NEW ALTERNATIVE STATISTIC FOR TESTING SEVERAL INDEPENDENT SAMPLES OF CORRELATION MATRICES IN HIGH DIMENSION DATA

TAREQ A. M. ATIANY

DOCTOR OF PHILOSOPHY
UNIVERSITI UTARA MALAYSIA
2018

Awang Had Salleh
Graduate School
of Arts And Sciences

Universiti Utara Malaysia

PERAKUAN KERJA TESIS / DISERTASI
(Certification of thesis / dissertation)

Kami, yang bertandatangan, memperakukan bahawa
(We, the undersigned, certify that)

TAREQ A.M. ATIANY

calon untuk Ijazah PhD
(candidate for the degree of)

telah mengemukakan tesis / disertasi yang bertajuk:
(has presented his/her thesis / dissertation of the following title):

"NEW ALTERNATIVE STATISTIC FOR TESTING SEVERAL INDEPENDENT SAMPLES OF CORRELATION MATRICES IN HIGH DIMENSION DATA"

seperti yang tercatat di muka surat tajuk dan kulit tesis / disertasi.
(as it appears on the title page and front cover of the thesis / dissertation).

Bahawa tesis/disertasi tersebut boleh diterima dari segi bentuk serta kandungan dan meliputi bidang ilmu dengan memuaskan, sebagaimana yang ditunjukkan oleh calon dalam ujian lisan yang diadakan pada : ***20 Ogos 2017.***
That the said thesis/dissertation is acceptable in form and content and displays a satisfactory knowledge of the field of study as demonstrated by the candidate through an oral examination held on:
August 20, 2017.

Pengerusi Viva: *(Chairman for VIVA)*	**Prof. Dr. Engku Muhammad Nazri Engku Abu Bakar**	Tandatangan *(Signature)*
Pemeriksa Luar: *(External Examiner)*	**Dr. Ani Shabri**	Tandatangan *(Signature)*
Pemeriksa Dalam: *(Internal Examiner)*	**Prof. Dr. Sharipah Soaad Syed Yahaya**	Tandatangan *(Signature)*
Nama Penyelia/Penyelia-penyelia: *(Name of Supervisor/Supervisors)*	**Dr. Shamshuritawati Sharif**	Tandatangan *(Signature)*

Tarikh:
(Date) ***August 20, 2017***

Permission to Use

In presenting this thesis in fulfilment of the requirements for a postgraduate degree from Universiti Utara Malaysia, I agree that the Universiti Library may make it freely available for inspection. I further agree that permission for the copying of this thesis in any manner, in whole or in part, for scholarly purpose may be granted by my supervisor(s) or, in their absence, by the Dean of Awang Had Salleh Graduate School of Arts and Sciences. It is understood that any copying or publication or use of this thesis or parts thereof for financial gain shall not be allowed without my written permission. It is also understood that due recognition shall be given to me and to Universiti Utara Malaysia for any scholarly use which may be made of any material from my thesis.

Requests for permission to copy or to make other use of materials in this thesis, in whole or in part, should be addressed to :

Dean of Awang Had Salleh Graduate School of Arts and Sciences
UUM College of Arts and Sciences
Universiti Utara Malaysia
06010 UUM Sintok

Abstrak

Statistik Jennrich adalah salah satu statistik sedia ada yang digunakan untuk menguji kesamaan bagi beberapa sampel bebas matriks korelasi. Statistik tersebut semakin mendapat perhatian dalam beberapa bidang ekonomi dan pasaran kewangan.Dalam bidang penyelidikan ini, kebiasaannya bilangan pemboleh ubah, *p*, lebih besar daripada saiz sampel, *n*, yang dikenali sebagai data berdimensi tinggi. Selanjutnya, penganggaran pekali penentu bagi matrik korelasi dan kovarians akan mengalami kegagalan akibat daripada masalah kesingularan. Apabila ini berlaku, statistik Jennrich tidak akan berfungsi kerana pengiraannya melibatkan songsangan matrik korelasi. Oleh itu, bagi mengatasi kelemahan tersebut, kajian ini membangunkan satu statistik alternatif untuk menguji kesamaan matriks korelasi bagi beberapa sampel bebas dalam data berdimensi tinggi. Untuk tujuan itu, pendekatan aljabar berdasarkan operator *vec*, matriks komutasi dan elemen norma Frobenius keluar-pepenjuru-atas digunakan untuk menerbitkan taburan asimptotik baharu bagi statistik alternatif, yang dikenali sebagai statistik Z^*. Kajian simulasi dilakukan dengan mengambil kira bilangan pembolehubah, saiz sampel, dan anjakan korelasi yang berbeza untuk menilai prestasi statistik baharu. Sebagai tambahan, data sebenar bagi struktur mata wang Asia Pasifik semasa gempa bumi Tohoku digunakan untuk mengesahkan statistik Z^* baharu. Kuasa bagi statistik Z^* dibandingkan dengan statistik Jennrich dan juga statistik T^* yang sedia ada melalui kajian simulasi. Hasilnya, kuasa statistik Z^* mendominasi kuasa statistik Jennrich dan statistik T^* dalam semua keadaan, terutamanya, apabila perubahan dalam matrik korelasi adalah sekurang-kurangnya 0.3. Kesimpulannya, hasil dari segi teori ataupun simulasi menunjukkan keputusan yang mantap disokong oleh kuasa ujian yang diperlukan. Manakala, kajian terhadap data sebenar menunjukkan statistik alternatif baharu boleh memenuhi kondisi data berdimensi tinggi.

Kata kunci:. Matrik korelasi, Operator *vec*, Matrik komutasi, Norma Frobenius.

Abstract

Jennrich Jennrich statistic is one of the existing statistics which is used for testing the equality of several independent samples of correlation matrices. The statistic is gaining considerable importance in several areas of economics and financial markets. In these research areas, the number of variables, *p*, is usually larger than the sample size, *n*, which is known as high dimension data $p > n$. Subsequently, the estimation of correlation and covariance determinant will breakdown due to singularity problem. When this happens, Jennrich statistic is unable to function as the calculation involves the inversion of correlation matrix. Therefore, to resolve the aforementioned problem, this study develops an alternative statistic for testing several independent samples of correlation matrices in high dimension data. For this reason, the algebraic approach on the basis of *vec* operator, commutation matrix and Frobenius norm of upper-off-diagonal elements are used to derive the new asymptotic distribution for the new alternative statistic, namely Z^* statistic. Simulation study was conducted by considering different number of variables, sample sizes, and correlation shifts to evaluate the performance of the new statistic. In addition, real data on Asia Pacific currencies structure during the Tohoku earthquake were applied to validate the new Z^* statistic. The power of the Z^* statistic is compared with the existing Jennrich statistic, and T^* statistic through simulation study. As a result, the power of Z^* statistic dominates the power of Jennrich statistic and T^* statistic in all conditions, especially, when the shift in correlation matrix is at least 0.3 As a conclusion, the theoretical and simulation results are established and supported by desirable power of test. Meanwhile, investigation on real data indicates that the new alternative statistic can accommodate high dimension data.

Keywords: Correlation matrix, *Vec* operator, Commutation matrix, Frobenius norm.

Acknowledgement

I am grateful to the Almighty Allah the most beneficent, and most merciful, for giving me the strength to pursue this academic thesis to a successful conclusion.

Since I started my Ph.D. journey through these years I have been very fortunate to encouragement, receive guidance, patience and trust from my mentors, colleagues, friends and family. This work could not have been possible without their support. First, I would like to express my most sincere gratitude to my advisor Dr. Shamshuritawati Sharif for her invaluable support and guidance. I have always been admiring her vast knowledge, deep insight and passion to research. Her trust of my abilities has provided me complete academic freedom to pursue my research interest and to develop my professional skills. Her encouragement has given me invaluable opportunities to interact with world-class scholars. I also would like to thank Associate Professor Sharipah Soaad the dean of school of quantitative science. Further, I would like to thank my dear parents, my brothers, my sisters, my wife and my son Ahmed and my daughter Toqa for their constant love and firm support throughout the years, which provides a never ending source of energy to my doctoral studies and future endeavors. Finally, I wish to express my sincere gratitude to all my friends for assistance provided during this journey.

Table of Contents

Permission to Use ii

Acknowledgement v

Table of Contents vi

List of Tables ix

List of Figures x

List of Appendices xi

List of Symbols xii

List of Tables

List of Figures

List of Appendices

List of Symbols

$\otimes$	Kronecker product
I_p	Identity matrix
∂	Partial differentiation opretor
r	Sample correlation cofficient
R	Sample correlation matrix
ρ	Population correlation cofficient
Ω	Population correlation matrix
K_{pp}	Commutation matrix
$(K_{p,p})_d$	Diagonal elements of commutation matrix
R_p	Pooled of sample correlation matrix
R_U	Upper off diagonal of sample correlation matrix
T	Transformation matrix
Σ	Population covariance matrix
S	Sample covariance matrix
S_d	Diagonal elements of sample covariance matrix
Σ_d	Diagonal elements of population covariance matrix
σ^2	Population variance
$v(R)$	Vectorization of R
$v(R_U)$	Vectorization of R_U
Tr	Trace
$\|\| \|\|$	Norm
$\xrightarrow{d}$	Convergence in distribution
$\|v(R)\|^2$	Squared Frobenius norm of sample correlation matrix
$\|v(\Omega)\|^2$	Squared Frobenius norm of population correlation matrix

$\left\|v(R_U)\right\|^2$	Squared Frobenius norm of upper-off-diagonal elements of sample correlation matrix
$\left\|v(\Omega_U)\right\|^2$	Squared Frobenius norm of upper-off-diagonal elements of population correlation matrix
$\xrightarrow{p}$	Convergence in probability

CHAPTER ONE

INTRODUCTION

1.1 Background of Study

Recently, testing the equality of several correlation matrices has become an important subject in the economic and financial industry research areas. Various researchers have implemented this test for diagnosing the structure of several independent samples of correlation matrices.

Applications have been found in equity markets, asset businesses, stock markets, real estate analysis, risk management, portfolio analysis and financial markets. However, when dealing with a large number of variables in real cases, understanding all their interrelationships simultaneously is a difficult job. Therefore, the correlation structure analysis it becomes very important when dealing with related variables where the number of variables is large.

Several researchers have studied the equality of several correlation matrices, including, for example, Cho and Taylor (1987), Tang (1995), Meric and Meric (1997), Lee (1998), Tang (1998) and Da Costa Jr, Nunes, Ceretta, and Da Silva (2005). All these researchers studied stability in stock returns to understand the behavior of a sequence of the correlation structures based on independent samples in certain time periods by applying Box's M statistic proposed by Box (1949) and Jennrich's statistic proposed by Jennrich (1970). Furthermore, Deblauwe and Le (2000) studied risk credit and portfolio analysis using the same analysis of correlation structures. Another example is that of Annaert, De Ceuster, and Claes (2003) who focused their study on the stability of covariance structures and correlation based on small samples.

In testing the equality of a correlation matrix, the earliest development was in 1898 by Pearson and Filon who obtained the asymptotic derivation for a covariance matrix of a set of correlations (Steiger, 1980). Following this line of logic, several other studies have tested correlation matrices from time to time. These include Hotelling (1940), Box (1949), Bartlett and Rajalakshman (1953), Lawley (1963), Kullback (1967), Aitkin, Nelson, and Reinfurt (1968), and Jennrich (1970).

Therefore, in the next section, we discussed the overview of Jennrich statistic for details.

1.2 Overview of Jennrich Statistic

There are many statistics that can be used for testing the equality of correlation matrices. The latest statistic was introduced by Jennrich (1970). Jennrich presented a counter example where Kullback's assertion fails and showed that this assertion was incorrect because the asymptotic distribution under the null hypothesis is in reality a linear combination of independent Chi-square variates with the weights depending on an unknown value of common correlation matrix. Furthermore, Jennrich's statistic has much better properties than Kullback's statistic in terms of computational and distributional properties.

Specifically, Jennrich's statistic is used to test the equality of m independent sample of correlation matrices. The null hypothesis is $H_0 : \Omega_1 = \Omega_2 = ... = \Omega_m \left(= \Omega_0\right)$ versus the alternative hypothesis is $H_1 : \Omega_i \neq \Omega_j$ for at least one $(i,\ j)$ is different. The independent m samples of size $n_1, n_2, ..., n_m$ is drawn from p-variate normal distribution $N_p\left(\mu_1, \Sigma_1\right), N_p\left(\mu_2, \Sigma_2\right), ..., N_p\left(\mu_m, \Sigma_m\right)$ and Σ_i denotes the covariance matrix of the i-th population where $i = 1, 2, ..., m$. The Jennrich statistic is as follows:

$$J = \sum_{i=1}^{m} \left\{ \frac{1}{2} Tr\left(Z_i^2\right) - \left(Z_d\right)^t W^{-1} Z_d \right\} \tag{1.1}$$

where:

$Z_i = n_i^{\frac{1}{2}} R_p^{-1} (R_i - R_p)$

R_i is the i- th sample correlation

R_p is the average of all sample correlation matrices

$W = I_p + R_p * R_p^{-1}$ the * is Hadamard product of two matrices

Z_d is a diagonal of Z_i

I_p is the identity matrix of size $\left(p \times p\right)$

J is asymptotically distributed as χ^2 with $df = \frac{1}{2}(m-1)p(p-1)$ degrees of freedom where p is the number of variables (dimensions). The null hypothesis is rejected at a level of significance α if $J > \chi^2_{\alpha,df}$ at the $\left(1-\alpha\right)^{th}$ quantile of Chi-square distribution.

Jennrich's statistic was basically developed for a large sample test. Therefore, the statistic performs poorly for a small sample size (Larntz & Perlman, 1985). Furthermore, Jennrich's statistic was developed based on a likelihood ratio test (LRT) approach.

Equation (1.4) clearly shows that the Jennrich statistic involves the inversion of the correlation matrix. Under this condition, the correlation matrix will be singular if the number of variables is larger than the sample size $\left(p > n\right)$ (Eichholtz, 1996; Gande & Parsley, 2005)

Schott (1996) and Schott (2007a) also constructed tests based on the same approach. A LRT is sensitive to non-normality but is also impossible to use when the number of variables is larger than the sample size (Anderson, 2006; Herdiani & Djauhari, 2012; Schott, 1996). Besides that, LRT is unsuitable for use if the number of the variables is much larger than $\frac{n_i}{2}$ (Schott, 2007a) and requires more time to calculate (Sul, Han, & Eskin, 2011). .

In multivariate setting, testing the correlation matrix with a high dimensional data is very difficult and challenging. However, some efforts have been undertaken to improve the problem of singularity that Jennrich faced. More than three decades after Jennrich had proposed his statistic, Djauhari and Herdiani (2008) introduced the vector variance standardized variable test. This statistic was developed based on the vector variance (VV) approach to increase computational efficiency, and this approach is very promising particularly when dealing with a high number of dimensions. Another advantage of the VV approach is the simplicity of its computation. The computation is based on the Frobenius norm, which is equal of the sum of all the square elements.

A few years later, Sharif and Djauhari (2014) presented the T^* statistic in. This statistic is used from a different approach which is the upper-off-diagonal elements to overcome the problems of singularity. The upper-off-diagonal elements of the correlation matrix approach were used in the derivation of the T^* statistic. From the power of test analysis, generally, this statistic is more sensitive than Jennrich's statistic. However, room for improvement still exists because the results of sensitivity analysis are inconsistent.

1.3 Problem Statement

Nowadays, data with a high number of dimensions is collected routinely in finance, genomics, biomedical imaging and tomography (Pourahmadi, 2013). Let X be an $n \times p$ data matrix where n is the sample size, and p is the number of variables. In a high dimensional data, $n < p$, any statistical method that relies on the inversion of correlation matrix will be singular or not well conditioned because the sample correlation is not full rank (Djauhari & Herdiani, 2008; Pourahmadi, 2013). The inversion matrix is also negligible and sparse for very large correlation matrices and is a poor estimator when $p = n$ (Djauhari & Gan, 2014; Khare, Oh, & Rajaratnam, 2015). In this case, the use of Jennrich's statistic is not appropriate anymore for testing the correlation matrix. When p is large, the computation of this statistic is tedious because the computational efficiency of the determinant and the inverse of the matrix is low. The larger the value of p, the higher the costs to compute the determinant and the inverse of covariance or correlation matrix. This is the first problem that arises in using Jennrich's statistic when involving high dimensional data.

Moreover, Jennrich's statistic was developed based on likelihood ratio test (LRT) for testing the equality of two correlation matrices (Jennrich, 1970). This statistic is theoretically and asymptotically derived under Chi-square distribution and its distributional behavior totally fails when normality assumptions cannot be met (Aslam & Rocke, 2005). The complexity of the computing methods for LRT are higher when more and more variables are involved. This is because this method involves the inversion of the correlation matrix.

For this reason and to overcome the drawbacks of singularity problem, the algebraic approach was investigated. The combination of the upper-off-diagonal elements approach (Sharif, 2013; Sharif & Djauhari, 2014) and the VV approach (Djauhari & Herdiani, 2008), which is called Frobenius norm of upper-off-diagonal elements, is expected to produce better alternative statistics than the existing statistics.

Therefore, the aim of this research is to focus on how to handle the drawbacks of Jennrich's statistic as well as the T^* statistic. Later, a statistical test for testing the equality of several independent samples of correlation matrices is derived to satisfy instances in which a high number of dimensions are present. The derivation is based on the condition that the samples are independent and identically distributed (i.i.d.).

1.4 Objectives of the Study

The main objective of this study is to propose an alternative statistical test for testing the equality of several independent samples of correlation matrices in the case of high number of dimensions. The sub-objectives of the study are as follows:

i. To derive new statistical test Z^* for testing the equality of several independent samples of correlation matrices in cases that have a high number of dimensions;

ii. To evaluate and compare the performance of the new alternative statistic based on the power of the test by using simulation study; and

iii. To validate the performance of the new alternative statistic by using real data from Asia Pacific currencies; two independent samples of correlation matrices and several independent samples of correlation matrices.

1.5 Limitation of the study

The main assumption in this study is that all random samples are drawn from a multivariate normal distribution. The sampling distributions are derived based on that assumption. Hence, all the results of this study are valid only under multivariate normal distribution. The second limitations of this study is the application of new statistic is applied using one set of financial data only. Thus, the ability of the real data performance is limited. However, it does not means that the statistic is not suitable for other case of data.

1.6 Significance of the Study

This study contributes towards knowledge development in multivariate hypothesis testing especially in the derivation of a correlation test in the cases that have a high number of dimensions. For example, a financial analyst could use this alternative statistic to validate a global financial crisis in a short period of time while including a large number of variables in the analysis. Moreover, the alternative test offers some improvement in the power of the test.

As is well known, data that fulfill all the assumption of the Jennrich's statistic are difficult to find. The benefit of this study is the development of an alternative statistic that can tolerate the condition of $p > n$. Additionally, researchers will not be constrained by $n > p$ as required by Jennrich’s statistic.

1.7 Thesis Organization

This thesis contains five chapters. Chapter One covers a general introduction of the study. This chapter, presents the background of this study, problem statements, objectives of the study, limitation of the study, significance of the study and thesis organization.

The reminder of the thesis organized as follows. Chapter Two, the literature review this chapter start with introduction and correlation matrix then we focus on some methodologies for testing the equality of correlation matrices and some of the applications and then we discuss methods used to overcome the drawback when the number of variables are larger than the sample size, then we presents vector operator, commutation matrix and power of statistical test. Later in this chapter we present T^* statistic.

In Chapter Three, we discuss the methodology and the theory used to achieve the objective of the study. This chapter, presents mathematical derivation of asymptotic distribution and discuss variables manipulated, significance level and performance evaluation based on simulation study. In the last of this chapter, we discusses validation of Tohoku earthquake on Asia Pacific currencies by using correlation structure. For that purpose we used 23 currencies from Asia Pacific countries.

Chapter Four presents the derivation of the new altrnative statistic and its asymptotic distribution and the computation of the covariance of correlation matrix. Then, we discusses the analysis power of test of the new alternative statistical test Z^*, Jennrich statistic and T^* statistic. In the last of this chapter we presents of real application by using two approaches for propose of validation.

Finally, this thesis closes with the conclusions and some suggestions for future research in Chapter Five.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This study, inspires to propose the alternative statistical test which is used for testing the equality independent sample of the correlation matrices. This test can be used when the number of variables is small. Interestingly, the new test is able to overcome the drawback of Jennrich statistic.

This chapter reviews some of the literatures to have a better understanding on testing the equality of correlation matrices. We start this chapter by introducing the correlation matrix, then we presented some of methodologies used for improvement of the theoretical of testing the equality of correlation matrices and introduced some of previous application on testing the equality of correlation matrices. The next section, aims to introduce some of methods used to overcome the drawback the case where number of variables are larger than the sample size, such as banding, tapering, thresholding and Frobenius norm of upper-off-diagonal elements. In section five and six, vector operator and commutation matrix are discussed, respectively. Next, the power of statistical test is elaborated. Finally, the details of T^* statistic presented in the last section close this chapter.

2.2 Correlation Matrix

Correlation is the most commonly used measure for describing the linear relationship between two variables (Sang, Dang, & Sang, 2016). The correlation coefficient, ρ_{ij} between two random variables of X say the i-th and j-th is,

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \text{ for all } i, j = 1, 2, \ldots, p \tag{2.1}$$

Where σ_{ii} and σ_{jj} are the standard deviations, and σ_{ij} is the covariance. When the covariance is divided by the two standard deviations, the range of the covariance is rescaled to the interval between -1 and +1. Therefore, correlation is a scaled version of covariance (El Karoui, 2007). The computation is simple and how well variables correlate with each other is quickly noticed. The correlation efficiency gives information on the degree of the relationship as well as its direction.

The coefficient of correlation that equals $\rho_{ij} = \pm$ is present if and only if the two variables are perfectly related, while $\rho_{ij} = 0$ is present if the two variables are perfectly unrelated. When the coefficient has a negative value, it implies a negative linear relationship between the i-th and j-th variables. In addition, when the coefficient has a positive value, it implies a positive linear relationship between the i-th and j-th variables.

Consequently, the sample correlation coefficient between the i-th and j-th variables is,

$$r_{ij} = \frac{s_{ij}}{\sqrt{s_{ii}s_{jj}}} \tag{2.2}$$

where the r_{ij} is the maximum likelihood estimate of ρ_{ij} under the multivariate normality assumption (Muirhead, 1982).

Furthermore, the correlation coefficient can be used to evaluate whether a linear relationship exists in the population. The null hypothesis for testing the bivariate correlation is $H_0 : \rho_{ij} = 0$, and three different hypotheses alternatives must be chosen from. These are $H_1 : \rho_{ij} \neq 0$ (two-tailed), $H_1 : \rho_{ij} > 0$ (one-tailed), or $H_1 : \rho_{ij} < 0$ (one-tailed). Next, the statistical test is computed using the following formula, $t = \frac{r_{ij}\sqrt{n-2}}{\sqrt{1-r_{ij}^2}}$. The null hypothesis is rejected if $|t| > t_{\frac{\alpha}{2},df}$ (two-tailed) or $|t| > t_{\alpha,df}$ (one-tailed), with $df = n-2$ degrees of freedom.

In recent years, more than two variables is often have been considered in the analysis. To estimate the correlation between all possible pairs of variables calls for a correlation matrix. The correlation matrix is a symmetric matrix in which the correlation between X_i and X_j is equal to X_j and X_i. The main advantage of using a correlation matrix is the ability to visualize all coefficients for a large number of variables in the same window. Interpretation is easy because the correlation matrix is a symmetric positive semi-definite matrix (Cui, 2010).

In the matrix form, the population and sample correlation matrices are denoted by

$$\Omega = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & \rho_{pp} \end{pmatrix} \text{ and } R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1p} \\ r_{21} & r_{22} & \cdots & r_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ r_{p1} & r_{p2} & \cdots & r_{pp} \end{pmatrix}, \text{ respectively.}$$

The correlation matrix can be obtained from the covariance matrix (Rencher, 2003), let $\Sigma_d = \text{diag}\left(\sqrt{\sigma_{11}},\ \sqrt{\sigma_{22}},\ ...,\ \sqrt{\sigma_{pp}}\right) = \text{diag}\left(\sigma_1,\ \sigma_2,\ ...,\ \sigma_p\right)$ the population correlation matrix is $\Omega = \Sigma_d^{-1/2} \times \Sigma \times \Sigma_d^{-1/2}$ and let $S_d = \text{diag}\left(\sqrt{s_{11}}, \sqrt{s_{22}}, ..., \sqrt{s_{pp}}\right)$ $= \text{diag}\left(s_1, s_2, ..., s_p\right)$ the sample correlation matrix is $R = S_d^{-1/2} \times S \times S_d^{-1/2}$ where Σ_d and S_d are the diagonal elements of Σ and S, respectively. Thus, the correlation matrix Ω and R is,

$$\Omega = \begin{pmatrix} \frac{1}{\sqrt{\sigma_{11}}} & 0 & ... & 0 \\ 0 & \frac{1}{\sqrt{\sigma_{22}}} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \frac{1}{\sqrt{\sigma_{pp}}} \end{pmatrix} \times \begin{pmatrix} \sigma_{11} & \sigma_{12} & ... & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & ... & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & ... & \sigma_{pp} \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{\sigma_{11}}} & 0 & ... & 0 \\ 0 & \frac{1}{\sqrt{\sigma_{22}}} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \frac{1}{\sqrt{\sigma_{pp}}} \end{pmatrix}$$

$$R = \begin{pmatrix} \frac{1}{\sqrt{s_{11}}} & 0 & ... & 0 \\ 0 & \frac{1}{\sqrt{s_{22}}} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \frac{1}{\sqrt{s_{pp}}} \end{pmatrix} \times \begin{pmatrix} s_{11} & s_{12} & ... & s_{1p} \\ s_{21} & s_{22} & ... & s_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{p1} & s_{p2} & ... & s_{pp} \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{s_{11}}} & 0 & ... & 0 \\ 0 & \frac{1}{\sqrt{s_{22}}} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \frac{1}{\sqrt{s_{pp}}} \end{pmatrix}$$

Therefore, the correlation matrix is equivalent to the covariance matrix of the standardized random variables. Under the multivariate normality assumption, the correlation coefficient is formally regarded as a measure of the linear dependence between those variables (Muirhead, 1982).

Since the 1970's, many studies involving correlation analysis can be found in various literatures. These began with simple correlation analysis, continuing with more advanced analysis such as correlation matrix analysis. This includes the research work of Makridakis and Wheelwright (1974), Watson (1980), Philippatos, Christofi, and Christofi (1983). For example, Maldonado and Saunders (1981) used correlation analysis for hypothesis testing related to bivariate correlation. They examined the inter-temporal stability of international return correlations over a 22-year period. The data included monthly returns on a United States stock index and four foreign stock market indices (Japan, Germany, Canada, and the United Kingdom). The results showed that in the short term of up to two quarters of a period, a comparatively predictable relationship existed between inter-country correlations, which is a very short-term strategy a good news for investors. Next, we discuss some methodologies for testing the equality of correlation matrices.

2.3 Some Methodologies for Testing the Equality of Correlation Matrices

Before Jennrich(1970) presented his test, testing the equality of correlation matrices had a long history. Nowadays, Jennrich's statistic has been widely used by researchers such as Deblauwe and Le (2000), Annaert et al. (2003) and Gan, Djauhari, and Ismail (2014) to examine the equality of correlation matrices.

From time to time, research on the correlation matrix test have been developed by scholars to overcome the problems encountered when using existing statistics. Generally, the three main problems that inspired researchers to enhance existing tests have been to: (1) accomplish non-normality distribution (2) overcome the drawbacks of LRT, and (3) solve the singularity problem of a covariance determinant matrix.

First, to handle the problem of non-normal distribution, Browne and Shapiro (1986) and Neudecker and Wesselman (1990) derive a general matrix expression for the asymptotic covariance matrix of correlation coefficients by using a linearization theorem with some matrix notation and its properties. In their research only requires that the observation vector are i.i.d. according to multivariate distribution with finite fourth moments.

Next, the asymptotic of covariance matrix and Wald statistic theorem were enhanced by Schott (2001) to derive a more general Wald statistic. This test was constructed under the assumption of elliptical distributions and compared with LRT. Analysis has shown that the LRT is conservative where the value of type I error is smaller than 0.025 with negative kurtosis for non-normal elliptical populations and, liberal where the value of Type I error is larger than 0.075 with positive kurtosis for non-normal elliptical populations. For that reason, the Wald statistic has been found to be asymptotically equivalent to the likelihood ratio criterion. This test is not appropriate unless the sample size is very large and number of variables is small.

In Schott (2007b) study, partitioning the correlation matrix into a submatrix is implemented to obtain Wald statistics under the assumption of multivariate normality as well as extensions that apply to elliptical distributions. The Wald tests produces sensible significance levels, except if the number of the variables is large and the number of groups is small. In this study the Schott construct his test for testing the equality of several dependent samples correlation matrices.

The next problem is to overcome the LRT drawback. The LRT approach is very sensitive to non-normal distribution. By using LRT, it is very difficult to use when $p > n$ and it requires more time to calculate. For example, Schott (1996) derived a Wald statistic for testing the equality of correlation matrices for several independent samples from a normal distribution for small sample size. Then, the estimated significance levels of the alternative statistic are compared with LRT. The simulation results indicate that in most cases that involve small sample sizes the Wald statistic yields actual significance levels closer to the nominal level than LRT does. In addition, the computation of the Wald statistic has advantages over LRT, it is easy to calculate (Sabharwal & Potter, 2002).

Last is the singularity of covariance determinant matrix problem. In order to solve this problem, Larntz and Perlman (1985) suggested a new procedure for small sample size and the singular matrix, through the application of the Fisher z-transformation formula to each sample coefficient. The result showed that the procedure has numerous advantages over the Jennrich statistic for a small sample size and is still valid when one or more correlation matrices are singular. This procedure is easily computed by a hand calculator. Additionally, this test is more sensitive than Jennrich's statistic and is asymptotically consistent as n goes to infinity. However, this test is not preferable to large sample size (Olkin, Lou, Stokes, & Cao, 2015).

In addition, Goetzmann, Li, and Rouwenhorst (2005) investigated the correlation structure of the main world equity markets over 150 years. They discovered that international equity correlation changes vary over time and are highest during of periods of financial and economic integration. Further, they also examined the derivation of asymptotic distribution of a correlation matrix that Browne and Shapiro (1986) and Neudecker and Wesselman (1990) introduced. The upper-off-diagonal elements approach is integrated into the asymptotic distribution to avoid the singularity problem. To achieve the best results, the validity of the asymptotic distribution requires that the observation vectors are i.i.d. according to a multivariate distribution with finite fourth moments.

Furthermore, Schott (2005) proposed a simple statistic for testing the complete independence of random variables in instances in which the variables have a multivariate normal distribution. This statistic is designed for a sample correlation matrix test in instances with a high number of data dimensions. To derive the statistic, the sum of squared of sample correlation coefficient approach is employed. This test was compared with LRT, and the simulation results of significance levels showed that the Chi-square approximation is particularly poor if $p = n$. For fixed p, the performance of the statistic improves as the sample size n increases, but the rate of improvement decreases as p increases. Based on the power of the test, the results showed that the power increases as the sample size n increases and increases as p increases. Additionally, the result showed that the distribution was asymptotic normal as number of variables and sample size tend to infinity.

In another study, Schott (2007a) suggested a simple statistic for testing the equality of covariance matrices for several multivariate normal populations when the number of dimensions is large relative to the sample size based on the Frobenius norm approach. The result showed that, when the sample size was small, the empirical results of the test were not close to the nominal level. The values are lower than the nominal level and they converged to the nominal level when the sample size and p are increased.

After the development of theoretical, some applications began testing correlation matrices. For example, Tang (1995) provided an extension of the theoretical for Box's M so it could be used for testing the correlation matrices directly, provided that the data were standardized in the first stage. In doing so, this study investigated the stability of correlation matrix of stock markets. The results showed that the correlation structure is very stable even when the holding interval varied among 12 stock markets. Furthermore, the results showed that the correlation matrix of returns was sensibly stable over time when short time periods are considered. While, the variance-covariance structure of stock returns is less stable that was empirically supported by the literature in this study by many researchers. When p is large the use of this procedure is cumbersome.

In addition, Kaplanis (1988) investigated the stability of the correlation and covariance matrices of monthly returns and compared the matrices estimated through sub periods by using Jennrich's statistic. The result discovered that the covariance was less stable than the correlation matrix, which becomes stable over time.

In another study, Deblauwe and Le (2000) used Jennrich's statistic based on a pairwise test to investigate the stability of correlation matrices for market risk and credit over different periods of time. The results showed that a pair-wise test for log returns calculated on a monthly basis was more stable than log returns calculated on a daily or weekly basis. Furthermore, the test pointed out that holding periods over one month were more stable than 2 or 4 month holding periods. Additionally, the researchers applied the Jennrich's statistic to determine stability of daily, weekly, and monthly log returns for holdout periods ranging from 1-12 months. The use of the Jennrich statistic confirmed the non-homogeneity of correlation matrices for all holding periods and all data frequency.

Similarly, Chesnay and Jondeau (2001) used weekly stocks returns series for the S&P, the DAX, and the FTSE over the period from 1988 to 1999 and found that international correlations significantly increased during high turbulent periods. Interestingly, Ragea (2003), who studied emerging markets in Europe (Czech Republic, Hungary, Poland, Russia, and Turkey) and counterparts from established markets in Europe and North America), found that the null hypothesis of constant correlation could not be rejected.

In contrast, Annaert et al. (2003) studied the inter-temporal stability of correlation and covariance matrices using the Jennrich statistic for small samples with normal distributions. The result showed that, for small sample sizes, the test was not well specified when the assumption of normality was relaxed. In other research, Annaert, Claes, and De Ceuster (2006) studied the intertemporal stability of the covariance and correlation matrices of credit spread changes on weekly data based on the EMU Broad Market indices. The Jennrich statistic was used for the equality of correlation matrices and Box *M* statistic was used for the equality of covariance matrices. A bootstrap-based statistical inference provided evidence that correlations and covariances between various (investment grade) credit spread changes were unstable over the 1998-2003 period.

Two years later, Djauhari and Herdiani (2008) reported that neither the instability nor the stability of the correlation structure accounted for either the instability or stability of the covariance structure. Therefore, the researchers used the multivariate statistical process control (MSPC) approach to eliminate the obstacles that the Jennrich statistic faces when p is large. This was implemented by using VV as a multivariate dispersion measure to overcome the limitations. In this study, the researchers used real data and compared the proposed method with Box's M statistic and Jennrich's statistic. The results for Box's M statistic and Jennrich's statistic were the same; however, the result given by the proposed multivariate statistical process control approach using vector variance as multivariate dispersion measure was different. The difference in the result was not surprising because the two statistics and the proposed approach used different measures of multivariate dispersion. Based on the simulation experiment, the result showed that, in general, vector variance standardized variables (VVSV) were better than Jennrich's statistic and more sensitive to the shift of the correlation structure. Moreover, the notion of MSPC used to monitor the stability of the correlation structure was equivalent to reduplicating the tests of significance of the hypothesis.

In brief, the Jennrich statistic as shown in equation (1.4) contains the inversion of pooled correlation matrix. In practice, however, it is not infrequent that the number of variables p is large. Consequently, when p is large the computation of this statistic is uninteresting because the computational efficiency is very low and makes the correlation matrix singular.

In the next section, some of the methods used to handle the drawback of $n < p$ are discussed.

2.4 Methods to Overcome the Drawback of $n < p$

When the number of variables is larger than the sample size the correlation matrix is not of full of rank so the inverse of the matrix will not exist (Bai & Shi, 2011). However, the data collected routinely in scientific investigations often have a high number of dimensions. For example, these studies include web-search problems, climate studies, gene expression arrays, risk management, and functional magnetic resonance (Cai, Zhang, & Zhou, 2010). Regularization methods that were originally developed for nonparametric estimation functions have been applied recently to estimate large covariance matrices. However, these matrices will have many elements of zero that are sparse. To overcome this problem, methods such as banding, tapering, thresholding, and upper-off-diagonal elements can be applied to estimate the covariance matrix.

2.4.1 Banding

Large covariance matrices are bound to have many elements with either zeros or small entries, which denote as a sparse, estimating or regularization the small entries by zeros appears like a natural thing to do. The banding of covariance matrix arises when the variables are ordered and serially dependent as in the data of climatology, time series or spectroscopy (Choi, Lim, Roy, & Park, 2016).

Banding is a simple and systematic way for estimating a large covariance matrix because for high dimension data the sample covariance matrix is singular Pourahmadi (2013). Banding starts by estimating the covariance matrix by a diagonal and then consecutively adds the other subdiagonals by estimating the first, second,…, ℓth subdiagonals, if warranted by the data or the application area. In implementing the banding method, the main thing is how to choose the tuning or the banding parameter ℓ. Banding is a simple approach to obtain nonsingular estimator and a well-conditioned estimator.

Let $S = (s_{ij})$ a sample covariance matrix of size $p \times p$ and any integer ℓ, $0 \leq \ell \leq p$, its ℓ banded version (Bickel & Levina, 2008b; Pourahmadi, 2013; Tulic, 2010) which is defined by,

$$B_\ell(S) = [s_{ij} 1(|i-j|) \leq \ell] \quad (2.3)$$

can serve as an estimator for Σ. The estimate matrix $\hat{\Sigma}_{\ell,p} = B_\ell(\hat{\Sigma}_p)$ for some ℓ. This regularization is perfect when the indices or the variables can be arranged so that items of the covariance matrix are farther away from the main diagonal are negligible

$$|i-j| > \ell \Rightarrow \sigma_{ij} = 0 \quad (2.4)$$

if Σ is covariance matrix for $X = \left(X_1, X_2, ..., X_p\right)^t$ where $X_1, X_2, ..., X_p$ are defined by the moving average process $\sum_{j=1}^{\ell} \varpi_{t,t-j} \varepsilon_j$, where ε_j are i.i.d. with the mean equal to 0 and finite inverse. The performance of the estimator depends on the optimal choice of the banding parameter. The method usually used for the choice is a cross validation method because banding an arbitrary covariance matrix does not assure positive definiteness (Bickel & Levina, 2008b).

