MODELING FINANCIAL ENVIRONMENTS USING GEOMETRIC FRACTIONAL BROWNIAN MOTION MODEL WITH LONG MEMORY STOCHASTIC VOLATILITY

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Abstrak

Model Pergerakan Pecahan Geometrik Brownian (GFBM) digunakan dengan meluas dalam persekitaran kewangan. Model ini mengandungi parameter penting iaitu min, ruapan, dan indeks Hurst, yang bererti kepada kebanyakan masalah dalam bidang kewangan terutamanya bagi menentukan harga opsyen, nilai pada risiko, kadar tukaran, dan insuran cagaran. Kebanyakan penyelidikan terkini mengkaji *GFBM* dengan mengandaikan ruapannya adalah malar disebabkan keringkasannya. Walau bagaimanapun, anggapan ini selalunya disangkal dalam kebanyakan kajian empirikal. Oleh itu, kajian ini membangunkan model GFBM baharu yang mampu menerangkan dan menggambarkan situasi sebenar dengan lebih baik terutamanya dalam senario kewangan. Kesemua parameter yang terlibat dalam model yang dibangunkan dianggar menggunakan algoritma inovasi. Kajian simulasi seterusnya dilakukan untuk menentukan prestasi model baharu. Hasil simulasi mendedahkan bahawa penganggar yang disyorkan adalah cekap berdasarkan kepada kepincangan, varians, dan min kuasa dua ralat. Seterusnya, dua teorem berkaitan kewujudan dan keunikan penyelesaian bagi model baharu dan pengitlakannya dibina. Pengesahan bagi model yang dibangunkan kemudiannya dilakukan dengan membandingkannya dengan beberapa model lain bagi meramal harga terlaras Standard and Poor's 500, Shanghai Stock Exchange Composite Index, dan FTSE Kuala Lumpur Composite Index. Kajian empirikal terhadap empat aplikasi kewangan terpilih, iaitu penentuan harga opsyen, nilai risiko, kadar pertukaran, dan insuran gadai janji, menunjukkan bahawa model baharu mempamerkan keputusan yang lebih baik berbanding model sedia ada. Justeru itu, model baharu amat berpotensi untuk dijadikan model pendasar bagi sebarang aplikasi kewangan yang berupaya mencerminkan keadaan sebenar dengan lebih tepat.

Kata kunci: Pergerakan Pecahan Geometrik Brownian, ruapan stokastik, memori panjang, senario kewangan.

Abstract

Geometric Fractional Brownian Motion (GFBM) model is widely used in financial environments. This model consists of important parameters i.e. mean, volatility, and Hurst index, which are significant to many problems in finance particularly option pricing, value at risk, exchange rate, and mortgage insurance. Most current works investigated GFBM under the assumption of its volatility that is constant due to its simplicity. However, such assumption is normally rejected in most empirical studies. Therefore, this research develops a new GFBM model that can better describe and reflect real life situations particularly in financial scenario. All parameters involved in the developed model are estimated by using innovation algorithm. A simulation study is then conducted to determine the performance of the new model. The results of simulation reveal that the proposed estimators are efficient based on the bias, variance, and mean square error. Subsequently, two theorems on existence and uniqueness of the solution for the new model and its generalisation are constructed. The validation of the developed model was then carried out by comparing with other models in forecasting adjusted prices of Standard and Poor 500, Shanghai Stock Exchange Composite Index, and FTSE Kuala Lumpur Composite Index. Empirical studies on four selected financial applications, i.e. option pricing, value at risk, exchange rate, and mortgage insurance, indicate that the new model performs better than the existing ones. Hence, the new model has strong potential to be employed as an underlying model for any financial applications that capable of reflecting the real situation more accurately.

Keywords: Geometric Fractional Brownian Motion, stochastic volatility, long memory, financial scenario.

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List of Abbreviations

AR	Autoregressive Process
ARCH	Autoregressive Conditional Heteroscedasticity
ARFIMA	Autoregressive Fractionally Integrated Moving Average
ARIMA	Autoregressive Integrated Moving Average
ARSV	Autoregressive Stochastic Volatility
BM	Brownian Motion
BS	Black–Scholes
CIR	Cox–Ingersoll–Ross
CMLE	Complete Maximum Likelihood Estimation
DE	Differential Evolution
EBSCO	Elton Bryson Stephens Company
EMM	Efficient Method of Moments
FBM	Fractional Brownian Motion
FBS	Fractional Black–Scholes
GFBM	Geometric Fractional Brownian Motion
FGN	Fractional Gaussian Noise
FOU	Fractional Ornstein Uhlenbeck
FSDE	Fractional Stochastic Differential Equation
GARCH	Generalized Autoregressive Conditional Heteroscedasticity
GBM	Geometric Brownian Motion
GMM	Generalized method of moment
GS	Google Scholar
H	Hurst
JP	Jump
KLCI	Kuala Lumpur Composite Index
LMSV	Long Memory Stochastic Volatility
MAPE	Mean Absolute Percentage Error
MBI	Moment Based Inference
MCMC	Markov Chain Monte Carlo

MLE	Maximum Likelihood Estimation
MM	Method of Moments
MSE	Mean Square Error
OU	Ornstein– Uhlenbeck
R/S	Rescale / Range
RS	Random Search
S &P 500	Standard and Poor's 500
S.V	SciVerse
SA	Simulated Annealing
SABR	Stochastic Alpha–Beta–Rho
SBI	Simulation Based Inference
SDE	Stochastic Differential Equation
SV	Stochastic Volatility
Var	Variance

CHAPTER ONE
INTRODUCTION

1.1 Research Background

Volatility has been actively discussed in time series econometrics and economic forecasting in recent years. Volatility explains the variations witnessed in some phenomena over time. In economics, it is used to describe variability of random component of a time series. In financial economics, volatility is defined as the standard deviation of a random Wiener driven component in a continuous time diffusion model.

