MODELING FINANCIAL ENVIRONMENTS USING GEOMETRIC FRACTIONAL BROWNIAN MOTION MODEL WITH LONG MEMORY STOCHASTIC VOLATILITY

MOHAMMED KAMEL ALHAGYAN

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Graduate School
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Abstrak

Model Pergerakan Pecahan Geometrik Brownian (GFBM) digunakan dengan meluas dalam persekitaran kewangan. Model ini mengandungi parameter penting iaitu min, ruapan, dan indeks Hurst, yang bererti kepada kebanyakan masalah dalam bidang kewangan terutamanya bagi menentukan harga opsyen, nilai pada risiko, kadar tukaran, dan insuran cagaran. Kebanyakan penyelidikan terkini mengkaji *GFBM* dengan mengandaikan ruapannya adalah malar disebabkan keringkasannya. Walau bagaimanapun, anggapan ini selalunya disangkal dalam kebanyakan kajian empirikal. Oleh itu, kajian ini membangunkan model GFBM baharu yang mampu menerangkan dan menggambarkan situasi sebenar dengan lebih baik terutamanya dalam senario kewangan. Kesemua parameter yang terlibat dalam model yang dibangunkan dianggar menggunakan algoritma inovasi. Kajian simulasi seterusnya dilakukan untuk menentukan prestasi model baharu. Hasil simulasi mendedahkan bahawa penganggar yang disyorkan adalah cekap berdasarkan kepada kepincangan, varians, dan min kuasa dua ralat. Seterusnya, dua teorem berkaitan kewujudan dan keunikan penyelesaian bagi model baharu dan pengitlakannya dibina. Pengesahan bagi model yang dibangunkan kemudiannya dilakukan dengan membandingkannya dengan beberapa model lain bagi meramal harga terlaras Standard and Poor’s 500, Shanghai Stock Exchange Composite Index, dan FTSE Kuala Lumpur Composite Index. Kajian empirikal terhadap empat aplikasi kewangan terpilih, iaitu penentuan harga opsyen, nilai risiko, kadar pertukaran, dan insuran gadai janji, menunjukkan bahawa model baharu mempamerkan keputusan yang lebih baik berbanding model sedia ada. Justeru itu, model baharu amat berpotensi untuk dijadikan model pendasar bagi sebarang aplikasi kewangan yang berupaya mencerminkan keadaan sebenar dengan lebih tepat.

Kata kunci: Pergerakan Pecahan Geometrik Brownian, ruapan stokastik, memori panjang, senario kewangan.

Abstract

Geometric Fractional Brownian Motion (GFBM) model is widely used in financial environments. This model consists of important parameters i.e. mean, volatility, and Hurst index, which are significant to many problems in finance particularly option pricing, value at risk, exchange rate, and mortgage insurance. Most current works investigated GFBM under the assumption of its volatility that is constant due to its simplicity. However, such assumption is normally rejected in most empirical studies. Therefore, this research develops a new GFBM model that can better describe and reflect real life situations particularly in financial scenario. All parameters involved in the developed model are estimated by using innovation algorithm. A simulation study is then conducted to determine the performance of the new model. The results of simulation reveal that the proposed estimators are efficient based on the bias, variance, and mean square error. Subsequently, two theorems on existence and uniqueness of the solution for the new model and its generalisation are constructed. The validation of the developed model was then carried out by comparing with other models in forecasting adjusted prices of Standard and Poor 500, Shanghai Stock Exchange Composite Index, and FTSE Kuala Lumpur Composite Index. Empirical studies on four selected financial applications, i.e. option pricing, value at risk, exchange rate, and mortgage insurance, indicate that the new model performs better than the existing ones. Hence, the new model has strong potential to be employed as an underlying model for any financial applications that capable of reflecting the real situation more accurately.

Keywords: Geometric Fractional Brownian Motion, stochastic volatility, long memory, financial scenario.

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List of Abbreviations

AR	Autoregressive Process
ARCH	Autoregressive Conditional Heteroscedasticity
ARFIMA	Autoregressive Fractionally Integrated Moving Average
ARIMA	Autoregressive Integrated Moving Average
ARSV	Autoregressive Stochastic Volatility
BM	Brownian Motion
BS	Black–Scholes
CIR	Cox–Ingersoll–Ross
CMLE	Complete Maximum Likelihood Estimation
DE	Differential Evolution
EBSCO	Elton Bryson Stephens Company
EMM	Efficient Method of Moments
FBM	Fractional Brownian Motion
FBS	Fractional Black–Scholes
GFBM	Geometric Fractional Brownian Motion
FGN	Fractional Gaussian Noise
FOU	Fractional Ornstein Uhlenbeck
FSDE	Fractional Stochastic Differential Equation
GARCH	Generalized Autoregressive Conditional Heteroscedasticity
GBM	Geometric Brownian Motion
GMM	Generalized method of moment
GS	Google Scholar
H	Hurst
JP	Jump
KLCI	Kuala Lumpur Composite Index
LMSV	Long Memory Stochastic Volatility
MAPE	Mean Absolute Percentage Error
MBI	Moment Based Inference
MCMC	Markov Chain Monte Carlo

MLE	Maximum Likelihood Estimation
MM	Method of Moments
MSE	Mean Square Error
OU	Ornstein– Uhlenbeck
R/S	Rescale / Range
RS	Random Search
S &P 500	Standard and Poor's 500
S.V	SciVerse
SA	Simulated Annealing
SABR	Stochastic Alpha–Beta–Rho
SBI	Simulation Based Inference
SDE	Stochastic Differential Equation
SV	Stochastic Volatility
Var	Variance

CHAPTER ONE
INTRODUCTION

1.1 Research Background

Volatility has been actively discussed in time series econometrics and economic forecasting in recent years. Volatility explains the variations witnessed in some phenomena over time. In economics, it is used to describe variability of random component of a time series. In financial economics, volatility is defined as the standard deviation of a random Wiener driven component in a continuous time diffusion model.

In the last decades, two main classes of volatility models have been developed: the generalized autoregressive conditional Heteroscedasticity (GARCH) and the stochastic volatility (SV) model. These classes were developed in order to capture time–varying autocorrelation, i.e. the correlation between values of the process at different points in time.

To begin, in 1982 Engle introduced autoregressive conditional heteroscedasticity (ARCH) model to estimate conditional variance of the sequence of increasing price of the United Kingdom's financial environment. This model was developed by prior assumption that the variance of random errors was related to the previous random, with inclusion of the conditional variance and mean in equation. Four years later, the extension of ARCH was proposed by Bollerslev (1986), known as generalized autoregressive conditional heteroscedasticity (GARCH) model. This model adds the memory of past variances to the model which is useful in modeling and forecasting time–varying variances of financial returns (Zhang and Hyvarinen, 2012). Thus, it is not surprising that GARCH model is extensively used as a tool to model financial data due to its capability in capturing volatility clustering of large price movements. However, Fleming and Kirby (2003) claimed that the popularity of GARCH model is because of its convenience, rather than its efficiency which was amongst the sought trait in finance.

Such weakness gave rise to another approach, stochastic volatility (SV) model in the financial environment. SV, stemmed from the idea of modeling volatility as a stochastic process, was motivated by an empirical study of stock price returns where the estimated volatility was observed to exhibit random characteristics (Fouque, Papanicolaou and Sircar, 2000).

In general, a SV model is a statistical method that plays a significant role in mathematical finance. This model refers to the volatility and common dependence between variables that are permitted to fluctuate over time, instead of remaining constant.

The main difference between the GARCH models and SV models is in terms of conditional volatility. The conditional volatility in GARCH models is a deterministic function of past observation, whereas SV models a random process. Furthermore, Asai (2008) discovered that SV models were commonly known to be more fit in describing the thickness of tail of financial returns, compared to the ARCH–type models.

SV models are typically analyzed by using advanced models which become more accurate and efficient as computer technology develops over time. Apart from that, SV models may also compensate current weaknesses in the standard Black–Scholes model whose volatility is assumed to be constant and unaffected by the fluctuations of price level over time for the underlying security. This assumption, however, was rejected by Bakshi, Cao, and Chen (2000), Aït–Sahalia and Lo (1998), and Stein (1989) to name just a few. It was observed that the inconsistent movements of stock prices were not exclusively described by constant volatility as evidently portrayed in empirical studies which included market crashes of Black–Monday in 1987, the Asian crisis in 1989, housing bubble and credit crisis in 2007 until 2009. In order to describe the stock prices more precisely, few alternative models such as generalized Lévy processes, fractional Brownian motion (FBM), diffusions model with jumps and SV models were proposed.

In this work that follows, we will in particular investigate SV models perturbed by FBM based on three reasons. First, SV models have a robust theoretical basis in option pricing theory. Second, the connection with state space approach under SV models is strong, in which its vector of quantities is able to offer attractive features with respect to generality, flexibility and transparency in order to describe a time-varying process (Durbin and Koopman, 2001). Third, SV models are able to describe the thickness of tail of financial returns better than ARCH–type as mentioned earlier. Therefore, SV models possess good characteristics which enable them to provide details of the empirical features of the joint time–series behavior of option prices and stock which cannot be captured by available limited models. Furthermore, by accommodating FBM into SV model, real market behaviors can be described more accurately since these models have memory, or dependency. For simplicity, the term ‘memory’ will be used throughout the thesis.

SV models are divided into two main streams. One is based on discrete time setting and the other is based on continuous time setting. The next subsection provides a discussion of these streams and some related topics that will be further utilized in the following chapters.

1.1.1 Discrete Stochastic Volatility and Continuous Stochastic Volatility

Discrete SV model was being proposed by Taylor (1986), whereas Hull and White (1987) introduced continuous time diffusion model. Autoregressive integrated moving average (ARIMA) models and autoregressive fractionally integrated moving average (ARFIMA) models are examples of the discrete SV models whereas Ornstein–Uhlenbeck (OU) model and the fractional Ornstein–Uhlenbeck (FOU) model are examples of continuous time SV model.

Discrete time setting is very much dominated by a variant of autoregressive conditionally heteroskedastic (ARCH) model while the representation of stochastic differential equations (SDE) represents the continuous time.

To date, works on SV in literature are mostly involved in discrete approximation of the continuous time SV models, such as works by Hull and White (1987), Hamilton (1989), Harvey (1998), Breidt, Crato, and de Lima (1998), Comte and Renault

(1998), Comte, Coutin and Renault (2012) and Chronopoulou and Viens (2012a, 2012b).

In the following subsections, we will discuss briefly about the development of long memory in financial environment, and how SV can be an improvement to current financial model.

1.1.2 Long Memory in Financial Modeling

Traditional financial modeling is based on semimartingale processes with stationary and independent increments, meaning that the process can be decomposed into a finite variation term and a local martingale term, though practical examination of financial data showed such assumption are contradictory. Most of real data showed dependency better known as self–similarity or long memory (long-range dependence). One of the models proposed to handle this issue is Fractional Brownian motion (FBM). FBM is a continuous Gaussian process with independent increments. The correlation between the increments of FBM varies consistently with its self–similarity parameter, H index, helps to capture the correlation dynamics of data and therefore should produce better forecasting results.

Initially, Kolmogorov (1940) introduced FBM within a Hilbert space framework. He considered continuous Gaussian processes with stationary increments coupled with self–similarity property. This process was later studied by Mandelbrot and Van Ness (1968) by presenting FBM in the form of a stochastic integral.

Based on empirical studies, volatilities and returns of stock prices habitually showed long memory property or long–range dependency. Such behavior is depicted in its autocovariance function, in which the values at different times decay slowly. Since FBM process is able to exhibit long memory phenomenon in the data, thus, it is only natural to extend the traditional works in financial modeling by adapting theoretical advantages of such process.

Currently, FBM popularity lies in various applications. Among applications include works that describe the widths of consecutive annual rings of a tree (Biagini, Hu, Oksendal and Zhang, 2008), temperature at particular places (Shiryaev, 1999), water level in a river (Alos, Mazet and Nualart, 2000), characters of solar activity (Massoulie and Simonian, 1999), values of the log returns of stocks (Narayan, 1998) and financial turbulence (Norros, 1995; 1997).

Below are some definitions that may be of use in the later chapters.

Definition 1.1 (Ash and Doleans–Dade, 2000)*: If* Ω *is a given set, then the* σ*–algebra* $\mathcal{F}$ *on* Ω *is a family of subsets of* Ω *such that:*

1. $\phi \in \mathcal{F}$.
2. *If* $A \in \mathcal{F}$ *then* $A^c \in \mathcal{F}$.
3. *If* $A_1, A_2, \ldots \in \mathcal{F}$ *then the infinite union* $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Further, the bilateral $(\Omega, \mathcal{F})$ *is called measurable space and the subsets* F *of* Ω *which belong to* $\mathcal{F}$ *are called* $\mathcal{F}$*–measurable sets.*

Definition 1.2 (Ash and Doleans–Dade, 2000): *A probability measure on a measurable space* $(\Omega, \mathcal{F})$ *is a function* $P: F \to [0,1]$ *satisfying the following conditions:*

1. $P(\phi) = 0$ *and* $P(\Omega) = 1$.
2. *If* $A_1, A_2, \ldots \in \mathcal{F}$ *are mutually disjoint then* $p(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} p(A_i)$ *and the triple* $(\Omega, \mathcal{F}, P)$ *is called a probability space.*

Definition 1.3 (Cinlar, 2013)*: A stochastic process is a parameterized collection of random variables* $\{X_t\}_{t\in[0,\infty)}$ *defined on probability space* $(\Omega, \mathcal{F}, P)$ *and assuming value in* $\mathbb{R}^n$.

Definition 1.4 (Coculescu and Nikeghbali, 2010): *A filtration on a measurable space* $(\Omega, \mathcal{F})$ *is a family* $\mathcal{M} = \{\mu_t\}_{t\geq 0}$ *of* σ*–algebras where* $\mu_t \in \mathcal{F}$ *such that* $0 \leq s \leq t$ *implies* $\mu_s \leq \mu_t$.

Definition 1.5 (Taylor and Karlin, 2014): *Let* $\{N_t\}_{t\geq 0}$ *be an* n*–dimensional stochastic process on a probability space* $(\Omega, \mathcal{F}, P)$*. Then* $\{N_t\}_{t\geq 0}$ *is called a martingale with respect to a filtration* $\{\mu_t\}_{t\geq 0}$ *if for all* t *and for all* $s > t$ *the following are hold:*

1. N_t *is* μ_t*–measurable.*
2. $\mathrm{E}[|N_t|] < \infty$.
3. $\mathrm{E}[N_s | N_t] = N_t$.

Definition 1.6 (Billingsley, 1999): *Let* $\{N_t\}_{t\geq 0}$ *be an increasing family of* σ*–algebras of subsets of* Ω.

1. *A process* $g(t, w): [0, \infty) \rightarrow \mathbb{R}^n$ *is called* N_t*–adapted if for each* $t \geq 0$ *the function* $w \rightarrow g(t, w)$ *is* N_t*–measurable.*
2. *A function* $\tau: \Omega \rightarrow [0, \infty)$ *is called stopping time with respect to* $\{N_t\}_{t \geq 0}$ *if* $\{w;\ \tau(w) \leq t\} \in N_t$ *for all* $t \geq 0$.

Definition 1.7 (Mörters and Peres, 2010): *Let* $M(t) \in \mathbb{R}^n$ *be an* N_t*–adapted stochastic process. Then* $M(t)$ *is called a local martingale with respect to some given filtration* $\{N_t\}_{t \geq 0}$ *if there exists an increasing sequence of* N_t *stopping time* τ_k *such that* $\tau_k \rightarrow \infty$ *almost surely as* $k \rightarrow \infty$, *and* $M(\tau_k \wedge t)$ *is an* N_t*–martingale for all* k.

Definition 1.8 (Patrick, 1995): *A cadlag function is a real function that is right continuous and has a left limit.*

Definition 1.9 (He and Yan, 1992): *A process* $\{X_t\}_{t \geq 0}$ *is called a semimartingale for a given filtration* $\{\mu_t\}_{t \geq 0}$ *if it can be decomposed as* $X_t = X_0 + M_t + A_t$ *where* X_0 *is* $\mathcal{F}$*–measurable,* M_t *is local martingale and* A_t *is cadlag adapted process.*

Definition 1.10 (Biagini et al., 2008): *Let* $X = \{X_t\}_{t \geq 0}$ *be an* $\mathbb{R}^d$*–valued random process. We say that* X *is self–similar or satisfies the property of self–similar if for every* $a > 0$ *there exists* $b > 0$ *such that:*

$$Law(X_{at}, t \geq 0) = Law\ (bX_t, t \geq 0).$$

Definition 1.11 (Mörters and Peres, 2010): *A stochastic process* $W(t)$ *is a Brownian motion (BM) if it satisfies the following properties:*

1. *$W\ (t)$ is a continuous function of time with $W\ (0)\ =\ 0$.*
2. *$W\ (t)$ has independent increments, i.e., for all $t > s,\ v > u$ and $u > t, W(t) - W(s)$ and $W(v) - W(u)$ are independent.*
3. *$W\ (t)$ has normal increments, i.e., for all $t\ >\ s, W\ (t) - W(s) \sim N\ (0, t - s)$.*

Definition 1.12 (Racine, 2011): *The Hurst parameter is a dimensionless estimator for the self-similarity of a time series.*

Definition 1.13 (Mishura, 2008): *The fractional Brownian motion (FBM), $\{B_H(t)\}$, with Hurst parameter $H \in (0,1)$ is a centered Gaussian process whose paths are continuous with probability 1 and its distribution is defined by the covariance structure:*

$$\mathrm{E}[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Remarks:

1. $\mathrm{E}(B_H(0)) = 0$ and $\mathrm{var}\left(B_H(t) - B_H(s)\right) = |t - s|^{2H}$.
2. We have three different families according to the value of H as in the following table :

Table 1.1

Representation of memory dependence families

H	**Memory dependence**
$0 < H < \frac{1}{2}$	Short memory dependence
$H = \frac{1}{2}$	No memory dependence
$\frac{1}{2} < H < 1$	Long memory dependence

Definition 1.14 (Biagini et al., 2008): *If* $\rho(n) = \mathrm{cov}(X_m, X_{m+n})/\mathrm{var}(X_m)$ *be the autocorrelated function, then a process* $\{X_m, m \in \mathbb{N}\}$ *is said to have long memory if* $\sum_{n=1}^{\infty} \rho(n) = +\infty$.

The next subsection introduces the geometric fractional Brownian motion, a financial model for underlying asset with long memory element.

1.1.3 Geometric Fractional Brownian Motion (GFBM)

Ross (1999) employed Brownian motion (BM) process for modeling stock price directly. However, his work has been under heavy criticism since BM process also takes into consideration of negative price because the price of a stock is assumed a normal random variable. To overcome this issue, a non–negative variation of BM known as geometric Brownian motion (GBM) was introduced to enhance the application of BM in finance. This new characteristic allows GBM to be widely used in financial mathematics since it is capable of describing the real situation better as in the famous Black–Scholes model. The definition of GBM is as follows:

Definition 1.15 (Wiersema, 2008): *A stochastic process* S_t *is said to follow a geometric Brownian motion (GBM) if it satisfies the following stochastic differential equation (SDE):*

$$dS_t = \mu S_t dt + \sigma S_t dW_{1t}, \tag{1.1}$$

where W_{1t} *is a BM,* μ *is a drift and* σ *is a volatility taking constants values. The solution of Equation (1.1) is of the form*

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_{1t}\right\},$$

where S_0 *is an arbitrary initial value.*

In a SV model, the constant volatility σ in Equation (1.1) is replaced by a deterministic function of a stochastic process or volatility process, Y_t

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t d\, W_{1t}. \tag{1.2}$$

In classical SV models, Y_t represents the solution of stochastic differential equation (SDE) that is driven by other noise denoted by W_{2t} or $B_H(t)$ which can either be correlated with W_{1t} or independent. Different models describe Y_t in different forms as shown in the Table 1.2.

Table 1.2

Models of stochastic processes describing Y_t in SV models

Name	**Model**
Log–normal process	$dY_t = \alpha Y_t dt + \beta Y_t dW_{2t}$
Cox–Ingersoll–Ross (CIR) process	$dY_t = \theta(\omega - Y_t)dt + \xi\sqrt{Y_t} dW_{2t}$
Ornstein–Uhlenbeck (OU) process	$dY_t = \alpha(m - Y_t)dt + \beta dW_{2t}$
Non mean reverting process	$dY_t = \alpha Y_t dW_{2t}$
Fractional Ornstein–Uhlenbeck (FOU) process	$dY_t = \alpha(m - Y_t)dt + \beta dB_H(t)$

The following Table 1.3 depicts some popular SV models which depend on the deterministic function $\sigma(\cdot)$ and the volatility process Y_t.

Table 1.3

Popular SV models

Model	$\sigma(\cdot)$	Y_t	Author
Scott	e^y	Mean–reverting OU process	Scott (1987)
Hull and White	$\sqrt{y}$	Log–normal process	Hull and White (1987)
Stein and Stein	$\|y\|$	Mean–reverting OU process	Stein and Stein (1991)
Heston	$\sqrt{y}$	CIR process	Heston (1993)
Hagan	$\|y\|$	Non mean reverting	Hagan et al. (2002)
Vasicek	y	Mean–reverting OU process	Vasicek (1977)
Comte and Renault	y	Fractional OU process	Comte and Renault (1998)

A geometric fractional Brownian motion (GFBM) model is obtained by replacing BM in the error term in GBM model with FBM to incorporate long memory properties to GBM model as stated in Definition 1.14. Below is the definition of GFBM.

Definition 1.16 (Biagini et al., 2008): *A stochastic process S_t is said to follow a geometric fractional Brownian motion (GFBM) if it satisfies the following stochastic differential equation (SDE):*

$$dS_t = \mu S_t dt + \sigma S_t \, dB_H(t), \qquad (1.3)$$

where $B_H(t)$ is FBM, σ and μ are constants parameters that represent volatility and drift respectively. The solution of Equation (1.2) is of the form

$$S(t) = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2 t^{2H-1}\right) t + \sigma B_H(t)\right],$$

where s_0 is an arbitrary initial value.

1.2 Problem Statement

Earlier works on geometric Brownian motion (GBM) were mainly concerned on constant volatility because of the simplicity in its derivation. However, this assumption does not describe the real situation accurately and therefore rejected by many empirical studies such as in Bakshi et al. (2000), Aït–Sahalia and Lo (1998), and Stein (1989). There has been much effort to tackle this issue by influencing the constant volatility in GBM model with a stochastic volatility (SV). Among the effort includes works carried out by Scott (1987), Hull and White (1987), Stein and Stein (1991), Heston (1993), Hagan et al. (2002), Comte and Renault (1998), Chronopoulou and Viens (2012a, 2012b), and Wang and Zhang (2014). Despite the improvement of this approach, a number of researchers such as Painter (1998), Willinger et al. (1999), Grau–Carles (2000) and Rejichi and Aloui (2012), to name just a few, argued that data of a time series governed by this model exhibited memory. Thus, the need to develop a model GBM which is capable of incorporating with long memory properties started gearing up in the literature. This improved model is known as geometric fractional Brownian motion (GFBM).

A GFBM model has important features as its parameters, i.e. mean (μ), volatility (σ) and Hurst parameter (H) are essential elements in many financial applications such as pricing the option by using fractional Black–Scholes model ((Misiran et al., 2010; 2012) and (Kukush et al., 2005)), forecasting index prices (Xiao et al., 2015), determining value at risk (Wang et al., 2017), identifying volatility in exchange rate (Gözgör, 2013) and determining premium in mortgage insurance (Bardhan et al., 2006; and Chen et al., 2013). Though the application of

GFBM is massive, only limited works in the literature estimate these parameters. This could be contributed by the difficulties in maximizing the likelihood function analytically due to the complexity of both the covariance function and its inverse involved. To date, it is discovered that only few researchers attempted to estimate the concerned parameters. Kukush et al. (2005) developed incomplete maximum likelihood estimation (IMLE) approach for volatility, where its Hurst index H is estimated by some other heuristic methods, e.g. variation analysis or rescaled range (R/S) analysis. Later, their work has been improved by Misiran et al. (2010; 2012) by introducing the complete maximum likelihood estimation (CMLE) which enables to estimate all parameters involved in GFBM simultaneously. However, the major setback of their work is the assumption of volatility being constant for the sake of simplicity in the calculation.

The drawbacks of the existing models motivate us to fill the gap by extending the current works to come up with a new model of GFBM having stochastic volatility that enables to estimate all the parameters involved in the model. As a result, the developed model is capable of describing and representing real life situations more accurately, particularly in financial scenario.

1.3 Research Objective

The main objective of this research is to develop a new geometric fractional Brownian motion model that can better describe and represent real life situations particularly in financial scenario which can be obtained through the following sub objectives:

i. To estimate all parameters in GFBM with stochastic volatility through simulation study.
ii. To establish conditions for the existence and uniqueness of the solution of the new model.
iii. To validate the proposed model.
iv. To apply the new model in financial applications.

1.4 Limitation of the Study

The simulation carried out in this study is very expensive in terms of computational time. Due to this constraint, the segmentation approach of time series data in simulation was employed in order to significantly reduce the time.

1.5 Significance of the Research

This study has the following contributions:

i. Provide better tool for estimating parameters involved in the underlying financial models to produce more accurate description of real world applications.
ii. Help investors to make better decision for their investments by comparing the forecasts of the future price using several methods.
iii. Develop theorems for existence and uniqueness of the solution of the new model.

1.6 Outline of the Thesis

This thesis comprises of seven chapters. Chapter One begins by discussing the background of stochastic volatility and geometric fractional Brownian motion together with related topics, problem statement, research objectives, limitation of research, and the significance of the research. Chapter Two reviews important literature related to stochastic volatility models and then identifies the advantages and disadvantages of these models. A review of content analysis on the stochastic volatility is also presented in this chapter.

Chapter Three presents the development of the new model which consists of two parts; the approximation of GFBM's covariance and estimation of its parameters. The description of a simulation study is also covered in this chapter.

Chapter Four establishes a theorem for the existence and the uniqueness solution of GFBM model. In addition, a theorem for the existence and the uniqueness solution of a general case of a class of fractional stochastic differential equation is also constructed here.

Chapter Five validates the proposed model through investigation on forecasting three different types of markets; Standard and Poor's 500 (S&P 500), Shanghai Stock Exchange (SSE) Composite Index, and Kuala Lumpur Composite index (KLCI). The validations are conducted through selected volatility measurements, i.e. simple volatility, high–low–closed volatility, log volatility and the stochastic volatility proposed in this study.

Chapter Six demonstrates the applications of the proposed model to financial applications which include European option pricing, value at risk, exchange rates and mortgage insurance. Finally, Chapter Seven concludes this study with a brief conclusion and recommendations for future researches.

CHAPTER TWO
PRELIMINARIES AND LITERATURE REVIEW

In this chapter, we provide basic concepts, definitions, lemmas and theorems needed in our work. An extensive literature review on stochastic volatility models in financial environment is also presented.

2.1 Preliminaries

In this subsection, basic concepts, definitions, lemmas and theorems are presented.

Definition 2.1 (Harris and Stocker, 1998): *A likelihood function $L(\theta)$ is the probability or probability density for the occurrence of a sample configuration $x_1, \dots, x_n$ given that the probability density $f(x;\theta)$ with parameter $\theta$is known,*

$$L(\theta) = f(x_1;\theta) \cdots f(x_n;\theta).$$

Definition 2.2 (Ross, 1979): *If X is a discrete (or continuous) random variable having probability mass function $p(x)$ (or probability density function $f(x)$) then the expected value of X defined by*

$$\mathrm{E}[X] = \begin{cases} \sum_x x\, p(x), & if\ X\ discret \\ \int_x \ x\, f(x), & if\ X\ continuous \end{cases} .$$

Definition 2.3(Ross, 1979): *The Variance of a random variable X is a tool to measure the expected square of the deviation of X from its expected value. i.e.*

$$\mathrm{var}(X) = \mathrm{E}[(X - \mathrm{E}[X])^2].$$

Definition 2.4 (Ross, 1997): *The covariance of any two random variable* X *and* Y*, denoted by* $\operatorname{cov}(X,Y)$*, is defined by*

$$\operatorname{cov}(X,Y) = \mathrm{E}[(X - \mathrm{E}[X])(Y - \mathrm{E}[Y])].$$

Definition 2.5 (Willinger, 1999): *For a given set of observation* $(X_i, i \geq 1)$*, with partial sum* $Y(n) = \sum_{i=1}^{n} X_i$*, and sample variance* $S^2(n) = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \frac{Y(n)}{n}\right)^2$*, the rescaled adjusted range statistic or R/S- statistic is defined by*

$$\frac{R}{S}(n) = \frac{1}{S(n)}\left[\max_{0\leq t<n}\left(Y(t) - \frac{t}{n}Y(n)\right) - \min_{0\leq t<n}\left(Y(t) - \frac{t}{n}Y(n)\right)\right] \quad , \quad n \geq 1.$$

Hurst (1951) found that many naturally accurring empirical records appear to be well represented by the relation $\mathrm{E}[R/S(n)] \sim c_1 n^H$, as $n \to \infty$, with typical values of the Hurst parameters H in the interval $(0.5, 1)$, and c_1 a finite positive constant that does not depend on n.On the other hand, if the observations X_i come from a short-range dependent model, then it is known that $\mathrm{E}[R/S(n)] \sim c_2 n^{0.5}$, as $n \to \infty$, where c_2 is independent of n, and finite and positive. The discrepancy between these two relations is generally referred to as the Hurst effect or the Hurst phenomena.

The following lemma state basic properties of random variables, Cauchy criterion and Euler discretization.

Lemma 2.1 (Ross, 1997): *For any random variables* X, Y, Z *and constant* a *and* b*, then*

i. $\mathrm{E}[aX + bY] = a\mathrm{E}[X] + b\mathrm{E}[Y]$.

ii. $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$ *iff* X *and* Y *are independent.*

iii. $\operatorname{var}(X + a) = \operatorname{var}(X)$.

iv. $\text{var}(aX) = a^2\text{var}(X)$.

v. $\text{var}(aX + bY) = a^2\text{var}(X) + b^2\text{var}(Y) + 2ab\,\text{cov}(X, Y)$.

vi. $\text{cov}(X, X) = \text{var}(X)$.

vii. $\text{cov}(aX, Y) = c^2\text{cov}(X, Y)$.

viii. $\text{cov}(X, Y + Z) = \text{cov}(X, Y) + \text{cov}(X, Z)$

Lemma 2.2 (Cauchy Criterion) : *A sequence $\{u_n\}$ converges if and only if for each $\epsilon > 0$, we can find a number N such that $|u_m - u_r| < \epsilon$ for all $m, n > N$.*

Lemma 2.3 (Euler's Discretization): *For the ordinary differential equation $\frac{du}{dt} = f(u)$, Euler's discretization is defined as follows:*

$$x_{n+1} = x_n + \Delta t f(x_n).$$

2.2 Stochastic Volatility Models in Financial Environment

Stochastic volatility (SV) models are considered the most appropriate approach to capture an implied volatility smile and fat tailed distribution of asset price return (Kim and Wee, 2014). Such properties can significantly improve the pricing of asset under the Black-Scholes model.

Apart from the previous benefit, SV models are also substantial for financial markets and decision making because they can capture the effect of time–varying volatility. For this reason, many studies on SV models have been carried out in financial environment such as option pricing, value at risk, risk assessment and portfolio allocation. In addition, SV models also provide alternatives to standard Black–Scholes assumption where observations to volatility do not need to be perfectly correlated with observations of the underlying asset price (Heston, 1993; and Stein and Stein, 1991). These SV type models can offer better information for the joint time–series behavior of option prices and stocks, which could not be captured by using other models.

In a SV model, the constant volatility σ in standard geometric Brownian motion (GBM) model is replaced by a deterministic function of a stochastic process $\sigma(Y_t)$ where Y_t represents the solution of stochastic differential equation (SDE) that is driven by other noise. This implies that SV model has two sources of randomness which can either be correlated or not.

There are two ways to describe SV; in discrete time setting and continuous time setting. Since the intuitive setting for market trading is normally continuous such as derivative pricing (Johnson and Shanno, 1987; Hull and White, 1987; Stein and Stein, 1991; Comte and Renault, 1998; and Chronopoulou and Viens, 2012a; 2012b) and portfolio optimization (Pakdel, 2016; and Vierthauer, 2010), it is natural to embark studying a continuous time setting in a financial environment.

SV models were first introduced by Taylor (1986) to account for inconsistency in implied volatility values. Taylor recommended modeling the logarithm of volatility as an autoregressive AR(1) process. This model is known as autoregressive stochastic volatility ARSV(1) and is given by

$$x_t = \exp\left(\frac{y_t}{2}\right) u_t, \quad u_t \sim N(0,1),$$

$$y_t = \mu + \phi(y_{t-1} - \mu) + \eta_t, \quad \eta_t \sim N(0, \sigma_{\eta_t}^2),$$

where x_t denotes the log return at time t, where $t = 1, \ldots, T$ and y_t is the log–volatility which is assumed to follow a stationary AR(1) process with persistence parameter $|\phi| < 1$. The error terms u_t and η_t are Gaussian white noise sequences. Although the Taylor model is simple and easy to use, it has some drawbacks such as the absence of the mean–reverting part, zero correlation assumption between stock price and volatility, and non-existence of memories in its returns series and volatility.

Subsequently, Johnson and Shanno (1987) used time changing volatility in option pricing where the deterministic function $\sigma(Y_t) = Y_t$ is defined by

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$

$$dY_t = \alpha Y_t dt + \beta\, Y_t\, d\, W_{2t},$$

where α and β are the mean and the volatility of a volatility process Y_t, respectively. In this model, Wiener processes W_{1t} and W_{2t} are correlated. The main advantage of this model is that the computational results of option prices are consistent with empirical observations. This model exhibits volatility smile and an increase in value towards expiry (Mitra, 2011). However, this model only provides numerical method to option pricing instead of in its closed form. The mean–reverting parameter as well as memories of both returns and volatility are also absent in this model.

Scott (1987) later developed the following option pricing model which allows the variance parameter to change randomly of an independent diffusion process,

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$

$$dY_t = \alpha(m - Y_t)dt + \beta dW_{2t},$$

where α, m and β represent mean reverting parameter, mean of volatility and volatility of volatility of process Y_t respectively. The instantaneous volatility parameter for stock prices is assumed to follow Ornstein–Uhlenbeck process. He also noticed that $\sigma(Y_t) = e^{Y_t}$ and the Wiener processes W_{1t} and W_{2t} were not correlated. This model is able to observe marginal improvement in option pricing's accuracy as compared to standard Black–Scholes option pricing (Mitra, 2011) and included mean reverting parameter into account. However, its returns series and volatility have no memory and its two sources of randomness are assumed uncorrelated, which is conflicting with the current literature.

Meanwhile, Hull and White (1987) represented option price by a form of series provided that stochastic volatility is independent of the stock price. They proposed a continuous time diffusion model where $\sigma(Y_t) = \sqrt{Y_t}$ and Y_t obeys log–normal process as in the following equations

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t d\, W_{1t},$$

$$dY_t = \alpha Y_t dt + \beta Y_t dW_{2t},$$

where α and β are the mean and the volatility of a volatility of process Y_t respectively, with both Wiener processes W_{1t} and W_{2t} are correlated. This model sets the price of volatility risk to be zero, which is contrary to Heston (1993) who presented a closed–form model with a non–zero price of volatility risk. This model is among the most significant in the literature since it presents closed form solution to European option pricing. Nevertheless, the absence of mean–reverting parameter and the non-existence of both memory of returns and volatility in this model are the flaws.

In the same year, Wiggins (1987) proposed stochastic volatility model under $\sigma(Y_t) = \sqrt{e^{Y_t}}$ and both Wiener processes, i.e W_{1t} and W_{2t} are correlated as given below

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$

$$dY_t = \alpha(m - Y_t)dt + \beta dW_{2t},$$

where α, m and β represent the mean reverting parameter, mean of volatility and volatility of volatility of process Y_t, respectively. Although this model has taken into account the mean reverting parameter, it fails including memories in returns and volatility.

In a bid to develop models which can describe the real financial environment better, Stein and Stein (1991) and Schöbel and Zhu (1999) considered stock price distributions that follow diffusion process with a stochastically varying volatility parameter as defined below

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$

$$dY_t = \alpha(m - Y_t)dt + \beta dW_{2t},$$

where α, m and β represent mean reverting parameter, mean of volatility and volatility of volatility of process Y_t, respectively, with assumption of $\sigma(Y_t) = |Y_t|$. The difference between the models of Stein and Stein (1991) and Schöbel and Zhu (1999) is in terms of the correlation between W_{1t} and W_{2t}. It was observed that in the former model, W_{1t} and W_{2t} are not correlated but both parameters are correlated in the latter model. Besides considering mean reverting parameter into account, both models also share the same disadvantages by omitting memory of returns or the memory of volatility.

An attempt to derive a closed–form solution for the pricing of a European call option was made by Heston (1993). In his approach, the deterministic function of volatility is assumed as $\sigma(Y_t) = \sqrt{Y_t}$, provided that Y_t obeys Cox–Ingersoll–Ross (CIR) process as follows

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$

$$dY_t = \theta(\omega - Y_t)dt + \xi\sqrt{Y_t} dW_{2t},$$

where θ, ω and ξ are mean reverting parameter, long variance parameter and volatility of volatility parameter, respectively. The Brownian processes W_{1t} and W_{2t} are correlated. Heston's model stands out from other SV models as it has analytical solution for European options under assumption of correlated Brownian motions. It can also describe the asymmetric smiles by instant correlation between returns series and its volatility. Furthermore, the empirical performance of Heston's model also outperforms other SV models. As a result, this model generates rich mathematical results and enjoys the positivity of the volatility process besides taking into account of mean reverting parameters (Kim and Wee, 2014). However, this model also omits the existence of memories for returns and volatility in which is considered as a drawback of this model.

Hagan et al. (2002) revealed that the market smile dynamics predicted by using local volatility models (i.e. volatility is merely a function of the current asset S_t and of time *t*) are contrary of observed market behavior. As a treatment of this issue, they proposed an extension of the local volatility model in which the volatility is assumed to be stochastic model and both asset price and volatility are correlated. This extension is called the stochastic alpha–beta–rho (SABR) model as written in the following:

$$dS_t = \sigma(Y_t){S_t}^{\beta} dW_{1t},$$

$$dY_t = vY_t dW_{2t},$$

where S_t, $\sigma(Y_t)$ and v are forward value, volatility of forward value and volatility of volatility, respectively. In this case, they assumed $\sigma(Y_t) = |Y_t|$ and Y_t follows a non-mean reverting process. The Wiener processes W_{1t} and W_{2t} are ρ correlated. This is the simplest stochastic volatility model which is homogeneous in S_t and α, which enables to accurately fit the implied volatility curves observed in the marketplace for any single exercise date. This model can also predict the correct dynamics of the implied volatility curves. However, this model also lack of memory in its return or volatility. Furthermore, mean reverting parameter is also not included in this model.

In summary, there are three main advantages that can be highlighted from the existing models. First, the mean reverting parameter is being taken into account in Scott (1987), Wiggins (1987), Stein and Stein (1991), Schobel and Zhu (1999) and Heston (1993). Second, a closed form of solution is established in Heston (1993) and Hull and White (1987). Finally, the correlation between the error terms existed in Johnson and Shanno (1987), Hull and White (1978), Wiggins (1987), Hagan (2002), Heston (1993) and Schobel and Zhu (1999).

Based on the previous discussion, it can be deduced that each model mentioned previously has at least one of three main drawbacks. First, the existence of zero correlation between stock price and volatility occurs in models proposed by Taylor

(1982), Stein and Stein (1987) and Scott (1987). Second, the absence of mean-reverting parameter that should be included into volatility dynamic; which is observed in the works of Taylor (1982), Johnson and Shanno (1987), Hull and White (1978) and Hagan (2002). Third, it is also noted that all previous models did not consider the existence of memory in neither its returns series nor its volatility component.

Previous literature also suggested that the third drawback pose serious concern in modeling financial asset (Willinger et al., 1999) and (Grau–Carles, 2000). This is strongly supported by the empirical investigations which reveal that volatilities and returns of stock prices habitually show long memory property or long range dependence (Painter, 1998; and Rejichi and Aloui, 2012). Such drawback motivates us to focus on investigating model with long memory property in this study.

In the following discussion, we will review some of the long memory stochastic volatility (LMSV) models.

2.3 Stochastic Volatility Models Perturbed by Long Memory

As we mentioned before, empirical studies showed that the volatility of many assets has long memory properties. Thus, taking long memory into account of volatility contribute in providing better understanding for financial transaction and then better forecasting of future risky asset's prices.

Bredit et al. (1998) introduced general case of LMSV model as follows

$$y_t = \sigma_t \, \xi_t \,,$$
$$\sigma_t = \sigma \exp\left\{\frac{v_t}{2}\right\},$$

where y_t is the return at time t, σ_t is stochastic volatility of return, σ is volatility of volatility, v_t a stationary long memory process, ξ_t is independent and identically distribution, and both $\{v_t\}$ and $\{\xi_t\}$ are independent.

In 1998, Harvey proposed the following equivalent model of LMSV:

$$y_t = \sigma_t \xi_t,$$
$$\sigma_t = \sigma^2 \exp\left\{\frac{\eta_t}{(1-L)^d}\right\},$$

where y_t is the return at time t, σ_t is stochastic volatility of return, σ is volatility of volatility, L is lag operator, ξ_t is independent identically distribution and η_t a stationary long memory process or η_t has normal independent distribution (nid) (i.e. $\eta_t \sim nid(0, \sigma_\eta)$), with $0 < d < 1$.

These two models share similar advantages in incorporating long memory into their volatility parameter, and both models are also simple in their nature. However, they ignore memory of the returns, mean reverting parameter and the correlation between stock price and volatility.

In the same year, Comte and Renault (1998) introduced long–memory mean reverting volatility processes in the setting of continuous time Hull and White model. They modeled the log of volatility as a fractionally integrated Brownian motion (i.e. $\sigma(Y_t) = e^{Y_t}$ in which Y_t follows fractional Ornstein–Uhlenbeck process for $H > 0.5$) as shown below:

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$

$$dY_t = \alpha(m - Y_t)dt + \beta dB_H(t).$$

According to them, not only this model could empirically capture observed strong smile effect for long maturity times, it also incorporated memory in volatility and considered mean reverting into account. However, this model lacks of memory in its return series in addition to lacks of correlation between stock price and volatility.

Subsequently, Comte et al. (2012) extended Heston's model by influencing long memory to its model based on the fractional integration of a square root volatility process. This approach has been approved by Chronopoulou and Viens (2012a; 2012b) as it succeeded in describing volatilities with strong memory in the long run. They also came up with a new LMSV model as follows

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$

$$dY_t = \alpha Y_t dt + \beta dB_H(t).$$

However, this model fails to consider memory in return series, as well as the assumption of zero correlation between stock price and volatility.

In the recent work that follows, Mishura and Swishchuk (2010) studied financial markets with stochastic volatilities driven by fractional Brownian motion with Hurst index $H > 0.5$. Firstly, they assumed that stock price S_t satisfies the following stochastic differential equation

$$dS_t = rS_t dt + \sigma(Y_t) S_t dW_t,$$

where r is an interest rate, $\sigma(Y_t)$ is a volatility, and W_t is a standard Brownian motion. Subsequently, they proposed four SV models in which all models incorporated strong memory into its volatility. First, LMSV model driven by fractional Ornstein–Uhlenbeck process where $\sigma(Y_t) = Y_t$ given by

$$dY_t = -aY_t dt + \gamma Y_t dB_H(t),$$

where $a > 0$ is mean–reverting parameter, $\gamma > 0$ is volatility of volatility and B_H is FBM with Hurst index $H > 0.5$ independent of W_t.

Second, LMSV model driven by continuous–time GARCH process where $\sigma(Y_t) = \sqrt{Y_t}$ is expressed as

$$dY_t = a(b - Y_t)dt + \gamma Y_t dB_H(t),$$

where $a > 0$ is mean–reverting parameter, b mean–reverting level, $\gamma > 0$ is volatility of volatility, and B_H is FBM with Hurst index $H > 0.5$, independent of W_t.

Third, LMSV model driven by Vasicek process where $\sigma(Y_t) = Y_t$ is given by

$$dY_t = a(b - Y_t)dt + \gamma Y_t dB_H(t),$$

where $a > 0$ is mean–reverting speed, b equilibrium level, $\gamma > 0$ is volatility of volatility, and B_H is FBM with Hurst index $H > 0.5$, independent of W_t.

The setbacks of these three models can be abridged into two points. These models ignore the existence of memory in return series and the assumption of zero correlation between stock price and volatility.

Finally, the fourth model is LMSV model driven by GFBM process where $\sigma(Y_t) = \sqrt{Y_t}$ is written as

$$dY_t = aY_t dt + \gamma Y_t dB_H(t),$$

where $a > 0$ is drift, $\gamma > 0$ is volatility of Y_t, and B_H is FBM with Hurst index $H > 0.5$ independent of W_t. This model also has no memory in its return, its mean reverting parameter does not exist and zero correlation is assumed between stock price and volatility.

Based from previous discussions, the common disadvantage shared in all LMSV models is that they assumed returns series of the stock price is independent, meaning no memory. This is contradictory to most empirical findings conducted by Painter (1998), Willinger et al. (1999), Grau–Carles (2000), and Rejichi and Aloui (2012), to name only a few. They also suggested GFBM model should be considered as underlying process for financial variables, due to its ability to incorporate long memory in the system under study.

Table 2.1 presents a summary of the existing SV models, correlation between error terms, advantages and disadvantages for easy read.

Table 2.1

Current Stochastic Volatility Models

Model	$\sigma(Y_t)$	Correlation between error terms	Advantages	Disadvantages
Taylor (1986) $x_t = \exp\left(\frac{\sigma(Y_t)}{2}\right) u_t$ $y_t = \mu + \phi(y_{t-1} - \mu) + \eta_t$	Y_t	not correlated	• Simple form	• Its returns has no memory • Its volatility has no memory • Absence of mean–reverting parameter • Zero correlation assumption between stock price and volatility
Johnson and Shanno (1987) $dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t}$ $dY_t = \alpha Y_t dt + \beta\, Y_t\, d\, W_{2t}$	Y_t	correlated	• Computational results of this model display that their option prices are consistent with empirically observations.	• Its returns has no memory • Its volatility has no memory • Absence of mean–reverting parameter • Provide numerical methods to option pricing instead of closed form
Wiggins(1987) $dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t}$ $dY_t = \alpha(m - Y_t)dt + \beta dW_{2t}$	$\sqrt{e^{Y_t}}$	correlated	• Mean reverting parameter is taken into account.	• Its returns has no memory • Its volatility has no memory

Table 2.1 (Continued)

Model	$\boldsymbol{\sigma(Y_t)}$	**Correlation between error terms**	**Advantages**	**Disadvantages**
Scott(1987) $dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t}$ $dY_t = \alpha(m - Y_t)dt + \beta dW_{2t}$	e^{Y_t}	not correlated	• This model observes a marginal improvement in option pricing accuracy compared to standard Black–Scholes option pricing. • Mean reverting parameter is taken into account.	• Its returns has no memory • Its volatility has no memory • Zero correlation assumption between stock price and volatility
Hull and White (1987) $dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t}$ $dY_t = \alpha Y_t dt + \beta Y_t dW_{2t}$	$\sqrt{Y_t}$	correlated	• This model presents a closed form solution to European option prices. • Volatility risk is set to be zero.	• Its returns has no memory • Its volatility has no memory • Absence of mean–reverting parameter.
Stein and Stein (1991) $dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t}$ $dY_t = \alpha(m - Y_t)dt + \beta dW_{2t}$	$\lvert Y_t \rvert$	not correlated	• Mean reverting parameter is taken into account.	• Its returns has no memory • Its volatility has no memory • Zero correlation assumption between stock price and volatility

Table 2.1 (Continued)

Model	$\sigma(Y_t)$	Correlation between error terms	Advantages	Disadvantages
Heston (1993) $dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t}$ $dY_t = \theta(\omega - Y_t)dt + \xi\sqrt{Y_t}dW_{2t}$	$\sqrt{Y_t}$	correlated	• Describe asymmetric smiles by an instant correlation between returns and volatility. • The empirical performance of Heston's model outperforms other stochastic volatility models • Generating rich mathematical results and enjoying the positivity of the volatility process. • This model stands out from other stochastic volatility models because the existence of analytical solution for European options under assumption of correlated Brownian motions. • Mean reverting parameter is taken into account. • Volatility risk is set to be non–zero	• Its returns has no memory • Its volatility has no memory

Table 2.1 (Continued)

Model	$\sigma(Y_t)$	Correlation between error terms	Advantages	Disadvantages
Schöbel and Zhu (1999) $dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t}$ $dY_t = \alpha(m - Y_t)dt + \beta dW_{2t}$	$\lvert Y_t \rvert$	correlated	• Mean reverting parameter is taken into account.	• Its returns has no memory • Its volatility has no memory
Hagan et al. (2002) $dS_t = \sigma(Y_t) S_t dW_{1t}$ $dY_t = \alpha Y_t dW_{2t}$	$\lvert Y_t \rvert$	correlated	• SABR model is simplest stochastic volatility model which is homogenous in S_t and α. • SABR model can be used to accurately fit the implied volatility curves observed in the marketplace for any single exercise date. • SABR model can predicts the correct dynamics of the implied volatility curves.	• Its returns has no memory • Its volatility has no memory • Absence of mean–reverting parameter.

Table 2.1 (Continued)

Model	$\sigma(Y_t)$	Correlation between error terms	Advantages	Disadvantages
Bredit (1998) $y_t = \sigma(Y_t)\xi_t$ $Y_t = \sigma \exp\left\{\frac{v_t}{2}\right\}$	Y_t	not correlated	• Incorporate long memory property into volatility. • Simple form	• Its returns has no memory • Absence of mean–reverting parameter • Zero correlation assumption between stock price and volatility
Harvey (1998) $y_t = \sigma(Y_t)\xi_t$ $Y_t = \sigma^2 \exp\left\{\frac{\eta_t}{(1-L)^d}\right\}$	$\sqrt{Y_t}$	not correlated	• Incorporate long memory property into volatility. • Simple form	• Its returns has no memory • Absence of mean–reverting parameter • Zero correlation assumption between stock price and volatility
Comte and Renault (1998) $dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t}$ $dY_t = \alpha(m - Y_t)dt + \beta dB_H(t)$	Y_t	not correlated	• Incorporate long memory property into volatility. • This model captures the empirically–observed strong smile effect for long maturity times. • Mean reverting parameter is taken into account.	• Its returns has no memory • Zero correlation assumption between stock price and volatility

Table 2.1 (Continued)

Model	$\sigma(Y_t)$	Correlation between error terms	Advantages	Disadvantages
Mishura and Swishchuk (2010) LMSV model driven by FOU $dS_t = rS_t dt + \sigma(Y_t) S_t dW_t$ $dY_t = -aY_t dt + \gamma Y_t dB_H(t)$	Y_t	not correlated	• Incorporate long memory property into volatility.	• Its returns has no memory • Zero correlation assumption between stock price and volatility
Mishura and Swishchuk (2010) LMSV model driven by Vasicek process. $dS_t = rS_t dt + \sigma(Y_t) S_t dW_t$ $dY_t = a(b - Y_t)dt + \gamma Y_t dB_H(t)$	Y_t	not correlated	• Incorporate long memory property into volatility. • Mean reverting parameter is taken into account.	• Its returns has no memory • Zero correlation assumption between stock price and volatility
Mishura and Swishchuk (2010) LMSV model driven by GFBM $dS_t = rS_t dt + \sigma(Y_t) S_t dW_t$ $dY_t = aY_t dt + \gamma Y_t dB_H(t)$	$\sqrt{Y_t}$	not correlated	• Incorporate long memory property into volatility.	• Its returns has no memory • Absence of mean–reverting parameter • Zero correlation assumption between stock price and volatility

Table 2.1 (Continued)

Model	$\sigma(Y_t)$	Correlation between error terms	Advantages	Disadvantages
Mishura and Swishchuk (2010) LMSV model driven by continuous–time GARCH process $dS_t = rS_t dt + \sigma(Y_t) S_t dW_t$ $dY_t = a(b - Y_t)dt + \gamma Y_t dB_H(t)$	$\sqrt{Y_t}$	not correlated	• Incorporate long memory property into volatility. • Mean reverting parameter is taken into account.	• Its returns has no memory • Zero correlation assumption between stock price and volatility
Chronopoulou and Viens (2012) $dS_t = \mu S_t dt + \sigma(Y_t) S_t d\, W_{1t}$ $dY_t = \alpha Y_t dt + \beta dB_H(t)$	Y_t	not correlated	• Incorporate long memory property into volatility.	• Its returns has no memory • Absence of mean–reverting parameter • Zero correlation assumption between stock price and volatility

To recap, there are three stages of evolutions for volatility in GBM model. First, GBM model with assumption of constant volatility. Second, GBM model with assumption of stochastic volatility. Third, GBM model with assumption of stochastic volatility influenced by long memory.

According to the best of our knowledge, the majority of scholars and researchers have investigated GFBM having constant volatility. To date, no one has ever studied GFBM under the assumption of LMSV. Therefore, in this thesis, we will propose a GFBM model involving LMSV.

The next section embarks on exploring SV in the literature and the discussion will be based on content analysis study.

2.4 Content Analysis on Stochastic Volatility in the Literature

In order to acquire the current state and development on works focusing on SV model in both discrete and continuous time settings, a systematic literature investigation in some selected academic databases from the year 2001 to 2017 was carried. Google scholar, EBSCOhost and SciVerse were used as search engines to find the following keywords:

i. Long memory stochastic volatility and continuous.

ii. Long memory stochastic volatility and discrete.

iii. Stochastic volatility and jumps and continuous.

iv. Stochastic volatility and jumps and discrete.

v. Stochastic volatility and moment-based inference and continuous.

vi. Stochastic volatility and moment-based inference and discrete.

vii. Stochastic volatility and simulation-based inference and continuous.

viii. Stochastic volatility and simulation-based inference and discrete.

The results of the search are shown in Table 2.2 and Figure 2.1 below.

Table 2.2

The evolution of stochastic volatility from 2001–2017

Keyword	**Continuous Setting**			**Discrete Setting**			**Total**
	GS	**EBSCO**	**S.V**	**GS**	**EBSCO**	**S.V**	
LMSV	490	6	88	403	7	76	**1070**
JP	14100	18	2497	10800	4	1531	**28950**
MBI	21	18	1	19	18	1	**78**
SBI	186	19	28	178	19	24	**454**
Total	**14797**	**61**	**2614**	**11400**	**48**	**1632**	**30552**

Indicators:

LMSV: Long memory SV **JP: Jump** **MBI: Moment–based Inference**

GS: Google Scholar **S.V: SciVerse** **SBI: Simulation–based inference**

From Table 2.2, it is discovered that there is a significant popularity difference between the works on SV with jumps in comparison to other works, i.e. long memory, moment based inference and simulation based inference. However, this popularity does not correlate with level of importance (Diebold and Nerlove, 1989; Barndorff-Nielsen, 2001; Jacquier, Polson, and Rossi, 1994; Andersen and Sorensen, 1996; Kermiche, 2014; and Fernández-Villaverde et al., 2015). Meanwhile, works on moment–based inference are very rare. Readers are encouraged to note that the popularity of SV works with long memory is the second highest in frequency, though it may base on the complexity of its derivations and computation (Abken and Nandi, 1996; Han et al., 2014; and Wu and Elliott, 2017), not due to its being less importance (Painter, 1998; Rejichi and Aloui, 2012; and Kim and Wee, 2014). Works involved with simulation–based inference also attracts the interest of researchers, at the third spot. The visualization of Table 2.2 can also be presented in terms of charts as depicted in Figure 2.1

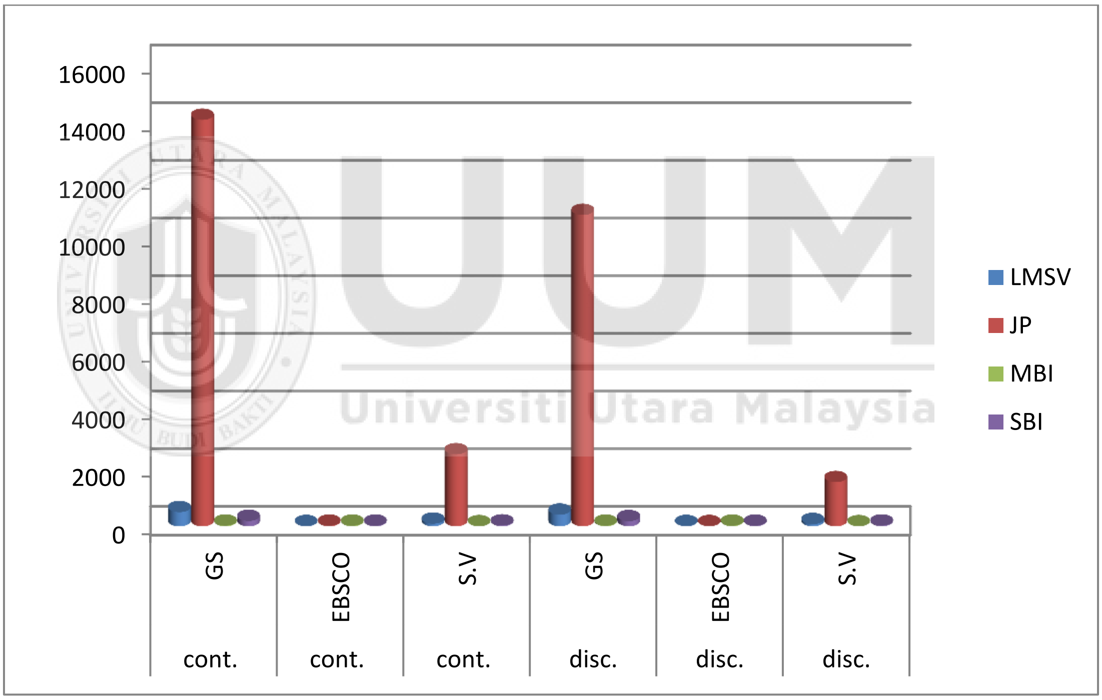


Figure 2.1. Numbers of articles in long memory stochastic, volatility, jumps, moment–based inference, and simulation–based inference.

We will present a brief summary of selected extension works on stochastic volatility model under study.

2.4.1 Jump

In empirical studies, most authors included jumps to volatility dynamics or price process to improve standard stochastic volatility model. Bates (1996) adding jumps to the stochastic volatility when the volatility is Markovian. Barndorff–Nielsen and Shephard (2001; 2002) designed the volatility model in which the price includes a continuous component and jumps which are time homogenous. Brockwell (2001) and Todorov and Tauchen (2006) later proposed an extension of Barndorf–Nielsen and Shephard model.

2.4.2 Multivariate Models

Mandelbrot (1963) proposed good description of volatility clustering as such "*large changes tend to be followed by large changes, of either sign and small changes tend to be followed by small changes*". Volatility clustering into standard factor models presented by Diebold and Nerjove (1989) were used in more than one field of asset pricing. Within continuous time, the same author introduced the following models

$$M_t = \sum_{i=1}^{J} B_{(j)s} dF_{(j)s} + G_i,$$

where G is a correlated multivariate Brownian motion (BM) and the factors $F_{(1)}, F_{(2)}, \dots \dots F_{(J)}$ are independent univariate stochastic volatility models. In the literature, related papers on this issue included Fiorentini et al. (2004) and King et al. (1994), among others, claiming that the factor loading vectors are constant over time. Meanwhile, Harvey, Ruiz, and Shephard (1994) proposed discrete time setting

$$M_t = C \int_0^T \sigma_s dW_s,$$

where C is a fixed matrix of constants such that the main diagonal of all units, W_s is BM, and σ is a diagonal matrix process. This implies that the price' risky part is just the rotation of p–dimensional vector of univariate stochastic volatility independent processes.