In cases in which asymptotic analysis for banded estimators ℓ, n and p are large, Pourahmadi (2013) suggested using the class of bandable covariance matrix

$$\eta(\upsilon,\varepsilon) = \left\{ \Sigma \in \varsigma(\varepsilon) : \max_j \sum_i \left\{ \left|\sigma_{ij}\right| ; \left|i - j\right| > \ell \right\} \le C\ell^{-\upsilon} \right\} \tag{2.5}$$

where $\varsigma(\varepsilon) = \left\{\Sigma_p : 0 < \varepsilon \le \lambda_{min}\left(\Sigma_p\right) \le \varepsilon^{-1}\right\}$ is the set of well-conditioned of covariance matrix and $C > 0$. The parameter υ controls the average of decay of the entries of the covariance matrix as one move away from the main diagonal. In addition, the optimal rate of the convergence of estimating a covariance matrix from this class depends critically on ℓ. The ℓ-banded is not necessarily positive definite. The idea of banding and regularizing the lower triangular matrix of the Cholesky decomposition of the inverse of covariance matrix has been studied by Wu and Pourahmadi (2003). In the next section, tapering is introduced as another method of regularization in estimation of the covariance matrices in instances of a high number of dimensions. In some cases, tapering is beneficial in including the positive definiteness.

2.4.2 Tapering

Tapering has a long history and has been used in time series analysis. Recently, it has been used to develop the performance of linear discriminant analysis (Bickel & Levina, 2004; Da Rocha, 2008). When the researchers used the banding of the covariance matrix the problem that is faced is the lack of assured positive definiteness (Bickel & Levina, 2008b). Therefore, Furrer and Bengtsson (2007) found that the positive definiteness can be protected by tapering the covariance matrix. A tapered estimator of the covariance matrix to a tapering matrix $W = w_{ij}$ replaces S by

$$S_W = S * W = s_{ij} w_{ij} \tag{2.6}$$

where $(*)$ is the Schur matrix multiplication. If W is a positive definite symmetric matrix, then S_W is ensured to be positive definite (Pourahmadi, 2013). The choice of W a smoother positive definite tapering with off diagonals elements progressively decaying to zero to ensure the positive definiteness (Pourahmadi, 2013; Xue & Zou, 2014). The tapering weights can be written as follows:

$$w_{ij} = \begin{cases} 1, & \text{if } |i - j| \leq \ell_h \\ 2 - \dfrac{|\mathrm{i - j}|}{\ell_{\mathrm{h}}}, & \text{if } \ell_h < |\mathrm{i - j}| < \ell \\ 0, & \text{otherwise,} \end{cases} \tag{2.7}$$

where ℓ is a tapering parameter with $\ell_k = \dfrac{\ell}{2}$. The tapering estimator for any even integer ℓ with $1 \leq \ell \leq p$, is defined in equation (2.4). The tapering estimator can be written as a sum of several small block matrices along the diagonal (Cai & Zhou, 2012). Tapering is different from banding, and the difference is how it progressively shrinks the off-diagonal entries on the band to zero (Chen, Wang, & McKeown, 2012). Originally, tapering was proposed by Cai et al. (2010). A simulation study was conducted to compare the performance of banding and tapering estimator. The simulation results showed that the tapering estimator is usually better than the banding estimator and has a good numerical performance. However, the suggested tapering estimator does not develop the bound under the Frobenius norm, so under this norm, the banding estimator performs as well as tapering (Pourahmadi, 2013).

2.4.3 Thresholding

It is reasonable in instances of a high number of dimensions that many of the entries of the covariance population matrix could be small, and thus the Σ could be sparse. Therefore, the development of an estimator other than S that can deal with additional information is needed.

Thresholding was developed in nonparametric function estimation. It has been used for the estimation of the large covariance matrices (Karoui, 2008; Rothman, Levina, & Zhu, 2009). As example, thresholding of the sample covariance matrix is used in time series (Song, 2011). This estimator does not require the variables to be ordered in sequence; therefore, the estimator is invariant to the permutation of the variables (Pourahmadi, 2013). For the details, a thresholding operator $\hbar_{\lambda}$ for an $\lambda \geq 0$ for the sample covariance matrix is defined as follows:

$$\hbar_{\lambda}(S) = \left[s_{ij} 1(s_{ij}) \geq \lambda\right] \tag{2.8}$$

thus thresholding S at λ amounts are replaced by zeros in all elements with an absolute value less than λ. Thresholding under the permutation of the variables labels maintains symmetry and is invariant. However, it does not necessarily maintain positive definiteness. Moreover, it is simplistic as it carries no main computational encumbrances compared to its competitors methods except for cross validation for the tuning parameter, and hard thresholding tends to do worse than more flexible estimators (Bickel & Levina, 2008a; Pourahmadi, 2013; Rothman et al., 2009). In practice, the choice of a threshold parameter is an important step in applying this procedure. Threshold selection is difficult to deal with analytically. Bickel and Levina (2008a) used the Frobenius norm to partially analyze numerical and theatrical performance. Thresholding has good properties in estimating large sparse covariance matrix but it frequently has negative eigenvalues in real data analysis (Xue, Ma, & Zou, 2012).

In addition, Bickel and Levina (2008b) suggested a regularization method by thresholding to estimate large covariance matrices. However, the major advantage is its simplicity, and hard thresholding carries no computational burdens, unlike several methods for covariance regularization (Rothman et al., 2009). A possible disadvantage is the loss of positive definiteness, but for appropriately sparse classes of matrices the estimators are consistent as long as they preserve the positive definite with the probability as it tends to one.

2.4.4 Frobenius Norm of Upper-off-Diagonal Elements

The upper off-diagonal elements can help to overcome the problem of singularity and reduce the computational complexity (Goetzmann et al., 2005). Goetzmann derived the statistic on the basis of the Wald test. This test allows the relaxation of restrictive assumptions on the correlation matrices. This test has advantages over the covariance matrix. It can work directly with correlation matrices and easily improved according to different hypotheses. Consequently, the upper-off-diagonal elements was used by Sharif (2013) and Sharif and Djauhari (2014) to derive the asymptotic distribution for testing several independent samples of correlation matrices. The development of the T^* statistic is based on *vec* operator and commutation matrix. This statistic shows that the singularity problem when high dimension data is solved. However, computation is very challenging.

In reducing the computational complexity, Djauhari and Herdiani (2008) implemented the VV approach as a measure of dispersion in developing VVSV. This measure is derived for testing the equality of several correlation matrices. Based on the MSPC approach, it also can be used to monitor the stability of correlation matrices using a control chart.

From the simulation results, it has been proven that the statistic can handle the singularity problem in cases in which the number of variables is larger than the sample size. The development of this statistic is also based on the commutation matrix and *vec* operator. However, it is different from the T^* statistic in terms of the correlation estimator.

The singularity problem can be avoided by examining the algebraic approach. As mentioned in Chapter One using the Frobenius norm of the upper-off-diagonal elements approach to derive alternative statistical test is expected to produce a test having better properties than the existing tests.

The following matrices show how to calculate Frobenius norm of the upper-off-diagonal elements of R .

Let $p = 3$, therefore, the correlation matrix of size 3×3as follows:

$$R = \begin{pmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{23} & 1 \end{pmatrix}$$

and the upper-off-diagonal element of R is

$$R_U = \begin{pmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{pmatrix}$$

The Frobenius norm of the upper-off-diagonal elements of R is

$$\|R_U\|^2 = r_{12}^2 + r_{13}^2 + r_{23}^2.$$

The development of the new alternative statistic test is also based on the commutation matrix and *vec* operator. Next, we illustrate the *vec* operator and the commutation matrix.

2.5 Vector operator

The technique of vector operator is very simple and it can be applied to a matrix for any order. There are situations in which it is very useful to transform a matrix to a *vec* one of such situation in statistics involves the study of the distribution of the sample covariance matrix (Schott, 1997). It is usually more convenient mathematically in distribution theory to express density functions and moments of jointly distributed random variables in terms of the vector with these random variables as its components (Schott, 2016). The vector operator alters a matrix into a vector by adding the columns of the matrix one underneath the other. The *vec* is defined for any matrix, not only for square matrix.

Let B be an $p \times p$ matrix and b_{ij} its j-th column, then $vec(B)$ is the $p \times 1$ vector

$$vec(B) = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix}$$

Let further F be a matrix; then the Kronecker product $B \otimes F$ can be defined as

$$B \otimes F = (b_{ij} F)$$

In certain case, the matrix algebra associated with the utilize of the *vec* operator and Kronecker product can be facilitated over the use of commutation matrix (Schott, 2003). Next, we deliver the discussion on the commutation matrix.

2.6 Commutation matrix

The commutation matrix is used in multivariate statistical analysis. This matrix is very useful when computing the moments of the multivariate normal and related distributions (Schott, 1997). Commutation matrix K is a square matrix containing only zeros and ones. The main property of the commutation matrix is that it transforms $vec(R)$ into $vec(R)^t$. The commutation matrix can be used for reversing the order of a Kronecker product (Magnus & Neudecker, 1980). This property indicates that there is important relationship with *vec* operator and Kronecker product (Schott, 1997) . Also, this property is very useful in the calculation of matrix derivatives.

Let G_{ij} be a matrix of $p \times p$ that has its nonzero element, a one, in the (i, j)-th position. Then the $pp \times pp$ commutation matrix, denoted by K_{pp}, as follows,

$$K_{pp} = \sum_{i=1}^{m} \sum_{j=1}^{n} G_{ij} \otimes G_{ij}^{t}$$

where G_{ij} is a matrix of size $(p \times p)$ having all elements are equal 0 except it is (i, j)-th element equals 1.

Next, we illustrate how to obtain the commutation matrix K_{pp} for $p = 2$, and in Table 4.1, show the commutation matrix K_{pp} for $p = 2,\ 3$ and 4.

Let $p = 2$, the commutation matrix $K_{pp} = \sum_{i=1}^{p} \sum_{j=1}^{p} G_{ij} \otimes G_{ij}^{t}$ of size $(p^2 \times p^2) = (2^2 \times 2^2) = (4 \times 4)$

$G_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, G_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, G_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, G_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$G_{11} \otimes G_{11}^{t} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_{12} \otimes G_{12}^{t} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_{21} \otimes G_{21}^{t} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_{22} \otimes G_{22}^{t} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ Therefore,

$$K_{pp} = \sum_{i=1}^{2}\sum_{j=1}^{2} G_{ij} \otimes G_{ij}^{t} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Table 2.1

Commutation Matrix K , for the Case $p = 2,\ 3\ and\ 4$

p	***K***
2	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
4	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

To simplify the mathematical derivation of the asymptotic distribution of the test the vec operator and commutation matrix are used. Since the exact distribution is impractical to use, an asymptotic distribution is our concern.

2.7 Power of statistical test

Power of statistical test is defined as $1-\beta$, where β is the Type II error. The Type II error is the probability of failing to reject the null hypothesis. As power of statistical test increase, the probability of a Type II error decreases, and vice versa. The minimum value of the power of statistical test is zero while the maximum value is one.

The power of a test is useful in research design and interpretation. During the design of research, the power test provides the ability to determine whether the study has a sensible chance of obtaining statistically significant results (Cohen, 1977). Therefore, it is important to consider the power of test when designing experiments, and the power of a test contributes to a more efficient research design that results in the saving of time, money and effort (Aktas, 2013; Baroudi & Orlikowski, 1989). The power analysis in the research enables a highly reliable and valid study and guarantees the validity and sensibility of the results in that research Sawyer (1982). It is also used to assess to what degree the decisions obtained as a result of a statistical test are reliable and valid in terms of probability values (Sullivan & Feinn, 2012).

It has been found the power of a test has not been accounted for by several researchers, and accordingly, their results run a high risk of Type II errors (Clark-

Carter, 1997). Most research studying the power of tests deals with the computation of the power statistics where normality assumptions are required use of the most powerful test but this power may not be provided when assumptions have been violated. In most parametric tools, the assumption of normality plays an important role, and, when these approaches are used for non-normal data, results are unreliable and inferences have low power (Gali, 2015).

A simulation study is a technique for conducting experiments on the computer that involves random sampling from the probability of distributions. Gilbert and Troitzsch (2005) stated that a simulation study can be used for getting a better understanding of a phenomenon of interest and for the purposes of prediction. In addition, they claimed that a simulation is worthy for social science as a tool for formalizing theory.

First, this current study generated data from multivariate normal distribution (MVN) to examine the performance of the three statistical tests. The data was generated as follows:

$$MVN_p(\mu, I_p) \tag{2.9}$$

where $\mu = 0$ is the mean vector and I_p is the identity of the covariance matrix. A simulation study is conducted for 10000 iterations to got best result Sharif, (2013) from the standard multivariate normal distribution $MVN_p(0, I_p)$. To test the hypothesis $H_0 : \Omega_i = \Omega_0$ versus $H_1 : \Omega_i \neq \Omega_0$. Next paragraph, the power of statistical test is discuss to compare the three statistical test.

The power of a test was used to compare the three different statistical tests, and the most powerful statistical test was defined as that which had the higher percentage of rejecting the null hypotheses. Thus, the power of a test can be defined as the probability of rejecting H_0 when it is false (Cohen, 1990) and quantifies the chance that the H_0 will be rejected when it is actually false. Therefore, the power of a test is the ability of a test to correctly reject H_0. It is known as $1-\beta$, where β is the Type II error that is defined as the probability of failing to reject the null hypothesis H_0 when it is false. If the power of statistical test is low, then a good possibility exists that the test will be indecisive (Syed-Yahaya, 2005).

In evaluating the performance of a test the power of that test must be large to ensure that shifts in the process can be detected. The power of the test gives a signal for the ability of the test to detect a shift in the correlation matrices. The closer the value of the test to the 100% the better is the performance of that test. The sensitivity of the test to the shifts in correlation structures increase as the power of the test increases. This means that a powerful test will be very sensitive to a shift in correlation structure. For example, if the power of the test reaches 100%, this would indicate that the test detected all shifts. However, in general, the minimum accepted level for the power of a test should be 0.8 or above to be considered suitable (McCrum-Gardner, 2010; Murphy, Myors, & Wolach, 2014).

Therefore, to evaluate the performance of a specific test, data are generated from a standard normal distribution with shifts in correlation matrix. A good statistic test should have strong power. As for the power of the test, the higher the percentage, the better the statistical test.

In this current study, simulations were run by using MATLAB (2016a) for the alternative statistic, the Jennrich's statistic and the T^* statistic to evaluate the power of each test. The power of statistical test was calculated for the three statistical test.

In this study, critical value is determined by using simulation method. A total of 10000 statstical values is generated for that purpose (Sharif, 2013) and then all the values is arranged in ascending order. Since, the hypothesis testing is implemented at the 5% significance level, the value at 95% is considered as critical value. This value is used to decide whether to reject or fail to reject the null hypothesis. Therefore, the $9500th$ value $(10000 \times 95\%)$ represents the simulated critical value. The process is run repeatedly for all combinations of p and n.

In the next section, we discussed the T^* statistic which introduced by Sharif and Djauhari (2014).

2.8 T^* Statistic

Testing the equality of a correlation structure has become a vital subject because the Jennrich statistic can only solve a problem when $p < n$. Because of the Jennrich statistic involving inversion of correlation matrix, this test cannot handle the case when $p > n$. Sharif and Djauhari (2014) proposed the T^* statistic constructed based on upper-off-diagonal elements to overcome the difficulty of the existing statistical test.

For testing the hypotheses $H_0 : \Omega_i = \Omega_0$ for all i, versus $H_1 : \Omega_i \neq \Omega_0$ for at least one i, where $i = 1, 2, ..., m$ the test is as follows:

$$T^* = n\{V(R_U) - V(\Omega_U)\}^t \psi^{*-1} \{V(R_U) - V(\Omega_U)\} \tag{2.10}$$

where

$\psi = V_1 - V_2 - V_2^t + V_3$

$V_1 = (I_{p^2} + K_{pp})(\Omega \otimes \Omega)$

$V_2 = (\Omega \otimes I_p + I_p \otimes \Omega)(K_{pp})_d (\Omega \otimes \Omega)$

$V_3 = \frac{1}{2}(\Omega \otimes I_p + I_p \otimes \Omega)(K_{pp})_d (\Omega \otimes \Omega)(K_{pp})_d (\Omega \otimes I_p + I_p \otimes \Omega)$

$K_{pp} = \sum_{i=1}^{p} \sum_{j=1}^{p} G_{ij} \otimes G_{ij}^t$ is the commutation matrix of size $(p^2 \times p^2)$

G_{ij} is a matrix of size $(p \times p)$ where all elements equal 0 except if $(i, j) - th$ elements equals 1.

$V(R_U)$ and $V(\Omega_U)$ are $T \times vec(R)$ and $T \times vec(\Omega)$, are the upper-off-diagonal for the matrix R and Ω respectively.

$\psi^* = T\psi T^t$

$V(R_U) = (r_{12}, r_{13}, r_{23}, r_{14}, r_{24}, r_{34}, ..., r_{1p}, r_{2p}, r_{3p}, ..., r_{qp})^t$

Such that $1 \leq q \leq p-1$.

$$t_{ij}^{a} = \begin{cases} 1 & \text{if } 1 < i < \dfrac{p(p-1)}{2}, j = ap + b \\ 0 & \text{otherwise} \end{cases}$$

The transformation T is presented in matrix form as a block matrix $T = \left(T_1 | T_1 | \ldots | T_p\right)$ of size $\left(k \times p^2\right)$ partitioned into p blocks where $k = \dfrac{p(p-1)}{2}$. $T_a = \left(t_{i,j}^{a}\right)$ each of size $\left(k \times p\right)$ where T_1 is zero matrix, where $1 \le a \le p-1; 1 \le b \le a.$

Under the null hypothesis, Sharif shows that $T^* \xrightarrow{d} \chi_v^2$ where v is the degree of freedom $v = \dfrac{1}{2} p(p-1)$. Therefore, at level of significance α the statistic is rejected if $T^* > \chi_{\alpha,v}^2$ at the $(1-\alpha)^{th}$ quantile of chi-square distribution with v degrees of freedom.

Sharif (2013) found that the power of the T^* statistic for a small sample size and when the number of variables was small $p = 3$, 4 and 5 are excellent only for a large shift of correlation matrices. On the other hand, the Jennrich statistic is dominate over the T^* statistic for a small shift of correlation matrices. When the sample size is large and $p = 3$, the T^* statistic is better than the Jennrich statistic. On the other hand for $p = 5$, the Jennrich statistic is better than the T^* statistic for a moderate shift. In cases in which $p = 10$ and 12, the T^* statistic dominates the Jennrich statistic only for a large shift in correlation matrices. When the number of variables is 15 and 20 the Jennrich statistic is dominate over the T^* statistic for small and moderate shifts. But, for a large shift in correlation matrices, the value of the power of the T^* statistic is better than the Jennrich statistic. For a large number of variables, the T^* statistic is better than the Jennrich statistic except for a moderate shift. From the results of the simulation study, the conclusion can be made that the sensitivity analysis is not consistent.

In what follows, the new altrnative statistical test Z^* is proposed based on linear transformation. Using that linear transformation, the correlation matrix is changed into a vector where its elements are the upper-off-diagonal of the correlation matrix. This transformation can help to ensure the non-singularity of the matrix. The alternative asymptotic distribution has been derived based on the commutation matrix and notation of the vector operator.

CHAPTER THREE
METHODOLOGY

3.1 Introduction

The purpose of this research is to derive the new alternative statistical test Z^*, to be used in testing correlation matrices. This test should be approparite for either when the number of dimension are low or high, the new alternative statistical test is Z^* expacted to have better performance than the Jennrich statistic test and T^* statistic test.

This chapter outlines the methodology to achieve the objective of the study. We beginning this chapter, firstly, we discuss the method to propose new alternative statistical test, which is constructed on *vec* operator, the commutation matrix and Forbenius norm of upper-off-diagonal elements. In order to assess the performance of the alternative statistical test, various conditions they are created by manipulating the number of dimensions (p), the number of observations (n), and shift in correlation matrix (ρ) and we illustrate the significance level. The performance of the alternative statistical test is assessed based on the assumption that there is a change (or shift) in elements of the correlation matrix. The performance evaluation is measured by power of test which represents the probability of not committing a Type II error. The value of power should be large enough to ensure that the statistical test can promptly detect and strongly sensitive to the change (or shift) in the correlation matrix. The power of the test, means that the sensitivity is high and the probability that it will reject a false null hypothesis is large.

This chapter begins with discuss the mathematical derivation of asymptotic distribution. In the next section, we introduce distribution of $vec(R)$. The variables manipulated will be explained in section four. In section five we introduce the significance level. In the next section, we introduce the performance evaluation of the statistical test based on simulation study. In the last section we discuss the validation of Tohoku earthquake on Asia Pacific currencies using the correlation structure.

3.2 Mathematical Derivation of Asymptotic Distribution

As mentioned in previous chapters, the Jennrich statistical test is the most frequently test used for testing the hypotheses of a correlation structure. It plays an important role in testing the stability of correlation structures (Deblauwe & Le, 2000). Indeed, this test has become a standard test in financial market analysis (Annaert et al., 2006; Ragea, 2003). However, this test is not free from drawbacks. To overcome the drawbacks researchers are focusing on improving this test.

The sample correlation is approximate by Wishart distribution (Kollo & Ruul, 2003). Wishart distribution is impractical (Sheppard, 2008). Due this limit the approximation of the distribution is needed.

In this study, new alternative statistical test is derived for testing the equality of several correlation matrices by using the notion of *vec* operator and the commutation matrix. To derive this test, first we have to derive the asymptotic distribution of sample correlation matrix. Therefore, in what follows, we derive the asymptotic distribution of sample correlation matrix. Then, based on that asymptotic distribution of sample correlation matrix we derive the asymptotic distribution of upper-off-diagonal elements. Then, we derive the asymptotic distribution of Frobenius norm of upper-off-diagonal elements. Based on that asymptotic distribution we derive the asymptotic distribution of the proposed test. To investigate the asymptotic distribution Frobenius norm of upper-off-diagonal elements we used Theorem 3.1 from Schott (2007b). To be formulated in next section, and the asymptotic distribution of $vec(R)$ developed by Browne and Shapiro (1986), and Neudecker and Wesselman (1990). This current study used Theorem 4.2.3 that is presented in Anderson (2003, p. 132) for finding the asymptotic of $v(R_U)$ to be formulated in next section.

3.3 Distribution of $vec(R)$

Let $Z_1, Z_2,..., Z_n$ be a random sample of size n having covariance matrix, Ω. The covariance matrix Ω of Z is so called correlation matrix of X. If R is a sample correlation matrix, then we call $vec(R)$ as the representation of R in vector form. This vector is obtained from R by arranging columns, one on top of other. the asymptotic distribution of the $vec(R)$ is stated by Browne and Shapiro (1986), Neudecker and Wesselman (1990), and Schott (2007b) as follows:

Theorem 3.1 (Schott, 2007b)

Let $X_1, X_2, ..., X_n$ be a random vector drawn from p-varite normal distribution of size n then

$$\sqrt{n-1}\left[vec(R)-vec(\Omega)\right]\xrightarrow{d}N(0,\Gamma)$$

where

$$\Gamma = 2M_p \phi M_p$$

$$M_p = \frac{1}{2}\left(I_{p^2} + K_{pp}\right)$$

$$\phi = \left\{I_{p^2} - \left(I_p \otimes \Omega\right)\Lambda_p\right\}\left(\Omega \otimes \Omega\right)\left\{I_{p^2} - \Lambda_p\left(I_p \otimes \Omega\right)\right\}$$

$K_{pp} = \sum_{i=1}^{p}\sum_{j=1}^{p} G_{ij} \otimes G_{ij}^t$ is the commutation matrix of size $\left(p^2 \times p^2\right)$

G_{ij} is a matrix of size $\left(p \times p\right)$ having all elements are equal 0 except it is $\left(i, j\right)$-th element equals 1.

$\Lambda_p = \sum_{i=1}^{p} h_i h_i^t \otimes h_i h_i^t$ where, h_i is the i-th column of I_p.

In this current study, the asymptotic distribution of $vec(R)$ will be examined from a particular case that is $p = 2$ to a general case where $p > 2$. For that purpose, a multivariate approach is used, and the basic tool utilized is the following Theorem 3.2 to identify the asymptotic distribution. Theorem 3.2 is presented in Anderson (2003, p. 132).

Theorem 3.2 (Anderson, 2003, p. 132)

Let $\left\{U\left(n\right)\right\}$ be a sequence of *p*-component random vectors and b a fixed vector such that $\sqrt{n}\left[U\left(n\right) - b\right] \xrightarrow{d} N\left(0, \gamma\right)$ as $n \to \infty$. Let $f\left(u\right)$ be a vector valued function of u such that each component $f_j\left(u\right)$ satisfies $\left.\frac{\partial f_j\left(u\right)}{\partial u_i}\right|_{u=b} \neq 0$. If

$\left.\frac{\partial f_j(u)}{\partial u_i}\right|_{u=b}$ is the $(i, j)^{th}$ component of ω. Then $\sqrt{n}\left[f\left(U(n)\right)-f(b)\right]\xrightarrow{d} N\left(0,\omega^t\gamma\omega\right)$.

Besides Theorem 3.1 and Theorem 3.2, another essential point is the integration method, i.e., the upper-off-diagonal element (Schott, 1997; Sharif & Djauhari, 2014) and the Frobenius norm (Djauhari & Herdiani, 2008). The Frobenius norm of upper-off-diagonal element is implemented to derive the alternative statistical test.

The following section will discuss the variables manipulated, which are used to calculate the power of test for the alternative statistic, the T^*statistic, and the Jennrich statistic.

3.4 Variables Manipulated

In this research, three variables are manipulated to investigate the strengths of the statistical tests: (1) the variables are the number of variables (p), (2) sample size (n), and (3) the shift in correlation matrix (ρ). The selection of variables was based on previous research from Sharif, (2013), Djauhari and Herdiani (2008), Mason, Chou, and Young (2009) and Alfaro and Ortega (2009) who utilized these variables in their research.

3.4.1 Number of Variables (p) and Sample Size (n)

In multivariate problems, sample size determination has always been slightly subjective and relies on the statistical instrument being used. Generally, when the sample size is large it is expected that it produce better estimation, the large sample size a rise the accuracy of the estimators (Chou, Mason, & Young, 2001). This current study considers different samples sizes including $n_1 = 3, 5, 10, 20, 30, 50$ and 100, the number variables $p = 3,\ 4,\ 5,\ 10,\ 15,\ 20$ and 30. The choice of the sample size and the number of variables are listed in Table 3.1 below

Table 3.1

The values of p and n

Classification	***p***	***n***
Small no. of. variable	3	3, 5, 10, 20, 30, 50, 100
	4	3, 5, 10, 20, 30, 50, 100
	5	3, 5, 10, 20, 30, 50, 100
Medium no. of. variable	10	3, 5, 10, 20, 30, 50, 100
	15	3, 5, 10, 20, 30, 50, 100
Large no. of. variable	20	3, 5, 10, 20, 30, 50, 100
	30	3, 5, 10, 20, 30, 50, 100

The sample size n and number of variables p, namely small no. of. variable ($p = 3,\ 4$ and 5), medium no. of. variable ($p = 10$ and 15) and large no. of variables ($p = 20$ and 30) in Table 3.1 are considered for calculating the power to evaluate the performance of the three statistical tests. Jennrich's statistic cannot be perform when $p > n$ because of singularity. So, that, in this study, the Jennrich statistic is calculated when $n > p$.

3.4.2 Shift in Correlation Matrices Ω_m

The power of test is used to assess the performance of the test. A simulation study will be focus on $H_0 : \Omega_m = \Omega_0$ versus $H_1 : \Omega_m \neq \Omega_0$ where $\Omega_0 = I_p$ and Ω_m is asymmetric matrix of size $(p \times p)$ with diagonal elements are 1's and all off the diagonal elements are from $\rho = 0, 0.1, ..., 0.8$ with increments of 0.1. Thus, the values of the ρ are specified in the range of values from no correlation to high correlation.

For example, for $p = 3$ and $\rho = 0.1$ the correlation matrix Ω_1, as follows:

$$\Omega_1 = \begin{pmatrix} 1 & 0.1 & 0.1 \\ 0.1 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{pmatrix}.$$

To obtain the best results, the study simulates 10000 datasets (Atiany & Sharif, 2016; Barnett & Onnela, 2016; Sharif, 2013) for different sample sizes n, and number of variables p in Table 3.1 by using Matlab (2016a) to calculate the power of the test.

3.5 Significance Level

In hypothesis testing, the significance level is also denoted as alpha α, is the criterion used for rejecting the null hypothesis and is defined as the probability of rejecting the null hypothesis when it is true. When the significance level 0.05 this indicates a 5% risk of concluding that a difference happens when no actual difference exists. To check the performance of the statistical tests, significance level $\alpha = 0.05$ was used.

To evaluating the performance of the proposed statistic test, the power of the test is calculated by using a simulation study, and the statistical test with a higher value of power is the better statistical test. Next, details of the performance evaluation are discussed in the following section.

3.6 Performance Evaluation Based on Simulation Study

To analyze the performance of the alternative statistical test, a simulation study was conducted. This study was designed to include two conditions: (1) to examine the weaknesses and strengths of the statistical test and (2) to evaluate the performance of the test. To illustrate the performance of the statistical test, a better understanding of its distribution is required to achieve the appropriateness of the statistical test. Thus, a simulation study was conducted to estimate the quantiles of the statistic. This study, focuses on the values of the sample size, n, the number of variables, p and correlation shift ρ.

Based on Theorem 3.1, an asymptotic distribution is implemented in derivation of new statistic. It means that, when the sample size is large, the statistic is asymptotically distributed to normal distribution. Therefore, CV=1.96 is used for large sample size, and the simulated CV is used for small sample size. Next, Algorithm 3.1 is presented to illustrate on how to determine the simulated critical value (CV). The value is identified before performing Algorithm 3.2 which identifying the power of test via simulation study (Sharif,; 2013; Haddad, 2013)

Algorithm 3.1

To compute the CV:

i. Set a variable, count = 0.

ii. Generate sample data related to condition to reflect the H_0 ($\alpha = 0.05$).

iii. Let the correlation shift ρ is equal zero, calculate the statistical test.

iv. Repeat step i *to* iii 10000 times.

v. Sort the 10000 values of statistical test.

vi. Identify simulated CV based on 95%.

Algorithm 3.2

To compute a power of test :

i. Generate sample data related to condition to reflect the H_0 ($\alpha = 0.05$).

ii. Change the correlation shift ρ to 0.1, calculate the statistical test.

iii. If the value of the statistic is larger than CV, then increase the count by one (count = count+1).

iv. Calculate the power of test by dividing the number of count with the number of replications.

v. Change the shift in correlation matrix ρ from 0.1 to 0.8 and repeat from step (i) to step (iv).

Based on Algorithm 3.2, all statistics; (1) the new alternative statistic, (2) the Jennrich statistic, and (3) the T^* statistic are computed. To illustrate the power of the test computational process for all statistics, Figure 3.1 is presented as follows.

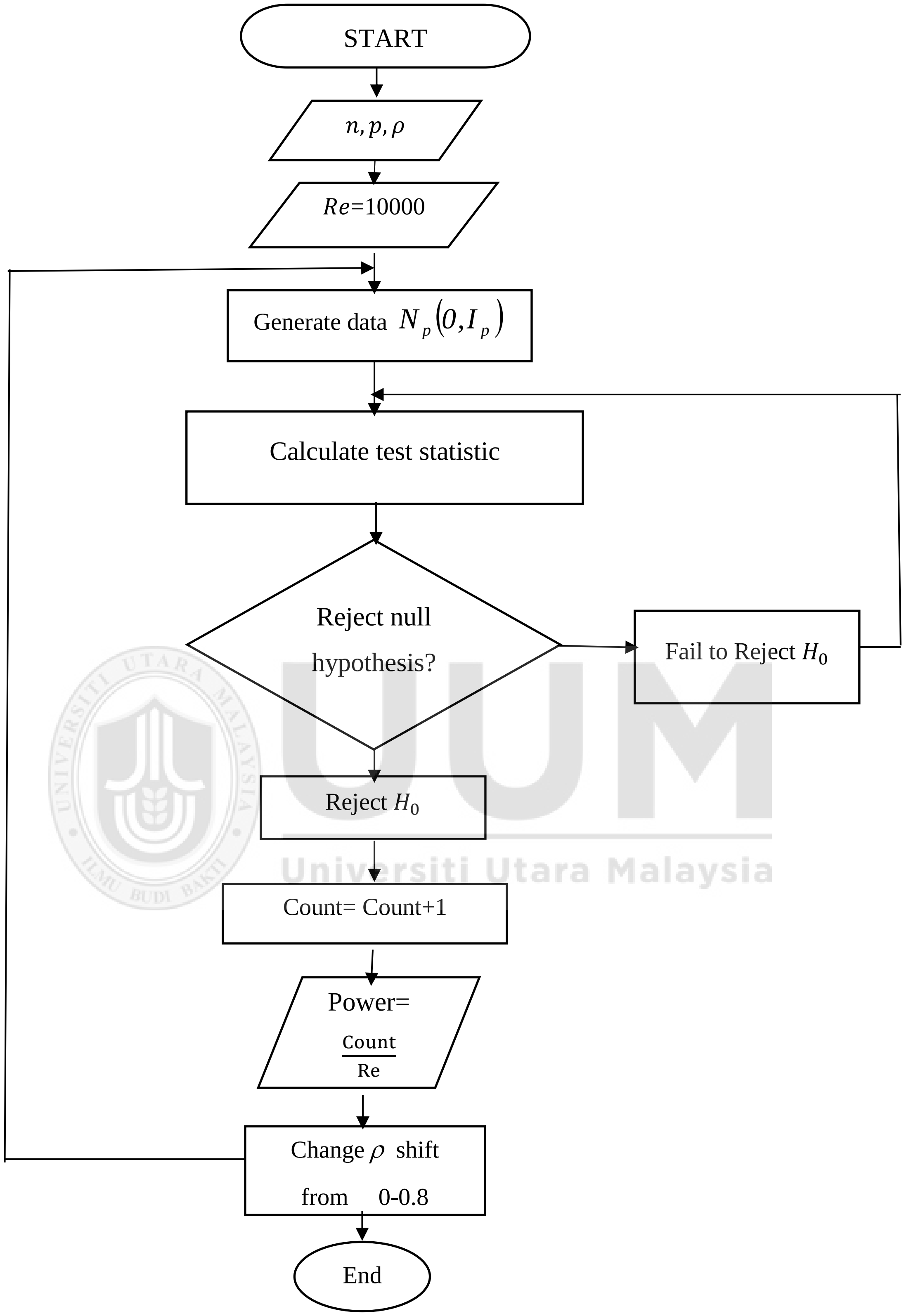

Figure 3.1. The flowchart for power of test

3.7 Validation of the Impact of Tohoku Earthquake on Asia Pacific Currencies Using Correlation Structure

The validation of the method is essential to determine that the method is fit for the proposed statistic and to ensure that the results are acceptable (Nickisch, Nockemann, Tillack, Murphy, & Sturges, 1997). To validate the alternative statistical test, the currencies from Asia Pacific countries were employed to analyze the differences in correlation matrices between before and after Tohoku earthquake incident.

The Tohoku earthquake ripped apart the seafloor on 11 March 2011, with a magnitude of 9.0. The earthquake shocked the Pacific coast of the northeastern part of Japan and included Miyagi, Fukushima, Ibaragi and Sanriku (Imamura & Anawat, 2011). It was very powerful and separated Japan by 8 feet from mainland Asia. The earthquake had a severe impact on the people of Japan. The number of deaths reached 15,883 and 2,671 were reported missing (Hood, Kamesaka, Nofsinger, & Tamura, 2013). The main cause of death was a tsunami and 57% of the deaths happened in Tohoku in Miyagi state, 33% in Iwate state and 9% in Fukushima state (Mori, Takahashi, Yasuda, & Yanagisawa, 2011). The early estimates of the insured losses were around 25 billion US dollars, and the total estimated economic lost exceeded 200 billion US dollars. The greatest damage was done to the three nuclear reactors that had failed and exploded. The waves of the tsunami spread throughout the Pacific and destroyed many port and harbor areas in Hawaii, Oregon and California and buildings along the coasts of Guam and Chile. However, the greatest devastation happened in Japan (Satake, 2013).

With respect to the Japanese economy, the earthquake also affected the Tokyo exchange, one of the biggest stock exchanges in the world. The everyday average market is 20 billion dollars, and the stock market felt of the influence of the earthquake and tsunami when trading opened (Hood et al., 2013). The earthquake not only affected stocks, but also the Japanese Yen (JPY) that is the national currency of Japan, which is the third largest economy in the world (Botman, de Carvalho Filho, & Lam, 2013; Cooper, Donnelly, & Johnson, 2011). Overall, the impact of the Tohoku earthquake reduced the value of JPY against the major currencies of the world and produced changes in stock market (Atiany & Sharif, 2015; Cooper et al., 2011).

Generally, a currency is used for payments of transactions within a country, and the exchange from one currency to another currency is needed for conducting business abroad or engaging in financial transactions with residents and businesses in other countries. So the exchange rates have direct impact on all markets because the price of any asset is expressed in terms of a currency (Górski, Drozdz, & Kwapien, 2008). The foreign exchange rate is considered to be a measure of the economic balance of a country and reflects the whole economic status of a country (Mizuno, Takayasu, & Takayasu, 2006). Therefore, the currency exchange rate plays an important role in economic growth.

This study focuses on the currencies of the Asia Pacific because they are nearest to Japan and the impact of Tohoku earthquake was felt in all countries in this region. In addition, the economy of each Asian country is considered to be part of the general economy of Asia region (Nguyen, 2012).

3.7.1 Data Preparation for the Case Study

Generally, financial data and currency exchange rate in particular is time series data that is dependent and autocorrelated (Sewell, 2011). For that reason, the tests were not performed on the original data but on the change of the data to logarithms of the price of currency, *j*. Mantegna and Stanley (2000) calculated the variable of interest as follows:

$$Y_j = ln\, P(t+1) - ln\, P_j(t), j = 1,2,\ldots,p \qquad (3.2\ )$$

where

Y_j is the daily change of the of the price of currency *j* at time *t*.

$P_j(t)$ is the dialy closure price of currency *j* at time *t*.

p is the number of variables (currencies).

The change of data to the logarithm price of the currencies as in (3.2) makes that data independent and stable (Fama, 1965).

In this example, the correlation matrices of exchange rate were analysed for 23 of the Asia Pacific exchange currencies $p = 23$; the data retrieved comprised daily data from January 1, 2010 to December 31, 2011. The data was downloaded from the University of British Columbia, Sauder School of Business (2011) Pacific Exchange Rate Service (http://fx.sauder.ubc.ca/data.html). Those currencies are listed in Table 3.2, and the study gathered two samples, one after and one before the Tohoku earthquake in March 2011. For the base currency, precious metal such as gold, silver and platinum can be used (Mizuno et al., 2006) because precious metals have been proven to offer a safe haven and are utilized to preserve fortunes and add safety to otherwise uncertain financial futures.

Furthermore, Jang, Lee, and Chang (2011) suggested using the Special Drawing Right (SDR) as a base, which is usable freely on the currencies of international monetary system. Therefore, the SDR is used as a base in this research. In general,

the data for currency exchange rates around 5 trading days per week, the data is monthly we have 24 samples. Equation 3.2 was used to change the data to the logarithm of the price. Table 3.2 below illustrates the Asia Pacific currencies used in validation the tests.

