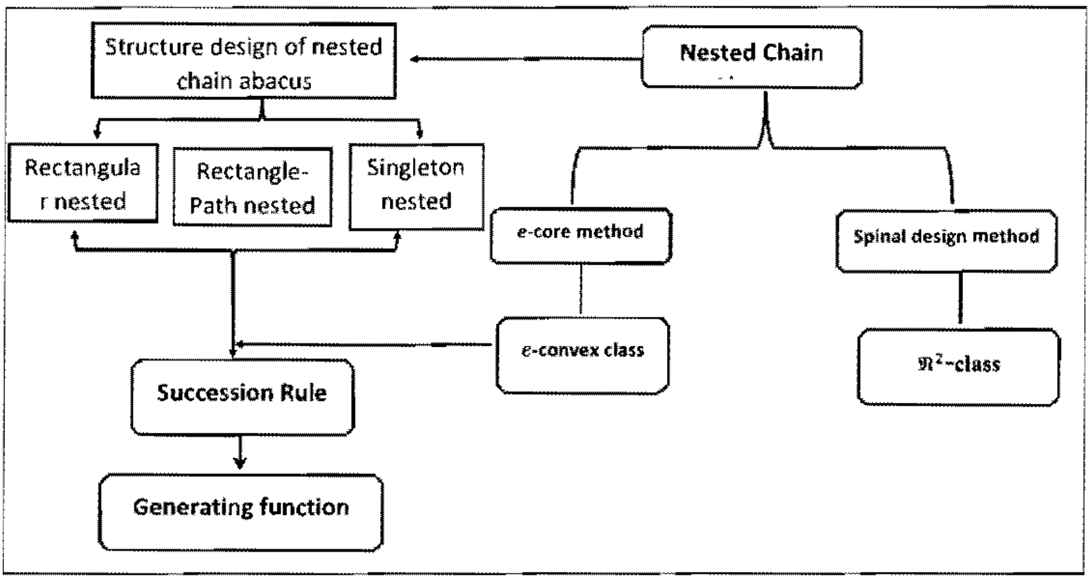

Figure 5.9. Methods to development classes in nested chain abacus and generating function

CHAPTER SIX

TILING WITH NESTED CHAIN ABACUS

6.1 Introduction

This chapter is devoted to apply classes of nested chain abacus of tiling in finite regions. Several mapping notations for generating two algorithms are presented. The mapping notations are used to move subsets of bead positions as well as empty bead positions after embedding the class of nested chain abacus in finite grid for tiling a rectangle.

This chapter begins with some definitions required for this chapter in Section 6.2. Next, in Section 6.3, two algorithms are derived for tiling a rectangle for the nested chain abacus. Finally, some theoretical results are proposed formulated and proved in Section 6.4.

6.2 Fundamental Definitions in Tiling

In this section we presented remark and some fundamental definitions for constructing tiling algorithm.

Remark 6.2.1. *Based on Definition 4.2.3 and Remark 4.2.4 the nested chain abacus is considered as a picture with rectangular forms, and the bead as well as empty bead positions are considered as squares in two colors where the bead positions constitute the image of the picture and the empty bead positions constitute the background. In this chapter, we consider the nested chain abacus with e columns and r rows as tiles used in tiling a finite region of grid with* e' *columns and* r' *rows for* $1 \leqslant e \leqslant e'$ *and* $1 \leqslant r \leqslant r'$.

Figure 6.1(a) illustrated an original nested chain abacus with bead and empty bead positions, colored in gray and blue respectively while Figure 6.1(b) illustrate the nested chain abacus embedded in a finite grid.

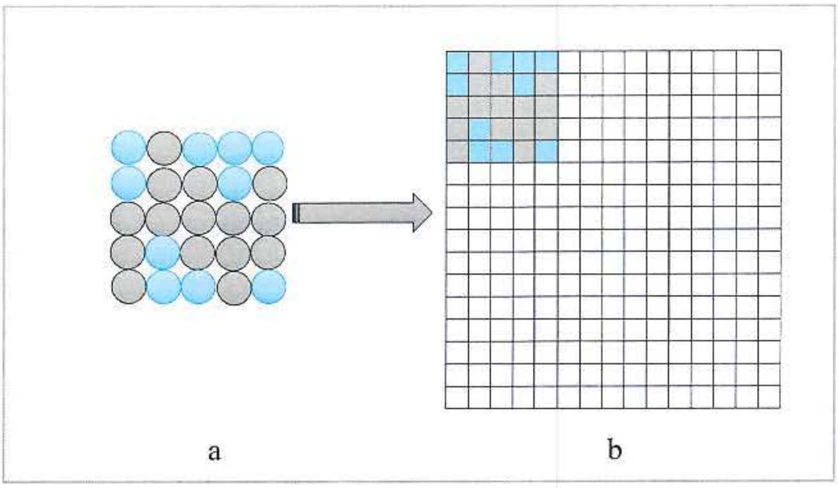

Figure 6.1. (a) Nested chain abacus for 15-connected beads and (b) The nested chain abacus for 15-connected beads embedded in a finite grid

Definition 6.2.2. *Let* $SR = \{me+j, me+j+1, ..., me+j+j'\}$ *be a set-row sequence of connected bead positions in the nested chain abacus with e columns and r rows.* $me+j$ *is a head row bead of the set-row sequence where* $0 \leqslant m \leqslant r-1$ *and* $0 \leqslant j \leqslant e-1$ *for* $0 \leqslant j' \leqslant e-j$.

Definition 6.2.3. *Let* $SC = \{me+j, (m+1)e+j, ..., (m+m')e+j\}$ *be a set-column sequence of connected bead positions in the nested chain abacus with e columns and r rows.* $me+j$ *is a head column bead of the set-column where* $0 \leqslant m \leqslant r-1$ *and* $0 \leqslant j \leqslant e-1$ *for* $0 \leqslant m' \leqslant r-m$.

Figure 6.2 illustrates the head column beads and head row beads.

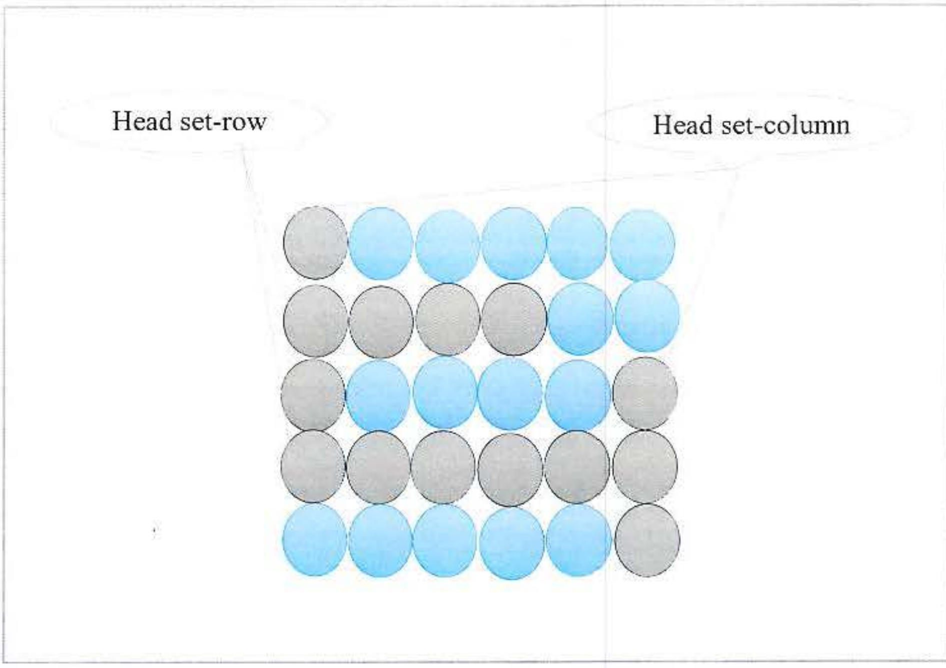

Figure 6.2. Head column beads and head row beads in nested chain abacus

Based on the connectedness in chapter two and chapter five, two families of nested chain abacus with respect to columns and rows are established in the following definition.

Definition 6.2.4. *An equivalent columns-convex nested chain abacus class* ($\mathfrak{N}_c$) *is a set of nested chain abacus with one set-column such that* $C_0 = C_1 = ... = C_{e-1}$ *and* $r = 2C_j$ *where* C_j *is the number of the beads in column* j *and* $0 \leqslant j \leqslant e-1$.

Definition 6.2.5. *An equivalent rows-convex nested chain abacus class* ($\mathfrak{N}_r$) *is a set of nested chain abacus with one set-row in each row such that* $R_0 = R_1 = ... = R_{r-1}$ *and* $r = 2R_m$ *where* R_m *is the number of the beads in row* m *and* $0 \leqslant m \leqslant r-1$.

Depending on the geometric characteristics of the nested chain abacus in Definition 6.2.4 and Definition 6.2.5, next we construct two algorithms for the tiling of a finite grid by translating the subset of positions.

6.3 Tiling Algorithm

In this section, we attempt to solve the problem of tiling in a finite region with $\mathfrak{N}_c$ and $\mathfrak{N}_r$ nested chain abacus. Next, two algorithms are proposed to tiling a finite region.

6.3.1 Algorithm with $\mathfrak{N}_c$

A class of nested chain abacus called $\mathfrak{N}_c$ with e columns and r rows will be used for tiling in a finite region with $e^{'}$ columns and $r^{'}$ rows:

Step 1: Creating $\mathfrak{N}_c$ nested chain abacus

1. Let set $S = [r, e, k]$ be an initial parameter such that each are the numbers of rows, columns and number of bead positions in column j where $0 \leqslant j \leqslant e-1$.
2. Identify the head column beads in column j by employing programming code in Appendix D, Code C and Code D.

   Consider $S = [4, 5, 3]$ be an initial parameter, the head column beads for all $\mathfrak{N}_c$ as shown in Table 6.1.
3. Identify the first and last head column beads, $k^{'}, k^{''}$ such that

   $\{(k^{'}, 0), (k^{'}+1, 0), \ldots, (k^{'}+k, 0)\}$ is set-column sequence in the first column,

   $\{(k^{''}, e-1), (k^{''}+1, e-1), \ldots, (k^{''}+k, e-1)\}$ is set-column sequence in the last column.
4. Calculate ρ where $\rho = k^{''} - k^{'}$.