2.4.3 Long Memory Stochastic Volatility

Harvey (1998) and Breidt et al. (1998) introduced the first long memory stochastic volatility (LMSV) as a discrete time model such as

$$X_t = \sigma(Y_t)\varepsilon_t,$$

where X_t is a stock returns, ε_t an independent and identically distributed random variables (i.i.d) presenting shocks and logarithm of $\{Y_t\}$ is described by autoregressive fractionally integrated moving average (ARFIMA). This model successfully described the long–range behavior of the log–squared returns of market indices. With respect to continuous time, Comte and Renault (1998) presented a model of the price process such that the dynamics of the volatility are designed using fractional Ornstein–Uhlenbeck (FOU) process. In a work by Comte, Coutin and Renault (2012), the model of square root that was driven by fractionally integrated Brownian motion was introduced. Comte et al. (2012) offered extension of Heston option pricing model to continuous time SV such that the volatility process is defined by a square root long memory process.

Meanwhile, Chronopoulou and Viens (2012b) studied the accuracy of three different types of LMSV. One of them was a continuous time stochastic volatility when the stock price is geometric Brownian motion where the volatility was introduced as a

FOU process. The others were discrete time models, a discretization of the previous continuous model and a discrete model when the returns are a zero mean independent identically distribution sequence with its volatility is a fractional ARIMA process. By working with simulated data and call option data of S&P 500 index, they found that the continuous time model was more accurate than the other discrete models. However, they acknowledged the main disadvantage of continuous time model is on its computation time that is expensive when applied to real data.

2.4.4 Simulation Based Inference

Researchers began to use simulation–based inference in the 1990s. Both Markov chain Monte Carlo (MCMC) and efficient methods of moments (EMM) are two popular simulation methods used to study the stochastic volatility models. To investigate the methods above, it is useful to discuss this with a simple discrete lognormal stochastic volatility given as follows:

$$m_i = \sigma_i \varepsilon_i \,,$$

$$h_{i+1} = \mu + \phi(h_i - \mu) + \eta_i,$$

where m_i is the risky part of returns, σ_i is non–negative process, ε_i follows autoregression with unit variance and zero mean, whereas h_i is a non–zero mean Gaussian linear process and η_i is white noise process with zero mean.

MCMC can be used to simulate high dimensional density data. Jacquier et al. (2004) applied MCMC algorithm in an attempt to solve this problem, while Kim, Shephard, and Chib (1998) presented an extensive discussion in different MCMC algorithms.

Although most papers based on MCMC are formulated in discrete time, exception to Eraker (2001), Elerian, Chib, and Shephard (2004), and Roberts and Stramer (2001) use adaptable general approach constructed to continuous time models. Kim et al. (1998) introduced filtering method for recursively sampling using the so-called particle filter. In addition to the significant role in decision making, filtering method permits computation of one–step–ahead predictions for model testing and marginal likelihood for model comparison.

2.4.5 Moment – Based Inference.

One of the disadvantages of the continuous time stochastic volatility models is that the moment y is not directly computed when using moments based estimators. Nonetheless, Meddahi (2001) presented an approach for generating moment conditions for the full range of models within the so called Eigenfunction stochastic volatility class. Barndorff–Nielsen and Shephard (2002) studied the case of no leverage and obtained the properties of y in the second order and their squares.

CHAPTER THREE

NEW MODEL OF GEOMETRIC FRACTIONAL BROWNIAN MOTION PERTURBED BY LONG MEMORY STOCHASTIC VOLATILITY

In this chapter, a geometric fractional Brownian motion (GFBM) model perturbed by long memory stochastic volatility model is proposed. We then estimate essential parameters in the proposed model and conduct simulation study.

3.1 Development of the Model

In this work, we aim to extend works in stochastic volatility (SV) that has been discussed in length in the previous chapters by introducing long memory SV model in the geometric fractional Brownian motion (GFBM) model. To begin, the next subsection presents the development of GFBM covariance.

3.1.1 Deriving Geometric Fractional Brownian Motion Covariance

Let $\{S_t; t \in [0, T]\}$ represents the stock price process with its dynamic as given by

$$dS_t = \mu\, S_t dt + \sigma(Y_t) S_t dB_{H_1}(t), \tag{3.1}$$

where μ is mean of return, Y_t is a stochastic process, $B_{H_1}(t)$ is a fractional Brownian motion (FBM) with Hurst index H_1, and $\sigma(\cdot)$ is a deterministic function. In this work, we set $\sigma(Y_t) = Y_t$ suggested by Vasicek (1977), Chronopoulou and Viens (2012b), Comte and Renault (1998) and Bredit (1998), for simplicity of computations.

Let the dynamics of volatility Y_t be described by fractional Ornstein–Uhlenbeck (FOU) process which is the solution of SDE

$$dY_t = \alpha(m - Y_t)dt + \beta dB_{H_2}(t), \tag{3.2}$$

where α, β and m are constant parameters that represent mean reverting of volatility, volatility of volatility, and mean of volatility, respectively. $B_{H_2}(t)$ is another FBM which is independent from $B_{H_1}(t)$ where both H_1 and H_2 are greater than $\frac{1}{2}$ with assumption that this model exhibits long memory.

Applying Euler's discretization (in Lemma 2.3) scheme in Equation (3.2) yields

$$Y(t + \Delta t) = Y(t) + \alpha\big(m - Y(t)\big)\Delta t + \beta[B_{H_2}(t + \Delta t) - B_{H_2}(t)]. \tag{3.3}$$

Substituting $t = k\Delta t$ in Equation (3.3) gives

$$\tilde{Y}_{1+k} = \tilde{Y}_k + \alpha\big(m - \tilde{Y}_k\big)\Delta t + \beta\eta_{1+k}\,, \tag{3.4}$$

where $\tilde{Y}_k = Y(k\Delta t)$ and $\eta_{1+k} = B_{H_2}((1+k)\Delta t) - B_{H_2}(k\Delta t)$.

Following the iteration process and Cauchy criterion (in Lemma 2.2), Equation (3.4) can be restated as:

$$\tilde{Y}_{1+k} = \sum_{i=0}^{\infty}(1 - \alpha\,\Delta t)^i(\alpha m\Delta t + \beta\eta_{k+1-i}\,). \tag{3.5}$$

Based on Equation (3.5) and Lemma 2.1, the covariance of $\tilde{Y}$ can be expressed as:

$$\begin{aligned}\gamma_{\tilde{Y}}(n) &= \operatorname{cov}\big(\tilde{Y}_k, \tilde{Y}_{k-n}\big) \\ &= \operatorname{cov}\{\textstyle\sum_{i=0}^{\infty}(1 - \alpha\,\Delta t)^i(\alpha m\Delta t + \beta\eta_{k-i})\,, \sum_{j=0}^{\infty}(1 - \alpha m)^j\big(\alpha m\Delta t + \beta\eta_{k-n-j}\big)\} \\ &= \beta^2 \textstyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\,(1 - \alpha\,\Delta t)^{i+j}\operatorname{cov}(\eta_{k-i}\,, \eta_{k-n-j}). \end{aligned} \tag{3.6}$$

For sufficiently large L, Equation (3.6) can be written as:

$$\gamma_{\tilde{Y}(n)} = \beta^2 \sum_{i=0}^{L} \sum_{j=0}^{L} \{(1-\alpha\, \Delta t)^{i+j} \gamma_\eta\, (n+i-j)\}, \tag{3.7}$$

where by using Definition 1.12,

$$\begin{aligned}\gamma_\eta(n+i-j) &= \mathrm{cov}(\eta_{k-i}, \eta_{k-n-j}) \\ &= \tfrac{1}{2}(|(k-i)\Delta t|^{2H_2} + |(k-n-j)\Delta t|^{2H_2} \\ &\quad -2|(n+i-j)\Delta t|^{2H_2}\,)\end{aligned} \tag{3.8}$$

Applying Euler's discretization (in Lemma 2.3) scheme in Equation (3.1), gives

$$S(t+\Delta t) = S(t) + \mu S(t)\Delta t + S(t)Y_t\, [B_{H_1}(t+\Delta t) - B_{H_1}(t)]. \tag{3.9}$$

Dividing Equation (3.9) with $S(t)$, we obtain

$$X(t+\Delta t) = 1 + \mu\Delta t + Y_t\, [B_{H_1}(t+\Delta t) - B_{H_1}(t)], \tag{3.10}$$

where $X(t+\Delta t) = \frac{S(t+\Delta t)}{S(t)}$.

When $t = k\Delta t$, Equation (3.10) can be expressed by

$$X((1+k)\Delta t) = 1 + \mu\Delta t + Y_t\, [B_{H_1}((1+k)\Delta t) - B_{H_1}(k\Delta t)], \tag{3.11}$$

which can be further simplified as

$$\tilde{X}_{k+1} = 1 + \mu\Delta t + Y_{k+1}\, \xi_{k+1}, \tag{3.12}$$

where $\tilde{X}_{k+1} = X((1+k)\Delta t)$ and $\xi_{k+1} = B_{H_1}((1+k)\Delta t) - B_{H_1}(k\Delta t)$.

Based on Equation (3.12), Equation (3.5) and Lemma 2.1, the covariance of $\tilde{X}$ can be expressed as

$$\begin{aligned}\gamma_{\tilde{X}}(n) &= \mathrm{cov}(\tilde{X}_k, \tilde{X}_{k-n}) \\ &= \mathrm{cov}(1 + \mu\Delta t + Y_k\, \xi_k, 1 + \mu\Delta t + Y_{k-n}\, \xi_{k-n})\end{aligned}$$

$$= \mathrm{cov}(Y_k\, \xi_k, Y_{k-n}\xi_{k-n})$$

$$= \mathrm{cov}\{\, \xi_k \textstyle\sum_{i=0}^{\infty}(1-\alpha\, \Delta t)^i(\alpha m\Delta t + \beta\eta_{k-\mathrm{i}}),$$

$$\xi_{k-n} \textstyle\sum_{j=0}^{\infty}(1-\alpha\, \Delta t)^i \;\; \left(\alpha m\Delta t + \beta\eta_{k-n-\mathrm{j}}\right)\}$$

$$= \textstyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(1-\alpha\, \Delta t)^{i+j}\, \mathrm{cov}\big(\, \alpha m\Delta t\, \xi_k + \beta\eta_{k-i}\, \xi_k\; , \alpha m\Delta t\, \xi_{k-n} +$$

$$\beta\eta_{k-n-\mathrm{j}}\xi_{k-n}\big)$$

$$= \textstyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(1-\alpha\, \Delta t)^{i+j}\; \{\, (\alpha m\Delta t)^2\mathrm{cov}(\xi_k, \xi_{k-n}) \; +$$

$$\alpha m\Delta t\beta\; \mathrm{cov}\big(\xi_k, \eta_{k-n-j}\xi_{k-n}\big) + \quad \alpha m\Delta t\beta\; \mathrm{cov}(\eta_{k-i}\xi_k, \xi_{k-n}) +$$

$$\beta^2\, \mathrm{cov}\big(\eta_{k-i}\xi_k, \eta_{k-n-j}\xi_{k-n}\big)\}. \qquad (3.13)$$

In order to determine the four covariance functions in Equations (3.13), the following calculation are conducted using Lemma 2.1:

First,

$$\mathrm{cov}(\xi_k, \xi_{k-n}) = \gamma_\xi\, (n+i-j\,), \qquad (3.14)$$

where by using Definition 1.12

$$\gamma_\xi(n+i-j) = \tfrac{1}{2}(|(n+i-j+1)\Delta t|^{2H_1} + |(n+i-j-1)\Delta t|^{2H_1} -$$

$$2|(n+i-j)\Delta t|^{2H_1}). \qquad (3.15)$$

Second,

$$\mathrm{cov}\big(\xi_k, \eta_{k-n-j}\xi_{k-n}\big) = \mathrm{E}[(\xi_k - \mathrm{E}[\xi_k])\left(\eta_{k-n-j}\xi_{k-n} - \mathrm{E}\big[\eta_{k-n-j}\xi_{k-n}\,\big]\right]. \qquad (3.16)$$

However, $\mathrm{E}[\xi_k] = 0$ since $\xi_k \sim N(0, \Delta t^{H_1})$ and $\mathrm{E}[k-n-j\xi_{k-n}] = 0$, since η_{k-n-j} and ξ_{k-n} are independent. Thus, Equation (3.16) becomes

$$\text{cov}(\xi_k, \eta_{k-n-j}\xi_{k-n}) = \text{E}[(\xi_k \eta_{k-n-j}\xi_{k-n}]$$

$$= \text{E}[(\xi_k \xi_{k-n}]\text{E}[\eta_{k-n-j}] = 0 \quad (3.17)$$

Similarly,

$$\text{cov}(\eta_{k-i}\xi_k, \xi_{k-n}) = 0 \quad (3.18)$$

since $\text{E}[\xi_k] = 0$ since $\xi_k \sim N(0, \Delta t^{H_1})$ and $\text{E}[\eta_{k+i-2}\xi_k] = 0$.

Finally, by using Lemma 2.1,

$$\text{cov}(\eta_{k-i}\xi_k, \eta_{k-n-j}\xi_{k-n}) = \text{E}[(\eta_{k-i}\xi_k - \text{E}[\eta_{k-i}\xi_k]) \quad (\eta_{k-n-j}\xi_{k-n} - \text{E}[\eta_{k-n-j}\xi_{k-n}])]$$

$$= \text{E}[\, \xi_{k-n}\, \xi_k\, \eta_{k-n-\text{j}}\eta_{k-i}]$$

$$= \text{E}[\, \xi_{k-n}\xi_k]\text{E}[\eta_{k-n-\text{j}}\eta_{k-i}]$$

$$= \text{cov}(\, \xi_{k-n}, \xi_k)\text{cov}(\eta_{k-n-\text{j}}, \eta_{k-i})$$

$$= \gamma_\xi(n) \cdot \gamma_\eta(n + j - i). \quad (3.19)$$

Thus $\gamma_{\tilde{X}}(n)$ in Equation (3.13) can be expressed as

$$\gamma_{\tilde{X}}(n) = \gamma_\xi(n) \cdot \gamma_{\tilde{Y}}(n) + (\alpha m \Delta t)^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \{(1 - \alpha\, \Delta t)^{i+j}\, \gamma_\xi(n + j - i)\,\}. \quad (3.20)$$

For sufficiently large L, Equation (3.20) can be written as:

$$\gamma_{\tilde{X}}(n) = \gamma_\xi(n) \cdot \gamma_{\tilde{Y}}(n) + (\alpha m \Delta t)^2 \sum_{i=0}^{L} \sum_{j=0}^{L} \{(1 - \alpha\, \Delta t)^{i+j}\, \gamma_\xi(n + j - i)\,\}. \quad (3.21)$$

Once all covariances have been derived, we can now estimate the parameters involved in Equation (3.1) and Equation (3.2).

3.1.2 Estimating Geometric Fractional Brownian Motion Parameters

In this section, all parameters involved in Equation (3.1) and Equation (3.2) will be estimated by optimizing a likelihood function (see Definition 2.1). For n random variables, likelihood function is represented by

$$L(\alpha,\beta,\mu,m,H_1,H_2) = (2\pi)^{-T/2}(\det(\mathbf{\Sigma_T}))^{-1/2}\exp\{-\tfrac{1}{2}\left(\widetilde{\mathbf{X}}-\mathbf{M}\right)'\mathbf{\Sigma_T^{-1}}\left(\widetilde{\mathbf{X}}-\mathbf{M}\right), \tag{3.22}$$

where $\det(\cdot)$ represents determinant function and $\mathbf{\Sigma}_{\boldsymbol{T}}$ the covariance function under study.

By maximizing Equation (3.22), we will be able to find efficient estimators for all parameters involved in this function. However, it is difficult to analytically maximize the likelihood function because both of the covariance function and its inverse are complicated. Alternatively, the innovation algorithm (Brockwell and Davis, 1991) will be applied to cater such difficulty.

The following definitions and brief description of innovation algorithm for the construction of autocovariance function are needed in order to estimate GFBM parameters.

Definition 3.1 (Shumway and Stoffer, 2006): *Given data $\tilde{X}_1, \dots \dots, \tilde{X}_n$ the best linear predictor of $\tilde{X}_{n+m}$ for $m \geq 1$ is $\tilde{X}_{n+m}^n = \alpha_0 + \sum_{k=1}^n \alpha_k \tilde{X}_k$ and it can be found by solving*

$$\mathrm{E}\left[\left(\tilde{X}_{n+m} - \tilde{X}_{n+m}^n\right)\tilde{X}_k\right] = 0, \qquad k = 0,1,\dots\dots\dots,n \tag{3.23}$$

where $\tilde{X}_0 = 1$ for $\alpha_0, \dots, \alpha_n$.

Equation (3.23) is called prediction equations and can determine the involved coefficients $\{\alpha_0, \dots, \alpha_n\}$.

In the definition that follows, we present the definition of innovation algorithm.

Definition 3.2 (Brockwell and Davis, 1991): *The one–step–ahead predictors,* x_{t+1}^t *and their mean–squared errors,* P_{t+1}^t*, can be calculated iteratively as*

$$x_1^0 = 0, \qquad P_1^0 = \gamma(0)$$

$$x_{t+1}^t = \sum_{j=0}^{t-1} \theta_{tj} \left(x_{t+1-j} - x_{t+1-j}^{t-j}\right), \qquad t = 1,2, \dots$$

$$P_{t+1}^t = \gamma(0) - \sum_{j=0}^{t-1} \theta_{t,t-j}{}^2 P_{j+1}^j, \qquad t = 1,2, \dots$$

where for $j = 0,1, \dots, t-1$,

$$\theta_{t,t-j} = \frac{\left\{\gamma(t-j) - \sum_{k=0}^{j-1} \theta_{j,j-k} \theta_{t,t-k} P_{k+1}^k\right\}}{P_{j+1}^j}.$$

Now, we employ the innovation algorithm to derive autocovariance function and its inversion.

Suppose that $\boldsymbol{X} = \{\tilde{X}_1, \dots, \tilde{X}_n\}$ is a stationary process and

$$\tilde{X}_{n+1}^n = \phi_{n1} (\tilde{X}_n - M) + \dots + \phi_{nn} (\tilde{X}_1 - M) + M. \qquad (3.24)$$

By Definition 3.1, the coefficients $\{\phi_{n1}, \dots, \phi_{nn}\}$ satisfy

$$\mathrm{E}[\tilde{X}_{n+1} \tilde{X}_{n+1-k}] - \sum_{j=1}^{n} \phi_{nj} \, \mathrm{E}\left[\tilde{X}_{n+1-j} \tilde{X}_{n+1-k}\right] = 0.$$

But $\gamma_{\tilde{X}}(k) = \mathrm{E}[\tilde{X}_{n+1} \tilde{X}_{n+1-k}]$ and $\gamma_{\tilde{X}}(k-j) = \mathrm{E}\left[\tilde{X}_{n+1-j} \tilde{X}_{n+1-k}\right]$, hence

$$\sum_{j=1}^{n} \phi_{nj} \gamma_{\tilde{X}} (k-j) = \gamma_{\tilde{X}}(k), \quad \text{where} \quad k = 1, \dots, n. \qquad (3.25)$$

Equation (3.25) can be written in a matrix form as follows:

$$\boldsymbol{\Gamma_n}\boldsymbol{\phi_n} = \boldsymbol{\gamma_n}, \tag{3.26}$$

where $\boldsymbol{\Gamma_n} = \{\gamma(k-j)\}_{j,k=1}^{n}$ is an $n \times n$ matrix, $\boldsymbol{\phi_n} = (\phi_{n1}, \dots, \phi_{nn})'$ is an $n \times 1$ vector, and $\boldsymbol{\gamma_n} = (\gamma(1), \dots, \gamma(n))'$ is an $n \times 1$ vector.

From Equation (3.26), we get

$$\boldsymbol{\phi_n} = \boldsymbol{\Gamma_n^{-1}}\boldsymbol{\gamma_n}. \tag{3.27}$$

Let $\tilde{\boldsymbol{\gamma}}_{\boldsymbol{n-1}} = (\gamma_{\tilde{X}}(n-1), \dots, \gamma_{\tilde{X}}(1))'$, then $\mathbf{\Gamma_n}$ can be written as

$$\boldsymbol{\Gamma_n} = \begin{bmatrix} \mathbf{\Gamma_{n-1}} & \tilde{\boldsymbol{\gamma}}_{\mathbf{n-1}} \\ \tilde{\boldsymbol{\gamma}}'_{\mathbf{n-1}} & \tilde{\gamma}_{\tilde{X}}(0) \end{bmatrix}. \tag{3.28}$$

The inverse of the block matrix $\boldsymbol{\Gamma_n}$ is given via the following form

$$\boldsymbol{\Gamma_n^{-1}} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{\Gamma_{n-1}^{-1}}\tilde{\boldsymbol{\gamma}}_{\boldsymbol{n-1}} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma_{n-1}^{-1}} & \mathbf{0} \\ \mathbf{0} & (\gamma_{\tilde{X}}(0) - \tilde{\boldsymbol{\gamma}}'_{\mathbf{n-1}}\mathbf{\Gamma_{n-1}^{-1}}\tilde{\boldsymbol{\gamma}}_{\mathbf{n-1}})^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ -\tilde{\boldsymbol{\gamma}}'_{\mathbf{n-1}}\mathbf{\Gamma_{n-1}^{-1}} & 1 \end{bmatrix}. \tag{3.29}$$

Let $\varepsilon_n = \tilde{X}_n - \tilde{X}_n^{n-1}$, then

$$\varepsilon_n = (\widetilde{X}_n - M) - \textstyle\sum_{k=1}^{n} \phi_{nk}\,(\tilde{X}_{k} - M), \tag{3.30}$$

where $\varepsilon_n \sim N(0, \upsilon_n^2)$, ϕ_{nk} is autoregressive parameter, and υ_n is standard deviation.

Equation (3.30) can be written in a matrix form as follows

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_T \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -\phi_{11} & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 1 & 0 \\ -\phi_{(T-1)1} & -\phi_{(T-1)2} & \cdots & -\phi_{(T-1)(T-1)} & 1 \end{bmatrix} \begin{bmatrix} \tilde{X}_1 - M \\ \tilde{X}_2 - M \\ \vdots \\ \tilde{X}_T - M \end{bmatrix}, \tag{3.31}$$

or

$$\boldsymbol{\varepsilon} = \boldsymbol{A}\mathbf{X}, \tag{3.32}$$

where $\mathbf{X} = [\tilde{X}_1 - M, \ldots, \tilde{X}_T - M]'$ and $\boldsymbol{\varepsilon} = [\varepsilon_1, \ldots . \varepsilon_T]$. Note that $\varepsilon_T \sim N(0, \upsilon_T^2)$, where υ_T^2 is standard deviation given by

$$\upsilon_T^2 = \gamma(0) - \boldsymbol{\gamma}_{\boldsymbol{T}'} \boldsymbol{\Gamma}_{\boldsymbol{T}}^{-\mathbf{1}} \boldsymbol{\gamma}_{\boldsymbol{T}}. \tag{3.33}$$

Further, we will make use of Equation (3.32) to get

$$\mathbf{X} = \boldsymbol{A}^{-\mathbf{1}} \boldsymbol{\varepsilon}. \tag{3.34}$$

Now the autocovariance function becomes

$$\begin{aligned}\boldsymbol{\Sigma}_{\boldsymbol{T}} &= \operatorname{cov}(\mathbf{X}, \mathbf{X}) = \mathrm{E}[\mathbf{X}\mathbf{X}'] = \boldsymbol{A}^{-\mathbf{1}} \mathrm{E}[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'](\boldsymbol{A}^{-\mathbf{1}})' \\ &= \boldsymbol{A}^{-\mathbf{1}} \begin{bmatrix} \mathrm{E}[\varepsilon_1]^2 & 0 & \cdots & 0 \\ 0 & \mathrm{E}[\varepsilon_2]^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \mathrm{E}[\varepsilon_T]^2 \end{bmatrix} (\boldsymbol{A}^{-\mathbf{1}})'\end{aligned} \tag{3.35}$$

and

$$\boldsymbol{\Sigma}_{\boldsymbol{T}}^{-\mathbf{1}} = \boldsymbol{A}' \begin{bmatrix} \frac{1}{\upsilon_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\upsilon_2^2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \frac{1}{\upsilon_T^2} \end{bmatrix} \boldsymbol{A}. \tag{3.36}$$

Referring to Equation (3.35), the determinant of the autocovariance function $\boldsymbol{\Sigma}_{\boldsymbol{T}}$ is given by

$$\det(\boldsymbol{\Sigma}_{\mathbf{T}}) = \prod_{i=1}^{T} \mathrm{E}\,[\varepsilon_i]^2 = \prod_{i=1}^{T} \upsilon_i^2. \tag{3.37}$$

We are now ready to find likelihood function for Equation (3.1) and Equation (3.2). The likelihood function is now being transformed into the following optimization problem.

Problem P

Maximizes the cost function

$$L(\boldsymbol{\theta}), \qquad \text{where} \quad \boldsymbol{\theta} = (\alpha, \beta, \mu, m, H_1, H_2), \tag{3.38}$$

subject to

$$\mathrm{E}[\widetilde{\mathbf{X}} - M])^2 \geq 0 \tag{3.39}$$

and

$$\upsilon^2 \geq 0. \tag{3.40}$$

Now,
$$\begin{aligned} \mathrm{E}[\widetilde{\mathbf{X}} - M]^2 &= \mathrm{E}[(\widetilde{\mathbf{X}} - M)(\widetilde{\mathbf{X}} - M)] \\ &= \mathrm{E}[(\widetilde{\mathbf{X}} - \mathrm{E}[\widetilde{\mathbf{X}}])(\widetilde{\mathbf{X}} - \mathrm{E}[\widetilde{\mathbf{X}}])] \\ &= \mathrm{cov}(\widetilde{\mathbf{X}}, \widetilde{\mathbf{X}}) \\ &= \gamma_{\widetilde{\mathrm{X}}}(n) \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} \upsilon^2 &= \mathrm{var}(1 + \mu t + Y_t\, \xi_t) \\ &= \mathrm{var}\,(Y_t\, \xi_t) \\ &= \mathrm{E}[Y_t]^2\, \mathrm{var}(\xi_t) + \mathrm{E}[\xi_t]^2\, \mathrm{var}(Y_t) + \mathrm{var}(Y_t)\mathrm{var}(\xi_t). \end{aligned}$$

But $\mathrm{E}[\xi_t] = 0$, this implies

$$\upsilon^2 = \mathrm{E}[Y_t]^2\, \mathrm{var}(\xi_t) + \mathrm{var}(Y_t)\mathrm{var}(\xi_t). \tag{3.42}$$

Now we will compute $\mathrm{E}[Y_t]^2$, $\mathrm{var}(\xi_t)$ and $\mathrm{var}(Y_t)$ individually.

For $\mathrm{E}[Y_t]^2$ we have

$$\mathrm{E}[Y_t]^2 = \mathrm{E}\left[\sum_{i=0}^{\infty}(1-\alpha\,\Delta t)^i(\alpha m\Delta t+\beta\eta_{k+1-i})\right]^2$$

$$= \left[\sum_{i=0}^{\infty}(1-\alpha\,\Delta t)^i\,(\alpha m\Delta t+\beta\,\mathrm{E}[\eta_{k+1-i}])\right]^2$$

$$= \left[\sum_{i=0}^{\infty}(1-\alpha\,\Delta t)^i\,\alpha m\Delta t\right]^2. \quad (3.43)$$

For $\mathrm{var}(Y_t)$ we have

$$\mathrm{var}(Y_t) = \mathrm{var}\left(\sum_{i=0}^{\infty}(1-\alpha\,\Delta t)^i(\alpha m\Delta t+\beta\eta_{k+1-i})\right)$$

$$= \mathrm{var}\left(\sum_{i=0}^{\infty}(1-\alpha\,\Delta t)^i\,\beta\,\eta_{k+1-i}\right)$$

$$= \sum_{i=0}^{\infty}(1-\alpha\,\Delta t)^{2i}\,\beta^2\mathrm{var}(\eta_{k+1-i})$$

$$= \sum_{i=0}^{\infty}(1-\alpha\,\Delta t)^{2i}\,\beta^2(\Delta t)^{2H_2}. \quad (3.44)$$

For $\mathrm{var}(\xi_t)$, using definition FBM implies

$$\mathrm{var}(\xi_t) = (\Delta t)^{2H_1}. \quad (3.45)$$

Substituting Equations (3.43 - 3.45) in Equation (3.42) leads to

$$\upsilon^2 = \left\{\sum_{i=1}^{L}(1-\alpha\Delta t)^i(\alpha m\Delta t)\right\}^2\Delta t^{2H1} + \sum_{i=1}^{L}(1-\alpha\Delta t)^{2i}\beta^2\Delta t^{2(H2+H1)}. \quad (3.46)$$

The constraints in this optimization problem are too involved with covariance functions, thus making the standard optimization problem difficult to solve. In order to simplify this problem, we use the constraint transcription method described in Jennings and Teo (1990) as follows.

Maximizes the cost function:

$$L(\boldsymbol{\theta}) \tag{3.47}$$

subject to

$$g_i(\boldsymbol{\theta}) \leq 0, \qquad i = 1, 2, \tag{3.48}$$

where g_i are the constraints in the original **Problem *P***. For each $i = 1, 2$ we approximate g_i with $G_{i,\varepsilon}(\boldsymbol{\theta})$, where

$$G_{i,\varepsilon}(\boldsymbol{\theta}) = \begin{cases} g_i \;, & g_i > \varepsilon \\ \frac{(g_i+\varepsilon)^2}{4\varepsilon} \;, & -\varepsilon < g_i < \varepsilon \\ 0 \;, & g_i < -\varepsilon \end{cases} \tag{3.49}$$

for small number ε. We now include the approximate functions $G_{i,\varepsilon}$ into the cost function $L(\boldsymbol{\theta})$ to an appended cost function given in the following **Problem $\boldsymbol{P_{\varepsilon,\gamma}}$**.

Problem $\boldsymbol{P_{\varepsilon,\gamma}}$

$$\hat{L}(\boldsymbol{\theta}) = -L(\boldsymbol{\theta}) - \gamma \sum_{j=1}^{m} G_{j,\varepsilon}(\boldsymbol{\theta}), \tag{3.50}$$

where $\gamma > 0$ is a penalty parameter (Jennings and Teo, 1990). **Problem $\boldsymbol{P_{\varepsilon,\gamma}}$** now becomes an unconstraint optimization problem. For any given $\varepsilon > 0$, there exists $\gamma(\varepsilon)$ such that for $\gamma > \gamma(\varepsilon)$. The solution of **Problem $\boldsymbol{P_{\varepsilon,\gamma}}$** will satisfy the constraint of **Problem *P***. Let $\hat{\gamma}(\varepsilon)$ be such a γ for each $\varepsilon > 0$. Furthermore, the solution of **Problem $\boldsymbol{P_{\varepsilon,\hat{\gamma}}}$** converges to the solution of **Problem *P***.

3.2 Simulation Study

To examine the performance of the proposed models and its parameters estimation, we carried out a simulation study. We divide this section in three parts. First, we validate the calculation for small sample sizes of $n = 1, 2$ and 3. Second, the simulation study for sample sizes $n = 100, 200, 300, 400$ and 500 are conducted. Finally, we illustrate the results of this simulation study with discussion.

3.2.1 Validation of Calculations

In this subsection, we present illustration of calculation for small sample size (n), i.e. $n = 1$, 2 and 3 based on the method proposed in this thesis. Note that this subsection is provided to illustrate the computation for accompanied simulation study in cases of small size n.

Let the initial values be: $\alpha = 2$, $\beta = 1$, $\mu = 2.5$, $m = 3$, $H_1 = 0.8$, $H_2 = 0.7$, $L = 3$, $\Delta t = 0.2$ and $\varepsilon = 0.2$. These parameters were selected for the purpose of demonstration on how the calculations being carried out. Furthermore, the size of data $n \leq 3$ was considered to ensure whether the simulation can be implemented.

For $\boldsymbol{n = 1}$, we adopt $H_1 = 0.83044$ and $H_2 = 0.72149$ based on maximum likelihood estimation. Two vectors of fractional Gaussian noise (FGN) are applied as follows:

$$\text{FGN}(0.83044) = \{-1.33984, -0.49147\}$$

$$\text{FGN}(0.72149) = \{0.08139, 0.15535\}$$

Table 3.1 illustrates the conducted calculations for $n = 1$. We computed the covariance matrix and minimized unconstraint optimization problem in Equation (3.50) and obtained the estimates in Table 3.2

Table 3.1

Calculation of the proposed method for sample size $n = 1$

Variables	Equation No.	$n = 0$	$n = 1$
$\gamma_{\xi}(n)$	3.15	0.0690387	0.0401149
$\gamma_{\eta}(n)$	3.8	0.0980384	0.0352364
$\gamma_{\tilde{X}}(n)$	3.21	0.0869248	0.121256
$\gamma_{\tilde{Y}}(n)$	3.7	0.355015	0.233826
γ_n	3.26		0.004941
Γ_n^{-1}	3.29		11.50420
ϕ_n	3.27		0.05684
A	3.31		[1]
v_T^2	3.33		0.49588
Σ_3^{-1}	3.36		[76.316]
$\det(\Sigma_3)$	3.37		0.01310

Table 3.2

Parameters estimates for sample size $n = 1$.

Min. L	H_1	H_2	α	β	μ	m
−1.24941	0.83044	0.72149	−394.90791	345	1818	1401

For $n = 2$, we adopt $H_1 = 0.79482$ and $H_2 = 0.69832$ based on maximum likelihood estimation. Two vectors of FGN are applied as follows:

$$\mathrm{FGN}(0.79482) = \{0.76666, -0.77830, 0.56310\}$$

$$\mathrm{FGN}(0.69832) = \{-0.39120, -0.64992, 0.47713\}$$

Table 3.3 illustrates the calculations for $n = 2$. We computed the covariance matrix and minimized the unconstraint optimization problem in Equation (3.50) and obtained the estimates in Table 3.4:

Table 3.3

Calculation of the proposed method for sample size $n = 2$

Variables	**Equation No.**	$\boldsymbol{n = 0}$	$\boldsymbol{n = 1}$	$\boldsymbol{n = 2}$
$\gamma_{\xi}(n)$	3.15	0.0774272	0.0390903	0.0276576
$\gamma_{\eta}(n)$	3.8	0.105631	0.0334257	0.0196806
$\gamma_{\tilde{X}}(n)$	3.21	0.0147577	0.00494102	0.00144452
$\gamma_{\tilde{Y}}(n)$	3.7	0.0661567	0.0434324	0.0201428
γ_n	3.26		0.00494	$\begin{bmatrix} 0.00494 \\ 0.00144 \end{bmatrix}$
Γ_n^{-1}	3.29		67.761365	$\begin{bmatrix} 59.20648 & 25.55146 \\ -25.55146 & 76.31625 \end{bmatrix}$
ϕ_n	3.27		0.33481	$\begin{bmatrix} 0.32945 \\ -0.01601 \end{bmatrix}$
A	3.31			$\begin{bmatrix} 1 & 0 \\ -0.33481 & 1 \end{bmatrix}$
υ_T^2	3.33		0.01310	0.01315
Σ_3^{-1}	3.36			$\begin{bmatrix} 84.83886 & -25.45504 \\ -25.45504 & 76.02828 \end{bmatrix}$
$\det(\Sigma_3)$	3.37			0.00017

Table 3.4

Parameters estimates for sample size $n = 2$.

Min. L	$\boldsymbol{H_1}$	$\boldsymbol{H_2}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\mu}$	$\boldsymbol{m}$
3.2567	0.794817	0.069832	10.0836	12.0005	100	95.9803

For $n = 3$, we adopt $H_1 = 0.764098$ and $H_2 = 0.699992$ based on maximum likelihood estimation. Two vectors of FGN are applied as follows:

$$\mathrm{FGN}(0.764098) = \{1.36911, 1.79033, 0.910268, 1.34508\}$$

$$\mathrm{FGN}(0.699992) = \{1.01456, -1.27671, -1.22756, -1.73576\}$$

Table 3.5 illustrates calculations for $n = 3$. We computed the covariance matrix and minimized the unconstraint optimization problem in Equation (3.50) and obtained the estimates in Table 3.6.

Table 3.5

Calculation of the proposed method for sample size $n = 3$

Variables	**Equation No.**	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$\gamma_{\xi}(n)$	3.15	0.08540	0.03779	0.02526	0.02068
$\gamma_{\eta}(n)$	3.8	0.10506	0.03357	0.01983	0.015357
$\gamma_{\tilde{X}}(n)$	3.21	0.54359	0.16104	0.08684	0.05654
$\gamma_{\tilde{Y}}(n)$	3.7	0.44961	0.32832	0.22702	0.15805
γ_n	3.26		0.16104	$\begin{bmatrix} 0.16104 \\ 0.08684 \end{bmatrix}$	$\begin{bmatrix} 0.16104 \\ 0.08684 \\ 0.05654 \end{bmatrix}$
Γ_n^{-1}	3.29		1.83962	$\begin{bmatrix} 2.01661 & -0.59742 \\ -0.59742 & 2.01661 \end{bmatrix}$	$\begin{bmatrix} 1.86311 & 0.09665 & -0.32625 \\ -0.508391 & 1.99023 & -0.50839 \\ -0.14701 & -0.60504 & 2.04235 \end{bmatrix}$
ϕ_n	3.27		0.29625	$\begin{bmatrix} 0.27287 \\ 0.07891 \end{bmatrix}$	$\begin{bmatrix} 0.28998 \\ 0.06221 \\ 0.03926 \end{bmatrix}$
A	3.31				$\begin{bmatrix} 1 & 0 & 0 \\ -0.29625 & 1 & 0 \\ -0.27288 & -0.07891 & 1 \end{bmatrix}$
υ_T^2	3.33		0.49588	0.49280	0.489271
Σ_3^{-1}	3.36				$\begin{bmatrix} 2.34690 & -0.55715 & -0.55771 \\ -0.55715 & 2.04190 & -0.16127 \\ -0.55771 & -0.16127 & 2.04380 \end{bmatrix}$
$\det(\Sigma_3)$	3.37				0.11956

Table 3.6

Parameters estimates for sample size $n = 3$.

Min. L	H_1	H_2	α	β	μ	m
1.68680	0.764098	0.69999	1.4×10^{-7}	-5.0×10^{-7}	3.18572	-4.2×10^{-7}

We now extend the calculation for case $n = 100$, 200, 300, 400, and 500 in the subsection that follows.

3.2.2 Results of the Simulation Study

This subsection will carried out simulation study to examine the performance of the proposed method. Sample sizes of $n = 100$, 200, 300, 400, and 500 will be considered. This study can highlight the efficiency and weakness found in the proposed model. First, we describe in brief the methods selected in this study, which are simulated annealing algorithm, Nelder–Mead algorithm, differential evolution algorithm and random search algorithm. We select these four methods since each belongs to different optimization algorithm. Simulated annealing is from the family of heuristic optimization algorithm, Nelder–Mead from simplex method (or downhill simplex), differential evolution from evolutionary algorithm (evolution strategies), and random search from random search algorithm. All of these methods aim to provide the best search for global optimization.

3.2.2.1 Algorithm of Simulation

We summarized the procedures of the simulation algorithms in Table 3.7 as follows.

Table 3.7

Brief summary of the procedure for selected simulation algorithms

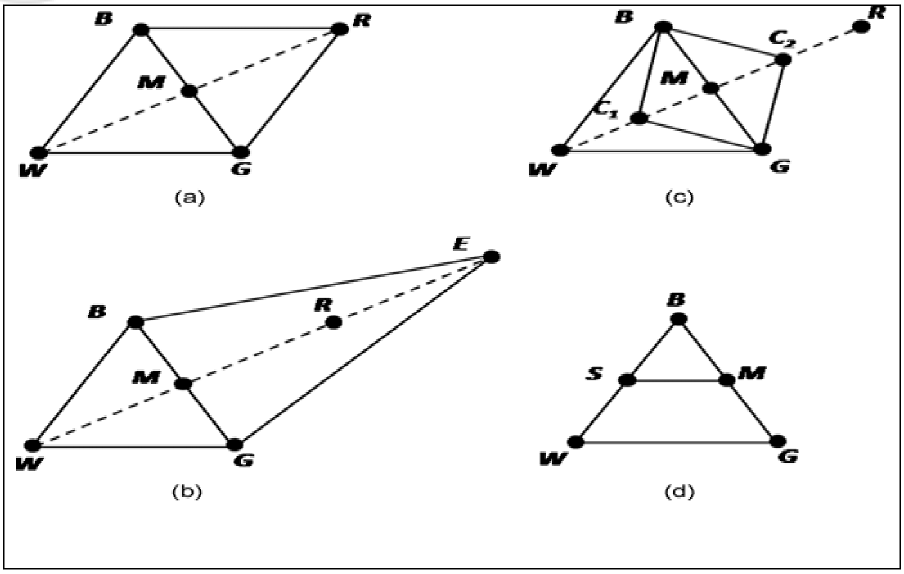
Algorithm	Brief Summary of the Procedure
Nelder–Mead **Nelder and Mead (1965)**	Nelder–Mead algorithm works by first assuming that we have two variables. In this case, the simplex is a triangle. Nelder–Mead method is a pattern search that compares the values of the function at the three vertices of a triangle. The vertex that has the largest value is considered the worst, and then rejected and replaced with a new vertex. So, there is a new triangle formed and the search continued. This process obtains a sequence of triangles which are not necessarily similar, for which the values of the function at the vertices get smaller and smaller. As a result, the triangles size is reduced, and the minimum point coordinates can be found. Now we can generalize this algorithm for n variables to find the minimum. Next figure explain the frame work of this algorithm.  (Wright, 2010)

Table 3.7 continued

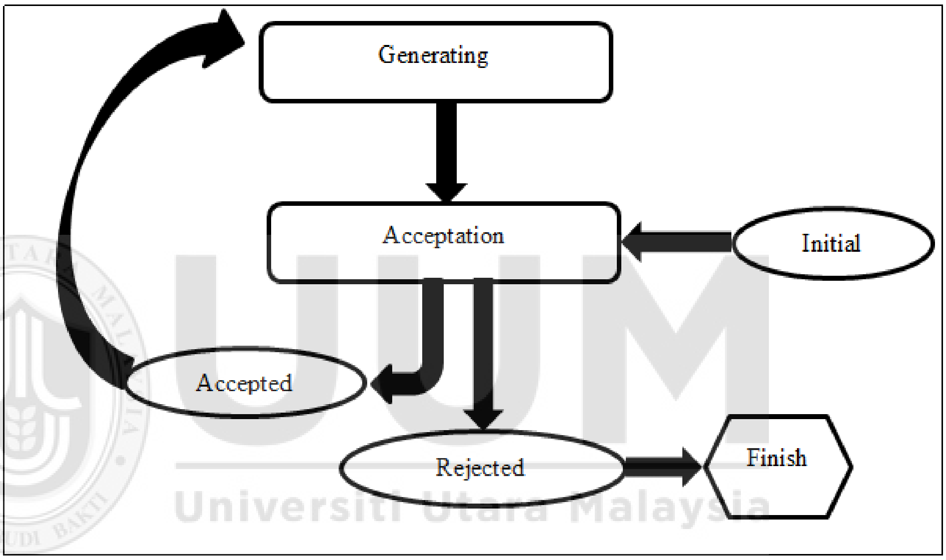
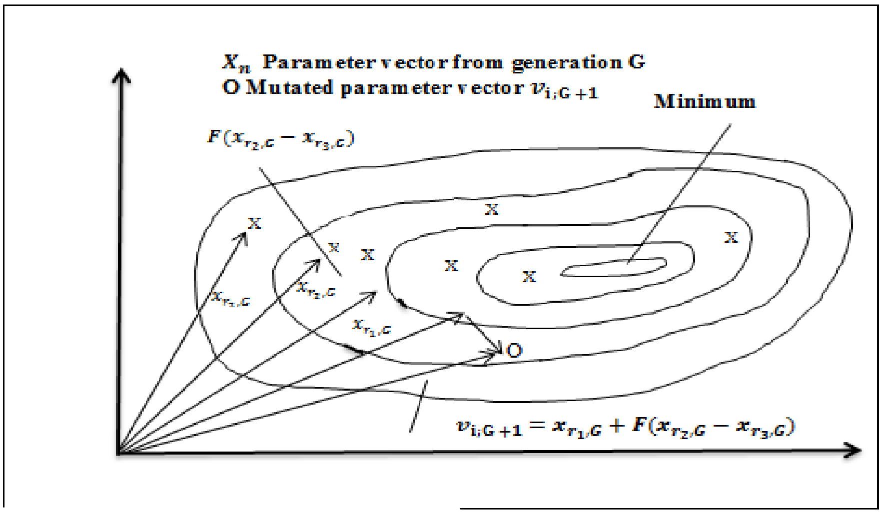
Algorithm	Brief Summary of the Procedure
Simulated Annealing **Weise (2009)**	Simulated Annealing start working at a given initial candidate point. Then, the generating distribution will use this initial point to generate possible minimums costs (or states) we need to explore. After that, the acceptance distribution will check these possible minimums to decide probabilistically whether to accept this new minimum or to reject it. The following figure illustrates the annealing algorithm framework. 
Random Search **Rastrigin (1964)**	Random search aims to create repeated jumps to better positions in the search–space, which are chosen from a hypersphere neighboring the current position. Meaning, the random search algorithm works by generating a population of random starting points and uses a local optimization method from each of the starting points to converge to a local minimum. The best local minimum is chosen to be the solution.

Table 3.7 continued

Algorithm	Brief Summary of the Procedure
Differential Evolution Storn and Price (1995; 1997).	Differential Evolution can be defined as a parallel direct search method which uses a whole parameter space. This algorithm assumes a uniform probability distribution for all random choices except if otherwise specified. In case a preliminary solution is available, the initial population might be generated by adding normally distributed random deviations to the nominal solution which is denoted by $x_{nom,0}$. This algorithm generates new parameter vectors by adding the weighted difference between the two population vectors to a third vector. This operation is called mutation. The parameters of the mutated vector are then mixed with the parameters of another predetermined vector, the target vector, to produce the so–called trial vector. The mixing of parameters is known as a "crossover" in the evolution strategies community. If the trial vector yields a lower cost function value than the target vector, the trial vector replaces the target vector in the following generation. This last operation is called selection. Next Figure illustrates an example of a two–dimensional cost function illustrating its contour lines and the process for generating $v_{i;G+1}$.



(Storn & Price, 1997)

The next subsection illustrates the results for the simulation study.

3.2.2.2 Numerical Results

We conducted a simulation study to investigate the performance of the proposed estimators by using Mathematica 10. In this study, we use Monte Carlo simulation (Nylund, Asparouhov and Muthén; 2007) with different sample sizes, specifically $n =$ 100, 200, 300, 400 and 500; with 100 replications were generated for each sample size to ensure that there was suffiecient reliability in the summary information calculated.

In order to choose initial parameters, the proposed methodology was applied several times with different values of initial parameters to estimate all parameters involved in the proposed model. All experiments results were convergent to the same estimators. Therefore, the initial parameters of the simulation were chosen to be close to the results of estimators. The selected initial values were $\varepsilon = 0.1$, $H_1 =$ 0.8, $H_2 = 0.7$, $\alpha = 0.02$, $\beta = 0.0001$, $\mu = 0.004$ and $m = 0.0003$. The values for H_1 and H_2 were estimated by maximum likelihood method, and we generated the data from Equation (3.5) and Equation (3.12).We then simulated time series from the proposed models in Equation (3.1) and Equation (3.2) to estimate parameters α, β, m and μ which were mean reverting, volatility of volatility, mean of volatility and drift respectively.

Four different algorithms which were simulated annealing, Nelder–Mead, random search and differential evolution were used to find the optimal parameters. The simulations were repeated one hundred times with different sample sizes $n = 100$, 200, 300, 400 and 500. The efficiency of the estimators is presented.

The standard simulation process took expensive computation time for large sample size n. For example, when $n = 100$, the required computations time exceeded 30 days per iteration. This scenario was unproductive as at least 100 iterations are needed to study its performance. Alternatively, the segmentation process is adopted as suggested by Keogh, Chu, Hart and Pazzani (2004). Interested reader can refer to Appendix A to see the flowchart of the program by using Mathematica 10 software detailing the algorithm of the proposed methodology. Appendix B and Appendix C which presented standard simulation program and simulation program by using segmentation technique respectively are also provided in this thesis.

Segmentation method divides original time series into a sequence of distinct segments with the aim of discovering their underlying properties (Keogh, Chu, Hart and Pazzani, 2004). Thus, we partitioned the series of FGN process into a sequence of its equal widths. Then we applied our methodology to each segment to get the estimations of all parameters involved. Finally, we evaluated the average for each estimator of the parameters.

Tables 3.8–3.11 show the findings of these simulations. The results of the simulations are considered in the following tables with their variance (var), bias (bias) and mean square error (MSE). These tables show results for different sample sizes via several optimization algorithms. The results from these tables indicate that the proposed methodology is efficient as most of the variances, biases, and mean square errors for the different algorithms are inversely proportional to the sample size n (meaning, when n gets larger, the values of biases and mean square errors becoming smaller).

Table 3.8

Simulation Based on Simulated Annealing Algorithm

Parameters	Error's Measurements Size	100	200	300	400	500
H_1		0.78242	0.78207	0.78453	0.78246	0.78210
	var	0.00041	0.00038	0.00031	0.00046	0.00030
	bias	0.01757	0.01793	0.01546	0.01753	0.01790
	MSE	0.00072	0.00070	0.00055	0.00077	0.00062
H_2		0.69162	0.69293	0.69265	0.68781	0.69279
	var	0.00038	0.00037	0.00039	0.00050	0.00032
	bias	0.08243	0.08207	0.08454	0.08246	0.08210
	MSE	0.00717	0.00711	0.00754	0.00730	0.00706
α		0.07649	0.02835	0.01626	0.01338	0.00834
	var	0.01520	0.00296	0.00110	0.00082	0.00027
	bias	0.05649	0.00835	0.00374	0.00662	0.01166
	MSE	0.01839	0.00289	0.00111	0.00086	0.00041
m		–0.00449	–0.00121	–0.00010	–0.00046	–0.00176
	var	0.00600	0.00077	0.00046	0.00021	0.00015
	bias	0.00479	0.00151	0.00040	0.00076	0.00206
	MSE	0.00602	0.00077	0.00046	0.00080	0.00015
β		–0.00481	–0.00154	0.00032	–0.00053	0.00039
	var	0.00091	0.00008	0.00002	0.00001	0.00002
	bias	0.00491	0.00164	0.00022	0.00063	0.00029
	MSE	0.00093	0.00008	0.00002	0.00001	0.00002
μ		0.05900	0.01463	0.00727	0.00445	0.00533
	var	0.01602	0.00434	0.00168	0.00065	0.00040
	bias	0.05500	0.01063	0.00327	0.00045	0.00133
	MSE	0.01905	0.00445	0.00169	0.00065	0.00040

Note: Initial values $H_1 = 0.8$, $H_2 = 0.7$, $\alpha = 0.02$, $\beta = 0.0001$, $\mu = 0.004$ and $m = 0.0003$

Table 3.9

Simulation Based on Nelder–Mead Algorithm

Parameters	Error's Measurements / Size	100	200	300	400	500
H_1		0.78056	0.78341	0.77845	0.78021	0.78681
	var	0.00031	0.00033	0.00040	0.00038	0.00046
	bias	0.01944	0.01659	0.02155	0.01980	0.01319
	MSE	0.00069	0.00061	0.00087	0.00077	0.00063
H_2		0.69298	0.69321	0.69465	0.69480	0.69186
	var	0.00048	0.00043	0.00034	0.00040	0.00048
	bias	0.08056	0.08341	0.07845	0.08021	0.08681
	MSE	0.00697	0.00738	0.00649	0.00683	0.00802
α		0.03282	0.02113	0.01118	0.00824	0.00735
	var	0.00187	0.00161	0.00018	0.00009	0.00007
	bias	0.01283	0.00113	0.00882	0.01176	0.01265
	MSE	0.00203	0.00161	0.00026	0.00023	0.00023
m		–0.01564	–0.00779	–0.00482	–0.00319	–0.00225
	var	0.00076	0.00016	0.00008	0.00003	0.00002
	bias	0.01594	0.00810	0.00512	0.00349	0.00255
	MSE	0.00101	0.00023	0.00011	0.00004	0.00003
β		0.00056	–0.00046	0.00035	1.4×10^{-6}	0.00003
	var	0.00001	0.00001	0.00001	1.7×10^{-10}	4.5×10^{-8}
	bias	0.00046	0.00056	0.00025	0.00010	0.00007
	MSE	0.00001	0.00001	0.00001	1.0×10^{-8}	5×10^{-8}
μ		0.04483	0.02525	0.01497	0.01145	0.00978
	var	0.00123	0.00063	0.00011	0.00005	0.00004
	bias	0.04083	0.02125	0.01097	0.00745	0.00578
	MSE	0.00290	0.00108	0.00023	0.00011	0.00007

Note: Initial values $H_1 = 0.8$, $H_2 = 0.7$, $\alpha = 0.02$, $\beta = 0.0001$, $\mu = 0.004$ and $m = 0.0003$

Table 3.10

Simulation Based on Random Search Algorithm

Parameters	Error's Measurements	Size 100	200	300	400	500
H_1		0.78273	0.78490	0.78577	0.78449	0.78469
	var	0.00028	0.00042	0.00034	0.00029	0.00049
	bias	0.01727	0.01510	0.01423	0.01551	0.01532
	MSE	0.00058	0.00065	0.00054	0.00053	0.00072
H_2		0.69673	0.69178	0.69287	0.69940	0.69373
	var	0.00033	0.00041	0.00041	0.00031	0.00045
	bias	0.08273	0.08491	0.08577	0.08449	0.08468
	MSE	0.00717	0.00762	0.00776	0.00745	0.00762
α		0.02561	0.01779	0.00684	0.00321	0.00389
	var	0.00272	0.00137	0.00025	0.00008	0.00007
	bias	0.00561	0.00221	0.01316	0.01679	0.01612
	MSE	0.00275	0.00137	0.00042	0.00036	0.00033
m		–0.01339	–0.00695	–0.00479	–0.00463	–0.00256
	var	0.00168	0.00037	0.00018	0.00013	0.00007
	bias	0.01369	0.00725	0.00510	0.00493	0.00286
	MSE	0.00187	0.00042	0.00021	0.00015	0.00007
β		$2.51\text{x}10^{-7}$	$3.6 \text{ x } 10^{-7}$	$-1.1 \text{ x } 10^{-6}$	$7.2 \text{ x } 10^{-7}$	$-7.9 \text{ x } 10^{-6}$
	var	$9.3\text{x}10^{-11}$	$1.2 \text{ x } 10^{-11}$	$1.0 \text{ x } 10^{-10}$	$7.5 \text{ x } 10^{-11}$	$4.2 \text{ x } 10^{-13}$
	bias	0.00010	0.00010	0.00010	0.00010	0.00010
	MSE	$1.0012\text{x}10^{-8}$	$1.0012\text{x}10^{-8}$	$1.01 \text{ x } 10^{-8}$	$1.0 \text{ x } 10^{-8}$	$1.0 \text{ x } 10^{-8}$
μ		0.00571	0.00568	0.00279	0.00150	0.00366
	var	0.00276	0.00070	0.00030	0.00012	0.00009
	bias	0.00171	0.00168	0.00121	0.00250	0.00035
	MSE	0.00276	0.00070	0.00030	0.00013	0.00009

Note: Initial values $H_1 = 0.8$, $H_2 = 0.7$, $\alpha = 0.02$, $\beta = 0.0001$, $\mu = 0.004$ and $m = 0.0003$

Table 3.11

Simulation Based on Differential Evolution Algorithm

Parameters	Error's Measurements	Size 100	200	300	400	500
H_1		0.78301	0.78440	0.78485	0.78463	0.77994
	var	0.00040	0.00035	0.00044	0.00046	0.00051
	bias	0.01699	0.01560	0.01515	0.01537	0.02006
	MSE	0.00069	0.00059	0.00067	0.00070	0.00091
H_2		0.69088	0.68792	0.69218	0.69405	0.69247
	var	0.00034	0.00044	0.00038	0.00040	0.00045
	bias	0.08301	0.08440	0.08485	0.08462	0.07994
	MSE	0.00723	0.00756	0.00758	0.00756	0.00684
α		0.18681	0.07082	0.04808	0.04731	0.03028
	var	0.03702	0.00636	0.00370	0.00225	0.00147
	bias	0.15747	0.05082	0.02808	0.02731	0.01028
	MSE	0.06181	0.00894	0.00449	0.00300	0.00158
m		0.00462	–0.00039	0.00027	0.00040	0.00025
	var	0.00281	0.00003	0.00004	0.00001	0.00004
	bias	0.00408	0.00069	0.00004	0.00010	0.00005
	MSE	0.00282	0.00003	0.00004	0.00001	0.00004
β		–0.00037	0.00014	–0.00001	0.00006	0.00006
	var	0.00004	6.8×10^{-6}	5.1×10^{-7}	8.4×10^{-7}	0.000001
	bias	0.00045	0.00004	0.00011	0.00004	0.00004
	MSE	0.00004	6.8×10^{-6}	5.0×10^{-7}	8.4×10^{-7}	0.000001
μ		0.20409	–0.03834	–0.03569	–0.02182	–0.01024
	var	1.0885	0.11378	0.85065	0.06178	0.03303
	bias	0.189881	0.04234	0.03969	0.02583	0.01424
	MSE	1.12445	0.11557	0.85223	0.06245	0.03323

Note: Initial values $H_1 = 0.8$, $H_2 = 0.7$, $\alpha = 0.02$, $\beta = 0.0001$, $\mu = 0.004$ and $m = 0.0003$

Tables 3.8–3.11 show the estimation of all parameters ($H_1, H_2, \alpha, \beta, \mu$ and m) by using four different algorithms-simulated annealing algorithm, Nelder–Mead algorithm, random search algorithm and differential evolution algorithm respectively to compare between these algorithms of optimization.

By focusing on the parameters that are estimated depending on our methodology in this work, i.e. α, β, m and μ, and observing the differences in values of var, bias and MSE between $n = 100$ and $n = 500$, one can see through the Tables 3.8-3.11 that all computed values of var, bias and MSE of the sample size $n = 500$ are less than the computed values of the sample size $n = 100$ for the same parameters. This showed that our methodology is satisfactory and acceptable. While there is no consistency in behavior of the differences in the computed values of var, bias and MSE between the sample sizes $n = 100$ and $n = 500$ with respect to the parameters H_1 and H_2 which were estimated by maximum likelihood method.

From the perspective of increasing and decreasing of the values of estimated parameters, one can see that all values of the mean reverting parameter α are decreasing through all optimization algorithms. While the values of the mean of volatility parameter m are increasing through Nelder-Mead and random search algorithms only while the value of m is fluctuating through simulated annealing and differential evolution algorithms. Further, the values of the mean return parameter μ are almost increasing through differential evolution and Nelder-Mead algorithms while it is almost decreasing through random search and simulated annealing algorithms. Finally, the values of the volatility of volatility parameter β are fluctuating through all algorithms.

Tables 3.12–3.15 show simulation of the parameters α, β, m and μ separately for different sizes n and methods by highlighting the smallest values of variance (in red color), bias (in green color) and mean sequare error MSE (in blue color) to determine the best algorithm on each parameter.