Table 3.2

Asia Pacific Currencies

No.	**Currency**	
1	AUD	Australia Dollar
2	CAD	Canadian Dollar
3	CNY	Chinese Yen
4	COP	Colombian Peso
5	FJD	Fijian Dollar
6	XPF	French- Pacific Francs
7	GHS	Ghanaian Cedis
8	HKD	Hong Kong Dollars
9	INR	Indian Rupiah
10	IDR	Indonesian Rupiah
11	JPY	Japanese Yen
12	MYR	Malaysian Ringgit
13	NZD	New Zealand Dollar
14	PKR	Pakistani Rupees
15	PEN	Peruvian Sol
16	PHP	Philippines Pesos
17	RUB	Russian Rubles
18	SGD	Singapore Dollars
19	LKR	Sir Lank Rupees
20	KRW	South Korean Won
21	THB	Thai Baht
22	USD	United States Dollar
23	VND	Vietnamese Dong

3.7.2 Testing the Currencies Correlation Matrices of Asia Pacific Currencies

This section presents two different approach validation of the new alternative statistic Z^* and the T^*statistic. This is as follows:

i. Testing two independent samples of correlation matrices; comparing before and after Tohoku earthquake. Specifically, February and April 2011 represent the periods before and after the Tohoku earthquake, respectively; and

ii. Testing several independent samples of correlation matrices using a control chart; comparing *i*-th sample and reference sample, $i = 1, 2, \dots, m$.

The hypothesis statement of the first approach is,

$$H_0 : \Omega_{before} = \Omega_{after} \text{ versus}$$

$$H_1 : \Omega_{before} \neq \Omega_{after} .$$

The hypothesis statement of the second approach is,

$$H_0 : \Omega_i = \Omega_0 \text{ versus}$$

$$H_1 : \Omega_i \neq \Omega_0$$

The second approach is known as the MSPC approach. However, by using this approach the stability is equivalent to testing the hypothesis of the similarity of the two correlation (or covariance) matrices done repeatedly that the approach correlation matrix is equal to a particular matrix of constants. Assume *m* independent samples are available, each of size $n_1, n_2, ..., n_m$ drawn from a p-variate normal distribution with positive definite covariance matrix. To test the hypothesis H_0 versus H_1(Montgomery, 2005) stated that testing the stability of approach correlation structure, H_0 versus H_1 is equivalent is equivalent to a repeated the tests of the hypothesis of correlation matrix is equal to a particular matrix of constants. In this approach Ω_i is the *i*-th monthly sample of correlation of size $p \times p$; from January 2010 until December 2011. Meanwhile, Ω_0 is the reference correlation sample that refer to the pooled correlation sample.

However, before the hypothesis testing is performed , Q-Q plot is presented in order to check the normality assumption of Z^* statistic. It is because Z^* statistic is developed on the basis of mutlivarite normal distribution. To confirm that, we test the hypothesis that the data are normally distributed. Therefore, the hypothesis are H_0: The data follow normal distribution versus H_1: The data do not follow normal distribution. Anderson-Darling (A-D) test is implemented to achieve the target. The test is used for checking the multivariate normality assumption (Rahman, Pearson, & Heien, 2006) as required by Z^* statistic.

CHAPTER FOUR
RESULTS AND ANALYSIS

4.1 Introduction

This chapter presents the results of analysis on the new alternative statistical test Z^*, which constructed based on *vec* operator, commutation matrix, and Forbenius norm of upper-off-diagonal elements. The search for alternative statistical test is justified by transforming the sample correlation matrix into vector of correlation where its elements is the upper-off-diagonal elements only which is called $v\left(R_U\right)$. The upper-off-diagonal elements is used to ensure the non-singularity problem Sharif, (2013) since the sample correlation matrix is a symmetric matrix and having many redundant elements (Schott, 1997). Futhermore, the commutation matrix can help to simplify the investigation of parameters (Djauhari & Herdiani, 2008).

To verify the claim, firstly, we derive the formulation of alternative statistical test using a theorem and asymptotic distribution. Next, in order to achieve the objective, the power of test is conducted for comparing three different statistics which are alternative statistical test, Jennrich statistic, and T^* statistic. The real data on financial study is used to validate the performance of the alternative statistic. Two different conditions of samples which are two independent samples of correlation matrices, and several independent samples of correlation matrices are presented at the end of this chapter.

4.2 Asymptotic Distribution of Correlation Matrix when $p = 2$

Findings are elaborated based on the asymptotic distribution of R investigated from particular case where $p = 2$ to general case where $p > 2$. To construct the new alternative statistical test Z^*, the asymptotic distribution of the correlation matrix developed by Browne and Shapiro (1986), and Neudecker and Wesselman (1990) are used.

Let $X_1, X_2, ..., X_n$ be a random sample drawn from p-variate normal distribution $N(\mu, \Sigma)$ of size n with mean vector μ and positive definite covariance matrix Σ. The sample mean vector and covariance matrix are respectively as follows,

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } S = \frac{1}{n-1}[B] \text{ where } B = \sum_{i=1}^{n}(X_i - \bar{X})(X_i - \bar{X})^t.$$

In this section, the asymptotic distributions of correlation matrix is examined from $p = 2$.

Firstly, the investigation on the asymptotic distribution for the correlation matrix R of size 2×2 is presented using a multivariate process approach and Theorem 3.2. When $p = 2,$ we denote $X_i = \begin{pmatrix} X_{i1} \\ X_{i2} \end{pmatrix}$ for bivariate normal distribution, assume that $\mathrm{E}\left(X_{ij}^4\right) < \infty,\ i = 1,\ 2,\ ...,\ n,$ and $j = 1, 2.$ Without loss of generality, assume that $\mathrm{E}(X_{ij}) = 0$. Therefore, to derive the asymptotic distribution of the sample correlation $r = \frac{s_{12}}{\sqrt{s_{11}s_{22}}}$, let consider the distribution of the next random vector,

$$\begin{pmatrix} M_1 \\ M_2 \\ M_{11} \\ M_{22} \\ M_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} X_{1i} \\ \frac{1}{n}\sum_{i=1}^{n} X_{2i} \\ \frac{1}{n}\sum_{i=1}^{n} X_{1i}^{2} \\ \frac{1}{n}\sum_{i=1}^{n} X_{2i}^{2} \\ \frac{1}{n}\sum_{i=1}^{n} X_{1i} X_{2i} \end{pmatrix} \tag{4.1}$$

Then, $s_1^2 = M_{11} - M_1, s_2^2 = M_{22} - M_2$ and $s_{12}^2 = M_{12} - M_1 M_2$

Since $\begin{pmatrix} M_1 \\ M_2 \\ M_{11} \\ M_{22} \\ M_{12} \end{pmatrix} \xrightarrow{p} \begin{pmatrix} 0 \\ 0 \\ \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$ according to central limit theorem,

$\begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} X_{1i} \\ \frac{1}{n}\sum_{i=1}^{n} X_{2i} \\ \frac{1}{n}\sum_{i=1}^{n} X_{1i}^{2} \\ \frac{1}{n}\sum_{i=1}^{n} X_{2i}^{2} \\ \frac{1}{n}\sum_{i=1}^{n} X_{1i} X_{2i} \end{pmatrix}$ converges to multivariate normal distribution, thereafter

$\sqrt{n}\left[\begin{pmatrix} M_1 \\ M_2 \\ M_{11} \\ M_{22} \\ M_{12} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}\right]$ converges to $N(0,\Sigma)$.

Where,

$$\Sigma = \begin{pmatrix} Cov\left(X_1, X_1\right) & Cov\left(X_1, X_2\right) & Cov\left(X_1, X_1^2\right) & Cov\left(X_1, X_2^2\right) & Cov\left(X_1, X_1 X_2\right) \\ Cov\left(X_2, X_1\right) & Cov\left(X_2, X_2\right) & Cov\left(X_2, X_1^2\right) & Cov\left(X_2, X_2^2\right) & Cov\left(X_2, X_1 X_2\right) \\ Cov\left(X_1^2, X_1\right) & Cov\left(X_1^2, X_2\right) & Cov\left(X_1^2, X_1^2\right) & Cov\left(X_1^2, X_2^2\right) & Cov\left(X_1^2, X_1 X_2\right) \\ Cov\left(X_2^2, X_1\right) & Cov\left(X_2^2, X_2\right) & Cov\left(X_2^2, X_1^2\right) & Cov\left(X_2^2, X_2^2\right) & Cov\left(X_2^2, X_1 X_2\right) \\ Cov\left(X_1 X_2, X_1\right) & Cov\left(X_1 X_2, X_2\right) & Cov\left(X_1 X_2, X_1^2\right) & Cov\left(X_1 X_2, X_2^2\right) & Cov\left(X_1 X_2, X_1 X_2\right) \end{pmatrix}$$

Moreover, since r depends on M_{11}, M_{22} and M_{12} the we define function from $R^5 \rightarrow R^3$ applied on (4.1) therefore, $\begin{pmatrix} M_{11} \\ M_{22} \\ M_{12} \end{pmatrix}$ and its distribution are investigated.

According to Sharif (2013), by using the delta then,

$$\sqrt{n}\left[\begin{pmatrix} M_{11} \\ M_{22} \\ M_{12} \end{pmatrix} - \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}\right] \xrightarrow{d} N(0,\Phi),$$

where

$$\gamma = \begin{pmatrix} Cov\left(X_1^2, X_1^2\right) & Cov\left(X_1^2, X_2^2\right) & Cov\left(X_1^2, X_1 X_2\right) \\ Cov\left(X_2^2, X_1^2\right) & Cov\left(X_2^2, X_2^2\right) & Cov\left(X_2^2, X_1 X_2\right) \\ Cov\left(X_1 X_2, X_1^2\right) & Cov\left(X_1 X_2, X_2^2\right) & Cov\left(X_1 X_2, X_1 X_2\right) \end{pmatrix}$$

By using Theorem 3.2, the vector $\sqrt{n}[U(n)-b]$ has normal distribution with mean 0 and covariance matrix γ, where $U(n)=\begin{pmatrix} M_{11} \\ M_{22} \\ M_{12} \end{pmatrix}$ and $b=\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$, the covariance matrix is as follows,

$$\gamma=\begin{pmatrix} 2\sigma_{11}^2 & 2\rho^2\sigma_{11}\sigma_{22} & 2\rho\sqrt{\sigma_{11}^3\sigma_{22}} \\ 2\rho^2\sigma_{11}\sigma_{22} & 2\sigma_{22}^2 & 2\rho\sqrt{\sigma_{22}^3} \\ 2\rho\sqrt{\sigma_{11}^3\sigma_{22}} & 2\rho\sqrt{\sigma_{11}\sigma_{22}^3} & (1+\rho)\sigma_{11}\sigma_{22} \end{pmatrix}.$$

To find the distribution of r, we define function f from R^3 to R, as follows,

$$f(v_1, v_2, v_3)=\frac{v_3}{\sqrt{v_1v_2}}=v_3v_1^{-\frac{1}{2}}v_2^{-\frac{1}{2}}$$

where $M_{11}=v_1, M_{22}=v_2, M_{12}=v_3$ and the function $f(\sigma_{12},\sigma_{11},\sigma_{22})=\rho$. Then, $f(U(n))=r$ and $f(b)=\rho$ and according to Theorem 3.2, $\sqrt{n}(f(U(n))-f(b))=\sqrt{n}(r-\rho)\xrightarrow{d}N(0,\Delta)$ where $\Delta=\omega^t\gamma\omega$ with $\omega^t=\left(\frac{\partial f}{\partial v_1}\frac{\partial f}{\partial v_2}\frac{\partial f}{\partial v_3}\right)$. Then,

$$\left.\frac{\partial r}{\partial v_1}\right|_{v_1=b}=-\frac{1}{2}v_3v_1^{-\frac{3}{2}}v_2^{-\frac{1}{2}}\Big|_{v_1=b}=-\frac{1}{2}\frac{v_3}{v_1^{\frac{3}{2}}v_2^{\frac{1}{2}}}=-\frac{1}{2}\frac{v_3}{\sqrt{v_1v_2}v_1}=-\frac{\rho}{2\sigma_{11}}$$

$$\left.\frac{\partial r}{\partial v_2}\right|_{v_2=b}=-\frac{1}{2}v_3v_2^{-\frac{3}{2}}v_1^{-\frac{1}{2}}\Big|_{v_2=b}=-\frac{\rho}{2\sigma_{22}}$$

$$\left.\frac{\partial r}{\partial v_3}\right|_{v_3=b} = \left. v_1^{-\frac{1}{2}} v_2^{-\frac{1}{2}} \right|_{v_3=b} = \frac{1}{\sqrt{\sigma_{11}\sigma_{22}}}$$

Thus $\omega = \begin{pmatrix} -\dfrac{\rho}{2\sigma_{11}} \\ -\dfrac{\rho}{2\sigma_{22}} \\ \dfrac{1}{\sqrt{\sigma_{11}\sigma_{22}}} \end{pmatrix}$

$$\Delta = \omega^t \gamma \omega = \begin{pmatrix} -\dfrac{\rho}{2\sigma_{11}} & -\dfrac{\rho}{2\sigma_{22}} & \dfrac{1}{\sqrt{\sigma_{11}\sigma_{22}}} \end{pmatrix} \times \begin{pmatrix} 2\sigma_{11}^2 & 2\rho^2\sigma_{11}\sigma_{22} & 2\rho\sqrt{\sigma_{11}^3\sigma_{22}} \\ 2\rho^2\sigma_{11}\sigma_{22} & 2\sigma_{22}^2 & 2\rho\sqrt{\sigma_{22}^3} \\ 2\rho\sqrt{\sigma_{11}^3\sigma_{22}} & 2\rho\sqrt{\sigma_{11}\sigma_{22}^3} & (1+\rho)\sigma_{11}\sigma_{22} \end{pmatrix} \times$$

$$\begin{pmatrix} -\dfrac{\rho}{2\sigma_{11}} \\ -\dfrac{\rho}{2\sigma_{22}} \\ \dfrac{1}{\sqrt{\sigma_{11}\sigma_{22}}} \end{pmatrix}$$

$$= \left(-\frac{\rho}{2\sigma_{11}} 2\sigma_{11}^2 - \frac{\rho}{2\sigma_{22}} 2\rho^2\sigma_{11}\sigma_{22} + \frac{1}{\sqrt{\sigma_{11}\sigma_{22}}} 2\rho\sqrt{\sigma_{11}^3\sigma_{22}} \quad -\frac{\rho}{2\sigma_{11}} 2\rho^2\sigma_{11}\sigma_{22} - \frac{\rho}{2\sigma_{22}} 2\sigma_{22}^2 + \right.$$

$$\left. \frac{1}{\sqrt{\sigma_{11}\sigma_{22}}} 2\rho\sqrt{\sigma_{11}\sigma_{22}^3} \quad -\frac{\rho}{2\sigma_{11}} 2\rho\sqrt{\sigma_{11}^3\sigma_{22}} - 2\rho\sqrt{\sigma_{11}\sigma_{22}^3} + \frac{1}{\sqrt{\sigma_{11}\sigma_{22}}} (1+\rho)\sigma_{11}\sigma_{22} \right) \times$$

$$= \left(-\rho\sigma_{11} - \rho^3\sigma_{11} + 2\rho\sigma_{11} \quad -\rho^3\sigma_{22} - \rho\sigma_{22} + 2\rho\sigma_{22} \quad -\rho^2\sqrt{\sigma_{11}\sigma_{22}} - (1+\rho^2)\sqrt{\sigma_{11}\sigma_{22}} \right) \times$$

$$\begin{pmatrix} -\dfrac{\rho}{2\sigma_{11}} \\ -\dfrac{\rho}{2\sigma_{22}} \\ \dfrac{1}{\sqrt{\sigma_{11}\sigma_{22}}} \end{pmatrix}$$

$$=\left(-\rho^3\sigma_{11}+\rho\sigma_{11}\right)\left(-\frac{\rho}{2\sigma_{11}}\right)+\left(-\rho^3\sigma_{22}+\rho\sigma_{22}\right)\left(-\frac{\rho}{2\sigma_{22}}\right)+\left(-2\rho^2\sqrt{\sigma_{11}\sigma_{22}}+\left(1+\rho^2\right)\sqrt{\sigma_{11}\sigma_{22}}\right)\times\frac{1}{\sqrt{\sigma_{11}\sigma_{22}}}$$

$$=\frac{\rho^4}{2}-\frac{\rho^2}{2}+\frac{\rho^4}{2}-\frac{\rho^2}{2}-2\rho^2+\left(1+\rho^2\right)$$

$$=\rho^4-\rho^2+2\rho^2+1+\rho^2$$

$$=\rho^4-2\rho^2+1$$

$$=1-2\rho^2+\rho^4$$

$$=\left(1-\rho^2\right)^2$$

Which implies that $\sqrt{n}(r-\rho)\xrightarrow{d}N\left(0,\left(1-\rho^2\right)^2\right)$. Next, we derive the asymptotic distribution of correlation matrix when $p>2$.

4.3 Asymptotic Distribution of Correlation Matrix When $p>2$

We presented the methodology for $p=2$ in the previous section. The correlation matrix when $p=2$ of size 2×2, is as follows,

$$R=\begin{pmatrix}1 & r_{12}\\ r_{21} & 1\end{pmatrix}$$

The matrix is transforms to r_{12} which is the upper-off-diagonal element. The generalization of this transformation matrix is a part of our contribution in this study. To simplify the sample correlation matrix into vector correlation, the notion of *vec* operator and commutation matrix are used. Following, we formulated the asymptotic distribution of R based on Theorem 3.1.

Theorem 3.1 (Schott, 2007b)

Let $X_1, X_2, ..., X_n$ be a random vector drawn from p-varite normal distribution of size n then $\sqrt{n-1}[vec(R) - vec(\Omega)] \xrightarrow{d} N_{p^2}(0, \Gamma)$, where $\Gamma = 2M_p \phi M_p$.

However, before performing the derivation of new statistic, the proving of covariance of *vec*(S) is shown in the next section

4.3.1 Covariance of $vec(S)$

The exact distribution of S under the normality is Wishart distribution. From Anderson (2003) we can write $S = \frac{1}{n-1}\sum_{i=1}^{n-1} Z_i Z_i^t$ where $Z_1, Z_2, ..., Z_n$ be an i.i.d. random vector from $N(0, \Sigma)$. The probability density function of $B = \sum_{i=1}^{n-1} Z_i Z_i^t$ is

$$f(A) = \frac{|A|^{\frac{1}{2}(n-p-1)} e^{-\frac{1}{2}Tr\Sigma^{-1}A}}{2^{\frac{1}{2}pn} \pi^{\frac{p(p-1)}{4}} |\Sigma|^{\frac{1}{2}n} \Pi_{i=1}^{n} \Gamma\left[\frac{1}{2}(n+1-i)\right]}$$

When $p = 1$, Wishart reduces to a central Chi-square distribution with (n-1) degrees of freedom (Kollo & Von Rosen, 2006). Conversely, when number of variables is larger than two, Wishart distribution is impractical (Sheppard, 2008). Due to this limitation, the distribution of S will be approximated.

When $n \to \infty$ the distribution of S is approximated by using multivariate central limit theorem. The asymptotic distribution of S is given in Proposition 4.1.

Proposition 4.1(Kollo & Von Rosen, 2006)

If $n \to \infty$, according to central limit theorem, the asymptotic distribution of S is equal to $\sqrt{n-1}vec\left(S-\Sigma\right)\xrightarrow{d} N_p\left(0, var(S)\right)$where the covariance of $vec(S)=\left(I_{p^2}+K_{pp}\right)\left(\Sigma \otimes \Sigma\right)$.

Proof:

(i) Since $S=\frac{1}{n-1}B$ and $B=\sum_{i=1}^{n}\left(X_i-\bar{X}\right)\left(X_i-\bar{X}\right)^t$, then $B=\sum_{i=1}^{n}X_iX_i^{\,t}-n\bar{X}\ \bar{X}^t$

where

$\sum_{i=1}^{n}X_iX_i^t=\sum_{i=1}^{n}Z_nZ_n^t$ and $\bar{X}=\frac{1}{\sqrt{n}}Z_n$

thus, $\sqrt{n}\bar{X}=Z_n$ and $\sqrt{n}\bar{X}^t=Z_n^t$.

$$B=\sum_{i=1}^{n}X_iX_i^{\,t}-n\bar{X}\ \bar{X}^t=\sum_{i=1}^{n}Z_iZ_i^{\,t}-\sqrt{n}\bar{X}\sqrt{n}\bar{X}^t=\sum_{i=1}^{n}Z_iZ_i^{\,t}-Z_nZ_n^t$$

$$=Z_1Z_1^t+Z_2Z_2^t+...+Z_nZ_n^t-Z_nZ_n^t$$

$$=\sum_{i=1}^{n-1}Z_nZ_n^t$$

Hence, $S=\frac{1}{n-1}\sum_{i=1}^{n-1}Z_iZ_n^t$.

(ii) Now we prove $E(S) = \Sigma$.

From Rencher (2003) suppose that $S = \frac{1}{n-1} B$, and $E(X_i) = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix} = \mu$, and

$$\Sigma = E\left[(X - \mu)(X - \mu)^t\right] = E\begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ \vdots \\ X_p - \mu_p \end{pmatrix}\begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ \vdots \\ X_p - \mu_p \end{pmatrix}^t$$

$$= E\begin{pmatrix} (X_1-\mu_1)^2 & (X_1-\mu_1)(X_2-\mu_2) & \cdots & (X_1-\mu_1)(X_p-\mu_p) \\ (X_1-\mu_1)(X_2-\mu_2) & (X_2-\mu_2)^2 & \cdots & (X_2-\mu_2)(X_p-\mu_p) \\ \vdots & \vdots & \vdots & \vdots \\ (X_p-\mu_p)(X_1-\mu_1) & (X_p-\mu_p)(X_2-\mu_2) & \cdots & (X_p-\mu_p)^2 \end{pmatrix}$$

$$= \begin{pmatrix} E(X_1-\mu_1)^2 & E(X_1-\mu_1)(X_2-\mu_2) & \cdots & E(X_1-\mu_1)(X_p-\mu_p) \\ E(X_1-\mu_1)(X_2-\mu_2) & E(X_2-\mu_2)^2 & \cdots & E(X_2-\mu_2)(X_p-\mu_p) \\ \vdots & \vdots & \vdots & \vdots \\ E(X_p-\mu_p)(X_1-\mu_1) & E(X_p-\mu_p)(X_2-\mu_2) & \cdots & E(X_p-\mu_p)^2 \end{pmatrix}, \text{ then}$$

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{pmatrix} = Cov(X_i)$$

$Cov(\bar{X}) = \frac{1}{n}\Sigma$, where $\Sigma = nE\left[(\bar{X} - \mu)(\bar{X} - \mu)^t\right]$

$B = \sum_{i=1}^{n}(X_i - \bar{X})(X_i - \bar{X})^t$ we can write it as

$\sum_{i=1}^{n}(X_i - \bar{X})(X_i - \bar{X})^t = \sum_{i=1}^{n}(X_i - \mu)(X_i - \mu)^t - n(\bar{X} - \mu)(\bar{X} - \mu)^t$ and

$$\sum_{i=1}^{n}(X_i-\mu)(X_i-\mu)^t = \sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(\overline{X}-\overline{X}\right)^t + n\left(\overline{X}-\mu\right)\left(\overline{X}-\mu\right)^t$$

$$\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(X_i-\overline{X}\right)^t = \sum_{i=1}^{n}\left[\left[\left(X_i-\overline{X}\right)+\left(\overline{X}-\mu\right)\right]\left[\left(X_i-\overline{X}\right)^t+\left(\overline{X}-\mu\right)^t\right]\right]$$

$$= \sum_{i=1}^{n}\left[\left(X_i-\overline{X}\right)\left(X_i-\overline{X}\right)^t + \left(X_i-\overline{X}\right)\left(\overline{X}-\mu\right)^t + \left(\overline{X}-\mu\right)\left(X_i-\overline{X}\right)^t + \left(\overline{X}-\mu\right)\left(\overline{X}-\mu\right)^t\right]$$

$$= \left[\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(X_i-\overline{X}\right)^t + \sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(\overline{X}-\mu\right)^t + \sum_{i=1}^{n}\left(\overline{X}-\mu\right)\left(X_i-\overline{X}\right)^t + \sum_{i=1}^{n}\left(\overline{X}-\mu\right)\left(\overline{X}-\mu\right)^t\right]$$

Since $\sum_{i=1}^{n}\left(X_i-\overline{X}\right)=\sum_{i=1}^{n}X_i-\sum_{i=1}^{n}\overline{X}, \overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$ and $n\overline{X}=\sum_{i=1}^{n}X_i$, then

$$= \sum_{i=1}^{n}X_i-\sum_{i=1}^{n}\overline{X} = n\overline{X}-n\overline{X}=0$$

$\sum_{i=1}^{n}(X_i-\mu)(X_i-\mu)^t = \sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(X_i-\overline{X}\right)^t + n\left(\overline{X}-\mu\right)\left(\overline{X}-\mu\right)^t$ hence

$$\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(X_i-\overline{X}\right)^t = \sum_{i=1}^{n}(X_i-\mu)(X_i-\mu)^t - n\left(\overline{X}-\mu\right)\left(\overline{X}-\mu\right)^t$$

Therefore, $E(B)=E\left(\sum_{i=1}^{n}(X_i-\mu)(X_i-\mu)^t\right)-E\left(n\left(\overline{X}-\mu\right)\left(\overline{X}-\mu\right)^t\right)$

$= n\Sigma - n\frac{1}{n}\Sigma = \Sigma(n-1)$, so the $E(S)=\frac{1}{n-1}E(B)$

$$= \frac{1}{n-1}\times\Sigma(n-1)=\Sigma.$$

(iii) According to Anderson (2003) from Theorem 3.4.4 the variance of S $var(S)=Cov(s_{ij},s_{kl})=\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk}$ for i, j, k and $l=1,2,...,p$.

Now for $p=2$ let $\Sigma=\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}$ and $S=\begin{pmatrix}s_{11} & s_{12}\\ s_{21} & s_{22}\end{pmatrix}$, where the

$vec(S) = (s_{11} \quad s_{21} \quad s_{12} \quad s_{22})$ and

$$Cov(vec(S)) = \begin{pmatrix} s_{11}s_{11} & s_{11}s_{21} & s_{11}s_{12} & s_{11}s_{22} \\ s_{21}s_{11} & s_{21}s_{21} & s_{21}s_{12} & s_{21}s_{22} \\ s_{12}s_{11} & s_{12}s_{21} & s_{12}s_{12} & s_{12}s_{22} \\ s_{22}s_{11} & s_{22}s_{21} & s_{22}s_{12} & s_{22}s_{22} \end{pmatrix} \text{ then,}$$

$$\Sigma \otimes \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \otimes \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{11}\sigma_{11} & \sigma_{11}\sigma_{12} & \sigma_{12}\sigma_{11} & \sigma_{12}\sigma_{12} \\ \sigma_{11}\sigma_{21} & \sigma_{11}\sigma_{22} & \sigma_{12}\sigma_{21} & \sigma_{12}\sigma_{22} \\ \sigma_{21}\sigma_{11} & \sigma_{21}\sigma_{12} & \sigma_{22}\sigma_{11} & \sigma_{22}\sigma_{12} \\ \sigma_{21}\sigma_{21} & \sigma_{21}\sigma_{22} & \sigma_{22}\sigma_{21} & \sigma_{22}\sigma_{22} \end{pmatrix}$$

Let, $K_{pp} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $I_{p^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

therefore $I_{p^2} + K_{pp} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$$(I_{p^2} + K_{pp})(\Sigma \otimes \Sigma) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \times \begin{pmatrix} \sigma_{11}\sigma_{11} & \sigma_{11}\sigma_{12} & \sigma_{12}\sigma_{11} & \sigma_{12}\sigma_{12} \\ \sigma_{11}\sigma_{21} & \sigma_{11}\sigma_{22} & \sigma_{12}\sigma_{21} & \sigma_{12}\sigma_{22} \\ \sigma_{21}\sigma_{11} & \sigma_{21}\sigma_{12} & \sigma_{22}\sigma_{11} & \sigma_{22}\sigma_{12} \\ \sigma_{21}\sigma_{21} & \sigma_{21}\sigma_{22} & \sigma_{22}\sigma_{21} & \sigma_{22}\sigma_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 2\sigma_{11}\sigma_{11} & 2\sigma_{11}\sigma_{12} & 2\sigma_{12}\sigma_{11} & 2\sigma_{12}\sigma_{12} \\ \sigma_{11}\sigma_{21} + \sigma_{21}\sigma_{11} & \sigma_{11}\sigma_{22} + \sigma_{21}\sigma_{12} & \sigma_{12}\sigma_{21} + \sigma_{22}\sigma_{11} & \sigma_{12}\sigma_{22} + \sigma_{22}\sigma_{12} \\ \sigma_{11}\sigma_{21} + \sigma_{21}\sigma_{11} & \sigma_{11}\sigma_{22} + \sigma_{21}\sigma_{12} & \sigma_{12}\sigma_{21} + \sigma_{22}\sigma_{11} & \sigma_{12}\sigma_{22} + \sigma_{22}\sigma_{12} \\ 2\sigma_{21}\sigma_{21} & 2\sigma_{21}\sigma_{22} & 2\sigma_{22}\sigma_{21} & 2\sigma_{22}\sigma_{22} \end{pmatrix}.$$

$$
= \begin{pmatrix} var(s_{11}s_{11}) & Cov(s_{11}s_{21}) & Cov(s_{11}s_{12}) & Cov(s_{11}s_{22}) \\ Cov(s_{21}s_{11}) & var(s_{21}s_{21}) & Cov(s_{21}s_{12}) & Cov(s_{21}s_{22}) \\ Cov(s_{12}s_{11}) & Cov(s_{12}s_{21}) & var(s_{12}s_{12}) & Cov(s_{12}s_{22}) \\ Cov(s_{22}s_{11}) & Cov(s_{22}s_{21}) & Cov(s_{22}s_{12}) & var(s_{22}s_{22}) \end{pmatrix}
$$

Now, for $p = 2$, it is showed that $var(S) = Cov(s_{ij}, s_{kl}) = \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}$. The proving process for the case $p > 2$, is similar. In the next example, we want to illustrate how to calculate the covariance of $vec(S) = (I_{p^2} + K_{pp})(\Sigma \otimes \Sigma)$.

Example 4.1

For $p = 2$, let $\Sigma = \begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}$ and $vec(S) = (s_{11} \quad s_{21} \quad s_{12} \quad s_{22})^t$

$$
var(S) = Cov(vec(S)) = \begin{pmatrix} s_{11}s_{11} & s_{11}s_{21} & s_{11}s_{12} & s_{11}s_{22} \\ s_{21}s_{11} & s_{21}s_{21} & s_{21}s_{12} & s_{21}s_{22} \\ s_{12}s_{11} & s_{12}s_{21} & s_{12}s_{12} & s_{12}s_{22} \\ s_{22}s_{11} & s_{22}s_{21} & s_{22}s_{12} & s_{22}s_{22} \end{pmatrix}
$$

Then,

$$
Cov(vec(S)) = \begin{pmatrix} 2\sigma_{11}\sigma_{11} & 2\sigma_{11}\sigma_{12} & 2\sigma_{12}\sigma_{11} & 2\sigma_{12}\sigma_{12} \\ \sigma_{11}\sigma_{21} + \sigma_{21}\sigma_{11} & \sigma_{11}\sigma_{22} + \sigma_{21}\sigma_{12} & \sigma_{12}\sigma_{21} + \sigma_{22}\sigma_{11} & \sigma_{12}\sigma_{22} + \sigma_{22}\sigma_{12} \\ \sigma_{21}\sigma_{11} + \sigma_{21}\sigma_{11} & \sigma_{11}\sigma_{22} + \sigma_{21}\sigma_{12} & \sigma_{12}\sigma_{21} + \sigma_{22}\sigma_{11} & \sigma_{22}\sigma_{12} + \sigma_{22}\sigma_{12} \\ 2\sigma_{21}\sigma_{21} & 2\sigma_{21}\sigma_{22} & 2\sigma_{22}\sigma_{21} & 2\sigma_{22}\sigma_{22} \end{pmatrix}
$$

$$
= \begin{pmatrix} 2\times2\times2 & 2\times2\times4 & 2\times4\times2 & 2\times4\times4 \\ (2\times4)+(4\times2) & (2\times3)+(4\times4) & (4\times4)+(3\times2) & (4\times3)+(3\times4) \\ (4\times2)+(4\times2) & (2\times3)+(4\times4) & (4\times4)+(3\times2) & (3\times4)+(3\times4) \\ 2\times4\times4 & 2\times4\times3 & 2\times3\times4 & 2\times3\times3 \end{pmatrix}
$$

$$= \begin{pmatrix} 8 & 16 & 16 & 32 \\ 16 & 22 & 22 & 24 \\ 16 & 22 & 22 & 24 \\ 32 & 24 & 24 & 18 \end{pmatrix}, \text{ and}$$

$$I_{p^2} + K_{pp} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \text{ Therefore,}$$

$$\left(I_{p^2} + K_{pp}\right)(\Sigma \otimes \Sigma) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \times \begin{pmatrix} 4 & 8 & 8 & 16 \\ 8 & 6 & 16 & 12 \\ 8 & 16 & 6 & 12 \\ 16 & 12 & 12 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 16 & 16 & 32 \\ 16 & 22 & 22 & 24 \\ 16 & 22 & 22 & 24 \\ 32 & 24 & 24 & 18 \end{pmatrix}$$

Hence $\left(I_{p^2} + K_{pp}\right)(\Sigma \otimes \Sigma) = Cov\left(vec\left(S\right)\right)$. From (i), (ii) and (iii) since, $S = \frac{1}{n-1} B = \frac{1}{n-1}\left(Z_1 Z_1^t + Z_2 Z_2^t + \ldots + Z_{n-1} Z_{n-1}^t\right)$ this implies

$$\frac{1}{\sqrt{n-1}}\left(B - E\left(B\right)\right) = \frac{1}{\sqrt{n-1}}\left(B - (n-1)\Sigma\right) = \sqrt{n-1}\left(S - \Sigma\right).$$

The asymptotic distribution of S based on central limit theorem, is $\sqrt{n-1}\left(S - \Sigma\right) \xrightarrow{d} N_p\left(0, var(S)\right)$. We proved that covariance of

$vec(S) = \left(I_{p^2} + K_{pp}\right)(\Sigma \otimes \Sigma)$, thus $\sqrt{n-1}\left(vec(S) - vec(\Sigma)\right) \xrightarrow{d} N_p\left(0, var(S)\right)$

where covariance of $vec(S) = \left(I_{p^2} + K_{pp}\right)(\Sigma \otimes \Sigma)$.

Next, we illustrate how to find the two parameters mean and variance in the distribution, based on the results of correlation matrix for *R*. To prove the covariance of $vec(R)$ is equal to $\frac{2}{n-1} M_p \phi M_p$ we use the following Proposition 4.2, from Schott (1997, p. 362) and Herdiani (2008).

4.3.2 Covariance of $vec(R)$

In this section, we illustrate how to proof the covariance of vector of correlation matrix $vec(R) \approx \frac{2}{n-1} M_p \phi M_p$. By using the following Proposition 4.2.

Proposition 4.2

If $A = S^* - \Omega$ where S^*a covariance matrix of $Z_1, Z_2, ..., Z_n$, , and suppose Ω is the corresponding population covariance matrix, has each of diagonal elements equal 1. Ω is the population matrix. Then, R it can be approximated by first order approximation as the following values

$$R \approx \Omega + A - \frac{1}{2}\left(\Omega D_A + D_A \Omega\right).$$

Where $D_A = \text{diag}\left(a_{11}, a_{22}, ..., a_{pp}\right)$.