In the last decades, two main classes of volatility models have been developed: the generalized autoregressive conditional Heteroscedasticity (GARCH) and the stochastic volatility (SV) model. These classes were developed in order to capture time–varying autocorrelation, i.e. the correlation between values of the process at different points in time.

To begin, in 1982 Engle introduced autoregressive conditional heteroscedasticity (ARCH) model to estimate conditional variance of the sequence of increasing price of the United Kingdom's financial environment. This model was developed by prior assumption that the variance of random errors was related to the previous random, with inclusion of the conditional variance and mean in equation. Four years later, the extension of ARCH was proposed by Bollerslev (1986), known as generalized autoregressive conditional heteroscedasticity (GARCH) model. This model adds the memory of past variances to the model which is useful in modeling and forecasting

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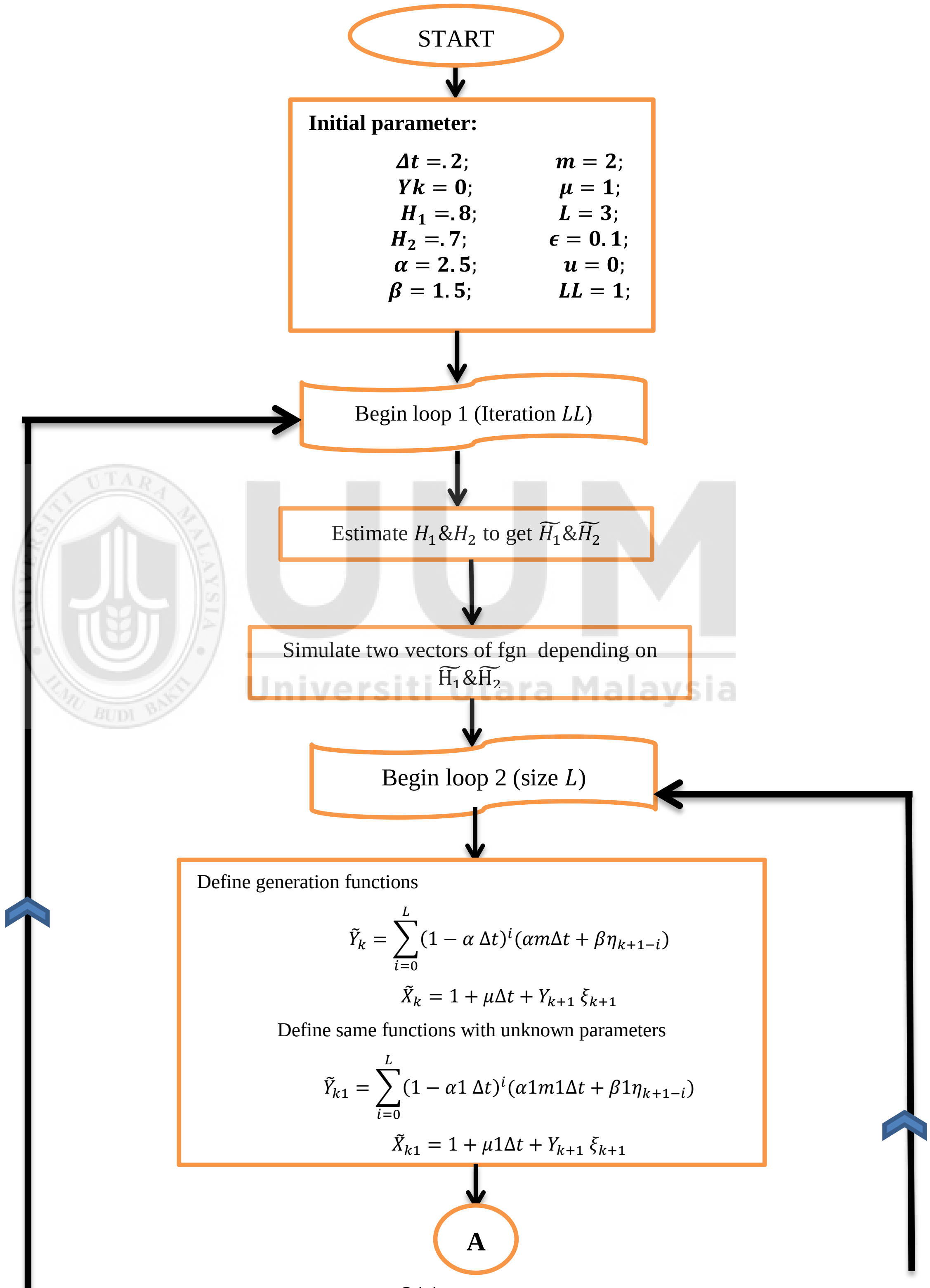
Appendix A

Flowchart for Parameters Estimation



A

Compute the following covariance function:

$$\gamma_{\xi}(n) = \frac{1}{2}(|(n+1)\Delta t|^{2H_1} + |(n-1)\Delta t|^{2H_1} - 2|n\Delta t|^{2H_1})$$

$$\gamma_{\eta}(n) = \frac{1}{2}(|(n+1)\Delta t|^{2H_2} + |(n-1)\Delta t|^{2H_2} - 2|n\Delta t|^{2H_2})$$

$$\gamma_{\tilde{Y}}(n) = \beta^2 \sum_{i=0}^{L}\sum_{j=0}^{L}(1-\alpha\,\Delta t)^{i+j}\gamma_{\eta}(n+i-j)$$