Table 3.12

Simulation of mean reverting parameter (α) for different sizes and different methods, with best variance, bias, and mean square error in $\{\cdot\}$, $(\cdot)$ *and* $[\cdot]$ *respectively.*

n	***Parameters***	***Simulated Annealing***	***Nelder–Mead***	***Random Search***	***Differential – Evolution***
	α	0.07649	0.03282	0.02561	0.18681
	var	0.01520	**{0.00187}**	0.00272	0.03702
100	bias	0.05649	0.01283	**(0.00561)**	0.15747
	MSE	0.01839	**[0.00203]**	0.00275	0.06181
	α	0.02835	0.02113	0.01779	0.07082
	var	0.00296	0.00161	**{0.00137}**	0.00636
200	bias	0.00835	**(0.00113)**	0.00221	0.05082
	MSE	0.00289	0.00161	**[0.00137]**	0.00894
	α	0.01626	0.01118	0.00684	0.04808
	var	0.00110	**{0.00018}**	0.00025	0.00370
300	bias	**(0.00374)**	0.00882	0.01316	0.02808
	MSE	0.00111	**[0.00026]**	0.00042	0.00449
	α	0.01338	0.00824	0.00321	0.04731
	var	0.00082	0.00009	**{0.00008}**	0.00225
400	bias	**(0.00662)**	0.01176	0.01679	0.02731
	MSE	0.00086	**[0.00023]**	0.00036	0.00300
	α	0.00834	0.00735	0.00389	0.03028
	var	0.00027	**{0.00007}**	0.00007	0.00147
500	bias	0.01166	0.01265	0.01612	**(0.01028)**
	MSE	0.00041	**[0.00023]**	0.00033	0.00158

Table 3.13

Simulation of mean of volatility parameter (m) for different sizes and different methods, with best variance, bias, and mean square error in $\{\cdot\}$, $(\cdot)$ *and* $[\cdot]$ *respectively.*

n	***Parameters***	***Simulated Annealing***	***Nelder–Mead***	***Random Search***	***Differential – Evolution***
	m	–0.00449	–0.01564	–0.01339	0.00462
	var	0.00600	**{0.00076}**	0.00168	0.00281
100	bias	0.00479	0.01594	0.01369	**(0.00408)**
	MSE	0.00602	**[0.00101]**	0.00187	0.00282
	m	–0.00121	–0.00779	–0.00695	–0.00039
	var	0.00077	0.00016	0.00037	**{0.00003}**
200	bias	0.00151	0.00810	0.00725	**(0.00069)**
	MSE	0.00077	0.00023	0.00042	**[0.00003]**
	m	–0.00010	–0.00482	–0.00479	0.00027
	var	0.00046	0.00008	0.00018	**{0.00004}**
300	bias	0.00040	0.00512	0.00510	**(0.00004)**
	MSE	0.00046	0.00011	0.00021	**[0.00004]**
	m	–0.00046	–0.00319	–0.00463	0.00040
	var	0.00021	0.00003	0.00013	**{0.00001}**
400	bias	0.00076	0.00349	0.00493	**(0.00010)**
	MSE	0.00080	0.00004	0.00015	**[0.00001]**
	m	–0.00176	–0.00225	–0.00256	0.00025
	var	0.00015	**{0.00002}**	0.00007	0.00004
500	bias	0.00206	0.00255	0.00286	**(0.00005)**
	MSE	0.00015	**[0.00003]**	0.00007	0.00004

Table 3.14

Simulation of volatility of volatility parameter (β) *for different sizes and different methods, with best variance, bias, and mean square error in* $\{\cdot\}$, $(\cdot)$ *and* $[\cdot]$ *respectively.*

n	***Parameters***	***Simulated Annealing***	***Nelder–Mead***	***Random Search***	***Differential – Evolution***
	β	–0.00481	0.00056	2.51x10^{-7}	–0.00037
	var	0.00091	0.00001	**{9.3x10^{-11}}**	0.00004
100	bias	0.00491	0.00046	**(0.00010)**	0.00045
	MSE	0.00093	0.00001	**[1.0x10^{-8}]**	0.00004
	β	–0.00154	–0.00046	3.6 x 10^{-7}	0.00014
	var	0.00008	0.00001	**{1.2 x 10^{-11}}**	6.8 x 10–6
200	bias	0.00164	0.00056	0.00010	**(0.00004)**
	MSE	0.00008	0.00001	**[1.0x10^{-8}]**	6.8 x 10^{-6}
	β	0.00032	0.00035	–1.1 x 10^{-6}	–0.00001
	var	0.00002	0.00001	**{1.0 x 10^{-10}}**	5.1 x 10–7
300	bias	0.00022	0.00025	**(0.00010)**	0.00011
	MSE	0.00002	0.00001	**[1.0x10^{-8}]**	5.0 x 10–7
	β	–0.00053	1.4 x 10^{-6}	7.2 x 10^{-7}	0.00006
	var	0.00001	1.7 x 10^{-10}	**{7.5 x 10^{-11}}**	8.4 x 10–7
400	bias	0.00063	0.00010	**(0.00010)**	0.00004
	MSE	0.00001	1.0x10^{-8}	**[1.0x10^{-8}]**	8.4 x 10–7
	β	0.00039	0.00003	–7.9 x 10^{-6}	0.00006
	var	0.00002	4.5 x 10^{-8}	**{4.2 x 10^{-13}}**	0.000001
500	bias	0.00029	0.00007	0.00010	**(0.00004)**
	MSE	0.00002	5 x 10^{-8}	**[1.0x10^{-8}]**	0.000001

Table 3.15

Simulation of drift parameter (μ) *for different sizes and different methods, with best variance, bias, and mean square error in* $\{\cdot\}$, $(\cdot)$ *and* $[\cdot]$ *respectively.*

n	***Parameters***	***Simulated Annealing***	***Nelder–Mead***	***Random Search***	***Differential – Evolution***
	μ	0.05900	0.04483	0.00571	0.20409
	var	0.01602	**{0.00123}**	0.00276	1.0885
100	bias	0.05500	0.04083	**(0.00171)**	0.189881
	MSE	0.01905	0.00290	**[0.00276]**	1.12445
	μ	0.01463	0.02525	0.00568	–0.03834
	var	0.00434	**{0.00063}**	0.00070	0.11378
200	bias	0.01063	0.02125	**(0.00168)**	0.04234
	MSE	0.00445	0.00108	**[0.00070]**	0.11557
	μ	0.00727	0.01497	0.00279	–0.03569
	var	0.00168	**{0.00011}**	0.00030	0.85065
300	bias	0.00327	0.01097	**(0.00121)**	0.03969
	MSE	0.00169	**[0.00023]**	0.00030	0.85223
	μ	0.00445	0.01145	0.00150	–0.02182
	var	0.00065	**{0.00005}**	0.00012	0.06178
400	bias	**(0.00045)**	0.00745	0.00250	0.02583
	MSE	0.00065	**[0.00011]**	0.00013	0.06245
	μ	0.00533	0.00978	0.00366	–0.01024
	var	0.00040	**{0.00004}**	0.00009	0.03303
500	bias	0.00133	0.00578	**(0.00035)**	0.01424
	MSE	0.00040	**[0.00007]**	0.00009	0.03323

From Table 3.12, we can see that the mean reverting parameter α shows that the best values of MSE and variance can be obtained by using Nelder–Mead algorithm. These values indicate that Nelder–Mead algorithm is the best algorithm to estimate α. Further, Table 3.12 shows that the differential evolution algorithm provides the largest value of α for all sample sizes. Meanwhile, the Random search algorithm provides the smallest value for all sample sizes.

In the case of mean of volatility m, Table 3.13 shows that the differential evolution algorithm produces smallest bias for all n sizes together with the smallest value of MSE, and variance for sample sizes $n = 200$, 300 and 400. However, Nelder-Mead algorithm has the smallest values of MSE and variance for $n = 100$ and 500. Moreover, Table 3.13 shows that the differential evolution algorithm provides the largest value of m for the majority of sample sizes. While, the smaller value of m obtained by Nelder-Mead algorithm for the small sample size and obtained by random search algorithm for big sample sizes.

Random search algorithm is the best when considering case of volatility of volatility β as it presents the smallest value of MSE, variance, and bias for all n sizes as shown in Table 3.14. But there is no clear trajectory to determine any algorithm that is able to provide the largest or smallest value of β.

For drift parameter μ in Table 3.15, we can see that Nelder–Mead algorithm always has the smallest value of variance according to all sizes. MSE of random search is the smallest for small sample sizes $n = 100$ and 200 while the Nelder-Mead algorithm presents the smallest MSE for the remaining sizes. Random search has less bias for sample size $n = 100$, 200 and 300. Furthermore, Table 3.15 shows that the Nelder-Mead algorithm provides the largest value of μ for most sample sizes. Whereas, the differential evolution algorithm provides the smallest value of μ for most sample sizes.

In general, reader can observe that Nelder-Mead presents the best algorithm since it provides the smallest values of MSE, in particular for cases with large sizes, while random search algorithm performs second best. We can observe that differential evolution algorithm is better than the simulated annealing algorithm with respect to m and β, but the situation is different for parameters α and σ.

Outcomes of this simulation show that our methodology is promising in obtaining statistical efficient estimators for GFBM model with long memory SV model obeys fractional Ornstein–Uhlenbeck process.

3.3 Discussion

In this chapter, the development of a new GFBM model with long memory stochastic volatility that obeys the fractional Ornstein-Uhlenbeck process was presented. By using Euler's discretization, Cauchy criterion, and covariance functions of fractional Brownian motion, the model was simplified, and the covariance functions $\gamma_\xi, \gamma_\eta, \gamma_X$ and γ_Y were computed.

Furthermore, the parameters involved in this model were estimated by utilizing innovation algorithm as an efficient alternative to standard maximization problem that is proven complicated. Then the complex likelihood function was then transformed into an optimization problem with known constraints. This problem was later transformed into unconditional optimization problem by using constraints transcription method. We solved this unconstrained optimization problem by using four different types of optimization algorithms; Simulated Annealing, Nelder–Mead, Differential Evolution and Random Search.

The simulation was conducted by applying segmentation method to the data. This approach has been used by Keogh, Chu, Hart and Pazzani (2004). The series of fractional Gaussian noise were partitioned into several segments of equal width. Then, the proposed methodology was applied to each segment to get the estimations of all the parameters. Finally, we evaluated the average for each estimator of the parameters. This approach is able to minimize substantial amount of time.

We presented in this chapter the findings of our method in in Tables 3.8-3.15. Most of the variances, biases and mean square errors are within acceptable range of tolerance. The outcomes of this simulation showed that our methodology is promising in obtaining statistically efficient estimators for GFBM with long memory SV model that obey fractional Ornstein–Uhlenbeck process.

CHAPTER FOUR

EXISTENCE AND UNIQUENESS SOLUTION OF FRACTIONAL STOCHASTIC DIFFERENTIAL MODEL

In this Chapter, we will prove for the existence and uniqueness of the solution of GFBM model, and subsequently generalize the proving for a class of fractional SDE model. These two theorems are able to conclude that there exists only one solution to a differential equation which satisfies a given initial conditions as described by Momani, Arqub, Al-Mezel, and Kutbi (2016). We begin the proving by introducing related definitions, lemmas and theorems that will be of use afterward.

4.1 Preliminaries Definitions and Theorems

Some necessary theorems and definitions are presented to show the existence and uniqueness of the solution for SDE driven by fractional Brownian motion (FBM).

Definition 4.1 (Oksendal, 2000): *Function $f: A \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$ satisfies a Lipschitz condition on the closed interval $[a;\ b]$ if there is a constant K such that*

$$|f(x) - f(y)| \leq K|x - y|, \tag{4.1}$$

for every pair of points x and $y \in A$. Further, function f is called locally Lipschitz if for each $x_0 \in A$ there exists constant $M > 0$ and $\delta > 0$ such that $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| \leq M|x - x_0|$.

Lemma 4.1 (Lin and Bai, 2011): *Let X be a random variable with $\mathrm{E}[X] < \infty$. For any convex function $f(X)$ such that $\mathrm{E}[|f(X)|] < \infty$ then*

$$\mathrm{E}[f(X)] \leq f(\mathrm{E}[X]). \tag{4.2}$$

Lemma 4.2 (Oguntuase, 2001): *Let $f(t)$ and $u(t)$ are nonnegative continuous functions on $[0,T]$ such that $f(t) \leq C + \int_0^t f(s)u(s)ds$ for $0 \leq t \leq T$ and for some nonnegative constant C then*

$$f(t) \leq C \exp[\int_0^t u(s)\, ds]. \tag{4.3}$$

Note: If $u(t) = A$ in Equation (4.3), we get $f(t) \leq C \exp[At]$.

Lemma 4.3 (Oksendal, 2000): *If $f(t,w)$ is bounded and elementary then*

$$\mathrm{E}[(\int_s^t f(t,w)dW_s)^2] = \mathrm{E}\left[\int_s^t f^2\ (s,t)\, dt\right], \tag{4.4}$$

where W_s is a standard Brownian motion.

Lemma 4.4 (Oksendal, 2000): *If M_t is a martingale such that $t \mapsto M_t(w)$ is continuous almost surely, then for all $p \geq 1, T \geq 0$ and all $\lambda > 0$,*

$$P[\sup_{0\leq t\leq T}|M_t| \geq \lambda] \leq \frac{1}{\lambda^p}\ \mathrm{E}[|M_t|^p]. \tag{4.5}$$

Lemma 4.5 (Chandra, 2012): *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_k\}_{k=1}^{\infty}$ be a sequence of events in $\mathcal{F}$. If $\sum_{k=1}^{\infty} P(A_k)$ converges then $P(\lim_{k\to\infty} \sup A_k)$ =0. If the events A_k are independent and $\sum_{k=1}^{\infty} P(A_k) = \infty$ then $P(\lim_{k\to\infty} \sup A_k) = 1$.*

Lemma 4.6 (Knapp, 2005): *If S is a measurable set and if $\{Y_n\}$ is sequence of non–negative measurable functions. Then,*

$$\int_S \lim_{n\to\infty} \inf Y_n\, dM \leq \lim_{n\to\infty} \inf \int_S Y_n\ dM. \tag{4.6}$$

In particular, if $Y(s) = \lim_{n\to\infty} \inf f_n(s)$, for all $s \in S$. Then Y is measurable and

$$\int_S Y\, dM \leq \lim_{n\to\infty} \inf \int_S f_n\,(s)\, dM \tag{4.7}$$

Lemma 4.7 (Oksendal, 2000): *Let* $T > 0$ *and* $b(\cdot,\cdot):[0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ *and* $\sigma(\cdot,\cdot):[0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ *be measurable functions satisfying the following* $|b(t,x)| + |\sigma(t,x)| \le C(1+|x|)$, *for some constant* C, $x \in \mathbb{R}^n$, $t \in [0,T]$ *and* $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K\,|x-y|\,)$;$x, y \in \mathbb{R}^n$, $t \in [0,T]$ *for some constant* K. *Let* Z *be a random variable which is independent of the* σ *−algebra* $\mathcal{F}$ *generated by* $w_s, s \ge 0$ *and such that* $\mathrm{E}[|Z|^2] < \infty$. *Then stochastic differential equation*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where W_t *represent Brownian motion process, has a unique solution* $X_t(w)$ *in* $t \in [0,T]$ *with property that* $X_t(w)$ *is adapted to the filtration* $\mathcal{M}$ *generated by* Z *and* $W_s(\cdot)$; $s<t$ *and* $\mathrm{E}\left[\int_0^T |X_t|^2 dt\right] < \infty$.

Approximation Approach

In terms of a practical approach to the theory, Thao (2006; 2014), Thao and Christine (2003), Thao, Sattayatham and Plienpanich (2008), Plienpanich, Sattayatham and Thao (2009), Dung (2011), Dung and Thao (2010), Tein (2013 a; 2013 b), and Intarasit and Sattayatham (2010) studied fractional stochastics driven by FBM of the Liouville form (LFBM) based on a crucial fact that any LFBM can be approximated in the space $L^2(\Omega, F, P)$ by semimartingales,

$$B_t^H = \int_0^t (t-s)^\alpha dW_s. \quad (4.8)$$

Also, Mozet and Nualart (2000) introduced the semimartingale

$$B_t^{H,\epsilon} = \int_0^t (t-s+\epsilon)^\alpha\, dW_s\,, \quad (4.9)$$

where $\alpha = H - 1/2$ and W_t is a standard Brownian motion. Furthermore

$$dB_t^{H,\epsilon} = \alpha \emptyset_t^{\epsilon}\, dt + \epsilon^{\alpha}\, dW_s\,, \tag{4.10}$$

where

$$\emptyset_t^{\epsilon} = \int_0^t (t - s + \epsilon)^{\alpha - 1}\, dW_s\,. \tag{4.11}$$

Thao (2006) proved that $B_t^{H,\epsilon}$ converges uniformly to B_t^H in $t \in [0, T]$, further lead to $\int_0^t f(s,w) dB_s^{H,\epsilon}$ converges to $\int_0^t f(s,w) dB_s^H$. Thao (2006) further compute the integral of $\emptyset_s^{\epsilon}$ as:

$$\begin{aligned} \int_0^t \emptyset_s^{\epsilon} ds &= \int_0^t \int_0^s (s - u + \epsilon)^{\alpha-1} ds\, dW_s \\ &= \int_0^t \left[\int_0^s (s - u + \epsilon)^{\alpha-1}\, ds \right] dW_s \\ &= \frac{1}{\alpha} \left[\int_0^t (t - s + \epsilon)^{\alpha}\, dW_u - \epsilon^{\alpha} W_t \right] \\ &= \frac{1}{\alpha} [B_t^{H,\epsilon} - \epsilon^{\alpha} W_t]. \end{aligned} \tag{4.12}$$

We are now ready to prove the theorem of existence and uniqueness for the new GFBM model.

4.2 Existence and Uniqueness Solution of Geometric Fractional Brownian Motion Model

We bring reader's attention to Definition 1.15 in Chapter One to revisit the solution of GFBM model with volatility assumption is assumed constant. In this section, we will further construct the existence and uniqueness theorem of the solution of our proposed GFBM model when the volatility is assumed to be stochastic with its function in time t by using approximation approach in $L^2(\Omega, F, P)$ in Thao (2013).

Let $(\Omega, \mathcal{F}, P)$ be a probability space where $\mathcal{F}$ is the σ- algebra of set Ω, and P is a probability measure. Let $\{X_t\}_{t\in[0,\infty]}$ be a stochastic process defined on $(\Omega, \mathcal{F}, P)$ such that for all $t \in [0, \infty]$ we have a random variable $w \in \Omega$ where $w \mapsto X_t(w)$ is a continuous function. $X_t(w)$ represents the result at time t of the experiment w.We can represent $X_t(w) = X(t, w)$ and define a function $T \times \Omega \mapsto \mathbb{R}^n$ as $(t, w) \mapsto X(t, w)$. Let $x \in \mathbb{R}^n$ then $x \mapsto (\sum_{k=1}^{n} x_k{}^2)^{½} \equiv \|x\|_{L^2}$ is the Euclidean norm.

Theorem 4.1: *Let $T > 0$ and $b(\cdot,\cdot) = \mu X_t: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot) = \sigma(t)X_t: [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying*

$$|\mu X_t(x) - \mu X_t(y)| + |\sigma(t)X_t(x) - \sigma(t)X_t(y)| \leq K|x - y|, \tag{4.13}$$

where $x, y \in \mathbb{R}^n, t \in [0, T]$ for some constant K and

$$|\mu X_t| + |\sigma(t)X_t| \leq C(1 + |x|) \text{ for some constant } C, x \in \mathbb{R}^n, t \in [0, T]. \tag{4.14}$$

Then GFBM given by

$$X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma(s)X_s dB_s^H, \tag{4.15}$$

has unique solution in $t \in [0, T]$ where X_0 is a given random variable such that $\mathrm{E}[X_0^2] < \infty$ *and B_t^H is a FBM.*

Proof:

First: The proof of the uniqueness for GFBM. The condition (4.13) guarantees that Equation (4.15) has a unique solution. The uniqueness means that if X_t^1 and X_t^2 are two t-continuous processes satisfying the hypothesis of the Theorem 4.1 then X_t^1 = X_t^2 for all $t \leq T$, almost surely.

Let X_t^1 and X_t^2 are two solutions of Equation (4.15) with equal initial value X_0. Suppose that

$$a = a(s,w) = \mu(X_s^1 - X_s^2)$$

and

$$\gamma = \gamma(s,w) = \sigma(s)(X_s^1 - X_s^2).$$

Define $\tau_n^1 = \inf \{ t \geq 0 \mid |X_t^1| \geq n\}$ and $\tau_n^2 = \inf \{ t \geq 0 \mid |X_t^2| \geq n\}$, and let $S_n = \min\{\tau_n^1, \tau_n^2\}$. We need to prove that $\mathrm{E}[|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2|^2] \to 0$ for all $t \in [0,T]$.

Now for all $t \in [0,T]$,

$$\mathrm{E}[|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2|^2] = \mathrm{E}\Big[| \int_0^{t\wedge S_n} \mu X_s^1 ds + \int_0^{t\wedge S_n} \sigma(s) X_s^1 \,) dB_s^H - \int_0^{t\wedge S_n} \mu X_s^2 ds - \int_0^{t\wedge S_n} \sigma(s) X_s^2 \,) dB_s^H \,|^2 \,\Big]$$

$$= \mathrm{E}\left[\left(\int_0^{t\wedge S_n} a\, ds + \int_0^{t\wedge S_n} \gamma\, dB_s^H \right)^2\right]. \tag{4.16}$$

Since, $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, leading to the last expression will be

$$\mathrm{E}\left[|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2|^2\right] \leq \mathrm{E}\left[2(\int_0^{t\wedge S_n} a\, ds\,)^2 + 2\left(\int_0^{t\wedge S_n} \gamma\, dB_s^H \right)^2 \right]. \tag{4.17}$$

By using Lemma 4.1, Equation (4.17) will become

$$\mathrm{E}\left[|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2|^2\right] \leq 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} a\, ds\right)^2\right] + 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\, dB_s^H \right)^2\right]. \tag{4.18}$$

However, $\int_0^t f(s,w) dB_s^{H,\epsilon}$ converge to $\int_0^t f(s,w) dB_s^H$, thus Equation (4.18) is approximately equal to

$$\mathrm{E}\left[\left|X^1_{t\wedge S_n} - X^2_{t\wedge S_n}\right|^2\right] \leq 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} a\, ds\right)^2\right] + 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\, dB_s^{H,\epsilon}\right)^2\right]. \qquad (4.19)$$

We substitute Equation (4.10) into Equation (4.19) to get

$$\mathrm{E}\left[\left|X^1_{t\wedge S_n} - X^2_{t\wedge S_n}\right|^2\right] \leq 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} a\, ds\right)^2\right] + 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\, (\alpha \emptyset_s^\epsilon ds + \epsilon^\alpha dW_s)\right)^2\right]$$

$$\leq 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} a\, ds\right)^2\right] + 2\mathrm{E}\left[\left(\alpha \int_0^{t\wedge S_n} \gamma\ \emptyset_s^\epsilon ds + \epsilon^\alpha \int_0^{t\wedge S_n} \gamma\ dW_s\right)^2\right]$$

$$\leq 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} a\, ds\right)^2\right] + 2\mathrm{E}\left[2\left(\alpha \int_0^{t\wedge S_n} \gamma\ \emptyset_s^\epsilon ds)\right)^2 + 2\left(\epsilon^\alpha \int_0^{t\wedge S_n} \gamma\ dW_s\right)^2\right]$$

$$\leq 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} a\, ds\right)^2\right] + 4\alpha^2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\ \emptyset_s^\epsilon ds\right)^2\right] + 4\epsilon^{2\alpha}\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\, dW_s\right)^2\right].$$

By Lemma 4.3,

$$\mathrm{E}\left[\left|X^1_{t\wedge S_n} - X^2_{t\wedge S_n}\right|^2\right] \leq$$

$$2t\mathrm{E}\left[\int_0^{t\wedge S_n} a^2\, ds\right] + 4\alpha^2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\ \emptyset_s^\epsilon ds\right)^2\right] + 4\epsilon^{2\alpha}\mathrm{E}\left[\int_0^{t\wedge S_n} \gamma^2\, ds\right]. \qquad (4.20)$$

Applying Equation (4.13) to (4.20) gives

$$\mathrm{E}\left[\left|X^1_{t\wedge S_n} - X^2_{t\wedge S_n}\right|^2\right] \leq$$

$$(2t + 4\epsilon^{2\alpha})K_n^2 \int_0^{t\wedge S_n} \mathrm{E}\left[\left|X^2_{s\wedge S_n} - X^1_{s\wedge S_n}\right|^2\right] ds + 4\alpha^2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\ \emptyset_s^\epsilon ds\right)^2\right]. \qquad (4.21)$$

Now, our aim is to show that $\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\ \emptyset_s^\epsilon ds\right)^2\right] = 0$ in Equation (4.21).

Since $\sigma(t)X_t$ is bounded, then

$$\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\, \emptyset_s^\epsilon ds\right)^2\right] \leq M^2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \emptyset_s^\epsilon ds\right)^2\right],$$

for some constant M.

It is known by Equation (4.12) that $\int_0^t \emptyset_s^\epsilon ds = \frac{1}{\alpha}[B_t^{H,\epsilon} - \epsilon^\alpha W_t]$, so

$$\left(\int_0^t \emptyset_s^\epsilon ds\right)^2 = \frac{1}{\alpha^2}\left[B_t^{H,\epsilon} - \epsilon^\alpha W_t\right]^2 \le \frac{1}{\alpha^2}\left[\left(B_t^{H,\epsilon}\right)^2 + (\epsilon^\alpha W_t)^2\right].$$

Then

$$\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \emptyset_s^\epsilon ds\right)^2\right] \le \frac{1}{\alpha^2}\left[\mathrm{E}\left[B_t^{H,\epsilon^2}\right] + \epsilon^{2\alpha}\mathrm{E}[W_t^2]\right]$$

$$\le \frac{1}{\alpha^2}\left[\left(\mathrm{E}\left[B_t^{H,\epsilon}\right]\right)^2 + \epsilon^{2\alpha}(\mathrm{E}[W_t])^2\right].$$

Since $\mathrm{E}\left[B_t^{H,\epsilon}\right] = 0$, $\mathrm{E}[W_t] = 0$, $\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \emptyset_s^\epsilon ds\right)^2\right] = 0$ and $\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\emptyset_s^\epsilon ds\right)^2\right] = 0$, Equation (4.21) becomes

$$\mathrm{E}\left[\left|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2\right|^2\right] \le (2t + 4\epsilon^{2\alpha})K_n^2 \int_0^{t\wedge S_n} \mathrm{E}\left[\left|X_{s\wedge S_n}^1 - X_{s\wedge S_n}^2\right|^2\right] ds$$

$$\le (2T + 4\epsilon^{2\alpha})K_n^2 \int_0^{t\wedge S_n} \mathrm{E}\left[\left|X_{s\wedge S_n}^1 - X_{s\wedge S_n}^2\right|^2\right] ds. \qquad (4.22)$$

Define $\psi(t) = \mathrm{E}\left[\left|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2\right|^2\right]$. Hence $\forall\ t \in [0, T]$, we have

$$\psi(t) \le (2T + 4\epsilon^{2\alpha})K_n^2 \int_0^t \psi(s)ds. \qquad (4.23)$$

Applying Lemma 4.2 gives $\psi(t) = 0$ or $\mathrm{E}\left[\left|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2\right|^2\right] = 0$ which implies that $X_{t\wedge S_n}^1 = X_{t\wedge S_n}^2\ \forall\ t \in [0, S_n]$. Since $t \mapsto X_t^1$ and $t \mapsto X_t^2$ are continuous, this implies the result for $t \in [0, S_n]$ as $n \to \infty$, so we obtain the uniqueness of solution on $[0, T]$ i.e. $X_t^1 = X_t^2$. This completes the proof of uniqueness for GFBM.

Second: The proof of the existence for GFBM. The Condition (4.14) ensures that the solution $X_t(w)$ of Equation (4.15) does not tend to ∞ in a finite time. We construct a sequence of continuous fuctions that is convergent to a certain limit. Then we prove that this limit satisfies Equation (4.15).

Consider a stochastic differential equation (SDE),

$$dX_t = \mu X_t dt + \sigma(t) X_t dB_t^H . \tag{4.24}$$

When $X_0^\epsilon = X_0$, the corresponding approximation equation of Equation (4.24) becomes

$$dX_t^\epsilon = \mu X_t^\epsilon dt + \sigma(t) X_t^\epsilon dB_t^{H,\epsilon} . \tag{4.25}$$

Using Equation (4.10), we can write Equation (4.25) as

$$\begin{aligned} dX_t^\epsilon &= \mu X_t^\epsilon dt + \sigma(t) X_t^\epsilon \{\alpha \emptyset_t^\epsilon dt + \epsilon^\alpha dW_s\} \\ &= (\mu X_t^\epsilon + \sigma(t) X_t^\epsilon \alpha \emptyset_t^\epsilon) dt + \epsilon^\alpha \sigma(t) X_t^\epsilon dW_s . \end{aligned} \tag{4.26}$$

Substituting $b_1(t, X_t^\epsilon) = \mu X_t^\epsilon + \sigma(t) X_t^\epsilon \alpha \emptyset_t^\epsilon$ and $\sigma_1(t, X_t^\epsilon) = \epsilon^\alpha \sigma(t) X_t^\epsilon$ in Equation (4.26) leads to

$$dX_t^\epsilon = b_1(t, X_t^\epsilon) dt + \sigma_1(t, X_t^\epsilon) dW_s. \tag{4.27}$$

Equation (4.27) can also be writtten as

$$X_t^\epsilon = X_0 + \int_0^t b_1(s, X_s^\epsilon) ds + \int_0^t \sigma_1(s, X_s^\epsilon) dW_s . \tag{4.28}$$

Equation (4.27) and Equation (4.28) represent stochastic differential equation with standard Brownian motion W_s where $b_1(\cdot\, , \,\cdot)$ and $\sigma_1(\cdot\, , \,\cdot)$ satisfy Definition 4.1, Equation (4.10) and Equation (4.14) i.e.

$$|b_1(t, x) - b_1(t, y)| + |\sigma_1(t, x) - \sigma_1(t, y)| \leq D\, |x - y|, \tag{4.29}$$

for some constant D and

$$|b_1(t,x)| + |\sigma_1\,(t,x)| \le L(1+|x|\,). \tag{4.30}$$

for some constant L.

Until now, the Equation (4.24) of SDE with fractional Brownian motion B_t^H has been converted to an equivalent SDE with standard Brownian motion W_s i.e. Equation (4.28). Thus, the existence of solution for Equation (4.28) implies the existence of solution for Equation (4.24) too. To prove the existence of Equation (4.28), we will follow the approach of Oksendal (2000).

We define $Y_t^0 = X_0$ and $Y_t^k = Y_t^k(w)$ such that

$$Y_t^{k+1} = X_0 + \int_0^t b_1(s, Y_s^k)ds + \int_0^t \sigma_1(s, Y_s^k\,)dW_s. \tag{4.31}$$

By similar computations as in the case of uniqueness, we have

$$\mathrm{E}\left[\left|Y_t^{k+1} - Y_t^k\right|^2\right] \le (2T + 4\epsilon^{2\alpha})D^2 \int_0^{t\wedge S_n} \mathrm{E}[|Y_s^k - Y_s^{k-1}|^2\,]\,ds. \tag{4.32}$$

Now mathematical induction is applied to Equation (4.32) as follows.

Let $k \ge 1$ and $t \le T$, then we have

$$\mathrm{E}[|Y_t^1 - Y_t^0|^2] \le \mathrm{E}\left[\left|\int_0^t b_1(s, X_0^\epsilon)ds + \int_0^t \sigma_1(s, X_0^\epsilon)dW_s\ \right|^2\right]$$

$$\le 2\mathrm{E}\left[\left|\int_0^t b_1(s, X_0^\epsilon)ds\right|^2\right] + \left[\left|\int_0^t \sigma_1(s, X_0^\epsilon)dW_s\ \right|^2\right].$$

By Lemma 4.3

$$\mathrm{E}[|Y_t^1 - Y_t^0|^2] \le 2t\mathrm{E}\left[\int_0^t b_1^2(s, X_0^\epsilon)ds\right] + 2\mathrm{E}\left[\int_0^t \sigma_1^2(s, X_0^\epsilon\,)ds\right]. \tag{4.33}$$

We employ Equation (4.30) into Equation (4.33) implies

$$\mathrm{E}[|Y_t^1 - Y_t^0|^2] \leq 2t\mathrm{E}\left[\int_0^t L^2(1+|X_0^\epsilon|)^2 ds\right] + 2\mathrm{E}\left[\int_0^t L^2(1+|X_0^\epsilon|)^2 ds\right]$$

$$\leq 2tL^2 \int_0^t \mathrm{E}(1+2|X_0^\epsilon| + |X_0^\epsilon|^2)ds + 2L^2 \int_0^t \mathrm{E}(1+2|X_0^\epsilon| \ + |X_0^\epsilon|^2)ds$$

$$\leq 2t^2L^2(1+\mathrm{E}[|X_0^\epsilon|^2]) + 2tL^2(1+\mathrm{E}[|X_0^\epsilon|^2])$$

$$\leq 2tL^2(1+\mathrm{E}[|X_0^\epsilon|^2])(t+1)$$

$$\leq 2TL^2(1+\mathrm{E}[|X_0^\epsilon|^2])t = A_1 t,$$

where $A_1 = 2TL^2(1+\mathrm{E}[|X_0^\epsilon|^2])$ is a constant that depends on T, L and $\mathrm{E}[|X_0|^2]$.

Now by induction on $k \geq 1$ and $t \leq T$ we get

$$\mathrm{E}\left[\left|Y_t^{k+1} - Y_t^k\right|^2\right] \leq \frac{A_2^{k+1}t^{k+1}}{(k+1)!}, \tag{4.34}$$

where $A_2 = (2T + 4\epsilon^{2\alpha})D^2 A_1$ is a constant that depends on T, L, D and $\mathrm{E}[|X_0|^2]$

Now,

$$\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| \leq$$

$$\int_0^T |b_1(s,Y_s^k) - b_1(s,Y_s^{k-1})|ds + \sup_{0\leq t\leq T}\left|\int_0^t (\sigma_1(s,Y_s^k) - \sigma_1(s,Y_s^{k-1}))dW_s\right|. \tag{4.35}$$

Applying the fact that $P(G > 2^{-k}) \leq P(G > 2^{-k-1})$ to Equation (4.35) gives

$$P\left[\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| > \ 2^{-k}\right]$$

$$\leq P\left[\left(\int_0^T |b_1(s,Y_s^k) - b_1(s,Y_s^{k-1})|ds\right)^2 > 2^{-2k-2}\right]$$

$$+P\left[\sup_{0\leq t\leq T}\left|\int_0^t (\sigma_1(s,Y_s^k) - \sigma_1(s,Y_s^{k-1}))dW_s\right| > 2^{-k-1}\right]. \tag{4.36}$$

Lemma 4.3 and Lemma 4.4 are applied to right hand side of Equation (4.36) gives

$$P\left[\left(\int_0^T |b_1(s,Y_s^k) - b_1(s,Y_s^{k-1})|ds\right)^2 > 2^{-2k-2}\right]$$

$$\leq 2^{2k+2} T \int_0^T \mathrm{E}[|b_1(s,Y_s^k) - b_1(s,Y_s^{k-1})|^2]ds \quad (4.37)$$

and

$$P\left[\sup_{0\leq t\leq T}\left|\int_0^t (\sigma_1(s,Y_s^k) - \sigma_1(s,Y_s^{k-1}))dW_s\right| > 2^{-k-1}\right]$$

$$\leq 2^{2k+2} \int_0^T \mathrm{E}(|\sigma_1(s,Y_s^k) - \sigma_1(s,Y_s^{k-1})|^2)ds. \quad (4.38)$$

Substituting Equation (4.37) and Equation (4.38) into Equation (4.36) obtains

$$P\left[\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| > 2^{-k}\right]$$

$$\leq 2^{2k+2} \int_0^T T\, \mathrm{E}(|b_1(s,Y_s^k) - b_1(s,Y_s^{k-1})|^2) + \mathrm{E}(|\sigma_1(s,Y_s^k) - \sigma_1(s,Y_s^{k-1})|^2)ds$$

$$= 2^{2k+2} \int_0^T \mathrm{E}(T|b_1(s,Y_s^k) - b_1(s,Y_s^{k-1})|^2 + |\sigma_1(s,Y_s^k) - \sigma_1(s,Y_s^{k-1})|^2)\, ds. \quad (4.39)$$

Applying Equation (4.29) into Equation (4.39) gives

$$P\left[\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| > 2^{-k}\right] \leq 2^{2k+2} D^2(T+1) \int_0^T \mathrm{E}(|Y_s^k - Y_s^{k-1}|^2)dt. \quad (4.40)$$

Substituting Equation (4.34) into Equation (4.40) yields

$$P\left[\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| > 2^{-k}\right] \leq 2^{2k+2} D^2(T+1) \int_0^T \frac{A_2^k t^k}{(k)!} dt$$

$$= 2^{2k+2} D^2(T+1) \frac{A_2^k T^{k+1}}{(k+1)!}.$$

However, $A_2 > D^2(T+1)$, thus

$$P\left[\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| > 2^{-k}\right] \leq \frac{2^{2k+2} A_2^{k+1} T^{k+1}}{(k+1)!} = \frac{(4A_2T)^{k+1}}{(k+1)!}. \quad (4.41)$$

This implies that $P\left[\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| > 2^{-k}\right]$ is bounded, so Lemma 4.5 can be used as follows

$$P\left[\lim_{n\to\infty}\sup_{0\le t\le T}\left|Y_t^{k+1}-Y_t^k\right| > 2^{-k}\right]=0. \tag{4.42}$$

This follows that for almost all w, there exists k_0 such that

$\sup_{0\le t\le T}\left|Y_t^{k+1}-Y_t^k\right| \le 2^{-k}$ for $k \ge k_0$. Therefore, the sequence

$$Y_t^n(w) = Y_t^0(w) + \sum_{k=0}^{n-1}\{Y_t^{k+1}(w) - Y_t^k(w)\}$$

is uniformly convergent in $[0, T]$ for almost all w.

If we denote $X_t^\epsilon = X_t^\epsilon(w) = \lim_{n\to\infty} Y_t^n(w)$, then X_t^ϵ is continuous on t almost all w since $Y_t^n(w)$ has the same property for all n.

As we know that every Cauchy sequence is convergent, by using Equation (4.34) we have

$$\mathrm{E}[|Y_t^m - Y_t^n|^2]^{1/2} = \|Y_t^m - Y_t^n\|_{L^2(p)} = \left\|\sum_{k=n}^{m-1}\{Y_t^{k+1}(w) - Y_t^k(w)\}\right\|_{L^2(p)}$$

$$\le \sum_{k=n}^{m-1}\left\|Y_t^{k+1}(w) - Y_t^k(w)\right\|_{L^2(p)} \le \sum_{k=n}^{m-1}\left|\frac{A_2^{k+1}t^{k+1}}{(k+1)!}\right|^{\frac{1}{2}} \to 0 \text{ as } n\to\infty \tag{4.43}$$

for $m > n \ge 0$.

Equation (4.43) proves that sequence $\{Y_t^n\}$ converges in $L^2(p)$ to certain limits say Y_t. Since subsequence $Y_t^n(w)$ converges to $Y_t(w)$ for all w, we must have $Y_t = X_t^\epsilon$ almost surely (i.e. $X_t^\epsilon(w) = \lim_{n\to\infty} Y_t^n(w) = Y_t(w)$).

Now, we will prove that X_t^ϵ satisfies Equation (4.24) and Equation (4.27). For all n,

$$Y_t^{n+1} = X_0^\epsilon + \int_0^t b_1(s, Y_s^n)ds + \int_0^t \sigma_1(s, Y_s^n)dW_s. \tag{4.44}$$

Now, for all $t \in [0, T]$, we have $Y_t^{n+1} \to X_t^\epsilon$ as $n \to \infty$ uniformly for almost all w. By equation (4.43) and the Lemma 4.6, we have

$$\mathrm{E}\left[\int_0^T |X_t^\epsilon - Y_t^n|^2 dt\right] \leq \lim_{m\to\infty} \sup \mathrm{E}\left[\int_0^T |Y_t^m - Y_t^n|^2 dt\right] \to 0 \text{ as } n \to \infty.$$

Using Lemma (4.3),

$$\mathrm{E}\left[\int_0^T |X_t^\epsilon - Y_t^n|^2 dt\right] = \mathrm{E}\left[\left(\int_0^T |X_t^\epsilon - Y_t^n| dW_s\right)^2\right] \to 0.$$

This implies that $X_t^\epsilon - Y_t^n \to 0$, and so

$$\int_0^t b_1(s, Y_s^n) ds \to \int_0^t b_1(s, X_t^\epsilon) ds \quad \text{and} \quad \int_0^t \sigma_1(s, Y_s^n) dW_s \to \int_0^t \sigma_1(s, X_s^\epsilon) dW_s .$$

By taking the limit for Equation (4.43) as $n \to \infty$,

$$\begin{aligned} X_t^\epsilon = \lim_{n\to\infty} Y_t^{n+1} &= X_0^\epsilon + \lim_{n\to\infty} \int_0^t b_1(s, Y_s^n) ds + \lim_{n\to\infty} \int_0^t \sigma_1(s, Y_s^n) dW_s \\ &= X_0^\epsilon + \int_0^t \lim_{n\to\infty} b_1(s, Y_s^n) ds + \int_0^t \lim_{n\to\infty} \sigma_1(s, Y_s^n) dW_s \\ &= X_0^\epsilon + \int_0^t b_1(s, X_s^\epsilon) ds + \int_0^t \sigma_1(s, X_s^\epsilon) dW_s . \end{aligned}$$

This ends the proof of the existence for GFBM.

4.3 Existence and Uniqueness Solution of Fractional Stochastic Differential Model

In this section, we construct the existence and uniqueness theorem for the generalized SDE driven by BM stated in Lemma 4.7 to the existence and uniqueness theorem for stochastic differential equations driven by FBM.

We start with X_t the stochastic process as below,

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s^H,$$

where $b(t, X_t)$ and $\sigma(t, X_t)$ are two continuous functions and X_0 is a given random variable such that $\mathrm{E}[X_0^2] < \infty$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where $\mathcal{F}$ is the σ-algebra of set Ω, and P is a probability measure. Let $\{X_t\}_{t\in[0,\infty]}$ be a stochastic process defined on $(\Omega, \mathcal{F}, P)$ such that for all $t \in [0, \infty]$ we have a random variable $w \in \Omega$ where $w \mapsto X_t(w)$ is a continuous function. $X_t(w)$ represents the result at time t of the experiment w. Write $X_t(w) = X(t, w)$ and define a function $T \times \Omega \mapsto \mathbb{R}^n$ as $(t, w) \mapsto X(t, w)$. Let $x \in \mathbb{R}^n$ then $x \mapsto (\sum_{k=1}^{n} {x_k}^2)^{½} \equiv \|x\|_{L^2}$ is Euclidean norm.

Theorem 4.2: *Let* $T > 0$ *and* $b(\cdot,\cdot): [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $\sigma(\cdot,\cdot): [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ *be measurable functions satisfying the following*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K\,|x - y|\,);x, y \in \mathbb{R}^n, t \in [0, T] \quad (4.45)$$

for some constant K *such that*

$$|b(t,x)| + |\sigma(t,x)| \le C(1 + |x|), \text{ for some constant } C, x \in \mathbb{R}^n, t \in [0, T]. \quad (4.46)$$

Then SDE driven by FBM

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t^H \quad (4.47)$$

has a unique solution in $t \in [0, T]$.

Proof:

First: The proof of uniqueness for the SDE driven by FBM. The condition (4.45) guarantees that Equation (4.47) has a unique solution. The uniqueness means that if X_t^1 and X_t^2 are two t-continuous processes satisfying the hypothesis of the Theorem 4.2 then $X_t^1 = X_t^2$ for all $t \le T$, almost surely.

Let X_t^1 and X_t^2 are two solutions of Equation (4.47) with same initial value X_0.

Suppose that:

$$a = a(s,w) = b(s, X_s^1\ ) - b(s, X_s^2\ )$$

and

$$\gamma = \gamma(s,w) = \sigma(s, X_s^1\ ) - \sigma(s, X_s^2\ ).$$

Define $\tau_n^1 = \inf\{\ t \geq 0 \mid |X_t^1| \geq n\}$ and $\tau_n^2 = \inf\{\ t \geq 0 \mid |X_t^2| \geq n\}$. Let $S_n = \min\{\tau_n^1, \tau_n^2\}$. We need to prove that $\mathrm{E}[\left|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2\right|^2] \to 0$ for all $t \in [0,T]$.

Now for all $t \in [0,T]$,

$$\begin{aligned}\mathrm{E}[\left|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2\right|^2] &= \mathrm{E}[|\int_0^{t\wedge S_n} b(s, X_s^1\ )ds + \int_0^{t\wedge S_n} \sigma(s, X_s^1\ )dB_s^H - \\ &\qquad \int_0^{t\wedge S_n} b(s, X_s^2\ )ds - \int_0^{t\wedge S_n} \sigma(s, X_s^2\ )dB_s^H\ |^2\ ] \\ &= \mathrm{E}[\left(\int_0^{t\wedge S_n} a\ ds + \int_0^{t\wedge S_n} \gamma\ dB_s^H\ \right)^2\ ]. \end{aligned} \tag{4.48}$$

However, it is known that $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, so the last expression will be

$$\mathrm{E}[\left|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2\right|^2]\ \leq \mathrm{E}\left[2\ \left(\int_0^{t\wedge S_n}\ a\ ds\right)^2 +\ 2\left(\int_0^{t\wedge S_n} \gamma\ dB_s^H\right)^2\ \right]. \tag{4.49}$$

Using Lemma 4.1, Equation (4.49) gives

$$\mathrm{E}\left[\left|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2\right|^2\right] \leq 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n}\ a\ ds\right)^2\right] + 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\ dB_s^H\right)^2\right]. \tag{4.50}$$

Since $\int_0^t f(s,w)dB_s^{H,\epsilon} \to \int_0^t f(s,w)dB_s^H$ then Equation (4.50) is approximately equal to

$$\mathrm{E}[|X^1_{t\wedge S_n} - X^2_{t\wedge S_n}|^2] \leq 2\mathrm{E}[\left(\int_0^{t\wedge S_n} a\, ds\right)^2] + 2\mathrm{E}[\left(\int_0^{t\wedge S_n} \gamma\, dB_s^{H,\epsilon}\right)^2]. \quad (4.51)$$

Replacing Equation (4.10) into Equation (4.50) gives

$$\mathrm{E}[|X^1_{t\wedge S_n} - X^2_{t\wedge S_n}|^2] \leq 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} a\, ds\right)^2\right] + 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\, (\alpha \emptyset_s^\epsilon ds + \epsilon^\alpha dW_s)\right)^2\right]$$

$$\leq 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} a\, ds\right)^2\right] + 2\mathrm{E}\left[\left(\alpha \int_0^{t\wedge S_n} \gamma\ \emptyset_s^\epsilon ds + \epsilon^\alpha \int_0^{t\wedge S_n} \gamma\ dW_s\right)^2\right]$$

$$\leq 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} a\, ds\right)^2\right] + 2\mathrm{E}\left[2\left(\alpha \int_0^{t\wedge S_n} \gamma\ \emptyset_s^\epsilon ds)\right)^2 + 2\ \left(\epsilon^\alpha \int_0^{t\wedge S_n} \gamma\ dW_s\right)^2\right]$$

$$\leq\ 2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} a\, ds\right)^2\right] + 4\alpha^2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\ \ \emptyset_s^\epsilon ds\right)^2\right] + 4\epsilon^{2\alpha}\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\, dW_s\right)^2\right].$$

By Lemma 4.3,

$$\mathrm{E}\left[|X^1_{t\wedge S_n} - X^2_{t\wedge S_n}|^2\right] \leq 2t\mathrm{E}\left[\int_0^{t\wedge S_n} a^2\, ds\right] + 4\alpha^2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\ \ \emptyset_s^\epsilon ds\right)^2\right] +$$

$$4\epsilon^{2\alpha}\mathrm{E}\left[\int_0^{t\wedge S_n} \gamma^2\, ds\right].$$

Applying Equation (4.45) we get

$$\mathrm{E}\left[|X^1_{t\wedge S_n} - X^2_{t\wedge S_n}|^2\right] \leq (2t + 4\epsilon^{2\alpha})K_n^2 \int_0^{t\wedge S_n} \mathrm{E}\left[|X^2_{s\wedge S_n} - X^1_{s\wedge S_n}|^2\right] ds\ +$$

$$4\alpha^2\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\ \ \emptyset_s^\epsilon ds\right)^2\right]. \quad (4.52)$$

Now we want to show that the last expression in Equation (4.52) is $\mathrm{E}\left[\left(\int_0^{\mathrm{t}\wedge \mathrm{S_n}} \gamma\ \ \emptyset_{\mathrm{s}}^\epsilon \mathrm{ds}\right)^2\right] = 0.$

Since $\sigma(t, x)$ is bounded, then for some constant M we have

$$\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\ \emptyset_s^{\epsilon} ds\right)^2\right] \leq M^2 \mathrm{E}\left[\left(\int_0^{t\wedge S_n} \emptyset_s^{\epsilon} ds\right)^2\right].$$

It is known by Equation (4.12) that $\int_0^t \emptyset_s^{\epsilon} ds = \frac{1}{\alpha}\left[B_t^{H,\epsilon} - \epsilon^{\alpha} W_t\right]$, so $\left(\int_0^t \emptyset_s^{\epsilon} ds\right)^2 = \frac{1}{\alpha^2}\left[B_t^{H,\epsilon} - \epsilon^{\alpha} W_t\right]^2 \leq \frac{1}{\alpha^2}\left[\left(B_t^{H,\epsilon}\right)^2 + (\epsilon^{\alpha} W_t)^2\right]$ and then,

$$\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \emptyset_s^{\epsilon} ds\right)^2\right] \leq \frac{1}{\alpha^2}\left[\mathrm{E}\left[B_t^{H,\epsilon^2}\right] + \epsilon^{2\alpha} \mathrm{E}[W_t^2]\right]$$

$$\leq \frac{1}{\alpha^2}\left[\left(\mathrm{E}\left[B_t^{H,\epsilon}\right]\right)^2 + \epsilon^{2\alpha}(\mathrm{E}[W_t])^2\right].$$

But $\mathrm{E}\left[B_t^{H,\epsilon}\right] = 0$ and $\mathrm{E}[W_t] = 0$, this implies that $\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \emptyset_s^{\epsilon} ds\right)^2\right] = 0$ and subsequently, $\mathrm{E}\left[\left(\int_0^{t\wedge S_n} \gamma\ \emptyset_s^{\epsilon} ds\right)^2\right] = 0$. Therefore, Equation (4.52) will become

$$\mathrm{E}\left[\left|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2\right|^2\right] \leq (2t + 4\epsilon^{2\alpha}) K_n^2 \int_0^{t\wedge S_n} \mathrm{E}\left[\left|X_{s\wedge S_n}^1 - X_{s\wedge S_n}^2\right|^2\right] ds. \tag{4.53}$$

Define $\psi(t) = \mathrm{E}\left[\left|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2\right|^2\right]$. For $\forall\ t \in [0, T]$ we have

$$\psi(t) \leq (2t + 4\epsilon^{2\alpha}) K_n^2 \int_0^t \psi(s) ds. \tag{4.54}$$

Applying Lemma 4.2, in Equation (4.54) gives $\psi(t) = \mathrm{E}\left[\left|X_{t\wedge S_n}^1 - X_{t\wedge S_n}^2\right|^2\right] = 0$ which implies that $X_{t\wedge S_n}^1 = X_{t\wedge S_n}^2\ \ \forall\ t \in [0, S_n]$. Since $t \mapsto X_t^1$ and $t \mapsto X_t^2$ are continuous, this implies the result for $t \in [0, S_n]$ as $n \to \infty$, so we obtain the uniqueness of solution on $[0, T]$ i.e. $X_t^1 = X_t^2$. This is the end of the proof of uniqueness for fractional stochastic differential equation.

Second: The proof of the existence SDE driven by FBM. The Condition (4.46) ensures that the solution $X_t(w)$ of Equation (4.47) does not tend to ∞ in a finite time. We construct a sequence of continuous fuctions that is convergent to a certain limit. Then we prove that this limit satisfies Equation (4.47).

Consider the equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t^H . \quad (4.55)$$

Set $X_0^\epsilon = X_0$. The corresponding approximation equation of Equation (4.55) is

$$dX_t^\epsilon = b(t, X_t^\epsilon)dt + \sigma(t, X_t^\epsilon)dB_t^{H,\epsilon} . \quad (4.56)$$

Substituting Equation (4.10) into Equation (4.56) gives

$$\begin{aligned} dX_t^\epsilon &= b(t, X_t^\epsilon)dt + \sigma(t, X_t^\epsilon)\{\alpha \emptyset_t^\epsilon dt + \epsilon^\alpha dW_s\} \\ &= (b(t, X_t^\epsilon) + \sigma(t, X_t^\epsilon)\alpha \emptyset_t^\epsilon)dt + \epsilon^\alpha \sigma(t, X_t^\epsilon)dW_s. \end{aligned} \quad (4.57)$$

Set $b_1(t, X_t^\epsilon) = b(t, X_t^\epsilon) + \sigma(t, X_t^\epsilon)\alpha \emptyset_t^\epsilon$ and $\sigma_1(t, X_t^\epsilon) = \epsilon^\alpha \sigma(t, X_t^\epsilon)$. Then

$$dX_t^\epsilon = b_1(t, X_t^\epsilon)dt + \sigma_1(t, X_t^\epsilon)dW_s. \quad (4.58)$$

Equivalently,

$$X_t^\epsilon = X_0 + \int_0^t b_1(s, X_s^\epsilon)ds + \int_0^t \sigma_1(s, X_s^\epsilon)dW_s . \quad (4.59)$$

Equation (4.58) and Equation (4.59) represent a SDE with standard BM W_s, where $b_1(t, X_s^\epsilon)$ and $\sigma_1(t, X_s^\epsilon)$ satisfy Equation (4.45) and Equation (4.46) i.e.

$$|b_1(t,x) - b_1(t,y)| + |\sigma_1(t,x) - \sigma_1(t,y)| \le D\,|x - y| \quad (4.60)$$

and

$$|b_1(t,x)| + |\sigma_1\,(t,x)| \le L(1+|x|\,). \tag{4.61}$$

Until now, the Equation (4.55) of SDE with FBM B_t^H has been converted to an equivalent SDE with standard BM W_s i.e. Equation (4.59). Thus, the existence of solution for Equation (4.59) implies the existence of solution for Equation (4.55) too. To prove the existence of Equation (4.59), we will follow the approach of Oksendal (2000).

We follow the approach of Oksendal (2000) to prove the existence of solution of Equation (4.59).

First, define $Y_t^0 = X_0$ and $Y_t^k = Y_t^k(w)$ such that

$$Y_t^{k+1} = X_0 + \int_0^t b_1(s, Y_s^k)ds + \int_0^t \sigma_1(s, Y_s^k\,)dW_s. \tag{4.62}$$

Applying similar approach of computations that carried out before to get the Equation (4.53) on Equation (4.62) gives

$$\mathrm{E}\left[\left|Y_t^{k+1} - Y_t^k\right|^2\right] \le (2T + 4\epsilon^{2\alpha})D^2 \int_0^{t\wedge S_n} \mathrm{E}[|Y_s^k - Y_s^{k-1}|^2]\,ds. \tag{4.63}$$

Now mathematical induction is applied to Equation (4.63),

For $k \ge 1$ and $t \le T$ we have

$$\mathrm{E}[|Y_t^1 - Y_t^0|^2] \le \mathrm{E}\left[\left|\int_0^t b_1(s, X_0^\epsilon)ds + \int_0^t \sigma_1(s, X_0^\epsilon)dW_s\,\right|^2\right]$$

$$\le 2\mathrm{E}\left[\left|\int_0^t b_1(s, X_0^\epsilon)ds\right|^2\right] + 2\mathrm{E}\left[\left|\int_0^t \sigma_1(s, X_0^\epsilon)dW_s\,\right|^2\right].$$

By Lemma 4.3,

$$\mathrm{E}[|Y_t^1 - Y_t^0|^2] \leq 2t\mathrm{E}\left[\int_0^t b_1^2(s, X_0^\epsilon)ds\right] + 2\mathrm{E}\left[\int_0^t \sigma_1^2\,(s, X_0^\epsilon)ds\right]. \qquad (4.64)$$

We employ Equation (4.61) into Equation (4.64) implies

$$\begin{aligned}
\mathrm{E}[|Y_t^1 - Y_t^0|^2] &\leq 2t\mathrm{E}\left[\int_0^t L^2(1 + |X_0^\epsilon|)^2 ds\right] + 2\mathrm{E}\left[\int_0^t L^2(1 + |X_0^\epsilon|)^2 ds\right] \\
&\leq 2tL^2 \int_0^t \mathrm{E}(1 + 2|X_0^\epsilon| + |X_0^\epsilon|^2)ds + 2L^2 \int_0^t \mathrm{E}(1 + 2|X_0^\epsilon| + |X_0^\epsilon|^2 ds) \\
&\leq 2t^2L^2(1 + \mathrm{E}[|X_0^\epsilon|^2]) + 2tL^2(1 + \mathrm{E}[|X_0^\epsilon|^2]) \\
&\leq 2tL^2(1 + \mathrm{E}[|X_0^\epsilon|^2])(t + 1) \\
&\leq 2TL^2(1 + \mathrm{E}[|X_0^\epsilon|^2])t = A_1 t,
\end{aligned}$$

where $A_1 = 2TL^2(1 + \mathrm{E}[|X_0^\epsilon|^2])$ is constant depends on T, L and $\mathrm{E}[|X_0|^2]$. Now by induction on $k \geq 0$ and $t \leq T$ we get

$$\mathrm{E}\left[\left|Y_t^{k+1} - Y_t^k\right|^2\right] \leq \frac{A_2^{k+1}t^{k+1}}{(k+1)!}, \qquad (4.65)$$

where $A_2 = (2T + 4\epsilon^{2\alpha})D^2A_1$ is constant depends on T, L, D and $\mathrm{E}[|X_0|^2]$.

Now,

$$\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| \leq$$

$$\int_0^T |b_1(s, Y_s^k) - b_1(s, Y_s^{k-1})|ds + \sup_{0\leq t\leq T}\left|\int_0^t (\sigma_1(s, Y_s^k) - \sigma_1(s, Y_s^{k-1}))dW_s\right|. \qquad (4.66)$$

Using $P(G > 2^{-k}) \leq P(G > 2^{-k-1})$ obtains

$$\begin{aligned}
&P\left[\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| > 2^{-k}\right] \\
&\leq P\left[\left(\int_0^T |b_1(s, Y_s^k) - b_1(s, Y_s^{k-1})|ds\right)^2 > 2^{-2k-2}\right] \\
&\quad + P\left[\sup_{0\leq t\leq T}\left|\int_0^t (\sigma_1(s, Y_s^k) - \sigma_1(s, Y_s^{k-1}))dW_s\right| > 2^{-k-1}\right]. \qquad (4.67)
\end{aligned}$$

Applying Lemma 4.4 and Lemma 4.3 to the right hand side of Equation (4.67) leads to

$$P\left[\left(\int_0^T |b_1(s,Y_s^k) - b_1(s,Y_s^{k-1})| ds\right)^2 > 2^{-2k-2}\right]$$

$$\leq 2^{2k+2} T \int_0^T \mathrm{E}(|b_1(s,Y_s^k) - b_1(s,Y_s^{k-1})|^2) ds, \quad (4.68)$$

and

$$P\left[\sup_{0\leq t\leq T}\left|\int_0^t (\sigma_1(s,Y_s^k) - \sigma_1(s,Y_s^{k-1})) dW_s\right| > 2^{-k-1}\right]$$

$$\leq 2^{2k+2} \int_0^T \mathrm{E}(|\sigma_1(s,Y_s^k) - \sigma_1(s,Y_s^{k-1})|^2) ds. \quad (4.69)$$

Substituting Equation (4.68) and Equation (4.69) into Equation (4.67)

$$P\left[\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| > 2^{-k}\right]$$

$$\leq 2^{2k+2} \int_0^T T\,\mathrm{E}(|b_1(s,Y_s^k) - b_1(s,Y_s^{k-1})|^2) + \mathrm{E}(|\sigma_1(s,Y_s^k) - \sigma_1(s,Y_s^{k-1})|^2) ds$$

$$= 2^{2k+2} \int_0^T \mathrm{E}(T|b_1(s,Y_s^k) - b_1(s,Y_s^{k-1})|^2 + |\sigma_1(s,Y_s^k) - \sigma_1(s,Y_s^{k-1})|^2)\, ds. \quad (4.70)$$

Applying Equation (4.60) into Equation (4.70) gives

$$P\left[\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| > 2^{-k}\right] \leq 2^{2k+2} D^2(T+1) \int_0^T \mathrm{E}\left(\left|Y_t^k - Y_t^{k-1}\right|^2\right) dt. \quad (4.71)$$

Replacing Equation (4.65) in Equation (4.71) gives

$$P\left[\sup_{0\leq t\leq T}\left|Y_t^{k+1} - Y_t^k\right| > 2^{-k}\right] \leq 2^{2k+2} D^2(T+1) \int_0^T \frac{A_2^k t^k}{(k)!} dt$$

$$= 2^{2k+2} D^2(T+1) \frac{A_2^k T^{k+1}}{(k+1)!}.$$

But $A_2 > D^2(T+1)$, this implies

$$P\left[\sup_{0\le t\le T}\left|Y_t^{k+1}-Y_t^k\right|> 2^{-k}\right]\le \frac{2^{2k+2}A_2^{k+1}T^{k+1}}{(k+1)!}=\frac{(4A_2T)^{k+1}}{(k+1)!}. \quad (4.72)$$

Since we have proven that $P\left[\sup_{0\le t\le T}\left|Y_t^{k+1}-Y_t^k\right|> 2^{-k}\right]$ is bounded we can use Lemma 4.5.

$$P\left[\lim_{k\to\infty}\sup_{0\le t\le T}\left|Y_t^{k+1}-Y_t^k\right|> 2^{-k}\right]=0. \quad (4.73)$$

This follows that for almost all w, there exists k_0 such that $\sup_{0\le t\le T}\left|Y_t^{k+1}-Y_t^k\right|\le 2^{-k}$ for $k\ge k_0$. Therefore, the sequence $Y_t^n(w)=Y_t^0(w)+\sum_{k=0}^{n-1}\{Y_t^{k+1}(w)-Y_t^k(w)\}$ is uniformly convergent in $[0,T]$, for almost all w.