Proof:

By using Proposition 4.2, an approach is attained for $vec(R)$ as follows,

$$vec(R) \approx vec\left(\Omega + A - \frac{1}{2}\left(\Omega D_A + D_A \Omega \right) \right)$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2} vec\left(\Omega D_A + D_A \Omega \right)$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2}\left\{ vec\left(\Omega D_A \right) + vec\left(D_A \Omega \right) \right\}$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2}\left\{ vec\left(\Omega D_A I_p \right) + vec\left(D_A \Omega I_p \right) \right\}$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2}\left\{ vec\left(\Omega D_A I_p \right) + vec\left(D_A \Omega I_p \right) \right\}$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2}\left\{ \left(I_p \otimes \Omega \right) vec\left(D_A \right) + \left(\Omega \otimes I_p \right) vec\left(D_A \right) \right\}$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2}\left\{ \left(I_p \otimes \Omega \right) + \left(\Omega \otimes I_p \right) \right\} vec\left(D_A \right)$$

Now $vec\left(D_A\right)$. For $D_A = \sum_{i=1}^{p} h_i a_{ii} h_i^t$ where h_i is the i-th column of I.

$$\left(D_A\right) = \sum_{i=1}^{p} h_i \left(h_i A h_i \right) h_i^t, \; a_{ii} = h_i^t A h_i$$

$$= \sum_{i=1}^{p} \left(h_i h_i^t \right) A \left(h_i h_i^t \right)$$

$$= \sum_{i=1}^{p} H_{ii} A H_{ii} \text{ where } H_{ii} = h_i h_i^t$$

Therefore, $vec\left(D_A\right) = \sum_{i=1}^{p} vec\left(H_{ii} A H_{ii} \right)$

$$= \sum_{i=1}^{p} \left(H_{ii}^t \otimes H_{ii} \right) vec(A), \text{ since } H_{ii} \text{ is symmetric}$$

$$= \Lambda_p vec(A), \; \Lambda_p = \sum_{i=1}^{p} \left(H_{ii}^t \otimes H_{ii} \right)$$

Therefore,

$$var\left(vec\left(A\right)\right)=var\left(vec\left(S^{*}-\Omega\right)\right)=var\left(vec\left(S^{*}\right)-vec\left(\Omega\right)\right)$$

$$=var\left(vec\left(S^{*}\right)\right)\ ,\ \Omega \text{ is constatnt matrix}$$

$$=var\left(vec\left(D_{\Sigma}^{-\frac{1}{2}}SD_{\Sigma}^{-\frac{1}{2}}\right)\right)=var\left(\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)vec\left(S\right)\right)$$

$$=E\left(\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)vec(S)\right)\left(\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)vec\left(S\right)\right)^{t}-$$

$$E\left(\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)vec(S)\right)E\left(\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)vec\left(S\right)\right)^{t}$$

$$=\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)E\left(vec\left(S\right)\left(vec\left(S\right)\right)^{t}\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)^{t}\right)-$$

$$\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)E\left(vec\left(S\right)\right)E\left(vec\left(S\right)\right)^{t}\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)^{t}$$

$$=\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)E\left(vec\left(S\right)\left(vec\left(S\right)\right)^{t}\right)\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)^{t}-\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)$$

$$E\left(vec\left(S\right)E\left(vec\left(S\right)\right)^{t}\right)\quad\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)^{t}$$

$$=\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)\left[E\left(vec\left(S\right)\left(vec\left(S\right)\right)^{t}\right)-E\left(vec\left(S\right)E\left(vec\left(S\right)\right)^{t}\right)\right]\times$$

$$\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)^{t}$$

$$=\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)var\left(vec\left(S\right)\right)\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)^{t}.$$

From above the covariance matrix of $vec(S)$ is equal to $var\left(vec\left(S\right)\right)=\frac{1}{n-1}\left(I_{p^2}+K_{pp}\right)\left(\Sigma\otimes\Sigma\right)$.

$$var\left(vec\left(A\right)\right)=\frac{1}{n-1}\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)\left(\left(I_{p^2}+K_{pp}\right)\left(\Sigma\otimes\Sigma\right)\right)\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)^t$$

$$=\frac{1}{n-1}\left(\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)I_{p^2}+\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)K_{pp}\right)\left(\Sigma\otimes\Sigma\right)\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)^t$$

Note $\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)I_{p^2}=I_{p^2}\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)$ and

$\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)K_{pp}=K_{pp}\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)$

$$var\left(vec\left(A\right)\right)=\frac{1}{n-1}\left(I_{p^2}+K_{pp}\right)\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)\left(\Sigma\otimes\Sigma\right)\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)^t$$

now $\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)^t=\left(\left(D_{\Sigma}^{-\frac{1}{2}}\right)^t\otimes\left(D_{\Sigma}^{-\frac{1}{2}}\right)^t\right)=\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)$

$$var\left(vec\left(A\right)\right)=\frac{1}{n-1}\left(I_{p2}+K_{pp}\right)\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)\left(\Sigma\otimes\Sigma\right)\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)$$

From Schott (1997)

$$\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)\left(\Sigma\otimes\Sigma\right)\left(D_{\Sigma}^{-\frac{1}{2}}\otimes D_{\Sigma}^{-\frac{1}{2}}\right)=\left(D_{\Sigma}^{-\frac{1}{2}}\Sigma D_{\Sigma}^{-\frac{1}{2}}\right)\otimes\left(D_{\Sigma}^{-\frac{1}{2}}\Sigma D_{\Sigma}^{-\frac{1}{2}}\right)$$

$$var(vec(A))=\frac{1}{n-1}\left(I_{p^2}+K_{pp}\right)\left(D_{\Sigma}^{-\frac{1}{2}}\Sigma D_{\Sigma}^{-\frac{1}{2}}\right)\otimes\left(D_{\Sigma}^{-\frac{1}{2}}\Sigma D_{\Sigma}^{-\frac{1}{2}}\right),\ D_{\Sigma}^{-\frac{1}{2}}\Sigma D_{\Sigma}^{-\frac{1}{2}}=\Omega$$

$$var(vec(A)) = \frac{1}{n-1}\left(I_{p^2} + K_{pp}\right)(\Omega \otimes \Omega) \text{since } M_p = \frac{1}{2}\left(I_{p^2} + K_{pp}\right)$$

then $2M_p = \left(I_{p^2} + K_{pp}\right)$

Based on commutation matrix as reported by Schott (1997, p. 403)

$$var(vec(A)) = \frac{2M_p}{n-1}(\Omega \otimes \Omega).$$

According to Herdiani and Djauhari (2012) $M_p(\Omega \otimes \Omega) = M_p(\Omega \otimes \Omega)M_p$, then

$$var(vec(A)) = \frac{2}{n-1}M_p(\Omega \otimes \Omega)M_p$$

By using the value approach for R from Proposition 4.2, to find the mean and variance for $vec(R)$.

In next section, we used Proposition 4.2 to determine the mean of $vec(R)$

Proof:

Suppose that $R \approx \Omega + A - \frac{1}{2}(\Omega D_A + D_A \Omega)$ from proposition 4.2 then using the procedures of *vec*, we set

$$vec(R) \approx vec(\Omega) + vec(A) - \frac{1}{2}\left(vec(\Omega D_A) + vec(D_A \Omega)\right)$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2}\left(\left(I_p \otimes \Omega\right) + \left(I_p \otimes \Omega\right)vec(D_A)\right)$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2}\left(\left(I_p \otimes \Omega\right) + \left(\Omega \otimes I_p\right)\right)vec\left(D_A\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)vec(A)\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)vec(A)\right)$$

Now

$$E\left(vec(R)\right)\approx E\left(vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)vec(A)\right)\right)$$

$$\approx E\left(vec(\Omega)\right)+E\left(I_{p^2}-\frac{1}{2}\left(\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)vec(A)\right)$$

Since Ω is constant $E\left(vec(\Omega)\right)=vec(\Omega)$.

$$E\left(vec(R)\right)\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)\right)E\left(vec(A)\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)\right)\times$$

$$E\left(vec\left(S^*-\Omega\right)\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)\right)\times$$

$$\left(E\left(vec\left(S^*\right)\right)-E\left(vec(\Omega)\right)\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_P\right)\right)\Lambda_p\right)\times$$

$$\left(E\left(vec\left(D_\Sigma^{-\frac{1}{2}}SD_\Sigma^{-\frac{1}{2}}\right)\right)-vec(\Omega)\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_P\right)\right)\Lambda_p\right)\times$$

$$\left(E\left(vec\left(D_\Sigma^{-\frac{1}{2}}SD_\Sigma^{-\frac{1}{2}}\right)\right)-vec(\Omega)\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)\times$$

$$\left(E\left(\left(D_\Sigma^{-\frac{1}{2}}\otimes D_\Sigma^{-\frac{1}{2}}\right)vec(S)\right)-vec(\Omega)\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)\times$$

$$\left(\left(D_\Sigma^{-\frac{1}{2}}\otimes D_\Sigma^{-\frac{1}{2}}\right)E\left(vec(S)\right)-vec(\Omega)\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)\times$$

$$\left(\left(D_\Sigma^{-\frac{1}{2}}\otimes D_\Sigma^{-\frac{1}{2}}\right)vec(\Sigma)-vec(\Omega)\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)\times$$

$$\left(\left(vec\left(D_\Sigma^{-\frac{1}{2}}\Sigma D_\Sigma^{-\frac{1}{2}}\right)-vec(\Omega)\right)\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)\times$$

$$\left(vec\left(D_\Sigma^{-\frac{1}{2}}\Sigma D_\Sigma^{-\frac{1}{2}}\right)-vec(\Omega)\right)$$

$$\approx vec(\Omega)+\left(I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)\left(vec(\Omega)-vec(\Omega)\right)$$

Therefore, $\left(vec(\Omega)-vec(\Omega)\right)=0.$

Thus we obtain $E\left(vec(R)\right)=vec(\Omega)$.

By using the value of R, the variance of $vec(R)$ is obtained.

$$vec(R) \approx vec(\Omega) + \left(I_{p^2} - \frac{1}{2}\left(\left(I_p \otimes \Omega\right) + \left(\Omega \otimes I_p\right)\right)\Lambda_p \right) vec(A)$$

$$var\left(vec\left(R\right)\right) \approx var\left(vec\left(\Omega\right) + \left(I_{p^2} - \frac{1}{2}\left(\left(I_p \otimes \Omega\right) + \left(\Omega \otimes I_p\right)\right)\Lambda_p \right) vec\left(A\right) \right)$$

$$\approx var\left(\left(I_{p^2} - \frac{1}{2}\left(\left(I_p \otimes \Omega\right) + \left(\Omega \otimes I_p\right)\right)\Lambda_p \right) vec\left(A\right) \right)$$

$$\approx \left(I_{p^2} - \frac{1}{2}\left(I_p \otimes \Omega\right) + \left(\Omega \otimes I_p\right)\Lambda_p \right) var\left(vec\left(A\right)\right) \times$$

$$\left(I_{p^2} - \frac{1}{2}\left(\left(I_p \otimes \Omega\right) + \left(\Omega \otimes I_p\right)\right)\Lambda_p \right)^t$$

$$\approx \left(I_{p^2} - \frac{1}{2}\left(I_p \otimes \Omega + \left(\Omega \otimes I_p\right)\right)\Lambda_p \right) \frac{2}{n-1} M_p (\Omega \otimes \Omega) M_p \times$$

$$\left(I_{p^2} - \frac{1}{2}\left(\left(I_p \otimes \Omega\right) + \left(\Omega \otimes I_p\right)\right)\Lambda_p \right)^t$$

Since $\left(I_{p^2} - \frac{1}{2}\left(\left(I_p \otimes \Omega\right) + \left(\Omega \otimes I_p\right)\right)\Lambda_p \right) M_p = M_p\left(\left(I_{p^2} - \left(I_p \otimes \Omega\right)\Lambda_p\right)\right)$ and M_p is symmetric matrix (Schott, 1997) then,

$$\approx \frac{2}{n-1}\left(\left(I_{p^2} - \frac{1}{2}\left(\left(I_p \otimes \Omega\right) + \left(\Omega \otimes I_p\right)\right)\Lambda_p \right) M_p \right)\left(\Omega \otimes \Omega\right) \times$$

$$\left(M_p \left(I_{p^2} - \frac{1}{2}\left(\left(I_p \otimes \Omega\right) + \left(\Omega \otimes I_p\right)\right)\Lambda_p \right)^t \right)$$

$$\approx \frac{2}{n-1}\left(M_p\left(I_{p^2} - \left(I_p \otimes \Omega\right)\Lambda_p\right)\right)\left(\Omega \otimes \Omega\right) \times$$

$$\left(\left(I_{p^2} - \frac{1}{2}\left(\left(I_p \otimes \Omega\right) + \left(\Omega \otimes I_p\right)\right)\Lambda_p \right) M_p \right)^t$$

$$\approx \frac{2}{n-1}\left(M_p\left(I_{p^2}-\left(I_p\otimes\Omega\right)\Lambda_p\right)\right)\left(\Omega\otimes\Omega\right)\times$$

$$\left(M_p\left(I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right)\right)^t$$

$$\approx \frac{2}{n-1}M_p\left(I_{p^2}-\left(I_p\otimes\Omega\right)\Lambda_p\right)\left(\Omega\otimes\Omega\right)\left(I_{p^2}-\left(I_p\otimes\Omega\right)\Lambda_p\right)^t M_p$$

Note, $\left(M_p\left(I_{p^2}-\left(I_p\otimes\Omega\right)\Lambda_p\right)\right)^t=\left(\left(I_{p^2}-\Lambda_p\left(I_p\otimes\Omega\right)M_p\right)\right)$

$$\approx \frac{2}{n-1}M_p\left(I_{p^2}-\left(I\otimes\Omega\right)\Lambda_p\right)\left(\Omega\otimes\Omega\right)\left(I_{p^2}-\Lambda_p\left(I\otimes\Omega\right)\right)M_p\,.$$

However, based on Theorem 3.1, $\phi=\left(I_{p^2}-\left(I_p\otimes\Omega\right)\Lambda_p\right)\left(\Omega\otimes\Omega\right)\left(I_{p^2}-\Lambda_p\left(I_p\otimes\Omega\right)\right)$

then,

$$var\left(vec\left(R\right)\right)\approx\frac{2}{n-1}M_p\phi M_p$$

Thus, we prove the covariance matrix of vector of correlation matrix R is $\frac{2}{n-1}M_p\phi M_p$.

The next example to illustrate how to calculate $\Gamma=2M_p\phi M_p$, for $p=2,\,3$.

Example 4.2

(1) Let $\Omega=\begin{pmatrix}1 & 0.4\\ 0.4 & 1\end{pmatrix}$

To calculate Γ we want to calculate M_p and ϕ

i. $M_p = \frac{1}{2}\left(I_{p^2} + K_{pp}\right)$, from Table 4.1, $K_{pp} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The identity matrix

of size $\left(p^2 \times p^2\right)$, $I_{p^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\Lambda_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Therefore, $M_p = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.50 & 0.50 & 0 \\ 0 & 0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

ii. $\phi = \left\{I_{p^2} - \left(I_p \otimes \Omega\right)\Lambda_p\right\}\left(\Omega \otimes \Omega\right)\left\{I_{p^2} - \Lambda_p\left(I_p \otimes \Omega\right)\right\}$

We start calculate ϕ by calculating the kronecker product for Ω

a) $\left(\Omega \otimes \Omega\right) = \begin{pmatrix} 1 & 0.4 \\ 0.4 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0.4 \\ 0.4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.40 & 0.40 & 0.16 \\ 0.40 & 1 & 0.16 & 0.40 \\ 0.40 & 0.16 & 1 & 0.40 \\ 0.16 & 0.40 & 0.40 & 1 \end{pmatrix}$

and thus $\left(I_p \otimes \Omega\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0.4 \\ 0.4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.40 & 0 & 0 \\ 0.40 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.40 \\ 0 & 0 & 0.40 & 1 \end{pmatrix}$

b) $$I_{p^2} - \left(I_p \otimes \Omega\right)\Lambda_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0.40 & 0 & 0 \\ 0.40 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.40 \\ 0 & 0 & 0.40 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.40 & 1 & 0 & 0 \\ 0 & 0 & 1 & -0.40 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

c) $$I_{p^2} - \Lambda_p\left(I_p \otimes \Omega\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0.40 & 0 & 0 \\ 0.40 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.40 \\ 0 & 0 & 0.40 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -0.40 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -0.40 & 0 \end{pmatrix}.$$

Therefore, from a, b and c we calculate ϕ

$$\phi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.40 & 1 & 0 & 0 \\ 0 & 0 & 1 & -0.40 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0.40 & 0.40 & 0.16 \\ 0.40 & 1 & 0.16 & 0.40 \\ 0.40 & 0.16 & 1 & 0.40 \\ 0.16 & 0.40 & 0.40 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & -0.40 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -0.40 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0.40 & 0 \\ 0 & 0.840 & -0.134 & 0 \\ 0 & -0.134 & 0.840 & 0 \\ 0 & 0.40 & 0.40 & 0 \end{pmatrix}$$

Consequently,

$$\Gamma = 2M_p \phi M_p = 2\times\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0.50 & 0.50 & 0\\ 0 & 0.50 & 0.50 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\times\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0.840 & -0.134 & 0\\ 0 & -0.134 & 0.840 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\times$$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0.50 & 0.50 & 0\\ 0 & 0.50 & 0.50 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0.706 & 0.706 & 0\\ 0 & 0.706 & 0.706 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

2) Suppose $\Omega = \begin{pmatrix}1 & 0.3 & 0.3\\ 0.3 & 1 & 0.3\\ 0.3 & 0.3 & 1\end{pmatrix}$ then $(\Omega\otimes\Omega)$ is equal

$$(\Omega\otimes\Omega) = \begin{pmatrix}
1 & 0.30 & 0.30 & 0.30 & 0.09 & 0.09 & 0.30 & 0.09 & 0.09\\
0.30 & 1 & 0.30 & 0.09 & 0.30 & 0.09 & 0.09 & 0.30 & 0.09\\
0.30 & 0.30 & 1 & 0.09 & 0.09 & 0.30 & 0.09 & 0.09 & 0.30\\
0.30 & 0.09 & 0.09 & 1 & 0.30 & 0.30 & 0.30 & 0.09 & 0.09\\
0.09 & 0.30 & 0.09 & 0.30 & 1 & 0.30 & 0.09 & 0.30 & 0.09\\
0.09 & 0.09 & 0.30 & 0.30 & 0.30 & 1 & 0.09 & 0.09 & 0.30\\
0.30 & 0.09 & 0.09 & 0.30 & 0.09 & 0.09 & 1 & 0.30 & 0.30\\
0.09 & 0.30 & 0.09 & 0.09 & 0.30 & 0.09 & 0.30 & 1 & 0.30\\
0.09 & 0.09 & 0.30 & 0.09 & 0.09 & 0.30 & 0.30 & 0.30 & 1
\end{pmatrix}$$

On the other hand, the commutation matrix for $p = 3$ from Table 2.1, K_{pp} is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and thus } M_p \text{ is}$$

$$M_p = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In addition, ϕ is equal to

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.910 & 0.210 & -0.082 & 0 & -0.019 & -0.019 & -0.254 & -0.630 \\ 0 & 0.210 & 1.910 & -0.019 & 0 & 0.554 & -0.172 & -0.109 & -0.573 \\ 0 & -0.082 & -0.019 & 0.910 & 0 & 0.210 & 0.254 & -0.019 & -0.063 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.019 & 0.554 & 0.210 & 0 & 1.910 & -0.019 & -0.172 & -0.573 \\ 0 & -0.019 & -0.172 & 0.254 & 0 & -0.109 & 2 & 0.600 & 0.600 \\ 0 & 0.254 & -0.109 & -0.019 & 0 & -0.172 & 0.600 & 2 & 0.600 \\ 0 & -0.063 & -0.573 & 0.063 & 0 & -0.573 & 0.600 & 0.600 & 4 \end{pmatrix}$$

Consequently, $\Gamma = 2M_p \phi M_p$

$$
\Gamma = 2 \times \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \times
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.910 & 0.210 & -0.082 & 0 & -0.019 & -0.019 & -0.254 & -0.63 \\
0 & 0.210 & 1.910 & -0.019 & 0 & 0.554 & -0.172 & -0.109 & -0.573 \\
0 & -0.082 & -0.019 & 0.910 & 0 & 0.210 & 0.254 & -0.019 & -0.063 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.019 & 0.554 & 0.210 & 0 & 1.910 & -0.109 & -0.172 & -0.573 \\
0 & -0.019 & -0.172 & 0.254 & 0 & -0.109 & 2 & 0.600 & 0.600 \\
0 & 0.254 & -0.109 & -0.019 & 0 & -0.172 & 0.600 & 2 & 0.600 \\
0 & -0.063 & -0.573 & 0.063 & 0 & -0.573 & 0.600 & 0.600 & 4
\end{pmatrix} \times
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.828 & 0.213 & 0.828 & 0 & 0.213 & 0.213 & 0.213 & -0.126 \\ 0 & 0.214 & 1.783 & 0.213 & 0 & 0.468 & 1.783 & 0.468 & 0.027 \\ 0 & 0.828 & 0.213 & 0.828 & 0 & 0.213 & 0.213 & 0.213 & -0.126 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.213 & 0.468 & 0.213 & 0 & 1.783 & 0.468 & 1.783 & 0.027 \\ 0 & 0.213 & 1.783 & 0.213 & 0 & 0.468 & 1.783 & 0.468 & 0.027 \\ 0 & 0.213 & 0.468 & 0.213 & 0 & 1.783 & 0.468 & 1.783 & 0.027 \\ 0 & 0.126 & 0.027 & -0.126 & 0 & 0.027 & 0.027 & 0.027 & 8 \end{pmatrix}$$

In this section, we proved two mathematical formulation which are the covariance of $vec(S) = (I_{p^2} + K_{pp})(\Sigma \otimes \Sigma)$, and the asymptotic distribution of correlation matrix is normally distributed with variance $\frac{2}{n-1} M_p \phi M_p$.

4.4 Asymptotic Distribution of $v(R_U)$

The correlation matrix is a symmetric and having some redundant elements. To eliminate those elements, we consider only the upper-off-diagonal elements in the correlation matrix, which is denoted $v(R_U)$.

For that purpose, we generalized the linear transformation matrix, which is denote by T in this study in order to remove the non-random elements in the correlation matrix R, by consider $p = 2,3$ and 4.

i. Let $p = 2$, and $R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$.

Therefore, the $vec(R) = \begin{pmatrix} 1 \\ r_{21} \\ r_{12} \\ 1 \end{pmatrix}$, where $T = \left(0 \quad 0 \middle| \quad 1 \quad 0\right)$.

So that $v(R_U) = T \times vec(R) = \left(0 \quad 0 \middle| \quad 1 \quad 0\right) \times \begin{pmatrix} 1 \\ r_{21} \\ r_{12} \\ 1 \end{pmatrix} = r_{12}$.

ii. Let $p = 3$, and $R = \begin{pmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{pmatrix}$.

Therefore, $vec(R) = \begin{pmatrix} 1 & r_{21} & r_{31} & r_{12} & 1 & r_{32} & r_{13} & r_{23} & 1 \end{pmatrix}^t$ so that

$v(R_U) = T \times vec(R)$, where $T = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$

So that $v(R_U) = \begin{pmatrix} r_{12} & r_{13} & r_{23} \end{pmatrix}^t$.

iii. Let $p = 4$, and $R = \begin{pmatrix} 1 & r_{12} & r_{13} & r_{14} \\ r_{21} & 1 & r_{23} & r_{24} \\ r_{31} & r_{32} & 1 & r_{34} \\ r_{41} & r_{42} & r_{43} & 1 \end{pmatrix}$.

Therefore,

$$vec\ (R) = (1 \quad r_{21} \quad r_{31} \quad r_{41} \quad r_{12} \quad 1 \quad r_{32} \quad r_{42} \quad r_{13} \quad r_{23} \quad 1 \quad r_{43} \quad r_{14} \quad r_{24} \quad r_{34} \quad 1)^t$$

Consequently, $T = \left(\begin{array}{cccc|cccc|cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$

So that $v(R_U) = T \times vec(R) = (r_{12} \quad r_{13} \quad r_{23} \quad r_{14} \quad r_{24} \quad r_{34})^t$.

Therefore, the transformation matrix T can be presented in the matrix form as a block matrix $T = (T_1 | T_2 | \ldots | T_p)$, of size $(k \times p^2)$ partitioned into p blocks where $k = \frac{p(p-1)}{2}$. $T_a = (t^a_{i,j})$, each of size $(k \times p)$, T_1 is zero matrix, where $a = 2, 3, \ldots, p$.

$$t^a_{i,j} = \begin{cases} 1; & (i,j) = (C^a_2 - a + b + 1, b) \text{ for } b = 1, 2, \ldots, a\text{-}1 \\ 0; & \text{otherwise} \end{cases}$$

Where C^a_2 is the number of combinations of 2 out of C objects. This transformation matrix is done by modification of transformation that has been used by (Sharif, Ismail, Omar, & Theng, 2016). Subsequently, we illustrate on how to generalize the transformation matrix T.

i. Let $p = 2$,

T partitioned into 2 blocks $T = (T_1 | T_2)$, where $T_1 = (0 \quad 0)$ is zero matrix, the size $k \times p = \frac{p(p-1)}{2} \times p = \frac{2(2-1)}{2} \times 2 = 1 \times 2$, and $T_2 = (1 \quad 0)$.

The entrance of matrix T_2, $a=2$, and $b=1$

$$\left(\binom{2}{2}-2+1+1,1\right)=(1,1), \text{ then } T_2=\begin{pmatrix}1 & 0\end{pmatrix}.$$

ii. Let $p=3$ so that $a=2,3$ and $b=1,2$

$a=2, b=1$ then $\left(\binom{2}{2}-2+1+1=1,1\right)=(1,1)$

$a=3, b=1$ then $\left(\binom{3}{2}-3+1+1=2,1\right)=(2,1)$

$a=3, b=2$ then $\left(\binom{3}{2}-3+2+1=3,2\right)=(3,2)$

$$T=\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right),$$

$$T_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, T_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \text{ and } T_3=\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}.$$

iii. Let $p=4$, so that $a=2,3$ and 4, $b=1,2$ and 3

$a=2, b=1$ then $\left(\binom{2}{2}-2+1+1,1\right)=(1,1)$

$a=3, b=1$ then $\left(\binom{3}{2}-3+1+1=2,1\right)=(2,1)$

$a=3, b=2$ then $\left(\binom{3}{2}-3+2+1=3,2\right)=(3,2)$

$a=4, b=1$ then $\left(\binom{4}{2}-4+1+1=4,1\right)=(4,1)$

$a = 4, b = 2$ then $\left(\binom{4}{2} - 4 + 2 + 1 = 5, 2\right) = (5, 2)$

$a = 4, b = 3$ then $\left(\binom{4}{2} - 4 + 3 + 1 = 6, 3\right) = (6, 3)$

$$T = \left(\begin{array}{cccc|cccc|cccc|cccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)$$

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{and}$$

$$T_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The summary of linear transformation matrix T, for $p = 2,\ 3,\ 4$ and 5 is summarized in Table 4.1.

Table 4.1

Linear Transformation T for $p = 2, 3, 4$ and 5

p	T
2	$\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \end{pmatrix}$
4	$\begin{pmatrix} 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix}$
5	$\begin{pmatrix} 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix}$

The following proposition is derived by Herdiani and Djauhari (2012)

Proposition 4.3

Let $X_1, X_2, ..., X_n$ is a random sample of size *n* from $N_p(\mu, \sigma^2)$. If Ω is correlation matrix then, $\sqrt{n-1}\left[v(R_U) - v(\Omega_U)\right] \xrightarrow{d} N(\mu, \sigma^2)$.

$$\sigma^2 = 4(v(\Omega_U))^t T\, \Gamma T^t v(\Omega_U)$$

$$= 8(v(\Omega_U))^t\ T\ M_p\ \varphi\ M_p T^t\ v(\Omega_U)$$

Where $v(\Omega_U) = T \times vec(\Omega)$ and $v(R_U) = T \times vec(R)$.

Based on Theorem 3.2 we have the following corollary 4.1. The proof on how to find the variance of $v(R_U)$ is delivered.

Corollary 4.1

Let $u(vec(R))$ a real value function of $vec(R)$ and u' exists and $u'(vec(R)) \neq 0$ for all R in the neighborhood of Ω. Therefore, $\sqrt{n-1}\left[u(vec(R) - vec(\Omega))\right] \xrightarrow{d} N(0, \sigma^2)$

where,

$$\sigma^2 = \left(\frac{\partial(vec(\Omega))}{\partial vec(R)}\right)^t \Gamma \left(\frac{\partial(vec(\Omega))}{\partial vec(R)}\right)$$

Based on corollary 4.1, define that $u\left(vec\left(R\right)\right)=\left\|vec\left(R\right)\right\|^{2}$, then the following proposition is produced.

Proposition 4.4

Let $X_1, X_2, ..., X_n$ is a random sample of size *n* from $N\left(\mu,\sigma^2\right)$. If Ω is population correlation matrix then, $\left\|vec\left(R\right)\right\|^{2} \xrightarrow{d} N\left(\mu_{\left\|vec\left(R\right)\right\|^{2}}, \sigma^2_{\left\|vec\left(R\right)\right\|^{2}}\right)$.

With $\mu_{\left\|vec(R)\right\|^{2}} \to \left\|vec(\Omega)\right\|^{2}$ and $\sigma^2_{\left\|vec\left(R\right)\right\|^{2}} \to \frac{8}{n-1}\left(vec\left(\Omega\right)\right)^{t} M_p\ \phi\ M_p vec\left(\Omega\right)$.

Proof:

Note that $u\left(vec\left(R\right)\right)=\left\|vec\left(R\right)\right\|^{2}$ and u' exist $u'\left(R^*\right)\neq 0$ for all R^* in the environment Ω. For $r_{ij} \xrightarrow{p} \rho_{ij}$, for all $i, j = 1, 2, ..., p$ and

$vec\left(R\right) \xrightarrow{d} N\left(vec\left(\Omega\right), \frac{2}{n-1} M_p \phi M_p\right)$, then

$$u\left(vec\left(R\right)\right)=\left\|vec\left(R\right)\right\|^{2} \xrightarrow{d} N\left(\mu_{\left\|vec\left(R\right)\right\|^{2}}, \sigma^2_{\left\|vec\left(R\right)\right\|^{2}}\right)$$

where

$$\mu_{\left\|vec(R)\right\|} = E\left(u\left(vec(R)\right)\right) \to E\left(u\left(vec(\Omega)\right)\right) = \left\|vec(\Omega)\right\|^{2}$$

$$\sigma^2_{\left\|vec(R)\right\|^{2}} \to \frac{2}{n-1}\left(\frac{\partial u\left(vec(\Omega)\right)}{\partial vec(R)}\right)^{t} M_p \phi M_p \left(\frac{\partial u\left(vec(\Omega)\right)}{\partial vec(R)}\right).$$

Then,

$$\frac{2}{n-1}\left(\frac{\partial u\left(vec\left(\Omega\right)\right)}{\partial vec\left(R\right)}\right)^{t} M_{p}\phi M_{p}\left(\frac{\partial u\left(vec\left(\Omega\right)\right)}{\partial vec\left(R\right)}\right)=8\left(vec\left(\Omega\right)\right)^{t} M_{p}\phi M_{p} vec\left(\Omega\right).$$

Therefore,

$$\sigma^{2}_{\|vec(R)\|^{2}} \rightarrow 8\left(vec\left(\Omega\right)\right)^{t} M_{p}\varphi M_{p} vec\left(\Omega\right)$$

The two parameters mean and variance in the above proposition derived directly based on description of Taylor real valued vector function as set out in Herdiani (2008). Taylor description $u\left(vec\left(R\right)\right)=\left\|vec\left(R\right)\right\|^{2}$.

$$u\left(vec\left(R\right)\right)=u\left(vec\left(\Omega\right)\right)+\left(\left.\frac{\partial u\left(vec\left(R\right)\right)}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}\left(vec\left(R\right)-vec\left(\Omega\right)\right)$$

Therefore,

a) Mean of $u\left(vec\left(R\right)\right)$ is

$$E\left(u\left(vec\left(R\right)\right)\right)\approx E\left(\left\|vec\left(\Omega\right)\right\|^{2}+\left(\left.\frac{\partial u\left(vec\left(R\right)\right)}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}\left(vec\left(R\right)-vec\left(\Omega\right)\right)\right)$$

$$\approx E\left(\left\|vec\left(\Omega\right)\right\|^{2}\right)+E\left(\left(\left.\frac{\partial\left\|\left(vec\left(R\right)\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}\left(vec\left(R\right)-vec\left(\Omega\right)\right)\right)$$

So, $E\left(\left(\left.\frac{\partial\left\|\left(vec(R)\right)\right\|^{2}}{\partial vec(R)}\right|_{R=\Omega}\right)^{t}\left(vec(R)-vec(\Omega)\right)\right)=0$

Then, $u(vec(R))\xrightarrow{p}E\left(\left\|vec(R)\right\|^{2}\right)=\left\|vec(\Omega)\right\|^{2}$

b) Variance $u\left(vec\left(R\right)\right)$

$$var\left(u\left(vec\left(R\right)\right)\right)\approx var\left(\left\|vec\left(\Omega\right)\right\|^{2}+\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}\left(vec\left(R\right)-vec\left(\Omega\right)\right)\right)$$

$$\approx var\left(\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}\left(vec\left(R\right)-vec\left(\Omega\right)\right)\right),$$

$\left\|vec\left(\Omega\right)\right\|^{2}$ is constant

$$\approx E\left(\left(\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}\left(vec\left(R\right)-vec\left(\Omega\right)\right)\right)\left(\left(\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right)^{t}\left(vec\left(R\right)-vec\left(\Omega\right)\right)\right)^{t}\right)-$$

$$\left(E\left(\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}\left(vec\left(R\right)-vec\left(\Omega\right)\right)\right)\right)\left(E\left(\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)\times\right.\right.$$

$$\left.\left.\left(vec\left(R\right)-vec\left(\Omega\right)\right)^{t}\right)\right)$$

The second term in the right side is equal 0 because $E\left(u\left(vec\left(R\right)\right)\right)=\left\|vec\left(\Omega\right)\right\|^{2}$.

Therefore,

$$var\left(u\left(vec\left(R\right)\right)\right)\xrightarrow{p}\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}E\left(vec\ \left(R\right)-vec\left(\Omega\right)\right)\times$$

$$\left(vec\left(R\right)-vec\left(\Omega\right)\right)^{t}\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)$$

$$var\left(u\left(vec\left(R\right)\right)\right)\xrightarrow{p}\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}E\left(vec(R)-vec(\Omega)\right)\left(vec(R)-vec(\Omega)\right)^{t}\times$$

$$\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)$$

$$=\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}\frac{\Gamma}{n\text{-}1}\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)$$

$$=\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}\frac{2}{n\text{-}1}M_{p}\phi M_{p}\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right),$$ from Theorem 3.1

$\Gamma=2M_{p}\phi M_{p}$ therefore, $\frac{2}{n\text{-}1}\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)^{t}M_{p}\ \phi M_{p}\left(\left.\frac{\partial\left\|vec\left(R\right)\right\|^{2}}{\partial vec\left(R\right)}\right|_{R=\Omega}\right)$

.

But $\dfrac{\partial u\left(\left(\Omega\right)\right)}{\partial vec\left(R\right)}$ is vector dimensionless p^2 where it is element is

$$\left.\frac{\partial \sum_{i=1}^{p}\sum_{i=1}^{p} r_{ij}^{2}}{\partial r_{ij}}\right|_{r_{ij}=\rho_{ij}} \quad \text{for each } i, j = 1, 2, ..., p.$$

$$\left.\frac{\partial \sum_{i=1}^{p}\sum_{i=1}^{p} r_{ij}^{2}}{\partial r_{ij}}\right|_{r_{ij}=\rho_{ij}} = \left.\frac{\partial\left(r_{ij}^{2}\right)}{\partial r_{ij}}\right|_{r_{ij}=\rho_{ij}} = \left.\frac{2r_{ij}\partial\left(r_{ij}\right)}{\partial r_{ij}}\right|_{r_{ij}=\rho_{ij}} = 2\rho_{ij}\,.$$

$$\frac{\partial u\left(v\left(\Omega\right)\right)}{\partial v\left(R\right)} = 2\begin{pmatrix} \rho^{1,1} \\ \rho^{1,2} \\ \vdots \\ \rho^{1,p} \\ \vdots \\ \vdots \\ \rho^{p,1} \\ \rho^{p,2} \\ \vdots \\ \rho^{p,p} \end{pmatrix} = 2v\left(\Omega\right).$$

So,

$$\frac{2}{n-1}\left(\frac{\partial u\left(vec\left(\Omega\right)\right)}{\partial vec\left(R\right)}\right)^{t} M_{p}\phi M_{p}\left(\frac{\partial u\left(vec\left(\Omega\right)\right)}{\partial vec\left(R\right)}\right)$$

$$=\frac{2}{n-1}\left(2vec\left(\Omega\right)\right)^{t} M_{p}\phi M_{p}\left(2vec\left(\Omega\right)\right)$$

$$\text{Variance } \|vec(R)\|^2 \rightarrow \frac{8}{n-1}(vec(\Omega))^t M_p \phi M_p vec(\Omega).$$

In this study, the upper-of-diagonal elements is used since the matrix is symmetric and have redundant elements. In the next proposition, by using corollary 4.1 and Proposition 4.4 we have the following proposition.

Proposition 4.5

Let $X_1, X_2, ..., X_n$ is a random sample of size *n* from $N(\mu, \Sigma)$. If Ω is correlation matrix then, $\sqrt{n-1}\left(\|v(R_U)\|^2 - \|v(\Omega_U)\|^2\right) \xrightarrow{d} N(\mu, \sigma^2)$.

$$\sigma^2 = 4(v(\Omega_U))^t T \; \Gamma T^t \; v(\Omega_U) = 8(v(\Omega_U))^t T \; M_p \; \phi M_p T^t \; v(\Omega_U)$$

Where $v(\Omega_U) = T \times vec(\Omega)$ and $v(R_U) = T \times vec(R)$.

In the next section, the proof on how to find the variance of $v(R_U)$ is derived.

4.4.1 Mean and Variance of $v(R_U)$

We used Proposition 4.2 to found the mean$v(R_U)$

$$R \approx \Omega + A - \frac{1}{2}(\Omega D_A + D_A \Omega)$$

$$vec(R) \approx vec\left(\Omega + A - \frac{1}{2}(\Omega D_A + D_A \Omega)\right)$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2} vec(\Omega D_A + D_A \Omega)$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2}\left(vec(\Omega D_A) + vec(D_A \Omega)\right)$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2}\left(vec(\Omega D_A I_p) + vec(D_A \Omega I_p)\right)$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2}\left((I_p \otimes \Omega) vec(D_A) + (\Omega \otimes I_p) vec(D_A)\right)$$

$$\approx vec(\Omega) + vec(A) - \frac{1}{2}\left((I_p \otimes \Omega) + (\Omega \otimes I_p)\right) vec(D_A)$$

Now,

$$E\left(T \times vec(R)\right) \approx E\left(T \times vec(\Omega)\right) + \left\{I_{p^2} - \frac{1}{2}\left((I_p \otimes \Omega) + (\Omega \otimes I)\right)\Lambda_p\right\}\left(T \times vec(A)\right)$$

$$\approx E\left(T \times vec(\Omega)\right) + E\left(\left\{I_{p^2} - \frac{1}{2}\left((I_p \otimes \Omega) + (\Omega \otimes I)\right)\Lambda_p\right\} T \times vec(A)\right)$$

$$\approx v(\Omega_U) + \left\{I_{p^2} - \frac{1}{2}\left((I_p \otimes \Omega) + (\Omega \otimes I)\right)\Lambda_p\right\} E\left(T \times vec(A)\right), A = S^* - \Omega$$

$$\approx v(\Omega_U) + \left(I_{p^2} - \frac{1}{2}\left((I_p \otimes \Omega) + (\Omega \otimes I)\right)\Lambda_p\right)\left(E\left(T \times vec(S^*)\right)\right.$$

$$\left. - E\left(T \times vec(\Omega)\right)\right)$$

$$\approx v(\Omega_U) + \left\{I_{p^2} - \frac{1}{2}\left((I_p \otimes \Omega) + (\Omega \otimes I)\right)\Lambda_p\right\} E\left(T \times vec\left(D_\Sigma^{\frac{-1}{2}} S D_\Sigma^{\frac{-1}{2}}\right)\right)$$

$$- E\left(v(\Omega_U)\right)$$

$$\approx v\left(\Omega_U\right)+\left\{I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right\}\times$$

$$E\left[T\times\left(D_\Sigma^{\frac{-1}{2}}\otimes D_\Sigma^{\frac{-1}{2}}\right)\left(vec\left(S\right)-v\left(\Omega_U\right)\right)\right]$$

$$\approx v\left(\Omega_U\right)+\left\{I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right\}T\times\left(D_\Sigma^{\frac{-1}{2}}\otimes D_\Sigma^{\frac{-1}{2}}\right)$$

$$E\left(vec\left(S\right)-v\left(\Omega_U\right)\right)$$

$$\approx v\left(\Omega_U\right)+\left\{I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right\}$$

$$\left(T\times vec\left(D_\Sigma^{\frac{-1}{2}}\Sigma D_\Sigma^{\frac{-1}{2}}\right)-v\left(\Omega_U\right)\right)$$

$$\approx v\left(\Omega_U\right)+\left\{I_{p^2}-\frac{1}{2}\left(\left(I_p\otimes\Omega\right)+\left(\Omega\otimes I_p\right)\right)\Lambda_p\right\}\left(v\left(\Omega_U\right)-v\left(\Omega_U\right)\right)$$

Noted that

$v\left(\Omega_U\right)-v\left(\Omega_U\right)=0$ then, $\left(I_{p^2}-\frac{1}{2}\left((I_p\otimes\Omega)+(\Omega\otimes I)\right)\Lambda_p\right)\times 0=0$

We prove mean of $v\left(R_U\right)=v\left(\Omega_U\right)$.

