SIMILARITY SOLUTIONS OF BOUNDARY LAYER FLOWS IN A CHANNEL FILLED BY NON-NEWTONIAN FLUIDS

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Abstrak

Penyelesaian keserupaan bagi bendalir tak-Newtonan semakin mendapat perhatian para penyelidik kerana kepentingan praktikal dalam bidang sains dan kejuruteraan. Pada masa ini, kebanyakan penyelidik menumpukan kajian terhadap bendalir tak-Newtonan pada permukaan helaian. Walau bagaimanapun, hanya segelintir penyelidik sahaja yang memberi perhatian kepada geometri saluran disebabkan kekompleksan persamaan menakluk. Oleh itu, kajian ini berhasrat untuk mengkaji penyelesaian berangka bagi masalah baharu dalam bendalir nano, bendalir Casson dan bendalir mikropolar tak termampat lamina di bawah pelbagai keadaan aliran bendalir. Setiap bendalir yang dipertimbang melibatkan dinding saluran berliang, dinding meregang atau mengecut, dan dinding mengembang atau menguncup dengan pengaruh pelbagai parameter fizikal. Rumusan matematik seperti hukum pemuliharaan, momentum atau momentum sudut, pemindahan haba dan jisim dilakukan terhadap masalah baharu. Setelah rumusan matematik dibangunkan, persamaan menakluk bagi aliran bendalir berbentuk persamaan pembezaan separa kemudiannya dijelmakan kepada masalah nilai sempadan (MNS) persamaan pembezaan biasa (PPB) tak linear dengan menggunakan penjelmaan keserupaan yang sesuai. Selepas menukarkan MNS peringkat tinggi kepada sistem MNS peringkat pertama yang setara, fungsi *shootlib* dalam perisian Maple 18 digunakan bagi mendapatkan penyelesaian keserupaan PPB tak linear. Keputusan berangka dalam kajian ini dibandingkan dengan penyelesaian sedia ada dalam kajian lepas bagi tujuan pengesahan. Keputusan yang diperolehi amat bertepatan sekali dengan penyelesaian sedia ada. Penyelesaian berbilang untuk beberapa masalah terutamanya dalam saluran berliang dengan dinding mengembang atau menguncup juga wujud untuk kes sedutan kuat. Kajian ini berjaya menemui penyelesaian berangka bagi masalah baharu untuk pelbagai keadaan aliran bendalir. Keputusan yang diperolehi daripada kajian ini boleh dijadikan rujukan teori dalam bidang berkaitan.

Kata Kunci: Bendalir Casson, Bendalir micropolar, Bendalir nano, Penyelesaian berbilang, Saluran berliang.

Abstract

Similarity solutions of non-Newtonian fluids are getting much attention to researchers because of their practical importance in the field of science and engineering. Currently, most of researchers focus their work on non-Newtonian fluids over a sheet. However, only a few of them pay their attention towards the geometry of channel due to the complexity of governing equations. Therefore, this study attempts to investigate the numerical solutions of new problems of laminar incompressible Nanofluids, Casson fluids and Micropolar fluids under various fluid flow conditions. Each considered fluid involves porous channel walls, stretching or shrinking walls, and expanding or contracting walls with the influence of various physical parameters. Mathematical formulations such as the law of conservation, momentum or angular momentum, heat and mass transfer are performed on the new problems. After the mathematical formulation is developed, the governing fluid flow equations of partial differential equations are then transformed into boundary value problems (BVPs) of nonlinear ordinary differential equations (ODEs) by using the suitable similarity transformations. After converting high order BVPs into the equivalent first order system of BVPs, *shootlib* function in Maple 18 software is employed to obtain the similarity solutions of nonlinear ODEs. The numerical results in this study are compared with the existing solutions in literature for the purpose of validation. The results are found to be in good agreement with the existing solutions. Multiple solutions of some of the problems particularly in porous channel with expanding or contracting walls also exist for the case of strong suction. This study has successfully find the numerical solutions of the new problems for various fluid flow conditions. The results obtained from this study can serve as a theoretical reference in related fields.

Keywords: Casson fluid, Micropolar fluid, Multiple solutions, Porous channel, Nanofluid.

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Nomenclature

$a(t)$	Height of the channel
$\bar{V}$	Velocity field
p	Pressure
$\bar{l}$	Body couple per unit mass
$\lambda, \mu, \alpha, \beta, \gamma$ and κ	Micropolar material constants
T	Temperature of the fluid
(x, y)	Coordinate axes
$\kappa_\circ$	Thermal conductivity
g	Component of micro-rotation
N	Micro-inertia spin parameter
θ	Dimensionless temperature
$B_\circ$	External uniform magnetic field
p	Pressure (Pa)
k_s	Thermal conductivity of the solid fraction (W/m.K)
k_{nf}	Thermal conductivity of the nanofluid (W/m.K)
ρ_s	Density of the solid fraction (Kg/m^3)
$\bar{v}$	Micro-rotation vector
ρ	Density
$\bar{f}$	Body force
j	Micro-inertia
t	Time
(u, v)	Velocity component of the fluid

Pr	Prandtl number
C_p	Specific heat
C_1	Vortex viscosity parameter
η	Similarity variable
R	Reynolds number
T	Fluid temperature (K or ^{0}C)
$v_\circ$	Injection/suction
k_f	Thermal conductivity of the fluid (W/m.K)
n	Shape factor through H-C Model
c_p	Specific heat at constant pressure (J/(kg K)
$(c_p)_{nf}$	Specific heat of nanofluid
(u, v)	Velocity component in Cartesian coordinate