If we denote $X_t^\epsilon=X_t^\epsilon(w)=\lim_{n\to\infty}Y_t^n(w)$, then X_t^ϵ is continuous on t almost all w since $Y_t^n(w)$ has the same property for all n.

Since every Cauchy sequence is convergent, so for $m>n\ge 0$ and by using Equation (4.65) we have

$$\mathrm{E}[|Y_t^m-Y_t^n|^2]^{1/2}=\|Y_t^m-Y_t^n\|_{L^2(p)}=\left\|\sum_{k=n}^{m-1}\{Y_t^{k+1}(w)-Y_t^k(w)\}\right\|_{L^2(p)}$$

$$\le \sum_{k=n}^{m-1}\left\|Y_t^{k+1}(w)-Y_t^k(w)\right\|_{L^2(p)}\le \sum_{k=n}^{m-1}\left|\frac{A_2^{k+1}t^{k+1}}{(k+1)!}\right|^{\frac{1}{2}}\to 0, \quad (4.74)$$

as $n\to\infty$.

Equation (4.74) proves that the sequence $\{Y_t^n\}$ converges in $L^2(p)$ to certain limits say Y_t. Since subsequence $Y_t^n(w)$ converges to $Y_t(w)$ for all w, we must have $Y_t=X_t^\epsilon$ almost surely.

Now, it remains to show that X_t^ϵ satisfies Equation (4.59) which is equivalent to Equation (4.57). For all n

$$Y_t^{n+1} = X_0^\epsilon + \int_0^t b_1(s, Y_s^n)ds + \int_0^t \sigma_1(s, Y_s^n)dW_s. \tag{4.75}$$

Now, for every $t \in [0, T]$, we have $Y_t^{n+1} \to X_t^\epsilon$ as $n \to \infty$ uniformly for almost all w.

Its follows from Equation (4.74) and the Lemma 4.6 that

$$\mathrm{E}\left[\int_0^T |X_t^\epsilon - Y_t^n|^2 dt\right] \leq \lim_{m\to\infty} \sup \mathrm{E}\left[\int_0^T |Y_t^m - Y_t^n|^2 dt\right] \to 0 \text{ as } n \to \infty.$$

By Lemma 4.3, we obtain

$$\mathrm{E}\left[\int_0^T |X_t^\epsilon - Y_t^n|^2 dt\right] = \mathrm{E}\left[\left(\int_0^T |X_t^\epsilon - Y_t^n| dW_s)\right)^2\right] \to 0.$$

This implies that $(X_t^\epsilon - Y_t^n) \to 0$ and then

$$\int_0^t b_1(s, Y_s^n)ds \to \int_0^t b_1(s, X_t^\epsilon)ds \quad \text{and} \quad \int_0^t \sigma_1(s, Y_s^n)dW_s \to \int_0^t \sigma_1(s, X_s^\epsilon)dW_s\ .$$

By taking the limit of $n \to \infty$ for Equation (4.75),

$$\begin{aligned} X_t^\epsilon &= \lim_{n\to\infty} Y_t^{n+1} \\ &= X_0^\epsilon + \lim_{n\to\infty} \int_0^t b_1(s, Y_s^n)ds + \lim_{n\to\infty} \int_0^t \sigma_1(s, Y_s^n)dW_s \\ &= X_0^\epsilon + \int_0^t \lim_{n\to\infty} b_1(s, Y_s^n)ds + \int_0^t \lim_{n\to\infty} \sigma_1(s, Y_s^n)dW_s \\ &= X_0^\epsilon + \int_0^t b_1(s, X_s^\epsilon)ds + \int_0^t \sigma_1(s, X_s^\epsilon)dW_s\ . \end{aligned}$$

This ends the proof of the existence for SDE driven by FBM.

4.4 Discussion

In this chapter, we present two theorems of the existence and uniqueness of the solution for GFBM and SDE driven by FBM model. In the case of GFBM, we prove the existence and uniqueness theorem of the solution when the volatility is assumed to be function in time t rather than to be constant. While in the case of a class of SDE driven by FBM, we generalize the theorem of SDE that stated in Oksendal (2000).

CHAPTER FIVE

VALIDATION OF THE DEVELOPED MODEL BASED ON DIFFERENT TYPES OF MARKET INDICES

The validation of the method is essential to determine the performance of the new proposed model and to ensure that the results are acceptable and can be applied in real financial environment. The validation is done by forecasting the index prices of three different types of markets and comparing the results with other existing models.

In what follows are comparison study of the performance between the proposed model and selected current GBM and GFBM models with some volatility formulation available in the literature. These GBM models include model with constant volatility computed by simple volatility formula (GBM-S), GBM with constant volatility computed by log volatility formula (GBM-L), GBM with constant volatility computed by high-low-closed volatility formula (GBM-HLC), GBM with stochastic volatility computed by a deterministic functions $\sigma(Y_t) = Y_t$ (GBM-STO).

Meanwhile, GFBM models include model with constant volatility computed by simple volatility formula (GFBM-S), GFBM with constant volatility computed by log volatility formula (GFBM-L), GFBM with constant volatility computed by high-low-closed volatility formula (GFBM-HLC), and GFBM with stochastic volatility computed by the deterministic functions $\sigma(Y_t) = Y_t$ (GFBM-STO) in term of their forecasted values of prices in three different types of markets indices. Table 5.1 shows formulations of these volatilities under study.

Table 5.1

Formulas of Volatility

Volatility	Formula
Simple volatility (S)	$\sigma = \sqrt{\frac{1}{(n-1)\Delta t}\sum_{i=1}^{n}(R_i - \bar{R})^2}$
Log volatility (L)	$\sigma = \sqrt{\frac{1}{(n-1)\Delta t}\sum_{i=1}^{n}(Log(S_i) - Log\ (S_{i-1}))^2}$
High–Low–Closed volatility (HLC)	$\sigma = \sqrt{\frac{1}{(n-1)\Delta t}\left(\sum_{i=1}^{n} 0.5(Log(H_i) - Log\ (L_i))^2 - \sum_{i=1}^{n} 0.3(Log(S_i) - Log\ (S_{i-1}))^2\right)}$
Stochastic volatility (STO)	$\sigma(Y_t) = Y_t$ $dY_t = \alpha(m - Y_t)dt + \beta dB_{H_2}(t)$

Next, to evaluate each model, we apply mean absolute percentage error (MAPE) to forecast stock indices as the works of Walsh (1998), Lam, Chang and Lee (2002), Omar and Jaffar (2012) and Abidin and Jaffar (2012; 2014). MAPE is a measure of prediction accuracy of a forecastiong method. It is the most commonly used measure of assessing forecasts in organizations (Tofallis; 2015). Accoding to Abidin and Jaffar (2012; 2014) the formula of MAPE is given by:

$$\text{MAPE} = \frac{\sum_{i=1}^{n}\frac{|Y_i - F_i|}{Y_i}}{n}, \tag{5.1}$$

where Y_i and F_i represent actual price and forecast price at day i, respectively. While, n the total of forecasting days.

Lawrence et al. (2009) determined the scale of judgment of forecast accuracy using MAPE as illustrated in Table 5.2.

Table 5.2

The scale of judgment of forecast accuracy using MAPE

Accuracy	**MAPE**
Highly accurate	$\text{MAPE} < 10\%$
Good accurate	$10\% \leq \text{MAPE} < 20\%$
Reasonable	$20\% \leq \text{MAPE} < 50\%$
Inaccurate	$\text{MAPE} \geq 50\%$

The next subsection discusses the results of forecasted values for selected indices in some selected markets and their characteristics.

5.1 Characteristic of Market Indices

In this study, we will select market indices, as market index reflects the performance of economic growth and financial stability of particular country.

We will focus our discussion to three stock market indices, i.e. the Standard and Poor's 500 (S&P 500) of USA, Shanghai Stock Exchange (SSE) Composite Index of China and FTSE Bursa Malaysia Kuala Lumpur Composite Index (KLCI) of Malaysia. These three indexes are selected as they represent different nature of the global markets. S&P 500 index is the most developed and efficient market (Malkiel, 1989), KLCI index comes from emerging market segment (Ibrahim, 1999), while SSE index is of interest due to its vibrancy and its exposure to high volatility and uncertainties of Chinese market (Tripathy and Rahman, 2013).

5.1.1 Standard and Poor's 500

The Standard and Poor's 500 (S&P 500) index is the most standout amongst recognized lists in the USA, and is regarded as the best gauge of the stock market's large public companies in US. It contains 500 top market leaders which mirror the most noteworthy level of aggregate conduct among its business sectors, in which its vast top is more than USD 10 billion. It is a market capitalization weighted index; along these lines the impact of changes in the cost of a vast company's stock numbers proportionately more than that of a littler firm. This index incorporates 379 modern, 74 budgetary, 37 utility, and 10 transportation firms which represents 70-80 percent of the aggregate market capitalization of all US firms exchanged the value showcase. As in February 2017, its market capital is US $21.4 trillion.

5.1.2 Shanghai Stock Exchange Composite Index

Shanghai Stock Exchange (SSE) is the biggest record in China, that is kept running by the China Securities Regulatory Commission (CRSC). The foundation of SSE stock trades in Shanghai and Shenzhen in the mid-1990s denoted the re-development of the Chinese securities exchange, which developed quickly and animated change of the money related framework and corporate administration (Yao et al., 2008). At exhibit, it involves 1,313 recorded organizations, 10,195 recorded securities and 1,357 recorded stocks and positions among the best five biggest stock trades of the world (Mansaku et al., 2016). The calculation of SSE Composite Index is weighted by the aggregate market estimation of recorded stocks where the monetary division stocks overwhelm the market (Yao et al., 2008). In spite of the fact that the record is open for residential and remote financial specialist, outside speculators need to experience firmly controlled qualified outside speculator framework. As in February 2016, its market capital is US $3.5 trillion.

5.1.3 FTSE Bursa Malaysia KLCI

FTSE Bursa Malaysia KLCI (KLCI) is a Malaysian stock, one of the greatest stock exchanges in Southeast Asia. The index is a partnership between FTSE Group and Bursa Malaysia designed to measure performance of major capital segments of the Malaysian market. KLCI includes 30 stocks tradable of index organizations by full market capitalizations that are representative, liquid and transparent. KLCI is figured from the costs of these organizations utilizing the market capitalization weighted technique and the arrival is regularly controlled by list variety every once in a while (Rahman et al., 2013; and Murthy et al., 2016). As in August 2017, its market capital is US $ 5.69 billion.

5.2 Validation of the Developed Model

To validate the performance of the developed model in this work, we will apply the proposed model to forecast the adjusted prices to these three markets, i.e. S&P 500, SSE and KLCI.

5.2.1 Forecasting the Performance of Standard and Poor's 500

5.2.1.1 Description of Data

The data is available online at *http://finance.yahoo.com*. The daily adjusted closed prices from 2nd January 2015 to 31st December 2015 are studied with total observations of 252 days. This duration is selected due to the appearance of long memory, i.e. $H > 0.5$. Return series (in logarithm) is considered to handle high volatility in the data. Figure 5.1 and Figure 5.2 show the adjusted prices and its return series.

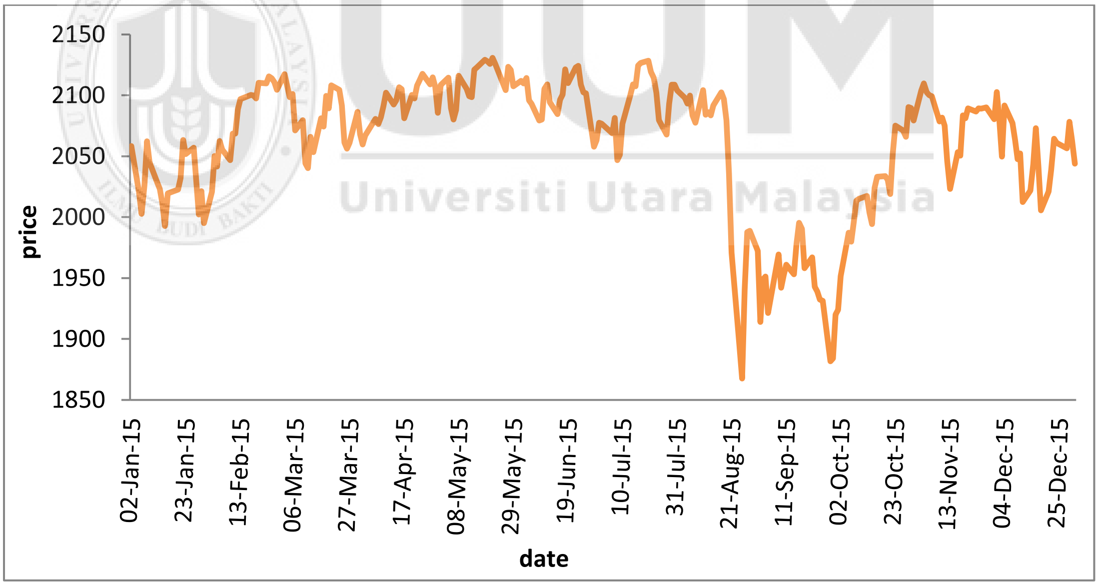


Figure 5.1. Daily adjusted price series of S&P 500 from 1st January 2015 to 31st December 2015

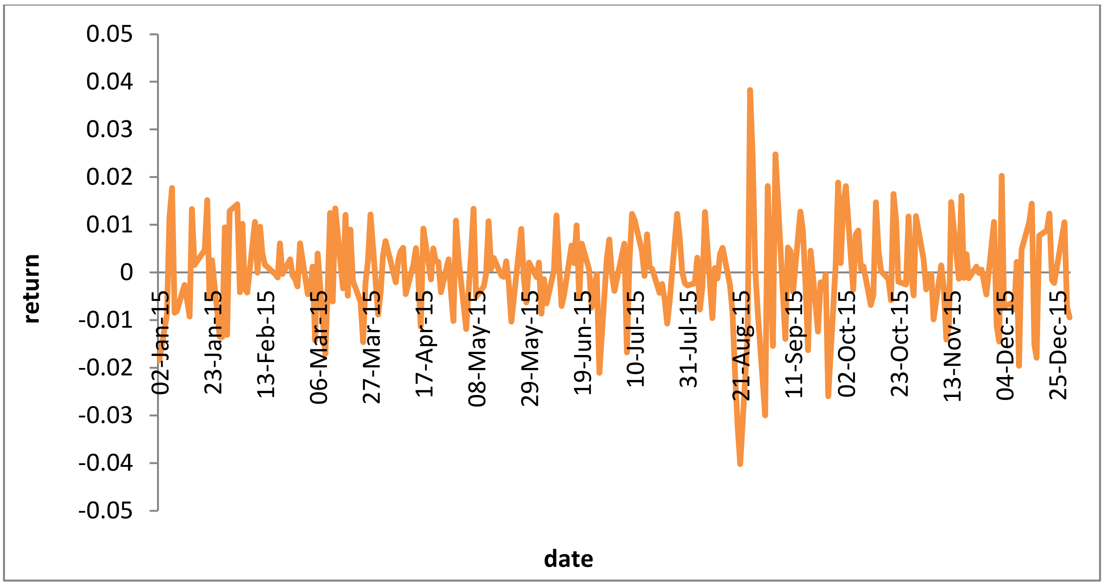


Figure 5.2. Daily returns series of S&P 500 from 1st January 2015 to 31st December 2015

5.2.1.2 Forecasting Standard and Poor's 500

In this subsection, we forecasted daily index prices of S&P 500 for year 2016. The intial parameters, i.e. H_1, H_2, μ, β, m, and αwere obtained from daily index prices in 2015. The values of H_1 =0.54 and H_2 =0.53 were obtained by using maximum likelihood function, while $\mu =$ -0.000028, $\beta =$ 0.00018, $m =$ 0.000096 and $\alpha =$1.0715 were determined by mean of return, volatility of volatility, mean of volatility and mean reverting parameter respectively.

First, we compute the value of volatility using four different formulas that are listed in Table 5.1, as presented in Table 5.3.

Table 5.3

The values of volatilities according to the formulas of Simple (S), Log (L), High-Low-Close (HLC) and Stochastic (STO)

Volatility type	**S**	**L**	**HLC**	**STO**
Value	0.098841	0.09141	0.04876	0.02394

Second, we forecast the adjusted prices by using both GBM and GFBM models (i.e. Definitions 1.15 and 1.16) as its underlying process via the volatility values listed in Table 5.3. The forecasted prices were computed by using eight models include GBM-S, GFBM-S, GBM-L, GFBM-L, GBM-HLC, GFBM-HLC, GBM-STO and GFBM-STO. MAPE values are compared where the smallest value is considered the best. Table 5.4 shows the forecasted prices using eight models and actual prices of S&P500. Table 5.5 represents the level of accuracy of the models.

Table 5.4

Forecasted Prices and Actual Prices of S&P 500 with MAPE

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
05 Jan	2046.37	2035.04	2049.95	2034.6	2039.81	2034.23	2038.4	2035.68	**2016.71**
06 Jan	2057.65	2048.99	2062.07	2047.48	2045.36	2041.02	2041.1	2038.96	**1990.26**
07 Jan	2047.08	2049.18	2051.15	2047.67	2039.9	2041.16	2038.38	2039.05	**1943.09**
08 Jan	2032.77	2044.03	2037.5	2042.97	2033.15	2038.83	2035.16	2037.96	**1922.03**
11 Jan	2039.12	2050.51	2043.24	2048.95	2036.28	2041.97	2036.68	2039.48	**1923.67**
12 Jan	2036.53	2040.66	2040.3	2039.78	2035.19	2036.91	2036.2	2036.96	**1938.68**
13 Jan	2056.03	2042.15	2060.61	2041.24	2044.62	2037.94	2040.76	2037.53	**1890.28**
14 Jan	2047.38	2060.19	2051.32	2057.94	2040.3	2046.94	2038.63	2041.95	**1921.84**
15 Jan	2054.93	2059.4	2059.34	2057.04	2043.86	2045.94	2040.33	2041.31	**1880.33**
19 Jan	2035.14	2057.15	2039.18	2055	2034.55	2044.94	2035.9	2040.85	**1881.33**
20 Jan	2038.81	2056.44	2042.58	2054.37	2036.3	2044.69	2036.74	2040.76	**1859.33**
21 Jan	2058.97	2042.77	2063.14	2041.75	2046.08	2038.01	2041.46	2037.51	**1868.99**
22 Jan	2045.8	2065.52	2049.84	2062.79	2039.59	2049.26	2038.3	2043	**1906.9**
25 Jan	2045.91	2066.55	2049.91	2063.68	2039.76	2049.56	2038.42	2043.11	**1877.08**
26 Jan	2040.7	2067.88	2044.52	2065.03	2037.16	2050.65	2037.14	2043.74	**1903.63**
27 Jan	2026.04	2046.33	2029.71	2045.19	2030.1	2040.3	2033.74	2038.76	**1882.95**
28 Jan	2047.9	2051.81	2052.85	2050.15	2040.64	2042.62	2038.82	2039.8	**1893.36**
29 Jan	2057.73	2038.39	2061.4	2037.72	2045.32	2035.91	2041.06	2036.51	**1940.24**
01 Feb	2030.52	2045.36	2034.97	2044.23	2032.18	2039.61	2034.72	2038.37	**1939.38**
02 Feb	2054.35	2047.81	2058.26	2046.48	2043.68	2040.79	2040.27	2038.94	**1903.03**
03 Feb	2042.68	2036.6	2047.08	2036.05	2038.25	2035.02	2037.7	2036.07	**1912.53**
04 Feb	2057.91	2054.72	2061.92	2052.85	2045.76	2044.13	2041.36	2040.55	**1915.45**
05 Feb	2058.4	2047.92	2062.54	2046.55	2045.82	2040.7	2041.34	2038.86	**1880.05**
08 Feb	2066.67	2035.49	2071.22	2035.07	2050.07	2034.64	2043.46	2035.92	**1853.44**
09 Feb	2044.45	2045.33	2048.27	2044.19	2039.32	2039.57	2038.26	2038.35	**1852.21**
10 Feb	2035.31	2055.26	2039.21	2053.42	2034.62	2044.62	2035.93	2040.84	**1851.86**
11 Feb	2048.1	2036.28	2052.35	2035.83	2041.13	2035.1	2039.15	2036.16	**1829.08**
12 Feb	2037.27	2039.34	2041.53	2038.62	2035.67	2036.45	2036.46	2036.77	**1864.78**
16 Feb	2065.58	2052.91	2069.81	2051.21	2049.51	2043.35	2043.18	2040.2	**1895.58**
17 Feb	2052.47	2045.78	2056.54	2044.53	2042.96	2039.5	2039.96	2038.23	**1926.82**
18 Feb	2039.22	2057.07	2042.88	2055.06	2036.58	2045.37	2036.89	2041.17	**1917.83**
19 Feb	2027.82	2032.71	2032.37	2032.49	2030.95	2033.21	2034.14	2035.21	**1917.78**
22 Feb	2038.64	2046.65	2043.38	2045.34	2036	2039.96	2036.54	2038.47	**1945.5**
23 Feb	2053.35	2054.23	2056.99	2052.43	2043.76	2043.96	2040.44	2040.48	**1921.27**
24 Feb	2048.87	2053.3	2052.93	2051.47	2041.56	2043.15	2039.37	2040	**1929.8**
25 Feb	2055.52	2053.43	2060.18	2051.78	2044.48	2043.91	2040.71	2040.54	**1951.7**

Table 5.4 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
26 Feb	2031.9	2056.95	2036.14	2054.99	2033.02	2045.48	2035.17	2041.26	**1948.05**
29 Feb	2062.9	2063.48	2066.31	2060.97	2047.99	2048.49	2042.39	2042.69	**1932.23**
01 Mar	2040.54	2064.86	2044.86	2062.22	2037.12	2049.07	2037.13	2042.95	**1978.35**
02 Mar	2057.08	2030.31	2061.07	2030.31	2045.65	2032.17	2041.38	2034.74	**1986.45**
03 Mar	2047.78	2044.77	2052.2	2043.77	2040.83	2039.61	2038.97	2038.44	**1993.4**
04 Mar	2041.08	2036.51	2045	2036.05	2037.77	2035.25	2037.54	2036.25	**1999.99**
07 Mar	2044.53	2052.64	2048.89	2050.93	2039.5	2043.06	2038.38	2040.02	**2001.76**
08 Mar	2048.46	2056.09	2052.6	2054.17	2041.04	2044.95	2039.04	2040.98	**1979.26**
09 Mar	2053.5	2054.52	2057.26	2052.7	2043.67	2044.12	2040.36	2040.56	**1989.26**
10 Mar	2046.85	2044.83	2050.54	2043.75	2040.59	2039.37	2038.91	2038.25	**1989.57**
11 Mar	2065.35	2059.34	2069.43	2057.24	2049.21	2046.79	2042.98	2041.93	**2022.19**
14 Mar	2025.32	2049.12	2029.54	2047.78	2029.83	2041.73	2033.63	2039.46	**2019.64**
15 Mar	2049.12	2038.59	2053.11	2038.01	2041.59	2036.41	2039.36	2036.84	**2015.93**
16 Mar	2039.66	2048.91	2043.88	2047.52	2036.64	2041.38	2036.89	2039.23	**2027.22**
17 Mar	2051.76	2063.4	2056.1	2060.93	2042.78	2048.58	2039.92	2042.76	**2040.59**
18 Mar	2046.2	2056.03	2050.19	2054.01	2040.27	2044.58	2038.74	2040.72	**2049.58**
21 Mar	2065	2041.17	2069.34	2040.37	2049.56	2037.6	2043.28	2037.4	**2051.60**
22 Mar	2034.98	2060.93	2039.23	2058.67	2034.59	2047.45	2035.94	2042.23	**2049.8**
23 Mar	2050.71	2049.54	2054.93	2048.1	2042.44	2041.71	2039.79	2039.4	**2036.71**
24 Mar	2075.67	2038.41	2079.32	2037.82	2054.38	2036.24	2045.52	2036.74	**2035.94**
28 Mar	2044.9	2059.18	2049.18	2056.98	2039.79	2046.35	2038.55	2041.63	**2037.05**
29 Mar	2041.14	2045.09	2046.46	2044.05	2037.67	2039.71	2037.46	2038.47	**2055.01**
30 Mar	2037.4	2051.09	2041.79	2049.52	2036.06	2042.4	2036.73	2039.72	**2063.95**
31 Mar	2032.79	2047.3	2037.01	2046.11	2033.79	2040.88	2035.63	2039.06	**2059.74**
01 Apr	2056.09	2053.15	2061.22	2051.44	2044.88	2043.47	2040.93	2040.26	**2072.78**
04 Apr	2043.18	2049.46	2047.67	2048.09	2038.87	2041.9	2038.08	2039.55	**2066.13**
05 Apr	2064.78	2044.47	2067.96	2043.49	2049.16	2039.45	2043.01	2038.35	**2045.17**
06 Apr	2059.07	2046.79	2063.79	2045.58	2046.38	2040.43	2041.66	2038.8	**2066.66**
07 Apr	2065.79	2040.03	2070.57	2039.37	2049.55	2037.23	2043.18	2037.27	**2041.91**
08 Apr	2028.47	2062.92	2032.24	2060.48	2031.53	2048.32	2034.49	2042.62	**2047.6**
11 Apr	2037.84	2052.33	2041.88	2050.78	2036.16	2043.4	2036.75	2040.3	**2041.99**
12 Apr	2053.08	2064.6	2057.3	2062.03	2043.61	2049.12	2040.36	2043	**2061.72**
13 Apr	2041.67	2047.45	2045.6	2046.29	2037.95	2041.12	2037.6	2039.21	**2082.42**
14 Apr	2050.81	2036.46	2055.6	2036.06	2042.36	2035.46	2039.72	2036.41	**2082.78**
15 Apr	2039.66	2048.92	2043.56	2047.64	2037.4	2041.79	2037.44	2039.53	**2080.73**
18 Apr	2045.15	2045.78	2049.26	2044.76	2039.85	2040.34	2038.57	2038.85	**2094.34**
19 Apr	2039.47	2038.69	2043.14	2038.19	2036.98	2036.79	2037.15	2037.1	**2100.8**

Table 5.4 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
20 Apr	2049.25	2035.22	2053.42	2035.04	2042.22	2035.29	2039.8	2036.42	**2049.25**
21 Apr	2049.5	2052.19	2052.94	2050.69	2042.03	2043.52	2039.63	2040.4	**2049.50**
22 Apr	2056.81	2052.43	2060.8	2050.91	2045.65	2043.6	2041.4	2040.43	**2056.81**
25 Apr	2046.79	2051.62	2050.61	2050.08	2040.91	2042.89	2039.15	2040.01	**2046.79**
26 Apr	2038.21	2057.93	2043.22	2056.02	2036.36	2046.4	2036.85	2041.82	**2038.21**
27 Apr	2045.66	2041.37	2049.47	2040.63	2040.15	2037.96	2038.72	2037.64	**2045.66**
28 Apr	2044.79	2038.65	2049.37	2038.13	2039.53	2036.68	2038.38	2037.03	**2044.79**
29 Apr	2050.53	2046.88	2054.15	2045.75	2042.88	2040.75	2040.13	2039.02	**2050.53**
02 May	2042.38	2057.32	2046.49	2055.44	2038.67	2046.05	2038.04	2041.63	**2042.38**
03 May	2022.62	2037.61	2026.28	2037.16	2028.8	2036.14	2033.19	2036.76	**2022.62**
04 May	2046.14	2046.54	2050.81	2045.37	2040.45	2040.37	2038.88	2038.78	**2046.14**
05 May	2043.08	2039.43	2047.04	2038.85	2038.81	2037.04	2038.05	2037.2	**2043.08**
06 May	2045.78	2044.41	2049.9	2043.53	2039.99	2039.79	2038.59	2038.61	**2045.78**
09 May	2057.96	2047.42	2061.85	2046.24	2046.2	2041	2041.67	2039.13	**2057.96**
10 May	2054.63	2045.54	2059.35	2044.51	2044.82	2040.1	2041.06	2038.7	**2054.63**
11 May	2017.53	2038.23	2021.46	2037.86	2026.44	2036.88	2032.08	2037.22	**2017.53**
12 May	2045.1	2046.59	2049.63	2045.52	2039.8	2040.77	2038.53	2039.07	**2045.10**
13 May	2045.92	2036.31	2049.84	2036.02	2040.48	2035.71	2038.93	2036.6	**2045.92**
16 May	2048.69	2048.61	2052.62	2047.42	2042	2041.89	2039.71	2039.64	**2048.69**
17 May	2049.51	2046.39	2052.77	2045.42	2042.31	2040.99	2039.84	2039.24	**2049.51**
18 May	2057.57	2037.15	2061.41	2036.82	2046.16	2036.21	2041.68	2036.87	**2057.57**
19 May	2043.83	2045.8	2048.1	2044.79	2039.44	2040.36	2038.42	2038.86	**2043.83**
20 May	2040.91	2023.4	2045.36	2024.08	2038.18	2029.34	2037.85	2033.49	**2040.91**
23 May	2058.61	2034.33	2062.93	2034.19	2046.54	2034.74	2041.84	2036.12	**2058.61**
24 May	2032.9	2045.54	2037.08	2044.62	2034.11	2040.48	2035.84	2038.98	**2032.90**
25 May	2051.23	2045.21	2055.31	2044.27	2043.08	2040.16	2040.19	2038.78	**2051.23**
26 May	2043.06	2046.46	2046.34	2045.54	2039.02	2041.19	2038.2	2039.38	**2043.06**
27 May	2047.95	2047.74	2052.23	2046.6	2041.73	2041.37	2039.6	2039.36	**2047.95**
31 May	2054.17	2054.47	2058.18	2052.81	2044.59	2044.66	2040.94	2040.96	**2054.17**
01 Jun	2040.94	2042.59	2044.92	2041.93	2038.09	2039.16	2037.78	2038.37	**2040.94**
02 Jun	2037.77	2049.45	2042.36	2048.19	2036.6	2042.25	2037.07	2039.8	**2037.77**
03 Jun	2043.56	2050.66	2048.25	2049.29	2039.43	2042.78	2038.45	2040.04	**2043.56**
06 Jun	2050.32	2032.57	2054.42	2032.62	2042.49	2034.09	2039.87	2035.87	**2050.32**
07 Jun	2051.23	2037.1	2054.9	2036.7	2042.92	2035.92	2040.07	2036.66	**2051.23**
08 Jun	2039.72	2050.2	2043.33	2048.93	2037.23	2042.81	2037.3	2040.12	**2039.72**
09 Jun	2027.55	2047.77	2031.44	2046.6	2031.65	2041.31	2034.68	2039.32	**2027.55**
10 Jun	2049.88	2040.04	2054.36	2039.51	2042.61	2037.72	2040.01	2037.62	**2049.88**

Table 5.4 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
13 Jun	2042.20	2047.20	2045.84	2046.11	2039.08	2041.16	2038.35	2039.28	**2079.06**
14 Jun	2046.13	2048.72	2050.23	2047.50	2041.06	2041.85	2039.33	2039.59	**2075.32**
15 Jun	2057.05	2057.48	2061.64	2055.71	2046.13	2046.56	2041.72	2041.99	**2071.50**
16 Jun	2037.23	2032.01	2041.74	2032.11	2036.56	2033.82	2037.11	2035.74	**2077.99**
17 Jun	2039.13	2040.16	2043.31	2039.68	2037.53	2037.98	2037.58	2037.79	**2071.22**
20 Jun	2043.88	2055.04	2047.23	2053.39	2039.67	2045.13	2038.59	2041.24	**2083.25**
21 Jun	2036.27	2049.68	2040.72	2048.44	2036.00	2042.53	2036.81	2039.97	**2088.90**
22 Jun	2036.36	2046.39	2040.47	2045.41	2035.98	2040.93	2036.79	2039.21	**2085.45**
23 Jun	2054.84	2046.99	2058.96	2045.91	2045.11	2041.02	2041.24	2039.19	**2113.32**
24 Jun	2025.01	2039.90	2029.17	2039.38	2030.33	2037.64	2034.02	2037.58	**2037.41**
27 Jun	2045.19	2048.01	2049.41	2046.91	2040.16	2041.75	2038.78	2039.61	**2000.54**
28 Jun	2033.26	2027.87	2037.37	2028.29	2034.57	2031.81	2036.13	2034.76	**2036.09**
29 Jun	2054.97	2038.67	2059.20	2038.34	2045.33	2037.36	2041.38	2037.53	**2070.77**
30 Jun	2040.95	2037.45	2044.68	2037.22	2038.38	2036.80	2037.99	2037.25	**2098.86**
01 July	2060.82	2043.88	2064.99	2043.14	2047.96	2039.86	2042.60	2038.72	**2102.95**
05 July	2045.68	2039.57	2049.40	2039.12	2040.72	2037.62	2039.13	2037.60	**2088.55**
06 July	2049.39	2036.09	2053.53	2035.95	2042.42	2036.09	2039.93	2036.90	**2099.73**
07 July	2045.23	2044.83	2050.08	2044.01	2040.32	2040.32	2038.89	2038.95	**2097.90**
08 July	2044.88	2054.20	2048.84	2052.63	2040.30	2044.77	2038.92	2041.07	**2129.90**
11 July	2041.33	2054.76	2045.48	2053.21	2038.61	2045.26	2038.11	2041.36	**2137.16**
12 July	2051.02	2045.62	2055.11	2044.73	2043.41	2040.66	2040.46	2039.10	**2152.14**
13 July	2030.83	2049.51	2034.37	2048.22	2033.46	2042.20	2035.61	2039.76	**2152.43**
14 July	2030.20	2035.69	2034.26	2035.52	2032.90	2035.68	2035.28	2036.65	**2163.75**
15 July	2051.12	2031.66	2054.42	2031.82	2043.29	2033.76	2040.36	2035.73	**2161.74**
18 July	2045.14	2044.80	2049.30	2043.96	2040.45	2040.22	2039.00	2038.87	**2166.89**
19 July	2038.86	2053.55	2042.43	2052.05	2037.43	2044.51	2037.55	2040.95	**2163.78**
20 July	2050.58	2036.31	2054.11	2036.11	2043.11	2036.03	2040.29	2036.83	**2173.02**
21 July	2032.56	2051.41	2035.95	2050.13	2034.24	2043.67	2035.97	2040.60	**2165.17**
22 July	2033.38	2051.89	2036.93	2050.54	2034.44	2043.77	2036.02	2040.61	**2175.03**
25 July	2058.23	2052.77	2062.70	2051.40	2046.59	2044.38	2041.91	2040.96	**2168.48**
26 July	2029.68	2039.77	2034.00	2039.30	2032.96	2037.71	2035.38	2037.64	**2169.18**
27 July	2039.87	2037.34	2043.83	2037.12	2038.10	2036.73	2037.92	2037.22	**2166.58**
28 July	2060.45	2044.39	2064.64	2043.60	2047.62	2040.07	2042.40	2038.81	**2170.06**
29 July	2035.40	2049.49	2040.12	2048.38	2035.74	2042.84	2036.72	2040.22	**2173.60**
01 Aug	2033.13	2037.21	2037.36	2037.07	2034.66	2036.93	2036.21	2037.38	**2170.84**
02 Aug	2037.02	2052.56	2040.93	2051.18	2036.48	2044.20	2037.08	2040.85	**2157.03**
03 Aug	2033.84	2049.94	2037.95	2048.79	2035.12	2043.00	2036.47	2040.28	**2163.79**

Table 5.4 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
04 Aug	2063.74	2044.46	2068.27	2043.70	2049.89	2040.24	2043.66	2038.93	**2164.25**
05 Aug	2032.05	2026.40	2036.27	2027.01	2034.29	2031.37	2036.07	2034.62	**2182.87**
08 Aug	2047.67	2040.88	2051.70	2040.47	2041.74	2038.78	2039.64	2038.29	**2180.89**
09 Aug	2045.59	2031.70	2049.96	2031.90	2040.86	2033.96	2039.25	2035.87	**2181.74**
10 Aug	2044.97	2044.48	2048.97	2043.73	2040.54	2040.32	2039.08	2038.98	**2175.49**
11 Aug	2033.58	2014.63	2037.27	2016.13	2034.90	2025.59	2036.33	2031.80	**2185.79**
12 Aug	2039.53	2043.80	2043.47	2043.11	2038.07	2039.99	2037.94	2038.82	**2184.05**
15 Aug	2041.78	2045.68	2045.62	2044.74	2039.01	2040.55	2038.35	2039.00	**2190.15**
16 Aug	2042.23	2041.97	2046.13	2041.40	2039.23	2039.04	2038.45	2038.35	**2178.15**
17 Aug	2040.91	2040.00	2046.02	2039.61	2038.42	2038.15	2038.02	2037.93	**2182.22**
18 Aug	2054.58	2048.09	2058.34	2047.16	2045.35	2042.39	2041.44	2040.06	**2187.02**
19 Aug	2034.51	2037.13	2038.81	2036.92	2035.87	2036.62	2036.93	2037.16	**2183.87**
22 Aug	2046.23	2036.06	2050.29	2036.05	2041.43	2036.55	2039.58	2037.24	**2182.64**
23 Aug	2042.16	2038.83	2046.35	2038.52	2039.39	2037.55	2038.57	2037.64	**2186.90**
24 Aug	2040.90	2045.85	2045.25	2045.03	2038.87	2041.08	2038.35	2039.37	**2175.44**
25 Aug	2044.04	2024.23	2047.85	2025.08	2039.88	2030.58	2038.71	2034.29	**2172.47**
26 Aug	2046.48	2038.43	2050.85	2038.15	2041.30	2037.36	2039.46	2037.55	**2169.04**
29 Aug	2049.32	2040.43	2053.86	2040.01	2042.74	2038.39	2040.17	2038.06	**2180.38**
30 Aug	2050.22	2042.41	2053.95	2041.85	2043.31	2039.41	2040.47	2038.56	**2176.12**
31 Aug	2047.52	2040.71	2052.01	2040.35	2041.65	2038.82	2039.59	2038.34	**2170.95**
01 Sep	2054.90	2059.96	2059.12	2058.05	2045.33	2047.97	2041.39	2042.71	**2170.86**
02 Sep	2050.76	2052.21	2054.17	2050.93	2043.78	2044.30	2040.75	2040.96	**2179.98**
06 Sep	2045.60	2039.66	2049.28	2039.36	2041.07	2038.25	2039.39	2038.05	**2186.48**
07 Sep	2048.57	2033.00	2052.87	2033.18	2042.39	2034.85	2040.01	2036.36	**2186.16**
08 Sep	2038.33	2042.87	2042.28	2042.26	2037.18	2039.57	2037.43	2038.63	**2181.30**
09 Sep	2031.27	2043.21	2035.30	2042.62	2033.82	2039.91	2035.82	2038.83	**2127.81**
12 Sep	2046.50	2033.88	2050.79	2034.09	2041.25	2035.62	2039.42	2036.82	**2159.04**
13 Sep	2050.36	2043.55	2054.44	2042.90	2043.45	2039.93	2040.56	2038.81	**2127.02**
14 Sep	2048.51	2040.93	2052.71	2040.53	2042.63	2038.81	2040.18	2038.30	**2125.77**
15 Sep	2034.66	2048.53	2038.78	2047.56	2035.68	2042.60	2036.77	2040.16	**2147.26**
16 Sep	2043.91	2058.17	2047.45	2056.47	2040.49	2047.34	2039.17	2042.47	**2139.16**
19 Sep	2014.59	2038.54	2019.06	2038.30	2025.81	2037.59	2031.96	2037.69	**2139.12**
20 Sep	2027.18	2037.98	2030.63	2037.78	2032.18	2037.30	2035.11	2037.55	**2139.76**
21 Sep	2041.92	2046.08	2045.70	2045.26	2039.52	2041.28	2038.70	2039.49	**2163.12**
22 Sep	2043.11	2025.71	2047.63	2026.46	2039.97	2031.35	2038.89	2034.68	**2177.18**
23 Sep	2031.43	2045.33	2035.84	2044.55	2033.99	2040.83	2035.93	2039.25	**2164.69**
26 Sep	2038.79	2039.34	2042.41	2038.97	2037.87	2037.74	2037.87	2037.71	**2146.10**

Table 5.4 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
27 Sep	2041.48	2042.58	2045.27	2042.03	2039.22	2039.57	2038.53	2038.66	**2159.93**
28 Sep	2052.52	2040.04	2056.55	2039.74	2044.56	2038.52	2041.11	2038.19	**2171.37**
29 Sep	2044.82	2041.78	2048.90	2041.33	2040.80	2039.29	2039.29	2038.55	**2151.13**
30 Sep	2024.96	2048.70	2028.86	2047.80	2031.07	2042.97	2034.56	2040.40	**2168.27**
03 Oct	2059.59	2037.91	2063.27	2037.76	2048.32	2037.44	2043.01	2037.66	**2161.20**
04 Oct	2017.75	2044.67	2021.75	2043.96	2027.37	2040.57	2032.72	2039.13	**2150.49**
05 Oct	2030.71	2040.72	2034.39	2040.36	2033.75	2038.83	2035.83	2038.34	**2159.73**
06 Oct	2047.09	2044.85	2051.08	2044.21	2041.93	2040.96	2039.84	2039.40	**2160.77**
07 Oct	2039.71	2024.55	2043.54	2025.43	2038.54	2030.93	2038.25	2034.51	**2153.74**
10 Oct	2045.59	2041.73	2049.84	2041.28	2041.66	2039.25	2039.82	2038.52	**2163.66**
11 Oct	2036.06	2047.36	2039.90	2046.52	2036.29	2042.16	2037.05	2039.98	**2136.73**
12 Oct	2036.90	2041.32	2041.16	2040.99	2036.87	2039.38	2037.37	2038.67	**2139.18**
13 Oct	2054.17	2053.64	2058.81	2052.35	2045.42	2045.34	2041.54	2041.55	**2132.55**
14 Oct	2049.66	2044.64	2054.24	2043.98	2043.33	2040.72	2040.55	2039.25	**2132.98**
17 Oct	2027.56	2035.15	2032.11	2035.23	2032.46	2036.13	2035.27	2037.04	**2126.50**
18 Oct	2043.58	2037.80	2047.23	2037.68	2040.14	2037.45	2038.95	2037.68	**2139.60**
19 Oct	2040.54	2041.63	2045.63	2041.19	2038.89	2039.25	2038.41	2038.53	**2144.29**
20 Oct	2028.41	2033.81	2032.40	2033.96	2033.04	2035.36	2035.59	2036.64	**2141.34**
21 Oct	2031.74	2032.38	2036.10	2032.64	2034.71	2034.69	2036.41	2036.31	**2141.16**
24 Oct	2037.59	2037.75	2041.22	2037.65	2037.32	2037.47	2037.61	2037.71	**2151.33**
25 Oct	2037.21	2052.69	2040.71	2051.42	2037.22	2044.69	2037.58	2041.18	**2143.16**
26 Oct	2034.89	2041.33	2038.79	2040.97	2036.24	2039.26	2037.15	2038.58	**2139.43**
27 Oct	2044.78	2053.49	2049.31	2052.22	2040.93	2045.32	2039.39	2041.55	**2133.04**
28 Oct	2044.78	2041.10	2049.36	2040.81	2041.10	2039.39	2039.51	2038.70	**2126.41**
31 Oct	2038.38	2031.96	2042.32	2032.33	2037.74	2034.75	2037.82	2036.41	**2126.15**
01 Nov	2032.75	2042.74	2037.28	2042.31	2034.95	2040.15	2036.46	2039.06	**2111.72**
02 Nov	2041.82	2052.41	2045.49	2051.26	2039.45	2044.89	2038.66	2041.36	**2097.94**
03 Nov	2038.59	2043.09	2042.54	2042.61	2038.04	2040.19	2038.02	2039.05	**2088.66**
04 Nov	2043.27	2056.24	2047.00	2054.73	2040.24	2046.53	2039.07	2042.10	**2085.18**
07 Nov	2037.25	2040.34	2041.92	2040.14	2037.42	2039.10	2037.73	2038.58	**2131.52**
08 Nov	2044.50	2038.89	2047.67	2038.77	2040.90	2038.27	2039.40	2038.15	**2139.56**
09 Nov	2050.86	2032.22	2054.85	2032.61	2044.15	2035.02	2041.01	2036.57	**2163.26**
10 Nov	2041.53	2048.11	2045.17	2047.26	2039.53	2042.70	2038.75	2040.27	**2167.48**
11 Nov	2041.74	2051.90	2046.26	2050.82	2039.78	2044.78	2038.91	2041.34	**2164.45**
14 Nov	2027.46	2026.93	2030.92	2027.64	2032.69	2032.12	2035.44	2035.10	**2164.20**
15 Nov	2038.61	2027.61	2042.57	2028.23	2037.86	2032.33	2037.89	2035.16	**2180.39**
16 Nov	2043.77	2023.02	2048.06	2024.14	2040.51	2030.62	2039.20	2034.47	**2176.94**

Table 5.4 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
17 Nov	2027.70	2044.46	2031.33	2043.92	2032.79	2041.02	2035.49	2039.49	**2187.12**
18 Nov	2036.69	2032.04	2041.07	2032.44	2037.33	2034.94	2037.72	2036.53	**2181.90**
21 Nov	2037.52	2046.79	2041.98	2046.14	2037.58	2042.41	2037.81	2040.22	**2198.18**
22 Nov	2050.68	2046.15	2054.84	2045.50	2044.00	2041.91	2040.92	2039.93	**2202.94**
23 Nov	2040.12	2031.27	2044.26	2031.76	2039.01	2034.63	2038.54	2036.41	**2204.72**
25 Nov	2031.92	2017.98	2035.82	2019.46	2034.74	2028.07	2036.41	2033.21	**2213.35**
28 Nov	2032.77	2038.89	2037.23	2038.71	2035.22	2038.08	2036.65	2038.01	**2201.72**
29 Nov	2026.34	2047.79	2030.59	2046.97	2032.07	2042.57	2035.12	2040.21	**2204.66**
30 Nov	2047.65	2032.09	2052.19	2032.46	2042.68	2034.87	2040.32	2036.48	**2198.81**
01 Dec	2034.00	2025.78	2037.75	2026.66	2035.67	2031.88	2036.84	2035.05	**2191.08**
02 Dec	2044.79	2010.35	2048.35	2012.40	2041.10	2024.31	2039.51	2031.37	**2191.95**
05 Dec	2046.37	2025.27	2050.17	2026.27	2041.85	2031.89	2039.87	2035.12	**2204.71**
06 Dec	2038.78	2057.55	2042.61	2056.03	2038.43	2047.49	2038.27	2042.64	**2212.23**
07 Dec	2053.03	2043.11	2056.30	2042.67	2045.32	2040.35	2041.60	2039.16	**2241.35**
08 Dec	2033.32	2034.36	2036.87	2034.60	2035.28	2036.12	2036.63	2037.12	**2246.19**
09 Dec	2035.50	2031.29	2039.17	2031.78	2036.60	2034.69	2037.33	2036.45	**2259.53**
12 Dec	2035.42	2034.57	2039.66	2034.75	2036.56	2036.04	2037.31	2037.03	**2256.96**
13 Dec	2052.59	2039.93	2056.43	2039.77	2045.12	2038.94	2041.50	2038.51	**2271.72**
14 Dec	2037.80	2046.56	2041.68	2045.85	2037.70	2042.03	2037.86	2039.97	**2253.28**
15 Dec	2040.45	2028.27	2044.15	2028.95	2039.60	2033.03	2038.93	2035.59	**2262.03**
16 Dec	2025.25	2021.89	2029.46	2023.12	2031.65	2030.15	2034.95	2034.25	**2258.07**
19 Dec	2048.45	2040.89	2052.42	2040.62	2043.45	2039.28	2040.78	2038.64	**2262.53**
20 Dec	2044.24	2042.32	2048.54	2041.98	2040.88	2040.09	2039.41	2039.06	**2270.76**
21 Dec	2040.06	2046.62	2044.42	2045.90	2038.87	2042.06	2038.44	2039.99	**2265.18**
22 Dec	2036.70	2035.46	2039.93	2035.68	2037.29	2036.90	2037.69	2037.56	**2260.96**
23 Dec	2054.30	2045.34	2058.11	2044.78	2045.66	2041.64	2041.69	2039.83	**2263.79**
27 Dec	2048.30	2046.63	2052.06	2045.90	2042.91	2042.01	2040.40	2039.95	**2268.88**
28 Dec	2052.95	2034.66	2057.81	2034.85	2045.26	2036.17	2041.57	2037.12	**2249.92**
29 Dec	2032.71	2035.21	2037.48	2035.39	2035.54	2036.54	2036.90	2037.32	**2249.26**
30 Dec	2033.33	2035.67	2037.88	2035.86	2035.83	2036.91	2037.03	2037.54	**2238.83**
MAPE	**4.7365%**	**4.7330%**	**4.7129%**	**4.7256%**	**4.7281%**	**4.7263%**	**4.7362%**	**4.7038%**	

Table 5.5

The level of accuracy ranking for forecasting model of S&P 500

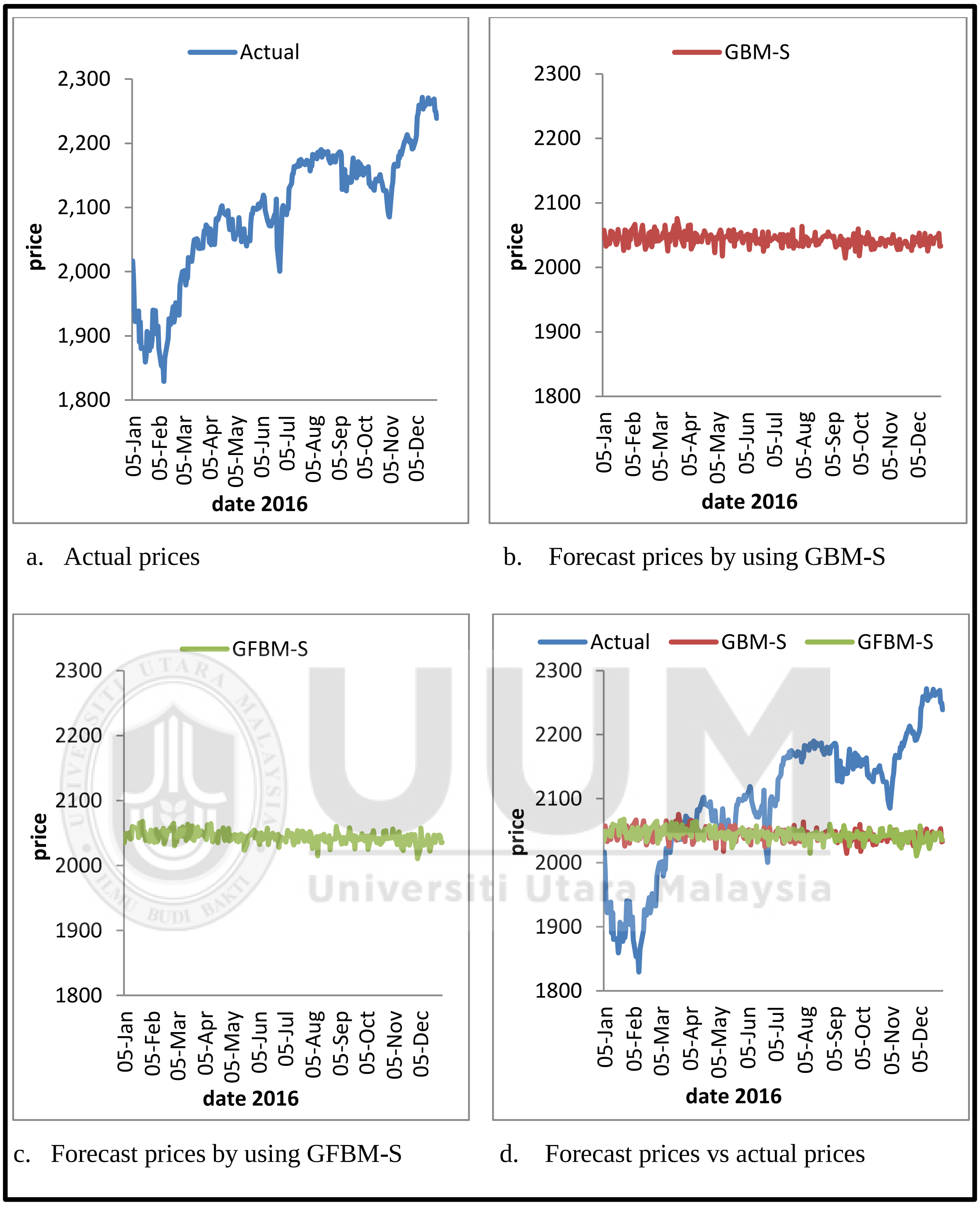
Rank	**Model**	**MAPE**
1	GFBM-STO*	4.7038%
2	GBM-L	4.7129%
3	GFBM-L	4.7256%
4	GFBM-HLC	4.7263%
5	GBM-HLC	4.7281%
6	GFBM-S	4.7330%
7	GBM-STO	4.7362%
8	GBM-S	4.7365%

* The proposed model

Table 5.4 and Table 5.5 proposed that all values of MAPE are relatively close, indicating that forecasts by both GBM and GFBM models are highly accurate (less than 10%). However, we can observe that GFBM models are more accurate than GBM models when the volatility computed by STO, HLC, and S. These findings are consistent with Painter (1998), Willinger et al. (1999), Grau–Carles (2000), and Rejichi and Aloui (2012) that suggesting long memory model are best suited in empirical analysis.

From the findings, the proposed model GFBM-STO demonstrates the most accurate in its performance, whereas GBM-S performed the worst. This result indicates to the efficiency and the ability of the proposed model to be applied in real financial environments.

Figures 5.3-5.6 compare the actual prices versus forecasted prices in GBM and GFBM model with four different volatilities in previously mentioned Table 5.1. These figures indicated that the forecasted prices are closer together and less fluctuated than the actual prices.