By using the corollary 4.1and the Proposition 4.5 arrive to the following Proposition about the asymptotic distribution of $v\left(R_U\right)$.

Proposition 4.6

Let $X_1, X_2, ..., X_n$ is a random sample of size n from $N\left(\mu,\ \sigma^2\right)$. If Ω is correlation matrix then, $\left\|v\left(R_U\right)\right\|^2 \xrightarrow{d} N\left(\mu_{\|R_U\|^2}, \sigma^2_{\|v(R_U)\|^2}\right)$

with $\mu_{\|v(R_U)\|^2} \to \left\|v\left(\Omega_U\right)\right\|^2$ and

$$\sigma^2_{\|v(R_U)\|^2} \to \frac{8}{n-1}\left(v\left(\Omega_U\right)\right)^t TM_p\phi M_p T^t v\left(\Omega_U\right)$$

Proof:

Note that $u\left(v\left(R_U\right)\right) = \left\|v\left(R_U\right)\right\|^2$ and u' exist $u'\left(R^*\right) \neq 0$ for all R^* in the environment Ω.

For $r_{ij} \xrightarrow{p} \rho_{ij}$, for all $i, j = 1, 2, ..., p$ and $v\left(R_U\right) \xrightarrow{d} N\left(v\left(\Omega_U\right), \sigma^2_{\|v(R_U)\|^2}\right)$, then

$u\left(v\left(R_U\right)\right) = \left\|v\left(R_U\right)\right\|^2 \xrightarrow{d} N\left(\mu_{\|v(R_U)\|^2}, \sigma^2_{\|v(R_U)\|^2}\right)$, where

$$\mu_{\|v(R_U)\|^2} = E\left(u\left(v\left(R_U\right)\right)\right) \to E\left(u\left(v\left(\Omega_U\right)\right)\right) = \left\|v\left(\Omega_U\right)\right\|^2$$

$\sigma^2_{\|v(R_U)\|} \to \frac{2}{n-1}\left(\frac{\partial u\left(v\left(\Omega_U\right)\right)}{\partial v\left(R_U\right)}\right)^t TM_p\phi M_p T^t \left(\frac{\partial u\left(v\left(\Omega_U\right)\right)}{\partial v\left(R_U\right)}\right)$. Then,

$$\frac{2}{n-1}\left(\frac{\partial u\left(v\left(\Omega_U\right)\right)}{\partial v\left(R_U\right)}\right)^t TM_p\phi M_p T^t \left(\frac{\partial u\left(v\left(\Omega_U\right)\right)}{\partial v\left(R_U\right)}\right) = 8\left(v\left(\Omega_U\right)\right)^t TM_p\phi M_p T^t v\left(\Omega_U\right)$$

Therefore, $\sigma^2_{\|v(R_U)\|^2} \to 8\left(v\left(\Omega_U\right)\right)^t TM_p \phi M_p T^t v\left(\Omega_U\right)$.

The two parameters mean and variance in the proposition 4.6 derived directly based on description of Taylor real valued vector function as set out in Herdiani (2008). Taylor description $u\left(v\left(R_U\right)\right) = \left\|v\left(R_U\right)\right\|^2$

$$u\left(v\left(R_U\right)\right) = u\left(v\left(\Omega_U\right)\right) + \left(\left.\frac{\partial u\left(v\left(\Omega_U\right)\right)}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t \left(v\left(R_U\right) - v\left(\Omega_U\right)\right).$$

Therefore,

a) Mean of $u\left(v\left(R_U\right)\right)$ is

$$E\left(u\left(v\left(R_U\right)\right)\right) \approx E\left(\left\|v\left(\Omega_U\right)\right\|^2 + \left(\left.\frac{\partial \left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t \left(v\left(R_U\right) - v\left(\Omega\right)\right)\right)$$

$$\approx E\left(\left\|v\left(\Omega_U\right)\right\|^2\right) + E\left(\left(\left.\frac{\partial \left\|\left(v\left(R_U\right)\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t \left(v\left(R_U\right) - v\left(\Omega_U\right)\right)\right)$$

So, $E\left(\left(\left.\frac{\partial \left\|\left(v\left(R_U\right)\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t \left(v\left(R_U\right) - v\left(\Omega_U\right)\right)\right) = 0$

Then, $u\left(v\left(R_U\right)\right)\xrightarrow{p}E\left(\left\|v\left(R_U\right)\right\|^2\right)=\left\|v\left(\Omega_U\right)\right\|^2$.

b) Variance of $u\left(v\left(R_U\right)\right)$

$$var\left(u\left(v\left(R_U\right)\right)\right)\approx var\left(\left\|v\left(\Omega_U\right)\right\|^2+\left(\left.\frac{\partial\left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t\left(v\left(R_U\right)-v\left(\Omega_U\right)\right)\right)$$

$$\approx var\left(\left(\left.\frac{\partial\left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t\left(v\left(R_U\right)-v\left(\Omega_U\right)\right)\right),\ \left\|v\left(\Omega_U\right)\right\|^2 \text{ is constant}$$

$$\approx E\left(\left(\left.\frac{\partial\left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t\left(v\left(R_U\right)-v\left(\Omega_U\right)\right)\right)\left(\left(\left.\frac{\partial\left\|vv\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t\left(v\left(R_U\right)-v\left(\Omega_U\right)\right)\right)^t-$$

$$\left(E\left(\left(\left.\frac{\partial\left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t\left(v\left(R_U\right)-v\left(\Omega_U\right)\right)\right)\left(E\left(\left.\frac{\partial\left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t\left(v\left(R_U\right)-v\left(\Omega_U\right)\right)\right)^t\right)^t.$$

The second term in the right side is equal 0 because $E\left(u\left(v\left(R_U\right)\right)\right)=\left\|v\left(\Omega_U\right)\right\|^2$.

Therefore,

$$var\left(u\left(v\left(R_U\right)\right)\right)\xrightarrow{p}\left(\left.\frac{\partial\left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t E\left(v\left(R_U\right)-v\left(\Omega_U\right)\right)\left(v\left(R_U\right)-v\left(\Omega_U\right)\right)^t\left(\left.\frac{\partial\left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)$$

$$var\left(u\left(v\left(R_U\right)\right)\right)\xrightarrow{p}\left(\left.\frac{\partial\left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t\frac{\Gamma^*}{n\text{-}1}\left(\left.\frac{\partial\left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)$$

$$var\left(u\left(v\left(R_U\right)\right)\right)\xrightarrow{p}\left(\left.\frac{\partial\left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t\frac{2}{n\text{-}1}TM_p\phi M_pT^t\left(\left.\frac{\partial\left\|v\left(R_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right),$$

$\Gamma^* = 2TM_p\phi M_pT^t$,therefore

$$\frac{2}{n-1}\left(\left.\frac{\partial\left\|v\left(\Omega_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t TM_p\phi M_pT^t\left(\left.\frac{\partial\left\|v\left(\Omega_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right).$$

But $\dfrac{\partial u\left(v\left(\Omega_U\right)\right)}{\partial v\left(R_U\right)} = 2v\left(\Omega_U\right)$

$$\frac{2}{n-1}\left(\left.\frac{\partial\left\|v\left(\Omega_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)^t TM_p\phi M_pT^t\left(\left.\frac{\partial\left\|v\left(\Omega_U\right)\right\|^2}{\partial v\left(R_U\right)}\right|_{R=\Omega}\right)$$

$$= \frac{2}{n-1}2v\left(\Omega_U\right)^t TM_p\phi M_pT^t\,2v\left(\Omega_U\right)$$

$$= \frac{8}{n-1}v\left(\Omega_U\right)^t TM_p\phi M_pT^t v\left(\Omega_U\right)$$

Variance $\left\|v\left(R_U\right)\right\|^2 \rightarrow \dfrac{8}{n-1}\left(v\left(\Omega_U\right)\right)^t TM_p\phi M_pT^t v\left(\Omega_U\right)$.

Proposition 4.7

If variance $\left\|v\left(R_U\right)\right\|^2 = \dfrac{8}{n-1}\left(v\left(\Omega_U\right)\right)^t TM_p\phi M_pT^t v\left(\Omega_U\right)$ then

$$\left(v\left(\Omega_U\right)\right)^t T M_p \phi M_p T^t v\left(\Omega_U\right) = \frac{1}{4}\Big[2Tr\left(\lambda^t \Omega\lambda\Omega\right) + 2Tr\left(\left(\lambda\Omega\right)^2\right) - 4Tr\left(\lambda^t \Omega\, D_{\lambda\Omega}\Omega\right)$$

$$-4Tr\left(\lambda\Omega\, D_{\Omega\lambda}\Omega\right) + 2Tr\left(D_{\Omega\lambda}\Omega\, D_{\lambda\Omega}\Omega\right) + Tr\left(\left(D_{\Omega\lambda}\Omega\right)^2\right) + Tr\left(\left(D_{\lambda\Omega}\Omega\right)^2\right)\Big]$$

Proof:

To prove Proposition 4.7, the left side $\left(v\left(\Omega_U\right)\right)^t T M_p \phi M_p T^t v\left(\Omega_U\right)$

Where $M_p = \frac{1}{2}\left(I_{p^2} + K_{pp}\right)$ and $\phi = \left(I_{p^2} - \left(I_p \otimes \Omega\right)\Lambda_p\right)\left(\Omega \otimes \Omega\right)\left(I_{p^2} - \Lambda_p\left(I_p \otimes \Omega\right)\right)$

$K_{pp} = \sum_{i=1}^{p}\sum_{j=1}^{p} G_{ij} \otimes G_{ij}^t$ is the commutation matrix of size $\left(p^2 \times p^2\right)$

G_{ij} is a matrix of size $\left(p \times p\right)$ having all elements are equal 0 except it is (i, j)-th element equals 1.

$\Lambda_p = \sum_{i=1}^{p} h_i h_i^t \otimes h_i h_i^t$ where, h_i is the i-th column of I_p.

$$\left(v\left(\Omega_U\right)\right)^t T M_p \phi M_p T^t v\left(\Omega_U\right) = \left(v\left(\Omega_U\right)\right)^t \times T \times \frac{1}{2}\left(I_{p^2} + K_{pp}\right)\left\{I_{p^2} - \left(I_p \otimes \Omega\right)\Lambda_p\right\} \times$$

$$\left(\Omega \otimes \Omega\right)\left\{I_{p^2} - \Lambda_p\left(I_p \otimes \Omega\right)\right\} \times \frac{1}{2}\left(I_{p^2} + K_{pp}\right) \times T^t \times v\left(\Omega_U\right)$$

$$\left(v\left(\Omega_U\right)\right)^t TM_p\phi M_p T^t v\left(\Omega_U\right)=\frac{1}{4}\Big[\left\{\left(v\left(\Omega_U\right)\right)^t\times T\times I_{p^2}+\left(v\left(\Omega_U\right)\right)^t\times T\times K_{pp}\right\}\times$$
$$\left\{I_{p^2}-\left(I_p\otimes\Omega\right)\Lambda_p\right\}\left(\Omega\otimes\Omega\right)\left\{I_{p^2}-\Lambda_p\left(I_p\otimes\Omega\right)\right\}\left(I_{p^2}+K_{pp}\right)\times T^t\times v\left(\Omega_U\right)\Big]$$

$$\left(v\left(\Omega_U\right)\right)^t TM_p\phi M_p T^t v\left(\Omega_U\right)=\frac{1}{4}\Big[\left\{\left(v\left(\Omega_U\right)\right)^t\times T\times I_{p^2}+\left(v\left(\Omega_U\right)\right)^t\times T\times K_{pp}\right\}\left\{I_{p^2}-\left(I_p\otimes\Omega\right)\Lambda_p\right\}\times$$
$$\left(\Omega\otimes\Omega\right)\left\{\left(I_{p^2}-\Lambda_p\left(I_{p^2}\otimes\Omega\right)\right)\right\}\left(I_{p^2}T^t\times v\left(\Omega_U\right)+K_{pp}T^t\times v\left(\Omega_U\right)\right)\Big].$$

Since $T\times I_p=T$, $K_{pp}\times v\left(\Omega_U\right)=v\left(\Omega_U\right),\left(v\left(\Omega_U\right)\right)^t\times K_{pp}=\left(v\left(\Omega_U\right)\right)^t$ then

$$\left(v\left(\Omega_U\right)\right)^t TM_p\phi M_p T^t v\left(\Omega_U\right)=\frac{1}{4}\Big[\left\{\left(v\left(\Omega_U\right)\right)^t\times T+\left(v\left(\Omega_U\right)\right)^t\times T\times K_{pp}\right\}\times\left\{I_{p^2}-\left(I_p\otimes\Omega\right)\Lambda_p\right\}\times$$
$$\left(\Omega\otimes\Omega\right)\left\{I_{p^2}-\Lambda_p\left(I_p\otimes\Omega\right)\right\}\left\{I_{p^2}\times T^t\times v\left(\Omega_U\right)+K_{pp}\times T^t\times v\left(\Omega_U\right)\right\}\Big].$$

Corollary 4.2

Let λ a matrix of size $(p\times p)$ that

$$v\left(\lambda\right)=T^t\times T\times vec\left(\Omega\right)$$

Then by using the corollary 4.2 we have the following

$$\left(v\left(\Omega_U\right)\right)^t TM_p\phi M_p T^t v\left(\Omega_U\right)=\frac{1}{4}\Big[\left\{\left(v\left(\lambda\right)\right)^t+\left(v\left(\lambda^t\right)\right)^t\right\}\left\{I_{p^2}-\left(I_p\otimes\Omega\right)\Lambda_p\right\}\left(\Omega\otimes\Omega\right)$$
$$\left\{I_{p^2}-\Lambda_p\left(I_p\otimes\Omega\right)\right\}\left\{v\left(\lambda\right)+v\left(\lambda^t\right)\right\}\Big]$$

$$=\frac{1}{4}\Big[\left\{\left(v\left(\lambda\right)\right)^t I_{p^2}+\left(v\left(\lambda^t\right)\right)^t I_{p^2}-\left(v\left(\lambda\right)\right)^t\left(I_p\otimes\Omega\right)\Lambda_p-\left(v\left(\lambda^t\right)\right)^t\left(I_p\otimes\Omega\right)\Lambda_p\right\}\left(\Omega\otimes\Omega\right)$$
$$\left\{I_{p^2}v\left(\lambda\right)+I_{p^2}v\left(\lambda^t\right)-\Lambda_p\left(I_p\otimes\Omega\right)v\left(\lambda\right)-\Lambda_p\left(I_p\otimes\Omega\right)v\left(\lambda^t\right)\right\}\Big]$$

$$
=\frac{1}{4}\Bigg[\Big\{\left(v\left(\lambda\right)\right)^{t}+\left(v\left(\lambda^{t}\right)\right)^{t}-\left(v\left(\lambda\right)\right)^{t}\left(I_{p}\otimes\Omega\right)\Lambda_{p}-\left(v\left(\lambda^{t}\right)\right)^{t}\left(I_{p}\otimes\Omega\right)\Lambda_{p}\Big\}\times
$$

$$
\left(\Omega\otimes\Omega\right)\Big\{v\left(\lambda\right)+v\left(\lambda^{t}\right)-\Lambda_{p}\left(I_{p}\otimes\Omega\right)v\left(\lambda\right)-\Lambda_{p}\left(I_{p}\otimes\Omega\right)v\left(\lambda^{t}\right)\Big\}\Bigg]
$$

we define D_W as a matrix the diagonal elements are the diagonal elements of W.

Now from Theorem 7.30 (Schott, 1997).

$$
\left(v\left(\lambda\right)\right)^{t}\left(I_{p}\otimes\Omega\right)\Lambda_{p}=v\left(D_{\Omega\lambda}\right)\quad\text{and}\quad\left(v\left(\lambda^{t}\right)\right)^{t}\left(I_{p}\otimes\Omega\right)\Lambda_{p}=v\left(D_{\lambda\Omega}\right)^{t}
$$

$$
=\frac{1}{4}\Big[\Big(v\left(\lambda\right)^{t}\left(\Omega\otimes\Omega\right)+v\left(\lambda^{t}\right)^{t}\left(\Omega\otimes\Omega\right)-v\left(D_{\Omega\lambda}\right)^{t}\left(\Omega\otimes\Omega\right)-v\left(D_{\lambda\Omega}\right)^{t}\left(\Omega\otimes\Omega\right)\Big)
$$

$$
\Big(v\left(\lambda\right)+v\left(\lambda^{t}\right)-v\left(D_{\Omega\lambda}\right)-v\left(D_{\lambda\Omega}\right)\Big)\Big]
$$

$$
=\frac{1}{4}\Big[v\left(\lambda\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda\right)+v\left(\lambda^{t}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda\right)-v\left(D_{\Omega\lambda}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda\right)-
$$

$$
v\left(D_{\lambda\Omega}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda\right)+v\left(\lambda\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda^{t}\right)+v\left(\lambda^{t}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda^{t}\right)-
$$

$$
v\left(D_{\Omega\lambda}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda^{t}\right)-v\left(D_{\lambda\Omega}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda^{t}\right)-v\left(\lambda\right)^{t}\left(\Omega\otimes\Omega\right)v\left(D_{\Omega\lambda}\right)-
$$

$$
v\left(\lambda^{t}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(D_{\Omega\lambda}\right)^{t}+v\left(D_{\Omega\lambda}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(D_{\Omega\lambda}\right)+v\left(D_{\lambda\Omega}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(D_{\Omega\lambda}\right)-
$$

$$
v\left(\lambda^{t}\right)\left(\Omega\otimes\Omega\right)v\left(D_{\lambda\Omega}\right)-v\left(\lambda^{t}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(D_{\lambda\Omega}\right)+v\left(D_{\Omega\lambda}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(D_{\lambda\Omega}\right)+
$$

$$
v\left(D_{\lambda\Omega}\right)^{t}\left(\Omega\otimes\Omega\right)v\left(D_{\lambda\Omega}\right)\Big]
$$

Note

$$
\left(v\left(\lambda\right)\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda\right)=\left(v\left(\lambda^{t}\right)\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda^{t}\right),
$$

$$
\left(v\left(\lambda^{t}\right)\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda\right)=\left(v\left(\lambda\right)\right)^{t}\left(\Omega\otimes\Omega\right)v\left(\lambda^{t}\right),
$$

$$v\left(D_{\Omega\lambda}\right)^t\left(\Omega\otimes\Omega\right)v\left(\lambda\right)=v\left(D_{\Omega\lambda}\right)^t\left(\Omega\otimes\Omega\right)v\left(\lambda^t\right)=v\left(\lambda^t\right)\left(\Omega\otimes\Omega\right)v\left(D_{\Omega\lambda}\right),$$

$$v\left(D_{\lambda\Omega}\right)^t\left(\Omega\otimes\Omega\right)v\left(\lambda\right)=v\left(D_{\lambda\Omega}\right)^t\left(\Omega\otimes\Omega\right)v\left(\lambda^t\right)=v\left(\lambda^t\right)\left(\Omega\otimes\Omega\right)v\left(D_{\lambda\Omega}\right)=$$

$$v\left(\lambda^t\right)^t\left(\Omega\otimes\Omega\right)v\left(D_{\lambda\Omega}\right)\ \text{and}\ v\left(D_{\lambda\Omega}\right)^t\left(\Omega\otimes\Omega\right)v\left(D_{\lambda\Omega}\right)=v\left(D_{\lambda\Omega}\right)^t\left(\Omega\otimes\Omega\right)v\left(D_{\lambda\Omega}\right)$$

Now by using Theorem 7.15, 7.16 and 7.17 from (Schott, 1997) and Proposition 1.3.14. from Kollo and Von Rosen (2006) we have the following

$$=\frac{1}{4}\Big[2Tr\left(\lambda^t\Omega\lambda\Omega\right)+2Tr\left(\left(\lambda\Omega\right)^2\right)-4Tr\left(\lambda^t\Omega D_{\lambda\Omega}\Omega\right)-4Tr\left(\lambda\Omega D_{\Omega\lambda}\Omega\right)+2Tr\left(D_{\Omega\lambda}\Omega D_{\lambda\Omega}\Omega\right)$$

$$+Tr\left(\left(D_{\Omega\lambda}\Omega\right)^2\right)+Tr\left(\left(D_{\lambda\Omega}\Omega\right)^2\right)\Big]$$

$$\left(v\left(\Omega_U\right)\right)^t TM_p\phi M_pT^t v\left(\Omega_U\right)=\frac{1}{4}\Big[2Tr\left(\lambda^t\Omega\lambda\Omega\right)+2Tr\left(\left(\lambda\Omega\right)^2\right)-4Tr\left(\lambda^t\Omega D_{\lambda\Omega}\Omega\right)$$

$$-4Tr\left(\lambda\Omega D_{\Omega\lambda}\Omega\right)+2Tr\left(D_{\Omega\lambda}\Omega D_{\lambda\Omega}\Omega\right)+Tr\left(\left(D_{\Omega\lambda}\Omega\right)^2\right)+Tr\left(\left(D_{\lambda\Omega}\Omega\right)^2\right)\Big]$$

Now, $8\left(v\left(\Omega_U\right)\right)^t TM_p\phi M_pT^t v\left(\Omega_U\right)$

$$=8\times\frac{1}{4}\Big[2Tr\left(\lambda^t\Omega\lambda\Omega\right)+2Tr\left(\left(\lambda\Omega\right)^2\right)-4Tr\left(\lambda^t\Omega D_{\lambda\Omega}\Omega\right)-4Tr\left(\lambda\Omega D_{\Omega\lambda}\Omega\right)+2Tr\left(D_{\Omega\lambda}\Omega D_{\lambda\Omega}\Omega\right)$$

$$+Tr\left(\left(D_{\Omega\lambda}\Omega\right)^2\right)+Tr\left(\left(D_{\lambda\Omega}\Omega\right)^2\right)\Big].$$

Thus the variance of $v\left(R_U\right)$ is

$$=2\Big[2Tr\left(\lambda^t\Omega\lambda\Omega\right)+2Tr\left(\left(\lambda\Omega\right)^2\right)-4Tr\left(\lambda^t\Omega D_{\lambda\Omega}\Omega\right)-4Tr\left(\lambda\Omega D_{\Omega\lambda}\Omega\right)+2Tr\left(D_{\Omega\lambda}\Omega D_{\lambda\Omega}\Omega\right)$$

$$+Tr\left(\left(D_{\Omega\lambda}\Omega\right)^2\right)+Tr\left(\left(D_{\lambda\Omega}\Omega\right)^2\right)\Big]$$

Corollary 4.3

Under H_0, then $\sqrt{n-1}\left\{\left\|v\left(R_{i,U}\right)\right\|^2-\left\|v\left(\Omega_{0,U}\right)\right\|^2\right\}\xrightarrow{d}N\left(0,\sigma^2\right)$

Where:

$R_{i,U}$ and $\Omega_{0,U}$ are the upper-off-diagonal elements of R_i and Ω_0, respectively.

$$\sigma^2=2\Big[2Tr\left(\lambda^t\Omega_0\lambda\Omega_0\right)+2Tr\left(\left(\lambda\Omega_0\right)^2\right)-4Tr\left(\lambda^t\Omega_0 D_{\lambda\Omega_0}\Omega_0\right)-4Tr\left(\lambda\Omega_0 D_{\Omega_0\lambda}\Omega_0\right)+$$
$$2Tr\left(D_{\Omega_0\lambda}\Omega_0 D_{\lambda\Omega_0}\Omega_0\right)+Tr\left(\left(D_{\Omega_0\lambda}\Omega_0\right)^2\right)+Tr\left(\left(D_{\lambda\Omega_0}\Omega_0\right)^2\right)\Big]$$

λ a matrix of size $\left(p\times p\right)$ such that $v\left(\lambda\right)=T^t\times T\times vec\left(\Omega\right)$.

We define $D_{\Omega_0\lambda}$, $D_{\lambda\Omega_0}$ as a matrix the diagonal elements are the diagonal elements of $\Omega_0\lambda$ and $\lambda\Omega_0$, recpectively.

However, when Ω_0 is unknown is commonly exist in real application, it must be estimated from *m* independent random samples. We presented it in Proposition 4.8.

Prpoposition 4.8

If Ω_0 is unknown, under the null hypothesis H_0, then we have $\sqrt{n-1}\left[\left\|v\left(R_{i,U}\right)\right\|^2-\left\|v\left(\hat{\Omega}_{0,U}\right)\right\|^2\right]\xrightarrow{d}N\left(0,\ \sigma^2\right)$

where:

$R_{i,U}$ and $\hat{\Omega}_{0,U}$ are the upper-off-diagonal elements of R_i and $\hat{\Omega}_0$, respectively

$$\sigma^2 = 2\Big[2Tr\left(\lambda^t \hat{\Omega}_0 \lambda \hat{\Omega}_0\right) + 2Tr\left(\left(\lambda \hat{\Omega}_0\right)^2\right) - 4Tr\left(\lambda^t \hat{\Omega}_0 D_{\lambda \hat{\Omega}_0} \hat{\Omega}_0\right) - 4Tr\left(\lambda \hat{\Omega}_0 D_{\hat{\Omega}_0 \lambda} \hat{\Omega}_0\right)$$

$$+ 2Tr\left(D_{\hat{\Omega}_0 \lambda} \hat{\Omega}_0 D_{\lambda \hat{\Omega}_0} \hat{\Omega}_0\right) + Tr\left(\left(D_{\hat{\Omega}_0 \lambda} \hat{\Omega}_0\right)^2\right) + Tr\left(\left(D_{\lambda \hat{\Omega}_0} \hat{\Omega}_0\right)^2\right)\Big]$$

$\hat{\Omega}_0 = \bar{R}$ is the avarge of correlation matrices of R_1, R_2, ..., R_m

λ a matrix of size $(p \times p)$ such that $v(\lambda) = T^t \times T \times vec\left(\hat{\Omega}_0\right)$.

We define $D_{\hat{\Omega}_0 \lambda}$, $D_{\lambda \hat{\Omega}_0}$ as a matrix the diagonal elements are the diagonal elements of $\hat{\Omega}_0 \lambda$ and $\lambda \hat{\Omega}_0$ recpectively.

4.4.2 Computation the Variance of $v(R_U)$

The computation of variance of $v(R_U)$ is complexs. It contains the application of Kronecker product and commutation matrix. However, it can be easily performed by using software. We illustrate the computation of covariance matrix for case $p = 2$ and $p = 3$ in next example for elaboration.

Example 4.3

i. Suppose $\Omega = \begin{pmatrix} 1 & 0.4 \\ 0.4 & 1 \end{pmatrix}$

Now, we want to identify $v(\lambda) = T^t \times T \times vec(\Omega)$. Based on Table 4.2, linear transformation matrix is $T = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$. Thus,

$$vec(\lambda) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0.4 \\ 0.4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0.4 \\ 0 \end{pmatrix}$$

Then,

$\lambda = \begin{pmatrix} 0 & 0.4 \\ 0 & 0 \end{pmatrix}$, $D_{\Omega\lambda}$ is a matrix where the diagonal elements of the matrix are the diagonal elements of $\Omega\lambda$,

$$\Omega \times \lambda = \begin{pmatrix} 1 & 0.4 \\ 0.4 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0.4 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0.4 \\ 0 & 0.16 \end{pmatrix}, \text{then}$$

$$D_{\Omega\lambda} = \begin{pmatrix} 0 & 0 \\ 0 & 0.16 \end{pmatrix}$$

$D_{\lambda\Omega}$ is a matrix the diagonal elements are the diagonal elements of $\lambda\Omega$,

$$D_{\lambda\Omega} = \begin{pmatrix} 0.16 & 0 \\ 0 & 0 \end{pmatrix} \text{thus,}$$

$$\sigma^2 = 2\Big[2Tr\left(\lambda^t \Omega_0 \lambda \Omega_0\right) + 2Tr\left(\left(\lambda\Omega_0\right)^2\right) - 4Tr\left(\lambda^t \Omega_0 D_{\lambda\Omega_0} \Omega_0\right) - 4Tr\left(\lambda\Omega_0 D_{\Omega_{0\lambda}} \Omega_0\right) +$$
$$2Tr\left(D_{\Omega_{0\lambda}} \Omega_0 D_{\lambda\Omega_0} \Omega_0\right) + Tr\left(\left(D_{\Omega_0\lambda} \Omega_0\right)^2\right) + Tr\left(\left(D_{\lambda\Omega_0} \Omega_0\right)^2\right)\Big]$$

$=0.4516$

ii. Suppose $\Omega = \begin{pmatrix} 1 & 0.3 & 0.3 \\ 0.3 & 1 & 0.3 \\ 0.3 & 0.3 & 1 \end{pmatrix}$

Based on Table 4.2, linear transformation matrix is,

$T = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, then

$$v(\lambda) = T^t \times T \times vec(\Omega) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0.3 \\ 0.3 \\ 0.3 \\ 1 \\ 0.3 \\ 0.3 \\ 0.3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.3 \\ 0 \\ 0 \\ 0.3 \\ 0.3 \\ 0 \end{pmatrix}$$

Then,

$\lambda = \begin{pmatrix} 0 & 0.3 & 0.3 \\ 0 & 0 & 0.3 \\ 0 & 0 & 0 \end{pmatrix}$ and

$$\Omega \times \lambda = \begin{pmatrix} 1 & 0.3 & 0.3 \\ 0.3 & 1 & 0.3 \\ 0.3 & 0.3 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0.3 & 0.3 \\ 0 & 0 & 0.3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0.3 & 0.39 \\ 0 & 0.09 & 0.39 \\ 0 & 0.09 & 0.18 \end{pmatrix}$$

Thus,

$$D_{\Omega\lambda} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.18 \end{pmatrix} \text{ and } D_{\lambda\Omega} = \begin{pmatrix} 0.18 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Consequently,

$$\sigma^2 = 2\Big[2Tr\left(\lambda^t \Omega_0 \lambda \Omega_0\right) + 2Tr\left(\left(\lambda\Omega_0\right)^2\right) - 4Tr\left(\lambda^t \Omega_0 D_{\lambda\Omega_0} \Omega_0\right) - 4Tr\left(\lambda \Omega_0 D_{\Omega_0\lambda} \Omega_0\right) +$$
$$2Tr\left(D_{\Omega_0\lambda} \Omega_0 D_{\lambda\Omega_0} \Omega_0\right) + Tr\left(\left(D_{\Omega_0\lambda} \Omega_0\right)^2\right) + Tr\left(\left(D_{\lambda\Omega_0} \Omega_0\right)^2\right)\Big]$$

$$\sigma^2 = 1.355.$$

4.5 New Alternative Test Z^* Statistic

By using the repeated tests as introduced by Montgomery (2005), the hypothesis is $H_0 : \Omega_i = \Omega_0$ for all i where $i = 1, 2, \ldots, m$ versus $H_1 : \Omega_i \neq \Omega_0$ for at least one i where Ω_0 is the reference matrix. In this study, the main assumption is that the data is drawn from multivariate normal distribution.

Under the null hypothesis H_0 the asymptotic distribution of new statistical test is,

$$\sqrt{n-1}\left[\left\| v\left(R_{iU}\right)\right\|^2 - \left\| v\left(\Omega_{iU}\right)\right\|^2\right] \xrightarrow{d} N\left(0,\ \sigma^2\right),$$

Therefore, the new alternative statistical test can be represented as follows,

$$Z_i^* = \frac{\left\| v\left(R_{iU}\right)\right\|^2 - \left\| v\left(\Omega_{iU}\right)\right\|^2}{\sqrt{\frac{1}{n-1}\sigma^2}} \xrightarrow{d} N\left(0,1\right) \tag{4.2}$$

$$\sigma^2 = 2\Big[2Tr\left(\lambda^t \Omega\lambda\Omega\right) + 2Tr\left(\left(\lambda\Omega\right)^2\right) - 4Tr\left(\lambda^t \Omega D_{\lambda\Omega}\Omega\right) - 4Tr\left(\lambda\Omega D_{\Omega_\lambda}\Omega\right) +$$

$$2Tr\left(D_{\Omega_\lambda}\Omega D_{\lambda\Omega}\Omega\right) + Tr\left(\left(D_{\Omega\lambda}\Omega\right)^2\right) + Tr\left(\left(D_{\lambda\Omega}\Omega\right)^2\right)\Big]$$

In testing the equality of several independent samples of correlation matrices, the null hypothesis H_0 will be rejected at the significance level α when $|Z^*| > z_{\alpha/2}$ with $(1-\alpha/2)^{th}$ quantile of standard normal distribution. However, in the case of Ω unknown, the value must be estimated from independent random sample $\hat{\Omega}$ where $\hat{\Omega} = \overline{R}$ the average of correlation matrices of R_1, R_2, ..., R_m .

4.6 Analysis Power of Test

In this section, we investigate the evaluating of the sensitivity analysis between Jennrich test, T^* statistic and Z^* statistic based on power of test.

The power of test is defined as the probability of the test which leads to the rejection of the null hypothesis when it is false (Cohen, 1977). Power of test is used to measure the sensitivity of the test to identify a real difference in parameter if one actually exists as we mentioned in section 2.7 the power of test $1-\beta$ where β is the Type II error, as the Type II error decrease the power increase, and vice versa. In general, the power value is between 0 to 1. The minimum accepted value of the power of the test is greater than 0.5 (Murphy et al., 2014), while the value smaller than 0.5 indicate the power is to be unexceptional. Otherwise, the test indicate to be unexceptional. When the power of the test is close to 1, the test is will be considered to have high power (Syed-Yahaya, 2005). In this study, if the test is able to detect small effect, that test is more sensitive to the others.

The results of the investigation are presented in Table 4.2 to 4.8. Each table represent the ascending number of variables which are namely small ($p = 3, 4$ and 5), medium ($p = 10$ and 15) and large number of variables ($p = 20$ and 30) with significance level $\alpha = 0.05$. The first column in each table shows the shift in the matrix ρ where its diagonal elements equal to 1 the shift from 0 to 0.8 with 0.1 increment. The following three columns presented the power of test for the three statistical test: Jennrich statistic, Z^* statistic, and T^* statistic, respectively. This analysis is conducting repeatedly for various sample size.

4.6.1 Power of theTest for a Small Number of Variables ($p = 3, 4$ and 5)

Table 4.2, 4.3 and 4.4 display the power of the statistical tests for a small number of variables, which are $p = 3, 4$ and 5.

According to the Table 4.2 for each sample size (n = 3, 5,10, 20, 30, 50 and 100), the value of the power for the two statistical tests (the Jennrich statistic and the T^* statistic) are always smaller than the Z^* statistic. In details, when $n = 3$ and 10 all the statistical tests, the Jennrich statistic, the T^* statistic and the Z^* statistic, are power when the shift of correlation matrix is $\rho = 0.6$ and above. While, when the sample size $n = 5$ the values of the power within the interval of power when the shifts are 0.7 and 0.8. In addition, as the sample size increase to 20, 30, 50 and 100 all the statistical tests, the Jennrich statistic, the T^* statistic and the Z^* statistic are mostly preduce high power with a small and large shift, from $\rho = 0.3$ and above. Afterwards, mostly the Jennrich statistic possesses a smaller value of power.

Subsequently, in Table 4.3, when the sample size $n = 3$ the values of the power of the T^* statistic, fall within the interval when the shift of correlation matrix are 0.4 and above. While, the Z^* statistic, the values fall within the interval when the shift is 0.2 and above. In this case we can not calculate Jennrich statistic because the number of variables is larger than the sample size. While, for sample size $n = 5$ and 10 the value of power for the Jennrich statistic and the T^* statistic fall within the interval when the shift is 0.6 and above. Meanwhile, the value of the power of the Z^* statistic are mostly sensitive in detecting the effect (power) at 0.2 and above. In addition, as we move to n=20 the value of the power for the Jennrich statistic and the T^* statistic fall within the interval when the shift in correlation matrix is 0.4 and above. While, the power value of the Z^* statistic fall within the interval when the shift is 0.3 and above. When the sample size 30 all the values of the of Jennrich statistic, the T^* statistic and the Z^* statistic are within the interval of power when the shift of correlation matrix is 0.3 and above. In addition, as the sample size increase to 50 the Jennrich statistic, the T^* statistic and the Z^* statistic are power at $\rho = 0.2$ and above. While, when the sample size 100 the Z^* statistic at $\rho = 0.1$ and above.

In, Table 4.4, when the sample size $n = 3$, the values of the power for the T^* statistic are power when the shift in correlation matrix $\rho = 0.4$ and above. When the sample increase to 5, 10 and 20 the two satistic the Jennrich statistic and the T^* statistic are power at 0.5 shift in correlation matrix and above. Meanwhile, the Z^* statistic at 0.2 and above when the sample size are 3, 5, 10 and 20. In addition, as the sample increase to 30, 50 and 100, majority of the values are power when $\rho = 0.3$ and above.

Drawing from Tables 4.2, 4.3 and 4.4, the conclusion can be made that the larger sample size the values of the power of statistical tests fall within the interval of the power. When the sample sizes are $n = 3$, 5, 10, 20, 30, 50 and 100 there are 38, 38, 43, 56, 56, 61 and 65 out of the 549, values of the power fall within the interval of the power respectively. Thus, the conclusion can be made from the results that, when the sample size is small, the value of the power of the statistical tests fall within the interval power with a large shift of the correlation matrix. Meanwhile, when the sample size is large, the values of the power for all the statistical tests fall within the interval of power from 0.3 shift in correlation matrix and above. In brief, the Z^* statistic dominates the Jennrich statistic and T^* statistic.