Table 6.1

Head column bead position of nested chain abacus with e = 5, r = 4 and k = 3

Nested chain abacus number	Head in column 1	Head in column 2	Head in column 3	Head in column 4	Head in column 5
1	0	1	2	3	9
2	0	1	2	8	4
3	0	1	2	8	9
4	0	1	7	4	4
5	0	1	7	8	4
6	0	1	7	8	9
7	0	6	2	8	4
8	0	6	2	4	9
9	0	6	2	8	4
10	0	1	7	3	4
11	0	6	2	8	9
12	0	6	7	3	9
13	0	6	7	3	9
14	0	6	7	8	4
15	0	6	7	8	9
16	5	1	2	3	4
17	5	1	2	3	9
18	5	1	2	8	4
19	5	1	7	8	9
20	5	1	7	3	4
21	5	1	7	3	9
22	5	1	7	8	4
23	5	1	2	8	9
24	5	1	2	3	4
25	5	6	2	3	9
26	5	6	2	8	4
27	5	6	2	8	9
28	5	6	7	3	4
29	5	6	7	3	9
30	5	6	7	8	4

Step 2: Finding the mapping

Propagate the nested chain abacus with e columns and r rows of class $\mathfrak{N}_c$ within the finite region with e' columns and r' rows for $1 \leqslant e \leqslant e'$ and $1 \leqslant r \leqslant r'$ by applying three mapping notations.

1. Γ_1 is mapping notation: $\mathbb{Z}\times\mathbb{Z}\rightarrow\mathbb{Z}\times\mathbb{Z}$ such that

$$\Gamma_1(me^{'}+j)\rightarrow(m+s\rho)e^{'}+(j+se)$$

where $0\leqslant m\leqslant r-1$, $0\leqslant j\leqslant e-1$ and $1\leqslant s\leqslant\lceil\frac{e^{'}}{e}\rceil-1$.

2. Γ_2 is a mapping notation: $\mathbb{Z}\times\mathbb{Z}\rightarrow\mathbb{Z}\times\mathbb{Z}$, such that $\forall\, s^{'}\ \exists\, s$. Then,

$$\Gamma_2(me^{'}+j)\rightarrow(m+s\rho+s^{'}k)e^{'}+(j+se)$$

where $r-k\leqslant m\leqslant r-1$, $0\leqslant j\leqslant e-1$ and $0\leqslant s\leqslant\lceil\frac{e^{'}}{e}\rceil-1$, $1\leqslant s^{'}\leqslant\lceil\frac{r^{'}}{r}\rceil-1$.

3. $\Gamma_3(m,j)$ is mapping notation $\mathbb{Z}\times\mathbb{Z}\rightarrow\mathbb{Z}\times\mathbb{Z}$, such that. Then,

$$\Gamma_3(me^{'}+j)\rightarrow(m+s\rho)e^{'}+(j+es)$$

where $0\leqslant m\leqslant r-1$, $0\leqslant j\leqslant e-1$ and $1\leqslant s\leqslant\lceil\frac{e^{'}}{e}\rceil$, $1\leqslant s_1\leqslant\lceil\frac{r^{'}}{r}\rceil$.

4. $\Gamma_4(m,j)$ is mapping notation $\mathbb{Z}\times\mathbb{Z}\rightarrow\mathbb{Z}\times\mathbb{Z}$, such that $\forall\, s^{''}\ \exists\, s$. Then,

$$\Gamma_3(me^{'}+j)\rightarrow(m+s\rho-s^{''}k)e^{'}+(j+es)$$

where $0\leqslant m\leqslant k-1$, $0\leqslant j\leqslant e-1$ and $1\leqslant s\leqslant\lceil\frac{e^{'}}{e}\rceil$, $1\leqslant s_1\leqslant\lceil\frac{r^{'}}{r}\rceil$.

Remark 6.3.1. *Let the nested chain abacus be a class of $\mathfrak{N}_c$ and let $\Gamma_v(m,j)$: $\mathbb{Z}\times\mathbb{Z}\rightarrow\mathbb{Z}\times\mathbb{Z}$ be the mapping notation. Then,*

1. *If $k^{''}\leqslant k^{'}$ then we apply two mapping notations Γ_1 and Γ_2.*
2. *If $k^{''}>k^{'}$ then we apply three mapping notations Γ_2, Γ_3 and Γ_4.*
3. *Γ_v(position with color gray (respectively, blue)) $\rightarrow$ (bead position with color gray (respectively, blue)) if $s+s^{'}+s^{''}\in\mathbb{Z}_e^+$.*
4. *Γ_v(bead position with color gray (respectively, blue)) $\rightarrow$ (bead position with color blue (respectively, gray) if $s+s^{'}+s^{''}\in\mathbb{Z}_o^+$, where $1\leqslant v\leqslant 3$.*

The result of the process of tiling rectangle by using $\mathfrak{N}_c$ is demonstrated by the following example.

Example 6.3.2. *Consider nested chain abacus number 20 in Table 6.1 where* $[r, e, k] = [4, 5, 3]$, *by employing programming code in Appendix C with head column beads* $\{5, 1, 7, 3, 4\}$ *then,*

in column 1, positions 5, 10, 15 are bead positions while position 0 is empty bead position,

in column 2, positions 1, 6, 11 are bead positions while position 16 is empty bead position,

in column 3, positions 7, 12, 17 are bead positions while position 2 is empty bead position,

in column 4, positions 3, 8, 13 are bead positions while position 18 is empty bead position and

in column 5, positions 4, 9, 14 are bead positions while position 19 is empty bead position, as shown in Figure 6.3.

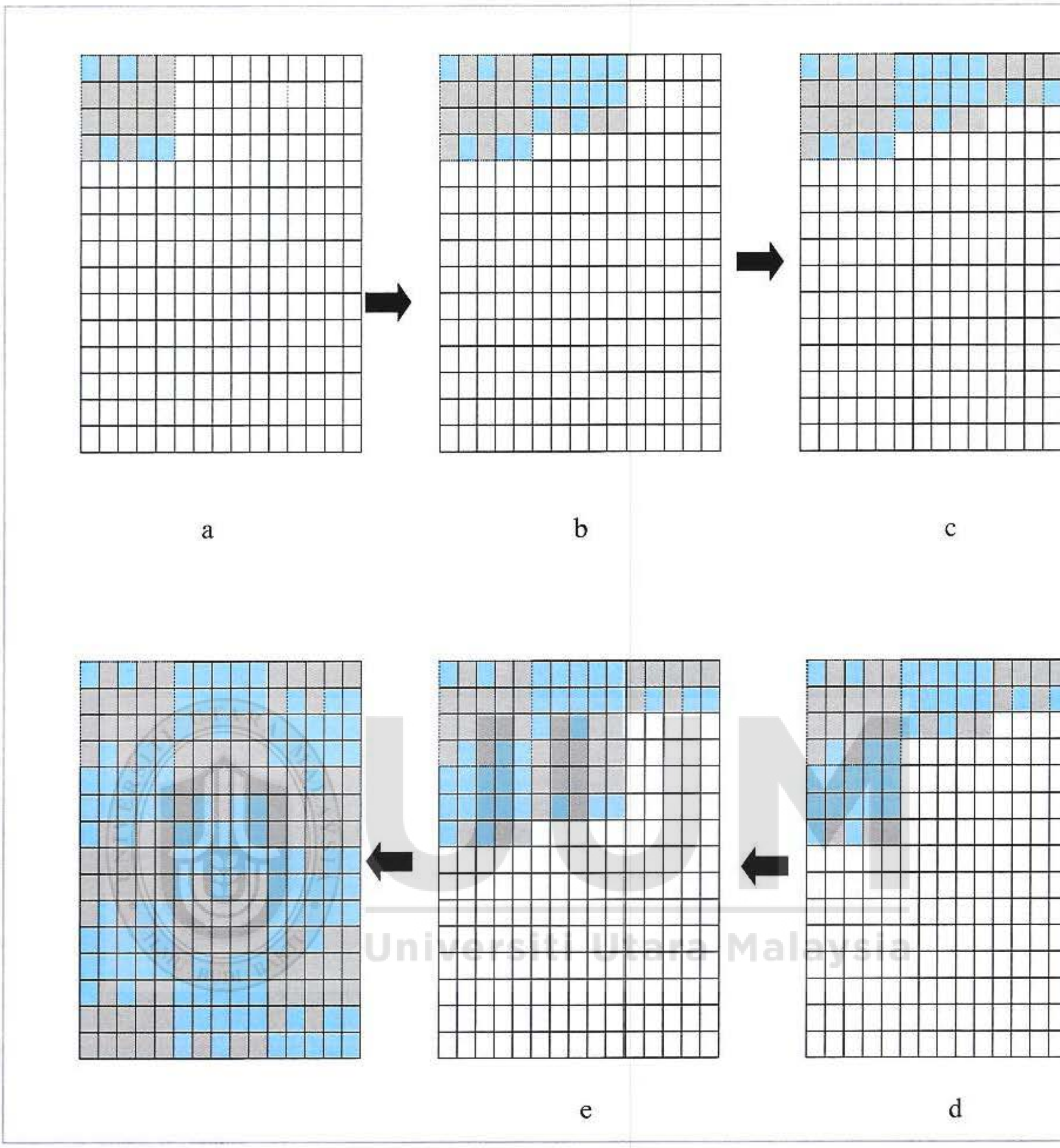

Figure 6.3. (a) Nested chain abacus where $e = 4$, $r = 4$,$k = 3$ (b) Γ_1 where $s= 1$ (c) Γ_1 where $s= 1,2$ (d) Γ_1 where $s =1, 2$ and Γ_2 where $s =0$ and $S' = 1$ (e) Γ_1 where $s =1, 2$ and Γ_2 where $s =0, 1$ and $S' = 1, 1$

6.3.2 Algorithm with $\mathfrak{N}_r$

In this section, a new algorithm for tiling in a finite region with e' columns and r' rows is proposed using $\mathfrak{N}_r$ with e columns and r rows as follows:

Step 1: Creating $\mathfrak{N}_r$ nested chain abacus

1. Let set $S = [r, e, L]$ be an initial parameter such that each are the numbers of rows, columns and bead positions in row m where $0 \leqslant m \leqslant r-1$.
2. Identify the head row beads in row j by employing programming code in Appendix C, Code **C** and Code D.

   Consider $S = [4, 4, 3]$ be an initial parameter, the head row beads for all $\mathfrak{N}_r$ nested chain abacus can be create as shown in Table 6.2.
3. Identify the first and last head row beads, L', L'' such that

   $\{(0, L'), (0, L'+1), \ldots, (0, L'+L)\}$ is set-row sequence in the first row,

   $\{(r-1, L''), (r-1, L''+1), \ldots, (r-1, L''+L)\}$ is set-row sequence in the last row.
4. Calculate the ε where $\varepsilon = L'' - L'$.

Table 6.2

Head column bead position of nested chain abacus with e = 4, r = 4 and k = 3

Nested chain abacus number	Head in column 1	Head in column 2	Head in column 3	Head in column 4
1	0	0	0	0
2	0	0	2	2
3	0	0	2	0
4	0	2	0	2
5	0	2	0	0
6	0	2	2	2
7	0	2	2	0
8	2	0	0	2
9	2	0	0	0
10	2	0	2	2
11	2	0	2	0
12	2	2	0	2
13	2	2	0	0
14	2	2	2	2

Step 2: Finding the mapping

Propagate the nested chain abacus of class $\mathfrak{N}_c$ with e columns and r rows within the finite grid with e' columns and r' rows for $1 \leqslant e \leqslant e'$ and $1 \leqslant r \leqslant r'$.