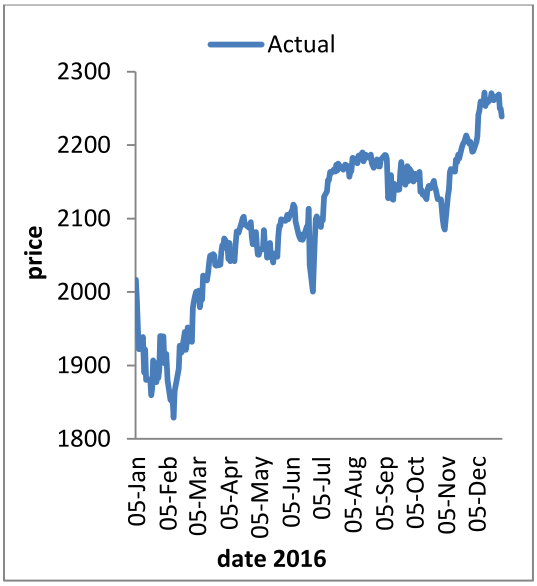
a. Actual prices

b. Forecast prices by using GBM-S

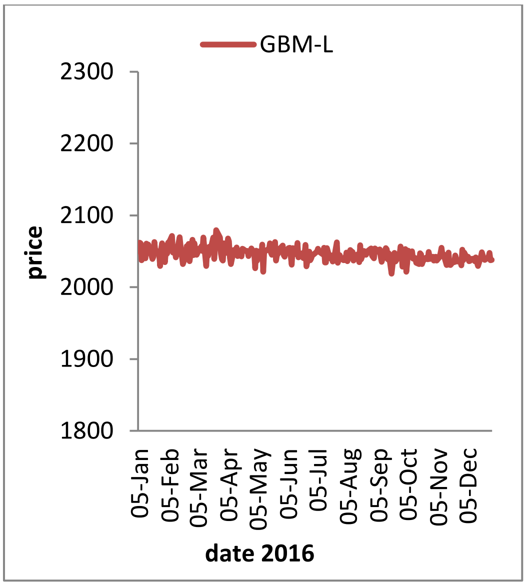
c. Forecast prices by using GFBM-S

d. Forecast prices vs actual prices

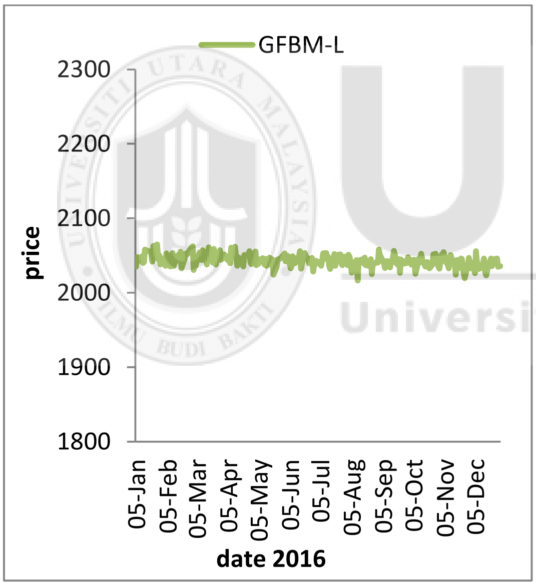
Figure 5.3. Forecast prices of S&P 500 by using GBM-S and GFBM-S vs actual prices



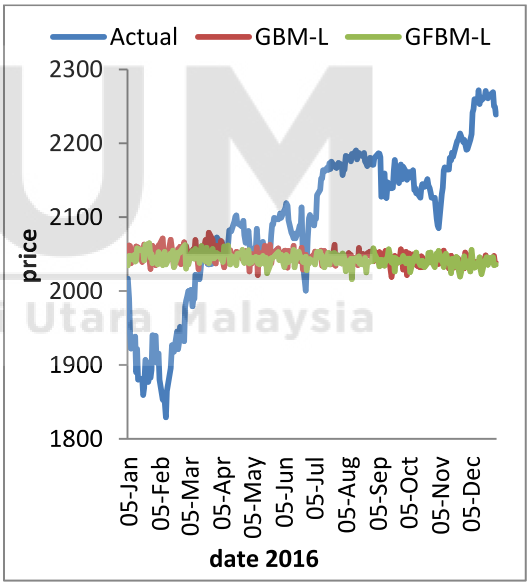
a. Actual prices



b. Forecast prices by using GBM-L

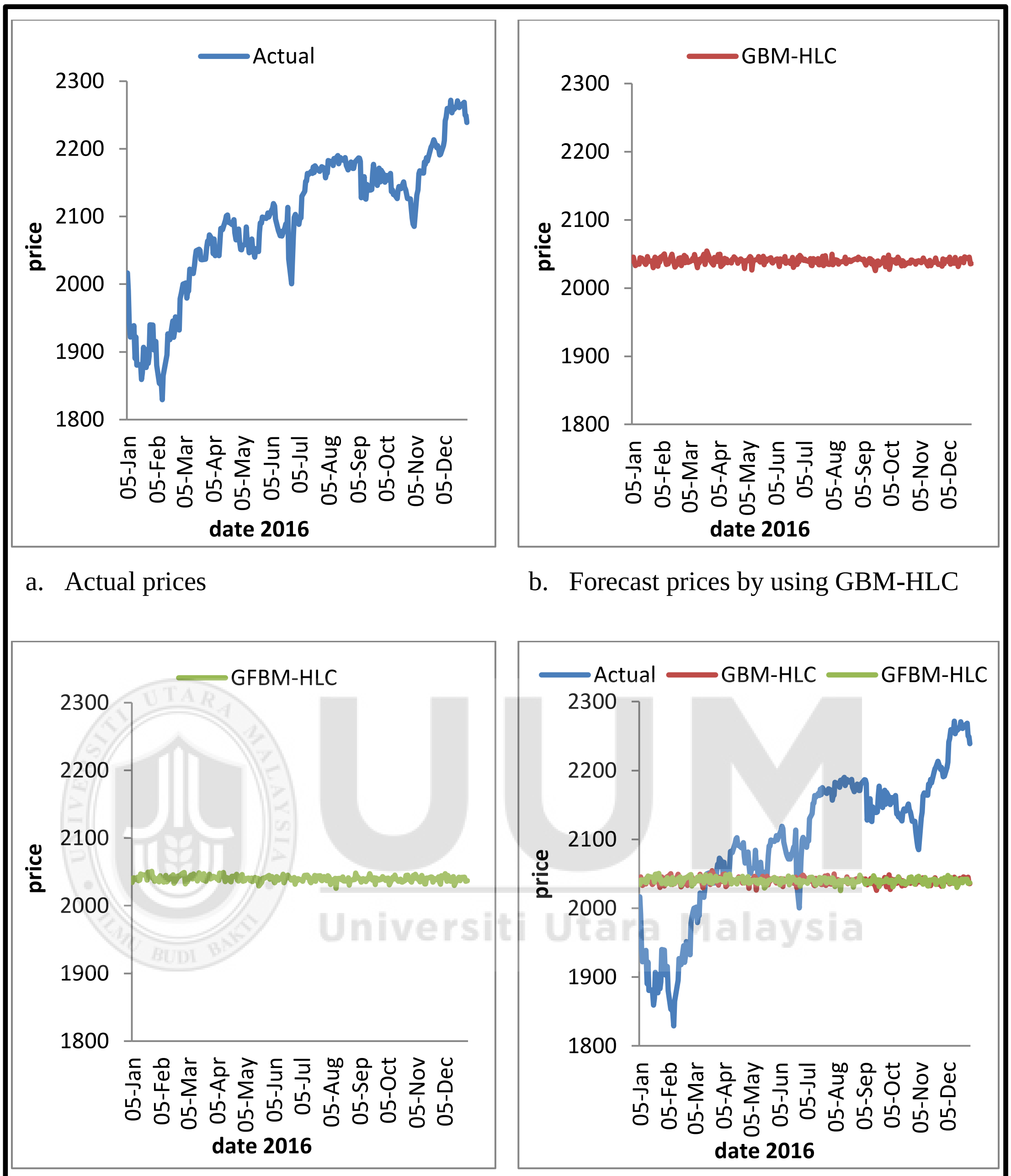


c. Forecast prices by using GFBM-L



d. Forecast prices vs actual prices

Figure 5.4. Forecast prices of S&P 500 by using GBM-L and GFBM-L vs actual prices



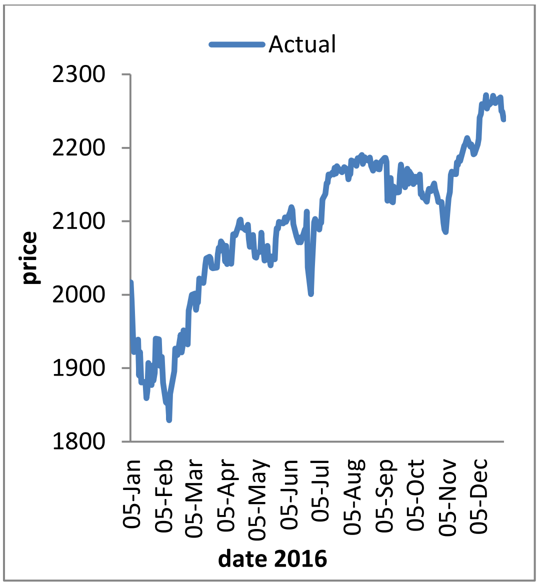
a. Actual prices

b. Forecast prices by using GBM-HLC

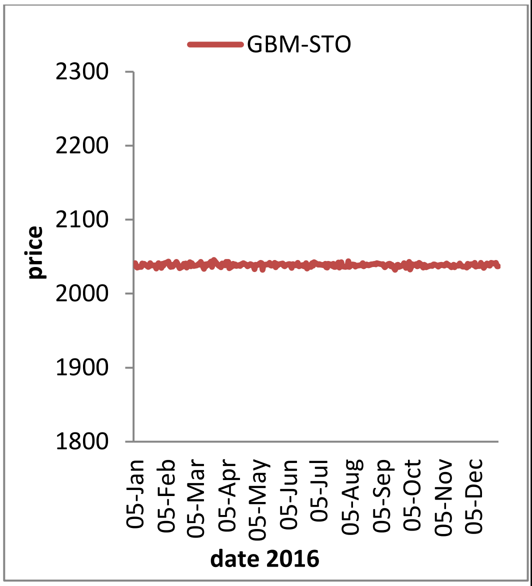
c. Forecast prices by using GFBM-HLC

d. Forecast prices vs actual prices

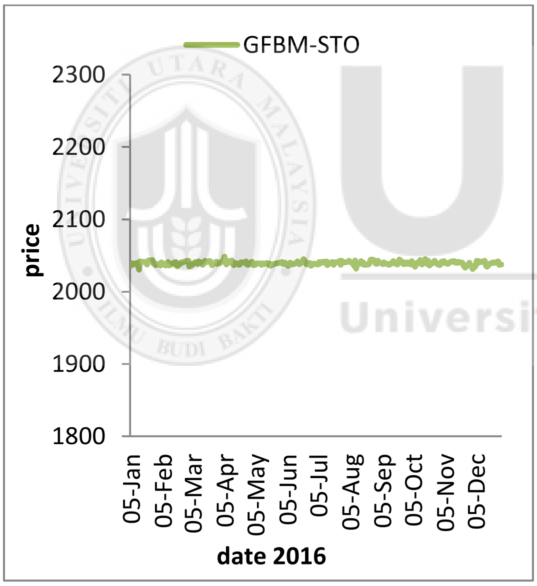
Figure 5.5. Forecast prices of S&P 500 by using GBM-HLC and GFBM-HLC vs actual prices



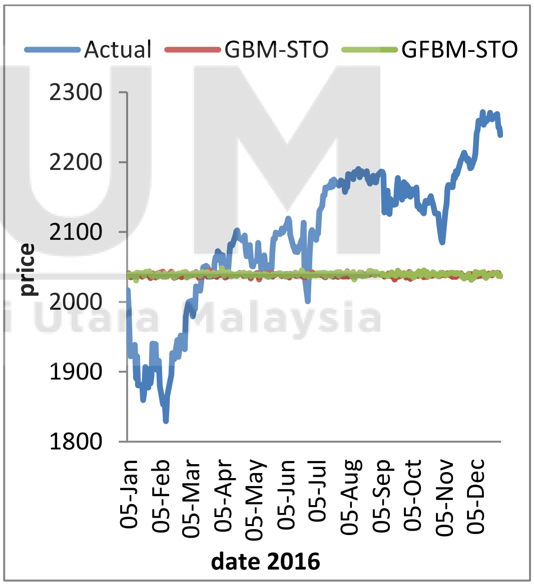
a. Actual prices



b. Forecast prices by using GBM-STO



c. Forecast prices by using GFBM-STO



d. Forecast prices vs actual prices

Figure 5.6. Forecast prices of S&P 500 by using GBM-STO and GFBM-STO vs actual prices

5.2.2 Forecasting the Performance of Shanghai Stock Exchange Composite Index

5.2.2.1 Description of data

The daily adjusted closed prices from 5th January 2015 to 31st December 2015 are studied with total observations of 233 days. The data is available online at *http://finance.yahoo.com*. This duration is selected because it exhibits long memory property. Return series (in logarithm) is considered to handle high volatility in the data. Figure 5.7 and Figure 5.8 show adjusted prices and its return series.

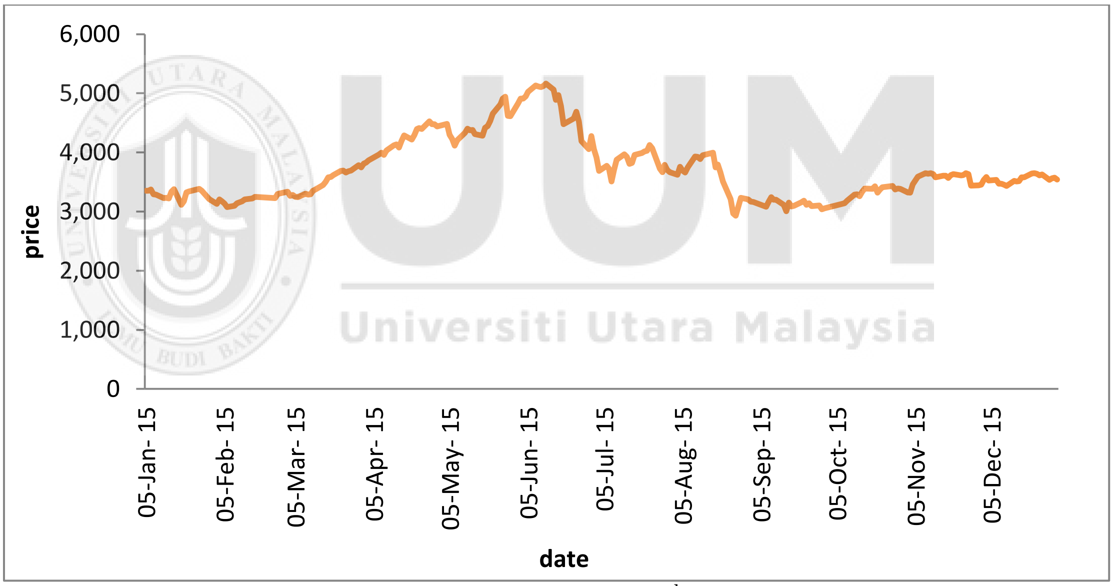


Figure 5.7. Daily adjust price series of SSE from 5th January 2015 to 31st December 2015

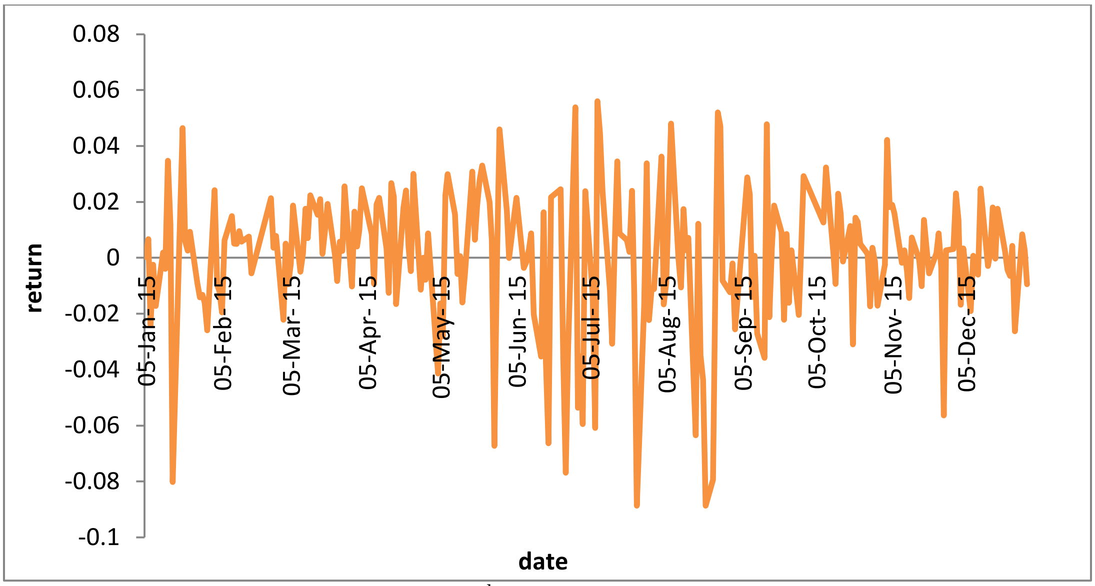


Figure 5.8. Daily returns SSE from 5th January 2015 to 31st December 2015

5.2.2.2 Forecasting the Shanghai Stock Exchange Composite Index

In this subsection, we forecasted daily index prices of SSE for year 2016 using similar procedures as discussed in 5.2.1.2 to the initial parameters of H_1 =0.5991, H_2 =0.6126, μ =0.00024, β =0.0013, m =0.00064 and α =1.6574.

First, we compute the value of volatility using three different formulas in Table 5.6 (S, L and STO), with exception of HLC since all of open, closed, high, low and adjusted prices have same value in same day. The values for all involved volatilities are presented in Table 5.6.

Table 5.6

The values of volatilities according to the formulas of Simple (S), Log (L), and Stochastic (STO)

Volatility type	**S**	**L**	**STO**
Value	0.100306	0.09438	0.14848

Second, we forecast the adjusted prices by using both GBM and GFBM models as its underlying process via all these volatility values listed in Table 5.6. The forecasted prices computed by using six models include GBM-S, GFBM-S, GBM-L, GFBM-L, GBM-STO and GFBM-STO. The comparison is conducted through the value MAPE where the smallest value is considered the best. Table 5.7 shows the forecasted prices in addition to actual prices of SSE, while Table 5.8 represents the level of accuracy of the models.

Table 5.7

Forecasted Prices and Actual Prices of SSE with MAPE

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-STO	GFBM-STO	**Actual**
05 Jan	3315.96	3307.67	3321.59	3306.12	3336.95	3324.49	**3287.71**
06 Jan	3324.84	3318.45	3332.37	3316.23	3350.28	3340.91	**3361.84**
07 Jan	3286.35	3316.85	3292.58	3314.68	3292.84	3338.98	**3125**
08 Jan	3326.33	3319.88	3331.91	3317.58	3352.91	3342.96	**3186.41**
11 Jan	3302.73	3321.81	3309.31	3319.35	3316.1	3364.45	**3016.7**
12 Jan	3293.72	3303.15	3299.42	3301.89	3304.18	3317.4	**3022.86**
13 Jan	3316.2	3329.71	3321.3	3326.87	3338.27	3356.94	**2949.6**
14 Jan	3295.54	3313.66	3301.4	3311.71	3305.99	3333.93	**3007.65**
15 Jan	3297.94	3311.27	3303.53	3309.47	3308.94	3330.27	**2900.97**
18 Jan	3314.59	3317.01	3322.1	3314.95	3334.95	3337.72	**2913.84**
19 Jan	3301.21	3355.41	3307.82	3350.95	3314.4	3396.37	**3007.74**
20 Jan	3319.03	3320	3325.77	3317.74	3340.95	3342.46	**2976.69**
21 Jan	3315.1	3289.8	3321.25	3289.38	3334.41	3297.07	**2880.48**
22 Jan	3311.45	3329.17	3316.73	3326.43	3329.72	3355.31	**2916.56**
25 Jan	3286.61	3332.36	3292.59	3329.46	3293.15	3359.68	**2938.51**
26 Jan	3311.44	3280.21	3317.98	3280.31	3329.44	3283.34	**2749.79**
27 Jan	3312.26	3275.87	3320.24	3276.26	3330.36	3276.47	**2735.56**
28 Jan	3293.07	3302.69	3300.78	3301.46	3301.32	3316.7	**2655.66**
29 Jan	3302.47	3315.44	3308.85	3313.41	3317.15	3336.25	**2737.6**
01 Feb	3307.66	3321.69	3314.39	3319.33	3324.07	3344.96	**2688.85**
02 Feb	3310.24	3308.79	3316.88	3307.24	3327.15	3325.18	**2749.57**
03 Feb	3306.92	3315.38	3313.74	3313.39	3321.65	3335.76	**2739.25**
04 Feb	3296.63	3332.65	3303.44	3329.6	3307.44	3361.87	**2781.02**
05 Feb	3312.2	3315.31	3318.05	3313.3	3330.9	3335.8	**2763.49**
15 Feb	3286.83	3306.96	3293.58	3305.56	3293.21	3322.07	**2746.2**
16 Feb	3336.85	3309.02	3343.9	3307.43	3366.69	3325.85	**2836.57**
17 Feb	3330.39	3318.97	3336.98	3316.84	3358.39	3340.07	**2867.34**
18 Feb	3324.51	3300.04	3331.67	3299.09	3348.81	3311.15	**2862.89**
22 Feb	3344.24	3312.16	3350.2	3310.54	3378.13	3328.62	**2927.18**
24 Feb	3270.14	3329.12	3276.49	3326.26	3267.74	3356.68	**2928.9**
25 Feb	3286.91	3300.64	3293.02	3299.6	3293.34	3312.84	**2741.25**
26 Feb	3312.6	3313.89	3317.93	3312.04	3331.26	3332.82	**2767.21**
29 Feb	3272.15	3296.76	3278.38	3295.9	3270.35	3307.6	**2687.98**
01 Mar	3317.09	3326.8	3323.99	3324.17	3337.48	3352.19	**2733.17**
02 Mar	3299.5	3294.97	3307.02	3294.27	3313.69	3304.36	**2849.68**
03 Mar	3328.95	3334.44	3335.37	3331.43	3353.79	3362.61	**2859.76**
04 Mar	3308.73	3332.8	3315.76	3329.82	3324.97	3361	**2874.15**
07 Mar	3305.07	3361.03	3311.24	3356.27	3319.09	3404.22	**2897.34**
08 Mar	3294.97	3284.86	3301.39	3284.68	3304.29	3290.41	**2901.39**
09 Mar	3329.41	3322.43	3335.3	3320.14	3355.53	3344.54	**2862.56**

Table 5.7 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-STO	GFBM-STO	Actual
10 Mar	3315.52	3302.5	3321.33	3301.43	3334.41	3314.63	**2804.73**
11 Mar	3323.72	3318	3330.03	3316.03	3347.14	3337.28	**2810.31**
14 Mar	3316.85	3336.97	3323.87	3333.83	3336.57	3366.07	**2859.5**
15 Mar	3334.75	3310.82	3340.36	3309.08	3362.91	3329.05	**2864.37**
16 Mar	3298.99	3304.45	3305.36	3303.2	3310.52	3318.3	**2870.43**
17 Mar	3303.45	3319.36	3310.72	3317.28	3315.13	3339.78	**2904.83**
18 Mar	3313.66	3306.58	3320.19	3305.22	3331.86	3321.22	**2955.15**
21 Mar	3296.48	3277.39	3302.99	3277.79	3305.39	3277.51	**3018.8**
22 Mar	3300.07	3320.04	3306.37	3317.92	3311.86	3340.72	**2999.36**
23 Mar	3295.74	3290.13	3301.67	3289.7	3304.35	3297.33	**3009.96**
24 Mar	3305.03	3300.04	3311.44	3299.03	3318.87	3312.07	**2960.97**
25 Mar	3350.71	3300.9	3358.06	3299.9	3386.94	3312.46	**2979.43**
28 Mar	3301.31	3288.96	3308.27	3288.68	3313.53	3294.54	**2957.82**
29 Mar	3295.23	3322.57	3301.34	3320.23	3304.37	3345.38	**2919.83**
30 Mar	3304.9	3331.97	3312.54	3329.11	3317.48	3358.93	**3000.65**
31 Mar	3337.39	3334.46	3343.45	3331.49	3366.12	3362.08	**3003.92**
01 Apr	3302.66	3305.55	3309.03	3304.29	3314.57	3319.26	**3009.53**
05 Apr	3316.88	3312.18	3322.53	3310.54	3336.26	3328.88	**3053.07**
06 Apr	3294.92	3311.4	3301.86	3309.74	3302.41	3328.52	**3050.59**
07 Apr	3307.42	3309.26	3313.8	3307.84	3320.49	3323.86	**3008.42**
08 Apr	3316.68	3319.41	3323.84	3317.34	3335.04	3339.72	**2984.96**
11 Apr	3298.61	3328.64	3304.07	3326.07	3309.36	3352.68	**3033.96**
13 Apr	3301.06	3298.39	3307.41	3297.59	3311.41	3308.12	**3066.64**
14 Apr	3272.76	3286.15	3279.58	3286.11	3269.94	3289.38	**3082.36**
15 Apr	3325.33	3298.73	3333.02	3297.94	3346.7	3308.31	**3078.12**
18 Apr	3339.37	3284.24	3345.64	3284.31	3369.81	3286.69	**3033.66**
19 Apr	3302.43	3335.09	3310.02	3332.09	3313.39	3362.87	**3042.82**
20 Apr	3286.22	3311.35	3292.12	3309.8	3289.5	3327.21	**2972.58**
21 Apr	3309.44	3307.22	3315.1	3306.04	3322.68	3319.33	**2952.89**
22 Apr	3305.91	3331.11	3313.24	3328.47	3319.36	3355.33	**2959.24**
25 Apr	3312.91	3290.87	3320.97	3290.5	3329.27	3297.17	**2946.67**
26 Apr	3294.49	3320.02	3302.08	3317.94	3302.41	3340.13	**2964.7**
27 Apr	3340.05	3308.48	3346.88	3307.18	3368.78	3321.82	**2953.67**
28 Apr	3295.7	3308.79	3302.37	3307.49	3303.08	3322.06	**2945.59**
29 Apr	3296.42	3289.78	3303.32	3289.51	3304.33	3294.95	**2938.32**
03 May	3298.32	3291.79	3304.49	3291.46	3306.99	3297.3	**2992.64**
04 May	3313.75	3297.76	3319.35	3297.03	3330.78	3306.67	**2991.27**
05 May	3295.85	3294.99	3301.23	3294.46	3303.16	3302.25	**2997.84**
06 May	3290.33	3301.32	3297.63	3300.43	3295.51	3311.37	**2913.25**
09 May	3330.6	3303.68	3337.44	3302.61	3355.39	3315.45	**2832.11**
10 May	3281.43	3301.83	3287.98	3300.95	3281.55	3311.72	**2832.59**

Table 5.7 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-STO	GFBM-STO	Actual
11 May	3317.48	3288.85	3323.97	3288.7	3335.6	3292.99	**2837.04**
12 May	3337.56	3302.5	3344.43	3301.51	3366.3	3313.41	**2835.86**
13 May	3318.64	3325.62	3324.63	3323.31	3336.65	3347.19	**2827.11**
16 May	3320.38	3307.38	3327.66	3306.12	3339.88	3320.59	**2850.86**
17 May	3298.38	3310.55	3304.44	3309.22	3306.52	3323.71	**2843.68**
18 May	3306.81	3285.61	3313.26	3285.75	3321.02	3286.76	**2807.51**
19 May	3336.94	3315.06	3343.88	3313.44	3364.66	3330.61	**2806.91**
20 May	3288.63	3314.91	3293.8	3313.27	3292.51	3330.97	**2825.48**
24 May	3322.56	3324.04	3330	3321.65	3342.54	3347.08	**2821.67**
25 May	3324.79	3327.89	3331.56	3325.43	3344.95	3350.71	**2815.09**
27 May	3287.8	3317.27	3294.47	3315.44	3291.36	3335	**2821.05**
30 May	3331.14	3275.23	3337.66	3275.96	3355.79	3271.75	**2822.45**
31 May	3327.24	3321.32	3333.47	3319.29	3349.37	3340.44	**2916.62**
01 June	3302.73	3324.93	3308.81	3322.84	3312.17	3343.81	**2913.51**
02 June	3306.58	3316.91	3312.94	3315.08	3317.89	3334.89	**2925.23**
03 June	3304	3332.52	3309.88	3329.72	3314.65	3358.47	**2938.68**
06 June	3299.5	3313.58	3305.28	3311.95	3308.01	3329.85	**2934.1**
07 June	3292.19	3292.5	3298.53	3292.2	3297.77	3297.48	**2936.04**
08 June	3287.98	3303.86	3297.12	3302.78	3290.89	3315.7	**2927.16**
13 June	3286.85	3300.51	3294.57	3299.63	3288.81	3310.63	**2833.07**
14 June	3326.85	3302.38	3334.6	3301.48	3349.25	3312.19	**2842.19**
15 June	3296.89	3305.73	3303.09	3304.6	3304.2	3317.68	**2887.21**
16 June	3298.43	3307.98	3304.16	3306.71	3306.3	3321.14	**2872.82**
17 June	3280.07	3315.62	3286.48	3314.04	3278.69	3330.57	**2885.1**
21 June	3318.56	3317.14	3325.81	3315.43	3336.85	3333.4	**2878.56**
22 June	3301.95	3298.89	3308.42	3298.3	3310.54	3305.8	**2905.55**
23 June	3279.25	3297.14	3284.86	3296.66	3276.27	3303.13	**2891.96**
24 June	3318.03	3299.95	3324.23	3299.19	3335.4	3308.8	**2854.29**
27 June	3312.72	3291.19	3319.62	3291.07	3327.32	3294.2	**2895.7**
28 June	3287.78	3309.34	3295.74	3307.91	3289.07	3324.11	**2912.56**
29 June	3289.7	3286.35	3296.99	3286.34	3292.25	3289.37	**2931.59**
30 June	3296.46	3304.97	3303.58	3303.97	3302.98	3315.37	**2929.61**
01 July	3317.33	3284.91	3323.57	3285	3335.67	3287.01	**2932.48**
04 July	3302.26	3303.12	3309.07	3302.22	3311.14	3312.94	**2988.6**
05 July	3305.91	3317.44	3312.03	3315.78	3318.11	3333.06	**3006.39**
06 July	3310.07	3315.76	3316.5	3314.18	3321.65	3330.83	**3017.29**
07 July	3300.5	3296.54	3306.36	3296.06	3309.29	3302.77	**3016.85**
08 July	3317.76	3321.59	3324.2	3319.48	3333.2	3341.77	**2988.09**
11 July	3296.89	3315.22	3303.31	3313.58	3302.05	3331.09	**2994.92**
12 July	3291.05	3289.47	3297.44	3289.42	3293.74	3292.03	**3049.38**
13 July	3310.16	3298.56	3316.66	3297.9	3323.37	3306.46	**3060.69**

Table 5.7 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-STO	GFBM-STO	Actual
14 July	3319.29	3340.41	3326.04	3337.4	3337.38	3366.94	**3054.02**
15 July	3297.66	3282.14	3304.16	3282.54	3304.56	3281.04	**3054.3**
18 July	3316.68	3295.55	3323.77	3295.16	3333.54	3300.88	**3043.56**
19 July	3309	3305.05	3315.64	3304.06	3320.95	3315.47	**3036.6**
20 July	3314.55	3314.94	3321.25	3313.4	3329.31	3329.75	**3027.9**
21 July	3307.15	3291.4	3312.85	3291.3	3317.51	3294.1	**3039.01**
22 July	3302.22	3301.17	3308.74	3300.4	3310.49	3309.85	**3012.82**
25 July	3287.96	3303.89	3294.52	3303.06	3289.4	3312.48	**3015.83**
26 July	3316.18	3305.87	3323.25	3304.98	3333.01	3314.81	**3050.17**
27 July	3287.41	3294.5	3294.6	3294.08	3287.62	3300.46	**2992**
28 July	3326.08	3307.5	3332.49	3306.53	3345.39	3316.91	**2994.32**
29 July	3322.64	3306.74	3330.07	3305.74	3340.76	3316.71	**2979.34**
01 Aug	3308.38	3292.31	3314.9	3292.03	3319.77	3297.03	**2953.39**
02 Aug	3320.66	3336.79	3327.21	3333.88	3337.13	3363.08	**2971.28**
03 Aug	3302.42	3291.47	3309.29	3291.37	3310.51	3294.19	**2978.46**
04 Aug	3291.93	3285.66	3297.64	3285.86	3293.87	3286.24	**2982.43**
05 Aug	3275.2	3309.57	3281.19	3308.37	3269.71	3321.37	**2976.7**
08 Aug	3333.08	3270.16	3339.98	3271.3	3356.11	3262.83	**3004.28**
09 Aug	3331.58	3308.69	3337.83	3307.64	3354.21	3318.88	**3025.68**
10 Aug	3295.09	3325.74	3301.79	3323.58	3299.36	3345.41	**3018.75**
11 Aug	3284.17	3294.93	3290.68	3294.67	3282.91	3298.8	**3002.64**
12 Aug	3308.25	3298.57	3313.94	3298.03	3318.34	3304.94	**3050.67**
15 Aug	3309.45	3294.71	3314.81	3294.46	3320.04	3298.43	**3125.2**
16 Aug	3313.69	3299.37	3319.47	3298.69	3327.08	3307.41	**3110.04**
17 Aug	3304.76	3277.39	3311.4	3278.22	3312.83	3272.15	**3109.55**
18 Aug	3311.56	3313.19	3318.11	3311.95	3323.38	3324.48	**3104.11**
19 Aug	3308.63	3285.06	3315.25	3285.38	3318.96	3284.14	**3108.1**
22 Aug	3292.23	3321.34	3298.5	3319.57	3294.78	3337.2	**3084.81**
23 Aug	3302.85	3322.66	3310.15	3320.81	3309.81	3339.3	**3089.71**
24 Aug	3294.28	3284.99	3300.47	3285.38	3297.88	3283.23	**3085.88**
25 Aug	3302.9	3332.77	3309.26	3330.37	3312.57	3353.59	**3068.33**
26 Aug	3311.36	3335.22	3317.22	3332.55	3323.48	3358.84	**3070.31**
29 Aug	3299.2	3330.61	3305.11	3328.19	3304.74	3352.32	**3070.03**
30 Aug	3284.27	3297.91	3290.16	3297.5	3282.12	3302.88	**3074.68**
31 Aug	3321.6	3301.57	3328.83	3300.96	3337.97	3308.07	**3085.49**
01 Sep	3339.25	3295.22	3345.48	3295.1	3363.98	3297.15	**3063.31**
05 Sep	3288.96	3318.07	3296.23	3316.5	3287.69	3332.38	**3072.1**
06 Sep	3311.83	3309.76	3318.22	3308.63	3322.54	3320.56	**3090.71**
07 Sep	3309.26	3280.54	3316.37	3281.15	3319.75	3277.28	**3091.93**
08 Sep	3282.6	3276.3	3289.36	3277.28	3279.51	3269.4	**3095.95**
09 Sep	3306.19	3312.73	3313.78	3311.46	3314.85	3324.52	**3078.85**

Table 5.7 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-STO	GFBM-STO	Actual
12 Sep	3307.67	3306.84	3314.27	3305.97	3316.42	3315.32	**3021.98**
13 Sep	3324.64	3309.99	3331.15	3308.9	3342.63	3320.28	**3023.51**
14 Sep	3297.98	3323.16	3303.79	3321.33	3303.16	3339.27	**3002.85**
19 Sep	3310.35	3303.9	3317.32	3303.15	3321.64	3311.46	**3026.05**
20 Sep	3301.64	3265.45	3308.46	3267	3308.61	3254.28	**3023**
21 Sep	3313.9	3311.8	3320.82	3310.57	3326.41	3323.44	**3025.87**
22 Sep	3290.62	3288.15	3296.33	3288.37	3290.55	3287.82	**3042.31**
23 Sep	3280.42	3297.63	3287.02	3297.23	3275.71	3302.61	**3033.9**
26 Sep	3315.87	3299.72	3322.53	3299.22	3328.35	3305.31	**2980.43**
27 Sep	3299.74	3310.55	3306.4	3309.59	3304.55	3319.07	**2998.17**
28 Sep	3324.28	3318.81	3330.78	3317.26	3340.85	3332.52	**2987.86**
29 Sep	3281.04	3271.74	3288.68	3272.93	3276.63	3263.48	**2998.48**
30 Sep	3287.92	3313.77	3293.76	3312.52	3287.99	3325.13	**3004.7**
10 Oct	3313.08	3292.45	3319.86	3292.46	3324.39	3293.55	**3048.14**
11 Oct	3313.94	3312.67	3321.87	3311.44	3325.44	3323.96	**3065.25**
12 Oct	3310.59	3287.41	3318.66	3287.69	3320.08	3286.47	**3058.5**
13 Oct	3294.54	3294.11	3300.56	3294.02	3296.26	3295.92	**3061.35**
14 Oct	3294.87	3312.54	3301.75	3311.38	3297.65	3323.01	**3063.81**
18 Oct	3313.75	3299.74	3320.9	3299.27	3325.33	3304.92	**3083.88**
19 Oct	3290.07	3277.32	3296.47	3278.28	3288.86	3270.43	**3084.72**
20 Oct	3307.74	3316.07	3314.08	3314.63	3314.38	3329.21	**3084.46**
21 Oct	3283.11	3280.43	3289.9	3281.23	3280.51	3274.68	**3090.94**
24 Oct	3292.11	3304.54	3298.15	3303.85	3291.04	3311.25	**3128.25**
25 Oct	3307.99	3320.64	3314.69	3318.91	3314.83	3336.27	**3131.94**
26 Oct	3322.81	3285.98	3330.44	3286.48	3338.18	3282.64	**3116.31**
27 Oct	3302.11	3302.47	3308.28	3301.93	3308.1	3307.9	**3112.35**
28 Oct	3302.1	3285.83	3308.43	3286.28	3307.79	3283.03	**3104.27**
31 Oct	3310.61	3285.05	3316.44	3285.63	3320.25	3280.91	**3100.49**
01 Nov	3316.02	3290.02	3323.06	3290.25	3328.31	3288.95	**3122.44**
02 Nov	3292.33	3297.15	3298.75	3296.95	3292.93	3299.63	**3102.73**
03 Nov	3306.25	3304.6	3313.2	3303.9	3314.3	3311.39	**3128.94**
04 Nov	3315.37	3279.71	3322.19	3280.41	3326.82	3275.49	**3125.32**
07 Nov	3326.32	3316.76	3333.01	3315.41	3343.01	3328.48	**3133.33**
08 Nov	3299.72	3337.43	3305.86	3334.89	3303.97	3358.82	**3147.89**
09 Nov	3312.68	3292.51	3319.2	3292.57	3322.66	3292.96	**3128.37**
10 Nov	3286.39	3314.58	3291.96	3313.33	3283.99	3325.73	**3171.28**
11 Nov	3312.83	3310.8	3318.66	3309.76	3322.86	3320.4	**3196.04**
14 Nov	3290.62	3328.95	3297.86	3326.93	3288.92	3346.03	**3210.37**
15 Nov	3304.27	3269.14	3312.07	3270.66	3310.83	3257.45	**3206.99**
16 Nov	3309.29	3301.48	3315.87	3301.03	3317.22	3306.05	**3205.06**
17 Nov	3297.15	3291.27	3304.55	3291.4	3298.45	3291.16	**3208.45**

Table 5.7 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-STO	GFBM-STO	Actual
18 Nov	3281.97	3307.52	3289.31	3306.72	3276.56	3314.84	**3192.86**
22 Nov	3290.88	3303.97	3297.51	3303.36	3290.95	3309.79	**3248.35**
23 Nov	3320.38	3306.45	3326.63	3305.75	3332.47	3312.87	**3241.14**
24 Nov	3264.48	3303.99	3270.43	3303.46	3250.27	3308.74	**3241.74**
25 Nov	3307.02	3295.15	3313.32	3295.09	3314.12	3296.47	**3261.94**
28 Nov	3319.75	3306.93	3326.03	3306.2	3332.3	3313.48	**3277**
29 Nov	3308.12	3288.63	3314.62	3289.04	3314.58	3285.64	**3282.92**
30 Nov	3295.2	3321.1	3301.26	3319.54	3296.1	3334.47	**3250.03**
01 Dec	3303.61	3307.21	3309.16	3306.59	3307.44	3312.28	**3273.31**
02 Dec	3309.08	3287.87	3315.43	3288.25	3317.03	3285.38	**3243.84**
05 Dec	3324.06	3300.06	3329.65	3299.68	3340.09	3304.03	**3204.71**
06 Dec	3308.68	3290.9	3315.7	3291.09	3315.66	3290.21	**3199.65**
07 Dec	3305.01	3301.58	3311.56	3301.16	3308.95	3305.69	**3222.24**
08 Dec	3300.72	3290.19	3306.97	3290.46	3303.63	3288.62	**3215.37**
09 Dec	3290.01	3292.66	3296.42	3292.84	3288.2	3291.56	**3232.88**
12 Dec	3306.52	3308.93	3311.6	3308.13	3312.23	3315.89	**3152.97**
13 Dec	3318.95	3304.39	3325.27	3303.84	3332.06	3309.4	**3155.04**
16 Dec	3278.11	3293.94	3284.33	3294.03	3270.27	3293.74	**3122.98**
19 Dec	3297.42	3300.21	3303.55	3299.93	3298.59	3302.96	**3118.08**
20 Dec	3319.91	3303.71	3326.65	3303.21	3332.53	3308.23	**3102.88**
21 Dec	3305.68	3301.76	3312.58	3301.42	3311.23	3304.76	**3137.43**
22 Dec	3290.88	3281.5	3297.35	3282.34	3289.99	3275.07	**3139.56**
23 Dec	3272.03	3307.08	3278.42	3306.34	3261	3313.71	**3110.15**
26 Dec	3293.64	3292.44	3300.66	3292.54	3292.34	3292.33	**3122.57**
27 Dec	3304.73	3284.21	3311.17	3284.83	3308.16	3279.76	**3114.66**
28 Dec	3312.13	3284.15	3318.45	3284.81	3319.83	3279.36	**3102.24**
29 Dec	3311.54	3297.44	3317.69	3297.37	3317.67	3298.24	**3096.1**
30 Dec	3304.34	3273.23	3309.86	3274.62	3309.36	3262	**3103.64**
MAPE	**10.315%**	**10.311%**	**10.531%**	**10.277%**	**10.669%**	**10.668%**	

Table 5.8

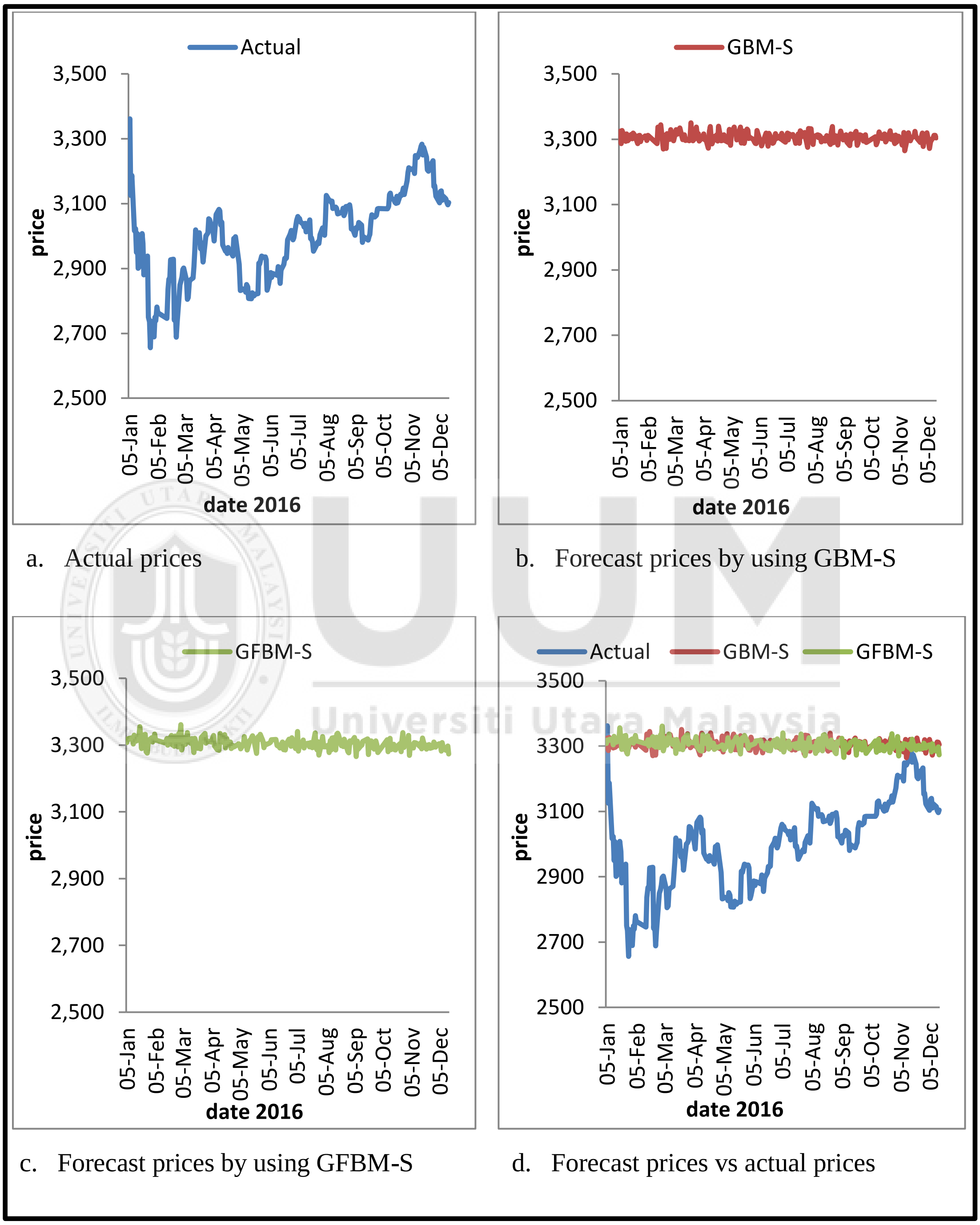
The level of accuracy ranking for forecasting model of SSE

Rank	**Model**	**MAPE**
1	GFBM-L	10.277%
2	GFBM-S	10.311%
3	GBM-S	10.315%
4	GBM-L	10.531%
5	GFBM-STO*	10.668%
6	GBM-STO	10.669%

* The proposed model

Table 5.7 and Table 5.8 suggested that the values of MAPE are relatively close when the volatility is computed by the formulas of simple volatility, log volatility or stochastic volatility provided that $\sigma(Y_t) = Y_t$, with the values between 10% and 11% indicating that both GBM and GFBM models have good accurate forecasts according to Lawrence et al. (2009) as stated in Table 5.3. We can also observe that GFBM model is significantly more accurate than GBM model in all cases involving different types of volatility, i.e. MAPE of GFBM is less than MAPE of GBM for all assumptions of volatility.

From the findings, GFBM-L demonstrates the most accurate in performance. The proposed model (GFBM-STO) has accuracy of less than 0.4% inferior than the best model. Such result indicates that the proposed model is efficient and can be applied in real financial environment. Figures 5.9-5.11 illustrate the comparison between the actual prices versus forecasted prices in GBM and GFBM model with three different volatilities in Table 5.1. These figures indicated that the forecasted prices are closer together and less fluctuated than the actual prices.



a. Actual prices

b. Forecast prices by using GBM-S

c. Forecast prices by using GFBM-S

d. Forecast prices vs actual prices

Figure 5.9. Forecast prices of SSE by using GBM-S and GFBM-S vs actual prices.

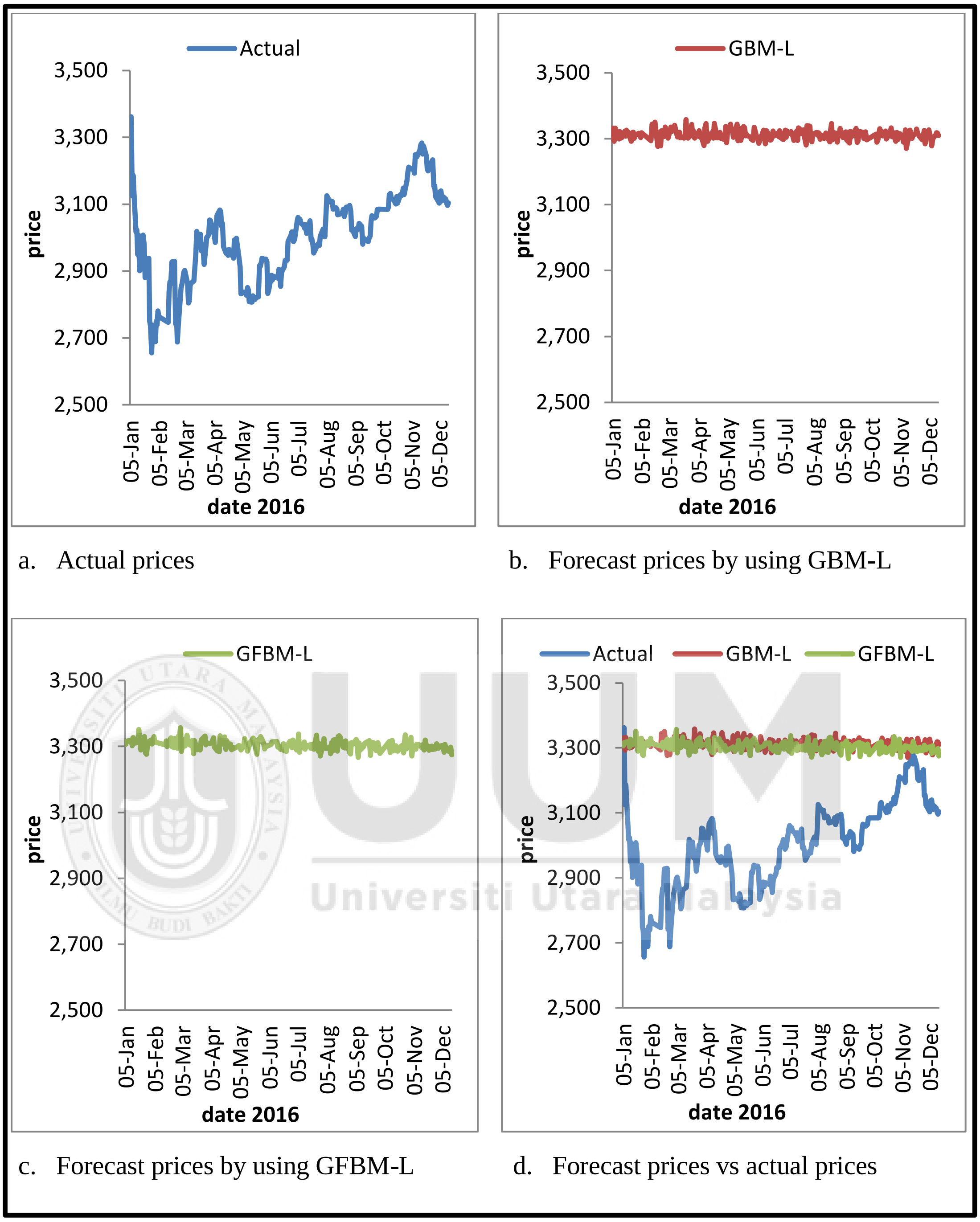
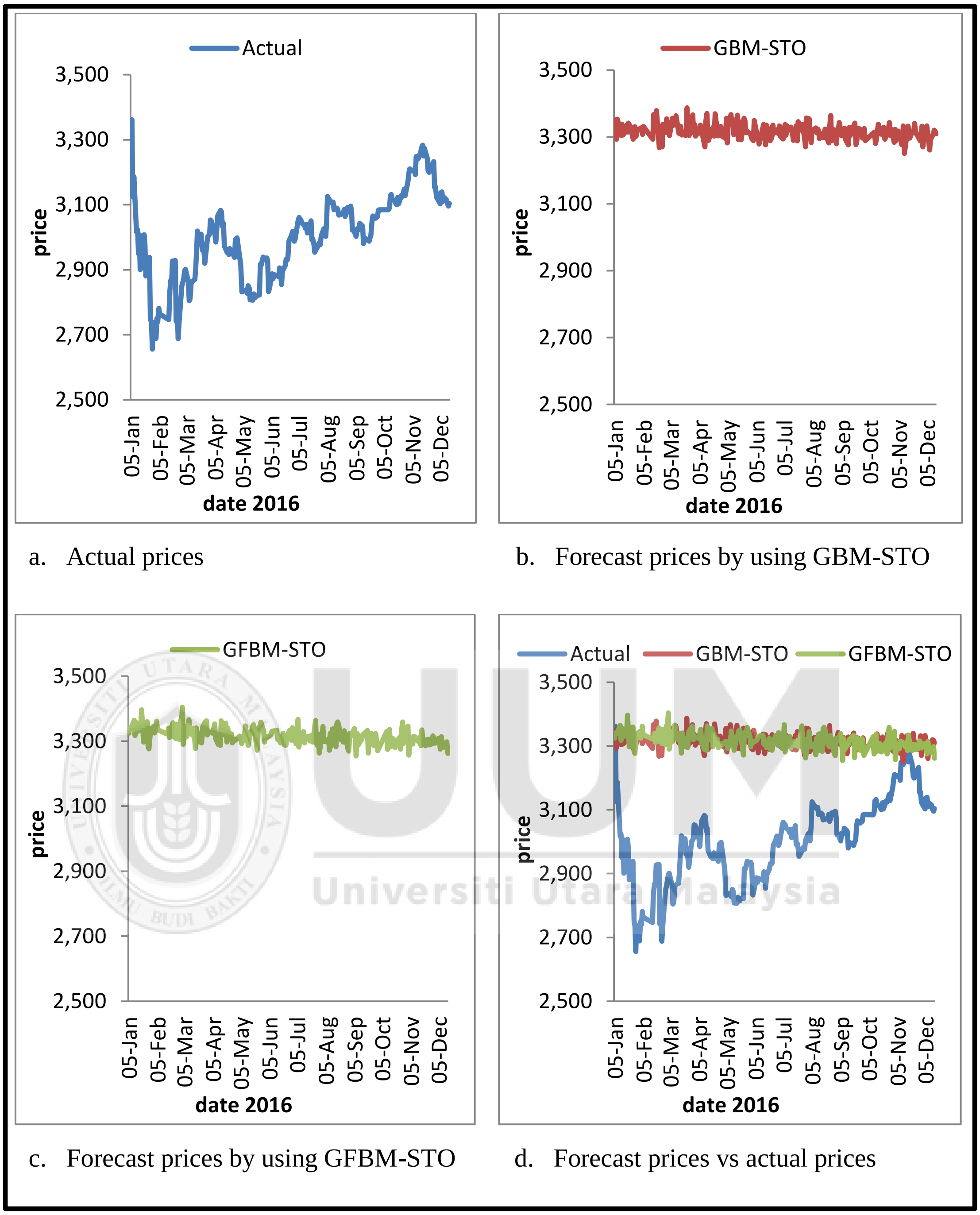


Figure 5.10. Forecast prices of SSE by using GBM-L and GFBM-L vs actual prices.



a. Actual prices

b. Forecast prices by using GBM-STO

c. Forecast prices by using GFBM-STO

d. Forecast prices vs actual prices

Figure 5.11. Forecast prices of SSE by using GBM-STO and GFBM-STO vs actual prices.

5.2.3 Forecasting the Performance Kuala Lumpur Composite Index

5.2.3.1 Description of data

The daily adjusted closed prices from 2nd January 2015 to 31st December 2015, available online at *https://www.investing.com/indices/ftse-malaysia-klci-historical-data* are studied with total observations of 246 days. This duration is selected as they exhibit long memory property. The return series are calculated (in logarithm) to deal high volatility in the data. Figure 5.12 and Figure 5.13 show the adjusted prices and its return series.

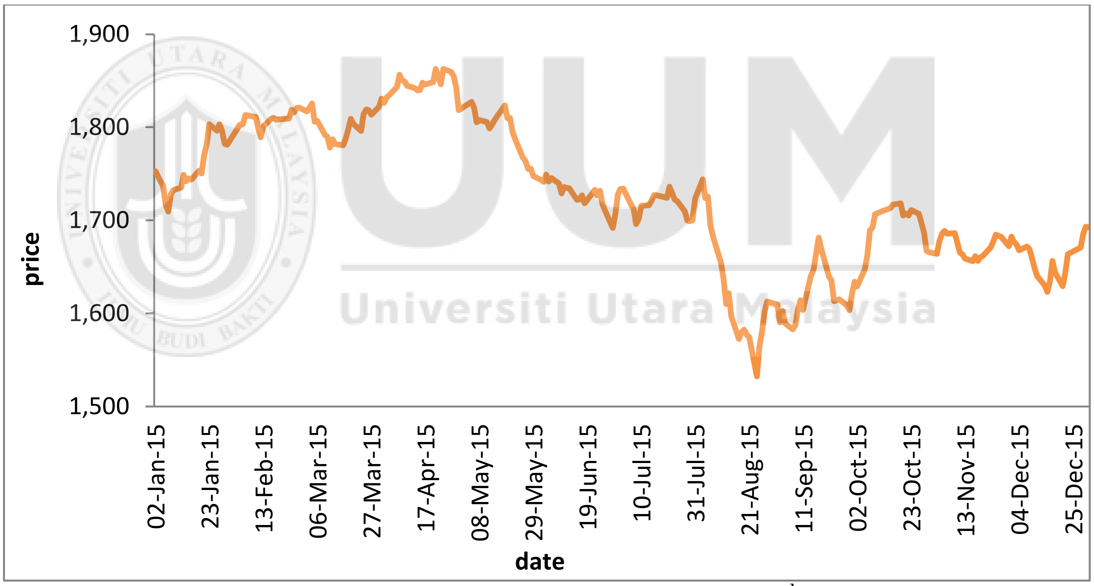


Figure 5.12. Daily adjust price series of KLCI from 2nd January 2015 to 31st December 2015

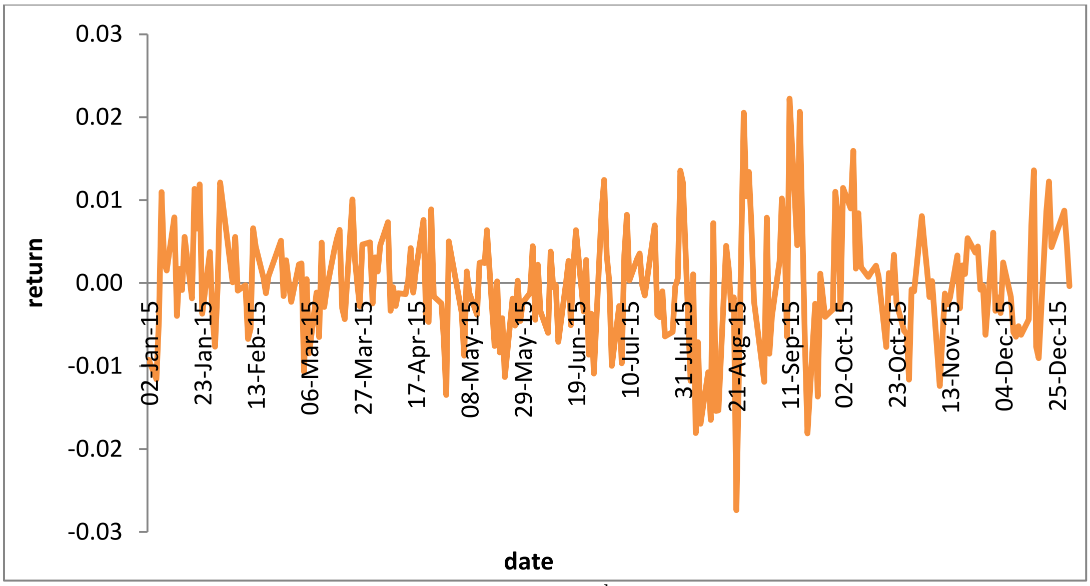


Figure 5.13. Daily returns SSE KLCI from 2nd January 2015 to 31st December 2015

5.2.3.2 Forecasting of Kuala Lumpur Composite Index

Using the same approach in 5.2.1.2, we obtained the values H_1 =0.5936, H_2 =0.5, μ=-0.00014, β =0.00009, m =0.00005 and α =1.6531 as initial parameters to forecast daily index prices for 2016.

Second, we compute the value of volatility in Table 5.1, as illustrates in Table 5.9.

Table 5.9

The values of volatilities according to the formulas of Simple (S), Log (L), High-Low-Close (HLC) and Stochastic (STO)

Volatility type	S	L	HLC	STO
Value	0.05441	0.05443	0.06650	0.01513

Third, we forecast the adjusted prices by using both GBM and GFBM models as its underlying process via the volatility values listed in Table 5.9. The forecasted prices computed by using eight models include GBM-S, GFBM-S, GBM-L, GFBM-L, GBM-HLC, GFBM-HLC, GBM-STO and GFBM-STO. The comparison is conducted through the value MAPE where the smallest value is considered the best. Table 5.10 shows the forecasted prices and actual prices of KLCI while Table 5.11 represents the level of accuracy of the models.