Table 4.2

Power of test for $p = 3$

ρ	$n=3$			$n=5$			$n=10$			$n=20$			$n=30$			$n=50$			$n=100$		
	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*
0.0	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.000	0.050	0.050	0.000	0.050	0.050	0.000
0.1	0.057	0.104	0.079	0.068	0.083	0.158	0.061	0.062	0.342	0.076	0.074	0.579	0.099	0.091	0.029	0.127	0.141	0.078	0.240	0.245	0.232
0.2	0.074	0.316	0.356	0.110	0.124	0.496	0.093	0.099	0.755	0.159	0.158	0.920	0.251	0.245	0.304	0.409	0.450	0.503	0.765	0.772	0.812
0.3	0.101	0.441	0.730	0.166	0.186	0.787	0.158	0.164	0.937	0.325	0.529	0.990	0.525	0.511	0.648	0.783	0.802	0.853	0.986	0.999	0.999
0.4	0.162	0.518	0.904	0.234	0.247	0.928	0.252	0.262	0.983	0.558	0.557	0.999	0.800	0.794	0.878	0.972	0.969	0.982	1	0.987	1
0.5	0.378	0.553	0.920	0.312	0.317	0.959	0.392	0.412	0.996	0.787	0.927	0.880	0.956	0.994	0.979	0.998	0.999	0.999	1	1	1
0.6	0.508	0.572	0.941	0.401	0.398	0.983	0.562	0.999	0.999	0.941	0.982	1	0.997	0.999	0.999	0.999	1	1	1	1	1
0.7	0.676	0.608	0.960	0.504	0.500	0.994	0.754	0.742	0.999	0.993	1	1	0.999	1	1	1	1	1	1	1	1
0.8	0.819	0.653	0.812	0.622	0.615	0.999	0.910	0.873	1	1	1	1	1	1	1	1	1	1	1	1	1

119

Table 4.3

Power of test for $p = 4$

ρ	$n=3$ T^*	$n=3$ Z^*	$n=5$ J	$n=5$ T^*	$n=5$ Z^*	$n=10$ J	$n=10$ T^*	$n=10$ Z^*	$n=20$ J	$n=20$ T^*	$n=20$ Z^*	$n=30$ J	$n=30$ T^*	$n=30$ Z^*	$n=50$ J	$n=50$ T^*	$n=50$ Z^*	$n=100$ J	$n=100$ T^*	$n=100$ Z^*
0.0	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.000	0.050	0.050	0.000	0.050	0.050	0.000
0.1	0.195	0.397	0.088	0.114	0.239	0.072	0.076	0.527	0.087	0.093	0.819	0.114	0.113	0.008	0.191	0.197	0.048	0.398	0.398	0.578
0.2	0.288	0.819	0.180	0.202	0.710	0.135	0.147	0.985	0.206	0.225	0.999	0.342	0.341	0.370	0.637	0.639	0.666	0.958	0.954	0.955
0.3	0.329	0.981	0.311	0.283	0.938	0.234	0.241	0.999	0.424	0.450	1	0.691	0.692	0.823	0.949	0.953	0.971	1	1	1
0.4	0.500	0.995	0.461	0.376	0.997	0.372	0.388	1	0.962	0.907	1	0.930	0.919	0.976	0.998	0.999	1	1	1	1
0.5	0.511	0.988	0.612	0.513	0.999	0.543	0.547	1	0.896	0.983	1	0.995	0.994	0.999	1	1	1	1	1	1
0.6	0.579	0.999	0.755	0.566	1	0.716	0.734	1	0.984	0.998	1	0.999	0.999	1	1	1	1	1	1	1
0.7	0.586	0.999	0.884	0.693	1	0.864	0.999	1	0.999	0.999	1	1	1	1	1	1	1	1	1	1
0.8	0.687	1	0.957	0.778	1	0.956	1	1	1	1	1	1	1	1	1	1	1	1	1	1

120

Table 4.4

Power of test for $p = 5$

ρ	$n=3$ T^*	$n=3$ Z^*	$n=5$ J	$n=5$ T^*	$n=5$ Z^*	$n=10$ J	$n=10$ T^*	$n=10$ Z^*	$n=20$ J	$n=20$ T^*	$n=20$ Z^*	$n=30$ J	$n=30$ T^*	$n=30$ Z^*	$n=50$ J	$n=50$ T^*	$n=50$ Z^*	$n=100$ J	$n=100$ T^*	$n=100$ Z^*
0.0	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.000	0.050	0.050	00000	0.050	0.050	0.000
0.1	0.168	0.459	0.063	0.073	0.233	0.070	0.073	0.410	0.083	0.085	0.573	0.103	0.102	0.002	0.160	0.155	0.039	0.323	0.334	0.561
0.2	0.268	0.799	0.098	0.120	0.776	0.120	0.124	0.912	0.195	0.299	0.923	0.300	0.300	0.583	0.542	0.541	0.823	0.903	0.905	0.995
0.3	0.340	0.960	0.155	0.205	0.956	0.201	0.212	0.992	0.396	0.406	0.989	0.622	0.614	0.929	0.899	0.902	0.996	0.999	0.999	1
0.4	0.500	0.981	0.237	0.313	0.991	0.319	0.339	0.999	0.649	0.658	0.999	0.886	0.882	0.996	0.994	0.993	1	1	1	1
0.5	0.510	0.988	0.372	0.410	0.998	0.478	0.504	0.999	0.869	0.865	0.860	0.986	0.998	1	0.999	1	1	1	1	1
0.6	0.570	0.992	0.525	0.509	0.999	0.662	0.680	1	0.976	0.970	0.999	0.999	0.999	1	1	1	1	1	1	1
0.7	0.617	0.996	0.663	0.999	0.999	0.831	0.835	1	0.998	0.994	1	1	1	1	1	1	1	1	1	1
0.8	0.667	0.998	0.780	1	1	0.945	0.939	1	1	0.999	1	1	1	1	1	1	1	1	1	1

4.6.2 Power of theTest for a Medium Number of Variables ($p = 10$ and 15)

Table 4.5 and Table 4.6 show the power of test for a medium number of variables. Drawing from the two tables, the conclusion can be made that power of statistical test increases compared with the previous section, and, the number of values that equal to 1 are increase.

In Table 4.5, most values of the power test of the Z^* statistic are greater than the Jennrich statistic and the T^* statistic for all shifts in the correlation matrix. When the sample size $n = 3$, the power values of T^* statistic fall within the interval when the shift in the correlation matrix is 0.1 and above. When the sample size n = 5 and 10 the values of power of the T^* statistic are very small and all the values fall outside the interval of the power. While, the values of Z^* statistic fall within the power interval at $\rho = 0.2$ and above. When the sample size n increases to 20, 30, 50 and 100 all the values of the Jennrich statistic, the T^* statistic and the Z^* statistic fall within the interval of the power when the shift in correlation matrix 0.3 and above. All the values of the power of the Z^* statistic were larger than the other two tests.

Based on Table 4.6, when $n = 3$, the T^* statistic and the Z^* statistic are power when the shift of correlation matrix is 0.2 and above. On the other hand, when the sample size is n = 5, 10, and 20 the values of power of the T^* statistic are very small, and outside the interval. The conclusion can be made that the T^* statistic is not power. For sample size 30, 50 and 100 the values of the three tests fall within the interval when the shift in correlation matrix 0.3 and above. Furthermore, from this table, all the values of the power test of the Z^* statistic are larger than the other two statistical tests, and the Z^* statistic dominates the other two statistical tests.

From the two tables we showed that there are 171 conditions for p = 10 involved to evaluate the performance of the power of statistical test, there are 53, 35 and 33 conditions for the Z^* statistic, the T^* statistic and the Jennrich statistic respectively fall within the interval of power. In addition, the Jennrich statistic cannot be calculated when $p > n$. For p = 15, there are 162 conditions in all the table. There are 53, 29, and 27 conditions belong to the Z^* statistic, the T^* statistic, and the Jennrich statistic respectively. From two tables, the value of the power of Z^* statistic fall within the interval of the power form the shift in the correlation matrix 0.2 and above. In brief, the two tables show that the results drawn from the Z^* statistic are more powerful than the other two statistical tests and dominate in the range of a medium number of variables.

Table 4.5

Power of test for $p = 10$

	$n=3$		$n=5$		$n=10$			$n=20$			$n=30$			$n=50$			$n=100$		
ρ	T^*	Z^*	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*
0.0	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.000	0.050	0.050	0.000	0.050	0.050	0.000
0.1	0.591	0.171	0.114	0.417	0.120	0.135	0.983	0.115	0.126	0.980	0.153	0.160	0.200	0.256	0.270	0.555	0.638	0.633	0.871
0.2	0.724	0.852	0.114	0.999	0.323	0.235	1	0.289	0.288	0.999	0.446	0.451	0.906	0.770	0.799	1	0.998	0.999	1
0.3	0.764	1	0.045	1	0.608	0.264	1	0.530	0.717	1	0.766	0.774	1	0.987	0.990	1	1	1	1
0.4	0.794	1	0.003	1	0.835	0.204	1	0.763	0.880	1	0.952	0.997	1	0.970	0.999	1	1	1	1
0.5	0.663	1	0.009	1	0.947	0.122	1	0.913	0.971	1	0.996	0.999	1	0.999	1	1	1	1	1
0.6	0.631	1	0.020	1	0.985	0.092	1	0.979	1	1	0.999	1	1	1	1	1	1	1	1
0.7	0.752	1	0.013	1	0.995	0.210	1	0.997	1	1	1	1	1	1	1	1	1	1	1
0.8	0.905	1	0.009	1	0.997	0.410	1	1	1	1	1	1	1	1	1	1	1	1	1

124

Table 4.6

Power of test for $p = 15$

	$n = 3$		$n = 5$		$n = 10$		$n = 20$			$n = 30$			$n = 50$			$n = 100$		
ρ	T^*	Z^*	T^*	Z^*	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*
0.0	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.000	0.050	0.050	0.0000	0.050	0.050	0.000
0.1	0.976	0.137	0.131	0.318	0.167	0.999	0.177	0.157	1	0.177	0.187	0.200	0.292	0.322	0.993	0.732	0.730	0.994
0.2	0.989	0.825	0.034	1	0.079	1	0.469	0.286	1	0.469	0.445	0.993	0.773	0.800	1	0.999	0.999	1
0.3	0.992	1	0.014	1	0.011	1	0.747	0.253	1	0.747	0.839	1	0.976	0.982	1	1	1	1
0.4	0.980	1	0.010	1	0.000	1	0.911	0.085	1	0.911	0.864	1	0.999	0.999	1	1	1	1
0.5	0.935	1	0.038	1	0.000	1	0.974	0.005	1	0.974	0.854	1	1	1	1	1	1	1
0.6	0.949	1	0.016	1	0.005	1	0.994	0.002	1	0.994	0.953	1	1	1	1	1	1	1
0.7	0.978	1	0.011	1	0.008	1	0.999	0.006	1	0.999	0.997	1	1	1	1	1	1	1
0.8	0.989	1	0.021	1	0.006	1	0.999	0.116	1	0.999	1	1	1	1	1	1	1	1

4.6.3 power of the Test for a Large Number of Variables ($p = 20$ and 30)

The power of tests for a larger number of variables for the three tests is presented in Table 4.7 and 4.8. For both tables, the value of the power of Z^* statistic is greater than the other two statistical tests for all sample sizes.

In table 4.7, when the sample size $n = 3$ the value of the power for the Z^* statistic and the T^* statistic fall within the interval when the shift of correlation matrix 0.2 and above. Furthermore, for $n = 10, 20, 30, 50$ and 100, the value of Z^* statistic within interval from 0.1 shift in correlation matrix and above for both tables. The values of the power of the T^* statistic when the sample size n = 5, 10, 20, and 30 are not within the interval in all shifts. Whereas, when n = 50 and 100, most the values are powerful when the shift is 0.2 and above. While, for the Jennrich statistic, no comparison can be made when $p > n$. When n = 20, 30, 50 and , 100 all the values of the Jennrich statistic are within the interval of power when the shift is 0.2 and above.

Based on Table 4.8, overall the Z^* statistic is powerful when the shift is 0.2 and above. When n = 3, the T^* statistic is powerful when the values are 0.1 and above. When the sample size is 5 the values of the T^* statistic are fall within the power interval for small shift 0.1 and 0.2. In cases when the sample size n = 10, 20, 30 and 50, all the values of the power of the T^* statistic are outside the interval . On the other hand, for n = 100 the values of the power of the T^* statistic are within the interval when the shift is 0.1 and above. For the Jennrich statistic, when the sample size equals the number of variable or the sample size is larger than number of variable, all values fall within the power interval when the shift is 0.2 and above. Furthermore, for Table 4.8 the value of the power of the Z^* statistic are within the interval when the shift of correlation is 0.1 and above for all sample size.

There are 162 and 153 conditions for p = 20 and 30, respectively involved to evaluate the power of the statistical tests. In the case when p = 20, there are 52 conditions of the Z^* statistic and 29 for the Jennrich statistic and 23 for the T^* statistic. However, for p = 30 for Z^* statistic, there are 53 and 22 conditions for the Jennrich statistic while the T^* statistic had 18 conditions. Thus, the conclusion can be made that the Z^* statistic dominated the other two tests.

Table 4.7

Power of test for $p = 20$

	$n = 3$		$n = 5$		$n = 10$		$n = 20$			$n = 30$			$n = 50$			$n = 100$		
ρ	T^*	Z^*	T^*	Z^*	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*
0.0	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.000	0.050	0.050	0.000	0.050	0.050	0.0000
0.1	0.999	0.140	0.328	0.292	0.340	1	0.273	0.060	0.999	0.224	0.213	0.300	0.323	0.315	0.378	0.750	0.767	1
0.2	0.999	0.911	0.033	1	0.037	1	0.737	0.003	1	0.580	0.326	0.999	0.750	0.709	1	0.999	0.999	1
0.3	0.999	0.990	0.040	1	0.010	1	0.963	0.001	1	0.837	0.161	1	0.947	0.896	1	1	1	1
0.4	0.995	1	0.001	1	0.018	1	0.998	0.000	1	0.949	0.110	1	0.993	0.963	1	1	1	1
0.5	0.986	1	0.051	1	0.046	1	0.999	0.027	1	0.984	0.212	1	0.999	0.987	1	1	1	1
0.6	0.987	1	0.017	1	0.049	1	1	0.005	1	0.992	0.111	1	1	0.999	1	1	1	1
0.7	0.989	1	0.007	1	0.057	1	1	0.009	1	0.993	0.122	1	1	1	1	1	1	1
0.8	0.996	1	0.002	1	0.091	1	1	0.010	1	0.977	0.131	1	1	1	1	1	1	1

Table 4.8

Power of test for $p = 30$

	$n=3$		$n=5$		$n=10$		$n=20$		$n=30$			$n=50$			$n=100$		
ρ	T^*	Z^*	T^*	Z^*	T^*	Z^*	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*	J	T^*	Z^*
0.0	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.000	0.050	0.050	0.000	0.050	0.050	0.000
0.1	1	0.131	0.998	0.340	0.034	1	0.042	1	0.466	0.201	0.300	0.390	0.312	0.514	0.724	0.707	1
0.2	1	0.987	0.876	1	0.014	1	0.010	1	0.949	0.318	1	0.820	0.274	1	0.991	0.980	1
0.3	1	1	0.020	1	0.100	1	0.008	1	1	0.152	1	0.965	0.043	1	1	0.999	1
0.4	1	1	0.137	1	0.005	1	0.004	1	1	0.020	1	0.994	0.010	1	1	0.999	1
0.5	1	1	0.053	1	0.003	1	0.004	1	1	0.010	1	0.998	0.010	1	1	1	1
0.6	1	1	0.021	1	0.010	1	0.012	1	1	0.010	1	0.999	0.004	1	1	1	1
0.7	1	1	0.008	1	0.009	1	0.002	1	1	0.001	1	0.995	0.002	1	1	1	1
0.8	1	1	0.010	1	0.000	1	0.000	1	1	0.029	1	0.915	0.000	1	1	1	1

129

4.6.4 Conclusion of the Power of Test

The following is the summary of the results. The power of test for Z^* statistic is show to be excellent when we have small sample size and large sample size. When comparing with the another two statistical tests the Z^* statistic is dominate the others statistical tests in any kind of variable size (small, medium or large). The sensitivity of the alternative statistical test to the correlation shits increase as the power of the new alternative statistical test Z^* increase. This means the new alternative statistical test Z^* is very sensitive to shift in the correlation structure.

4.7 Examples of Real Application

The Asia Pacific currencies is used to validate the new alternative statistical test Z^* and T^* statistic by using two approaches. Firstly, we performed the testing of two independent samples of correlation matrices, followed by the testing of several independent samples of correlation matrices using control chart. However, we cannot perform the Jennrich statistic due to the occurance of singularity problem.

We start this section by presenting the Q-Q plot to check the normality assumption since the Z^* statistic is developed on basis of multivariate normal distribution. When the points lie very nearly along a straight line, the normality assumption remains tenable. However, nonormality is suspected if the points deviate from a straight line.

Figure 4.1 (a) to Figure 4.1 (d) is performed for checking assumption of normality. We calculate the coeeficient of determination, R-square to evaluate how good the model fits the data. The values of R-square is lie between 0 and 1, and it becomes larger as the model fits better. Next, the Q-Q plot for January 2010 until December 2011 are presented

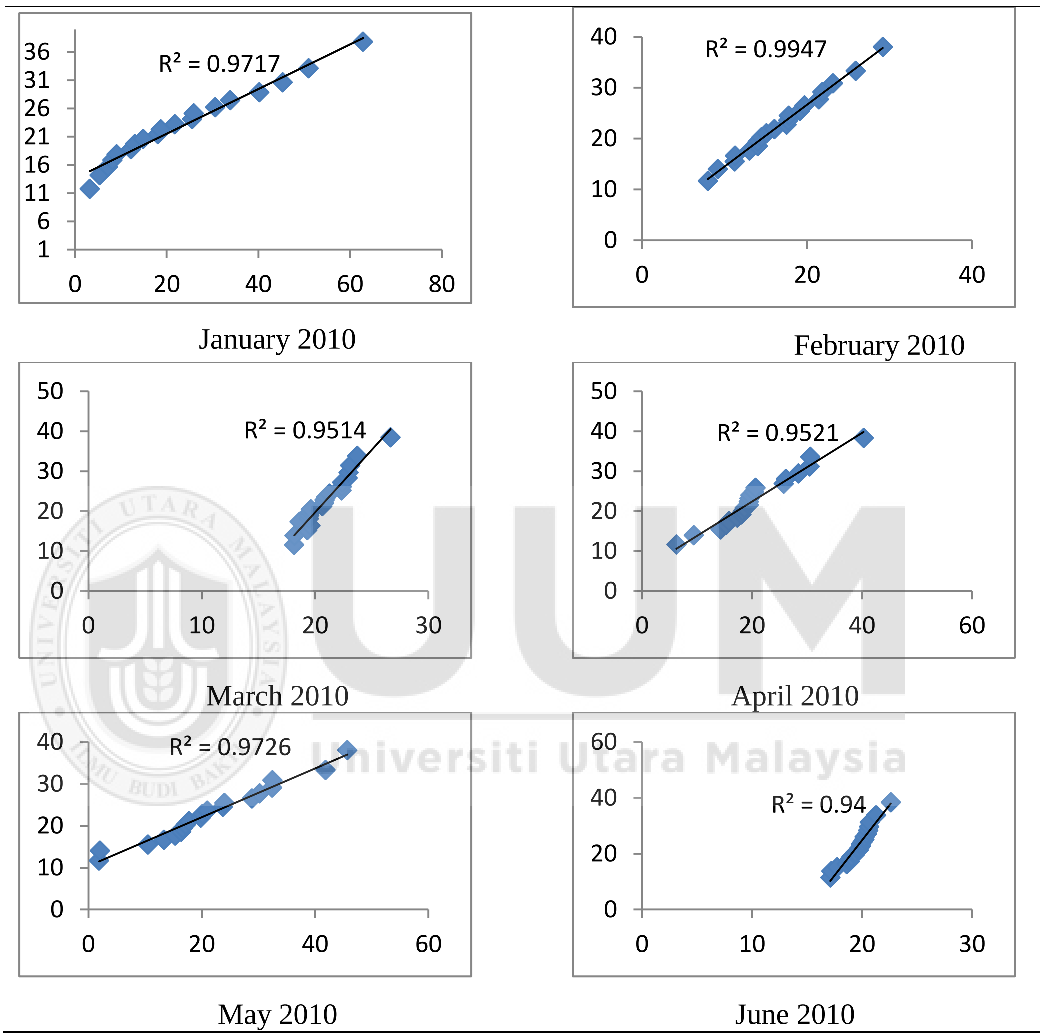

Figure 4.1 (a). The Q-Q plot for 6 months from January 2010-June 2010

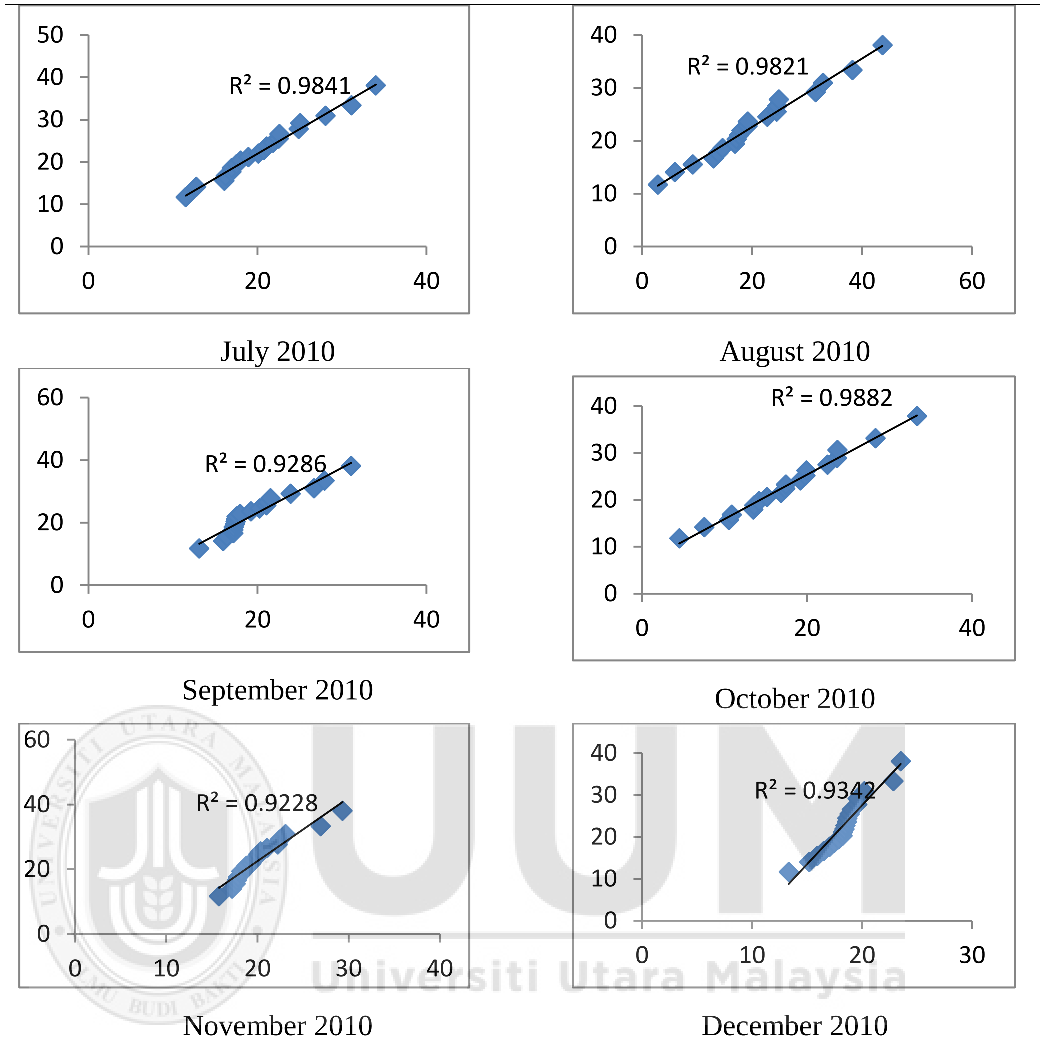

Figure 4.1 (b). The Q-Q plot for 6 months from July 2010-December 2010

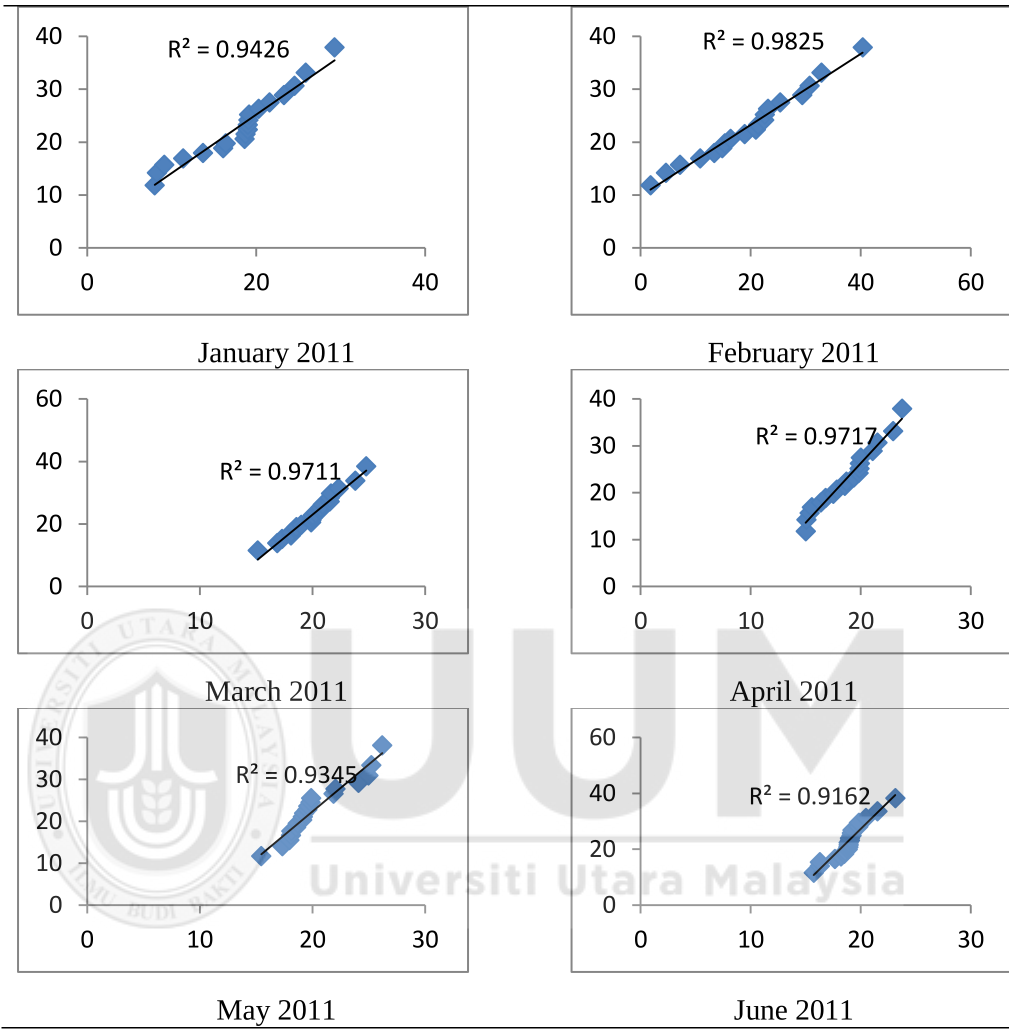

Figure 4.1 (c). The Q-Q plot for 6 months from January 2011-June 2011

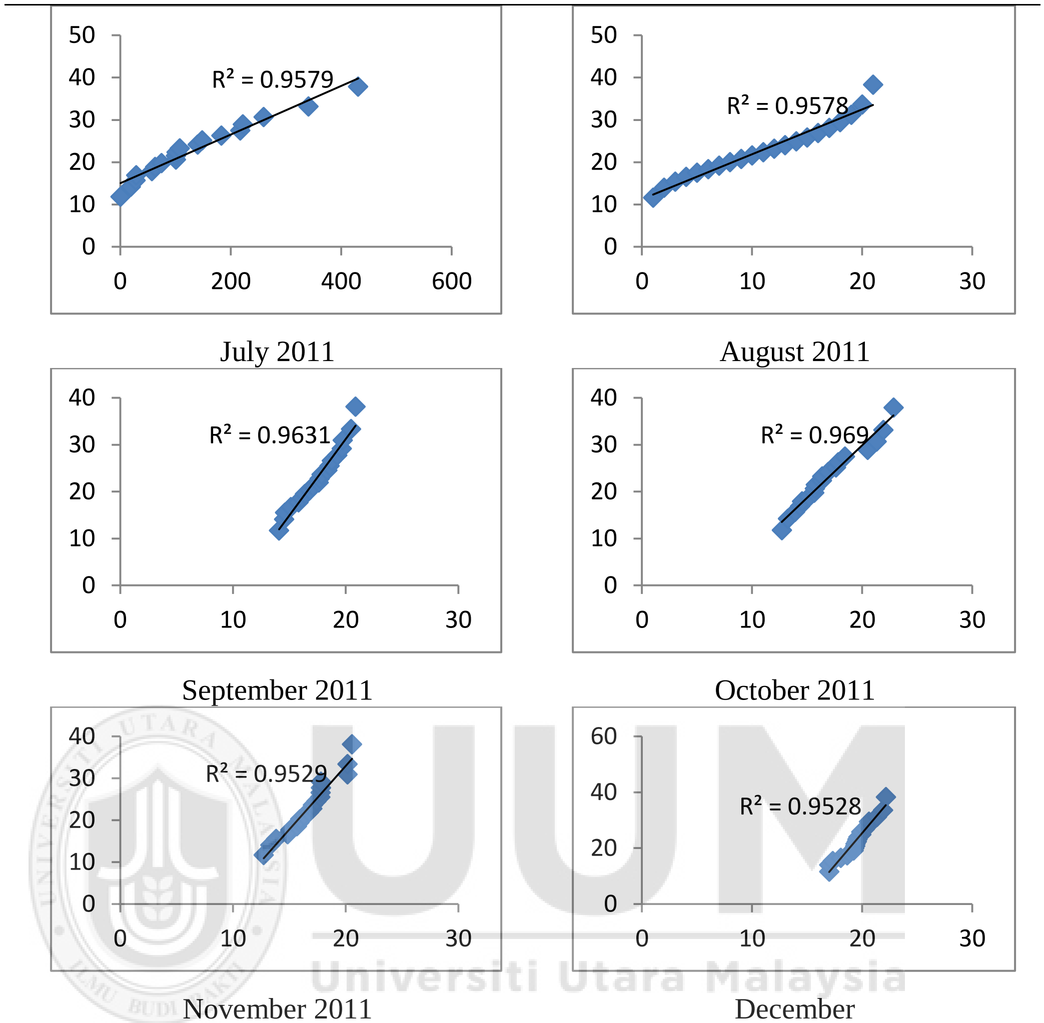

Figure 4.1 (d). The Q-Q plot for 6 months from July 2011-December 2011

The points in Figure 4.1 (a) to (d) are reasonably straight also it shows that the Q-Q plot fits the trend line this indicates the data are normal distribution. The values of R square are presented in the Table 4.9.

Table 4.9

R-square fro 24 samples

Sample	Month	R^2
1	January 2010	0.9717
2	February 2010	0.9947
3	March 2010	0.9541
4	April 2010	0.9521
5	May 2010	0.9726
6	June 2010	0.9400
7	July 2010	0.9841
8	August 2010	0.9821
9	September 2010	0.9286
10	October 2010	0.9882
11	November 2010	0.9228
12	December 2010	0.9342
13	January 2011	0.9426
14	February 2011	0.9825
15	March 2011	0.9711
16	April 2011	0.9717
17	May 2011	0.9345
18	June 2011	0.9162
19	July 2011	0.9579
20	August 2011	0.9405
21	September 2011	0.9631
22	October 2011	0.9690
23	November 2011	0.9529
24	December 2011	0.9528

The table shows that all the value of R-square is close to 1. Therefore, we conclude that all the data follow normal distribution. Next, we used A-D test to test the normality. The smaller the value of A-D test, the faster the speed of convergence in distribution. Table 4.10 shows the result of A-D test

Table 4.10

Anderson Darling test

Sample	**A-D test**	***p*-value**
1	0.1994	0.8750
2	0.5074	0.1782
3	0.2896	0.5802
4	0.8090	0.0548
5	0.3301	0.4970
6	0.6583	0.0736
7	0.6952	0.0594
8	0.5075	0.1782
9	0.3758	0.3783
10	0.5328	0.1504
11	0.3949	0.3399
12	0.3638	0.4070
13	0.2601	0.6741
14	0.1209	0.9856
15	0.4746	0.2171
16	0.3301	0.4970
17	0.4066	0.3181
18	0.6529	0.0760
19	0.2404	0.7428
20	0.3064	0.5370
21	0.2725	0.6306
22	0.5202	0.1622
23	0.5341	0.1504
24	0.2247	0.7976

Table 4.10, it is shown the best value of A-D is 0.8 and below. Since p value is large than 0.05, we fail to reject H_0 such that the data follow normal disribution.

4.7.1 Testing the Equality of Two Correlation Matrices

In this section, we present the results of Asia Pacific currencies using T^* statistic and the alternative statistical test called Z^* statistic, for the propose of validation. The currencies from Asia Pacific countries were employed to analyze the defferances in correlation matrices between February and April 2011, where it was one month before and one month after the Tohoku earthquake incident. To study the effect before and after the crisis. There are 23 currencies from the Asia Pacific countries and the sample size of the two samples equal to 19. To test the following hypothesis we used the significance level $\alpha = 0.05$.

Firstly, the results of the equality test between two correlation matrices by using T^* statistic (equation 2.10) is presented. To determine T^* statistic, the sample correlation matrix of February, R_1, the sample correlation matrix of April, R_2, the upper-off-diagonal elements of correlation matrix of February, R_{1U}, the upper-off-diagonal elements of correlation matrix of April, R_{2U}, the pooled correlation matrix, Ω are prepared.

$$
R_1 = \begin{pmatrix}
1 & 0.5 & -0.3 & 0.02 & 0.36 & 0.35 & -0.22 & 0.59 & 0.41 & 0.2 & 0.62 & 0.35 & 0.2 & 0.5 & 0.12 & 0.27 & 0.36 & 0.43 & 0.09 & 0.17 & 0.39 & 0.44 & 0.4 \\
0.5 & 1 & -0.8 & 0.21 & 0.46 & 0.42 & -0.26 & 0.73 & -0.2 & 0.68 & 0.84 & 0.48 & 0.6 & 0.62 & 0.31 & 0.89 & 0.85 & 0.98 & -0.3 & 0.06 & 0.67 & 0.74 & 0.32 \\
-0.3 & -0.8 & 1 & -0.2 & -0.4 & -0.3 & -0.13 & -0.5 & 0.21 & -0.5 & -0.6 & -0.5 & -0.5 & -0.6 & -0.2 & -0.9 & -0.6 & -0.9 & 0.34 & -0.3 & -0.5 & -0.6 & -0.3 \\
0.02 & 0.21 & -0.2 & 1 & 0.39 & 0.37 & -0.06 & 0.15 & 0.43 & -0.01 & 0.24 & 0.41 & 0.3 & 0.35 & 0.35 & 0.14 & 0.37 & 0.19 & -0.2 & 0.46 & 0.16 & 0.07 & 0.46 \\
0.36 & 0.46 & -0.4 & 0.39 & 1 & 0.3 & -0.26 & 0.73 & 0.28 & 0.08 & 0.26 & 0.65 & 0.3 & 0.5 & 0.5 & 0.4 & 0.66 & 0.36 & -0.2 & 0.1 & 0.23 & 0.48 & 0.37 \\
0.35 & 0.42 & -0.3 & 0.37 & 0.3 & 1 & -0.26 & 0.5 & 0.27 & 0.24 & 0.36 & 0.58 & 0.4 & 0.44 & 0.44 & 0.29 & 0.34 & 0.4 & 0.06 & 0.1 & 0.21 & 0.59 & 0.4 \\
-0.2 & -0.3 & -0.1 & -0.1 & -0.3 & -0.3 & 1 & -0.4 & 0.01 & -0.3 & -0.1 & -0.2 & 0.1 & -0.1 & -0.1 & -0.1 & -0.3 & -0.2 & -0.2 & 0.49 & -0.01 & -0.2 & -0.1 \\
0.59 & 0.73 & -0.5 & 0.15 & 0.73 & 0.5 & -0.42 & 1 & 0.18 & 0.39 & 0.52 & 0.75 & 0.5 & 0.64 & 0.61 & 0.51 & 0.69 & 0.62 & 0.06 & 0.02 & 0.43 & 0.71 & 0.25 \\
0.41 & -0.2 & 0.21 & 0.43 & 0.28 & 0.27 & 0.01 & 0.18 & 1 & -0.4 & 0.18 & 0.28 & 0.3 & 0.3 & 0.44 & -0.4 & -0.01 & -0.3 & 0.13 & 0.6 & 0.2 & 0.02 & 0.48 \\
0.2 & 0.68 & -0.5 & -0.01 & 0.08 & 0.24 & 0.33 & 0.39 & -0.4 & 1 & 0.45 & 0.21 & 0.2 & 0.12 & -0.1 & 0.69 & 0.48 & 0.73 & -0.1 & -0.3 & 0.18 & 0.29 & 0.03 \\
0.62 & 0.84 & -0.6 & 0.24 & 0.26 & 0.36 & -0.14 & 0.52 & 0.18 & 0.45 & 1 & 0.25 & 0.6 & 0.6 & 0.22 & 0.62 & 0.7 & 0.81 & -0.4 & 0.26 & 0.85 & 0.62 & 0.45 \\
0.35 & 0.48 & -0.5 & 0.41 & 0.65 & 0.58 & -0.23 & 0.75 & 0.28 & 0.21 & 0.25 & 1 & 0.4 & 0.54 & 0.57 & 0.39 & 0.53 & 0.37 & 0.18 & 0.28 & 0.23 & 0.58 & 0.2 \\
0.22 & 0.56 & -0.5 & 0.32 & 0.33 & 0.42 & 0.09 & 0.48 & 0.31 & 0.19 & 0.6 & 0.37 & 1 & 0.53 & 0.51 & 0.38 & 0.57 & 0.49 & -0.2 & 0.42 & 0.63 & 0.64 & 0.28 \\
0.5 & 0.62 & -0.6 & 0.35 & 0.5 & 0.44 & -0.14 & 0.64 & 0.3 & 0.12 & 0.6 & 0.54 & 0.5 & 1 & 0.71 & 0.56 & 0.64 & 0.56 & -0.01 & 0.5 & 0.48 & 0.48 & 0.39 \\
0.12 & 0.31 & -0.2 & 0.45 & 0.67 & 0.18 & -0.11 & 0.61 & 0.44 & -0.1 & 0.22 & 0.57 & 0.5 & 0.71 & 1 & 0.21 & 0.55 & 0.18 & 0.1 & 0.44 & 0.29 & 0.26 & 0.2 \\
0.27 & 0.89 & -0.9 & 0.14 & 0.4 & 0.29 & -0.1 & 0.51 & -0.4 & 0.69 & 0.62 & 0.39 & 0.4 & 0.56 & 0.21 & 1 & 0.78 & 0.92 & -0.3 & 0.06 & 0.45 & 0.53 & 0.26 \\
0.36 & 0.85 & -0.6 & 0.37 & 0.66 & 0.34 & -0.33 & 0.69 & -0.01 & 0.48 & 0.7 & 0.53 & 0.6 & 0.64 & 0.55 & 0.78 & 1 & 0.79 & -0.3 & 0.06 & 0.6 & 0.62 & 0.37 \\
0.43 & 0.98 & -0.9 & 0.19 & 0.36 & 0.4 & -0.2 & 0.62 & -0.3 & 0.73 & 0.81 & 0.37 & 0.5 & 0.56 & 0.18 & 0.92 & 0.79 & 1 & -0.4 & 0.01 & 0.62 & 0.68 & 0.3 \\
0.09 & -0.3 & 0.34 & -0.2 & -0.2 & 0.06 & -0.2 & 0.06 & 0.13 & -0.1 & -0.4 & 0.18 & -0.2 & -0.01 & 0.1 & -0.3 & -0.3 & -0.4 & 1 & -0.01 & -0.5 & -0.2 & -0.2 \\
0.17 & 0.06 & -0.3 & 0.46 & 0.1 & 0.1 & 0.49 & 0.02 & 0.6 & -0.3 & 0.26 & 0.28 & 0.4 & 0.5 & 0.44 & 0.06 & 0.06 & 0.01 & -0.01 & 1 & 0.28 & 0.05 & 0.46 \\
0.39 & 0.67 & -0.5 & 0.16 & 0.23 & 0.21 & -0.04 & 0.43 & 0.2 & 0.18 & 0.85 & 0.23 & 0.6 & 0.48 & 0.29 & 0.45 & 0.6 & 0.62 & -0.5 & 0.28 & 1 & 0.61 & 0.35 \\
0.44 & 0.74 & -0.6 & 0.07 & 0.48 & 0.59 & -0.17 & 0.71 & 0.02 & 0.29 & 0.62 & 0.58 & 0.6 & 0.48 & 0.26 & 0.53 & 0.62 & 0.68 & -0.2 & 0.05 & 0.61 & 1 & 0.26 \\
0.4 & 0.32 & -0.3 & 0.46 & 0.37 & 0.4 & -0.06 & 0.25 & 0.48 & 0.03 & 0.45 & 0.2 & 0.3 & 0.39 & 0.2 & 0.26 & 0.37 & 0.3 & -0.2 & 0.46 & 0.35 & 0.26 & 1
\end{pmatrix}
$$