1. τ_1 is mapping notation $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, such that

$$\tau_1(me' + j) \to (m + sr)e' + (j + s\varepsilon)$$

where $0 \leqslant m \leqslant r - 1, 0 \leqslant j \leqslant e - 1$ and $1 \leqslant s \leqslant \lceil \frac{e'}{r} \rceil - 1$.

2. τ_2 is mapping notation $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ such that $\forall s' \exists s$. Then

$$\tau_2(me' + j) \to (m + sr)e' + (j + s\varepsilon + s'k)$$

where $0 \leqslant m \leqslant r - 1, H - 1 \leqslant j \leqslant e - 1$ and $0 \leqslant s \leqslant \lceil \frac{r'}{r} \rceil - 1, 1 \leqslant s' \leqslant \lceil \frac{e'}{e} \rceil - 1$.

3. τ^3 is mapping notation $\mathbb{Z}\times \mathbb{Z} \rightarrow \mathbb{Z}\times \mathbb{Z}$ such that $\forall s'' \; \exists s$. Then

$$\tau_3(me' + j) \rightarrow (m + sr)e' + (j + s\varepsilon)$$

where $0 \leqslant m \leqslant r-1$, $0 \leqslant j < e-1$ and $0 \leqslant s \leqslant \lceil \frac{e'}{e} \rceil$ -1.

4. is mapping notation $\mathbb{Z}\times \mathbb{Z} \rightarrow \mathbb{Z}\times \mathbb{Z}$ such that $\forall s'' \; \exists s$. Then

$$\tau_3(me' + j) \rightarrow (m + sr)e' + (j + s\varepsilon - s'' k)$$

where $0 \leqslant m \leqslant r-1$, $0 \leqslant j < e-1$ and $0 \leqslant s \leqslant \lceil \frac{e'}{e} \rceil$ -1, $1 \leqslant s''$.

Remark 6.3.3. *Let the nested chain abacus be a class of* $\mathfrak{N}_r$ *and let*

$_1$ *and* τ_2

2. *If* $L' > L$, *then we will apply three mapping notations* τ_2, τ_3 *and* τ_4.

The result of the process of tiling rectangle by using $\mathfrak{N}_r$ is demonstrated by the following example.

Example 6.3.4. *Consider nested chain abacus number 12 in Table 6.2 where* $[r,\ e,\ k] = [4,\ 4,\ 3]$, *by employing programming code in Appendix B with head row beads* $\{1, 5, 8, 12\}$ *then,*

in row 1, positions 1, 2, 3 are bead positions while position 0 is empty bead position,

row 2, positions 5, 6, 7 are bead positions while position 4 is empty bead position,

row 3, positions 8, 9, 10 are bead positions while position 11 is empty bead position and

row 4, positions 12, 13, 14 are bead positions while position 15 is empty bead position, as shown in Figure 6.4.

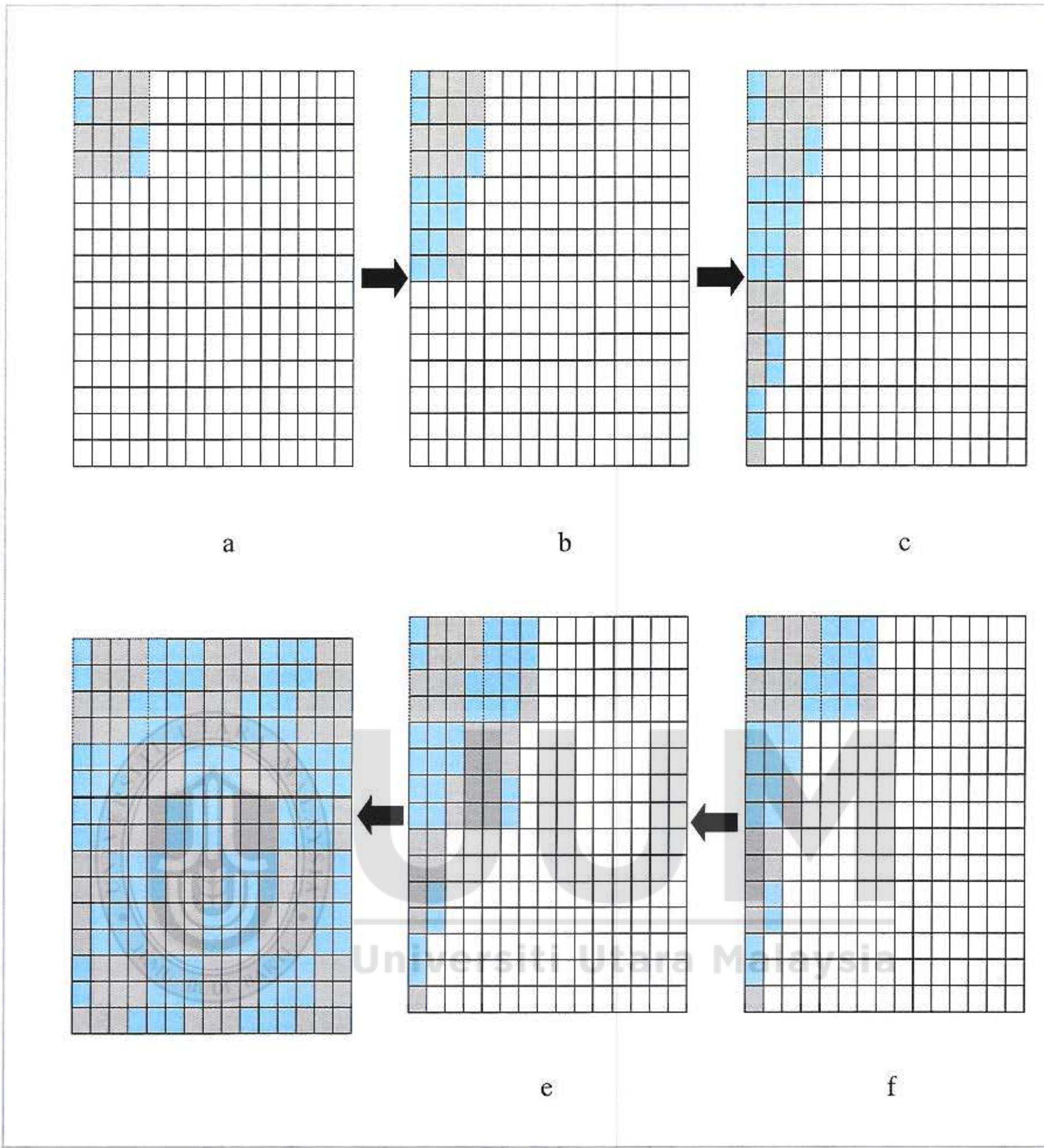

Figure 6.4. (a) Nested chain abacus where $e = 4$, $r = 4$,$k = 3$ (b) τ_1 where $s= 1$ (c) τ_1 where $s= 1,2$ (d) τ_2 where $s =1, 2$ and τ_2 where $s =0$ and $S' = 1$ (e) τ_1 where $s =1, 2$ and τ_2 where $s =0, 1$ and $S' = 1, 1$

Consider Figure 6.3 and Figure 6.4, if the position is marked with color gray (R) then the translation of it with color blue ($R^{'}$) is as shown in Table 6.3.

Table 6.3

Translation color R to color $R^{'}$

$s^{'}$ / s	0	1	2	3	4	5
0		$R^{'}$	R	$R^{'}$	R	$R^{'}$
1	$R^{'}$	R	$R^{'}$	R	$R^{'}$	R
2	R	$R^{'}$	R	$R^{'}$	R	$R^{'}$
4	$R^{'}$	R	$R^{'}$	R	$R^{'}$	R

6.4 Theoretical Result

Theorem 6.4.1. *Let C_j be the number of beads in column j. The number of $\mathfrak{N}_c$ of nested chain abacus with a C_j beads j is*

$$\prod_{j=1}^{e-1}(2C_j - 1)$$

where SC_j is the number of beads in column j.

Proof.

Basic step:

If $e = 1$ then the nested chain abacus with 1 column. Hence, there is only one nested chain abacus.

Inductive step:

consider adding a column with C_j beads to the right of column e. Then, there are $2C_j - 1$ ways to connect column $e - 1$ with column e. Thus, the number of nested chain abacus with a SC_j beads in column j is

$$(2C_j-1)\prod_{j=1}^{e-1}(2C_j-1).$$

Thus,

$$\prod_{j=1}^{e}(2C_j-1).$$

□

In the next theorem we will found a generating function of $\mathfrak{N}_c$ class

Theorem 6.4.2. *Let $b^{'}$ be the number of empty bead positions. The generating function for the number of $\mathfrak{N}_c$ has the following ordinary form:*

$$\frac{xy(2n-1)}{(1-x(2n-1))(1-y)}.$$

Proof. Based on Definition 6.2.4 the number of beads in e columns are equal. If $e=1$ then the n beads location in one column. Since the grows of $\mathfrak{N}_c$ nested chain abacus by adding one column and one row with n beads then, the number of beads in column j (SC_j) is equal to n where $1\leqslant j\leqslant e$. Based on Theorem 6.4.1 the number of $\mathfrak{N}_c$ is

$$\prod_{j=1}^{e-1}(2C_j-1)$$

where $C_j=n$. Based on the the ordinary form the generating function of $\mathfrak{N}_c$ is

$$\sum_{e,r\geqslant 1}(2n-1)^{e-1}x^e y^r = xy\sum_{e,r\geqslant 1}(2n-1)^{e-1}x^{e-1}y^{r-1}.$$

Since

$$\sum_{e\geqslant 1}x^{e-1}=1+x+x^2+\ldots=\frac{1}{1-x}.$$

Then,

$$xy\sum_{e\geqslant 1}(2n-1)^{e-1}x^{e-1}\sum_{r\geqslant 1}y^{r-1}=\frac{xy(2n-1)}{(1-x(2n-1))(1-y)}. \qquad \square$$

Theorem 6.4.3. *Let R_m be the number of beads in row m . Then, the number of $\mathfrak{N}_c$ of nested chain abacus is*

$$\prod_{R_m=1}^{r-1}(2R_m-1).$$

Proof. See proof Theorem 6.4.1 $\square$

Theorem 6.4.4. *Let $b^{'}$ be the number of empty bead positions. The generating function for the number of $\mathfrak{N}_r$ has the following ordinary form:*

$$\frac{xy(2n-1)}{(1-x(2n-1))(1-y)}.$$

Proof. see the proof of Theorem 6.4.2. $\square$

Theorem 6.4.5. *Let $\mathfrak{N}$ be a class of $\mathfrak{N}_r$ with e columns and r rows where L and $L^{'}$ are the head row beads in row 1 and row r respectively for $L\leqslant L^{'}$. Then, the number of nested chain abacus needed for tiling a rectangle with area $e^{'}\times r^{'}$ are*

$$\begin{cases} 1+\left\lceil\dfrac{e^{'}-e}{H}\right\rceil+\displaystyle\sum_{g=1}^{u}1+\left\lceil\dfrac{e^{'}-e+g\left|\varepsilon^{'}\right|}{H}\right\rceil & If \quad g\leqslant\left\lceil\dfrac{r^{'}-r}{r\left|\varepsilon^{'}\right|}\right\rceil \\ 1+\left\lceil\dfrac{e^{'}-e}{H}\right\rceil+\displaystyle\sum_{g=1}^{u}1+\left\lceil\dfrac{e^{'}-e+g\left|\varepsilon^{'}\right|}{H}\right\rceil+\left(\left\lceil\dfrac{r^{'}}{r}\right\rceil-e+1\right)\left\lceil\dfrac{e^{'}}{H}\right\rceil & If \quad g>\left\lceil\dfrac{r^{'}-r}{r\left|\varepsilon^{'}\right|}\right\rceil \end{cases}$$

where $u=\left\lceil\dfrac{r^{'}-r}{r\left|\varepsilon^{'}\right|}\right\rceil$, $\varepsilon^{'}=L^{'}-L.$

Proof.