Construct the following matrices:

$$\Gamma_{n-1} = \{\gamma(k-j)\}_{j,k=1}^{n-1} \quad \& \quad {\Gamma_{n-1}}^{-1}$$

$$\tilde{\gamma}_{n-1} = (\gamma_{\tilde{X}}(n-1), \dots, \gamma_{\tilde{X}}(1))' \quad \& \quad {\tilde{\gamma}_{n-1}}'$$

$$\gamma_n = (\gamma(1), \dots, \gamma(n))'$$

$$\Gamma_n = \begin{bmatrix} \Gamma_{n-1} & \tilde{\gamma}_{n-1} \\ {\tilde{\gamma}_{n-1}}' & \tilde{\gamma}_{\tilde{X}}(0) \end{bmatrix}$$

$$\phi_n = \Gamma_n^{-1}\gamma_n$$

$$\Gamma_n^{-1} = \begin{bmatrix} I & -\Gamma_{n-1}^{-1}\tilde{\gamma}_{n-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Gamma_{n-1}^{-1} & 0 \\ 0 & (\gamma_{\tilde{X}}(0) - \tilde{\gamma}'_{n-1}\Gamma_{n-1}^{-1}\tilde{\gamma}_{n-1})^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ -\tilde{\gamma}'_{n-1}\Gamma_{n-1}^{-1} & 1 \end{bmatrix}$$

Compute $v_T^2 = \gamma(0) - \gamma_{T'}\Gamma_T^{-1}\gamma_T$

B

B

Construct the following matrices

$$A = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ -\phi_{11} & 1 & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ -\phi_{(T-1)1} & -\phi_{(T-1)2} & \cdots & -\phi_{(T-1)(T-1)} & 1 \end{bmatrix}$$

$$KK = \begin{bmatrix} \frac{1}{v_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{v_2^2} & & 0 \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{v_T^2} \end{bmatrix}$$

Compute:

$$\Sigma_T^{-1} = A'.KK.A$$

$$\det(\Sigma_T) = \prod_{i=1}^{T} E(\varepsilon_i)^2 = \prod_{i=1}^{T} v_i^2$$

Find the value of:

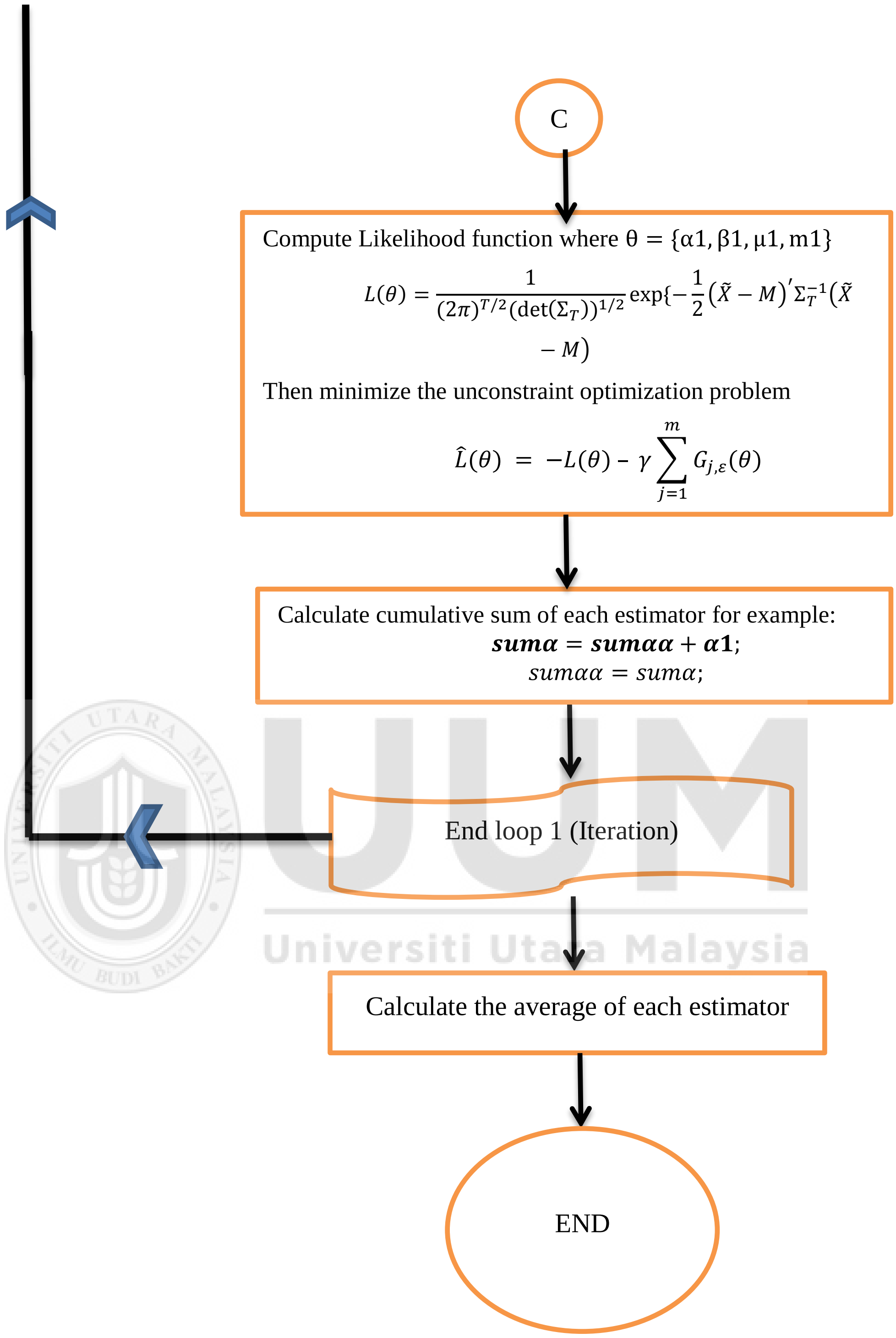
$$g_1(\theta) = -E(\tilde{X} - \mu)^2 = -\gamma_{\tilde{X}}(n);$$