Table 5.10

Forecasted Prices and Actual Prices of KLCI with MAPE

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
Jan-05	1,661.84	1,649.93	1,665.31	1,649.93	1,664.39	1,649.86	1,656.54	1,652.89	**1,665.70**
Jan-06	1,651.51	1,660.10	1,654.64	1,660.10	1,651.76	1,662.35	1,653.65	1,655.69	**1,667.97**
Jan-07	1,659.66	1,648.48	1,662.77	1648.48	1661.74	1648.05	1655.92	1652.5	**1,655.13**
Jan-08	1653.96	1657.48	1657.3	1657.48	1654.77	1659.07	1654.33	1655.01	**1,657.61**
Jan-11	1653.47	1659.14	1656.7	1659.14	1654.11	1661.07	1654.24	1655.5	**1,637.59**
Jan-12	1654.91	1656.57	1658.27	1656.57	1655.88	1657.95	1654.63	1654.77	**1,641.37**
Jan-13	1659.53	1655.85	1662.77	1655.85	1661.52	1657.08	1655.94	1654.55	**1,642.54**
Jan-14	1656.13	1650.63	1659.23	1650.63	1657.38	1650.66	1654.96	1653.13	**1,633.44**
Jan-15	1652.99	1651.39	1656.62	1651.39	1653.51	1651.69	1654.1	1653.26	**1,628.55**
Jan-18	1653.46	1659.33	1656.8	1659.33	1654.12	1661.26	1654.22	1655.59	**1,622.64**
Jan-19	1658.83	1653.17	1662.16	1653.17	1660.71	1653.71	1655.7	1653.87	**1,629.22**
Jan-20	1659.68	1655.09	1663.62	1655.09	1661.77	1656.07	1655.92	1654.4	**1,618.83**
Jan-21	1651.22	1647.71	1654.68	1647.71	1651.36	1647.12	1653.6	1652.27	**1,600.92**
Jan-22	1661.38	1655.14	1664.76	1655.15	1663.78	1656.11	1656.45	1654.43	**1,625.21**
Jan-25	1655.01	1651.05	1658.57	1651.05	1656.01	1651.12	1654.65	1653.27	**1,625.21**
Jan-26	1661.29	1657.78	1664.66	1657.78	1663.76	1659.33	1656.35	1655.18	**1,626.66**
Jan-27	1656.6	1648.66	1659.7	1648.66	1658.01	1648.25	1655.05	1652.57	**1,631.54**
Jan-28	1653.63	1652.53	1657	1652.53	1654.34	1652.93	1654.25	1653.69	**1,634.53**
Jan-29	1658.34	1656.9	1655.59	1656.9	1660.03	1658.33	1655.61	1654.87	**1,667.80**
Feb-02	1659.37	1655.64	1662.7	1655.65	1661.28	1656.71	1655.92	1654.58	**1,653.18**
Feb-03	1657.74	1649.02	1661.03	1649.02	1659.33	1648.71	1655.43	1652.65	**1,633.30**
Feb-04	1648.45	1659.23	1651.92	1659.23	1647.97	1661.16	1652.83	1655.54	**1,656.77**
Feb-05	1655.35	1656.61	1658.71	1656.61	1656.44	1658.02	1654.73	1654.75	**1,662.46**
Feb-10	1656.79	1660.79	1660.27	1660.79	1658.17	1663.05	1655.16	1655.98	**1,644.41**
Feb-11	1647.52	1658.2	1650.75	1658.21	1646.84	1659.86	1652.56	1655.28	**1,643.95**
Feb-12	1661.54	1652.17	1664.52	1652.17	1663.98	1652.51	1656.48	1653.56	**1,643.74**
Feb-15	1655.22	1657.01	1658.94	1657.01	1656.2	1658.43	1654.76	1654.93	**1,649.96**
Feb-16	1655.47	1652.55	1658.97	1652.55	1656.48	1652.97	1654.84	1653.67	**1,664.99**
Feb-17	1657.22	1658.09	1660.7	1658.09	1658.66	1659.76	1655.3	1655.22	**1,664.32**
Feb-18	1657.59	1653.13	1660.71	1653.13	1659.18	1653.62	1655.35	1653.88	**1,680.02**
Feb-19	1655.96	1651.51	1659.5	1651.51	1657.1	1651.65	1654.96	1653.42	**1,674.88**
Feb-22	1655.66	1654.43	1659.1	1654.43	1656.83	1655.19	1654.8	1654.25	**1,674.59**
Feb-23	1655.08	1655.29	1658.24	1655.29	1656.1	1656.27	1654.65	1654.48	**1,677.28**
Feb-24	1654.17	1653.73	1657.34	1653.73	1654.93	1654.4	1654.44	1654.02	**1,664.17**
Feb-25	1654.79	1647.51	1658.24	1647.51	1655.72	1646.83	1654.59	1652.24	**1,658.16**
Feb-26	1651.18	1661	1654.83	1661	1651.28	1663.35	1653.6	1656	**1,663.44**
Feb-29	1653.02	1659.7	1656.3	1659.7	1653.51	1661.73	1654.13	1655.66	**1,654.75**
Mar-01	1655.98	1658.33	1659.29	1658.34	1657.08	1660	1654.99	1655.33	**1,670.82**
Mar-02	1654.03	1658.31	1657.14	1658.31	1654.72	1660.05	1654.42	1655.26	**1,691.03**

Table 5.10 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
Mar-03	1653.54	1663.23	1656.94	1663.23	1654.16	1666.01	1654.26	1656.68	**1,688.20**
Mar-04	1655.47	1652.6	1659.17	1652.6	1656.4	1652.99	1654.89	1653.71	**1,692.49**
Mar-07	1660.02	1654.08	1663.39	1654.08	1662.06	1654.69	1656.1	1654.21	**1,697.93**
Mar-08	1655.86	1660.86	1659.04	1660.86	1656.95	1663.11	1654.95	1656.02	**1,687.86**
Mar-09	1656.23	1662.3	1659.73	1662.3	1657.31	1664.87	1655.12	1656.41	**1,686.35**
Mar-10	1651.08	1652.83	1654.41	1652.83	1651.05	1653.26	1653.65	1653.79	**1,690.91**
Mar-11	1658.52	1653.6	1661.76	1653.61	1660.26	1654.18	1655.65	1654.03	**1,696.54**
Mar-14	1655.17	1656.28	1658.26	1656.28	1656.18	1657.42	1654.7	1654.8	**1,700.31**
Mar-15	1653.37	1660.29	1656.3	1660.29	1653.93	1662.39	1654.23	1655.87	**1,690.92**
Mar-16	1657.19	1660.48	1660.25	1660.49	1658.59	1662.66	1655.3	1655.9	**1,693.43**
Mar-17	1651.57	1654.05	1654.76	1654.05	1651.69	1654.75	1653.75	1654.13	**1,703.19**
Mar-18	1653.16	1654.83	1656.15	1654.83	1653.69	1655.69	1654.15	1654.36	**1,716.34**
Mar-21	1657.74	1651.61	1661.05	1651.61	1659.2	1651.75	1655.5	1653.45	**1,718.36**
Mar-22	1656.9	1654.4	1660.21	1654.4	1658.25	1655.15	1655.22	1654.24	**1,724.75**
Mar-23	1659.77	1657.82	1662.88	1657.82	1661.71	1659.38	1656.05	1655.17	**1,724.55**
Mar-24	1656.4	1655.23	1659.8	1655.23	1657.65	1656.17	1655.06	1654.47	**1,715.53**
Mar-25	1659.53	1652.65	1662.68	1652.65	1661.37	1652.98	1656.01	1653.78	**1,703.79**
Mar-28	1659.66	1651.6	1663.26	1651.6	1661.61	1651.73	1656	1653.46	**1,702.41**
Mar-29	1656.43	1658.26	1659.83	1658.26	1657.62	1659.95	1655.11	1655.26	**1,715.04**
Mar-30	1651.55	1657.72	1655.26	1657.72	1651.61	1659.2	1653.79	1655.17	**1,717.82**
Mar-31	1655.56	1650.77	1658.66	1650.77	1656.58	1650.65	1654.86	1653.26	**1,717.58**
Apr-01	1661.93	1653.33	1665.28	1653.33	1664.31	1653.83	1656.69	1653.94	**1,710.55**
Apr-04	1659.75	1659.53	1658.16	1659.53	1661.67	1661.37	1656.06	1655.71	**1,725.24**
Apr-05	1654.2	1658.08	1657.16	1658.08	1654.89	1659.61	1654.5	1655.3	**1,718.08**
Apr-06	1661.2	1659.52	1664.45	1659.52	1663.43	1661.42	1656.47	1655.66	**1,717.01**
Apr-07	1657.18	1666.8	1660.79	1666.8	1658.56	1670.28	1655.31	1657.73	**1,724.29**
Apr-08	1662.22	1649.39	1665.78	1649.39	1664.74	1648.93	1656.7	1652.9	**1,718.40**
Apr-11	1663.48	1650.09	1666.75	1650.09	1666.24	1649.75	1657.08	1653.12	**1,715.28**
Apr-12	1656.57	1653.14	1659.84	1653.14	1657.76	1653.53	1655.17	1653.94	**1,715.00**
Apr-13	1661.34	1655.65	1664.84	1655.65	1663.64	1656.66	1656.47	1654.61	**1,723.11**
Apr-14	1655.15	1655.56	1658.48	1655.56	1656.03	1656.47	1654.78	1654.63	**1,723.78**
Apr-15	1653	1648.69	1656.37	1648.69	1653.4	1648.12	1654.17	1652.67	**1,727.99**
Apr-18	1661.29	1656.09	1664.43	1656.09	1663.55	1657.16	1656.48	1654.75	**1,717.68**
Apr-19	1656.02	1655.65	1659.26	1655.65	1657.13	1656.62	1654.99	1654.62	**1,711.15**
Apr-20	1655.4	1649.17	1658.85	1649.17	1656.32	1648.77	1654.85	1652.77	**1,708.91**
Apr-21	1654.73	1661.23	1658.17	1661.24	1655.49	1663.52	1654.67	1656.14	**1,721.47**
Apr-22	1651.65	1652.93	1655.1	1652.93	1651.78	1653.23	1653.77	1653.92	**1,717.96**
Apr-25	1649.54	1658.08	1653.11	1658.08	1649.11	1659.62	1653.24	1655.28	**1,714.51**
Apr-26	1654.68	1648.67	1657.93	1648.67	1655.35	1648.09	1654.71	1652.67	**1,692.50**
Apr-27	1658.31	1659.2	1661.82	1659.2	1659.84	1660.96	1655.69	1655.62	**1,692.34**
Apr-28	1660.31	1652.35	1663.54	1652.35	1662.34	1652.5	1656.21	1653.76	**1,674.76**
Apr-29	1658.29	1653.63	1661.5	1653.63	1659.78	1654.13	1655.71	1654.07	**1,672.72**
May-03	1657.69	1650.44	1661	1650.44	1659.05	1650.2	1655.54	1653.19	**1,651.44**

Table 5.10 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
May-04	1656.99	1658.73	1660.1	1658.73	1658.23	1660.35	1655.32	1655.52	**1,657.58**
May-05	1651.3	1645.58	1654.78	1645.58	1651.27	1644.35	1653.72	1651.78	**1,645.09**
May-06	1652	1654.28	1655.45	1654.28	1652.08	1654.9	1653.95	1654.27	**1,649.36**
May-09	1657.1	1648.65	1660.03	1648.65	1658.39	1647.98	1655.33	1652.72	**1,632.19**
May-10	1655.92	1660.63	1659.08	1660.64	1656.91	1662.76	1655.02	1655.98	**1,635.84**
May-11	1659.2	1654.55	1662.83	1654.55	1660.95	1655.18	1655.92	1654.38	**1,644.58**
May-12	1658.99	1654.14	1662.2	1654.14	1660.74	1654.84	1655.83	1654.15	**1,648.98**
May-13	1658.37	1654.05	1661.76	1654.05	1659.93	1654.7	1655.69	1654.14	**1,628.26**
May-16	1659.03	1650.96	1662.26	1650.96	1660.77	1650.89	1655.85	1653.3	**1,621.21**
May-17	1656.43	1653.46	1665.23	1653.46	1657.53	1653.89	1655.17	1654.04	**1,633.39**
May-18	1653.11	1653.77	1656.45	1653.77	1653.49	1654.35	1654.23	1654.07	**1,635.72**
May-19	1647.07	1655.95	1650.41	1655.95	1646.06	1656.95	1652.57	1654.73	**1,633.76**
May-20	1649	1650.68	1652.47	1650.68	1648.47	1650.48	1655.07	1653.27	**1,628.79**
May-23	1655.91	1656.77	1659.22	1656.77	1656.86	1657.91	1655.04	1654.99	**1,634.89**
May-24	1650.61	1650.38	1653.72	1650.38	1650.43	1650.07	1653.52	1653.22	**1,625.84**
May-25	1656.52	1655.86	1659.78	1655.86	1657.61	1656.81	1655.22	1654.73	**1,630.96**
May-26	1667.06	1650.3	1670.36	1650.3	1670.5	1650.09	1658.16	1653.1	**1,631.09**
May-27	1651.23	1655.51	1654.35	1655.51	1651.16	1656.38	1653.72	1654.63	**1,637.19**
May-30	1650.84	1656.04	1654.16	1656.04	1650.71	1657.04	1653.59	1654.76	**1,629.87**
May-31	1657.55	1656.74	1660.88	1656.74	1658.9	1657.95	1655.47	1654.92	**1,626.00**
Jun-01	1654.96	1648.4	1658.32	1648.4	1655.72	1647.69	1654.76	1652.63	**1,626.50**
Jun-02	1653.51	1660.27	1656.7	1660.27	1653.94	1662.26	1654.35	1655.91	**1,630.53**
Jun-03	1654.13	1657.08	1657.46	1657.08	1654.7	1658.26	1654.53	1655.1	**1,636.46**
Jun-06	1655.53	1651.06	1658.91	1651.06	1656.42	1650.95	1654.92	1653.36	**1,648.99**
Jun-07	1651.49	1652.09	1655.15	1652.09	1651.5	1652.18	1653.77	1653.67	**1,660.62**
Jun-08	1651.35	1650.45	1654.61	1650.45	1651.27	1650.18	1653.77	1653.21	**1,657.85**
Jun-09	1655.67	1654.54	1659.05	1654.54	1656.58	1655.15	1654.97	1654.38	**1,650.51**
Jun-10	1656.04	1657.05	1659.36	1657.06	1657	1658.26	1655.08	1655.06	**1,641.22**
Jun-13	1659.64	1656.42	1663.12	1656.42	1661.41	1657.53	1656.08	1654.85	**1,629.77**
Jun-14	1657.48	1654.7	1660.67	1654.7	1658.77	1655.32	1655.49	1654.45	**1,626.11**
Jun-15	1656.12	1652.43	1663.31	1652.43	1657.12	1652.57	1655.09	1653.78	**1,627.96**
Jun-16	1651.37	1647.38	1654.51	1647.38	1651.32	1646.38	1653.76	1652.38	**1,614.90**
Jun-17	1650.76	1662.42	1654.13	1662.42	1650.51	1664.77	1653.64	1656.59	**1,624.18**
Jun-20	1658.07	1650.5	1661.38	1650.5	1659.55	1650.28	1655.61	1653.19	**1,634.23**
Jun-21	1656.81	1649.38	1660.11	1649.38	1657.92	1648.88	1655.32	1652.91	**1,637.69**
Jun-23	1657.3	1656.38	1660.74	1656.38	1658.56	1657.46	1655.42	1654.85	**1,639.98**
Jun-24	1652.9	1655.33	1656.4	1655.33	1653.14	1656.26	1654.22	1654.49	**1,634.05**
Jun-27	1657.98	1652.58	1661.2	1652.58	1659.38	1652.81	1655.62	1653.79	**1,629.52**
Jun-28	1650.97	1654.37	1654.42	1654.38	1650.76	1654.98	1653.69	1654.3	**1,634.04**
Jun-29	1654.24	1650.89	1657.71	1650.89	1654.82	1650.75	1654.57	1653.3	**1,642.21**
Jun-30	1654.28	1640.18	1657.63	1640.17	1654.77	1637.64	1654.65	1650.32	**1,654.08**
Jul-01	1656.61	1652.58	1660.25	1652.58	1657.77	1652.77	1655.19	1653.81	**1,646.22**
Jul-04	1653.03	1646.02	1656.26	1646.01	1653.36	1644.72	1654.21	1651.99	**1,654.84**

Table 5.10 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
Jul-05	1654.86	1657.51	1658.25	1657.51	1655.59	1658.75	1654.73	1655.22	**1,650.71**
Jul-08	1656.89	1655.05	1660.19	1655.05	1657.98	1655.76	1655.37	1654.52	**1,644.54**
Jul-11	1653.49	1647.11	1656.95	1647.11	1653.86	1646.02	1654.39	1652.32	**1,653.87**
Jul-12	1660.29	1661.36	1663.6	1661.37	1662.14	1663.54	1656.32	1656.25	**1,653.97**
Jul-13	1648.11	1649.57	1651.47	1649.57	1647.29	1649.05	1652.87	1653	**1,660.39**
Jul-14	1652.55	1649.81	1656.19	1649.81	1652.7	1649.4	1654.12	1653.02	**1,654.78**
Jul-15	1652.13	1661.41	1655.25	1661.41	1652.21	1663.59	1653.99	1656.27	**1,668.40**
Jul-18	1652.98	1654.17	1656.28	1654.18	1653.18	1654.72	1654.28	1654.25	**1,670.84**
Jul-19	1651.97	1649.32	1655.34	1649.32	1652.03	1648.76	1653.93	1652.91	**1,670.55**
Jul-20	1652.19	1650.46	1655.64	1650.46	1652.34	1650.17	1651.97	1653.22	**1,669.61**
Jul-21	1659.52	1657.5	1662.72	1657.5	1661.22	1658.75	1656.07	1655.21	**1,657.54**
Jul-22	1659.92	1660.41	1663.11	1660.42	1661.74	1662.25	1656.16	1656.07	**1,657.42**
Jul-25	1654.01	1654.43	1657.31	1654.43	1654.48	1654.92	1654.53	1654.41	**1,668.26**
Jul-26	1646.45	1649.81	1649.51	1649.81	1645.24	1649.36	1652.41	1653.05	**1,661.42**
Jul-27	1656.87	1656.72	1660.03	1656.72	1657.94	1657.79	1655.37	1654.99	**1,663.56**
Jul-28	1654.76	1651.74	1657.99	1651.74	1655.47	1651.66	1654.69	1653.63	**1,658.50**
Jul-29	1654.98	1657.51	1658.36	1657.51	1655.58	1658.78	1654.87	1655.2	**1,653.26**
Aug-01	1657.44	1652.58	1658.8	1652.58	1658.71	1652.71	1655.46	1653.85	**1,665.23**
Aug-02	1655.35	1652.46	1658.48	1652.46	1656.09	1652.6	1654.92	1653.79	**1,660.23**
Aug-03	1653.78	1651.34	1656.87	1651.34	1654.19	1651.16	1654.47	1653.53	**1,648.50**
Aug-04	1652.57	1646.8	1655.93	1646.8	1652.72	1645.59	1654.12	1652.27	**1,655.29**
Aug-05	1659.64	1656.79	1663.02	1656.79	1661.3	1657.85	1656.16	1655.03	**1,664.04**
Aug-08	1643.18	1655.72	1646.64	1655.72	1641.23	1656.55	1651.5	1654.72	**1,672.68**
Aug-09	1659.88	1651.67	1663.11	1651.67	1661.59	1651.61	1656.22	1653.58	**1,671.71**
Aug-10	1651.68	1656.39	1654.63	1656.39	1651.56	1657.32	1653.92	1654.95	**1,673.03**
Aug-11	1657.02	1658.79	1660.55	1658.79	1658.06	1660.26	1655.45	1655.62	**1,678.80**
Aug-12	1654.26	1657.82	1657.31	1657.82	1654.67	1659.15	1654.68	1655.28	**1,684.15**
Aug-15	1653.43	1649.55	1656.55	1649.55	1653.74	1649.01	1654.4	1653	**1,690.33**
Aug-16	1659.34	1660.5	1662.7	1660.5	1660.98	1662.47	1656.03	1656	**1,699.89**
Aug-17	1650.09	1657.27	1653.45	1657.27	1649.56	1658.38	1653.53	1655.2	**1,694.32**
Aug-18	1655.58	1646.1	1658.85	1646.1	1656.32	1644.77	1655.03	1652.05	**1,694.87**
Aug-19	1657.51	1650.29	1660.73	1650.28	1658.7	1649.78	1655.55	1653.29	**1,687.68**
Aug-22	1658.21	1646.8	1661.39	1646.8	1659.55	1645.56	1655.75	1652.29	**1,691.07**
Aug-23	1647.48	1653.9	1650.88	1653.9	1646.43	1654.26	1652.74	1654.26	**1,683.07**
Aug-24	1657.99	1663	1661.27	1663	1659.3	1665.45	1655.67	1656.75	**1,682.06**
Aug-25	1655.64	1652.65	1659.11	1652.65	1656.39	1652.73	1655.04	1653.9	**1,680.30**
Aug-26	1658.35	1658.6	1661.88	1658.6	1659.7	1660.05	1655.81	1655.54	**1,683.09**
Aug-29	1653.62	1655.39	1657.16	1655.39	1653.92	1656.14	1654.47	1654.63	**1,681.60**
Aug-30	1663.84	1646.93	1667.22	1646.93	1666.37	1645.81	1657.37	1652.25	**1,678.06**
Sep-01	1653.32	1652.89	1656.73	1652.89	1653.59	1653.03	1654.36	1653.97	**1,670.55**
Sep-02	1657.24	1652.58	1660.45	1652.58	1658.38	1652.64	1655.46	1653.88	**1,671.79**
Sep-05	1653.5	1655.39	1656.73	1655.39	1653.82	1656.02	1654.41	1654.71	**1,678.08**
Sep-06	1658.67	1647.24	1661.82	1647.24	1660.06	1646.22	1655.9	1652.31	**1,689.92**

Table 5.10 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
Sep-07	1655.07	1653.24	1658.58	1653.24	1655.64	1653.49	1654.91	1654.04	**1,689.57**
Sep-08	1648.03	1655.06	1651.53	1655.06	1647.09	1655.68	1652.9	1654.57	**1,691.38**
Sep-09	1648.24	1652.75	1651.64	1652.75	1647.42	1652.86	1652.9	1653.92	**1,686.44**
Sep-13	1651.38	1657.36	1655.05	1657.36	1651.11	1658.45	1653.89	1655.25	**1,677.18**
Sep-14	1653.61	1655.42	1657.31	1655.42	1653.93	1656.05	1654.46	1654.72	**1,661.39**
Sep-15	1650.42	1656.74	1653.71	1656.74	1650.01	1657.66	1653.56	1655.1	**1,652.99**
Sep-19	1659.9	1661.56	1665.31	1661.56	1661.65	1663.61	1656.18	1656.41	**1,651.71**
Sep-20	1655.11	1653.68	1658.42	1653.68	1655.74	1653.94	1654.88	1654.23	**1,655.78**
Sep-21	1648.26	1649.57	1651.5	1649.57	1647.31	1648.99	1653.01	1653.02	**1,658.73**
Sep-22	1664.61	1650.44	1668.19	1650.44	1667.32	1649.95	1657.58	1653.33	**1,669.66**
Sep-23	1656.02	1650.16	1659.67	1650.16	1656.83	1649.73	1655.16	1653.17	**1,670.99**
Sep-26	1654.47	1647.93	1657.77	1647.93	1654.99	1646.94	1654.68	1652.58	**1,669.50**
Sep-27	1650.67	1656.16	1654.09	1656.16	1650.27	1656.93	1653.67	1654.95	**1,664.72**
Sep-28	1649.7	1655.78	1653.03	1655.78	1649.06	1656.61	1653.42	1654.73	**1,664.96**
Sep-29	1659.9	1651.58	1663.23	1651.58	1661.59	1651.4	1653.23	1653.62	**1,669.64**
Sep-30	1656.23	1652.81	1659.78	1652.81	1657.06	1652.9	1655.23	1653.96	**1,652.55**
Oct-04	1654.67	1656.35	1658.06	1656.35	1655.13	1657.27	1654.81	1654.92	**1,661.25**
Oct-05	1648.1	1654.83	1651.41	1654.83	1647.11	1655.39	1652.97	1654.51	**1,662.92**
Oct-06	1651.44	1652.72	1654.97	1652.72	1651.25	1652.7	1653.85	1653.99	**1,666.73**
Oct-07	1659.8	1657.29	1663.08	1657.29	1661.51	1658.38	1656.17	1655.21	**1,665.38**
Oct-10	1651.93	1647.06	1654.89	1647.06	1651.91	1645.88	1653.95	1652.34	**1,665.32**
Oct-11	1648.59	1656.36	1651.5	1656.36	1647.71	1657.17	1653.11	1655	**1,668.72**
Oct-12	1655.75	1661.85	1653.19	1661.85	1656.47	1663.93	1655.1	1656.5	**1,667.03**
Oct-13	1653.22	1657.67	1656.46	1657.67	1653.34	1658.84	1654.41	1655.31	**1,665.02**
Oct-14	1649.84	1652.49	1653.19	1652.49	1649.28	1652.5	1653.42	1653.88	**1,658.97**
Oct-17	1649.24	1649.37	1650.32	1649.37	1648.52	1648.7	1653.26	1652.99	**1,653.71**
Oct-18	1655.93	1644.51	1659.22	1644.51	1656.69	1642.7	1655.15	1651.67	**1,667.57**
Oct-19	1656.51	1653.3	1659.9	1653.3	1657.33	1653.42	1655.36	1654.15	**1,668.27**
Oct-20	1655.87	1653.5	1658.87	1653.5	1656.6	1653.68	1655.14	1654.19	**1,667.18**
Oct-21	1651.2	1655.98	1655	1655.98	1650.87	1656.78	1653.84	1654.83	**1,669.98**
Oct-24	1655.37	1651.24	1659.05	1651.24	1656.02	1650.93	1654.97	1653.55	**1,677.76**
Oct-25	1659.63	1653.5	1662.78	1653.5	1661.27	1653.69	1656.13	1654.18	**1,677.43**
Oct-26	1655.78	1657.59	1659.3	1657.59	1656.54	1658.76	1655.07	1655.28	**1,673.92**
Oct-27	1656.58	1652.13	1659.94	1652.13	1657.42	1651.99	1655.37	1653.82	**1,669.03**
Oct-28	1653.55	1653.87	1656.83	1653.87	1653.77	1654.11	1654.48	1654.31	**1,670.27**
Oct-31	1654.58	1646.02	1657.96	1646.02	1655.06	1644.53	1654.75	1652.1	**1,672.46**
Nov-01	1657.13	1659.67	1651.53	1659.67	1658.09	1661.23	1655.52	1655.92	**1,670.93**
Nov-02	1648.33	1658.64	1652.31	1658.65	1647.36	1659.94	1653.04	1655.65	**1,659.60**
Nov-03	1651.66	1655.23	1654.92	1655.23	1651.5	1655.8	1653.93	1654.66	**1,648.08**
Nov-04	1656.94	1653.5	1660.33	1653.5	1657.82	1653.7	1658.5	1654.17	**1,648.24**
Nov-07	1657.08	1651.56	1660.32	1651.56	1658.08	1651.35	1655.48	1653.62	**1,650.59**
Nov-08	1656.17	1656.1	1659.37	1656.1	1656.99	1656.82	1655.2	1654.94	**1,663.82**
Nov-09	1657.86	1653.14	1661.18	1653.14	1658.95	1653.16	1655.75	1654.14	**1,647.62**

Table 5.10 (Continued)

Date 2016	GBM-S	GFBM-S	GBM-L	GFBM-L	GBM-HLC	GFBM-HLC	GBM-STO	GFBM-STO	Actual Price
Nov-10	1654.25	1649.68	1657.79	1649.68	1654.62	1648.97	1654.68	1653.14	**1,652.74**
Nov-11	1650.11	1656.77	1653.62	1656.77	1649.57	1657.66	1653.51	1655.12	**1,634.19**
Nov-14	1652.87	1652.49	1656.26	1652.49	1652.96	1652.49	1654.27	1653.86	**1,616.64**
Nov-15	1657.83	1654.87	1661.06	1654.87	1658.95	1655.36	1655.71	1654.56	**1,630.56**
Nov-16	1661.42	1652.03	1664.76	1652.03	1663.29	1651.84	1656.75	1653.8	**1,627.63**
Nov-17	1656.81	1650.27	1660.18	1650.27	1657.67	1649.68	1655.45	1653.31	**1,626.77**
Nov-18	1654.96	1656.99	1658.03	1656.99	1655.37	1657.9	1654.97	1655.19	**1,623.80**
Nov-21	1650.53	1648.24	1653.75	1648.24	1650	1647.23	1653.48	1652.73	**1,627.28**
Nov-22	1651.85	1647.62	1655.16	1647.62	1651.61	1646.43	1653.86	1652.58	**1,629.32**
Nov-23	1661.07	1661.46	1664.32	1661.46	1662.91	1663.39	1656.42	1656.42	**1,630.38**
Nov-24	1659.88	1654.1	1663.43	1654.1	1661.48	1654.39	1656.06	1654.37	**1,624.21**
Nov-25	1660.24	1648.91	1663.85	1648.91	1661.88	1648.08	1656.19	1652.89	**1,627.26**
Nov-28	1651.1	1656.37	1654.44	1659.37	1650.72	1660.76	1653.62	1655.89	**1,628.66**
Nov-29	1654.16	1658.43	1657.71	1658.43	1654.48	1659.63	1654.46	1655.62	**1,626.93**
Nov-30	1664.33	1652.2	1667.68	1652.2	1666.86	1652.08	1657.36	1653.82	**1,619.12**
Dec-01	1646.04	1653.11	1649.07	1653.11	1644.46	1653.14	1652.26	1654.12	**1,626.44**
Dec-02	1650.35	1650.92	1653.55	1650.92	1655.79	1650.48	1653.43	1653.49	**1,628.96**
Dec-05	1651.23	1652.47	1660.56	1652.47	1650.81	1652.35	1653.72	1653.94	**1,624.97**
Dec-06	1649.04	1659.91	1652.31	1659.91	1648.1	1661.48	1653.12	1655.99	**1,629.73**
Dec-07	1657.25	1651.47	1660.59	1651.47	1658.1	1651.11	1657.45	1653.67	**1,632.47**
Dec-08	1643.07	1655.39	1646.48	1655.39	1640.87	1655.98	1651.4	1654.7	**1,643.75**
Dec-09	1650.6	1646.12	1654.17	1646.12	1650.06	1644.6	1653.51	1652.15	**1,641.42**
Dec-13	1649.63	1653.47	1655.94	1653.47	1650.94	1653.55	1653.2	1654.23	**1,645.28**
Dec-14	1653.4	1657.02	1657.07	1657.02	1653.52	1650.92	1655.28	1655.2	**1,643.29**
Dec-15	1653.03	1652.15	1656.45	1652.15	1653.08	1652.02	1654.16	1653.8	**1,636.99**
Dec-16	1654.79	1651.92	1658.37	1651.92	1655.22	1651.72	1654.66	1653.74	**1,637.79**
Dec-19	1657.89	1656.4	1661.42	1656.4	1658.97	1657.15	1655.56	1655.05	**1,634.30**
Dec-20	1657.4	1649.9	1658.55	1649.9	1648.55	1649.19	1653.21	1653.23	**1,634.52**
Dec-21	1650.84	1652.08	1654.14	1652.08	1650.36	1651.86	1653.58	1653.84	**1,629.59**
Dec-22	1663.03	1651.18	1666.24	1651.18	1665.16	1650.85	1657.06	1653.51	**1,623.20**
Dec-23	1654.64	1650.49	1658.13	1650.49	1654.91	1649.92	1654.7	1653.39	**1,617.15**
Dec-27	1650.23	1659.35	1653.78	1659.35	1649.52	1660.68	1653.47	1655.92	**1,619.68**
Dec-28	1656.19	1654.6	1659.56	1654.6	1657.9	1654.86	1655.07	1654.6	**1,630.30**
Dec-29	1649.73	1652.19	1652.9	1652.19	1648.98	1651.93	1653.29	1653.92	**1,637.93**
Dec-30	1650.98	1659.58	1654.32	1659.58	1650.51	1661.02	1653.63	1655.94	**1,641.73**
MAPE	**1.40962 %**	**1.40802 %**	**1.40814 %**	**1.40878 %**	**1.40721 %**	**1.40576 %**	**1.40744 %**	**1.40717 %**	

Table 5.11

The level of accuracy ranking for forecasting model

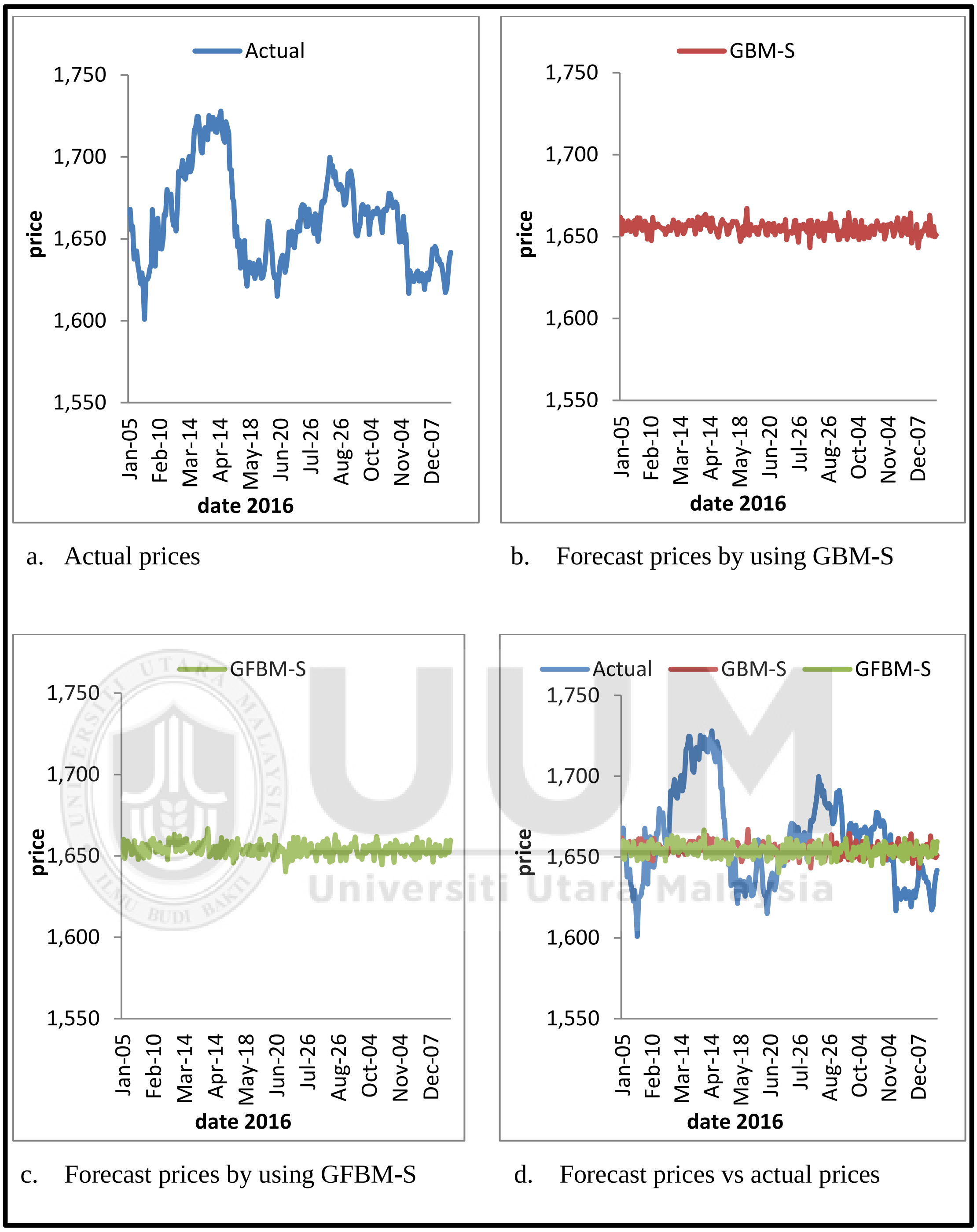
Rank	**Model**	**MAPE**
1	GFBM-HLC	1.40576%
2	GFBM-STO*	1.40717%
3	GBM-HLC	1.40721%
4	GBM-STO	1.40744%
5	GFBM-S	1.40802%
6	GBM-L	1.40814%
7	GFBM-L	1.40878%
8	GBM-S	1.40962%

* The proposed model

Table 5.10 and Table 5.11 showed that all values of MAPE are relatively close (less than 10%). These values indicate again that both GBM and GFBM models have highly accurate forecasts, as all value of MAPE <10%. However, with the exception of GFBM-S model, we can observe that stochastic models are often more accurate than the other types of volatilities considered in this work.

From the findings, the GFBM-HLC reveals the most accurate in performance, whereas GBM-S performed the worst. The proposed model performed the second in accuracy with trivial difference (< 0.00141%) from the first level. This result indicates that the proposed model is efficient.

Figures 5.14–5.17 illustrate the comparison between the actual prices versus forecasted prices in GBM and GFBM models with four different volatilities. In general, these figures indicated that the forecasted prices are closer together and less fluctuated than the actual prices. Moreover, one can observe that the forecasted models of GBM-STO and GFBM-STO are very close together and more steadiness than other forecasted models in addition to actual price.



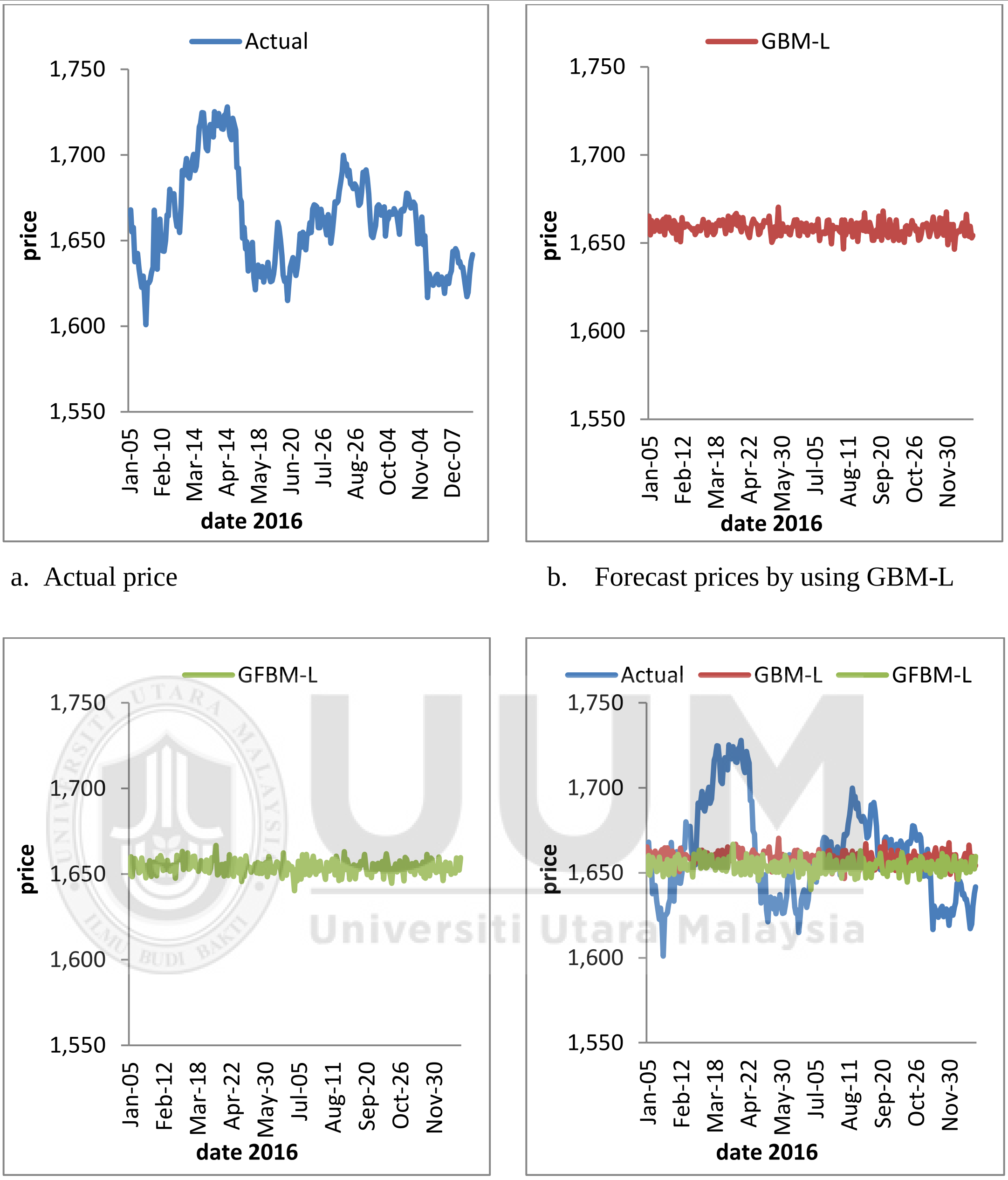
a. Actual prices

b. Forecast prices by using GBM-S

c. Forecast prices by using GFBM-S

d. Forecast prices vs actual prices

Figure 5.14. Forecast prices of KLCI by using GBM-S and GFBM-S vs actual prices.



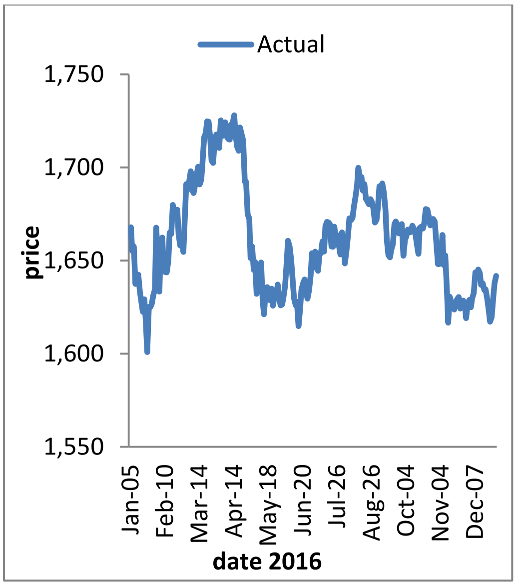
a. Actual price

b. Forecast prices by using GBM-L

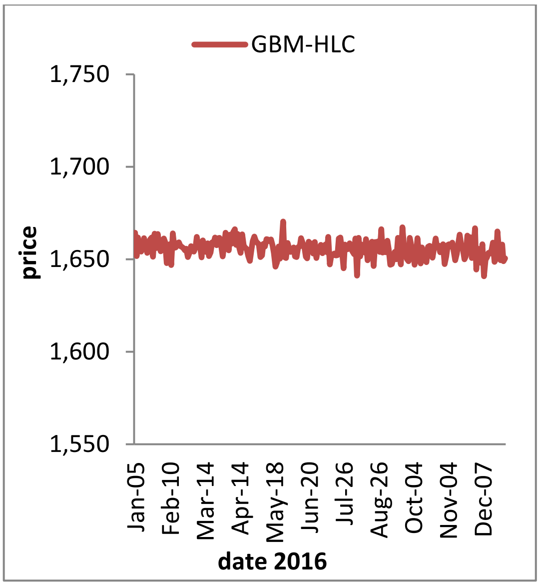
c. Forecast prices by using GFBM-L

d. Forecast prices vs actual prices

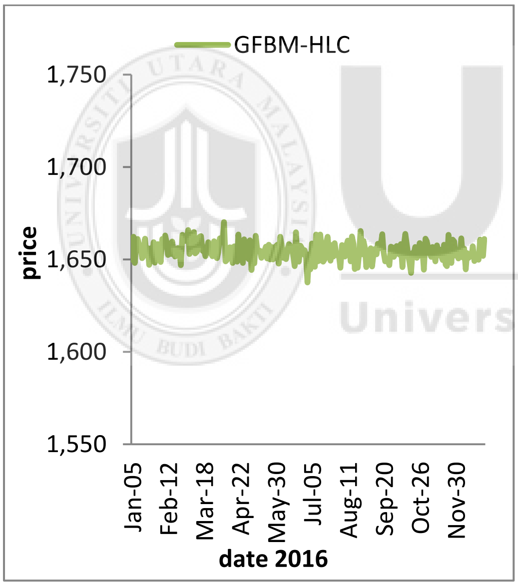
Figure 5.15. Forecast prices of KLCI by using GBM-L and GFBM-L vs actual prices.



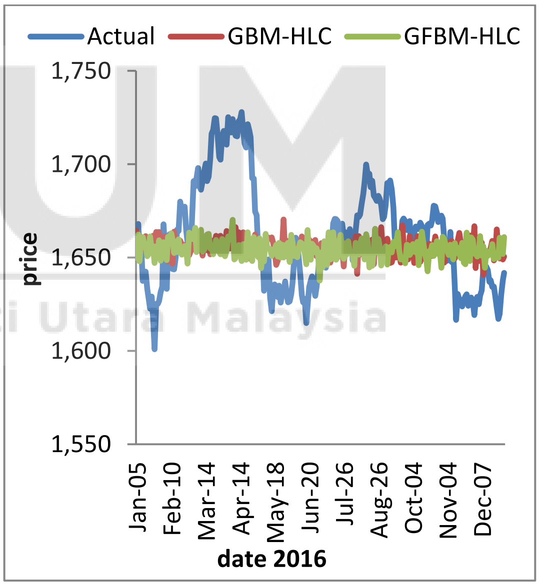
a. Actual price



b. Forecast prices by using GBM-HLC

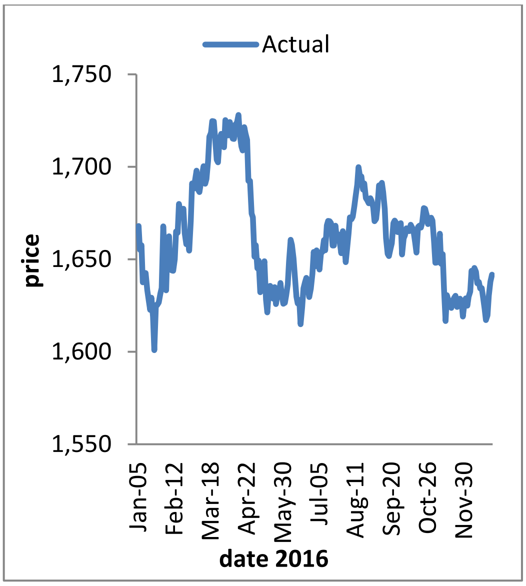


c. Forecast prices by using GFBM-HLC

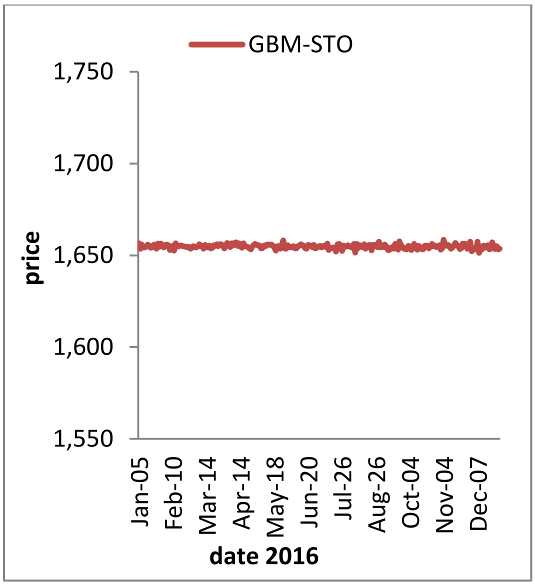


d. Forecast prices vs actual prices

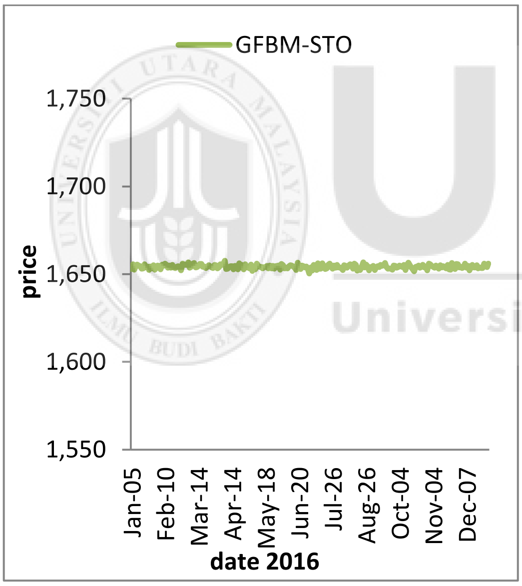
Figure 5.16. Forecast prices of KLCI using by GBM-HLC and GFBM-HLC vs actual prices.



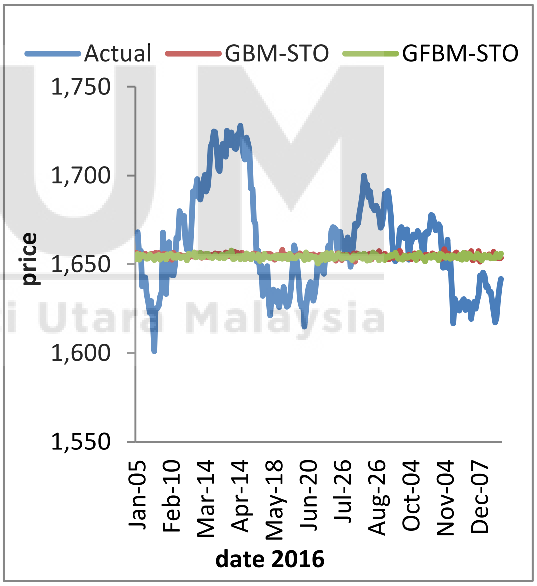
a. Actual price



b. Forecast prices by using GBM-STO



c. Forecast prices by using GFBM-STO



d. Forecast prices vs actual prices

Figure 5.17. Forecast prices of KLCI by using GBM-STO and GFBM-STO vs actual prices.

5.3 Discussion

In this chapter, we forecasted the adjusted prices of three global indices, the Standard and Poor's 500 (S&P 500), Shanghai Stock Exchange Composite Index (SSE) and Kuala Lumpur Composite Index (KLCI), each representing its own segment of markets, by using GBM and GFBM model with different volatility models as underlying process. The performance of the proposed models and its counterparts are investigated. Different volatility models under study include simple volatility (S), log volatility (L), high–low–closed volatility (HLC) and stochastic volatility obeying fractional Ornstein–Uhlenbeck process (STO). We use MAPE value to evaluate each model.

The findings of S&P 500 showed that the proposed method in this thesis provides the best forecast as seen from its minimum value of MAPE. While, the findings of SSE showed that the proposed model comes in the fifth level of accuracy. However, the difference in MAPE value is small not exceed 0.391%. Meanwhile, the findings of KLCI show that the proposed model comes in the second level of accuracy with trivial difference of 0.00141% in MAPE value. The performance of the proposed model into the three markets reveals that the proposed model is efficient and can be applied in real financial environment.

Furthermore, the findings also suggested that GFBM model is significantly more accurate than the GBM model due to its long memory property which is consistence with Painter (1998), Willinger et al. (1999), Grau–Carles (2000), and Rejichi and Aloui (2012).

Majority of the forecasting methods also portray high accuracy since most MAPE values are less than 10%. Such promising findings motivate more extensive future works on promoting stochastic volatility, GFBM model and long memory in the financial environments.

The next chapter shows our effort to collaborate the proposed model into some other financial applications, in particular in option pricing, risk valuation, exchange rate and mortgage insurance.

CHAPTER SIX

APPLICATIONS OF DEVELOPED LONG MEMORY STOCHASTIC VOLATILITY MODEL IN FINANCE

This chapter will investigate the significant use of the proposed model to selected financial application. In the subsections that follow, long memory stochastic volatility (LMSV) model is applied to the problem of option pricing, value at risk, exchange rate and mortgage insurance.

6.1 Pricing the Options

Option can be defined as an agreement that gives holder the right in buying or selling certain amount of an underlying asset at specified future time at specified price, though holder is under no obligation to exercise the contract. There are two classes of options, which are classified according to its expiry date, i.e. European option and American option. The European option allows holder to exercise his stock at the expiration date only, while American option is more flexible in allowing holder to exercise his stock before or at the expiration date. In this study, however, we only focus on European option for simplicity in calculation.

To date, many option pricing models available in the literature circling around the family of binomial model (Tian, 1999; Gianin and Sgarra, 2013) and Black-Scholes model and their extension (Cox et al., 1979; Chan and Wong, 2006). However, the Black-Scholes (BS) model and fractional Black-Scholes (FBS) model are the most widely used. BS and FBS models are considered as a strategy for investor to buy and sell their assets continuously without any loss. Besides, they are also simple in nature and mathematically understandable compared to other models (Yalincak, 2012). In this study, we will focus on fractional Black-Scholes model as it is a natural extension of Black-Scholes model to accommodate long memory property with GFBM model as its underlying process.

6.1.1 Fractional Black–Sholes Model for European Option Pricing

In this section, we will apply the proposed model to price option based on standard BS model and FBS model. European option pricing is use for its simplicity in calculations as the time to maturity is fixated. We first define the standard BS model and the FBS model as follow.

Definition 6.1 (Black and Scholes, 1973): *The price at time $t\epsilon[0,T]$ of a European call option with the strike price K and maturity T is given by*

$$C_0 = S_0\psi(D_1) - Ke^{-rT}\psi(D_2) \tag{6.1}$$

where

$$D_1 = \frac{ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{6.2}$$

and

$$D_2 = D_1 - \sigma\sqrt{\mathrm{T}} \tag{6.3}$$

where S_0 is the underlying stock price at time t, r is the risk–free interest rate, $\psi(\cdot)$ is the cumulative distribution function of the standard normal distribution and σ is the standard deviation of the stock price.

Definition 6.2 (Mishura, 2008): *The price at time $t\epsilon[0,T]$ of a European call option with the strike price K and maturity T is given by*

$$C(t,S) = S\phi(D_1) - Ke^{-r(T-t)}\phi(D_2), \tag{6.4}$$

where

$$D_1 = \frac{ln\left(\frac{S}{K}\right)+r(T-t)+(T^{2H}-t^{2H})\frac{\sigma^2}{2}}{\sigma\sqrt{T^{2H}-t^{2H}}}, \tag{6.5}$$

and

$$D_2 = \frac{ln\left(\frac{S}{K}\right)+r(T-t)-(T^{2H}-t^{2H})\frac{\sigma^2}{2}}{\sigma\sqrt{T^{2H}-t^{2H}}} \tag{6.6}$$

where S is the underlying stock price at time t, r is the risk free interest rate, and $\phi(\cdot)$ is the cumulative function of a standard normal distribution.

6.1.2 Description of Data

In this work, we selected Kuala Lumpur Composite Index (KLCI) to reflect Malaysian market.

The data set of KLCI is available online on *http://quotes.wsj.com* . Daily close price data set of KLCI from 3rd January, 2005 to 29th December, 2006 was studied with total observations of 494 data points. This same data set in Misiran et al. (2010) is selected to make a comparison study. The return series were then calculated in its logarithm. We consider its return series to handle high volatility in the data. In order to compute all the parameters contained in the fractional geometric Brownian motion and fractional Ornstein–Uhlenbeck, we obtained the log return of the adjusted closed price, daily volatility of log return and daily volatility of adjusted closed price. Figure 6.1-6.2 show the price and its return series.

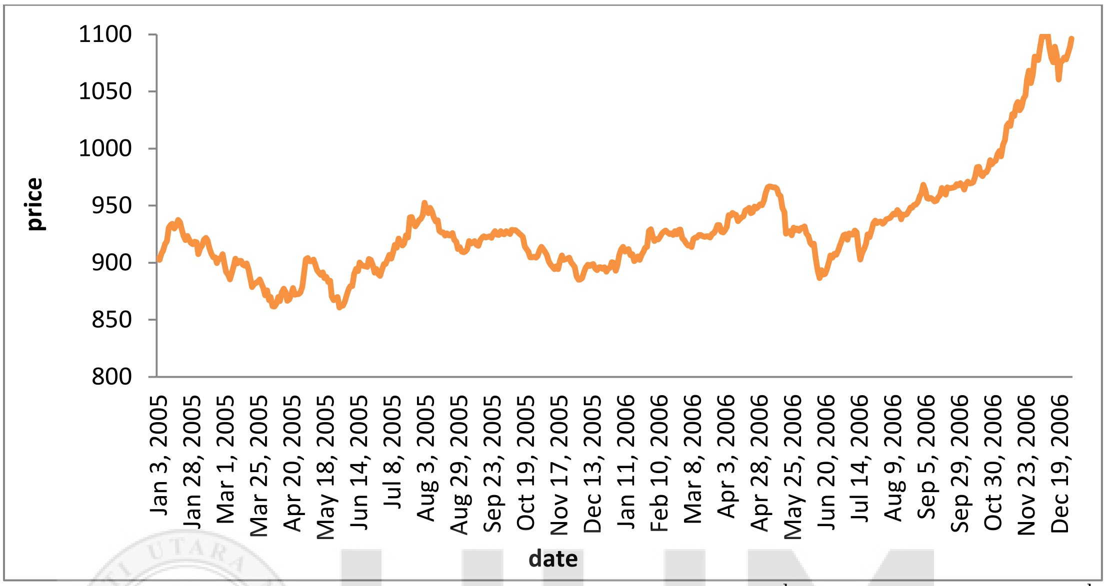


Figure 6.1. Daily closed price series of KLCI from 3rd January 2005 to 29th December 2006

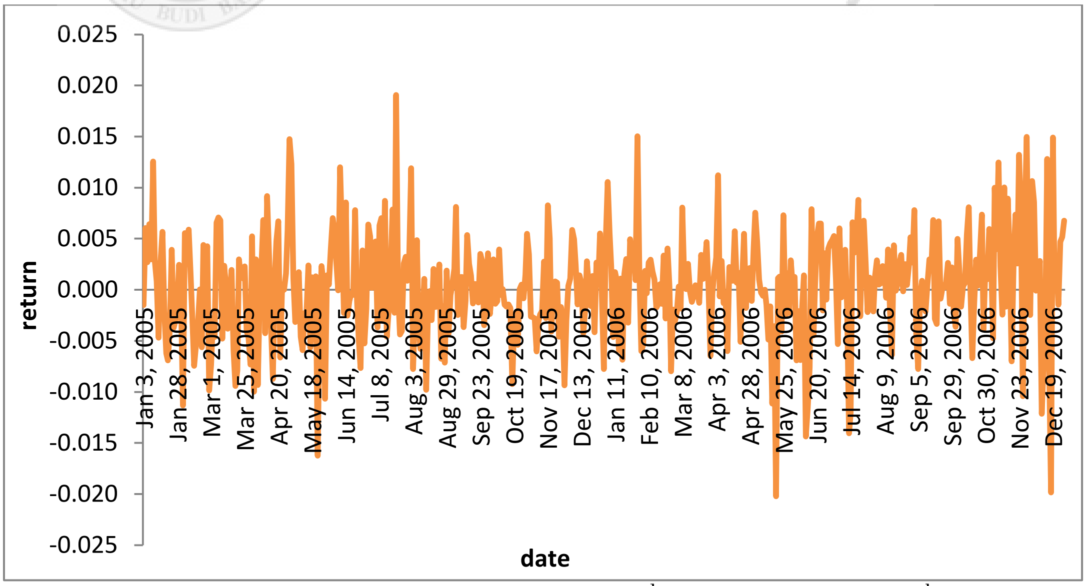


Figure 6.2. Daily return series of KLCI from 3rd January 2005 to 29th December 2006.

The values of all the parameters of the return series were obtained by using Mathematica 10 software as presented in Table 6.1.

Table 6.1

Summary of parameters

Parameter	Value
H_1: Hurst index of adjust closed price	0.57497
H_2: Hurst index of daily volatility of adjust closed price	0.50981
μ: mean of log returns	0.000391
β: volatility of volatility	2.33×10^{-9}
m: mean of daily volatility of log return	0.00002578
α: mean reverting of daily volatility of log return	2.219378

Substituting the values in Table 6.1 into the proposed model (i.e. Equations (3.1) and (3.2)) produces the estimators $\widehat{H}_1 = 0.5734$ and $\hat{\sigma}^2 = 2.578 \times 10^{-5}$. These estimators will be used in the approximation study in the next subsection.

6.1.3 Pricing the Options

In this study, we adopted Misiran et al. (2010) to compare the values of European call option price by using three different methods, i.e. complete maximum likelihood estimation (FBS-Misiran) by Misiran et al. (2010; 2012), incomplete maximum likelihood estimation (FBS-Kukush) by Kukush et al. (2005) and the standard BS (Standard-BS) model by Black and Scholes (1973) with our proposed model (FBS-Alhagyan) by using the same data set.

Several maturity times (in days) for traded option are used to calculate European call option. It is noted that the conventional interest rate on 29th December, 2006 was fixed at 3.5% per annum. We use MYR 1096.24 as the underlying price similar to Misiran et al. (2010). The volatility and Hurst exponent are estimated based on our proposed method for historical daily return data of KLCI, with listed estimates in Table 6.1. The comparison between the proposed methods, Misiran et al. (2010; 2012), Kukush et al. (2005) and standard BS European option price are then being made (refer to Table 6.2).

Table 6. 2

Comparison of the European call option prices using different methods with H in ( ) and σ^2 in [ ].

T–t	**K Strike Price**	**FBS–Alhagyan** (0.5734) [2.573×10^{-5}]	**FBS–Misiran** (0.575) [2.576×10^{-5}]	**FBS–Kukush** (0.6615) [2.59×10^{-5}]	**Standard BS** (0.5) [2.589×10^{-5}]
15	1070	28.1256	28.1162	27.854	28.744
	1080	19.068	19.048	18.1328	20.233
	1090	11.3859	11.3525	10.0388	13.0495
	1100	5.7765	5.7395	4.2417	7.5811
	1110	2.4117	2.3826	1.2854	3.9081
30	1070	30.8044	30.7824	30.0116	31.9587
	1080	22.5347	22.5021	21.2564	24.1353
	1090	15.4517	15.4105	13.758	17.3926
	1100	9.8296	9.7583	7.9795	11.8935
	1110	5.74902	5.7082	4.0753	7.6799
45	1070	33.5933	33.5642	32.4808	35.0049
	1080	25.7466	25.7087	24.2271	27.518
	1090	18.9178	18.8732	17.0777	20.9548
	1100	13.2623	13.2147	11.2814	15.4134
	1110	8.83295	8.7870	6.9319	10.9238
60	1070	36.3218	36.2886	35.5027	37.8585
	1080	28.7365	28.6958	27.1004	30.5724
	1090	20.0468	22.0004	20.1482	24.1036
	1100	16.3567	16.3076	14.3287	18.5210
	1110	11.7060	11.6575	9.7079	13.8486

From Table 6.2, we can see that longer expiry time means higher value of the call price, and the higher value in strike price means lower value of the call price in the same period of time. The call prices of FBS-Alhagyan and FBS-Misiran are relatively close. While there is a significant difference between the call prices of the FBS-Alhagyan and of FBS-Kukush and Standard BS.

The call prices obtained by FBS-Kukush with rescaled range (R/S) analysis displayed the lowest values. The highest value is calculated by Standard BS model, without long memory. In general, the difference in call prices between the four methods increase as the strike price increases. Figures 6.3–6.6 show European call option prices using different methods with different maturity times.