Case one: If $g < \left\lceil \frac{r' - r}{r\left|\varepsilon'\right|} \right\rceil$.

Suppose that H is the number of bead positions in each row. Based on mapping notation τ^1 where only one nested chain abacus with r rows and $e - g\left|\varepsilon'\right|$ columns will be translated where $1 \leqslant g < u$. Based on mapping notation τ^2 the nested chain abacus with r rows and H columns well be translated. Since τ^2 depends on the mapping τ^1.

Thus, there is

$$1 + \left\lceil \frac{e' - e}{H} \right\rceil + \sum_{g=1}^{u} 1 + \left\lceil \frac{e' - e + g\left|\varepsilon'\right|}{H} \right\rceil$$

nested chain abacus to be translated.

Case two: If $g \geqslant \left\lceil \frac{r' - r}{r\left|\varepsilon'\right|} \right\rceil$.

Since the mapping notation $\tau^{1(g)}$ out the region then, using τ^2 the number of nested chain abacus is

$$\left(\left\lceil \frac{r'}{r} \right\rceil - e + 1 \right) \lceil \frac{e'}{H} \rceil.$$

Thus, the number of nested chain abacus for tiling in a finite region with class $\mathfrak{N}_r$ is

$$1 + \left\lceil \frac{e' - e}{H} \right\rceil + \sum_{g=1}^{u} 1 + \left\lceil \frac{e' - e + g\left|\varepsilon'\right|}{H} \right\rceil + \left(\left\lceil \frac{r'}{r} \right\rceil - e + 1 \right) \lceil \frac{e'}{H} \rceil.$$

□

Theorem 6.4.6. *Let nested chain abacus be a class of $\mathfrak{N}_c$ with e columns and r rows and $k' \leqslant k''$ such that (k,d) is the lower head column bead position, $(k',1)$ and (k'',e) are the head column beads in columns 1 and e respectively. Then, the number of $\mathfrak{N}_c$ which can be used for tiling a rectangle with area $e'r'$ is*

$$\begin{cases} 1 + \left\lceil \frac{r' - r}{k} \right\rceil + \sum_{h=1}^{v} 1 + \left\lceil \frac{r' - r + h\left|\rho'\right|}{k} \right\rceil & if \quad h < \left\lceil \frac{e' - e}{e\left|\rho'\right|} \right\rceil \\ 1 + \left\lceil \frac{r' - r}{k} \right\rceil + \sum_{h=1}^{v} 1 + \left\lceil \frac{r' - r + h\left|\rho'\right|}{k} \right\rceil + \left(\left\lceil \frac{e'}{e} \right\rceil - r + 1 \right) \lceil \frac{r'}{k} \rceil & if \quad h \geqslant \left\lceil \frac{e' - e}{e\left|\rho'\right|} \right\rceil \end{cases}$$

where $\rho = r - k, \rho' = k'' - k'$ *and* $er < e'r'$.

Proof. Similar proof of Theorem 6.4.6 □

6.5 Conclusion

In this chapter, we apply classes of nested chain abacus in developing algorithm for tiling.

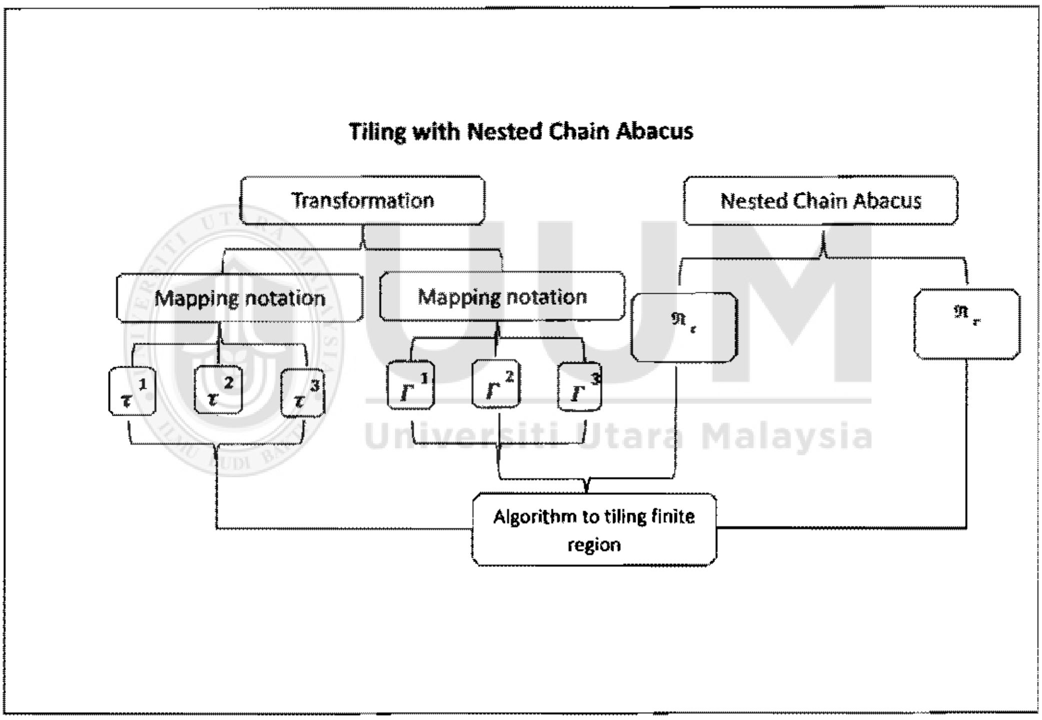

Figure 6.5. Tiling with nested chain abacus

CHAPTER SEVEN

CONCLUSION

In this chapter, we highlight research contributions which were developed and proven.

7.1 Research contributions

- **Constructing new abacus**:
  In Section 2.3, we develop an algorithm to construct a new representation for n-connected ominoes called nested chain abacus. We also show that the new development provides a direct representation to the n-connected ominoes and associated each form of n-connected ominoes with a partition called connected partition.

- **Uniqueness of the n-connected ominoes' representation**:
  Theorem 2.3.3 showed that the connected partition is a unique representation to n-connected ominoes by using a nested chain abacus.

- **Connectedness**:
  In Section 2.4, we formulated the connectedness of the beads with respect to columns and rows. The condition of connecting beads is achieved in Lemma 2.4.6, Lemma 2.4.9, Lemma 2.4.11, Lemma 2.4.14 and Theorem 2.4.15.

- **Topological structure of nested chain abacus**:
  We established five different structures of nested chain abacus based on three types of chains which are
  (i) vertical rectangular nested chain abacus, (ii) horizontal rectangular nested chain abacus, (iii) vertical rectangle-path nested chain abacus, (iv) horizontal rectangle-path nested chain abacus and (v) singleton nested chain abacus. Based on these structures, we produced the following:
  (i) the number of chains in each structural design of nested chain abacus (see in Lemma 2.5.7, Lemma 2.5.15 and Lemma 2.5.20)

(ii) the number of positions in the three type of chains (see in Theorem 2.5.8 and Theorem 2.5.16).

(iii) a series of sequence for *n*-connected ominoes (see in Theorem 2.5.10, Theorem 2.5.23 and Theorem 2.5.24).

- **Transformation of nested chain abacus**:
  In Chapter Three, we develop three different types of chain transformations which are SNC2-Transformation, SNC-Transformation, and MNC-Transformation (see Lemma 3.3.1, Theorem 3.3.3, Corollary 3.3.4, Theorem 3.3.5, Corollary 3.3.6, Lemma 3.3.8, Theorem 3.3.9, Theorem 3.3.10, Lemma 3.3.11, Theorem 3.3.12, Theorem 3.3.13 and Lemma 3.3.14).

- **Construction of classes of nested chain abacus based on**:

  (i) **Transformation**: (in Lemma 4.3.1, Theorem 4.3.3, Theorem 4.3.4, Lemma 4.4.1, Theorem 4.4.2, Lemma 4.4.4, Theorem 4.4.5, Lemma 4.4.6, Theorem 4.4.7, Theorem 4.4.10, Lemma 4.4.12, Theorem 4.4.13, Lemma 4.5.1, Theorem 4.5.2, Corollary 4.5.3, Theorem 4.5.4, Corollary 4.5.5, Corollary 4.5.6 and Theorem 4.5.7).

  (ii) ***e*-core method**: (in Lemma 5.3.6, Lemma 5.3.7, Theorem 5.3.8, Theorem 5.3.9, Theorem 5.3.10, Theorem 5.3.11, Theorem 5.3.12)

  (iii) **Spinal Design**: (in Theorem 5.5.2 and Theorem 5.5.3).

- **Generating functions**:
  We constructed two generating functions based on chain concept and class of *e*-convex nested chain abacus.

- **Application of the nested chain abacus**:
  In Chapter Six, we apply the nested chain abacus for tiling in a finite region (see in Theorem 6.4.1, Theorem 6.4.3, Theorem 6.4.5 and Theorem 6.4.6).

7.2 Future Work

- Developing new algorithms to represent *n*-connected hexagon (polyhex) using nested chain abacus.
- Developing new algorithms to represent *n*-connected triakis (polykite) using nested chain abacus.
- Developing new algorithms to represent *n*-connected triangle (polyiamond) using nested chain abacus.
- Construct other classes of nested chain abacus such as where $e > 2$ or parallel classes.
- Developed the idea for three dimensional objects. Also we can form chains which may not be rectangular in shape and can start from any place and make sure that all the chains will not intersect each other.
- Used two different nested chain abacus to tilling infinite region.
- Used factorization method to translate classes of nested chain abacus to tiling the finite grid depended only on the beads positions.
- Apply nested chain abacus to solve conjecture related with Tangle series

  Conjecture:

  Let Tanglegram configuration be an element in a class, then the element in a class from a cyclic class.