$$g_2(\theta) = -v^2 = -\{\textstyle\sum_{i=1}^{L}(1-\alpha\Delta t)^i(\alpha m\Delta t)\}^2 \Delta t^{2H1} - \textstyle\sum_{i=1}^{L}(1-\alpha\Delta t)^{2i}\beta^2\Delta t^{2(H2+H1)};$$

$$G_{i,\varepsilon}(\theta) = \begin{cases} g_i & , \quad g_i > \varepsilon \\ \frac{(g_i-\varepsilon)^2}{4\varepsilon} & , \quad -\varepsilon < g_i < \varepsilon \\ 0 & , \quad g_i < -\varepsilon \end{cases}$$

End loop 2 (size L)

C


C
Compute Likelihood function where $\theta = \{\alpha 1, \beta 1, \mu 1, m1\}$
$L(\theta) = \frac{1}{(2\pi)^{T/2}(\det(\Sigma_T))^{1/2}} \exp\{-\frac{1}{2}(\tilde{X} - M)'\Sigma_T^{-1}(\tilde{X} - M)$
Then minimize the unconstraint optimization problem
$\hat{L}(\theta) = -L(\theta) - \gamma \sum_{j=1}^{m} G_{j,\varepsilon}(\theta)$
Calculate cumulative sum of each estimator for example:
$\boldsymbol{suma} = \boldsymbol{sumaa} + \boldsymbol{\alpha 1}$;
$sumaa = suma$;
End loop 1 (Iteration)
Calculate the average of each estimator
END

Appendix B

Standard Simulation for Parameters Estimation

```
Δt=.2;
H1=.8;
H2=.7;
α=2.5;
m=2;
β=1.5;
μ=1;
ϵ=0.1;
l=0;
t=0;
u=0;
v=0;
c=0;
e=0;
f=0;
RR=100;
NN=10;
sumαhut1=0;
summhut1=0;
sumβhut1=0;
sumμhut1=0;
H11=0.8;
H22=0.7;
For[r=1,r<=RR,r++,
"Simulate a fractional Brownian motion process–1st one  ";
data1=RandomFunction[FractionalBrownianMotionProcess[H1],{0,1,0.001}];
"finds the parameter estimates for the fbm process(H1) from data1";
EH1=FindProcessParameters[data1,FractionalBrownianMotionProcess[h]];
H11=EH1[[1,2]];
"Simulate a fractional Brownian motion process–2nd one ";
data2=RandomFunction[FractionalBrownianMotionProcess[H2],{0,1,0.001}];
"finds the parameter estimates for the fbm process(H2) from data2";
```

```
EH2=FindProcessParameters[data2,FractionalBrownianMotionProcess[h]];
H22=EH2[[1,2]];
“simulate fgn”;
SimulateFGN[H_,n_]:=Module[{𝓝,ac}, 𝓝 =2^Ceiling[Log[2,n–1]];
ac=Table[FGNAcf[k,H],{k,0, 𝓝 }];
Take[SimulateGLP[ac],n]];SimulateGLP[γ_]:=Module[{m=Length[γ],n,c,g,Z,
Ncap},n=2^Ceiling[Log[2,m–1]];acvf=If[n==m–
1,γ,PadRight[γ,n+1]];Ncap=2*n;
c=Join[acvf,Rest[Reverse[Rest[acvf]]]];
g=Re[Fourier[c,FourierParameters–>{1,–1}]];
Z=RandomVariate[NormalDistribution[0,1],Ncap–2];
Z=(Complex[Sequence@@#1]&)/@Partition[Z,2];
Z=Flatten[{RandomVariate[NormalDistribution[0,Sqrt[2]]],Z,RandomVariate
[NormalDistribution[0,Sqrt[2]]],Reverse[Conjugate[Z]]}];
Take[Re[InverseFourier[Sqrt[g]*Z,FourierParameters–>{0,–1}]],m]/Sqrt[2]];
FGNAcf[k_,H_]:=Module[{},0.5*(Abs[k+1.0]^(2.0*H)–
2*Abs[k]^(2.0*H)+Abs[k–1.0]^(2.0*H))];
“fgn at H=.65 and n=20”;
SmH111=SimulateFGN[H11,RR+1];
“fgn at H=.9 and n=20”;
SmH222=SimulateFGN[H22,RR+1];


n=2;
While[n<=NN,
Clear[α1,m1,β1,μ1];
sumαα=0;
sumββ=0;
sumμμ=0;
summm=0;
```

Yk11[n_]:= $\sum_{\mathbf{i=0}}^{\infty}((\mathbf{1-\alpha 1\ \Delta t})^{\mathbf{i}}(\mathbf{\alpha 1\ m1\ \Delta t+\beta 1\ SmH2}[[\mathbf{Abs}[\mathbf{i+1}]]])$;