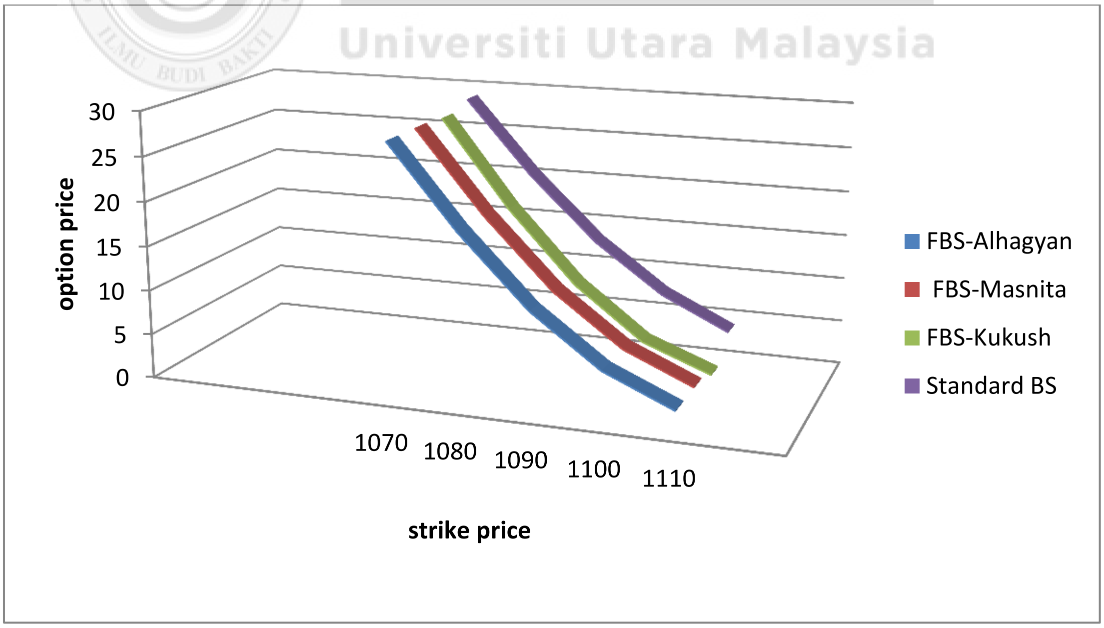


Figure 6.3. European call option prices using different methods with maturity time 15 days.

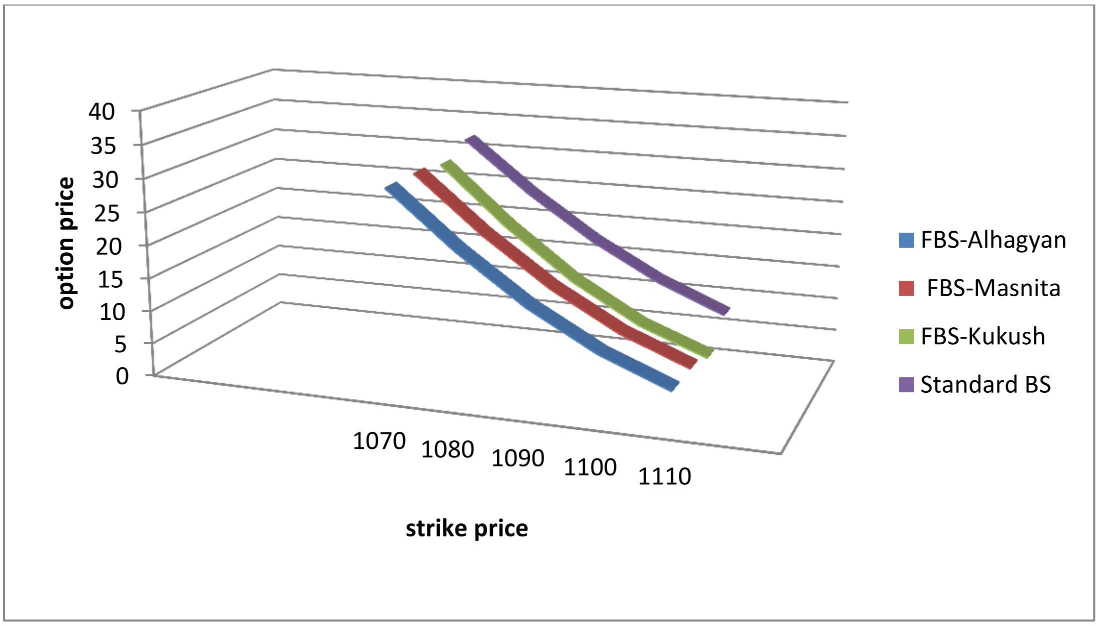


Figure 6.4. European call option prices using different methods with maturity time 30 days.

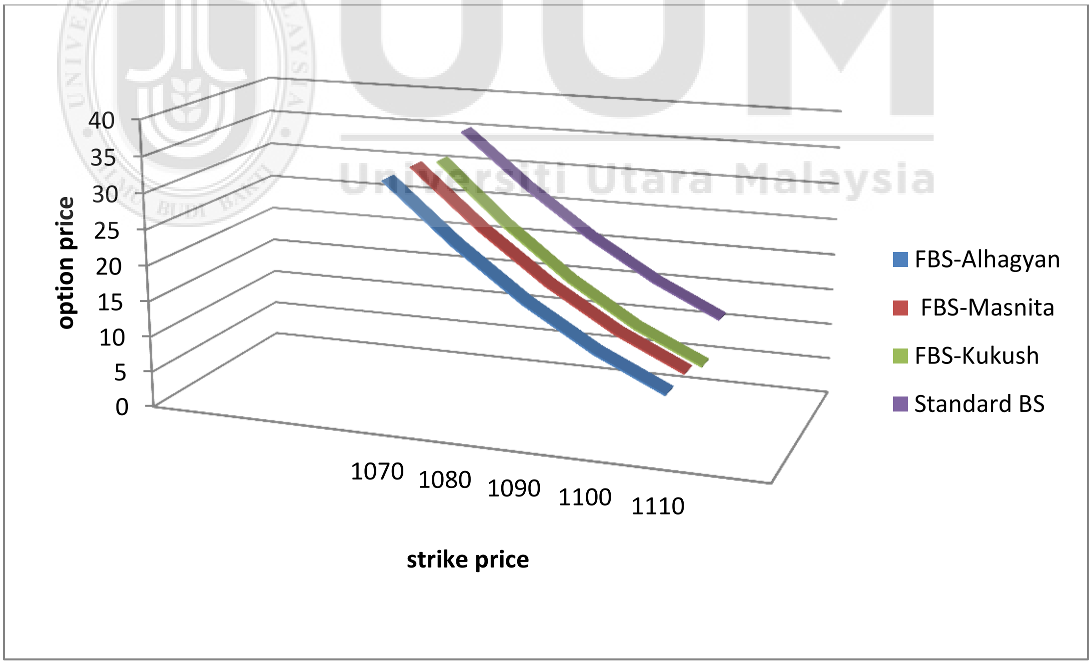


Figure 6.5. European call option prices using different methods with maturity time 45 days.

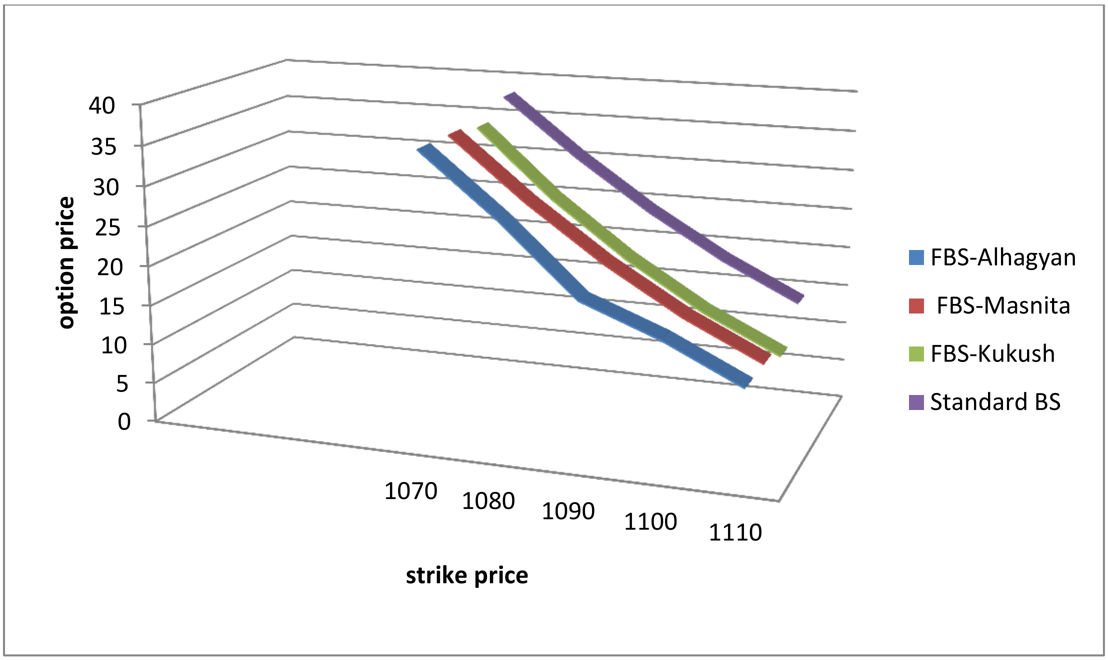


Figure 6.6. European call option prices using different methods with maturity time 60 days.

The prices computed by FBS-Alhagyan and FBS-Misiran lie between the values of the prices of FBS-Kukush and Standard BS. FBS-Misiran is based on theoretical reasoning, but it considered volatility is assumed constant, which was rejected by empirical studies as discussed extensively in previous Chapter. However, our proposed model (i.e. FBS–Alhagyan) assumes that the volatility is stochastic which mostly aligned empirical studies. Thus, the proposed model is significant to be of used in future works, in particular in modeling financial assets.

According to discussion above, we can observe that results obtained from our proposed model are practically acceptable, in which the long memory is taken into account, and the volatility is assumed to be stochastic.

The next subsection illustrates application of the proposed model to another financial application that is value at risk perturbed by long memory stochastic volatility.

6.2 Value at Risk and Long Memory

Value at risk (VaR) is a benchmark to measure market risk, in particular risk of unexpected changes in prices or rates. Investment banks and firms habitually use VaR modeling due to the potential for independent trading desks to expose the firm to highly correlated assets unintentionally. Applying VaR model in firms is considered as the main instrument that allows determining the cumulative risks through aggregated positions of different trading desks and departments within the institution. Further, the results of VaR model help financial institutions to determine the sufficient capital that should be reserve to face and to cover potential losses (Cordell and King, 1995; Gjerde and Semmen, 1995; Dimson and Marsh, 1995; and Mabrouk, 2017).

Menkens (2007) suggested that when changes in the value of portfolio follow Brownian motion, VaR figure of d days is derived by multiplying the VaR figure of one to one day with $\sqrt{d}$. However, when distributions of the changes in portfolio's values follow a long memory process with Hurst parameter H, the VaR figure of d days is obtained by multiplying VaR figure of one to one day with d^H. In work that follows, we will use historical simulation to investigate the behavior of VaR when perturbed by long memory stochastic volatility.

To begin, Menkens (2007) defined VaR as $q-$quantile of the distribution of the change of value for a given portfolio with its potential loss with probability q will not exceed the value of VaR.

Quantiles are referred as the cut points that divide the range of a probability distribution into adjacent intervals that has equal probabilities. Moreover, the observations in the sample can also be divided. For any distribution, the number of quantiles is one less than the number of groups created. Thus, q-quantiles separate a finite set of values into nearly equal sizes into q subsets.

Definition 6.3 (Hyndman and Fan, 1996): *The quantile of a distribution is defined by*

$$Q(p) = F^{-1}(p) = \inf\{x: F(x) \geq p\},\ 0 < p < 1.$$

Definition 6.4 (Menkens, 2007): *If P^d represents the return of the changing of values for a given portfolio over d days, i.e*

$$P^d\,(t) = \frac{P(t)-P(t-d)}{P(t-d)}\,, \tag{6.7}$$

where $P(t)$ is a value of portfolio at time t, and F_{P^d} represents distribution function of P^d, then

$$VaR_{1-q}(P^d) = -F_{P^d}^{-1}\,(q). \tag{6.8}$$

Note that F^{-1} represents the quantile function such that

$$F^{-1}(q) = \inf\{x: F(x) \geq q\} \quad \text{for } 0 < q < 1. \tag{6.9}$$

If P^d represents a normal distribution with stationary and independent increments with standard deviation $\sigma\sqrt{d}$ ($P^d \sim N(0, \sigma^2 d)$), then we can calculate VaR for d days depending on the VaR for 1 day through the following relation.

$$VaR_{1-q}(P^d) = \sqrt{d}\, VaR_{1-q}\,(P^1). \tag{6.10}$$

Note that in Equation (6.10), Menkens set $\sigma^2 = 1$ to simplify the notation.

Now, we attempt to generalize Menkens's formula through deriving formula of VaR under the assumptions of long memory and stochastic volatility.

If the changes in portfolio values exhibit long memory with Hurst index H, then

$$F_{P^d}(x) = \int_{-\infty}^{x} \frac{1}{d^H \sigma\sqrt{2\pi}} \exp\left(\frac{-z^2}{2\sigma^2 d^{2H}}\right) dz, \tag{6.11}$$

where F_{P^d} is distribution function of the return of the changing of values for a given portfolio over days d, Hurst index H, and volatility of returns σ. In other words, $P^d \sim N(0, \sigma^2 d^{2H})$.

Assume $z = w\sigma d^H$. Differentiating both sides gives $dz = \sigma d^H dw$. When $z \to -\infty$ implies $w \to -\infty$ and as $z \to x$ implies $w \to \frac{x}{\sigma d^H}$. Based on this assumption, replacing $z = w\sigma d^H$ in Equation (6.11) gives

$$F_{P^d}(x) = \int_{-\infty}^{\frac{x}{\sigma d^H}} \frac{1}{d^H \sigma\sqrt{2\pi}} \exp\left(\frac{-w^2}{2}\right) \sigma d^H dw$$

$$= \int_{-\infty}^{\frac{x}{\sigma d^H}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-w^2}{2}\right) dw.$$

$$= F_{P^1}\left(\frac{x}{\sigma d^H}\right)$$

$$= F_{P^1}\left(d^{-H} \frac{x}{\sigma} \right). \quad (6.12)$$

By using Equation (6.9) and Equation (6.12) we can define $F_{P^d}^{-1}$ as follows:

$$F_{P^d}^{-1}(q) = \inf\{x: F_{P^d}(x) \geq q\}$$

$$= \inf\{x: F_{P^1}\left(d^{-H} \frac{x}{\sigma} \right) \geq q\}.$$

Assuming that $\delta = d^{-H} \frac{x}{\sigma}$ implies $x = \sigma\, d^H \delta$ which then leads to

$$F_{P^d}^{-1}(q) = \inf\{\sigma\, d^H \delta: F_{P^1}(\delta) \geq q\}$$

$$= \sigma d^H \inf\{\delta: F_{P^1}(\delta) \geq q\}$$

$$= \sigma d^H\, F_{P^1}^{-1}(q). \quad (6.13)$$

Hence, by utilizing Equation (6.8) and Equation (6.13)

$$VaR_{1-q}(P^d) = \sigma d^H . VaR_{1-q}(P^1). \quad (6.14)$$

Equation (6.14) represents VaR model perturbed by long memory with constant volatility σ (VaR-STD). In this subsection, we extend the constant volatility in Equation (6.14) to stochastic volatility $\sigma(\cdot)$ obeys fractional Ornstein–Uhlenbeck (FOU) process i.e.

$$VaR_{1-q}(P^d) = \sigma(Y_t) d^{H_1} . VaR_{1-q}(P^1) \quad (6.15)$$

$$dY_t = \alpha(m - Y_t)dt + \beta dB_{H_2}(t), \quad (6.16)$$

where α, β and m represent mean reverting of volatility, volatility of volatility, and mean of volatility respectively. $B_{H_2}(t)$ is a fractional Brownian motion process. Note that we name the proposed model in Equations (6.15) and (6.16) by VaR-LMSV.

The next subsection illustrates this extension model to portfolio with long memory properties and investigates its performance.

6.2.1 Description of Data

We used data set of the Permanent Portfolio Permanent I (PRPFX) which is available online at *http://finance.yahoo.com*. PRPFX is an American portfolio that invests in gold, silver, Swiss franc assets, stocks of United State, foreign real estate and natural resource companies, aggressive growth stocks and dollar assets such as U.S. treasury bills and bonds.

We considered daily adjusted closed prices for PRPFX data from 1st of January 2015 to 31st December 2015 with total observation of 252 days. We choose PRPFX data set as an example of long memory time series to reveal the effect of memory on VaR.

The return series were then being calculated in logarithm. Figure 6.7-6.8 show PRPFX's adjusted prices and its return series.

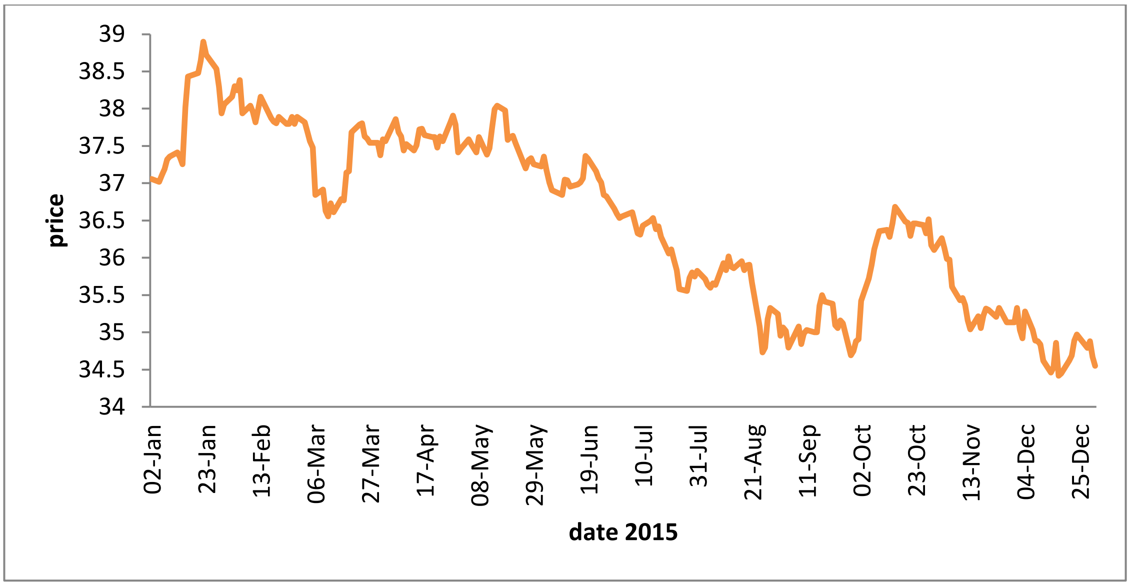


Figure 6.7. Daily adjust price series of PRPFX from 1st January 2015 to 31st December 2015.

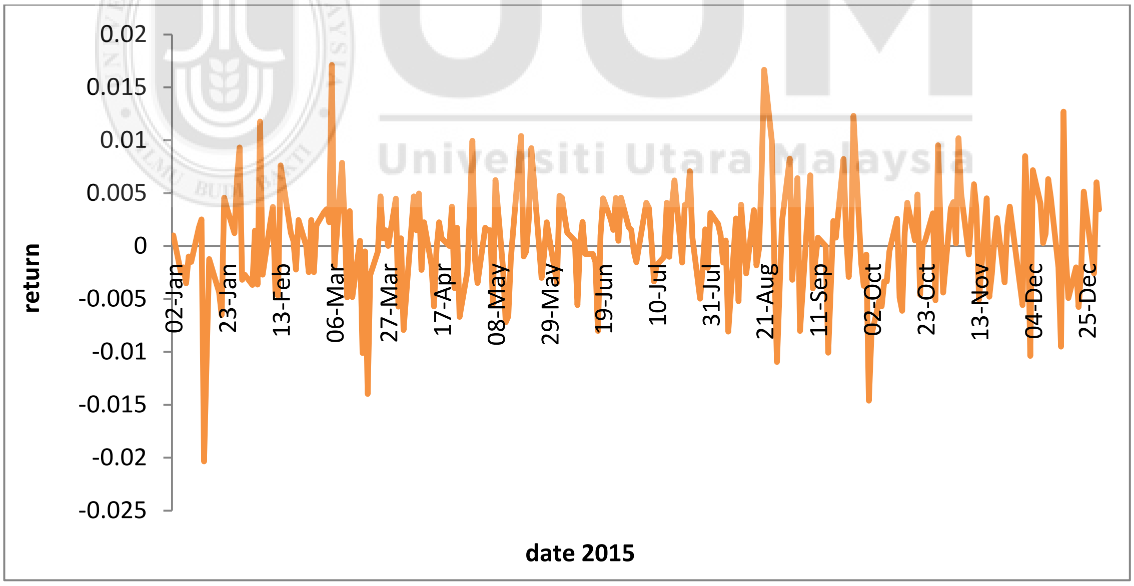


Figure 6.8. Daily returns series of PRPFX from 1st January 2015 to 31st December 2015.

6.2.2 Calculating Value at Risk with Self-Similarity

In this subsection, we then calculated log returns and daily volatility of log returns by using adjusted prices of PRPFX. Then we computed all parameters involved in VaR-STD and VaR-LMSV by using Mathematica 10. Table 6.3 summarizes the parameters for PRPFX.

Table 6.3

Parameters summary of PRPFX

Parameter		**Value**
H_1:	Hurst index of adjusted closed price	0.549766
H_2:	Hurst index of daily volatility of adjusted closed price	0.541527
β:	volatility of volatility	2.27×10^{-9}
m:	mean of daily volatility of log return	2.5×10^{-5}
α:	mean reverting of daily volatility of log return	2.700
$\sigma(Y_t)$:	stochastic volatility	0.0050014
σ:	constant volatility	0.0050013

VaR was then calculated by VaR-STD with $H = 0.5$ and VaR-LMSV model with $H = 0.55$. Next, comparison study was conducted via three different values of quantile q over different days d as shown in Table 6.4 and Figures 6.9-6.11.

Table 6.4

PRPFX :VaR model with memory and stochastic volatility versus VaR model with no memory and constant volatility

d	$q = 0.01$		$q = 0.05$		$q = 0.1$	
	VaR-LMSV	**VaR-STD**	**VaR-LMSV**	**VaR-STD**	**VaR-LMSV**	**VaR-STD**
1	1.14 %	1.14 %	0.80 %	0.80 %	0.61 %	0.61 %
50	10.0 %	8.22 %	7.05 %	5.80 %	5.48 %	4.51 %
100	14.62 %	11.63 %	10.33 %	8.22 %	8.04 %	6.39 %
150	18.28 %	14.25 %	12.92 %	10.07 %	10.06 %	7.84 %
200	21.42 %	16.46 %	15.14 %	11.63 %	11.79 %	9.05 %
250	24.22 %	18.41 %	17.12 %	13.01 %	13.33 %	10.13 %

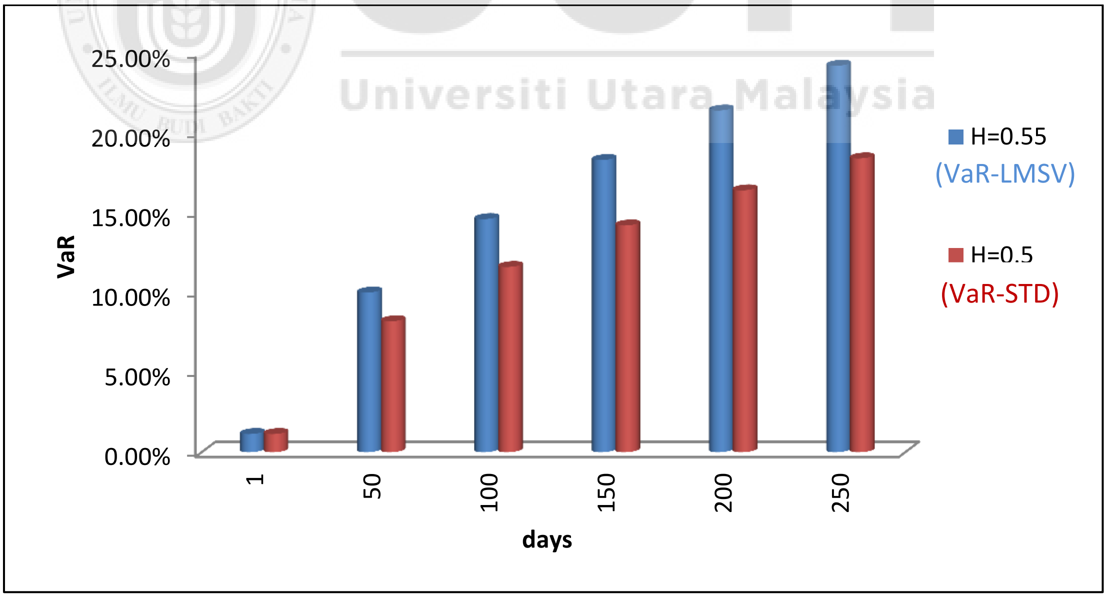


Figure 6.9. VaR of PRPFX with $q = 0.01$

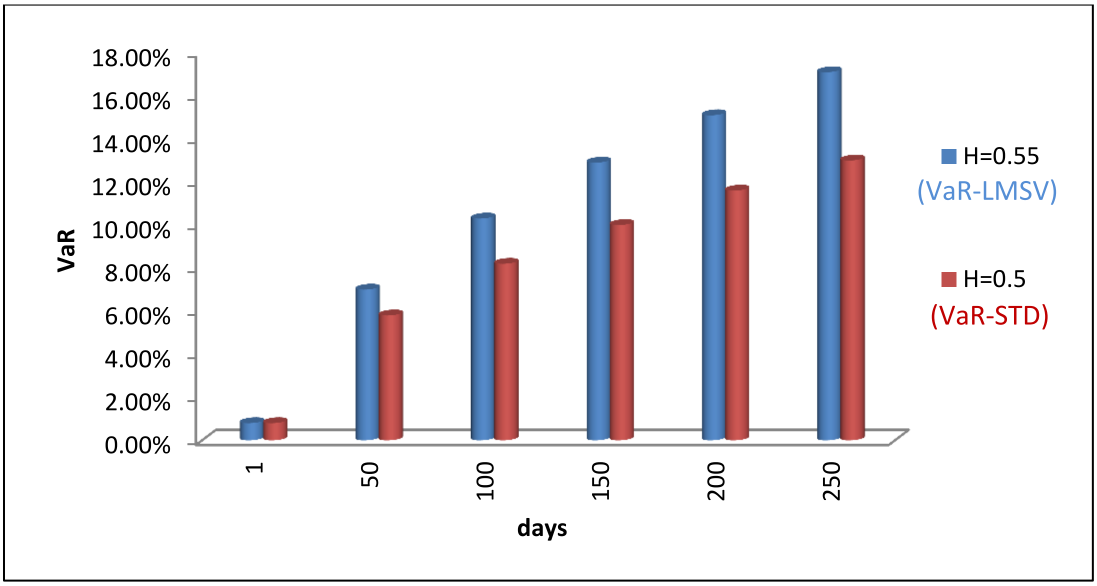


Figure 6.10. VaR of PRPFX with $q = 0.05$

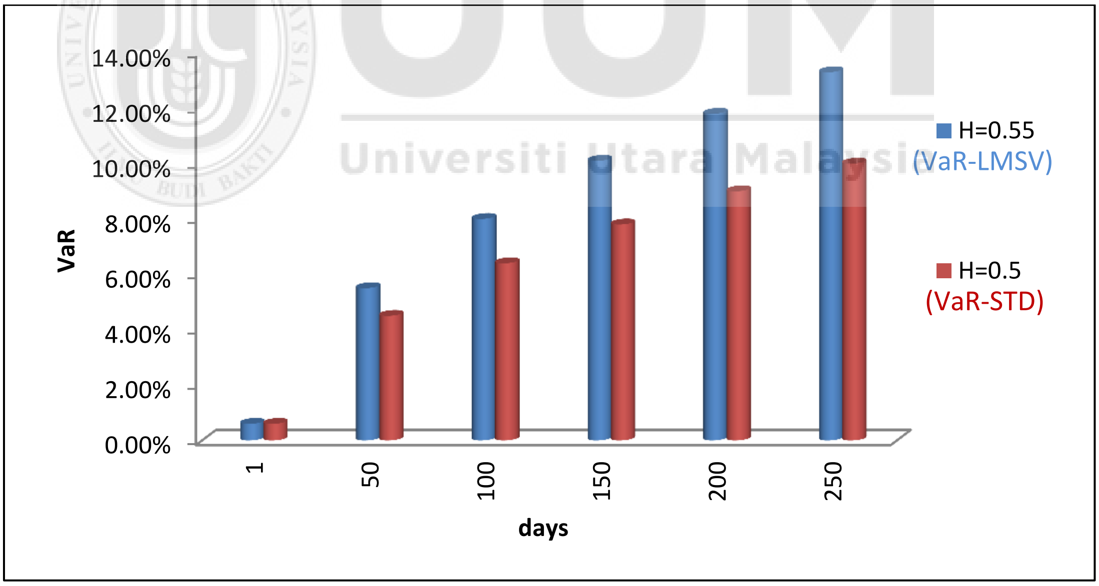


Figure 6.11. VaR of PRPFX with $q = 0.1$

Table 6.4 and Figures 6.9-6.11 show that all values of VaR calculated in VaR-LMSV produced higher percentages. These findings suggested that when hedging portfolio in a risky environment (dependent historical data that includes shocks, e.g. long memory), investors are encouraged to adopt the proposed model. This effort will minimize potential loss faced by investors, as higher percentage resulting to lower loss of investment (with $100(1-q)\%$ certainty).

From Table 6.4, we can observe that the gap between VaR-STD and VaR-LMSV increases proportionally with number of days d via all values of quantile q. Also, VaR computed by both models decreases when the value of quantile increases.

PRPFX's data exhibit close value between constant volatility and stochastic volatility. Thus, there is no significant affects to assume stochastic volatility in the future study.

In consequent, Hurst index H has significant control on VaR. Higher value in H will provide a higher VaR and vice versa.

6.3 Exchange Rate

The exchange rate plays a significant role on the performance of financial trading, specifically international trade activity in any country. Exchange rate changes every day, and it is determined by buying and selling of foreign currencies on world currency market. Immediate effect of these changes can be felt by different stake holders in different ways, depending on the direction of the change. Thus, it is very important to forecast the direction of exchange rate. In the existing literature, to forecast exchange rate involve some models and processes including random walk process, Brownian motion process, jump diffusion process, GBM, ARIMA, and Ornstein-Uhlenbeck mean reverting process (Gozgor, 2013).

The volatility of currencies and relative valuations always have important reflects on overall economic performance, the balance of payments and international trade (Nicita, 2013). Andersen et al. (2001) showed through empirical work that the volatility of exchange rate has well description by using long memory process. Recently, there are many scholars investigate empirically the effect of stochastic volatility on exchange rate such as Gong and Zhuang (2017) and Ahlip and Rutkowski (2016).

In this section we will forecast exchange rates (USD/MYR) using GBM and GFBM models under assumptions of constant volatility and stochastic volatility.

6.3.1 Description of Data

The data of USD/MYR is available online at Bank Negara Malaysia website, *http://www.bnm.gov.my*. The daily exchange rate from 2nd January 2015 to 31st December 2015 are studied with total observations of 246 days. This time frame is selected due to the exhibition of a long memory property. The return series are calculated in logarithm to avoid high volatility in the data. Figure 6.12 and Figure 6.13 show the exchange rates and its return series.

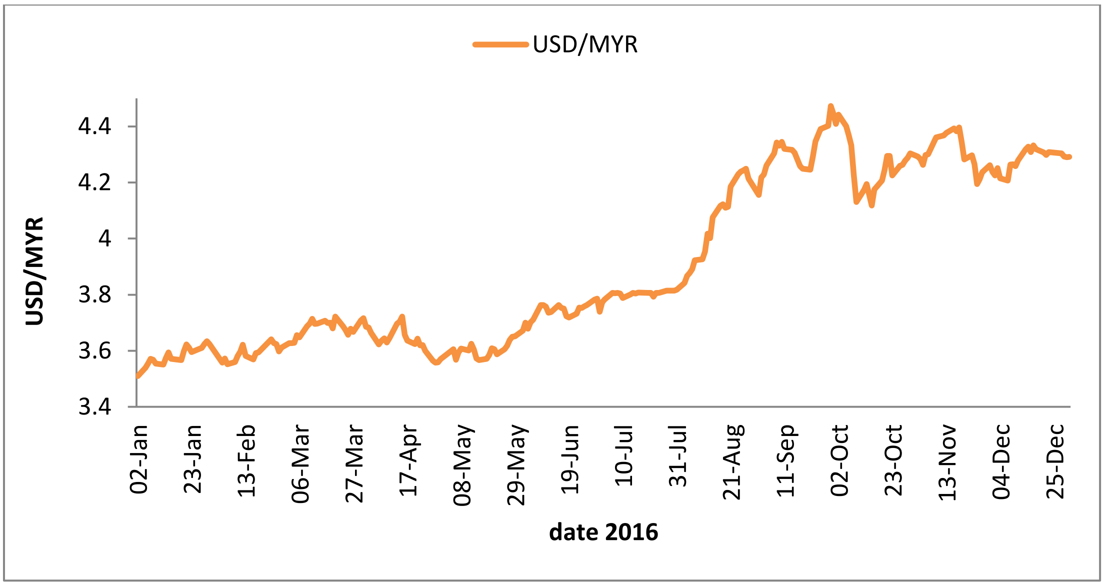


Figure 6.12. Historical exchange rates between USD and MYR from 2nd January 2015 to 31st December 2015

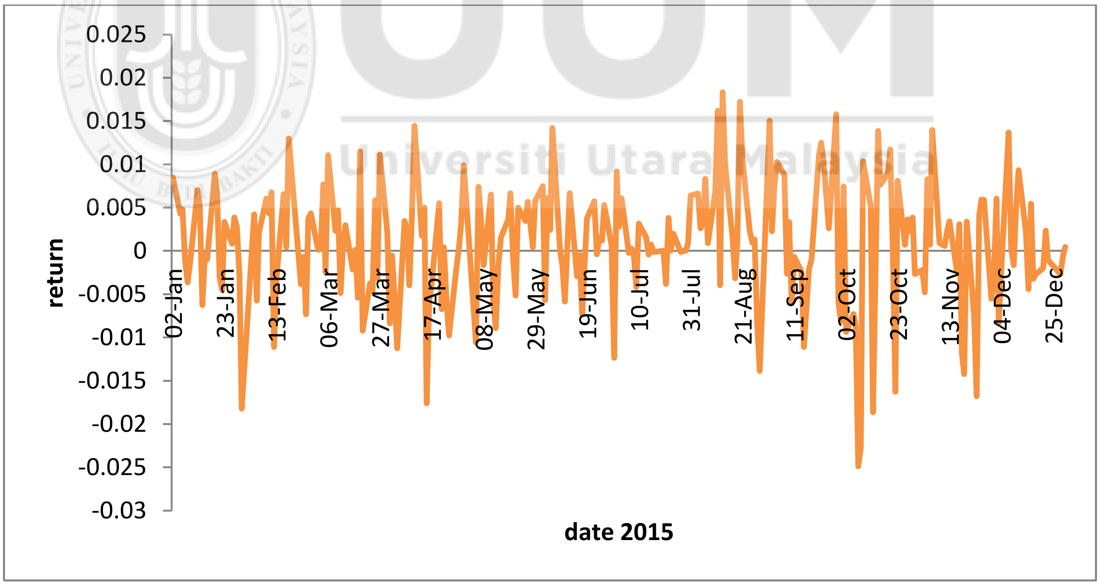


Figure 6.13. Daily returns series of exchange rate between USD and MYR from 2nd January 2015 to 31st December 2015

6.3.2 Forecasting Exchange Rates

In this subsection, we forecasted USD/MYR value for the first six months of year 2016 using the initial parameters, i.e., $H_1 = 0.5717$, $H_2 = 0.6031$, $\mu = 0.0008$, β =0.00009, m =0.00005 and α =1.4331 obtained from daily index price data of 2015. The forecasted USD/MYR values were computed by using GBM model, i.e., Equation (1.1), and by using GFBM model, i.e., Equation (3.1). We calculated volatility by using four different formulas as in Table 6.5.

Table 6.5

Formulas of volatility

Volatility	**Deterministic function**
Constant volatility (Con)	$\sigma = \sqrt{\frac{1}{(n-1)\Delta t}\sum_{i=1}^{n}(S_i - \bar{S})^2}$
Stochastic volatility (STO-1)	$\sigma(Y_t) = Y_t$
Stochastic volatility (STO-2)	$\sigma(Y_t) = \sqrt{Y_t}$
Stochastic volatility (STO-3)	$\sigma(Y_t) = e^{Y_t}$

Note that in all these stochastic volatility models, Y_t is assumed to obey fractional Ornstein–Uhlenbeck process, i.e. Equation (3.2). The computed volatility values are available in the following Table 6.6.

Table 6.6

Volatility values according to different formulas

Volatility type	Con	STO-1	STO-2	STO-3
Value	0.007	0.000049	0.007	1.00005

Next, to determine the performance of the proposed model, MSE, as adopted in Benavides (2004), Shen, Chao and Zhao (2015), and Ye (2017), was employed.

Table 6.7 shows the forecasted values of USD/MYR exchange rate using GBM and GFBM models via four formulas of stochastic volatility in addition to actual exchange rate. While Table 6.8 represents the level of accuracy ranking of all models depending on the values of MSE.

Table 6.7

Forecast value for exchange rate USD/MYR with MSE

Date 2016	GBM-Con	GBM – STO-1	GBM - STO-2	GBM – STO-3	GFBM-Con	GFBM – STO-1	GFBM - STO-2	GFBM – STO-3	Exact
4 Jan	4.28502	4.29536	4.28502	1.84791	4.2691	4.29525	4.26909	1.0854	**4.3235**
5 Jan	4.28992	4.2954	4.28992	2.17577	4.31045	4.29554	4.31045	4.30419	**4.3425**
6 Jan	4.31149	4.29555	4.3115	4.45617	4.31487	4.29557	4.31488	4.98377	**4.3695**
7 Jan	4.28138	4.29534	4.28137	1.63652	4.34141	4.29576	4.34143	11.9737	**4.414**
8 Jan	4.3049	4.2955	4.3049	3.58089	4.28123	4.29534	4.28123	1.62855	**4.3705**
11 Jan	4.31262	4.29556	4.31263	4.62599	4.30209	4.29548	4.3021	3.26205	**4.3985**
12 Jan	4.35382	4.29584	4.35384	18.0032	4.3126	4.29556	4.31261	4.62335	**4.404**
13 Jan	4.26132	4.2952	4.2613	0.836357	4.30078	4.29547	4.30079	3.12315	**4.3945**
14 Jan	4.26682	4.29523	4.26681	1.00563	4.32049	4.29561	4.3205	6.00262	**4.393**
15 Jan	4.31328	4.29556	4.31329	4.72849	4.30983	4.29554	4.30984	4.21735	**4.3775**
18 Jan	4.30416	4.2955	4.30416	3.49373	4.28558	4.29537	4.28558	1.88274	**4.4085**
19 Jan	4.29866	4.29546	4.29866	2.90994	4.29484	4.29543	4.29484	2.56291	**4.389**
20 Jan	4.26807	4.29524	4.26806	1.0486	4.29975	4.29547	4.29975	3.01778	**4.3795**
21 Jan	4.25979	4.29519	4.25978	0.794568	4.22952	4.29497	4.22949	0.286694	**4.376**
22 Jan	4.31193	4.29555	4.31193	4.52085	4.29246	4.29541	4.29246	2.36768	**4.337**
26 Jan	4.30701	4.29552	4.30702	3.84112	4.33762	4.29573	4.33764	10.5677	**4.2915**
27 Jan	4.29601	4.29544	4.29601	2.66488	4.27787	4.29531	4.27786	1.45537	**4.2615**
28 Jan	4.35111	4.29582	4.35114	16.4727	4.28862	4.29539	4.28861	2.08337	**4.235**
29 Jan	4.2297	4.29497	4.22967	0.288481	4.32838	4.29567	4.3284	7.79213	**4.1475**
2 Feb	4.3045	4.2955	4.3045	3.53348	4.30536	4.29551	4.30537	3.6363	**4.2045**
3 Feb	4.23204	4.29499	4.23201	0.312208	4.22327	4.29493	4.22324	0.232121	**4.2375**
4 Feb	4.33034	4.29568	4.33035	8.31228	4.33516	4.29571	4.33518	9.74544	**4.1485**
5 Feb	4.28392	4.29535	4.28391	1.78116	4.31306	4.29556	4.31306	4.69316	**4.1395**
10 Feb	4.31049	4.29554	4.3105	4.31119	4.35799	4.29587	4.35801	20.641	**4.161**
11 Feb	4.27433	4.29529	4.27433	1.29327	4.32919	4.29567	4.3292	8.00237	**4.1085**
12 Feb	4.29738	4.29545	4.29738	2.78884	4.30567	4.29551	4.30567	3.6735	**4.165**
15 Feb	4.31068	4.29554	4.31069	4.33759	4.27936	4.29532	4.27936	1.52996	**4.146**
16 Feb	4.28073	4.29533	4.28072	1.60114	4.33624	4.29572	4.33626	10.0988	**4.1475**
17 Feb	4.24603	4.29509	4.246	0.500288	4.24891	4.29511	4.24889	0.551287	**4.2065**
18 Feb	4.34385	4.29577	4.34388	12.976	4.25313	4.29514	4.25311	0.635301	**4.167**
19 Feb	4.27844	4.29532	4.27843	1.48344	4.34654	4.29579	4.34656	14.1752	**4.225**
22 Feb	4.349	4.29581	4.34903	15.37	4.30348	4.29549	4.30348	3.416	**4.2095**
23 Feb	4.34531	4.29578	4.34534	13.6141	4.30301	4.29549	4.30302	3.36333	**4.186**
24 Feb	4.28761	4.29538	4.2876	2.01423	4.3267	4.29565	4.32671	7.37121	**4.241**
25 Feb	4.30032	4.29547	4.30032	3.07555	4.31976	4.29561	4.31977	5.86056	**4.212**
26 Feb	4.2841	4.29536	4.2841	1.79224	4.28928	4.29539	4.28928	2.12976	**4.221**
29 Feb	4.26417	4.29522	4.26415	0.920162	4.30874	4.29553	4.30874	4.06698	**4.2195**
1 Mar	4.28105	4.29533	4.28105	1.61879	4.33807	4.29573	4.33809	10.7263	**4.194**
2 Mar	4.25692	4.29516	4.25691	0.721685	4.31922	4.2956	4.31923	5.75505	**4.162**
3 Mar	4.2627	4.29521	4.26269	0.876063	4.32046	4.29561	4.32047	5.99735	**4.143**

Table 6.7 (Continued)

Date 2016	GBM-Con	GBM – STO-1	GBM - STO-2	GBM – STO-3	GFBM-Con	GFBM – STO-1	GFBM - STO-2	GFBM – STO-3	Exact
4 Mar	4.32899	4.29567	4.329	7.94901	4.28452	4.29536	4.28452	1.81744	**4.129**
7 Mar	4.30919	4.29553	4.3092	4.12934	4.22122	4.29491	4.22119	0.21656	**4.097**
8 Mar	4.28614	4.29537	4.28613	1.91799	4.29031	4.2954	4.29031	2.20423	**4.1045**
9 Mar	4.33528	4.29571	4.3353	9.78384	4.25202	4.29513	4.25201	0.612133	**4.1355**
10Mar	4.31929	4.2956	4.3193	5.76994	4.28281	4.29535	4.2828	1.71638	**4.1195**
11 Mar	4.31252	4.29556	4.31252	4.60984	4.29769	4.29545	4.29769	2.81775	**4.105**
14 Mar	4.28134	4.29534	4.28134	1.63452	4.30662	4.29551	4.30663	3.79142	**4.092**
15 Mar	4.28064	4.29533	4.28064	1.59684	4.30535	4.29551	4.30536	3.63486	**4.1175**
16 Mar	4.30094	4.29547	4.30094	3.13958	4.26108	4.29519	4.26107	0.82972	**4.1355**
17 Mar	4.32227	4.29562	4.32229	6.36779	4.31392	4.29556	4.31393	4.82909	**4.083**
18 Mar	4.31835	4.2956	4.31836	5.5928	4.28156	4.29534	4.28156	1.6466	**4.0575**
21 Mar	4.28183	4.29534	4.28182	1.66104	4.32986	4.29568	4.32988	8.1818	**4.071**
22 Mar	4.29679	4.29545	4.29679	2.73498	4.27929	4.29532	4.27929	1.52638	**4.0435**
23 Mar	4.28377	4.29535	4.28376	1.77223	4.28652	4.29537	4.28651	1.94235	**3.982**
24 Mar	4.25893	4.29518	4.25891	0.771941	4.24367	4.29507	4.24364	0.462071	**4.027**
25 Mar	4.32951	4.29567	4.32952	8.08654	4.22484	4.29494	4.22481	0.244755	**4.047**
28 Mar	4.26345	4.29521	4.26344	0.898288	4.31494	4.29557	4.31495	4.99616	**4.0375**
29 Mar	4.29112	4.29541	4.29112	2.26452	4.30364	4.29549	4.30365	3.43442	**4.0025**
30 Mar	4.32325	4.29563	4.32326	6.57714	4.26935	4.29525	4.26934	1.09479	**3.959**
31 Mar	4.27093	4.29526	4.27092	1.15406	4.29009	4.2954	4.29009	2.18826	**3.922**
1 Apr	4.34259	4.29576	4.34261	12.4455	4.29812	4.29545	4.29812	2.85866	**3.901**
4 Apr	4.27463	4.29529	4.27462	1.30593	4.29215	4.29541	4.29215	2.34345	**3.88**
5 Apr	4.30714	4.29552	4.30715	3.85774	4.30893	4.29553	4.30893	4.09286	**3.9205**
6 Apr	4.28437	4.29536	4.28437	1.80826	4.28928	4.29539	4.28928	2.12971	**3.914**
7 Apr	4.30192	4.29548	4.30192	3.24315	4.29113	4.29541	4.29113	2.26522	**3.905**
8 Apr	4.34252	4.29576	4.34254	12.4194	4.26914	4.29525	4.26913	1.08696	**3.935**
11 Apr	4.29897	4.29546	4.29897	2.94043	4.21318	4.29485	4.21314	0.164871	**3.8807**
12 Apr	4.25673	4.29516	4.25671	0.716906	4.25718	4.29517	4.25716	0.727877	**3.892**
13 Apr	4.29133	4.29541	4.29133	2.28027	4.3203	4.29561	4.32031	5.96559	**3.865**
14 Apr	4.22821	4.29496	4.22818	0.274331	4.32865	4.29567	4.32866	7.86018	**3.9035**
15 Apr	4.32336	4.29563	4.32337	6.59959	4.26346	4.29521	4.26345	0.898663	**3.9055**
18 Apr	4.31833	4.2956	4.31834	5.58902	4.31671	4.29558	4.31672	5.29789	**3.934**
19 Apr	4.27823	4.29531	4.27822	1.47312	4.30218	4.29548	4.30218	3.27165	**3.8955**
20 Apr	4.28037	4.29533	4.28037	1.58244	4.2896	4.29539	4.2896	2.15267	**3.874**
21 Apr	4.29353	4.29542	4.29353	2.45355	4.32369	4.29563	4.3237	6.67262	**3.881**
22 Apr	4.31319	4.29556	4.3132	4.71434	4.27774	4.29531	4.27773	1.44915	**3.9035**
25 Apr	4.34172	4.29576	4.34174	12.0943	4.33047	4.29568	4.33049	8.34918	**3.9135**
26 Apr	4.34152	4.29576	4.34154	12.0175	4.27163	4.29527	4.27162	1.18133	**3.937**
27 Apr	4.27208	4.29527	4.27207	1.19932	4.29464	4.29543	4.29464	2.54597	**3.9315**
28 Apr	4.33416	4.29571	4.33418	9.43021	4.31295	4.29556	4.31296	4.67713	**3.917**
29 Apr	4.2793	4.29532	4.27929	1.5267	4.28494	4.29536	4.28494	1.84319	**3.9045**
3 May	4.30599	4.29551	4.306	3.71337	4.31105	4.29554	4.31106	4.39112	**3.9225**

Table 6.7 (Continued)

Date 2016	GBM-Con	GBM – STO-1	GBM - STO-2	GBM – STO-3	GFBM-Con	GFBM – STO-1	GFBM - STO-2	GFBM – STO-3	Exact
4 May	4.29881	4.29546	4.29881	2.92516	4.33496	4.29571	4.33497	9.67934	**3.9765**
5 May	4.31958	4.2956	4.31959	5.82576	4.30774	4.29552	4.30774	3.93444	**4.003**
6 May	4.29745	4.29545	4.29745	2.79535	4.30834	4.29553	4.30835	4.01407	**4.0095**
9 May	4.34702	4.29579	4.34704	14.3995	4.3155	4.29558	4.31551	5.08983	**4.0005**
10 May	4.30402	4.2955	4.30403	3.47799	4.35522	4.29585	4.35525	18.8516	**4.059**
11 May	4.28767	4.29538	4.28766	2.01832	4.30787	4.29552	4.30787	3.9514	**4.0545**
12 May	4.28055	4.29533	4.28054	1.59175	4.32896	4.29567	4.32897	7.94185	**4.0265**
13 May	4.29452	4.29543	4.29452	2.5357	4.26275	4.29521	4.26273	0.877352	**4.0275**
16 May	4.25042	4.29512	4.2504	0.580053	4.30026	4.29547	4.30027	3.06965	**4.0355**
17 May	4.30401	4.2955	4.30402	3.47679	4.29597	4.29544	4.29597	2.66095	**4.014**
18 May	4.2408	4.29505	4.24078	0.419532	4.29342	4.29542	4.29342	2.44478	**4.035**
19 May	4.29477	4.29543	4.29477	2.55721	4.28219	4.29534	4.28218	1.68126	**4.08**
20 May	4.31577	4.29558	4.31577	5.13396	4.28836	4.29539	4.28836	2.06584	**4.0795**
23 May	4.3175	4.29559	4.31751	5.43773	4.30411	4.2955	4.30412	3.4884	**4.068**
24 May	4.30588	4.29551	4.30589	3.69937	4.29016	4.2954	4.29016	2.19333	**4.1185**
25 May	4.27073	4.29526	4.27072	1.14643	4.26191	4.2952	4.2619	0.85315	**4.106**
26 May	4.28546	4.29537	4.28546	1.87536	4.2602	4.29519	4.26019	0.80563	**4.0825**
27 May	4.29272	4.29542	4.29272	2.38854	4.29807	4.29545	4.29807	2.85397	**4.0855**
30 May	4.27726	4.29531	4.27725	1.42608	4.35882	4.29588	4.35885	21.2158	**4.1095**
31 May	4.24103	4.29505	4.241	0.422753	4.31852	4.2956	4.31853	5.62422	**4.1195**
1 Jun	4.29808	4.29545	4.29808	2.85476	4.28947	4.29539	4.28947	2.14374	**4.1345**
2 Jun	4.31681	4.29559	4.31682	5.3141	4.29473	4.29543	4.29473	2.55394	**4.1625**
3 Jun	4.34243	4.29576	4.34245	12.3822	4.2942	4.29543	4.2942	2.50947	**4.15**
6 Jun	4.24536	4.29508	4.24534	0.489242	4.28634	4.29537	4.28634	1.93127	**4.1055**
7 Jun	4.31192	4.29555	4.31192	4.51934	4.27934	4.29532	4.27934	1.52894	**4.077**
8 Jun	4.30499	4.2955	4.305	3.59203	4.23548	4.29501	4.23546	0.350653	**4.0625**
9 Jun	4.30116	4.29548	4.30116	3.16246	4.25619	4.29516	4.25617	0.704005	**4.0355**
10 Jun	4.30145	4.29548	4.30145	3.19292	4.27476	4.29529	4.27475	1.3116	**4.0705**
13 Jun	4.24882	4.29511	4.2488	0.54968	4.28203	4.29534	4.28202	1.6722	**4.0945**
14 Jun	4.31205	4.29555	4.31206	4.54007	4.26879	4.29525	4.26877	1.07414	**4.0925**
15 Jun	4.36833	4.29594	4.36836	28.9674	4.29385	4.29542	4.29385	2.48016	**4.101**
16 Jun	4.31927	4.2956	4.31928	5.76622	4.25016	4.29512	4.25014	0.574931	**4.096**
17 Jun	4.33075	4.29568	4.33076	8.42484	4.3129	4.29556	4.31291	4.6696	**4.0985**
20 Jun	4.3049	4.2955	4.30491	3.58123	4.35823	4.29587	4.35825	20.8059	**4.077**
21 Jun	4.2943	4.29543	4.2943	2.51725	4.3135	4.29556	4.31351	4.7626	**4.0555**
23 Jun	4.27029	4.29526	4.27027	1.12945	4.30971	4.29554	4.30971	4.19981	**4.02**
24 Jun	4.24951	4.29511	4.24949	0.562441	4.37491	4.29599	4.37494	35.9166	**4.115**
27 Jun	4.30421	4.2955	4.30421	3.49981	4.30844	4.29553	4.30845	4.02774	**4.109**
28 Jun	4.31862	4.2956	4.31863	5.64311	4.282	4.29534	4.28199	1.67049	**4.0915**
29 Jun	4.31019	4.29554	4.3102	4.26803	4.2819	4.29534	4.28189	1.66514	**4.0575**
30 Jun	4.26973	4.29526	4.26972	1.10854	4.30866	4.29553	4.30866	4.05655	**4.0225**
MSE	**0.059017**	**0.057518**	**0.059018**	**18.98506**	**0.058116**	**0.057511**	**0.058117**	**24.39695**	

Table 6.8

The level of accuracy ranking for forecasting models.

Rank	**Model**	**MSE**
1	GFBM-STO-1*	0.057511
2	GBM-STO-1	0.057518
3	GFBM-Con	0.058116
4	GFBM-STO-2	0.0581176
5	GBM-Con	0.059017
6	GBM-STO-2	0.0589018
7	GBM-STO-3	18.98506
8	GFBM-STO-3	24.3970

** The proposed model*

From the findings, GFBM-STO-1 (the proposed model) demonstrates the most accurate in performance, whereas GFBM-STO3 model with exponential stochastic volatility performed worst.

Table 6.7 and Table 6.8 suggested that all values of MSE computed based on GBM-Con, GBM-STO-1, GBM-STO-2, GFBM-Con, GFBM-STO-1 and GFBM-STO-2 are relatively close. Meanwhile the MSE computed based on GBM-STO-3 and GFBM-STO-3 are very large. This huge gap refers to the large relative difference between the values of stochastic volatilities as shown in Table 6.7. Thus, large volatility implies large fluctuation.

However, if we exclude the case of STO-3, we can observe that GFBM model is significantly more accurate than GBM model in all cases involving different types of volatility. These findings are consistent with Painter (1998), Willinger et al. (1999), Grau–Carles (2000), and Rejichi and Aloui (2012) as previously stated.

Figures 6.14-6.17 illustrate the comparison between the actual exchange rates versus forecasted exchange rates in GBM and GFBM model with four different volatilities.

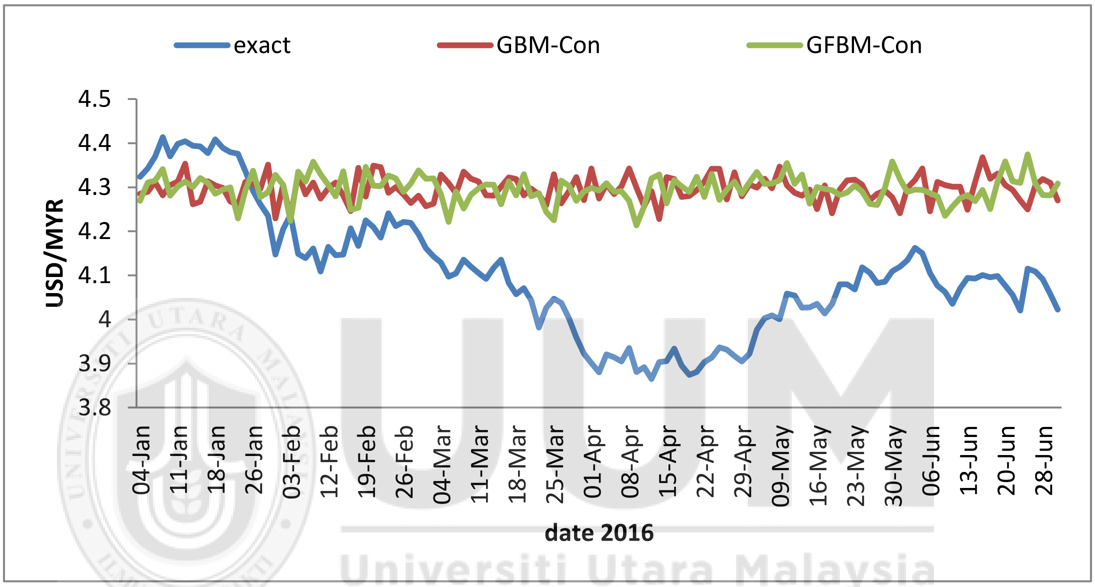


Figure 6.14. Forecast exchange rates vs actual price (constant volatility case).