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APPENDIX A

GENERATING FUNCTION W.R.T CHAINS

Input Rows and columns

```
r=input('Input the number of rows:');

e=input('Input the number of columns:');

* Classify cases

* Case 1(If r < e and r is odd, then p1=e-r + 1  p2=2p1+6)

if r<e  mod(r,2)==1

P1=e-r+1;

P2=2*P1+6;

end * Case 2(If e < r and e is odd. then p1= r-e + 1 and p2 = 2 p1 + 6)

if e<r  mod(e,2)==1

P1=r-e+1;

P2=2*P1+6;

end * Case 3(If e < r and e is odd. then p1= r-e + 1 and p2 = 2 p1 + 6)

if e==r mod(r,2)==1  mod(e,2)==1

P1=1;

P2=8;

end

* Case 4(If r < e and r is even then p1= 2r + 2e - 4 ( 2c- 1)=2r-2e+4, where c= r/2 and p2=p1 + 8.)

if r<=e mod(r,2)==0

P1=2*r-2*e+4;

P2=P1+8;

end

* Case 5(If e <= r and e is even then p1= 2r + 2e - 4 ( 2c- 1)=2e-2r+4, where c=e/2 and p2 = p1 + 8)
```

if e<=r mod(e,2)==0

P1=2*e-2*r+4

P2=P1+8

end

** Compute the generating function syms x

syms y

* f(x,y) fprintf('================f(x,y)=================')

f(x,y)=-exp(1/(8*x*y^8)) $* int(y^(P2 - 9) *exp(-1/8 *x*y^8), y)$

· $Generating function f(x)$

$fprintf(^I============= Generating function f(x) ================$

)

$f(x, 1)$

· $PolynomialFrom$

$ff(1) = 1; ff(2) = P2;$

$forn = 3 : 10$

$ff(n) = ff(n-1) * (P2 + 8);$

end

$* Writethisresult$

$fori = 1 : 10$

$fprintf(^I f(*d) = *d^I, i, ff(i))$

end

APPENDIX B

TILLING ALGORITHM W.R.T ROW

theta$_{c}ol = 25;$

$theta_{r}ow = 25;$

$H1 = 3;$

$r = 5;$

$e = 5;$

$Computetheuppero f s, s^{I}, s^{II}$

$s_{l}im = (ceil(theta_{r}ow/r)) - 1;$

$s1_{l}im = (ceil((theta_{c}ol - e)/H1)) + 3;$

$s2_{l}im = ceil(theta_{c}ol/H1);$

$Setcolors$

$color2 = [255, 217, 102]/255;$

$color1 = [131, 59, 10]/255;$

$\cdot\ Generatinginitialabacus$

$*temp = ceil(rand(1, e) * (r - num_{b}ead + 1));$

$temp = [2, 3, 2, 1, 2];$

$rect = zeros(theta_{r}ow, theta_{c}ol);$

$f ori = 1 : r$

$rect(i, temp(i) : temp(i) + H1 - 1) = 1;$

end

$Draw_{s}hape(rect, theta_{r}ow, theta_{c}ol, color1, color2)$

$title(^{I}InitialRectangle^{I})$

$H = max(temp);$

$L = temp(1, 1);$

$L1 = temp(1, end);$

$p = H1;$

$p1 = L1 - L;$

$\cdot IfL1 > L, thenapplicatethemapping1andmapping2, mapping3$
$ifL1 > L$

$form = 1 : r$

$forj = 1 : e$

$fors = 1 : s_lim$

$ifrect(m, j) == 1and(j + s * p1) > 0and(j + s * p1) <= theta_colandmod(s, 2)$

$1rect(m + s * r, j + s * p1) = 0;$

$elseifrect(m, j) == 0and(j + s * p1) > 0and(j + s * p1) <= theta_coland$

$mod(s, 2) == 1$

$rect(m + s * r, j + s * p1) = 1;$

$elseifrect(m, j) == 1and(j + s * p1) > 0and(j + s * p1) <= theta_coland$

$mod(s, 2) == 0$

$rect(m + s * r, j + s * p1) = 1;$

$elseifrect(m, j) == 0and(j + s * p1) > 0and(j + s * p1) <= theta_coland$

$mod(s, 2) == 0$

$rect(m + s * r, j + s * p1) = 0;$

end

end

end

end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$

$title(^I RectangleafterMapping1^I)$

$form = 1 : r$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseifrect(m, j) == 0and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 1andmod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

end

end

end

end

end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$

$title({}^{I}Rectangleafter Mapping2^{I})$

end

$if L1 > L$

$for m = 1 : r$

$for j = 1 : e$

$for s = 1 : s_lim$

$if rect(m, j) == 1 and (j + s * p1) > 0 and (j + s * p1) <= theta_col and$

$mod(s, 2) == 1$

$rect(m + s * r, j + s * p1) = 0;$

$else if rect(m, j) == 0 and (j + s * p1) > 0 and (j + s * p1) <= theta_col and$

$mod(s, 2) == 1$

$rect(m + s * r, j + s * p1) = 1;$

$else if rect(m, j) == 1 and (j + s * p1) > 0 and (j + s * p1) <= theta_col and$

$mod(s, 2) == 0$

$rect(m + s * r, j + s * p1) = 1;$

$else if rect(m, j) == 0 and (j + s * p1) > 0 and (j + s * p1) <= theta_col and$

$mod(s, 2) == 0$

$rect(m + s * r, j + s * p1) = 0;$

end

end

end

end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$

$title(^I RectangleafterMapping1^I)$

$form = 1 : r$

$forj = H : e$

$fors = 0 : s_lim$

$fors1 = 1 : s1_lim$

$ifrect(m, j) == 1and(m + s * p1 + s1 * p) > 0and(m + s * p1 + s1 * p) <= theta_col$

$andmod(s, 2) == 0andmod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

$elseifrect(m, j) == 0and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 0andmod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseifrect(m, j) == 1and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 0andmod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseifrect(m, j) == 0and(m + s * p1 + s1 * p) > 0and (m + s * p1 + s1 * p) <$

$0andmod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

$elseifrect(m, j) == 1and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 1andmod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

$elseifrect(m, j) == 0and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 1andmod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseifrect(m, j) == 1and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_{c}olandmod(s, 2) == 1andmod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseif rect(m, j) == 0and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_{c}olandmod(s, 2) == 1andmod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

end

end

end

end

end

$Draw_{s}hape(rect, theta_{r}ow, theta_{c}ol, color1, color2)$

$title({}^{I}Rectangleafter Mapping2^{I})$

$for m = 1 : r$

$for j = 1 : H1$

$for s = 1 : s1_{l}im$

$for s2 = 1 : s2_{l}im$

$for j = H : e$

$for s = 0 : s_{l}im$

$for s1 = 1 : s1_{l}im$

$ifrect(m, j) == 1and(m + s * p1 + s1 * p) > 0and(m + s * p1 + s1 * p) <= theta_{c}ol$

$andmod(s, 2) == 0andmod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

$elseif rect(m, j) == 0and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_{c}olandmod(s, 2) == 0andmod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseif rect(m, j) == 1and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_{c}olandmod(s, 2) == 0andmod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseif rect(m, j) == 0 and (m + s * p1 + s1 * p) > 0 and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 0 andmod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

$elseif rect(m, j) == 1 and (m + s * p1 + s1 * p) > 0 and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 1 andmod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

$elseif rect(m, j) == 0 and (m + s * p1 + s1 * p) > 0 and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 1 andmod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseif rect(m, j) == 1 and (m+s*p1+s1*p) > 0 and (m+s*p1+s1*p) <$

$1 andmod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseif rect(m, j) == 0 and (m + s * p1 + s1 * p) > 0 and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 1 andmod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

end

end

end

end

end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$

$title(^{I}RectangleafterMapping2^{I})$

$form = 1 : r$

$forj = 1 : H1$

$fors = 1 : s1_lim$

$fors2 = 1 : s2_lim$

$functionDraw_shape(M, a, b, color1, color2)$

$figure$

$axis([0b0a])$

$holdon$

$fori = 1 : a$

$forj = 1 : b$

$ifM(i, j) == 1$

$rectangle(^IPosition^I, [j - 1, a - i, 1, 1], ^I FaceColor^I, color1); elseifM(i, j) == 0$

$rectangle(^IPosition^I, [j - 1, a - i, 1, 1], ^IFaceColor^I, color2);$

end

end

end

end

end

APPENDIX C

TILLING ALGORITHM W.R.T COLUMN

Initial conditions $theta_col = 30;$

$theta_row = 30;$

$d1 = 3;$

$r = 5;$

$e = 5;$

$s1_lim = ceil(theta_col/e) - 1;$

$s_lim = ceil((r - theta_row)/d1);$

$s2_lim = ceil(theta_col/e) - 1;$

$color1 = [56, 85, 34]/255;$

$color2 = [156, 194, 228]/255;$

$temp = [2, 1, 2, 3, 2];$

$rect = zeros(theta_row, theta_col);$

$f ori = 1 : e$

$rect(temp(i) : temp(i) + d1 - 1, i) = 1;$
end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$
$title(^I InitialRectangle^I)$