Xk1[n_]:=1+μ1 Δt+SmH1[[n]]*

$\sum_{\mathbf{i=0}}^{\infty}((\mathbf{1-\alpha 1\ \Delta t})^{\mathbf{i}}(\mathbf{\alpha 1\ m1\ \Delta t+\beta 1\ SmH2}[[\mathbf{Abs}[\mathbf{i+1}]]])$;

Yk[n_]:= $\sum_{\mathbf{i=0}}^{\infty}(\mathbf{1-\alpha\ \Delta t})^{\mathbf{i}}(\mathbf{\alpha m \Delta t+\beta\ SmH2}[[\mathbf{Abs}[\mathbf{i+1}]]])$;

```
Xk[n_]:=1+μ Δt+Yk[n]*SmH1[[n]];
mu[n_]:=Mean[Table[Xk1[i],{i,1,n}]];
X[n_]:=Table[Xk1[i],{i,1,n}]; XX[n_]:=Table[Xk[i],{i,1,n–
1}]~Join~{Xk1[n]};
γξ[n_]:=1/2 (Abs[(n–1) Δt]^(2 H11)+Abs[(n+1) Δt]^(2 H11)–2 Abs[n Δt]^(2 H11));
γη[n_]:=1/2 (Abs[(n–1) Δt]^(2 H22)+Abs[(n+1) Δt]^(2 H22)–2 Abs[n Δt]^(2 H22));
γX[n_]:=Yk[n]^2 γξ[n];
γY[n_]:= β^2 ∑_{i=0}^∞ ∑_{j=0}^∞ (1 − α Δt)^(i+j) γη[n + 1 − j];

Γn1[n_]:=Table[γX[i–j],{i,1,n–1},{j,1,n–1}];"Γn–1";
γn1[n_]:=Table[γX[n–i],{i,1,n–1},{j,1,1}];"γn–1";
γTn1[n_]:=Transpose[γn1[n]];"γn–1 transpose";
γX0=γX[0];"γX(0)";
Γ[n_]:=ArrayFlatten[{{Γn1[n],γn1[n]},{γTn1[n],γX0}}];
invΓ[n_]:=Inverse[Γ[n]];
γ[n_]:=Table[{γX[i]},{i,1,n}] ;"γT in 3.27";
VT2[n_]:=Abs[γX[0]–Transpose[γ[n]].invΓ[n].γ[n]];"3.27";
VT22[n_]:=ToExpression[StringReplace[ToString[VT2[n]],{"{"–>"","}"–
>""}]];
invΓ[1]=1/γX0;
For[i=1,i<=n,i++,
ϕ[i_]:=invΓ[i].γ[i];
ϕ1[i_]:=Flatten[ϕ[i]];
For[j=1,j<=n,j++,
If[i==j,a[i,j]=1];
If[i<j,a[i,j]=0];
If[i>j,a[i,j]=– ϕ1[i–1][[j]]]
    ];
a[1,1]=1;
a[2,1]=–γX[1]/γX0;
   ];
A=Table[a[i,j],{i,1,n},{j,1,n}];
For[i=2,i<=n,i++,
For[j=2,j<=n,j++,
VT1=Abs[γX0–γX[1]* 1/γX0 *γX[1]];
d[1,1]=(1/VT1);
```

If[i==j,d[i,j]=1/VT22[i]];
If[i<j,d[i,j]=0];
If[i>j,d[i,j]=0];
]
];