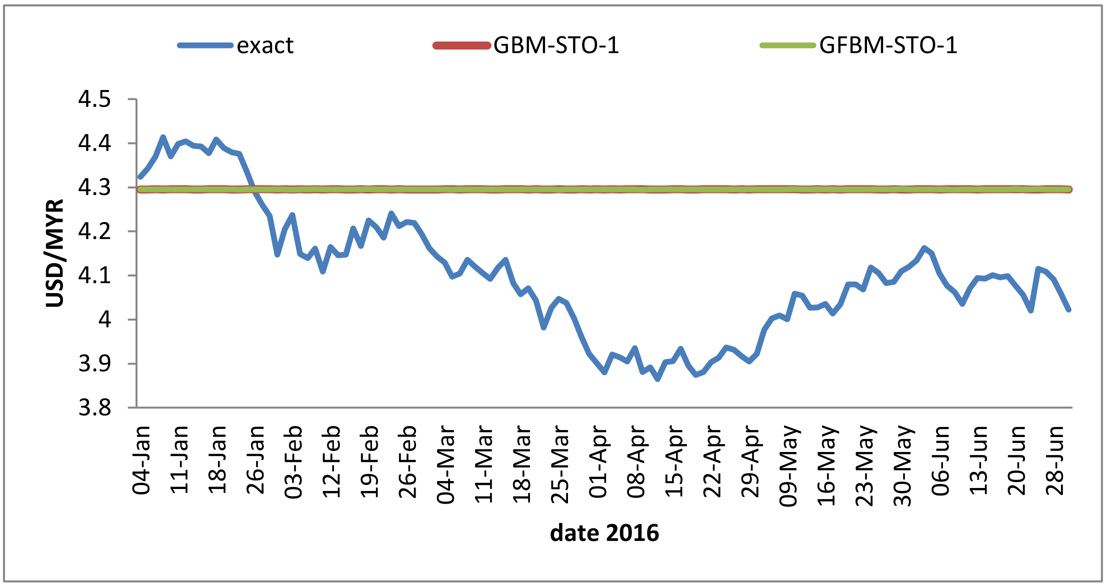


Figure 6.15. Forecast exchange rates vs actual price (STO-1 case)

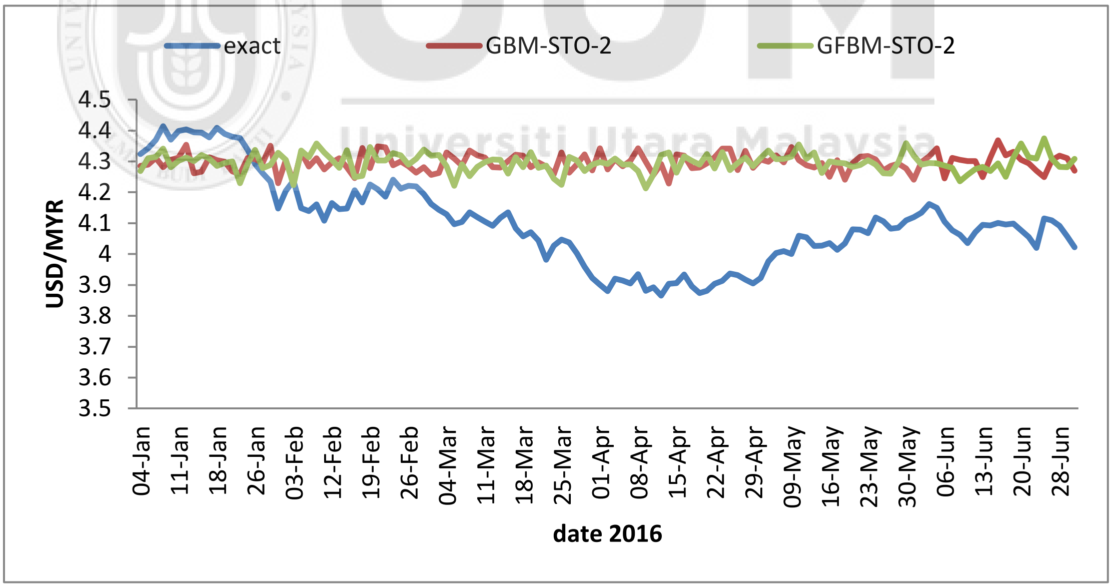


Figure 6.16. Forecast exchange rates vs actual price (STO-2 case)

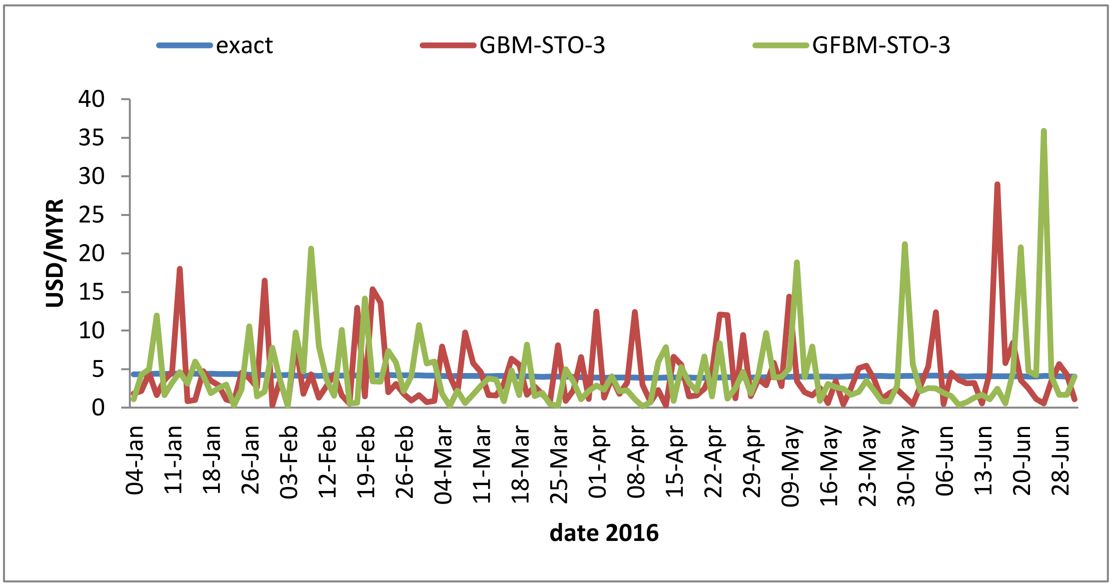


Figure 6.17. Forecast exchange rates vs actual price (STO-3 case)

Figures 6.14 - 6.16 indicate that the forecasted exchange rates are closer together and less fluctuated than the actual exchange rates. While Figure 6.17 indicates that the forecasted exchange rates are more fluctuated than actual exchange rates. However, the GFBM-STO-1 is more stable than other forecasted methods since this model provides the smallest value of volatility.

Meanwhile, value affected by STO-3 has large MSE, meaning a large fluctuation existed in exchange rates of USD/MYR, as illustrated in Figure 6.17. Thus, GBM-STO-3 and GFBM-STO-3 are inappropriate to forecast exchange rate between USD and MYR. Whereas, all other forecasting methods can be used in forecasting exchange rates between USD and MYR.

Tables 6.7-6.8 and Figures 6.14-6.17, show that our model (GFBM-STO-1) is efficient and can be used to forecast exchange rates in real market.

6.4 Mortgage Insurance

Mortgage insurance is an insurance policy that protects the lender in the event that the borrower defaults on payments, dies, or is unable to meet the contractual obligation of the mortgage.

It is a tool to mitigate exposure of risk among lenders as this risk is transferred from lenders to insurers. A good mortgage insurance model will contribute to the growth in house financing. Among challenges faced by mortgage insurance models include heavy dependency toward inflation, other risk faced by insurer and unexpected change in collateral prices.

Changing in collateral (risky asset) price plays a critical role in the pricing of mortgage insurance contracts since the amount that the insurer has to pay lender significantly depends on the price of collateral. Current literatures assume the change in collateral prices to follow a GBM model, such as in Bardhan et al. (2006) and Chen et al. (2013). According to the best of our knowledge, there has yet work on the change in collateral prices that follow a GFBM model. Thus, we will make use of long memory stochastic volatility properties to be included in this empirical study and investigate further of the expected loss for lenders.

According to a typical mortgage insurance contract, the insurer has to pay lender certain amount (*Loss*) if standard default occurs at time t by following the model

$$Loss(t) = \max(0, \min(B(t-1) - V(t), L_R\, B(t-1))) \tag{6.17}$$

where $B(t) = \frac{y}{c}\left(1 - \frac{1}{(1+c)^{T-t}}\right)$ is a loan balance with installment y and mortgage rate c during period T. L_R represents loss ratio while $V(t)$ is a collateral (risky asset) price.

Equation (6.17) implies that if the collateral value is greater than the remaining loan balance, then the insurer will not pay to the lender and then the $Loss$ is zero. While, if the value of the collateral is less than the loan balance then the maximum $loss$ is equal to $L_R B(t-1)$.

6.4.1 Description of Data

For investigation purposes, we use available data online at *http://www.nationwide.co.uk.* This data represents total house price index in the UK. The quarterly house price index from fourth quarter of 1973 (4Q73) to first quarter of 2017 (1Q17) are considered with total observation of 174 quarters. These data reveal long time dependency with Hurst parameter of $H = 0.85$. The return series are calculated in logarithm to avoid high volatility in the data. Figure 6.18 and Figure 6.19 show the house price index and its return series.

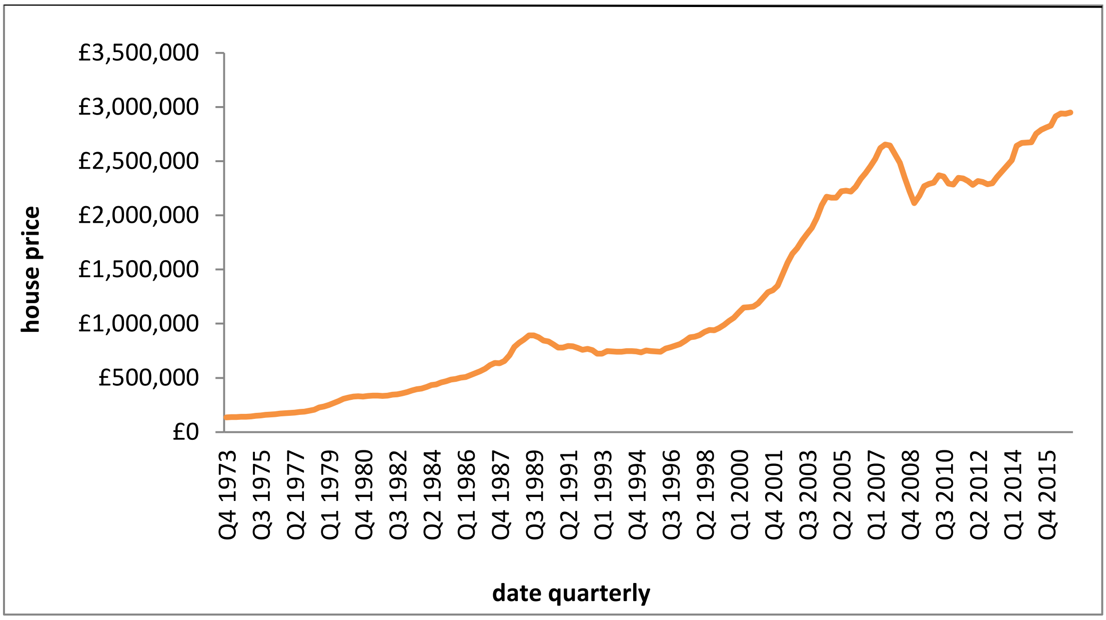


Figure 6.18. Quarterly house price index in the UK from 4Q73 to 1Q17.

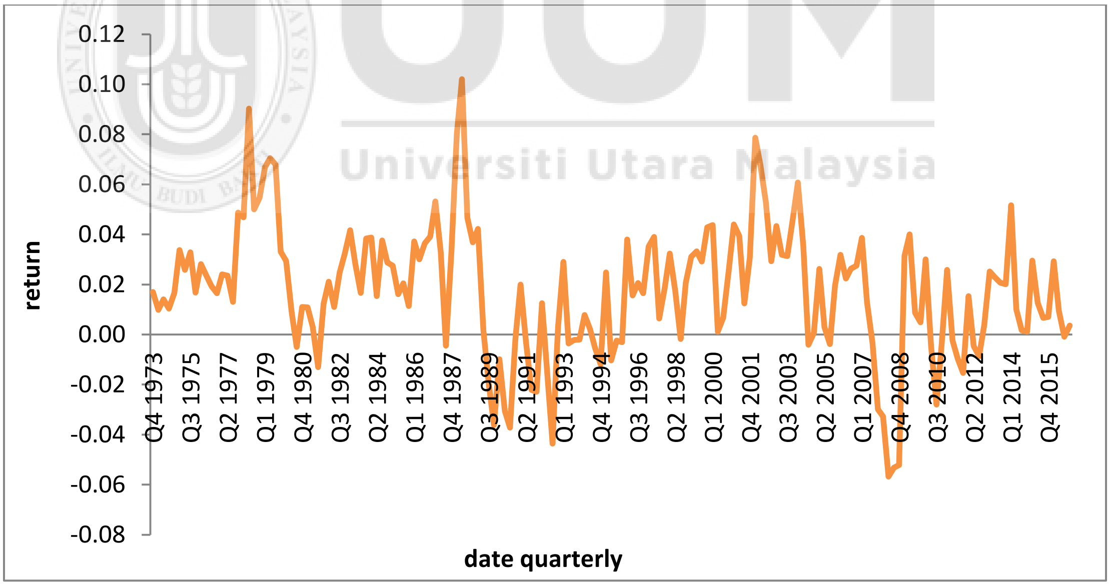


Figure 6.19. Quarterly return of house price index in the UK from 4Q73 to 1Q17.

6.4.2 Valuing Insurer Potential Loss

In this subsection, we will compute collateral values using four models i.e. GBM with constant volatility (GBM-Con), GBM with stochastic volatility that obeys fractional Ornstein-Uhlenbeck process (GBM-STO), GFBM with constant volatility (GFBM-Con) and GFBM with stochastic volatility that obeys fractional Ornstein-Uhlenbeck process (GFBM-STO). Do note that GFBM-STO is the proposed model in this thesis. We revisit these models in Table 6.9 for easier reference. The comparison for insurer's loss will be investigated further.

Table 6.9

The models under consideration

Model	**Formula**
GBM-Con	$dV_t = \mu V_t dt + \sigma V_t dW(t)$
GBM-STO	$dV_t = \mu V_t dt + \sigma(Y_t) V_t dW(t)$ $d\,Y_t = \alpha(m - Y_t)dt + \beta dB_{H_2}(t)$
GFBM-Con	$dV_t = \mu V_t dt + \sigma V_t dB_{H_1}(t)$
GFBM-STO	$dV_t = \mu V_t dt + \sigma(Y_t) V_t dB_{H_1}(t)$ $d\,Y_t = \alpha(m - Y_t)dt + \beta dB_{H_2}(t)$

The involved parameters computed in models listed in Table 6.10 are as follow

Table 6.10

Involved parameters value

Parameter	**Value**
μ	0.01782
α	0.6936
m	0.00066
β	0.00115
H_1	0.8538
H_2	0.8541
constant volatility σ	0.02586
stochastic volatility $\sigma(Y_t)$	0.000659

The annual insurer's potential loss is computed by following these parameters: insured property, $V_0 = £\ 100000$; annual installment, $y = £\ 15000$; mortgage rate, $c = 0.042$; loss ratio, $L_R = 0.75$; and time period, T=15 years. These parameters were obtained from Chuang,Yang, Chen, and Lin (2017).

Table 6.11 shows the computed values of loan balance, collaterals via different models in Table 6.9 and their corresponding insurer's potential loss in Equation (6.17). While, Figure 6.20 illustrates the level of computed potential losses for the first six years. Potential losses after six years are zeros.

Table 6.11

Collaterals values and their corresponding potential loss.

Time t	$B(t-1)$	$V_1(t)$	$V_2(t)$	$V_3(t)$	$V_4(t)$	$Loss_1(t)$	$Loss_2(t)$	$Loss_3(t)$	$Loss_4(t)$
1	164467.	108997.	101976.	103242.	101835.	55470.3	62491.3	61225.1	62632.1
2	156375.	110920	103810.	105063.	103666.	45455.3	52565.4	51311.6	52708.8
3	147943.	112876.	105676.	106917.	105530.	35066.5	42266.7	41026.1	42412.7
4	139156.	114868.	107576.	108803.	107428.	24289.	31580.3	30353.8	31728.9
5	130001.	116894.	109510.	110722.	109359.	13107.4	20490.7	19279.1	20642.
6	120461.	118956.	111479.	112675.	111325.	1505.47	8981.84	7786.13	9135.82
7	110520.	121054.	113484.	114662.	113327.	0	0	0	0
8	100162.	123189.	115524.	116685.	115364.	0	0	0	0
9	89369.1	125362.	117601.	118743.	117439.	0	0	0	0
10	78122.7	127574.	119715.	120838.	119550.	0	0	0	0
11	66403.8	129824.	121868.	122969.	121699	0	0	0	0
12	54192.8	132114.	124059.	125139.	123888.	0	0	0	0
13	41468.9	134444.	126290.	127346.	126115.	0	0	0	0
14	28210.6	136816.	128560.	129592.	128383.	0	0	0	0
15	14395.4	139229.	130872.	131878.	130691.	0	0	0	0

$V_1(t)$: Collateral computed by GBM-Con
$V_2(t)$: Collateral computed by GBM-STO
$V_3(t)$: Collateral computed by GFBM-Con
$V_4(t)$: Collateral computed by GFBM-STO
$B(t-1)$: Loan balance at time $t-1$

$Loss_1(t)$: potential loss corresponding to $V_1(t)$
$Loss_2(t)$: potential loss corresponding to $V_2(t)$
$Loss_3(t)$: potential loss corresponding to $V_3(t)$
$Loss_4(t)$: potential loss corresponding to $V_4(t)$

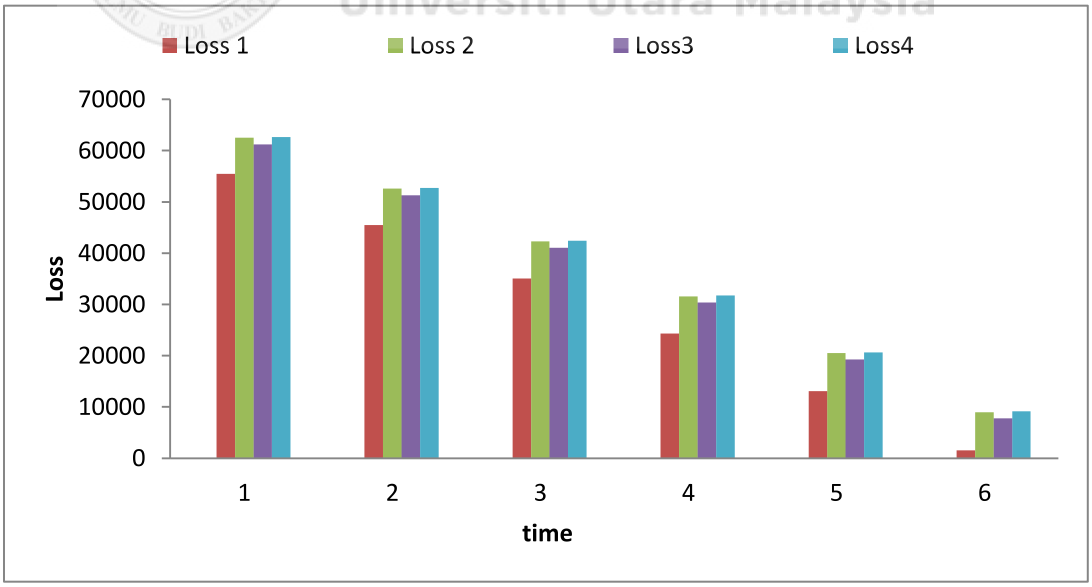


Figure 6.20. Comparison between the levels of potential losses in the first six years.

The findings in Table 6.11 reveal an inverse relationship between collateral value and their potential losses, i.e. as collateral value increases, the loss decreasing. In the first six years, the value of loan balance $B(t-1)$ is greater than the computed values of collateral that computed by $V_1(t), V_2(t), V_3(t)$, and $V_4(t)$. Thus the insurers have to pay a certain amount equal to $Loss_1(t)$, $Loss_2(t)$, $Loss3(t)$,and $Loss_4(t)$ corresponding to collateral values $V_1(t), V_2(t), V_3(t)$, and $V_4(t)$ respectively. On the seventh year onward, collateral values are greater than loan balances thus insurer's loss equal to zero.

The proposed model (GFBM-STO) in this thesis provides the greatest value of insurer's loss ($Loss_4(t)$). While, GBM-Cos provides the smallest value of insurer's loss ($Loss_1(t)$). These findings implied two perspectives, from the insurer, and the loaner's perspective. From the insurer's perspective, the loss computed by GBM-Con is the best, while the loss computed GFBM-STO the worst. While, from the loaner's perspective, the loss computed by GFBM-STO is the best, while the loss computed by GBM-Con the worst.

Table 6.11 also shows a significant difference between the potential loss computed via GBM-Con, and the other three potential computed losses (GBM-STO, GFBM-Con and GFBM-STO). These results reflect the level of affection of memory and stochastic volatility assumption on potential loss of insurer. Therefore, we strongly recommended taking memory and stochastic volatility into account in mortgage insurance contracts.

These findings ensure the ability of the proposed model in this thesis to be applied in the real financial environments.

6.5 Discussion

In this chapter, we investigated four applications of long memory stochastic volatility models in finance - the fractional Black–Scholes model in option pricing, value at risk and long memory, exchange rates and mortgage insurance.

For option pricing, the results of empirical study reveals a practically acceptable performance of the proposed model (FBM-Alhagyan) where long memory is taken into account and the volatility is assumed stochastic. The prices obtained from FBM-Alhagyan lie between those obtained by FBM-Kukush and FBM-standard. There is a small difference in values between the results FBM-Misiran and FBM-Alhagyan. FBM-Misiran is based on the assumption of constant volatility and corporation of long memory properties with drawback which we revisited in details in previous subsection. While FBM-Alhagyan assumed stochastic volatility that is more favorably agreed with current empirical studies in the literature.

For value at risk (VaR), we influenced its standard model of VaR with memory parameter and stochastic volatility. We made a comparison study between the standard model and model that exhibited long memory with stochastic volatility model.

This study demonstrates that higher value of Hurst parameter implies a higher VaR and then greater probability to lose, i.e. there is a positive relation between the Hurst parameter and VaR. Consequently, the investors are advised to adopt the proposed model in the case of long memory environment. However, there are no significant effects of stochastic volatility since its value is very close to constant volatility.

For exchange rate, we forecasted the currency's exchange rates between United State Dollar (USD) and Malaysian Ringgit (MYR) by using GBM and GFBM depending on the assumptions of constant volatility (GBM-Con and GFBM-Con) and three deterministic functions of stochastic volatility that obeys fractional Ornstein-Uhlenbeck process; i.e. $\sigma(Y_t) = Y_t$ (GBM-STO1, GFBM-STO1), $\sigma(Y_t) = \sqrt{Y_t}$ (GBM-STO2, GFBM-STO2), and $\sigma(Y_t) = e^{Y_t}$ (GBM-STO3, GFBM-STO3). The evaluation is conducted depending on the value of mean square error (MSE) as the smallest MSE is the best. The findings show that the proposed model (GFBM-STO1) has the most accurate performance as it satisfied the smallest value of MSE. While, GFBM-STO3 has the worst performance. Further, the findings showed GFBM model is significantly more accurate than GBM model in all cases except the case of exponential stochastic volatility. The forecasting models GBM-STO3 and GFBM-STO3 have large amounts of MSE, so the assumption of exponential deterministic function for stochastic volatility process is inappropriate in this case. Meanwhile, the other forecasting models reveal high accuracy according to the values of MSE.

For mortgage insurance, we computed the collateral values by using four models include GBM with constant volatility (GBM-Con), GBM with stochastic volatility that obeys fractional Ornstein-Uhlenbeck process (GBM-STO), GFBM with constant volatility (GFBM-Con) and the proposed model of GFBM with stochastic volatility that obeys fractional Ornstein-Uhlenbeck process (GFBM-STO). Then, we computed the corresponding potential loss of insurer.

The insurer's loss computed by using the proposed model (GFBM-STO) provides the greatest value while GBM-Con provides the smallest value. These results can be read from two viewpoints insurer's viewpoint and loaner's viewpoint. From the insurer's viewpoint, the loss calculated by GBM-Con is the best, while the loss calculated by GFBM-STO is the worst. While, from the loaner's viewpoint, the loss calculated by GFBM-STO is the best, while the loss calculated by GBM-Con is the worst.

Further, the findings indicate to a significant difference between the potential loss calculated depending on GBM-Con and the other three potential losses calculated depending on GBM-STO, GFBM-Con and GFBM-STO. These results reveal the level of affection of memory and stochastic volatility assumption on potential loss of insurer. Consequently, we strongly recommended taking memory and stochastic volatility into account in mortgage insurance contracts.

In general, according to all results of all applications that studied in this thesis we insure the ability of the new model to apply and to exercise in different real financial environments.

CHAPTER SEVEN
CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusion

Long memory stochastic volatility (LMSV) in modeling financial asset is the subject in this thesis. The element of long memory process better known as fractional Brownian motion (FBM) that is able to capture memory in historical data is exploited, thus improve current underlying assumptions (i.e. no memory of historical shocks in most finance models) that does not reflect the actual financial scenario. Therefore, this study proposed a new geometric fractional Brownian motion (GFBM) model by incorporating LMSV that obeys fractional Ornstein–Uhlenbeck (FOU) process.

In Chapter One, we underlined some discussion on the background of stochastic volatility (SV) accompanied by its related topics with key definitions, problem statement, research objectives, significance of the research and the limitation posed in this study. Subsequently, in Chapter Two we conducted extensive review on the existing models and determined the research gaps. Findings from content analysis highlighted the popularity of LMSV where its development stage was mainly focus on theoretical contribution. Further works suggested that there was a need to bridge theoretical works with practical estimation of important parameters that are useful in financial environment.

To respond to this challenge, we developed a new geometric fractional Brownian motion (GFBM) model perturbed by LMSV model that obeys fractional Ornstein-Uhlenbeck (FOU) process in Chapter Three. The likelihood function in this model was maximized by utilizing innovation algorithm and optimization to obtain estimators in the developed model. These estimators were proven efficient after undergoing simulation study, thus bestow positive feedback to the importance of our proposed model. Moving forward, we presented Chapter Four which is more theoretical in nature. It delved in the development of two new theorems namely the existence and uniqueness of the solution of GFBM equation and the generalized case of fractional stochastic differential equation driven by FBM.

Further in Chapter Five, our proposed model was validated in forecasting price indices of Standard and Poor's 500 (S&P 500), Shanghai Stock Exchange Composite Index (SSE), and FTSE Kuala Lumpur Composite Index (KLCI). These three distinct markets produced different performances using different models, in which our proposed model excelled in comparison to others except in SSE index (though the proposed model differs with the best model by only 0.4%). These encouraging findings motivated us to apply the proposed model to some selected applications in option pricing, value at risk, exchange rate and mortgage insurance, as discussed in details in Chapter Six. It was observed that by incorporating LMSV in these problems, their performances were improved significantly. The proposed model performed the best in aforementioned financial applications.

7.2 Future Research Problems and Recommendations

There are some potential problems that arise while working in this thesis.

- In the proposed model, we use $\sigma(Y_t) = Y_t$ to represent stochastic volatility. The extension to this model can be done by considering other deterministic functions, such that $\sigma(Y_t) = e^y$, $\sigma(Y_t) = \sqrt{y}$ or $\sigma(Y_t) = |y|$ that may improve the performance of the model.

- We also use fractional Ornstein-Uhlenbeck process in this model to represent LMSV model. This process can be replaced by other models such as log–normal or Cox–Ingersoll–Ross (CIR) process to better portray a financial problem.

- In this thesis, we made prior estimation of H_1 and H_2 , and then we further estimate other involved parameters(α, β, μ and m) due to heavy computational effort. More efficient way to improve the computation time by simultaneously estimates all involved is deemed to be a significant contribution.

- We assumed there is no correlation between error terms ($B_{H_1}$and B_{H_2}). Further investigation should be carried out if these two error terms are correlated.

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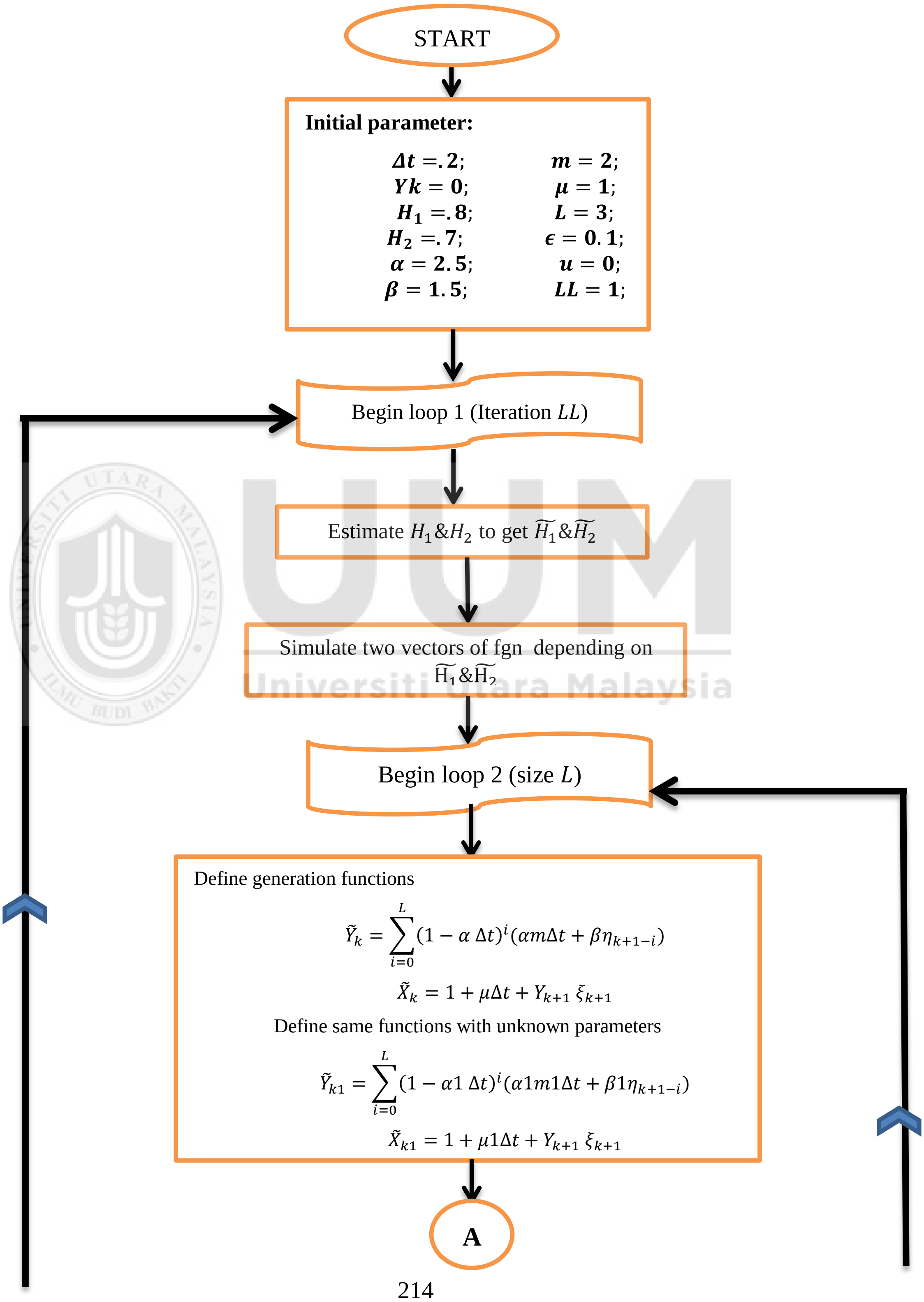
Appendix A

Flowchart for Parameters Estimation



A

Compute the following covariance function:

$$\gamma_{\xi}(n) = \frac{1}{2}\left(|(n+1)\Delta t|^{2H_1} + |(n-1)\Delta t|^{2H_1} - 2|n\Delta t|^{2H_1}\right)$$

$$\gamma_{\eta}(n) = \frac{1}{2}\left(|(n+1)\Delta t|^{2H_2} + |(n-1)\Delta t|^{2H_2} - 2|n\Delta t|^{2H_2}\right)$$

$$\gamma_{\tilde{Y}}(n) = \beta^2 \sum_{i=0}^{L}\sum_{j=0}^{L}(1-\alpha\,\Delta t)^{i+j}\gamma_{\eta}(n+i-j)$$

Construct the following matrices:

$$\Gamma_{n-1} = \{\gamma(k-j)\}_{j,k=1}^{n-1} \quad \& \quad {\Gamma_{n-1}}^{-1}$$

$$\tilde{\gamma}_{n-1} = (\gamma_{\tilde{X}}(n-1), \dots, \gamma_{\tilde{X}}(1))' \quad \& \quad {\tilde{\gamma}_{n-1}}'$$

$$\gamma_n = (\gamma(1), \dots, \gamma(n))'$$

$$\Gamma_n = \begin{bmatrix} \Gamma_{n-1} & \tilde{\gamma}_{n-1} \\ \tilde{\gamma}_{n-1}' & \gamma_{\tilde{X}}(0) \end{bmatrix}$$

$$\phi_n = \Gamma_n^{-1}\gamma_n$$

$$\Gamma_n^{-1} = \begin{bmatrix} I & -\Gamma_{n-1}^{-1}\tilde{\gamma}_{n-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Gamma_{n-1}^{-1} & 0 \\ 0 & (\gamma_{\tilde{X}}(0) - \tilde{\gamma}_{n-1}'\Gamma_{n-1}^{-1}\tilde{\gamma}_{n-1})^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ -\tilde{\gamma}_{n-1}'\Gamma_{n-1}^{-1} & 1 \end{bmatrix}$$

Compute $v_T^2 = \gamma(0) - \gamma_T{}'\Gamma_T^{-1}\gamma_T$

B

B

Construct the following matrices

$$A=\begin{bmatrix} 1 & 0 & \cdots & & 0 \\ -\phi_{11} & 1 & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ -\phi_{(T-1)1} & -\phi_{(T-1)2} & \cdots & -\phi_{(T-1)(T-1)} & 1 \end{bmatrix}$$

$$KK=\begin{bmatrix} \frac{1}{v_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{v_2^2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{v_T^2} \end{bmatrix}$$

Compute:

$$\Sigma_T^{-1}=A'.KK.A$$

$$\det(\Sigma_T)=\prod_{i=1}^{T}E(\varepsilon_i)^2=\prod_{i=1}^{T}v_i^2$$

Find the value of:

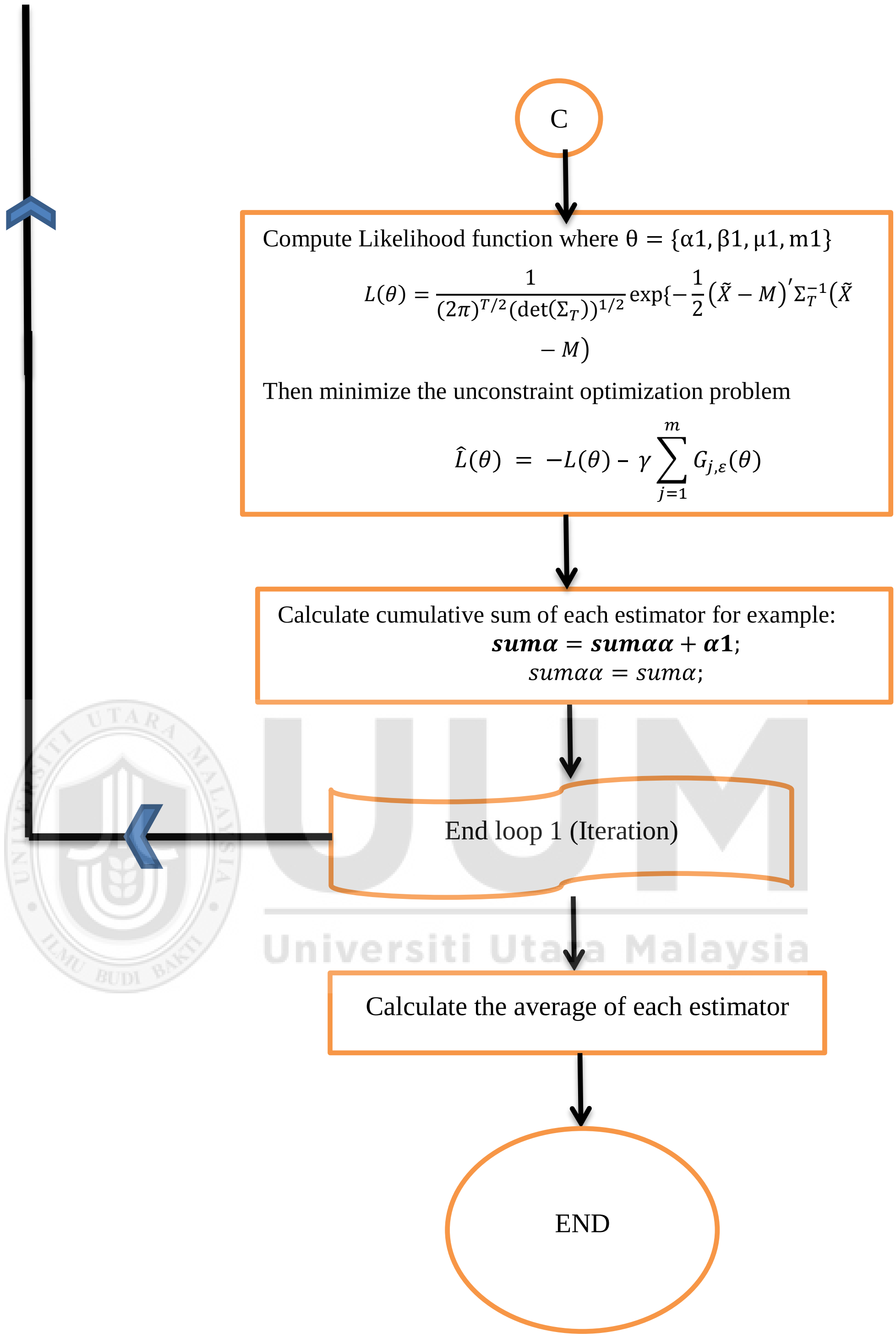
$$g_1(\theta)=-E(\tilde{X}-\mu)^2=-\gamma_{\tilde{X}}(n);$$

$$g_2(\theta)=-v^2=-\{\textstyle\sum_{i=1}^{L}(1-\alpha\Delta t)^i(\alpha m\Delta t)\}^2\,\Delta t^{2H1}-\textstyle\sum_{i=1}^{L}(1-\alpha\Delta t)^{2i}\beta^2\Delta t^{2(H2+H1)};$$

$$G_{i,\varepsilon}(\theta)=\begin{cases} g_i, & g_i>\varepsilon \\ \frac{(g_i-\varepsilon)^2}{4\varepsilon}, & -\varepsilon<g_i<\varepsilon \\ 0, & g_i<-\varepsilon \end{cases}$$

End loop 2 (size L)

C


C
Compute Likelihood function where $\theta = \{\alpha 1, \beta 1, \mu 1, m1\}$
$L(\theta) = \frac{1}{(2\pi)^{T/2}(\det(\Sigma_T))^{1/2}} \exp\{-\frac{1}{2}(\tilde{X} - M)' \Sigma_T^{-1}(\tilde{X} - M)$
Then minimize the unconstraint optimization problem
$\hat{L}(\theta) = -L(\theta) - \gamma \sum_{j=1}^{m} G_{j,\varepsilon}(\theta)$
Calculate cumulative sum of each estimator for example:
$\boldsymbol{suma = sumaa + \alpha 1}$;
$sumaa = suma$;
End loop 1 (Iteration)
Calculate the average of each estimator
END

Appendix B

Standard Simulation for Parameters Estimation

```
Δt=.2;
H1=.8;
H2=.7;
α=2.5;
m=2;
β=1.5;
μ=1;
ϵ=0.1;
l=0;
t=0;
u=0;
v=0;
c=0;
e=0;
f=0;
RR=100;
NN=10;
sumαhut1=0;
summhut1=0;
sumβhut1=0;
sumμhut1=0;
H11=0.8;
H22=0.7;
For[r=1,r<=RR,r++,
"Simulate a fractional Brownian motion process–1st one  ";
data1=RandomFunction[FractionalBrownianMotionProcess[H1],{0,1,0.001}];
"finds the parameter estimates for the fbm process(H1) from data1";
EH1=FindProcessParameters[data1,FractionalBrownianMotionProcess[h]];
H11=EH1[[1,2]];
"Simulate a fractional Brownian motion process–2nd one ";
data2=RandomFunction[FractionalBrownianMotionProcess[H2],{0,1,0.001}];
"finds the parameter estimates for the fbm process(H2) from data2";
```

```
EH2=FindProcessParameters[data2,FractionalBrownianMotionProcess[h]];
H22=EH2[[1,2]];
"simulate fgn";
SimulateFGN[H_,n_]:=Module[{𝓝,ac}, 𝓝 =2^Ceiling[Log[2,n–1]];
ac=Table[FGNAcf[k,H],{k,0, 𝓝 }];
Take[SimulateGLP[ac],n]];SimulateGLP[γ_]:=Module[{m=Length[γ],n,c,g,Z,
Ncap},n=2^Ceiling[Log[2,m–1]];acvf=If[n==m–
1,γ,PadRight[γ,n+1]];Ncap=2*n;
c=Join[acvf,Rest[Reverse[Rest[acvf]]]];
g=Re[Fourier[c,FourierParameters–>{1,–1}]];
Z=RandomVariate[NormalDistribution[0,1],Ncap–2];
Z=(Complex[Sequence@@#1]&)/@Partition[Z,2];
Z=Flatten[{RandomVariate[NormalDistribution[0,Sqrt[2]]],Z,RandomVariate
[NormalDistribution[0,Sqrt[2]]],Reverse[Conjugate[Z]]}];
Take[Re[InverseFourier[Sqrt[g]*Z,FourierParameters–>{0,–1}]],m]/Sqrt[2]];
FGNAcf[k_,H_]:=Module[{},0.5*(Abs[k+1.0]^(2.0*H)–
2*Abs[k]^(2.0*H)+Abs[k–1.0]^(2.0*H))];
"fgn at H=.65 and n=20";
SmH111=SimulateFGN[H11,RR+1];
"fgn at H=.9 and n=20";
SmH222=SimulateFGN[H22,RR+1];


n=2;
While[n<=NN,
Clear[α1,m1,β1,μ1];
sumαα=0;
sumββ=0;
sumμμ=0;
summm=0;
```

Yk11[n_]:= $\sum_{\mathbf{i=0}}^{\infty}((\mathbf{1-\alpha 1\ \Delta t})^{\mathbf{i}}(\mathbf{\alpha 1\ m1\ \Delta t+\beta 1\ SmH2}[[\mathbf{Abs}[\mathbf{i+1}]]]);$

Xk1[n_]:=1+μ1 Δt+SmH1[[n]]*

$\sum_{\mathbf{i=0}}^{\infty}((\mathbf{1-\alpha 1\ \Delta t})^{\mathbf{i}}(\mathbf{\alpha 1\ m1\ \Delta t+\beta 1\ SmH2}[[\mathbf{Abs}[\mathbf{i+1}]]])$;

Yk[n_]:= $\sum_{\mathbf{i=0}}^{\infty}(\mathbf{1-\alpha\ \Delta t})^{\mathbf{i}}(\mathbf{\alpha m\Delta t+\beta\ SmH2}[[\mathbf{Abs}[\mathbf{i+1}]]]);$

```
Xk[n_]:=1+μ Δt+Yk[n]*SmH1[[n]];
mu[n_]:=Mean[Table[Xk1[i],{i,1,n}]];
X[n_]:=Table[Xk1[i],{i,1,n}]; XX[n_]:=Table[Xk[i],{i,1,n–
1}]~Join~{Xk1[n]};
γξ[n_]:=1/2 (Abs[(n–1) Δt]^(2 H11)+Abs[(n+1) Δt]^(2 H11)–2 Abs[n Δt]^(2 H11));
γη[n_]:=1/2 (Abs[(n–1) Δt]^(2 H22)+Abs[(n+1) Δt]^(2 H22)–2 Abs[n Δt]^(2 H22));
γX[n_]:=Yk[n]^2 γξ[n];
γY[n_]:= β^2 ∑_{i=0}^∞ ∑_{j=0}^∞ (1 − α Δt)^(i+j) γη[n + 1 − j];

Γn1[n_]:=Table[γX[i–j],{i,1,n–1},{j,1,n–1}];”Γn–1”;
γn1[n_]:=Table[γX[n–i],{i,1,n–1},{j,1,1}];”γn–1”;
γTn1[n_]:=Transpose[γn1[n]];”γn–1 transpose”;
γX0=γX[0];”γX(0)”;
Γ[n_]:=ArrayFlatten[{{Γn1[n],γn1[n]},{γTn1[n],γX0}}];
invΓ[n_]:=Inverse[Γ[n]];
γ[n_]:=Table[{γX[i]},{i,1,n}] ;”γT in 3.27”;
VT2[n_]:=Abs[γX[0]–Transpose[γ[n]].invΓ[n].γ[n]];”3.27”;
VT22[n_]:=ToExpression[StringReplace[ToString[VT2[n]],{”{”–>””,”}”–
>””}]];
invΓ[1]=1/γX0;
For[i=1,i<=n,i++,
ϕ[i_]:=invΓ[i].γ[i];
ϕ1[i_]:=Flatten[ϕ[i]];
For[j=1,j<=n,j++,
If[i==j,a[i,j]=1];
If[i<j,a[i,j]=0];
If[i>j,a[i,j]=– ϕ1[i–1][[j]]]
    ];
a[1,1]=1;
a[2,1]=–γX[1]/γX0;
   ];
A=Table[a[i,j],{i,1,n},{j,1,n}];
For[i=2,i<=n,i++,
For[j=2,j<=n,j++,
VT1=Abs[γX0–γX[1]* 1/γX0 *γX[1]];
d[1,1]=(1/VT1);
```

If[i==j,d[i,j]=1/VT22[i]];
If[i<j,d[i,j]=0];
If[i>j,d[i,j]=0];
]
];

KK=Table[If[i==1,d[i,j]=0,d[i,j]],{i,1,n},
{j,1,n}];
KK=Table[If[j==1,d[i,j]=0,d[i,j]],{i,1,n},
{j,1,n}];
d[1,1]=Abs[1/VT1];
KK=Table[d[i,j],{i,1,n},{j,1,n}];
segmainv[n_]:=SetPrecision[Transpose[A].KK.A ,5];
"segma[n_]:=PaddedForm[SetPrecision[Inverse[segmainv[n]],5],{5,5}]";
DetKKK[n_]:=VT1 *$\prod_{i=1}^{T}\mathbf{VT2}[\mathbf{i}]$;
"define penalty function ";
g1[n_]:=–γX[n];
g2[n_]:= $-\sum_{i=1}^{L}(\mathbf{1}-\boldsymbol{\alpha}\boldsymbol{\Delta}\mathbf{t})^{\mathbf{i}}(\boldsymbol{\alpha}\mathbf{m}\boldsymbol{\Delta}\mathbf{t})\}^{2}\ \boldsymbol{\Delta}\mathbf{t}^{\mathbf{2H11}}-\sum_{i=1}^{L}(\mathbf{1}-\boldsymbol{\alpha}\boldsymbol{\Delta}\mathbf{t})^{\mathbf{2i}}\boldsymbol{\beta}^{\mathbf{2}}\boldsymbol{\Delta}\mathbf{t}^{\mathbf{2(H22+H11)}}$;
G1[n_]:=Piecewise[{{g1[n],g1[n]>ϵ},{(g1[n]–ϵ)^2/(4 ϵ),–
ϵ<g1[n]<ϵ},{0,g1[n]<– ϵ}}];"3.36 w.r.t g1";
G2[n_]:=Piecewise[{{g2[n],g2[n]>ϵ},{(g2[n]–ϵ)^2/(4 ϵ),–
ϵ<g2[n]<ϵ},{0,g2[n]<– ϵ}}];"3.36w.r.t g2";
;n++];
"lhood[NN_]:=Log[PDF[MultinormalDistribution[Flatten[Table[mu[i],{i,1,N
N}]],segma[NN]],X[NN]]]";
lhood1[NN_]:=Log[Exp[–0.5* Transpose[Table[Xk1[i]–
mu[i],{i,1,NN}],{1}].segmainv[NN].Table[Xk1[i]–
mu[i],{i,1,NN}]]/ToExpression[StringReplace[ToString[$(2\ Pi)^{\frac{NN}{2}}$
$DetKKK\ [NN]^{\frac{1}{2}}$],{"{"–>"","}"–>""}]]]];
op=NMinimize[– lhood1[NN]–ϵ/4 $\sum_{i=1}^{NN}\mathbf{G1}[\mathbf{i}]$ –ϵ/4 $\sum_{i=1}^{NN}\mathbf{G2}[\mathbf{i}]$
,{α1,m1,β1,μ1},WorkingPrecision–>15,Method–>"DifferentialEvolution"];
dd=op;
answer=op[[1]];
optimall$_{u=u+1}$=answer;

```
var=dd[[2]];"the answer";

αα=var[[1]];
ααα=αα[[2]];
sumα=sumαα+ααα;
sumαα=sumα;

mm=var[[2]];
mmm=mm[[2]];
summ=summm+mmm;
summm=summ;

ββ=var[[3]];
βββ=ββ[[2]];
sumβ=sumββ+βββ;
sumββ=sumβ;

μμ0=var[[4]];
μμμ=μμ0[[2]];
sumμ=sumμμ+μμμ;
sumμμ=sumμ;

"Print[dd];";
sepvar=Table[var[[i,2]],{i,1,4}];
{ α1,m1,β1,μ1}=sepvar;
Clear[ α1,m1,β1,μ1];
Table[optimall_i,{i,1,RR}];
H1hut=H11;
H1hutt_{l=l+1}=H1hut;
H2hut=H22;
H2hutt_{t=t+1}=H2hut;
αhut=sumαα/RR;
αhutt_{v=v+1}=αhut;
mhut=summm/RR;
mhutt_{c=c+1}=mhut;
βhut=sumββ/RR;
```

```
βhutt_{e=e+1}=βhut;
μhut=sumμμ/RR;
μhutt_{f=f+1}=μhut;
Print[H1hut];
Print[H2hut];
Print[αhut];
Print[mhut];
Print[βhut];
Print[μhut];
Print[">>>>>>>>>>>>>>>>>>>>>>>>>>>>>"];
  ];
```

Appendix C

Simulation with Segmentation for Parameters Estimation

```
Δt=.2;
H1=.8;
H2=.7;
α=2.5;
m=2;
β=1.5;
μ=1;
ϵ=0.1;
l=0;
t=0;
u=0;
v=0;
c=0;
e=0;
f=0;
RR=100;
JJ=10;
NN=10;
sumαhut1=0;
summhut1=0;
sumβhut1=0;
sumμhut1=0;
H11=0.8;
H22=0.7;
For[r=1,r<=RR,r++,
"Simulate a fractional Brownian motion process –1st one ";
data1=RandomFunction[FractionalBrownianMotionProcess[H1],{0,1,0.001}];
"finds the parameter estimates for the fbm process(H1) from data1";
EH1=FindProcessParameters[data1,FractionalBrownianMotionProcess[h]];
H11=EH1[[1,2]];
"Simulate a fractional Brownian motion process–2nd one ";
```

```
data2=RandomFunction[FractionalBrownianMotionProcess[H2],{0,1,0.001}];
”finds the parameter estimates for the fbm process(H2) from data2”;
EH2=FindProcessParameters[data2,FractionalBrownianMotionProcess[h]];
H22=EH2[[1,2]];
“simulate fgn”;
SimulateFGN[H_,n_]:=Module[{𝓝,ac}, 𝓝 =2^Ceiling[Log[2,n–1]];
ac=Table[FGNAcf[k,H],{k,0, 𝓝 }];
Take[SimulateGLP[ac],n]];SimulateGLP[γ_]:=Module[{m=Length[γ],n,c,g,Z,
Ncap},n=2^Ceiling[Log[2,m–1]];acvf=If[n==m–
1,γ,PadRight[γ,n+1]];Ncap=2*n;
c=Join[acvf,Rest[Reverse[Rest[acvf]]]];
g=Re[Fourier[c,FourierParameters–>{1,–1}]];
Z=RandomVariate[NormalDistribution[0,1],Ncap–2];
Z=(Complex[Sequence@@#1]&)/@Partition[Z,2];
Z=Flatten[{RandomVariate[NormalDistribution[0,Sqrt[2]]],Z,RandomVariate
[NormalDistribution[0,Sqrt[2]]],Reverse[Conjugate[Z]]}];
Take[Re[InverseFourier[Sqrt[g]*Z,FourierParameters–>{0,–1}]],m]/Sqrt[2]];
FGNAcf[k_,H_]:=Module[{},0.5*(Abs[k+1.0]^(2.0*H)–
2*Abs[k]^(2.0*H)+Abs[k–1.0]^(2.0*H))];
“fgn at H=.65 and n=20”;
SmH111=SimulateFGN[H11,110];
“fgn at H=.9 and n=20”;
SmH222=SimulateFGN[H22,110];
Do[
sumαα=0;
sumββ=0;
sumμμ=0;
summm=0;
w[j_]:=j+10(j–1);
SmH11[j_]:=Table[SmH111[[i]],{i,w[j],w[j]+10}];
SmH22[j_]:=Table[SmH222[[i]],{i,w[j],w[j]+10}];
SmH1=SmH11[j];
SmH2=SmH22[j];

n=2;
While[n<=NN,
```

```
Clear[α1,m1,β1,μ1];
Yk11[n_]:= ∑_{i=0}^∞((1 − α1 Δt)^i(α1 m1 Δt + β1 SmH2[[Abs[i + 1]]]);
Xk1[n_]:=1+μ1 Δt+SmH1[[n]]*
∑_{i=0}^∞((1 − α1 Δt)^i(α1 m1 Δt + β1 SmH2[[Abs[i + 1]]]) ;
Yk[n_]:= ∑_{i=0}^∞(1 − α Δt)^i(αmΔt + β SmH2[[Abs[i + 1]]]);
Xk[n_]:=1+μ Δt+Yk[n]*SmH1[[n]];
mu[n_]:=Mean[Table[Xk1[i],{i,1,n}]];
X[n_]:=Table[Xk1[i],{i,1,n}]; XX[n_]:=Table[Xk[i],{i,1,n–
1}]~Join~{Xk1[n]};
γξ[n_]:=1/2 (Abs[(n–1) Δt]^(2 H11)+Abs[(n+1) Δt]^(2 H11)–2 Abs[n Δt]^(2 H11));
γη[n_]:=1/2 (Abs[(n–1) Δt]^(2 H22)+Abs[(n+1) Δt]^(2 H22)–2 Abs[n Δt]^(2 H22));
γX[n_]:=Yk[n]^2 γξ[n];
γY[n_]:= β^2 ∑_{i=0}^∞ ∑_{j=0}^∞ (1 − α Δt)^(i+j) γη[n + 1 − j];

Γn1[n_]:=Table[γX[i–j],{i,1,n–1},{j,1,n–1}];"Γn–1";
γn1[n_]:=Table[γX[n–i],{i,1,n–1},{j,1,1}];"γn–1";
γTn1[n_]:=Transpose[γn1[n]];"γn–1 transpose";
γX0=γX[0];"γX(0)";
Γ[n_]:=ArrayFlatten[{{Γn1[n],γn1[n]},{γTn1[n],γX0}}];
invΓ[n_]:=Inverse[Γ[n]];
γ[n_]:=Table[{γX[i]},{i,1,n}] ;"γT in 3.27";
VT2[n_]:=Abs[γX[0]–Transpose[γ[n]].invΓ[n].γ[n]];"3.27";
VT22[n_]:=ToExpression[StringReplace[ToString[VT2[n]],{"{"–>"","}"–
>""}]];
invΓ[1]=1/γX0;
For[i=1,i<=n,i++,
ϕ[i_]:=invΓ[i].γ[i];
ϕ1[i_]:=Flatten[ϕ[i]];
For[j=1,j<=n,j++,
If[i==j,a[i,j]=1];
If[i<j,a[i,j]=0];
If[i>j,a[i,j]=– ϕ1[i–1][[j]]]
      ];
a[1,1]=1;
a[2,1]=–γX[1]/γX0;
```

```
  ];
A=Table[a[i,j],{i,1,n},{j,1,n}];
For[i=2,i<=n,i++,
For[j=2,j<=n,j++,
VT1=Abs[γX0–γX[1]* 1/γX0 *γX[1]];
d[1,1]=(1/VT1);
If[i==j,d[i,j]=1/VT22[i]];
If[i<j,d[i,j]=0];
If[i>j,d[i,j]=0];
   ]
  ];

KK=Table[If[i==1,d[i,j]=0,d[i,j]],{i,1,n},
{j,1,n}];
KK=Table[If[j==1,d[i,j]=0,d[i,j]],{i,1,n},
{j,1,n}];
d[1,1]=Abs[1/VT1];
KK=Table[d[i,j],{i,1,n},{j,1,n}];
segmainv[n_]:=SetPrecision[Transpose[A].KK.A ,5];
"segma[n_]:=PaddedForm[SetPrecision[Inverse[segmainv[n]],5],{5,5}]";
DetKKK[n_]:=VT1 *$\prod_{i=1}^{T} \mathbf{VT2[i]}$;
"define penalty function ";
g1[n_]:=–γX[n];
g2[n_]:= $-\sum_{i=1}^{L}(\mathbf{1-\alpha\Delta t})^{i}(\boldsymbol{\alpha m\Delta t})\}^{2}\,\boldsymbol{\Delta t^{2H11}}-\sum_{i=1}^{L}(\mathbf{1-\alpha\Delta t})^{2i}\boldsymbol{\beta^{2}\Delta t^{2(H22+H11)}}$;
G1[n_]:=Piecewise[{{g1[n],g1[n]>ϵ},{(g1[n]–ϵ)^2/(4 ϵ),–
ϵ<g1[n]<ϵ},{0,g1[n]<– ϵ}}];"3.36 w.r.t g1";
G2[n_]:=Piecewise[{{g2[n],g2[n]>ϵ},{(g2[n]–ϵ)^2/(4 ϵ),–
ϵ<g2[n]<ϵ},{0,g2[n]<– ϵ}}];"3.36w.r.t g2";
;n++];
"lhood[NN_]:=Log[PDF[MultinormalDistribution[Flatten[Table[mu[i],{i,1,N
N}]],segma[NN]],X[NN]]]";
lhood1[NN_]:=Log[Exp[–0.5* Transpose[Table[Xk1[i]–
mu[i],{i,1,NN}],{1}].segmainv[NN].Table[Xk1[i]–
```

```
mu[i],{i,1,NN}]]/ToExpression[StringReplace[ToString[(2 Pi)^(NN/2)
DetKKK [NN]^(1/2) ],{"{"–>"","}"–>""}]]];
op=NMinimize[– lhood1[NN]–ϵ/4 ∑_{i=1}^{NN} G1[i] –ϵ/4 ∑_{i=1}^{NN} G2[i]
,{α1,m1,β1,μ1},WorkingPrecision–>15,Method–>"DifferentialEvolution"];
dd=op;
answer=op[[1]];
optimall_{u=u+1}=answer;
var=dd[[2]];"the answer";

αα=var[[1]];
ααα=αα[[2]];
sumα=sumαα+ααα;
sumαα=sumα;

mm=var[[2]];
mmm=mm[[2]];
summ=summm+mmm;
summm=summ;

ββ=var[[3]];
βββ=ββ[[2]];
sumβ=sumββ+βββ;
sumββ=sumβ;

μμ0=var[[4]];
μμμ=μμ0[[2]];
sumμ=sumμμ+μμμ;
sumμμ=sumμ;

"Print[dd];";
sepvar=Table[var[[i,2]],{i,1,4}];
{ α1,m1,β1,μ1}=sepvar;
Clear[ α1,m1,β1,μ1];
,{j,JJ}];
Table[optimall_i,{i,1,JJ}];
```

```
H1hut=H11;
H1hutt_{l=l+1}=H1hut;
H2hut=H22;
H2hutt_{t=t+1}=H2hut;
αhut=sumαα/JJ;
αhutt_{v=v+1}=αhut;
mhut=summm/JJ;
mhutt_{c=c+1}=mhut;
βhut=sumββ/JJ;
βhutt_{e=e+1}=βhut;
μhut=sumμμ/JJ;
μhutt_{f=f+1}=μhut;
Print[H1hut];
Print[H2hut];
Print[αhut];
Print[mhut];
Print[βhut];
Print[μhut];
Print[">>>>>>>>>>>>>>>>>>>>>>>>>>>>"];
  ];
H1est=Table [H1hutt_i,{i,1,RR}];
H2est=Table [H2hutt_i,{i,1,RR}];
αest=Table[αhutt_i,{i,1,RR}];
mest=Table[mhutt_i,{i,1,RR}];
βest=Table[βhutt_i,{i,1,RR}];
μest=Table[μhutt_i,{i,1,RR}];
H1hutav=Mean[H1est];
H2hutav=Mean[H2est];
αhutav=Mean[αest];
mhutav=Mean[mest];
βhutav=Mean[βest];
μhutav=Mean[μest];
"αvar=Variance[αest];
mvar=Variance[mest];
βvar=Variance[βest];
μvar=Variance[μest]";
```

```
Print["H1hut–average = ",H1hutav];
Print["H2hut–average = ",H2hutav];
Print["αhut–average = ",αhutav];
Print["mhut–average = ",mhutav];
Print["βhut–average = ",βhutav];
Print["μhut–average = ",μhutav];
Print["H1–variance = ",Variance[H1est]];
Print["H2–variance = ",Variance[H2est]];
Print["α–variance = ",Variance[αest]];
Print["m–variance = ",Variance[mest]];
Print["β–variance = ",Variance[βest]];
Print["μ–variance = ",Variance[μest]];
Print["H1–bias = ",Abs[H1–H1hutav]];
Print["H2–bias = ",Abs[H2–H1hutav]];
Print["α–bias = ",Abs[α–αhutav]];
Print["m–bias = ",Abs[m–mhutav]];
Print["β–bias = ",Abs[β–βhutav]];
Print["μ–bias = ",Abs[μ–μhutav]];
Print["H1–MSE = ",Variance[H1est]+(Abs[H1–H1hutav])^2];
Print["H2–MSE = ",Variance[H2est]+(Abs[H2–H2hutav])^2];
Print["α–MSE = ",Variance[αest]+(Abs[α–αhutav])^2];
Print["m–MSE = ",Variance[mest]+(Abs[m–mhutav])^2];
Print["β–MSE = ",Variance[βest]+(Abs[β–βhutav])^2];
Print["μ–MSE = ",Variance[μest]+(Abs[μ–μhutav])^2];
```