$k = max(temp);$

$k1 = temp(1, 1);$

$k2 = temp(1, end);$

$p = d1;$

$p1 = k2 - k1;$

$i f k2 <= k1$

$f orm = 1 : r$

$f orj = 1 : e$

$for s1 = 1 : s1_{l}im$

$if rect(m, j) == 1 and (m + s1 * p1) > 0 and$

$(m + s1 * p1) <= theta_{r}ow and mod(s1, 2) == 1$

$rect(m + s1 * p1, j + s1 * e) = 0;$

$else if rect(m, j) == 0 and (m + s1 * p1) > 0$

$and (m + s1 * p1) <= theta_{r}ow and mod(s1, 2) == 1$

$rect(m + s1 * p1, j + s1 * e) = 1;$

$else if rect(m, j) == 1 and (m + s1 * p1) > 0$

$and (m + s1 * p1) <= theta_{r}ow and mod(s1, 2) == 0$

$rect(m + s1 * p1, j + s1 * e) = 1;$

$else if rect(m, j) == 0 and (m + s1 * p1) > 0$

$and (m + s1 * p1) <= theta_{r}ow and mod(s1, 2) == 0$

$rect(m + s1 * p1, j + s1 * e) = 0;$

end

end

end

end

$Draw_{s}hape(rect, theta_{r}ow, theta_{c}ol, color1, color2)$

$for m = r - p + 1 : r$

$for j = 1 : e$

$for s = 0 : 6$

$for s1 = 1 : 12$

$if rect(m, j) == 1 and (m + s * p1 + s1 * p) > 0$

$and (m + s * p1 + s1 * p) <= theta_{c}ol and mod(s, 2) == 0 and$

$mod(s1, 2) == 1$

$rect(m + s * p1 + s1 * p, j + s * e) = 0;$

$else if rect(m, j) == 0 and$

$(m+s*p1+s1*p) > 0and(m+s*p1+s1*p) <= theta_oland$

$mod(s,2) == 0andmod(s1,2) == 1$

$rect(m+s*p1+s1*p, j+s*e) = 1;$

$elseifrect(m, j) == 1and$

$(m+s*p1+s1*p) > 0and(m+s*p1+s1*p) <= theta_oland$

$mod(s,2) == 0andmod(s1,2) == 0$

$rect(m+s*p1+s1*p, j+s*e) = 1;$

$elseifrect(m, j) == 0and$

$(m+s*p1+s1*p) > 0and(m+s*p1+s1*p) <= theta_oland$

$mod(s,2) == 0andmod(s1,2) == 0$

$rect(m+s*p1+s1*p, j+s*e) = 0;$

$elseifrect(m, j) == 1and$

$(m+s*p1+s1*p) > 0and(m+s*p1+s1*p) <= theta_oland$

$mod(s,2) == 1andmod(s1,2) == 0$

$rect(m+s*p1+s1*p, j+s*e) = 0;$

$elseifrect(m, j) == 0and$

$(m+s*p1+s1*p) > 0and(m+s*p1+s1*p) <= theta_oland$

$mod(s,2) == 1andmod(s1,2) == 0$

$rect(m+s*p1+s1*p, j+s*e) = 1;$

$elseifrect(m, j) == 1and$

$(m+s*p1+s1*p) > 0and(m+s*p1+s1*p) <= theta_oland$

$mod(s,2) == 1andmod(s1,2) == 1$

$rect(m+s*p1+s1*p, j+s*e) = 1;$

$elseifrect(m, j) == 0and$

$(m+s*p1+s1*p) > 0and(m+s*p1+s1*p) <= theta_oland$

$mod(s,2) == 1andmod(s1,2) == 1$

$rect(m+s*p1+s1*p, j+s*e) = 0;$

end

end

end

end

end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$

$title({}^{I}Rectangleafter Mapping2^{I})$

end

$if k2 > k1$

$for m = 1 : r$

$for j = 1 : e$

$for s1 = 1 : s1_lim$

$if rect(m, j) == 1 and (m + s1 * p1) > 0 and (m + s1 * p1) <= theta_row$

$and mod(s1, 2) == 1$

$rect(m + s1 * p1, j + s1 * e) = 0;$

$else if rect(m, j) == 0 and (m + s1 * p1) > 0$

$and (m + s1 * p1) <= theta_row and mod(s1, 2) == 1$

$rect(m + s1 * p1, j + s1 * e) = 1;$

$else if rect(m, j) == 1 and (m + s1 * p1) > 0$

$and (m + s1 * p1) <= theta_row and mod(s1, 2) == 0$

$rect(m + s1 * p1, j + s1 * e) = 1;$

$else if rect(m, j) == 0 and (m + s1 * p1) > 0$

$and (m + s1 * p1) <= theta_row and mod(s1, 2) == 0$

$rect(m + s1 * p1, j + s1 * e) = 0;$

end

end

end

end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$

$form = r - p + 1 : r$

$forj = 1 : e$

$fors = 0 : 6$

$fors1 = 1 : 12$

$ifrect(m, j) == 1and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 0and$

$mod(s1, 2) == 1$

$rect(m + s * p1 + s1 * p, j + s * e) = 0;$

$elseifrect(m, j) == 0and$

$(m + s * p1 + s1 * p) > 0and(m + s * p1 + s1 * p) <= theta_coland$

$mod(s, 2) == 0andmod(s1, 2) == 1$

$rect(m + s * p1 + s1 * p, j + s * e) = 1;$

$elseifrect(m, j) == 1and$

$(m + s * p1 + s1 * p) > 0and(m + s * p1 + s1 * p) <= theta_coland$

$mod(s, 2) == 0andmod(s1, 2) == 0$

$rect(m + s * p1 + s1 * p, j + s * e) = 1;$

$elseifrect(m, j) == 0and$

$(m + s * p1 + s1 * p) > 0and(m + s * p1 + s1 * p) <= theta_coland$

$mod(s, 2) == 0andmod(s1, 2) == 0$

$rect(m + s * p1 + s1 * p, j + s * e) = 0;$

$elseifrect(m, j) == 1and$

$(m + s * p1 + s1 * p) > 0and(m + s * p1 + s1 * p) <= theta_coland$

$mod(s, 2) == 1andmod(s1, 2) == 0$

$rect(m + s * p1 + s1 * p, j + s * e) = 0;$

$elseifrect(m, j) == 0and$

$(m+s*p1+s1*p) > 0and(m+s*p1+s1*p) <= theta_coland$

$mod(s,2) == 1andmod(s1,2) == 0$

$rect(m+s*p1+s1*p, j+s*e) = 1;$

$elseifrect(m, j) == 1and$

$(m+s*p1+s1*p) > 0and(m+s*p1+s1*p) <= theta_coland$

$mod(s,2) == 1andmod(s1,2) == 1$

$rect(m+s*p1+s1*p, j+s*e) = 1;$

$elseifrect(m, j) == 0and$

$(m+s*p1+s1*p) > 0and(m+s*p1+s1*p) <= theta_coland$

$mod(s,2) == 1andmod(s1,2) == 1$

$rect(m+s*p1+s1*p, j+s*e) = 0;$

end

end

end

end

end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$

$title({}^{I}RectangleafterMapping2^{I})$

$form = 1 : d1$

$forj = 1 : e$

$fors = 0 : 6$

$fors2 = 1 : 12$

$ifrect(m, j) == 1and(m+s*p1-s2*p) > 0and(m+s*p1-s2*p) <= theta_coland$

$mod(s,2) == 0andmod(s2,2) == 1rect(m+s*p1-s2*p, j+s*e) = 0;$

$elseifrect(m, j) == 0and(m+s*p1-s2*p) > 0and$

$(m+s*p1-s2*p) <= theta_colandmod(s,2) == 0andmod(s2,2) == 1$

$rect(m+s*p1-s2*p, j+s*e) = 1;$

$elseifrect(m, j) == 1and(m + s * p1 - s2 * p) > 0and$

$(m + s * p1 - s2 * p) <= theta_{c}olandmod(s, 2) == 0andmod(s2, 2) == 0$

$rect(m + s * p1 - s2 * p, j + s * e) = 1;$

$elseifrect(m, j) == 0and(m + s * p1 - s2 * p) > 0and$

$(m + s * p1 - s2 * p) <= theta_{c}olandmod(s, 2) == 0andmod(s2, 2) == 0$

$rect(m + s * p1 - s2 * p, j + s * e) = 0;$

$elseifrect(m, j) == 1and(m + s * p1 - s2 * p) > 0and$

$(m + s * p1 - s2 * p) <= theta_{c}olandmod(s, 2) == 1andmod(s2, 2) == 0$

$rect(m + s * p1 - s2 * p, j + s * e) = 0;$

$elseifrect(m, j) == 0and(m + s * p1 - s2 * p) > 0and$

$(m + s * p1 - s2 * p) <= theta_{c}olandmod(s, 2) == 1andmod(s2, 2) == 0$

$rect(m + s * p1 - s2 * p, j + s * e) = 1;$

$elseifrect(m, j) == 1and(m + s * p1 - s2 * p) > 0and$

$(m + s * p1 - s2 * p) <= theta_{c}olandmod(s, 2) == 1andmod(s2, 2) == 1$

$rect(m + s * p1 - s2 * p, j + s * e) = 1;$

$elseifrect(m, j) == 0and(m + s * p1 - s2 * p) > 0and$

$(m + s * p1 - s2 * p) <= theta_{c}olandmod(s, 2) == 1andmod(s2, 2) == 1$

$rect(m + s * p1 - s2 * p, j + s * e) = 0;$

end

end

end

end

end

$Draw_{s}hape(rect, theta_{r}ow, theta_{c}ol, color1, color2)$

$title(^{I}RectangleafterMapping2^{I})$

end

$functionDraw_{s}hape(M, a, b, color1, color2)$

$figure$

$axis([0b0a])$

$holdon$

$fori = 1 : a$

$forj = 1 : b$

$ifM(i, j) == 1$

$rectangle(^IPosition^I, [j-1, a-i, 1, 1], ^IFaceColor^I, color1);$

$elseifM(i, j) == 0$

$rectangle(^IPosition^I, [j-1, a-i, 1, 1], ^IFaceColor^I, color2);$

end

end

end

end

end

APPENDIX D

GENERATING N_C AND N_R NESTED CHAIN ABACUS

Code A

```
r=input('Enter a number of rows:');
e=input('Enter a number of columns:');
theta_row = input('Enteranumberofcolmnsofonutputrectangle(r < theta_row) :')
theta_col = input('Enteranumberofcolmnsofonutputrectangle(e < theta_col) :');
d1 = input('Enterthesamenumberofbeadposition :');
con = Find_connected_abacus(e, r−d1 + 1, r, e, d1, 'col'); num_initialrect = size(con, 1);
if num_initialrect == 0
error('Error : Thereisnoconnectedabacusefortheseparameters.')
end
temp = con(randi([1, num_initialrect], 1), :)
rect = zeros(theta_row, theta_col);
for i = 1 : e
rect(temp(i)/2 + 1 : temp(i)/2 + d1, i) = 1;
end
s_lim = ceil(theta_row/r) − 1;
s1_lim = ceil(theta_col/e);
s2_lim = ceil(theta_col/e);
s_color1 = [56, 85, 34]/255;
s_color2 = [156, 194, 228]/255;
k = min(temp);
k1 = temp(1, 1)
k2 = temp(1, end)
p = d1;
p1 = k2 − k1;
```

```
rect = zeros(theta_row,theta_col);
fori = 1 : e
rect(temp(i) : temp(i)+ d1 −1, i) = 1;
end
Draw_shape(rect,theta_row,theta_col,color1,color2)
title('InitialRectangle')


if k2>k1
for m=1:r
for j=1:e
for s1=1:s1_lim
ifrect(m, j) == 1and(m + s1 ∗ p1) > 0and(m + s1 ∗ p1) <= theta_row
andmod(s1, 2) == 1
rect(m + s1 ∗ p1, j + s1 ∗ e) = 0;
elseifrect(m, j) == 0and(m + s1 ∗ p1) > 0
and(m + s1 ∗ p1) <= theta_rowandmod(s1, 2) == 1
rect(m + s1 ∗ p1, j + s1 ∗ e) = 1;
elseifrect(m, j) == 1and(m + s1 ∗ p1) > 0
and(m + s1 ∗ p1) <= theta_rowandmod(s1, 2) == 0
rect(m + s1 ∗ p1, j + s1 ∗ e) = 1;
elseifrect(m, j) == 0and(m + s1 ∗ p1) > 0
and(m + s1 ∗ p1) <= theta_rowandmod(s1, 2) == 0
rect(m + s1 ∗ p1, j + s1 ∗ e) = 0;
end
end
end
```