KK=Table[If[i==1,d[i,j]=0,d[i,j]],{i,1,n},
{j,1,n}];
KK=Table[If[j==1,d[i,j]=0,d[i,j]],{i,1,n},
{j,1,n}];
d[1,1]=Abs[1/VT1];
KK=Table[d[i,j],{i,1,n},{j,1,n}];
segmainv[n_]:=SetPrecision[Transpose[A].KK.A ,5];
“segma[n_]:=PaddedForm[SetPrecision[Inverse[segmainv[n]],5],{5,5}]”;
DetKKK[n_]:=VT1 *$\prod_{\mathbf{i=1}}^{\mathbf{T}} \mathbf{VT2[i]}$;
“define penalty function “;
g1[n_]:=–γX[n];
g2[n_]:= $-\sum_{\mathbf{i=1}}^{\mathbf{L}}(\mathbf{1}-\boldsymbol{\alpha\Delta t})^{\mathbf{i}}(\boldsymbol{\alpha m \Delta t})\}^{\mathbf{2}}\,\boldsymbol{\Delta t}^{\mathbf{2H11}} - \sum_{\mathbf{i=1}}^{\mathbf{L}}(\mathbf{1}-$
$\boldsymbol{\alpha\Delta t})^{\mathbf{2i}}\boldsymbol{\beta}^{\mathbf{2}}\boldsymbol{\Delta t}^{\mathbf{2(H22+H11)}}$;
G1[n_]:=Piecewise[{{g1[n],g1[n]>ϵ},{(g1[n]–ϵ)^2/(4 ϵ),–
ϵ<g1[n]<ϵ},{0,g1[n]<– ϵ}}];”3.36 w.r.t g1”;
G2[n_]:=Piecewise[{{g2[n],g2[n]>ϵ},{(g2[n]–ϵ)^2/(4 ϵ),–
ϵ<g2[n]<ϵ},{0,g2[n]<– ϵ}}];”3.36w.r.t g2”;
;n++];
“lhood[NN_]:=Log[PDF[MultinormalDistribution[Flatten[Table[mu[i],{i,1,N
N}]],segma[NN]],X[NN]]]”;
lhood1[NN_]:=Log[Exp[–0.5* Transpose[Table[Xk1[i]–
mu[i],{i,1,NN}],{1}].segmainv[NN].Table[Xk1[i]–
mu[i],{i,1,NN}]]/ToExpression[StringReplace[ToString[$(2\,\mathrm{Pi})^{\frac{NN}{2}}$
$\mathbf{DetKKK}\,[\mathbf{NN}]^{\frac{1}{2}}$],{”{”–>””,”}”–>””}]]];
op=NMinimize[– lhood1[NN]–ϵ/4 $\sum_{\mathbf{i=1}}^{\mathbf{NN}}\mathbf{G1[i]}$ –ϵ/4 $\sum_{\mathbf{i=1}}^{\mathbf{NN}}\mathbf{G2[i]}$
,{α1,m1,β1,μ1},WorkingPrecision–>15,Method–>”DifferentialEvolution”];
dd=op;
answer=op[[1]];
optimall$_{u=u+1}$=answer;

```
var=dd[[2]];"the answer";

αα=var[[1]];
ααα=αα[[2]];
sumα=sumαα+ααα;
sumαα=sumα;

mm=var[[2]];
mmm=mm[[2]];
summ=summm+mmm;
summm=summ;

ββ=var[[3]];
βββ=ββ[[2]];
sumβ=sumββ+βββ;
sumββ=sumβ;

μμ0=var[[4]];
μμμ=μμ0[[2]];
sumμ=sumμμ+μμμ;
sumμμ=sumμ;

"Print[dd];";
sepvar=Table[var[[i,2]],{i,1,4}];
{ α1,m1,β1,μ1}=sepvar;
Clear[ α1,m1,β1,μ1];
Table[optimall_i,{i,1,RR}];
H1hut=H11;
H1hutt_l=l+1=H1hut;
H2hut=H22;
H2hutt_t=t+1=H2hut;
αhut=sumαα/RR;
αhutt_v=v+1=αhut;
mhut=summm/RR;
mhutt_c=c+1=mhut;
βhut=sumββ/RR;
```

```
βhutt_{e=e+1}=βhut;
μhut=sumμμ/RR;
μhutt_{f=f+1}=μhut;
Print[H1hut];
Print[H2hut];
Print[αhut];
Print[mhut];
Print[βhut];
Print[μhut];
Print[">>>>>>>>>>>>>>>>>>>>>>>>>>>>>>"];
  ];
```

Appendix C

Simulation with Segmentation for Parameters Estimation

```
Δt=.2;
H1=.8;
H2=.7;
α=2.5;
m=2;
β=1.5;
μ=1;
ϵ=0.1;
l=0;
t=0;
u=0;
v=0;
c=0;
e=0;
f=0;
RR=100;
JJ=10;
NN=10;
sumαhut1=0;
summhut1=0;
sumβhut1=0;
sumμhut1=0;
H11=0.8;
H22=0.7;
For[r=1,r<=RR,r++,
“Simulate a fractional Brownian motion process –1st one “;
data1=RandomFunction[FractionalBrownianMotionProcess[H1],{0,1,0.001}];
“finds the parameter estimates for the fbm process(H1) from data1”;
EH1=FindProcessParameters[data1,FractionalBrownianMotionProcess[h]];
H11=EH1[[1,2]];
“Simulate a fractional Brownian motion process–2nd one “;
```

```
data2=RandomFunction[FractionalBrownianMotionProcess[H2],{0,1,0.001}];
"finds the parameter estimates for the fbm process(H2) from data2";
EH2=FindProcessParameters[data2,FractionalBrownianMotionProcess[h]];
H22=EH2[[1,2]];
"simulate fgn";
SimulateFGN[H_,n_]:=Module[{𝓝,ac}, 𝓝 =2^Ceiling[Log[2,n–1]];
ac=Table[FGNAcf[k,H],{k,0, 𝓝 }];
Take[SimulateGLP[ac],n]];SimulateGLP[γ_]:=Module[{m=Length[γ],n,c,g,Z,
Ncap},n=2^Ceiling[Log[2,m–1]];acvf=If[n==m–
1,γ,PadRight[γ,n+1]];Ncap=2*n;
c=Join[acvf,Rest[Reverse[Rest[acvf]]]];
g=Re[Fourier[c,FourierParameters–>{1,–1}]];
Z=RandomVariate[NormalDistribution[0,1],Ncap–2];
Z=(Complex[Sequence@@#1]&)/@Partition[Z,2];
Z=Flatten[{RandomVariate[NormalDistribution[0,Sqrt[2]]],Z,RandomVariate
[NormalDistribution[0,Sqrt[2]]],Reverse[Conjugate[Z]]}];
Take[Re[InverseFourier[Sqrt[g]*Z,FourierParameters–>{0,–1}]],m]/Sqrt[2]];
FGNAcf[k_,H_]:=Module[{},0.5*(Abs[k+1.0]^(2.0*H)–
2*Abs[k]^(2.0*H)+Abs[k–1.0]^(2.0*H))];
"fgn at H=.65 and n=20";
SmH111=SimulateFGN[H11,110];
"fgn at H=.9 and n=20";
SmH222=SimulateFGN[H22,110];
Do[
sumαα=0;
sumββ=0;
sumμμ=0;
summm=0;
w[j_]:=j+10(j–1);
SmH11[j_]:=Table[SmH111[[i]],{i,w[j],w[j]+10}];
SmH22[j_]:=Table[SmH222[[i]],{i,w[j],w[j]+10}];
SmH1=SmH11[j];
SmH2=SmH22[j];

n=2;
While[n<=NN,
```