139

$$
R_2 = \begin{pmatrix}
1 & 0.38 & -0.5 & -0.1 & 0.13 & 0.13 & 0.32 & 0.27 & -0.2 & 0.47 & 0.18 & 0.22 & -0.2 & 0.42 & -0.5 & 0.27 & 0.17 & 0.34 & -0.2 & -0.2 & -0.3 & 0.46 & -0.2 \\
0.38 & 1 & -0.8 & 0.13 & 0.46 & 0.51 & 0.32 & 0.68 & -0.2 & 0.79 & 0.73 & 0.53 & -0.2 & 0.28 & -0.3 & 0.85 & 0.84 & 0.99 & 0.16 & -0.4 & -0.2 & 0.74 & 0.42 \\
-0.5 & -0.8 & 1 & -0.3 & -0.2 & -0.3 & 0.76 & -0.5 & 0.47 & -0.6 & -0.7 & -0.4 & 0.4 & -0.2 & 0.47 & -0.8 & -0.6 & -0.8 & 0.1 & 0.47 & 0.26 & -0.8 & -0.2 \\
-0.1 & 0.13 & -0.3 & 1 & -0.2 & 0.01 & 0.44 & 0.08 & -0.3 & -0.01 & 0.36 & -0.1 & -0.3 & -0.1 & -0.2 & 0.29 & 0.14 & 0.16 & 0.29 & 0.17 & 0.18 & -0.1 & -0.01 \\
0.13 & 0.46 & -0.2 & -0.2 & 1 & 0.68 & 0.05 & 0.18 & 0.18 & 0.33 & 0.15 & 0.65 & 0.1 & 0.15 & -0.1 & 0.27 & 0.45 & 0.41 & -0.2 & -0.1 & -0.3 & 0.5 & 0.46 \\
0.13 & 0.51 & -0.3 & 0.01 & 0.68 & 1 & 0.11 & 0.65 & -0.1 & 0.42 & 0.29 & 0.66 & 0.2 & 0.03 & 0.04 & 0.45 & 0.45 & 0.49 & 0.12 & -0.1 & -0.5 & 0.32 & 0.37 \\
0.32 & 0.32 & -0.8 & 0.44 & 0.05 & 0.11 & 1 & 0.14 & -0.5 & 0.24 & 0.33 & 0.12 & -0.5 & 0.01 & -0.6 & 0.36 & 0.11 & 0.35 & -0.3 & -0.4 & -0.4 & 0.45 & -0.1 \\
0.27 & 0.68 & -0.5 & 0.08 & 0.18 & 0.65 & 0.14 & 1 & -0.4 & 0.44 & 0.53 & 0.44 & 0.2 & 0.12 & -0.01 & 0.77 & 0.49 & 0.69 & 0.3 & -0.2 & -0.2 & 0.4 & 0.21 \\
-0.2 & -0.2 & 0.47 & -0.3 & 0.18 & -0.1 & -0.45 & -0.4 & 1 & -0.01 & -0.4 & -0.2 & 0.3 & 0.33 & 0.1 & -0.4 & -0.1 & -0.2 & -0.01 & 0.41 & 0.18 & -0.3 & -0.1 \\
0.47 & 0.79 & -0.6 & -0.01 & 0.33 & 0.42 & 0.24 & 0.44 & -0.01 & 1 & 0.51 & 0.34 & -0.3 & 0.13 & -0.4 & 0.64 & 0.58 & 0.77 & -0.1 & -0.2 & -0.4 & 0.66 & 0.24 \\
0.18 & 0.73 & -0.7 & 0.36 & 0.15 & 0.29 & 0.33 & 0.53 & -0.4 & 0.51 & 1 & 0.41 & 0.01 & 0.02 & -0.2 & 0.73 & 0.76 & 0.72 & 0.24 & -0.1 & 0.12 & 0.51 & 0.45 \\
0.22 & 0.53 & -0.4 & -0.1 & 0.65 & 0.66 & 0.12 & 0.44 & -0.2 & 0.34 & 0.41 & 1 & 0.1 & 0.18 & -0.01 & 0.36 & 0.6 & 0.49 & 0.05 & -0.1 & -0.1 & 0.52 & 0.15 \\
-0.2 & -0.2 & 0.43 & -0.3 & 0.12 & 0.2 & -0.55 & 0.22 & 0.29 & -0.3 & 0.01 & 0.1 & 1 & 0.25 & 0.45 & -0.2 & 0.02 & -0.2 & 0.26 & 0.54 & 0.2 & -0.3 & 0.21 \\
0.42 & 0.28 & -0.2 & -0.1 & 0.15 & 0.03 & 0.01 & 0.12 & 0.33 & 0.13 & 0.02 & 0.18 & 0.2 & 1 & 0.16 & 0.06 & 0.33 & 0.23 & 0.17 & 0.19 & 0.33 & 0.19 & 0.07 \\
-0.5 & -0.3 & 0.47 & -0.2 & -0.1 & 0.04 & -0.6 & -0.01 & 0.1 & -0.4 & -0.2 & -0.01 & 0.5 & 0.16 & 1 & -0.3 & 0.08 & -0.3 & 0.44 & 0.33 & 0.4 & -0.3 & 0.22 \\
0.27 & 0.85 & -0.8 & 0.29 & 0.27 & 0.45 & 0.36 & 0.77 & -0.4 & 0.64 & 0.73 & 0.36 & -0.2 & 0.06 & -0.3 & 1 & 0.64 & 0.88 & 0.17 & -0.3 & -0.1 & 0.64 & 0.45 \\
0.17 & 0.84 & -0.6 & 0.14 & 0.45 & 0.45 & 0.11 & 0.49 & -0.1 & 0.58 & 0.76 & 0.6 & 0.01 & 0.33 & 0.08 & 0.64 & 1 & 0.79 & 0.17 & -0.1 & 0.12 & 0.66 & 0.53 \\
0.34 & 0.99 & -0.8 & 0.16 & 0.41 & 0.49 & 0.35 & 0.69 & -0.2 & 0.77 & 0.72 & 0.49 & -0.2 & 0.23 & -0.3 & 0.88 & 0.79 & 1 & 0.21 & -0.5 & -0.1 & 0.7 & 0.38 \\
-0.2 & 0.16 & 0.1 & 0.29 & -0.2 & 0.12 & -0.32 & 0.3 & -0.01 & -0.1 & 0.24 & 0.05 & 0.3 & 0.17 & 0.44 & 0.17 & 0.17 & 0.21 & 1 & -0.01 & 0.53 & -0.5 & 0.12 \\
-0.2 & -0.4 & 0.47 & 0.17 & -0.1 & -0.1 & -0.4 & -0.2 & 0.41 & -0.2 & -0.1 & -0.1 & 0.5 & 0.19 & 0.33 & -0.3 & -0.1 & -0.5 & -0.01 & 1 & 0.3 & -0.3 & -0.1 \\
-0.3 & -0.2 & 0.26 & 0.18 & -0.3 & -0.5 & -0.43 & -0.2 & 0.18 & -0.4 & 0.12 & -0.1 & 0.2 & 0.33 & 0.4 & -0.1 & 0.12 & -0.1 & 0.53 & 0.3 & 1 & -0.3 & -0.01 \\
0.46 & 0.74 & -0.8 & -0.1 & 0.5 & 0.32 & 0.45 & 0.4 & -0.3 & 0.66 & 0.51 & 0.52 & -0.3 & 0.19 & -0.3 & 0.64 & 0.66 & 0.7 & -0.5 & -0.3 & -0.3 & 1 & 0.29 \\
-0.2 & 0.42 & -0.2 & -0.01 & 0.46 & 0.37 & -0.05 & 0.21 & -0.1 & 0.24 & 0.45 & 0.15 & 0.2 & 0.07 & 0.22 & 0.45 & 0.53 & 0.38 & 0.12 & -0.1 & -0.01 & 0.29 & 1
\end{pmatrix}
$$

$$R_{1U} = \left(\begin{array}{ccccccccccccccccccccccc}
0 & 0.5 & -0.3 & 0.02 & 0.36 & 0.35 & -0.22 & 0.59 & 0.41 & 0.2 & 0.62 & 0.35 & 0.2 & 0.5 & 0.12 & 0.27 & 0.36 & 0.43 & 0.09 & 0.17 & 0.39 & 0.44 & 0.4 \\
0 & 0 & -0.8 & 0.21 & 0.46 & 0.42 & -0.26 & 0.73 & -0.2 & 0.68 & 0.84 & 0.48 & 0.6 & 0.62 & 0.31 & 0.89 & 0.85 & 0.98 & -0.3 & 0.06 & 0.67 & 0.74 & 0.32 \\
0 & 0 & 0 & -0.2 & -0.4 & -0.3 & -0.13 & -0.5 & 0.21 & -0.5 & -0.6 & -0.5 & -0.5 & -0.6 & -0.2 & -0.9 & -0.6 & -0.9 & 0.34 & -0.3 & -0.5 & -0.6 & -0.3 \\
0 & 0 & 0 & 0 & 0.39 & 0.37 & -0.06 & 0.15 & 0.43 & -0.01 & 0.24 & 0.41 & 0.3 & 0.35 & 0.35 & 0.14 & 0.37 & 0.19 & -0.2 & 0.46 & 0.16 & 0.07 & 0.46 \\
0 & 0 & 0 & 0 & 0 & 0.3 & -0.26 & 0.73 & 0.28 & 0.08 & 0.26 & 0.65 & 0.3 & 0.5 & 0.5 & 0.4 & 0.66 & 0.36 & -0.2 & 0.1 & 0.23 & 0.48 & 0.37 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.26 & 0.5 & 0.27 & 0.24 & 0.36 & 0.58 & 0.4 & 0.44 & 0.44 & 0.29 & 0.34 & 0.4 & 0.06 & 0.1 & 0.21 & 0.59 & 0.4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.4 & 0.01 & -0.3 & -0.1 & -0.2 & 0.1 & -0.1 & -0.1 & -0.1 & -0.3 & -0.2 & -0.2 & 0.49 & -0.01 & -0.2 & -0.1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.18 & 0.39 & 0.52 & 0.75 & 0.5 & 0.64 & 0.61 & 0.51 & 0.69 & 0.62 & 0.06 & 0.02 & 0.43 & 0.71 & 0.25 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.4 & 0.18 & 0.28 & 0.3 & 0.3 & 0.44 & -0.4 & -0.01 & -0.3 & 0.13 & 0.6 & 0.2 & 0.02 & 0.48 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.21 & 0.2 & 0.12 & -0.1 & 0.69 & 0.48 & 0.73 & -0.1 & -0.3 & 0.18 & 0.29 & 0.03 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.6 & 0.6 & 0.22 & 0.62 & 0.7 & 0.81 & -0.4 & 0.26 & 0.85 & 0.62 & 0.45 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.54 & 0.57 & 0.39 & 0.53 & 0.37 & 0.18 & 0.28 & 0.23 & 0.58 & 0.2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.53 & 0.51 & 0.38 & 0.57 & 0.49 & -0.2 & 0.42 & 0.63 & 0.64 & 0.28 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.71 & 0.56 & 0.64 & 0.56 & -0.01 & 0.5 & 0.48 & 0.48 & 0.39 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.21 & 0.55 & 0.18 & 0.1 & 0.44 & 0.29 & 0.26 & 0.2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.78 & 0.92 & -0.3 & 0.06 & 0.45 & 0.53 & 0.26 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.79 & -0.3 & 0.06 & 0.6 & 0.62 & 0.37 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.4 & 0.01 & 0.62 & 0.68 & 0.3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.01 & -0.5 & -0.2 & -0.2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.28 & 0.05 & 0.46 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.61 & 0.35 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.26 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

140

$$R_{2U} = \begin{pmatrix}
0 & 0.38 & -0.5 & -0.1 & 0.13 & 0.13 & 0.32 & 0.27 & -0.2 & 0.47 & 0.18 & 0.22 & -0.2 & 0.42 & -0.5 & 0.27 & 0.17 & 0.34 & -0.2 & -0.2 & -0.3 & 0.46 & -0.2 \\
0 & 0 & -0.8 & 0.13 & 0.46 & 0.51 & 0.32 & 0.68 & -0.2 & 0.79 & 0.73 & 0.53 & -0.2 & 0.28 & -0.3 & 0.85 & 0.84 & 0.99 & 0.16 & -0.4 & -0.2 & 0.74 & 0.42 \\
0 & 0 & 0 & -0.3 & -0.2 & -0.3 & 0.76 & -0.5 & 0.47 & -0.6 & -0.7 & -0.4 & 0.4 & -0.2 & 0.47 & -0.8 & -0.6 & -0.8 & 0.1 & 0.47 & 0.26 & -0.8 & -0.2 \\
0 & 0 & 0 & 0 & -0.2 & 0.01 & 0.44 & 0.08 & -0.3 & -0.01 & 0.36 & -0.1 & -0.3 & -0.1 & -0.2 & 0.29 & 0.14 & 0.16 & 0.29 & 0.17 & 0.18 & -0.1 & -0.01 \\
0 & 0 & 0 & 0 & 0 & 0.68 & 0.05 & 0.18 & 0.18 & 0.33 & 0.15 & 0.65 & 0.1 & 0.15 & -0.1 & 0.27 & 0.45 & 0.41 & -0.2 & -0.1 & -0.3 & 0.5 & 0.46 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.11 & 0.65 & -0.1 & 0.42 & 0.29 & 0.66 & 0.2 & 0.03 & 0.04 & 0.45 & 0.45 & 0.49 & 0.12 & -0.1 & -0.5 & 0.32 & 0.37 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.14 & -0.5 & 0.24 & 0.33 & 0.12 & -0.5 & 0.01 & -0.6 & 0.36 & 0.11 & 0.35 & -0.3 & -0.4 & -0.4 & 0.45 & -0.1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.4 & 0.44 & 0.53 & 0.44 & 0.2 & 0.12 & -0.01 & 0.77 & 0.49 & 0.69 & 0.3 & -0.2 & -0.2 & 0.4 & 0.21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.01 & -0.4 & -0.2 & 0.3 & 0.33 & 0.1 & -0.4 & -0.1 & -0.2 & -0.01 & 0.41 & 0.18 & -0.3 & -0.1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.51 & 0.34 & -0.3 & 0.13 & -0.4 & 0.64 & 0.58 & 0.77 & -0.1 & -0.2 & -0.4 & 0.66 & 0.24 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.41 & 0.01 & 0.02 & -0.2 & 0.73 & 0.76 & 0.72 & 0.24 & -0.1 & 0.12 & 0.51 & 0.45 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0.18 & -0.01 & 0.36 & 0.6 & 0.49 & 0.05 & -0.1 & -0.1 & 0.52 & 0.15 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.45 & -0.2 & 0.02 & -0.2 & 0.26 & 0.54 & 0.2 & -0.3 & 0.21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.16 & 0.06 & 0.33 & 0.23 & 0.17 & 0.19 & 0.33 & 0.19 & 0.07 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3 & 0.08 & -0.3 & 0.44 & 0.33 & 0.4 & -0.3 & 0.22 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.64 & 0.88 & 0.17 & -0.3 & -0.1 & 0.64 & 0.45 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.79 & 0.17 & -0.1 & 0.12 & 0.66 & 0.53 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.21 & -0.5 & -0.1 & 0.7 & 0.38 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.01 & 0.53 & -0.5 & 0.12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & -0.3 & -0.1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3 & -0.01 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.29 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

141

142

$$
\Omega = \begin{pmatrix}
1 & 0.44 & -0.4 & 0.04 & 0.25 & 0.24 & 0.05 & 0.43 & 0.11 & 0.33 & 0.4 & 0.29 & 0.04 & 0.46 & -0.18 & 0.27 & 0.26 & 0.38 & -0.1 & -0.01 & 0.06 & 0.45 & 0.13 \\
0.44 & 1 & -0.83 & 0.17 & 0.46 & 0.46 & 0.03 & 0.7 & -0.2 & 0.73 & 0.79 & 0.51 & 0.2 & 0.45 & 0.03 & 0.87 & 0.84 & 0.99 & -0.08 & -0.17 & 0.26 & 0.74 & 0.37 \\
-0.4 & -0.83 & 1 & -0.25 & -0.31 & -0.30 & -0.45 & -0.5 & 0.34 & -0.57 & -0.68 & -0.4 & -0.03 & -0.37 & 0.12 & -0.84 & -0.63 & -0.84 & 0.22 & 0.07 & -0.14 & -0.71 & -0.26 \\
0.04 & 0.17 & -0.25 & 1 & 0.1 & 0.19 & 0.19 & 0.12 & 0.08 & -0.01 & 0.3 & 0.15 & -0.01 & 0.14 & 0.13 & 0.22 & 0.26 & 0.18 & 0.03 & 0.31 & 0.17 & -0.03 & 0.23 \\
0.25 & 0.46 & -0.31 & 0.10 & 1 & 0.49 & -0.1 & 0.45 & 0.23 & 0.21 & 0.21 & 0.65 & 0.22 & 0.32 & 0.27 & 0.33 & 0.56 & 0.38 & -0.16 & -0.01 & -0.04 & 0.49 & 0.31 \\
0.24 & 0.46 & -0.29 & 0.19 & 0.49 & 1 & -0.07 & 0.57 & 0.07 & 0.33 & 0.33 & 0.62 & 0.31 & 0.24 & 0.11 & 0.37 & 0.39 & 0.45 & 0.09 & 0.02 & -0.14 & 0.45 & 0.38 \\
0.05 & 0.03 & -0.45 & 0.19 & -0.1 & -0.1 & 1 & -0.1 & -0.2 & -0.05 & 0.09 & -0.1 & -0.23 & -0.07 & -0.35 & 0.13 & -0.11 & 0.07 & -0.26 & 0.05 & -0.23 & 0.13 & -0.06 \\
0.43 & 0.7 & -0.52 & 0.12 & 0.45 & 0.57 & -0.14 & 1 & -0.1 & 0.42 & 0.52 & 0.59 & 0.35 & 0.38 & 0.29 & 0.64 & 0.59 & 0.65 & 0.18 & -0.07 & 0.1 & 0.56 & 0.23 \\
0.11 & -0.19 & 0.34 & 0.08 & 0.23 & -0.07 & -0.22 & -0.1 & 1 & -0.22 & -0.1 & 0.05 & 0.30 & 0.31 & 0.27 & -0.41 & -0.07 & -0.26 & 0.05 & 0.51 & 0.19 & -0.13 & 0.21 \\
0.33 & 0.74 & -0.57 & -0.01 & 0.21 & 0.33 & -0.05 & 0.42 & -0.2 & 1 & 0.48 & 0.28 & -0.05 & 0.13 & -0.26 & 0.66 & 0.53 & 0.75 & -0.13 & -0.28 & -0.09 & 0.48 & 0.13 \\
0.4 & 0.79 & -0.68 & 0.3 & 0.21 & 0.33 & 0.09 & 0.52 & -0.1 & 0.48 & 1 & 0.33 & 0.30 & 0.31 & 0.02 & 0.67 & 0.73 & 0.77 & -0.09 & 0.8 & 0.49 & 0.56 & 0.45 \\
0.29 & 0.51 & -0.43 & 0.15 & 0.65 & 0.62 & -0.05 & 0.59 & 0.05 & 0.28 & 0.33 & 1 & 0.23 & 0.36 & 0.28 & 0.38 & 0.57 & 0.43 & 0.11 & 0.1 & 0.06 & 0.55 & 0.18 \\
0.04 & 0.20 & -0.03 & 0.01 & 0.22 & 0.31 & -0.23 & 0.35 & 0.3 & -0.05 & 0.30 & 0.23 & 1 & 0.39 & 0.48 & 0.08 & 0.29 & 0.13 & -0.1 & 0.48 & 0.42 & 0.18 & 0.24 \\
0.46 & 0.45 & -0.37 & 0.14 & 0.32 & 0.23 & -0.07 & 0.38 & 0.31 & 0.13 & 0.31 & 0.36 & 0.39 & 1 & 0.44 & 0.31 & 0.49 & 0.39 & 0.03 & 0.34 & 0.4 & 0.33 & 0.23 \\
-0.2 & 0.03 & 0.12 & 0.13 & 0.27 & 0.11 & -0.35 & 0.29 & 0.27 & -0.26 & 0.02 & 0.28 & 0.48 & 0.44 & 1 & -0.02 & 0.31 & -0.04 & 0.08 & 0.39 & 0.35 & -0.03 & 0.21 \\
0.27 & 0.87 & -0.84 & 0.22 & 0.33 & 0.37 & 0.13 & 0.64 & -0.4 & 0.66 & 0.67 & 0.38 & 0.08 & 0.31 & -0.02 & 1 & 0.71 & 0.9 & 0.27 & 0.15 & 0.16 & 0.58 & 0.35 \\
0.26 & 0.84 & -0.63 & 0.26 & 0.56 & 0.39 & -0.11 & 0.59 & -0.1 & 0.53 & 0.73 & 0.57 & 0.29 & 0.49 & 0.31 & 0.71 & 1 & 0.79 & -0.06 & -0.01 & 0.36 & 0.64 & 0.45 \\
0.38 & 0.98 & -0.84 & 0.18 & 0.38 & 0.45 & 0.07 & 0.65 & -0.3 & 0.75 & 0.77 & 0.43 & 0.13 & 0.39 & -0.04 & 0.9 & 0.79 & 1 & -0.09 & -0.22 & 0.24 & 0.69 & 0.34 \\
-0.1 & -0.08 & 0.22 & 0.03 & -0.16 & 0.09 & -0.26 & 0.18 & 0.05 & -0.13 & -0.09 & 0.11 & 0.03 & 0.08 & 0.27 & -0.08 & -0.06 & -0.09 & 1 & -0.03 & 0.03 & -0.31 & -0.06 \\
-0.01 & -0.17 & 0.07 & 0.31 & -0.01 & 0.02 & 0.05 & -0.1 & 0.51 & -0.28 & 0.08 & 0.1 & 0.48 & 0.34 & 0.39 & -0.15 & -0.01 & -0.23 & -0.02 & 1 & 0.29 & -0.13 & 0.2 \\
0.06 & 0.26 & -0.14 & 0.17 & -0.04 & -0.1 & -0.23 & 0.1 & 0.19 & -0.09 & 0.49 & 0.06 & 0.42 & 0.4 & 0.35 & 0.16 & 0.35 & 0.24 & 0.03 & 0.29 & 1 & 0.14 & 0.16 \\
0.45 & 0.74 & -0.71 & -0.03 & 0.48 & 0.45 & 0.14 & 0.56 & -0.1 & 0.48 & 0.56 & 0.55 & 0.18 & 0.33 & -0.03 & 0.58 & 0.64 & 0.69 & -0.31 & -0.13 & 0.14 & 1 & 0.27 \\
0.13 & 0.37 & -0.26 & 0.23 & 0.42 & 0.38 & -0.06 & 0.23 & 0.21 & 0.13 & 0.45 & 0.18 & 0.24 & 0.23 & 0.21 & 0.35 & 0.45 & 0.34 & -0.06 & 0.20 & 0.16 & 0.27 & 1
\end{pmatrix}
$$

We compute the $V_{0,1}, V_{0,2}, V_{0,3}$ and Ψ_0^*. Then, the T^* statistic is computed. The statistical test is compared to the critical value for making a decision. The degree of freedom is, $k = \frac{p(p-1)}{2} = \frac{23(23-1)}{2} = 253$. Therefore, the critical value of T^* is $\chi^2_{0.05,253} = 291.101$. Based on $T^* = 1015.82$ and p-value $= 0.050$, the null hypothesis is rejected.

Secondly, the results of the equality test between two correlation matrices, Z^* statistic is presented. To determine Z^* statistic, the $R_1, R_2, R_{1U}, R_{2U}, \Omega$ the upper-off-diagonal elements of pooled correlation matrix is $\lambda\ (\lambda = \Omega_U)$, the matrix where the diagonal elements are the diagonal elements of $\Omega\lambda$ is $D_{\Omega\lambda}$, the matrix where the diagonal elements are the diagonal elements of $\lambda\Omega$ is $D_{\lambda\Omega}$, are prepared.

$$
\lambda = \begin{pmatrix}
0 & 0.44 & -0.4 & 0.04 & 0.25 & 0.24 & 0.05 & 0.43 & 0.11 & 0.33 & 0.4 & 0.29 & 0.04 & 0.46 & -0.18 & 0.27 & 0.26 & 0.38 & -0.1 & -0.01 & 0.06 & 0.45 & 0.13 \\
0 & 0 & -0.83 & 0.17 & 0.46 & 0.46 & 0.03 & 0.7 & -0.2 & 0.73 & 0.79 & 0.51 & 0.2 & 0.45 & 0.03 & 0.87 & 0.84 & 0.99 & -0.08 & -0.17 & 0.26 & 0.74 & 0.37 \\
0 & 0 & 0 & -0.25 & -0.31 & -0.30 & -0.45 & -0.5 & 0.34 & -0.57 & -0.68 & -0.4 & -0.03 & -0.37 & 0.12 & -0.84 & -0.63 & -0.84 & 0.22 & 0.07 & -0.14 & -0.71 & -0.26 \\
0 & 0 & 0 & 0 & 0.1 & 0.19 & 0.19 & 0.12 & 0.08 & -0.01 & 0.3 & 0.15 & -0.01 & 0.14 & 0.13 & 0.22 & 0.26 & 0.18 & 0.03 & 0.31 & 0.17 & -0.03 & 0.23 \\
0 & 0 & 0 & 0 & 0 & 0.49 & -0.1 & 0.45 & 0.23 & 0.21 & 0.21 & 0.65 & 0.22 & 0.32 & 0.27 & 0.33 & 0.56 & 0.38 & -0.16 & -0.01 & -0.04 & 0.49 & 0.31 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.07 & 0.57 & 0.07 & 0.33 & 0.33 & 0.62 & 0.31 & 0.24 & 0.11 & 0.37 & 0.39 & 0.45 & 0.09 & 0.02 & -0.14 & 0.45 & 0.38 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1 & -0.2 & -0.05 & 0.09 & -0.1 & -0.23 & -0.07 & -0.35 & 0.13 & -0.11 & 0.07 & -0.26 & 0.05 & -0.23 & 0.13 & -0.06 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1 & 0.42 & 0.52 & 0.59 & 0.35 & 0.38 & 0.29 & 0.64 & 0.59 & 0.65 & 0.18 & -0.07 & 0.1 & 0.56 & 0.23 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.22 & -0.1 & 0.05 & 0.30 & 0.31 & 0.27 & -0.41 & -0.07 & -0.26 & 0.05 & 0.51 & 0.19 & -0.13 & 0.21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.48 & 0.28 & -0.05 & 0.13 & -0.26 & 0.66 & 0.53 & 0.75 & -0.13 & -0.28 & -0.09 & 0.48 & 0.13 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.33 & 0.30 & 0.31 & 0.02 & 0.67 & 0.73 & 0.77 & -0.09 & 0.8 & 0.49 & 0.56 & 0.45 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.23 & 0.36 & 0.28 & 0.38 & 0.57 & 0.43 & 0.11 & 0.1 & 0.06 & 0.55 & 0.18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.39 & 0.48 & 0.08 & 0.29 & 0.13 & -0.1 & 0.48 & 0.42 & 0.18 & 0.24 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.44 & 0.31 & 0.49 & 0.39 & 0.03 & 0.34 & 0.4 & 0.33 & 0.23 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.02 & 0.31 & -0.04 & 0.08 & 0.39 & 0.35 & -0.03 & 0.21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.71 & 0.9 & 0.27 & 0.15 & 0.16 & 0.58 & 0.35 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.79 & -0.06 & -0.01 & 0.36 & 0.64 & 0.45 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.09 & -0.22 & 0.24 & 0.69 & 0.34 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.03 & 0.03 & -0.31 & -0.06 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.29 & -0.13 & 0.2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.14 & 0.16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.27 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

144

$$D_{\Omega\lambda} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.38 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.52 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.29 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.36 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.38 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.54 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.13 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.63 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.38 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.07 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.24 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.39 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.69
\end{pmatrix}$$

145

146

$$D_{\lambda\Omega} = \left(\begin{array}{ccccccccccccccccccccccc}
1.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6.54 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4.41 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.58 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.97 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.46 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.76 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.93 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.02 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.49 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.83 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.36 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.71 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.14 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.07 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

We compute $\|v(R_{1U})\|^2 = 48.341$, $\|v(R_{2U})\|^2 = 34.472$ and the variance of Z^*,

$$= 2\Big[2Tr\left(\lambda^t \Omega\lambda\Omega\right) + 2Tr\left((\lambda\Omega)^2\right) - 4Tr\left(\lambda^t \Omega D_{\lambda\Omega}\Omega\right) - 4Tr\left(\lambda\Omega D_{\Omega\lambda}\Omega\right) + 2Tr\left(D_{\Omega\lambda}\Omega D_{\lambda\Omega}\Omega\right)$$
$$+ Tr\left((D_{\Omega\lambda}\Omega)^2\right) + Tr\left((D_{\lambda\Omega}\Omega)^2\right)\Big]$$

Therefore, $n = n_1 + n_2 = 19 + 19 = 38$. Then Z^* statistic is computed.

$$Z^* = \frac{\|v(R_{1U})\|^2 - \|v(R_{2U})\|^2}{\sqrt{\frac{1}{n-1}\sigma^2}} = \frac{48.341 - 34.472}{\sqrt{\frac{1}{37}1171.964}} = 1.765$$

The statistical test is compared to the critical value for making a decision. Therefore, the critical value of Z^* is $Z^*_{0.025} = \pm 1.96$. Based on $Z^* = 1.765$ and p-value$= 0.084$, we failed to reject the null hypothesis.

Therefore, by using T^*, we conclude that the two correlation samples are not equal. While, by using Z^* the two samples are equal.

4.7.2 Testing Several Correlation Matrices Using Control Chart

In this example, we want to use the two test for several independent samples. By utilizing this approach the stability is same to testing the hypothesis of the similarity of the two correlation matrices is done repeatedly

$$H_0 : \Omega_i = \Omega_0$$

$$H_1 : \Omega_i \neq \Omega_0$$

where i=1, 2,…, m, and Ω_0 is the reference sample. In this example, we have 24 samples (months), we retrieved the daily data of 23 currencies from Asia Pacific currencies. Hence there are 24 corresponding correlation matrices, with different size. In general, there are 5 trading days per week, and about 19 to 22 days for a month excluding holiday and weekend. In Table 4.11, illustrate the sample size.

Table 4.11

The Sample Size of the Foreign Exchange Rate Data

No.	$\boldsymbol{n}$	**size**	**No.**	$\boldsymbol{n}$	**size**
Jan 10	$n1$	19	Jan 11	$n13$	19
Fub10	$n2$	19	Fub11	$n14$	19
Mar10	$n3$	22	Mar11	$n15$	22
Apr10	$n4$	20	Apr11	$n16$	19
May10	$n5$	19	May11	$n17$	20
Jun10	$n6$	21	Jun11	$n18$	21
Jul10	$n7$	20	Jul11	$n19$	19
Aug10	$n8$	20	Aug11	$n20$	21
Sep10	$n9$	20	Sep11	$n21$	20
Oct10	$n10$	19	Oct11	$n22$	19
No10	$n11$	20	No10	$n23$	20
Dec10	$n12$	20	Dec10	$n24$	19

4.7.2.1 T^* Control Chart

To construct the corresponding control chart, we define R_{mU} and Ω_{0U}. Then, we compute the $V_{0,1}, V_{0,2}, V_{0,3}$ and Ψ_0^*. Next, the T^* statistic is computed to every 24 months of samples and the results are shown in Table 4.12.

Table 4.12 represent the values of T_m^* statistic where $m = 1, 2, ..., 24$, the first column represent 24 samples from January 2010 until December 2011, the second column represent the values of T_m^* statistic for 24 samples while, the third column present the tabulated value for χ^2, the fourth column represent the *p*-value and the last column is the results.

In Figure 4.2, we present the Chi-square control chart. The significance level $\alpha = 0.05$, the upper limits of control chart (*UCL*), $UCL = \chi_k^2 = 291.101$ with degree of freedom, $k = \frac{p(p-1)}{2} = \frac{23(23-1)}{2} = 253.$ All months in Table 4.12 are plotted in blue colour and *UCL* in orange colour. The vertical axis is the T_m^* statistic, where $m = 1, 2, ..., 24$ and the horizontal axis is 24 months.

Table 4.12

The values of T^* statistic

m	T^* *10^2	$\chi^2_{0.05,253}$*10^2	*p*-Value	Result
1	70.64994	2.91101	0.0000	Reject H_0
2	58.90507	2.91101	0.0000	Reject H_0
3	47.87139	2.91101	0.0000	Reject H_0
4	58.37806	2.91101	0.0000	Reject H_0
5	39.70816	2.91101	0.0000	Reject H_0
6	42.67845	2.91101	0.0000	Reject H_0
7	58.74452	2.91101	0.0000	Reject H_0
8	37.54839	2.91101	0.0000	Reject H_0
9	51.51666	2.91101	0.0000	Reject H_0
10	91.46684	2.91101	0.0000	Reject H_0
11	56.06998	2.91101	0.0000	Reject H_0
12	109.19585	2.91101	0.0000	Reject H_0
13	85.56412	2.91101	0.0000	Reject H_0
14	111.70083	2.91101	0.0000	Reject H_0
15	72.65498	2.91101	0.0000	Reject H_0
16	142.51274	2.91101	0.0000	Reject H_0
17	44.03481	2.91101	0.0000	Reject H_0
18	51.8249	2.91101	0.0000	Reject H_0
19	51.14576	2.91101	0.0000	Reject H_0
20	67.99281	2.91101	0.0000	Reject H_0
21	57.40211	2.91101	0.0000	Reject H_0
22	52.04419	2.91101	0.0000	Reject H_0
23	73.09234	2.91101	0.0000	Reject H_0
24	103.58171	2.91101	0.0000	Reject H_0

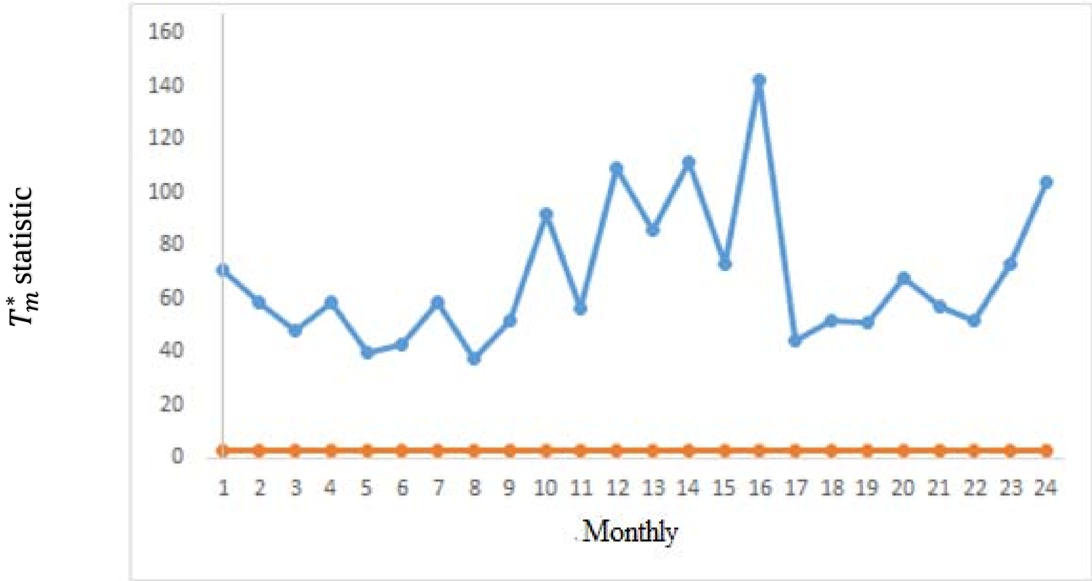

Figure 4.2. Foreign Exchange Chart for T^*

Therefore, according to Table 4.12 and the Figure 4.2. For the T^* statistic we reject all the hypotheses we conclude by using T^* all the sample are out-of-control. We learn that at all months give a signal. This signal shows that there is a change in correlation structure at that particular month, and the *p*-value to all samples are equal 0 is less than α for that reason we reject the null hypothesis for all samples. *p*-value is defined as the smallest significance level that the results in rejection of the null hypothesis this definition is the more useful definition since it can be applied to any collection test (Wright, 1992). A *p*-value is used to provides information about whether a statistical test is significant or not and also, it is indicates something about how significant the result is i.e. the smaller the *p*-value, it is the stronger the evidence against the null hypothesis.

4.7.2.2 Z^* Control Chart

To construct the corresponding control chart, for Z^* statistic we define R_{mU} and Ω_{0U}. Then, we compute Ω pooled correlation matrix, $\lambda = \Omega_U$, $D_{\Omega\lambda}$, $D_{\lambda\Omega}$, $\|v(R_{iU})\|^2$, $\|v(\Omega_{iU})\|^2$ and σ^2. To every 24 months of samples. Next, the Z_m^* statistic is computed to every 24 months of samples and the results are shown in Table 4.12.