end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$

$form = r - p + 1 : r$

$forj = 1 : e$

$fors = 0 : 6$

$fors1 = 1 : 12$

$ifrect(m, j) == 1and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 0and$

$mod(s1, 2) == 1$

$rect(m + s * p1 + s1 * p, j + s * e) = 0;$

$elseifrect(m, j) == 0and$

$(m + s * p1 + s1 * p) > 0and(m + s * p1 + s1 * p) <= theta_coland$

$mod(s, 2) == 0andmod(s1, 2) == 1$

$rect(m + s * p1 + s1 * p, j + s * e) = 1;$

$elseifrect(m, j) == 1and$

$(m + s * p1 + s1 * p) > 0and(m + s * p1 + s1 * p) <= theta_coland$

$mod(s, 2) == 0andmod(s1, 2) == 0$

$rect(m + s * p1 + s1 * p, j + s * e) = 1;$

$elseifrect(m, j) == 0and$

$(m + s * p1 + s1 * p) > 0and(m + s * p1 + s1 * p) <= theta_coland$

$mod(s, 2) == 0andmod(s1, 2) == 0$

$rect(m + s * p1 + s1 * p, j + s * e) = 0;$

$elseifrect(m, j) == 1and$

$(m + s * p1 + s1 * p) > 0and(m + s * p1 + s1 * p) <= theta_coland$

$mod(s, 2) == 1andmod(s1, 2) == 0$

$rect(m + s * p1 + s1 * p, j + s * e) = 0;$

$elseifrect(m, j) == 0and$

$(m+s*p1+s1*p)>0and(m+s*p1+s1*p)<=theta_coland$

$mod(s,2)==1andmod(s1,2)==0$

$rect(m+s*p1+s1*p,j+s*e)=1;$

$elseifrect(m,j)==1and$

$(m+s*p1+s1*p)>0and(m+s*p1+s1*p)<=theta_coland$

$mod(s,2)==1andmod(s1,2)==1$

$rect(m+s*p1+s1*p,j+s*e)=1;$

$elseifrect(m,j)==0and$

$(m+s*p1+s1*p)>0and(m+s*p1+s1*p)<=theta_coland$

$mod(s,2)==1andmod(s1,2)==1$

$rect(m+s*p1+s1*p,j+s*e)=0;$

end

end

end

end

end

$Draw_shape(rect,theta_row,theta_col,color1,color2)$

$title({}^{I}RectangleafterMapping2^{I})$

$form=1:d1$

$forj=1:e$

$fors=0:6$

$fors2=1:12$

$ifrect(m,j)==1and(m+s*p1-s2*p)>0and(m+s*p1-s2*p)<=theta_coland$

$mod(s,2)==0andmod(s2,2)==1rect(m+s*p1-s2*p,j+s*e)=0;$

$elseifrect(m,j)==0and(m+s*p1-s2*p)>0and$

$(m+s*p1-s2*p)<=theta_colandmod(s,2)==0andmod(s2,2)==1$

$rect(m+s*p1-s2*p,j+s*e)=1;$

$elseif rect(m, j) == 1 and (m + s * p1 - s2 * p) > 0 and$

$(m + s * p1 - s2 * p) <= theta_c ol and mod(s, 2) == 0 and mod(s2, 2) == 0$

$rect(m + s * p1 - s2 * p, j + s * e) = 1;$

$elseif rect(m, j) == 0 and (m + s * p1 - s2 * p) > 0 and$

$(m + s * p1 - s2 * p) <= theta_c ol and mod(s, 2) == 0 and mod(s2, 2) == 0$

$rect(m + s * p1 - s2 * p, j + s * e) = 0;$

$elseif rect(m, j) == 1 and (m + s * p1 - s2 * p) > 0 and$

$(m + s * p1 - s2 * p) <= theta_c ol and mod(s, 2) == 1 and mod(s2, 2) == 0$

$rect(m + s * p1 - s2 * p, j + s * e) = 0;$

$elseif rect(m, j) == 0 and (m + s * p1 - s2 * p) > 0 and$

$(m + s * p1 - s2 * p) <= theta_c ol and mod(s, 2) == 1 and mod(s2, 2) == 0$

$rect(m + s * p1 - s2 * p, j + s * e) = 1;$

$elseif rect(m, j) == 1 and (m + s * p1 - s2 * p) > 0 and$

$(m + s * p1 - s2 * p) <= theta_c ol and mod(s, 2) == 1 and mod(s2, 2) == 1$

$rect(m + s * p1 - s2 * p, j + s * e) = 1;$

$elseif rect(m, j) == 0 and (m + s * p1 - s2 * p) > 0 and$

$(m + s * p1 - s2 * p) <= theta_c ol and mod(s, 2) == 1 and mod(s2, 2) == 1$

$rect(m + s * p1 - s2 * p, j + s * e) = 0;$

end

end

end

end

end

$Draw_s hape(rect, theta_r ow, theta_c ol, color1, color2)$

$title(^I Rectangle after Mapping2^I)$

end

$function Draw_s hape(M, a, b, color1, color2)$

$figure$

$axis([0b0a])$

$holdon$

$fori = 1 : a$

$forj = 1 : b$

$ifM(i, j) == 1$

$rectangle({}^IPosition^I, [j-1, a-i, 1, 1], {}^IFaceColor^I, color1);$

$elseifM(i, j) == 0$

$rectangle({}^IPosition^I, [j-1, a-i, 1, 1], {}^IFaceColor^I, color2);$

end

end

end

end

end

Code B

r=input('Enter a number of rows:');

e=input('Enter a number of columns:');

theta$_r ow = input({}^IEnteranumberofcolmnsofonutputrectangle(r < theta_r ow) :^I);$

$theta_c ol = input({}^IEnteranumberofcolmnsofonutputrectangle(e < theta_c ol) :^I);$

$H1 = input({}^IEnterthesamenumberofbeadposition :^I);$

$con = Find_c onnected_a bacus(r, e-H1+1, r, e, H1, {}^Irow^I);$

$num_i nitialrect = size(con, 1);$

$ifnum_i nitialrect == 0$

$error({}^IError : Thereisnoconnectedabacusefortheseparameters.^I)$

end

$temp = con(randi([1, num_initialrect], 1), :);$

$rect = zeros(theta_row, theta_col);$

$for i = 1 : r$

$rect(i, temp(i)/2 + 1 : temp(i)/2 + H1) = 1;$

end

$s_lim = ceil(theta_row/r) - 1;$

$s1_lim = ceil(theta_col/e);$

$s2_lim = ceil(theta_col/e);$

$s_color1 = [131, 59, 10]/255;$

$s_color2 = [255, 217, 102]/255;$

$H = min(temp);$

$L = temp(1, 1);$

$L1 = temp(1, end);$

$p = e - H + 1;$

$p1 = L - L1;$

$figure$

$Draw_shape(rect, s_color1, s_color2)$

$title('InitialRectangle')$

$\cdot IfL1 <= L, applicatethemapping1andmpping1$

$if L1 > L$

$for m = 1 : r$

$for j = 1 : e$

$for s = 1 : s_lim$

$if rect(m, j) == 1 and (j + s * p1) > 0 and (j + s * p1) <= theta_col and$

$mod(s, 2) == 1 rect(m + s * r, j + s * p1) = 0;$

$elseif rect(m, j) == 0 and (j + s * p1) > 0 and (j + s * p1) <= theta_col and$

$mod(s, 2) == 1$

$rect(m + s * r, j + s * p1) = 1;$

$elseifrect(m, j) == 1and(j + s * p1) > 0and(j + s * p1) <= theta_coland$

$mod(s, 2) == 0$

$rect(m + s * r, j + s * p1) = 1;$

$elseifrect(m, j) == 0and(j + s * p1) > 0and(j + s * p1) <= theta_coland$

$mod(s, 2) == 0$

$rect(m + s * r, j + s * p1) = 0;$

end

end

end

end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$

$title(^IRectangleafterMapping1^I)$

$form = 1 : r$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseifrect(m, j) == 0and(m + s * p1 + s1 * p) > 0and$

$(m + s * p1 + s1 * p) <= theta_colandmod(s, 2) == 1andmod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

end

end

end

end

end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$

$title(^IRectangleafterMapping2^I)$

end

$ifL1 > L$

$for m = 1 : r$

$for j = 1 : e$

$for s = 1 : s_l im$

$if rect(m, j) == 1 and (j + s * p1) > 0 and (j + s * p1) <= theta_c oland$

$mod(s, 2) == 1$

$rect(m + s * r, j + s * p1) = 0;$

$elseif rect(m, j) == 0 and (j + s * p1) > 0 and (j + s * p1) <= theta_c oland$

$mod(s, 2) == 1$

$rect(m + s * r, j + s * p1) = 1;$

$elseif rect(m, j) == 1 and (j + s * p1) > 0 and (j + s * p1) <= theta_c oland$

$mod(s, 2) == 0$

$rect(m + s * r, j + s * p1) = 1;$

$elseif rect(m, j) == 0 and (j + s * p1) > 0 and (j + s * p1) <= theta_c oland$

$mod(s, 2) == 0$

$rect(m + s * r, j + s * p1) = 0;$

end

end

end

end

$Draw_s hape(rect, theta_r ow, theta_c ol, color1, color2)$

$title({}^I Rectangle after Mapping1^I)$

$for m = 1 : r$

$for j = H : e$

$for s = 0 : s_l im$

$for s1 = 1 : s1_l im$

$if rect(m, j) == 1 and (m + s * p1 + s1 * p) > 0 and (m + s * p1 + s1 * p) <= theta_c ol$

$and mod(s, 2) == 0 and mod(s1, 2) == 1$

$rect(m+s*r, j+s*p1+s1*p) = 0;$

$elseif rect(m, j) == 0 and (m+s*p1+s1*p) > 0 and$

$(m+s*p1+s1*p) <= theta_col and mod(s, 2) == 0 and mod(s1, 2) == 1$

$rect(m+s*r, j+s*p1+s1*p) = 1;$

$elseif rect(m, j) == 1 and (m+s*p1+s1*p) > 0 and$

$(m+s*p1+s1*p) <= theta_col and mod(s, 2) == 0 and mod(s1, 2) == 0$

$rect(m+s*r, j+s*p1+s1*p) = 1;$

$elseif rect(m, j) == 0 and (m+s*p1+s1*p) > 0 and$

$(m+s*p1+s1*p) <= theta_col and mod(s, 2) == 0 and mod(s1, 2) == 0$

$rect(m+s*r, j+s*p1+s1*p) = 0;$

$elseif rect(m, j) == 1 and (m+s*p1+s1*p) > 0 and$

$(m+s*p1+s1*p) <= theta_col and mod(s, 2) == 1 and mod(s1, 2) == 0$

$rect(m+s*r, j+s*p1+s1*p) = 0;$

$elseif rect(m, j) == 0 and (m+s*p1+s1*p) > 0 and$

$(m+s*p1+s1*p) <= theta_col and mod(s, 2) == 1 and mod(s1, 2) == 0$

$rect(m+s*r, j+s*p1+s1*p) = 1;$

$elseif rect(m, j) == 1 and (m+s*p1+s1*p) > 0 and$

$(m+s*p1+s1*p) <= theta_col and mod(s, 2) == 1 and mod(s1, 2) == 1$

$rect(m+s*r, j+s*p1+s1*p) = 1;$

$elseif rect(m, j) == 0 and (m+s*p1+s1*p) > 0 and$

$(m+s*p1+s1*p) <= theta_col and mod(s, 2) == 1 and mod(s1, 2) == 1$

$rect(m+s*r, j+s*p1+s1*p) = 0;$

end

end

end

end

end

$Draw_shape(rect, theta_row, theta_col, color1, color2)$

$title('Rectangleafter Mapping2')$

$for m = 1 : r$

$for j = 1 : H1$

$for s = 1 : s1_lim$

$for s2 = 1 : s2_lim$

$for j = H : e$

$for s = 0 : s_lim$

$for s1 = 1 : s1_lim$

$if rect(m, j) == 1 and (m + s * p1 + s1 * p) > 0 and (m + s * p1 + s1 * p) <= theta_col$