Clear[α1,m1,β1,μ1];
Yk11[n_]:= $\sum_{\mathbf{i=0}}^{\infty}((\mathbf{1-\alpha1\ \Delta t})^{\mathbf{i}}(\mathbf{\alpha1\ m1\ \Delta t+\beta1\ SmH2}[[\mathbf{Abs}[\mathbf{i+1}]]])$;
Xk1[n_]:=1+μ1 Δt+SmH1[[n]]*
$\sum_{\mathbf{i=0}}^{\infty}((\mathbf{1-\alpha1\ \Delta t})^{\mathbf{i}}(\mathbf{\alpha1\ m1\ \Delta t+\beta1\ SmH2}[[\mathbf{Abs}[\mathbf{i+1}]]])$;
Yk[n_]:= $\sum_{\mathbf{i=0}}^{\infty}(\mathbf{1-\alpha\ \Delta t})^{\mathbf{i}}(\mathbf{\alpha m\Delta t+\beta\ SmH2}[[\mathbf{Abs}[\mathbf{i+1}]]])$;
Xk[n_]:=1+μ Δt+Yk[n]*SmH1[[n]];
mu[n_]:=Mean[Table[Xk1[i],{i,1,n}]];
X[n_]:=Table[Xk1[i],{i,1,n}]; XX[n_]:=Table[Xk[i],{i,1,n–1}]~Join~{Xk1[n]};
γξ[n_]:=1/2 (Abs[(n–1) Δt]$^{2\ H11}$+Abs[(n+1) Δt]$^{2\ H11}$–2 Abs[n Δt]$^{2\ H11}$);
γη[n_]:=1/2 (Abs[(n–1) Δt]$^{2\ H22}$+Abs[(n+1) Δt]$^{2\ H22}$–2 Abs[n Δt]$^{2\ H22}$);
γX[n_]:=Yk[n]^2 γξ[n];
γY[n_]:= $\boldsymbol{\beta^2}\sum_{\boldsymbol{i=0}}^{\infty}\sum_{\boldsymbol{j=0}}^{\infty}\ (\mathbf{1}-\boldsymbol{\alpha\ \Delta t})^{\boldsymbol{i+j}}\ \boldsymbol{\gamma\eta}[\mathbf{n+1-j}]$;

Γn1[n_]:=Table[γX[i–j],{i,1,n–1},{j,1,n–1}];”Γn–1”;
γn1[n_]:=Table[γX[n–i],{i,1,n–1},{j,1,1}];”γn–1”;
γTn1[n_]:=Transpose[γn1[n]];”γn–1 transpose”;
γX0=γX[0];”γX(0)”;
Γ[n_]:=ArrayFlatten[{{Γn1[n],γn1[n]},{γTn1[n],γX0}}];
invΓ[n_]:=Inverse[Γ[n]];
γ[n_]:=Table[{γX[i]},{i,1,n}] ;”γT in 3.27”;
VT2[n_]:=Abs[γX[0]–Transpose[γ[n]].invΓ[n].γ[n]];”3.27”;
VT22[n_]:=ToExpression[StringReplace[ToString[VT2[n]],{”{”–>””,”}”–>””}]];
invΓ[1]=1/γX0;
For[i=1,i<=n,i++,
ϕ[i_]:=invΓ[i].γ[i];
ϕ1[i_]:=Flatten[ϕ[i]];
For[j=1,j<=n,j++,
If[i==j,a[i,j]=1];
If[i<j,a[i,j]=0];
If[i>j,a[i,j]=– ϕ1[i–1][[j]]]
];
a[1,1]=1;
a[2,1]=–γX[1]/γX0;

```
     ];
A=Table[a[i,j],{i,1,n},{j,1,n}];
For[i=2,i<=n,i++,
For[j=2,j<=n,j++,
VT1=Abs[γX0–γX[1]* 1/γX0 *γX[1]];
d[1,1]=(1/VT1);
If[i==j,d[i,j]=1/VT22[i]];
If[i<j,d[i,j]=0];
If[i>j,d[i,j]=0];
      ]
     ];

KK=Table[If[i==1,d[i,j]=0,d[i,j]],{i,1,n},
{j,1,n}];
KK=Table[If[j==1,d[i,j]=0,d[i,j]],{i,1,n},
{j,1,n}];
d[1,1]=Abs[1/VT1];
KK=Table[d[i,j],{i,1,n},{j,1,n}];
segmainv[n_]:=SetPrecision[Transpose[A].KK.A ,5];
"segma[n_]:=PaddedForm[SetPrecision[Inverse[segmainv[n]],5],{5,5}]";
```