Table 4.13 represent the values of Z_m^* statistic the first column represent 24 samples from January 2010 until December 2011, the second column represent the values of Z^*statistic for 24 samples while, the third column represent the upper control limit (*UCL*)= $Z_{0.05/2} = Z_{0.025} = 1.96$, the fourth column represent the *p*-value and the last column is the results.

In Figure 4.3, we present the Z^* control chart. The significance level $\alpha = 0.05$. All months in Table 4.13 are plotted in blue colour, *UCL* in orange colour and the *LCL* is gray colour. The vertical axis is the Z_m^* statistic, where $m = 1, 2, ..., 24$ and the horizontal axis is 24 months.

Table 4.13

The values of Z_m^* statistic

m	Z_m^*	$\boldsymbol{Z_{0.025}}$	***p-value***	***Result***
1	7.109	1.96	4.22×10^{-12}	Reject H_0
2	8.772	1.96	7.78×10^{-18}	Reject H_0
3	12.251	1.96	1.02×10^{-33}	Reject H_0
4	0.689	1.96	0.315	Fail to Reject H_0
5	18.489	1.96	2.34×10^{-75}	Reject H_0
6	7.895	1.96	1.16×10^{-14}	Reject H_0
7	16.037	1.96	5.65×10^{-57}	Reject H_0
8	18.489	1.96	2.35×10^{-75}	Reject H_0
9	20.444	1.96	6.92×10^{-92}	Reject H_0
10	7.766	1.96	3.21×10^{-14}	Reject H_0
11	5.158	1.96	6.67×10^{-7}	Reject H_0
12	9.506	1.96	9.54×10^{-21}	Reject H_0
13	15.551	1.96	1.23×10^{-53}	Reject H_0
14	6.194	1.96	1.86×10^{-9}	Reject H_0
15	7.448	1.96	3.59×10^{-13}	Reject H_0
16	−0.038	1.96	0.399	Fail to Reject H_0
17	8.291	1.96	4.74×10^{-16}	Reject H_0
18	23.176	1.96	9.30×10^{-118}	Reject H_0
19	17.309	1.96	3.49×10^{-66}	Reject H_0
20	10.495	1.96	4.85×10^{-25}	Reject H_0
21	16.779	1.96	2.93×10^{-62}	Reject H_0
22	11.728	1.96	5.40×10^{-31}	Reject H_0
23	−0.050	1.96	0.398	Fail to Reject H_0
24	−1.998	1.96	0.0452	Reject H_0

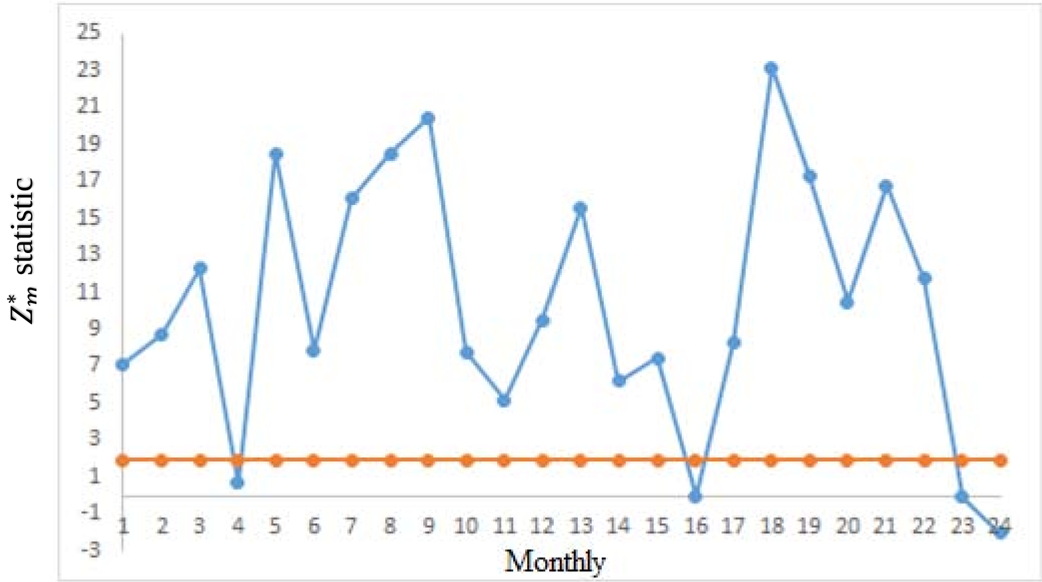

Figure 4.3. Foreign Exchange Chart for Z^*

Therefore, according to Table 4.13 and the Figure 4.3, we learn that all months give a signal except the 4th (April 2010), 16th (April 2011), 23rd (November 2011) sample. This three months failed to reject the null hypothesis. This signal shows there is change in correlation structure at that particular month. We conclude from the result these three samples are stable and the others are not stable.

Based on this example, testing the hypothesis repeatedly that is more advantageous since we can investigate all the corresponding correlation matrices independently. Thus, we can know which correlation matrix is challenging. In this situation, in order to determine which result is more trustworthy is by looking to the power of test of the two statistical test. These two examples is given to show the new alternative statistic can accommodate condition of high dimension data.

According to Schermelleh-Engel, Moosbrugger, and Müller (2003) Chi square is sensitive to the sample size, this may make a weak relationship statistically significant if the sample size is large enough.

CHAPTER FIVE
CONCLUSION AND FUTURE RESEARCH

5.1 Conclusion

The main goal of this research was to derive an alternative statistical test for solving the problem faced by Jennrich statistic, which is the singularity problem. In achieving this goal, an alternative statistical test, namely, the Z^* statistic, is derived based on *vec* operator, commutation matrix, and Frobenius norm of upper-off-diagonal elements. Later, the performance of the alternative statistical test, the Z^* statistic, is evaluated in terms of the power of test. By using a simulation study, the alternative statistical test, Z^* statistic is compared with the Jennrich statistic, and the T^* statistic. The result of simulation study shows that the Z^* statistic possesses good properties and was more powerful than the Jennrich statistic and the T^* statistic. Additionally, this new statistic can be used in both condition which are when the number of variables is larger than the sample size, and when the number of variables is smaller than the sample size. In general, the results showed that the alternative statistical test, the Z^*. Moreover, the following asymptotic distributions have been proven in this thesis,

i. The asymptotic distribution of sample correlation matrix for $p = 2$ has been proven as the following $\sqrt{n}(r - \rho) \xrightarrow{d} N\left(0, \left(1 - \rho^2\right)^2\right)$.

ii. The asymptotic distribution of sample correlation matrix are derived by using Theorem 3.1 $\sqrt{n-1}[vec(R)-vec(\Omega)] \xrightarrow{d} N_{p^2}(0,\Gamma)$, where $\Gamma = 2M_p\phi M_p$ while, $M_p = \frac{1}{2}\left(I_{p^2} + K_{pp}\right)$ and $\phi = \left[\left\{I_{p^2} - \left(I_p \otimes \Omega\right)\Lambda_p\right\}\left(\Omega \otimes \Omega\right)\left\{I_{p^2} - \Lambda_p\left(I_p \otimes \Omega\right)\right\}\right]$; K_{pp} is the commutation matrix of size $\left(p^2 \times p^2\right)$ and $\Lambda_p = \sum_{i=1}^{p} h_i h_i^t \otimes h_i h_i^t$ where, h_i is the i-th column of I_p.

iii. The asymptotic distribution of $\left\|v\left(R_U\right)\right\|^2$ is normally distributed with the mean $\left\|v\left(\Omega_U\right)\right\|^2$, and the variance $\sigma^2 = 4\left(v\left(\Omega_U\right)\right)^t T\ \Gamma\ T^t\ v\left(\Omega_U\right) = 8\left(v\left(\Omega_U\right)\right)^t T\ M_p\ \phi\ M_p T^t\ v\left(\Omega_U\right)$

iv. The asymptotic distribution of the new alternative statistical test, the Z^* statistic, in the following proposition $\sqrt{n-1}\left(\left\|v\left(R_U\right)\right\|^2 - \left\|v\left(\Omega_U\right)\right\|^2\right)$ is normal distribution with mean $\left\|v\left(\Omega_U\right)\right\|^2$ and variance

$$\sigma^2 = 2\Big[2Tr\left(\lambda^t\Omega\lambda\Omega\right) + 2Tr\left(\left(\lambda\Omega\right)^2\right) - 4Tr\left(\lambda^t\Omega D_{\lambda\Omega}\Omega\right) - 4Tr\left(\lambda\Omega D_{\Omega\lambda}\Omega\right) +$$
$$2Tr\left(D_{\Omega\lambda}\Omega D_{\lambda\Omega}\Omega\right) + Tr\left(\left(D_{\Omega\lambda}\Omega\right)^2\right) + Tr\left(\left(D_{\lambda\Omega}\Omega\right)^2\right)\Big]$$

The new alternative statistic is proposed in the following propositions and corollary.

i. The new alternative statistic Z^* is proposed in the following proposition: If $p > 2$, under H_0, then $\sqrt{n-1}\left[\left\|v\left(R_{i,U}\right)\right\|^2 - \left\|v\left(\Omega_{0,U}\right)\right\|^2\right] \xrightarrow{d} N\left(0,\sigma^2\right)$.

ii. If Ω_0 is unknown, under H_0, then we have the following corollary

$$\sqrt{n-1}\left[\left\|v\left(R_{i,U}\right)\right\|^2 - \left\|v\left(\hat{\Omega}_{0,U}\right)\right\|^2\right] \xrightarrow{d} N\left(0,\sigma^2\right)$$

where

$R_{i,U}$ and $\hat{\Omega}_{0,U}$ are the upper-off-diagonal elements of R_i and $\hat{\Omega}_0$, respectively.

$$\sigma^2 = 2\Big[2Tr\left(\lambda^t\hat{\Omega}_0\lambda\hat{\Omega}_0\right) + 2Tr\left(\left(\lambda\hat{\Omega}_0\right)^2\right) - 4Tr\left(\lambda^t\hat{\Omega}_0 D_{\lambda\hat{\Omega}_0}\hat{\Omega}_0\right) -$$
$$4Tr\left(\lambda\hat{\Omega}_0 D_{\hat{\Omega}_0\lambda}\hat{\Omega}_0\right) + 2Tr\left(D_{\hat{\Omega}_0\lambda}\hat{\Omega}_0 D_{\lambda\hat{\Omega}_0}\hat{\Omega}_0\right) + Tr\left(\left(D_{\hat{\Omega}_0\lambda}\hat{\Omega}_0\right)^2\right) + Tr\left(\left(D_{\lambda\hat{\Omega}_0}\hat{\Omega}_0\right)^2\right)\Big]$$

$\hat{\Omega}_0 = \overline{R}$ is the average of correlation matrices of $R_1, R_2, \ldots, R_m$.

λ a matrix of size $(p \times p)$ such that $v(\lambda) = T^t \times T \times vec\left(\hat{\Omega}_0\right)$.

We define $D_{\hat{\Omega}_0\lambda}$, $D_{\lambda\hat{\Omega}_0}$ as a matrix the diagonal elements are the diagonal elements of $\hat{\Omega}_0\lambda$ and $\lambda\hat{\Omega}_0$ respectively.

iii. For testing the hypothesis repeatedly $H_0 : \Omega_i = \Omega_0$ versus $H_1 : \Omega_i \neq \Omega_0$ i=1, 2, *…, m.* The proposed test is

$$Z_i^* = \frac{\left\| v\left(R_{iU}\right)\right\|^2 - \mu_0}{\sqrt{\frac{1}{n-1}\sigma^2}}$$

where

$$\mu_0 = \left\| v(\Omega_U)\right\|^2$$

$$\sigma^2 = 2\Big[2Tr\left(\lambda^t \Omega\lambda\Omega\right) + 2Tr\left((\lambda\Omega)^2\right) - 4Tr\Omega\left(\lambda^t \Omega D_{\lambda\Omega}\Omega\right) -$$
$$4Tr\left(\lambda\Omega D_{\Omega\lambda}\Omega\right) + 2Tr\left(D_{\Omega\lambda}\hat{\Omega}_0 D_{\lambda\Omega}\Omega\right) + Tr\left((D_{\Omega\lambda}\Omega)^2\right) + Tr\left((D_{\lambda\Omega}\Omega)^2\right)\Big]$$

H_0 is rejected if $\left|Z_i^*\right| > z_{\alpha/2}$

After the derivation, by using simulation study via Matlab (2016a), the results showed that

i. The power of the test for the small number of variables ($p = 3, 4$ and 5) showed that the Z^* statistic dominated the T^* statistic and the Jennrich statistic. In detail, the total number of conditions to evaluate the power of test for $p = 3, 4$ and 5 were 189, 180 and 180 conditions, respectively. For $p = 3$, there were 47, 35 and 33 conditions for the Z^* statistic, the T^* statistic, and the Jennrich statistic respectively that fell within the interval of power. While, for $p = 4$, there were 51, 37, and 31 conditions for the Z^* statistic, the T^* statistic, and the Jennrich statistic respectively that fell within the interval of power. For $p = 5$, there were 51, 38 and 33 conditions for the Z^* statistic, the T^* statistic, and the Jennrich statistic respectively that fell within the interval of power.

ii. The power of test for the medium number of variables, (p = 10 and p =15) showed that the Z^* dominated T^* statistic, and Jennrich statistic. In detail, the total number of conditions to evaluate the power of test were 171 and 162 conditions respectively. For $p = 10$ there were 53, 35 and 33 conditions for the Z^* statistic, the T^* statistic and the Jennrich statistic respectively that fell within the interval of power. While, for $p = 15$, there were 53, 29 and 27 conditions for the Z^* statistic, the T^* statistic and the Jennrich statistic respectively that fell within interval of the power.

iii. The power for a large number of variables ($p = 20$ and 30) shows that the Z^* statistic still dominated the T^* statistic and the Jennrich statistic. In detail, the total number of conditions to evaluate the power of test for $p = 20$ and 30 were 162 and 153 conditions respectively. For $p = 20$ there were 52, 29 and 23 conditions for the Z^* statistic, the Jennrich statistic and the T^* statistic respectively that fell within the interval of power. While, for $p = 30$, there were 53, 22 and 18 conditions that fell within the interval of power of test for the Z^* statistic, the Jennrich statistic and the T^* statistic respectively. For a large number of variables, the Z^* statistic still dominated the other two tests.

In general, the power of test of the alternative statistical test, the Z^* statistic, was better than the Jennrich statistic and the T^* statistic in all conditions $p < n$ and $p > n$.

Furthermore, to validate the new alternative statistical test, the currencies from Asia Pacific countries were employed to analyze the difference in the correlation matrices before and after Tohoku earthquake incident. This statistical test can be applied for testing hypotheses in a high dimension data using different approaches. The first approach with two independent samples of correlation matrices presented the data by using the T^* statistic and the new alternative the statistic Z^*. The result showed that by using T^* statistic the null hypothesis was rejected, and the two correlation samples were not equal. By using Z^* statistic, the null hypothesis was not rejected, meaning that the two correlation samples were equal. The second approach had several independent samples of correlation matrices by using a control chart. This approach tested the hypothesis repeatedly, which offers more advantages by testing all the corresponding correlation matrices independently. The results by using the T^* statistic showed that all the samples were out-of-control. Using the new alternative statistic Z^* we fail to reject the null hypothesis for three months. The result of validation showed that a difference exists between the results by using two statistical tests. To decide which result is more trustworthy the power of the test must be examined. The empirical study demonstrated show that the new alternative statistic can accommodate the condition of a high dimension data.

In conclusion, the new alternative statistical test Z^* statistic holds an advantage in that the test can handle cases in which the number of variables is larger than the sample size. For that reason, the alternative statistical test Z^* statistic is deemed to be more suitable in various real applications.

5.2 Future Research

This work has not yet answered all problems related to good statistical test. The direction for future research are summarized as follows:

i. To investigate the asymptotic distribution of the proposed test when the data are nonnormal; and

ii. To investigate the improvement of covariance estimators such as banding, tapering and thresholding.

REFERENCES

Aitkin, M., Nelson, W., & Reinfurt, K. H. (1968). Tests for correlation matrices. *Biometrika*, 327-334.

Aktas, B. K. A. (2013). STATISTICAL POWER ANALYSIS. *The 7th International Days of Statistics and Economics* (pp. 578-587). Prague.

Alfaro, J., & Ortega, J. F. (2009). A comparison of robust alternatives to Hotelling's T 2 control chart. *Journal of Applied Statistics, 36*(12), 1385-1396.

Anderson. (2006). Distance-based tests for homogeneity of multivariate dispersions. *Biometrics, 62*(1), 245-253. doi: 10.1111/j.1541-0420.2005.00440.x

Anderson, T. W. (2003). *Introduction to Multivariate Statistical Analysis*. New York: John Wiley & Sons, Inc.

Annaert, J., Claes, A. G., & De Ceuster, M. J. (2006). Inter-temporal Stability of the European Credit Spread Co-movement Structure. *The European Journal of Finance, 12*(1), 23-32.

Annaert, J., De Ceuster, M. J., & Claes, A. G. (2003). Inter-temporal Stability of the European Credit Spread Co-movement Structure. University of Antwerp: Faculty of Applied Economics.

Aslam, S., & Rocke, D. M. (2005). A robust testing procedure for the equality of covariance matrices. *Computational statistics & data analysis, 49*(3), 863-874.

Atiany, T. A. M., & Sharif, S. (2015). Asia-Pacific Currencies Structure Aftermath Tohoku Earthquake. *Research Journal of Applied Sciences, Engineering and Technology, 11*(2), 215-220.

Atiany, T. A. M., & Sharif, S. (2016). New Statistical Test for Testing Several Correlation Matrices. *Global Journal of Pure and Applied Mathematics, 12*(5), 4285-4298.

Bai, J., & Shi, S. (2011). Estimating high dimensional covariance matrices and its applications.

Barnett, I., & Onnela, J.-P. (2016). Change point detection in correlation networks. *Scientific reports, 6*.

Baroudi, J. J., & Orlikowski, W. J. (1989). The problem of statistical power in MIS research. *MIS Quarterly*, 87-106.

Bartlett, M., & Rajalakshman, D. (1953). Goodness of fit tests for simultaneous autoregressive series. *Journal of the Royal Statistical Society. Series B (Methodological)*, 107-124.

Bickel, P. J., & Levina, E. (2004). Some theory for Fisher's linear discriminant function,'naive Bayes', and some alternatives when there are many more variables than observations. *Bernoulli*, 989-1010.

Bickel, P. J., & Levina, E. (2008a). Covariance regularization by thresholding. *The Annals of Statistics*, 2577-2604.

Bickel, P. J., & Levina, E. (2008b). Regularized estimation of large covariance matrices. *The Annals of Statistics*, 199-227.

Botman, D. P., de Carvalho Filho, I., & Lam, W. R. (2013). The curious case of the yen as a safe haven currency: a forensic analysis. Working Paper 13/228.

Box, G. E. (1949). A general distribution theory for a class of likelihood criteria. *Biometrika*, *36*(3/4), 317-346.

Browne, M., & Shapiro, A. (1986). The asymptotic covariance matrix of sample correlation coefficients under general conditions. *Linear Algebra and its Applications*, *82*, 169-176.

Cai, T. T., Zhang, C.-H., & Zhou, H. H. (2010). Optimal rates of convergence for covariance matrix estimation. *The Annals of Statistics, 38*(4), 2118-2144.

Cai, T. T., & Zhou, H. H. (2012). Minimax estimation of large covariance matrices under ℓ 1-norm. *Statistica Sinica*, *12*(4), 1319-1349.

Chen, X., Wang, Z. J., & McKeown, M. J. (2012). Shrinkage-to-tapering estimation of large covariance matrices. *Signal Processing, IEEE Transactions on, 60*(11), 5640-5656.

Chesnay, F., & Jondeau, E. (2001). Does correlation between stock returns really increase during turbulent periods? *Economic Notes*, *30*(1), 53-80.

Cho, D. C., & Taylor, W. M. (1987). The seasonal stability of the factor structure of stock returns. *The Journal of Finance, 42*(5), 1195-1211.

Choi, Y.-G., Lim, J., Roy, A., & Park, J. (2016). Positive-definite modification of covariance matrix estimators via linear shrinkage. *arXiv preprint arXiv:1606.03814*.

Chou, Y.-M., Mason, R. L., & Young, J. C. (2001). The control chart for individual observations from a multivariate non-normal distribution. *Communications in statistics-Theory and methods, 30*(8-9), 1937-1949.

Clark-Carter, D. (1997). The account taken of statistical power in research published in the British Journal of Psychology. *British Journal of Psychology, 88*(1), 71-83.

Cohen, J. (1977). Statistical power analysis for the behavioral sciences (revised ed.): New York: Academic Press.

Cohen, J. (1990). Things I have learned (so far). *American psychologist, 45*(12), 1304.

Cooper, W. H., Donnelly, J. M., & Johnson, R. (2011). Japan's 2011 earthquake and tsunami: economic effects and implications for the United States. *Congressional research service*.

Cui, X. (2010). *Computing the Nearest Correlation Matrix using Difference Map Algorithm.* University of Waterloo, Canada.

Da Costa Jr, N., Nunes, S., Ceretta, P., & Da Silva, S. (2005). Stockmarket comovements revisited. *Economics Bulletin, 7*(3).

Da Rocha, G. N. V. (2008). *Sparsity and Model Selection Through Convex Penalties: Structured Selection, Covariance Selection and Some Theory*. University of California, Berkeley.

Deblauwe, F., & Le, H. (2000). *Stability of correlation between credit and market risk over different holding periods.* Ph. D. Thesis, University of Antwerp Management School, Antwerp, Belgium.

Djauhari, M. A. (2007). A measure of multivariate data concentration. *Journal of Applied Probability and Statistics, 2*(2), 139-155.

Djauhari, M. A., & Gan, S. L. (2014). Dynamics of correlation structure in stock market. *Entropy*, *16*(1), 455-470.

Djauhari, M. A., & Herdiani, E. (2008). Monitoring the Stability of Correlation Structure in Drive Rib Production Process: An MSPC Approach. *Open Industrial & Manufacturing Engineering Journal, 1*, 8-18.

Eichholtz, P. M. (1996). Does international diversification work better for real estate than for stocks and bonds? *Financial analysts journal*, *52*(1), 56-62.

El Karoui, N. (2007). On spectral properties of large dimensional correlation matrices and covariance matrices computed from elliptically distributed data. University of California, Berkeley: Technical report from Department of Statistics.

Fama, E. F. (1965). The behavior of stock-market prices. *The journal of Business, 38*(1), 34-105.

Furrer, R., & Bengtsson, T. (2007). Estimation of high-dimensional prior and posterior covariance matrices in Kalman filter variants. *Journal of Multivariate Analysis*, *98*(2), 227-255.

Gali, S. (2015, 3-5 April). *On Importance of Normality Assumption in Using a T-Test: OneSample and Two Sample Cases.* Paper presented at the Proceedings of the International Symposium on Emerging Trends in Social Science Research Chennai-India.

Gan, S. L., Djauhari, M. A., & Ismail, Z. (2014). *Monitoring correlation structures stability in foreign exchange market.* Paper presented at the 2014 IEEE International Conference on Industrial Engineering and Engineering Management.

Gande, A., & Parsley, D. C. (2005). News spillovers in the sovereign debt market. *Journal of Financial Economics, 75*(3), 691-734.

Gilbert, N., & Troitzsch, K. (2005). *Simulation for the social scientist*: McGraw-Hill Education (UK).

Goetzmann, W. N., Li, L., & Rouwenhorst, K. G. (2005). Long-term global market correlations. *Journal of Business*, *78*(1).

Górski, A., Drozdz, S., & Kwapien, J. (2008). Minimal spanning tree graphs and power like scaling in FOREX networks. *Acta Physica Polonica A, 114*(3), 531-538.

Haddad, F. S. F. (2013). *Statistical Process Control Using Modified Robust Hotelling's T² Control Charts.* Universiti Utara Malaysia.

Herdiani, E. T. (2008). *Statistik Penguji Kestabilan Barisan Matriks Korelasi*. Institut Teknologi Bandung.

Herdiani, E. T., & Djauhari, M. A. (2012). Distribution Sampling of Vector Variance without Duplications. *World Academy of Science, Engineering and Technology, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 6*(12), 1673-1676.

Hood, M., Kamesaka, A., Nofsinger, J., & Tamura, T. (2013). Investor response to a natural disaster: Evidence from Japan's 2011 earthquake. *Pacific-Basin Finance Journal, 25*, 240-252.

Hotelling, H. (1940). The selection of variates for use in prediction with some comments on the general problem of nuisance parameters. *The Annals of Mathematical Statistics, 11*(3), 271-283.

Imamura, F., & Anawat, S. (2011). *Damage due to the 2011 Tohoku earthquake tsunami and its lessons for future mitigation.* Paper presented at the Proceedings of the international symposium on engineering lessons learned from the 2011 Great East Japan Earthquake, March 1-4, 2012,, Tokyo, Japan.

Jang, W., Lee, J., & Chang, W. (2011). Currency crises and the evolution of foreign exchange market: Evidence from minimum spanning tree. *Physica A: Statistical Mechanics and its Applications, 390*(4), 707-718.

Jennrich, R. I. (1970). An asymptotic χ2 test for the equality of two correlation matrices. *Journal of the American Statistical Association, 65*(330), 904-912.

Kaplanis, E. C. (1988). Stability and forecasting of the comovement measures of international stock market returns. *Journal of international Money and Finance, 7*(1), 63-75.

Karoui, N. E. (2008). Operator norm consistent estimation of large-dimensional sparse covariance matrices. *The Annals of Statistics*, 2717-2756.

Khare, K., Oh, S. Y., & Rajaratnam, B. (2015). A convex pseudolikelihood framework for high dimensional partial correlation estimation with convergence guarantees. *Journal of the Royal Statistical Society: Series B (Statistical Methodology), 77*(4), 803-825.

Kollo, T., & Ruul, K. (2003). Approximations to the distribution of the sample correlation matrix. *Journal of multivariate analysis, 85*(2), 318-334.

Kollo, T., & Von Rosen, D. (2006). *Advanced multivariate statistics with matrices* (Vol. 579): Springer Science & Business Media.

Kullback, S. (1967). On testing correlation matrices. *Applied Statistics*, 80-85.

Larntz, K., & Perlman, M. D. (1985). A simple test for the equality of correlation matrices. *Rapport technique, Department of Statistics, University of Washington*, 141.

Lawley, D. (1963). On testing a set of correlation coefficients for equality. *The Annals of Mathematical Statistics, 34*(1), 149-151.

Lee, S. (1998). *The inter-temporal stability of real estate returns: an empirical investigation*. Paper presented at the International Real Estate Conference Maastricht The Netherlands.

Magnus, J. R., & Neudecker, H. (1980). The elimination matrix: some lemmas and applications. *SIAM Journal on Algebraic Discrete Methods, 1*(4), 422-449.

Makridakis, S. G., & Wheelwright, S. C. (1974). An analysis of the interrelationships among the major world stock exchanges. *Journal of Business Finance & Accounting, 1*(2), 195-215.

Maldonado, R., & Saunders, A. (1981). International portfolio diversification and the inter-temporal stability of international stock market relationships, 1957-78. *Financial Management*, 54-63.

Mantegna, R. N., & Stanley, H. E. (2000). An introduction to econophysics: correlation and complexity in finance. *Cambridge, UK: Cambridge University*.

Mason, R. L., Chou, Y.-M., & Young, J. C. (2009). Monitoring variation in a multivariate process when the dimension is large relative to the sample size. *Communications in Statistics—Theory and Methods, 38*(6), 939-951.

McCrum-Gardner, E. (2010). Sample size and power calculations made simple. *International Journal of Therapy and Rehabilitation, 17*(1), 10.

Meric, I., & Meric, G. (1997). Co-movements of European equity markets before and after the 1987 crash. *Multinational Finance Journal, 1*(2), 137-152.

Mizuno, T., Takayasu, H., & Takayasu, M. (2006). Correlation networks among currencies. *Physica A: Statistical Mechanics and its Applications, 364*, 336-342.

Montgomery, D. C. (2005). *Introduction to statistical quality control. Fifth Edition*: John Wiley & Sons.

Mori, N., Takahashi, T., Yasuda, T., & Yanagisawa, H. (2011). Survey of 2011 Tohoku earthquake tsunami inundation and run-up. *Geophysical research letters, 38*(7).

Muirhead, R. J. (1982). Aspects of multivariate statistical analysis. *JOHN WILEY & SONS, INC., 605 THIRD AVE., NEW YORK, NY 10158, USA, 1982, 656*.

Murphy, K. R., Myors, B., & Wolach, A. (2014). *Statistical power analysis: A simple and general model for traditional and modern hypothesis tests*: Routledge.

Neudecker, H., & Wesselman, A. M. (1990). The asymptotic variance matrix of the sample correlation matrix. *Linear Algebra and its Applications, 127*, 589-599.

Nguyen, H. M. (2012). Natural disasters and the economies of Asian countries: The case of earthquakes and tsunamis in Japan, storms and floods in Vietnam. Vietnam National University, Hanoi: University of Economics and Business.

Nickisch, S., Nockemann, C., Tillack, G.-R., Murphy, J., & Sturges, D. (1997). Alternative approaches to the validation of nondestructive testing methods *Review of Progress in Quantitative Nondestructive Evaluation* (pp. 2037-2044): Springer.

Olkin, I., Lou, Y., Stokes, L., & Cao, J. (2015). Analyses of wine-tasting data: A tutorial. *Journal of Wine Economics, 10*(1), 4-30.

Philippatos, G. C., Christofi, A., & Christofi, P. (1983). The inter-temporal stability of international stock market relationships: Another view. *Financial Management*, 63-69.

Pourahmadi, M. (2013). *High-dimensional covariance estimation: with high-dimensional data*: John Wiley & Sons.

Ragea, V. (2003). Testing correlation stability during hectic financial markets. *Financial Markets and Portfolio Management, 17*(3), 289-308.

Rahman, M., Pearson, L. M., & Heien, H. C. (2006). A modified anderson-darling test for uniformity. *Bulletin of the Malaysian Mathematical Sciences Society, 29*(1).

Rencher, A. C. (2003). *Methods of multivariate analysis* (Vol. 492): John Wiley & Sons.

Rothman, A. J., Levina, E., & Zhu, J. (2009). Generalized thresholding of large covariance matrices. *Journal of the American Statistical Association, 104*(485), 177-186.

Sabharwal, A., & Potter, L. (2002). Wald statistic for model order selection in superposition models. *IEEE transactions on signal processing, 50*(4), 956-965.

Sang, Y., Dang, X., & Sang, H. (2016). Symmetric Gini Covariance and Correlation. *arXiv preprint arXiv:1605.02332*.

Satake, K. (2013). Tohoku, Japan (2011 Earthquake and Tsunami) *Encyclopedia of Natural Hazards* (pp. 1015-1018): Springer.

Sawyer, A. G. (1982). Statistical power and effect size in consumer research. *NA-Advances in Consumer Research Volume 09*.

Schermelleh-Engel, K., Moosbrugger, H., & Müller, H. (2003). Evaluating the fit of structural equation models: Tests of significance and descriptive goodness-of-fit measures. *Methods of psychological research online, 8*(2), 23-74.

Schott, J. R. (1996). Testing for the equality of several correlation matrices. *Statistics & probability letters, 27*(1), 85-89.

Schott, J. R. (1997). *Matrix Analysis for Statistics*. New York: John Wiley & Sons.

Schott, J. R. (2001). Some tests for the equality of covariance matrices. *Journal of Statistical Planning and Inference, 94*(1), 25-36.

Schott, J. R. (2003). Kronecker product permutation matrices and their application to moment matrices of the normal distribution. *Journal of multivariate analysis, 87*(1), 177-190.

Schott, J. R. (2005). Testing for complete independence in high dimensions. *Biometrika, 92*(4), 951-956.

Schott, J. R. (2007a). A test for the equality of covariance matrices when the dimension is large relative to the sample sizes. *Computational statistics & data analysis, 51*(12), 6535-6542.

Schott, J. R. (2007b). Testing the equality of correlation matrices when sample correlation matrices are dependent. *Journal of Statistical Planning and Inference, 137*(6), 1992-1997.

Schott, J. R. (2016). *Matrix analysis for statistics*: John Wiley & Sons.

Sewell, M. (2011). Characterization of financial time series. *RN, 11*(01), 01.

Sharif, S. (2013). *A new statistic to the theory of correlation stability testing in financial market.* Universiti Teknologi Malaysia, Faculty of Science.

Sharif, S., & Djauhari, M. A. (2014). *Asymptotic derivation of T* statistic.* Paper presented at the International Conference on Quantitative Sciences and its Applications (ICOQSIA 2014): Proceedings of the 3rd International Conference on Quantitative Sciences and Its Applications.

Sharif, S., Ismail, S., Omar, Z., & Theng, L. H. (2016). Validation of Global Financial Crisis on Bursa Malaysia Stocks Market Companies via Covariance Structure. *American Journal of Applied Sciences, 13*(11), 1091-1095.

Sheppard, K. (2008). Forecasting Covariances using High-Frequency Data and Positive Semi-Definite Matrix Multiplicative Error Models

Song, S. (2011). Dynamic Large Spatial Covariance Matrix Estimation in Application to Semiparametric Model Construction via Variable Clustering: the SCE approach. *arXiv preprint arXiv:1106.3921*.

Steiger, J. H. (1980). Tests for comparing elements of a correlation matrix. *Psychological bulletin, 87*(2), 245-251.

Sul, J. H., Han, B., & Eskin, E. (2011). Increasing power of groupwise association test with likelihood ratio test. *Journal of Computational Biology, 18*(11), 1611-1624.

Sullivan, G. M., & Feinn, R. (2012). Using effect size-or why the P value is not enough. *Journal of graduate medical education, 4*(3), 279-282.

Tang, G. Y. (1995). Stability of international stock market relationships across month of the year and different holding intervals. *The European Journal of Finance, 1*(3), 207-218.

Tang, G. Y. (1998). The intertemporal stability of the covariance and correlation matrices of Hong Kong stock returns. *Applied financial economics, 8*(4), 359-365.

Tulic, M. (2010). *Estimation of large covariance matrices using results on large deviations.* na, Vienna University of Technology.

Watson, J. (1980). THE STATIONARITY OF INTER-COUNTRY CORRELATION COEFFICIENTS: A NOTE. *Journal of Business Finance & Accounting, 7*(2), 297-303.

Wright, S. P. (1992). Adjusted p-values for simultaneous inference. *Biometrics*, 1005-1013.

Wu, W. B., & Pourahmadi, M. (2003). Nonparametric estimation of large covariance matrices of longitudinal data. *Biometrika, 90*(4), 831-844.

Xue, L., Ma, S., & Zou, H. (2012). Positive-definite ℓ1-penalized estimation of large covariance matrices. *Journal of the American Statistical Association, 107*(500), 1480-1491.

Xue, L., & Zou, H. (2014). Rank-based tapering estimation of bandable correlation matrices. *Statistica Sinica, 24*, 83-100.

Yahaya, S. S. S. (2005). Robust statistical procedure for testing the equality of central tendency parameter under skewed distributions. *Unpublished Ph. D. thesis, Universiti Sains Malaysia*.

Appendix A

Matlab Programing Code of Performance Z* Statistic

```
clear all
clc
tic
format long
disp ('rho - value T_power ')
for j=0:8

n1=50; n2=n1; n=n1+n2;  %sample size
p=10;                    %variables

rho=j/10;                %Covariance shift
k=1.0;                  %Constant value
I=eye(p,p);
R2=eye(p);
T=Tp(p);
mu=repmat([0],1,p);
               %Number of contaminated
Sigma0=eye(p);
R1 = ones(p)*k*rho;
R1(logical(eye(size(R1)))) = 1;
Sigma=R1;
Re=10000;               %replication
alpha=0.05;              %Significance level
```

```
my_stat1=[ ];
for r1=1:Re

    Z2=mvnrnd(mu,Sigma0,n1);

%     OM=corr(Z2);

    RA2=corr(Z2);
    RA1=(n1*R1+n2*RA2)/n;
    rr1=T'*T*vec(RA1);
    OU=reshape(rr1,p,p);
    rr2=T'*T*vec(R1);
    R2U=reshape(rr2,p,p);
    l=T'*T*vec(RA1);
    L=reshape(l,p,p);
    d=diag(RA1*L);
    D=diag(d);
    dd=diag(L*RA1);
    DD=diag(dd);

sigma3=2*[2*trace(L'*RA1*L*RA1)+2*trace(L'*RA1*L'*RA1)
        -4*trace(L'*RA1*DD*RA1)-4*trace(L*RA1*D*RA1)

+2*trace(D'*RA1*DD*RA1)+trace(D'*RA1*D*RA1)+trace(DD*RA1
*DD*RA1)];
    b=sqrt((1/(n-1))*sigma3);
    Zi1=(sumsqr(R2U)-sumsqr(OU))/b;
    my_stat1=[my_stat1;Zi1];

end
```

```
y1=sort(my_stat1);%To arrange the values in ascending
order

%       CV1=y1(9500)

%To find Type I error rate and power
  C1=0;

 for i=1:Re
     if (y1(i)>   CV1)
         C1=C1+1;
    end

 end
  T_POWE1=C1/Re;
fprintf( '%2.7f % 7.6f\n',rho,T_POWE1 )
 end

 toc
```

Appendix B

Matlab Programing Code of Z* Statistic

```
 tic
close all
clear all
clc

load matlab
%    m=m1;%change m=m1 refer to what we need
k=2;
[r,s]=size(m1);
[t,u]=size(m0);
n1=r;
n2=t;
p=s;
 n=n1+n2;  %sample size
 I=eye(p,p);
 T=Tp(p);
 R1=corr(m1);
 R2=corr(m0);
 RA1=(n1*R1+n2*R2)/n;
 rr1=T'*T*vec(RA1);
 OU=reshape(rr1,p,p);
 rr2=T'*T*vec(R1);
 R2U=reshape(rr2,p,p);
 l=T'*T*vec(RA1);
 L=reshape(l,p,p);
 d=diag(RA1*L);
 D=diag(d);
 dd=diag(L*RA1);
```

```
    DD=diag(dd);

sigma3=2*[2*trace(L'*RA1*L*RA1)+2*trace(L'*RA1*L'*RA1)
        -4*trace(L'*RA1*DD*RA1)-4*trace(L*RA1*D*RA1)
+2*trace(D'*RA1*DD*RA1)+trace(D'*RA1*D*RA1)+trace(DD*RA1
*DD*RA1)];
    b=sqrt((1/(n-1))*sigma3);
    Zi1=(sumsqr(R2U)-sumsqr(OU))/b;
```