$and mod(s, 2) == 0 and mod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

$elseif rect(m, j) == 0 and (m + s * p1 + s1 * p) > 0 and$

$(m + s * p1 + s1 * p) <= theta_col and mod(s, 2) == 0 and mod(s1, 2) == 1$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseif rect(m, j) == 1 and (m + s * p1 + s1 * p) > 0 and$

$(m + s * p1 + s1 * p) <= theta_col and mod(s, 2) == 0 and mod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseif rect(m, j) == 0 and (m + s * p1 + s1 * p) > 0 and$

$(m + s * p1 + s1 * p) <= theta_col and mod(s, 2) == 0 and mod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

$elseif rect(m, j) == 1 and (m + s * p1 + s1 * p) > 0 and$

$(m + s * p1 + s1 * p) <= theta_col and mod(s, 2) == 1 and mod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 0;$

$elseif rect(m, j) == 0 and (m + s * p1 + s1 * p) > 0 and$

$(m + s * p1 + s1 * p) <= theta_col and mod(s, 2) == 1 and mod(s1, 2) == 0$

$rect(m + s * r, j + s * p1 + s1 * p) = 1;$

$elseif rect(m, j) == 1 and (m+s*p1+s1*p) > 0 and (m+s*p1+s1*p) <=$

$1 and mod(s1, 2) == 1$

$rect(m + s*r, j + s*p1 + s1*p) = 1;$

$elseif rect(m, j) == 0 and (m + s*p1 + s1*p) > 0 and$

$(m + s*p1 + s1*p) <= theta_c ol and mod(s, 2) == 1 and mod(s1, 2) == 1$

$rect(m + s*r, j + s*p1 + s1*p) = 0;$

end

end

end

end

end

$Draw_s hape(rect, theta_r ow, theta_c ol, color1, color2)$

$title({}^I Rectangle after Mapping2^I)$

$for m = 1 : r$

$for j = 1 : H1$

$for s = 1 : s1_l im$

$for s2 = 1 : s2_l im$

$function Draw_s hape(M, a, b, color1, color2)$

$figure$

$axis([0 b 0 a])$

$hold on$

$for i = 1 : a$

$for j = 1 : b$

$if M(i, j) == 1$

$rectangle({}^I Position^I, [j-1, a-i, 1, 1], {}^I FaceColor^I, color1); elseif M(i, j) == 0$

$rectangle({}^I Position^I, [j-1, a-i, 1, 1], {}^I FaceColor^I, color2);$

end

end

end

end

end

Code C

function Draw$_{s}hape(M, a, b, color1, color2)$

$figure$

$axis([0b0a])$

$holdon$

$fori = 1 : a$

$forj = 1 : b$

$ifM(i, j) == 1$

$rectangle({}^{I}Position^{I}, [j-1, a-i, 1, 1], {}^{I}FaceColor^{I}, color1);$

$elseifM(i, j) == 0$

$rectangle({}^{I}Position^{I}, [j-1, a-i, 1, 1], {}^{I}FaceColor^{I}, color2);$

end

end

end

end

end

** Find all Initial rect

function com=create$_c ombination(n, k)$

$f ori = 1 : n$

$tmp = [];$

$f or j = 1 : k$

$tmp = [tmp; ones(k^(n-i), 1) * j];$

end

$rr = [];$

$f or j = 1 : k^(i-1)$

$rr = [rr; tmp];$

end

$com(:, i) = rr;$

end

$com = com - 1$

end

Code D

** Find the connected Abacus

function con=Find$_c onnected_a bacus(n, k, r, e, H, str)$

$con = [];$

$\cdot Findallpossibleinitialrects$

$all = create_c ombination(n, k);$

$\cdot *Findtheconnectedabacus$

$\cdot Algorithmforcolumn$

$ifstrcmp(str,^I col^I)$

$f ornn = 1 : size(all, 1)$

$temp = all(nn, :);$

$flag = 1;$

$rectisconnectedabacus)$

$rect = zeros(r, e);$

$for i = 1 : e$

$rect(temp(i) + 1 : temp(i) + H, i) = 1;$

end

$\cdot * Testtheconnection$

$t = sum(rect^I);$

$for i = 1 : r$

$flag = flag * (t(1, i) >= 1);$

end

$for i = 1 : e - 1$

$A = temp(i) : temp(i) + H - 1;$

$B = temp(i + 1) : temp(i + 1) + H - 1;$

$C = intersect(A, B);$

$flag = flag * (isempty(C));$

end

$if flag == 1$

$con = [con; temp];$

end

APPENDIX E

CHAIN TRANSFORMATION

File Number one

```
clc
kkk=0;
clear;
nt=0;
v=1;
global Tmat;
r = input(' numbers of rows ')
c = input(' numbers of columns ')
mat = ones(r,c);
Tmat=mat;
r,c
= size(mat) ;
ch=21;
path=0;
tn=0;
fl=[r,c] ;
tmpv=min(fl)/2;
tpn1=ceil(tmpv)
while path <tpn1
path=path+1
v=v-.15
si=path;sj=path;
```

```
i=si;j=sj;
vv=path;
n=0;
sti=2;
stj=2;
str=('r-si-1');
nr=r-si;
nc=c-sj;
vs=1/ch;
vs=0;
nt=nt+1;
while 1
if (i==si ) (j==sj )( n>0)
break
end
n=n+1;
if ( tn>=totn)
break
end
mat(i,j)=v;
tn=tn+1;
pt(tn,4)=j;
pt(tn,3)=i;
pt(tn,1)=tn;
pt(tn,2)=path;
if ((i<nr+1)(sti>0))
i=i+1;
```

```
else if((j<nc+1)(stj>0));
j=j+1;
sti=-1;
else
if (i>si)
i=i-1;
stj=-1;
else if (j>sj)
j=j-1;
if ((i==si)(j==sj)) sti=1;stj=1;
end
end
end
end
end
i;
j;
A1 = path;
A2 = n;
end
end
s=size(pt);
c=s(1);
cp=1
i=1;
clc
k=1;
```

```
pathStart(1)=1;
for i=1:s
if (i>1)
if pt(i,2) =pt(i-1,2) k=k+1;
pathStart(k)=i;
end
end
end
k=k+1;
pathStart(k)=s(1)
k(1:tpn1)=0;
clc
p01=1
No=0;
pt(:,5)=1;
totalN ss
=size(pt)
tpn=max(pt(:,2))
for i=1:tpn
i
pn(i)=input('ÚÏÏ ÇáÚäÇÕÑ ááãÓÇÑ ')
end
for p=1:length(pn)
for i=1:totalN
if(pt(i,2)==p)(k(p)<pn(p)) pt(i,5)=11;
k(p)=k(p)+1;
end
```

```
ii=pt(i,3);
jj=pt(i,4);
mat(ii,jj)= pt(i,5);
end
end
Ax=0
ii=0;
jj=0;
**ccc=length(x1)
Ax(1:pathStart(2)-1,1:5,1:3) =-1
k=0
for i=1:tpn
i1=pathStart(i)
if i<s(1)
end
if (i==tpn)
i2=pathStart(i+1)
else
i2=pathStart(i+1)-1
end
* x1=pt(pathStart(1):pathStart(2)-1,:)
x=pt(i1:i2,:)
k=k+1;
rrr=size(x)
pL(k)=rrr(1)
Ax(1:pL(k),:,i)=x;
* Ax(1:length(x),:,i)=x;
```

```
end
Ax(1:pL(i),:,i)
clc
** x1=Ax(1:pL(1),:,1);
** x2=Ax(1:pL(2),:,2);
** x3=Ax(1:pL(3),:,3);
s=0
N=0
mTem=0
i=1
global Nx1
Nx1=0;
LoopF(i,Ax,pL,mat)
F3
```

File Number Two

```
global Nx1
global Tmat
for t=1:sxv
ii=xt(t,3);
jj=xt(t,4);
tt=xt(t,5)
mat(ii,jj)= tt;
end
```

```
mat

r,c

= size(mat) ;

if  (isequal(mTem,mat))

dlmwrite('Rtxt',mat,'-append','delimiter',' ','roffset',1);

****************************************************************

imagesc((1:c)+0.5,(1:r)+0.5,mat);

colormap(winter);

axis equal ;

N=N+1 ;

set(gca,'XTick',1:(c),'YTick',1:(r),...

'XLim',[1 c+1],'YLim',[1 r+1],...

'GridLineStyle','-','XGrid','on','YGrid','on');

rndd1 = 1

rndd2 = 1

Nx1=Nx1+1

Tmat(:,:,Nx1)=mat;

s=sprintf('000

saveas(gcf,s);

********************************************************************* end

mTem=mat;
```

File Number Three

```
a b w

=size(Tmat)

Tmat2=Tmat(:,:,:)
```

```
Tmat3=Tmat(:,:,1)
k=0;
m1=Tmat(:,:,1);
m2=Tmat2(:,:,1);
kk=1
for i=1:w
t=1
t=0
for j=i+1:w
kk=kk+1
m1=Tmat(:,:,i);
m2=Tmat2(:,:,j);
if Tmat(:,:,i)== Tmat2(:,:,j)
t=1;
end
if k==150
nnn=2
end
end
if (t==0)
k=k+1
Tmat3(:,:,k)= Tmat(:,:,i)
```

File Number Four

```
function it=LoopF(i,Ax,pL,mat)

global Nx1;

Nx1=Nx1*1

LpL=length(pL)

if i>LpL

return;

end

pLt=pL(i);

xt=Ax(1:pLt,:,i);

sx(i)=size(xt,1);

sxv=sx(i)

mTem=0;

N=0;

clc

for j=1:sxv

sh=1;

Y1 = circshift(xt(:,5),sh);

xt(:,5)=Y1;

F2

Ax(1:pL(i),5,i)=Y1;

*************************************************************** tt=i+1

Ax2=Ax;pL2=pL;mat2=mat;

LoopF(tt,Ax2,pL2,mat2)

end

end
```

APPENDIX F

GENERATING FUNCTION

```
clc;clear all;close all;

key=3;

tmp=key;

fprintf('==================================')

for n=1:7

tmp1=[];

for i=1:size(tmp,2)

tmp1=[tmp1,ones(1,tmp(i)-1)*tmp(i),tmp(i)+2];

end

tmp=[];tmp=tmp1;

fn=size(tmp,2);

fprintf('f

(end
```