DetKKK[n_]:=VT1 * $\prod_{i=1}^{T} \mathbf{VT2[i]}$;

```
"define penalty function ";
g1[n_]:=–γX[n];
```

g2[n_]:= $-\sum_{i=1}^{L}(\mathbf{1}-\boldsymbol{\alpha\Delta t})^{\mathbf{i}}(\boldsymbol{\alpha m \Delta t})\}^{2}\,\boldsymbol{\Delta t}^{\mathbf{2H11}} - \sum_{i=1}^{L}(\mathbf{1}- \boldsymbol{\alpha\Delta t})^{\mathbf{2i}}\boldsymbol{\beta}^{\mathbf{2}}\boldsymbol{\Delta t}^{\mathbf{2(H22+H11)}}$;

```
G1[n_]:=Piecewise[{{g1[n],g1[n]>ϵ},{(g1[n]–ϵ)^2/(4 ϵ),–
ϵ<g1[n]<ϵ},{0,g1[n]<– ϵ}}];"3.36 w.r.t g1";
G2[n_]:=Piecewise[{{g2[n],g2[n]>ϵ},{(g2[n]–ϵ)^2/(4 ϵ),–
ϵ<g2[n]<ϵ},{0,g2[n]<– ϵ}}];"3.36w.r.t g2";
;n++];
"lhood[NN_]:=Log[PDF[MultinormalDistribution[Flatten[Table[mu[i],{i,1,N
N}]],segma[NN]],X[NN]]]";
lhood1[NN_]:=Log[Exp[–0.5* Transpose[Table[Xk1[i]–
mu[i],{i,1,NN}],{1}].segmainv[NN].Table[Xk1[i]–
```

mu[i],{i,1,NN}]]/ToExpression[StringReplace[ToString[$(2\ \mathrm{Pi})^{\frac{NN}{2}}$
DetKKK [NN]$^{\frac{1}{2}}$],{”{”–>””,”}”–>””}]]];
op=NMinimize[– lhood1[NN]–ϵ/4 $\sum_{i=1}^{NN} \boldsymbol{G1}[\boldsymbol{i}]$ –ϵ/4 $\sum_{i=1}^{NN} \boldsymbol{G2}[\boldsymbol{i}]$
,{α1,m1,β1,μ1},WorkingPrecision–>15,Method–>”DifferentialEvolution”];
dd=op;
answer=op[[1]];
$\mathrm{optimall}_{u=u+1}$=answer;
var=dd[[2]];”the answer”;

αα=var[[1]];
ααα=αα[[2]];
sumα=sumαα+ααα;
sumαα=sumα;

mm=var[[2]];
mmm=mm[[2]];
summ=summm+mmm;
summm=summ;

ββ=var[[3]];
βββ=ββ[[2]];
sumβ=sumββ+βββ;
sumββ=sumβ;

μμ0=var[[4]];
μμμ=μμ0[[2]];
sumμ=sumμμ+μμμ;
sumμμ=sumμ;

“Print[dd];”;
sepvar=Table[var[[i,2]],{i,1,4}];
{ α1,m1,β1,μ1}=sepvar;
Clear[α1,m1,β1,μ1];
,{j,JJ}];
Table[$\mathrm{optimall}_{i}$,{i,1,JJ}];

H1hut=H11;
H1hutt$_{l=l+1}$=H1hut;
H2hut=H22;
H2hutt$_{t=t+1}$=H2hut;
αhut=sumαα/JJ;
αhutt$_{v=v+1}$=αhut;
mhut=summm/JJ;
mhutt$_{c=c+1}$=mhut;
βhut=sumββ/JJ;
βhutt$_{e=e+1}$=βhut;
μhut=sumμμ/JJ;
μhutt$_{f=f+1}$=μhut;
Print[H1hut];
Print[H2hut];
Print[αhut];
Print[mhut];
Print[βhut];
Print[μhut];
Print[”>>>>>>>>>>>>>>>>>>>>>>>>>>>”];
];
H1est=Table [H1hutt$_{i}$,{i,1,RR}];
H2est=Table [H2hutt$_{i}$,{i,1,RR}];
αest=Table[αhutt$_{i}$,{i,1,RR}];
mest=Table[mhutt$_{i}$,{i,1,RR}];
βest=Table[βhutt$_{i}$,{i,1,RR}];
μest=Table[μhutt$_{i}$,{i,1,RR}];
H1hutav=Mean[H1est];
H2hutav=Mean[H2est];
αhutav=Mean[αest];
mhutav=Mean[mest];
βhutav=Mean[βest];
μhutav=Mean[μest];
“αvar=Variance[αest];
mvar=Variance[mest];
βvar=Variance[βest];
μvar=Variance[μest]”;

```
Print["H1hut–average = ",H1hutav];
Print["H2hut–average = ",H2hutav];
Print["αhut–average = ",αhutav];
Print["mhut–average = ",mhutav];
Print["βhut–average = ",βhutav];
Print["μhut–average = ",μhutav];
Print["H1–variance = ",Variance[H1est]];
Print["H2–variance = ",Variance[H2est]];
Print["α–variance = ",Variance[αest]];
Print["m–variance = ",Variance[mest]];
Print["β–variance = ",Variance[βest]];
Print["μ–variance = ",Variance[μest]];
Print["H1–bias = ",Abs[H1–H1hutav]];
Print["H2–bias = ",Abs[H2–H1hutav]];
Print["α–bias = ",Abs[α–αhutav]];
Print["m–bias = ",Abs[m–mhutav]];
Print["β–bias = ",Abs[β–βhutav]];
Print["μ–bias = ",Abs[μ–μhutav]];
Print["H1–MSE = ",Variance[H1est]+(Abs[H1–H1hutav])^2];
Print["H2–MSE = ",Variance[H2est]+(Abs[H2–H2hutav])^2];
Print["α–MSE = ",Variance[αest]+(Abs[α–αhutav])^2];
Print["m–MSE = ",Variance[mest]+(Abs[m–mhutav])^2];
Print["β–MSE = ",Variance[βest]+(Abs[β–βhutav])^2];
Print["μ–MSE = ",Variance[μest]+(Abs[μ–μhutav])^2];
```