Dimensionless number

$R = \frac{a^2 b}{v_f}$	Stretching Reynolds number
$Pr = \frac{a^2 b (\rho C_p)_f}{\kappa_f}$	Prandtl number
$\rho_{nf} = \rho_f (1 - \varphi) + \rho_s$	Density of the nanofluid
$M^2 = \frac{\sigma B_\circ^2 a^2}{\mu_f}$	Magnetic parameter
$\mu_{nf} = \frac{\mu_f}{(1 - \varphi)^{2.5}}$	Dynamic viscosity of the nanofluid (Pa.s)
$\frac{\sigma_{nf}}{\sigma_f} = 1 + \frac{3\left(\frac{\sigma_s}{\sigma_f} - 1\right)\varphi}{\left(\frac{\sigma_s}{\sigma_f} + 2\right) - \left(\frac{\sigma_s}{\sigma_f} - 1\right)\varphi}$	Ratio of effective electrical conductivity of nanofluid to the base fluid

Greek symbols

η	Scaled boundary layer coordinate
σ_{nf}	effective electrical conductivity of nanofluid

μ	Dynamic viscosity
k_{nf}	Thermal conductivity of the nanofluid (W/m.K)
θ	Self-similar temperature
φ	Nanoparticle volume fraction parameter
μ_{nf}	Effective dynamic viscosity of nanofluid
ρ	Density (kg/m^3)

Subscripts

nf	Nanofluid
s	Solid phase
2	Upper wall
f	Fluid phase
1	Lower wall

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CHAPTER ONE

INTRODUCTION

1.1 Background

Mechanics is a branch of science that deals with the motion and properties of the rigid bodies that are either at rest or in motion. Mechanics is divided into two main categories: classic and quantum mechanics. In the classical mechanics, which is introduced by the Newton's laws of motion, demonstrates the theory of motion, the energy conservative principle, the forces, the movement of heavy bodies (comets, galaxies, planets, stars), the features of rigid bodies, the movement of fluids; gases as well as liquids, spacecraft navigation and soils mechanical behavior. On the other hand, the quantum mechanics explores the structure, responses and movement of particles (Marsden & Ratiu, 1999).

Nonetheless, fluid is a substance that cannot sustain its shape when shear stresses are applied on it. Therefore, fluid mechanics is the study of gases and liquids at rest or in motion. This area of physics is divided into two parts: fluid statics, the study of the behavior of stationary fluids; and fluid dynamics, the study of the behavior of moving and flowing fluids. Hauke and Moreau (2008) on the other hand examined the behavior of stationary fluid and dynamics of fluid that is dissimilar from the two-aforementioned behavior. Lastly, the flow of fluid is branched into hydrodynamics as well. This part concerns about the mechanical properties of the fluid and has many applications in flight science, air flow analysis and water stream exploration.

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APPENDIX-A

DERIVATION OF THE PROBLEM 3.3

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (1)$$

$$\left(\rho_f(1-\varphi)+\rho_s\right)\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\left(\frac{\mu_f}{(1-\varphi)^{2.5}}\right)\frac{\partial^2 u}{\partial y^2}-\sigma_{nf}B_\circ^2 u \quad (2)$$

$$\left(\rho_f(1-\varphi)+\rho_s\right)\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\left(\frac{\mu_f}{(1-\varphi)^{2.5}}\right)\frac{\partial^2 v}{\partial x^2} \quad (3)$$

$$u\frac{\partial T}{\partial x}+ v\frac{\partial T}{\partial y}= \frac{k_f}{\left(\rho C_p\right)_{nf}}\left(\frac{k_s+2k_f-2\varphi\left(k_f-k_s\right)}{k_s+2k_f+2\varphi\left(k_f-k_s\right)}\right)\frac{\partial^2 T}{\partial y^2} \quad (4)$$

where u and v are the velocity component along x and y axes respectively, σ_{nf} is effective electrical conductivity of nanofluid, ρ_{nf} is effective density, μ_{nf} is the effective dynamic viscosity, $\left(\rho C_p\right)_{nf}$ is heat capacitance and k_{nf} thermal conductivity of the nanofluid. These physical quantities described mathematically as:

$$\rho_{nf}= \rho_f(1-\varphi)+\rho_s \quad (5)$$

$$\mu_{nf}= \frac{\mu_f}{(1-\varphi)^{2.5}} \quad (6)$$

$$\left(\rho C_p\right)_{nf}= \left(\rho C_p\right)_f(1-\varphi)+\left(\rho C_p\right)_s\varphi \quad (7)$$

$$\frac{k_{nf}}{k_f}=\frac{k_s+2k_f-2\varphi\left(k_f-k_s\right)}{k_s+2k_f+2\varphi\left(k_f-k_s\right)} \quad (8)$$

$$\frac{\sigma_{nf}}{\sigma_f}=1+\frac{3\left(\frac{\sigma_s}{\sigma_f}-1\right)\varphi}{\left(\frac{\sigma_s}{\sigma_f}+2\right)-\left(\frac{\sigma_s}{\sigma_f}-1\right)\varphi} \quad (9)$$

Here φ is the solid volume fraction, φ_s is for nanosolid-particles and φ_f is for base fluid. The associated wall conditions are of the form:

$$u=0,\ v=\frac{V}{2},\ T= T_w, C= C_w \qquad \text{at} \qquad y=0 \quad (10)$$

$$\frac{\partial u}{\partial y}=0,\ v=0,\ T= T_H,\ C=C_H \qquad \text{at} \qquad y= H. \quad (11)$$

Introduce the following similarity transformation,

$$x^*=\frac{x}{H}, y^*= \frac{y}{H}, u=-Vx^*f'(y^*), v=Vf(y^*), \theta(y^*)=\frac{T-T_H}{T_w-T_H}, \vartheta(y^*)=\frac{C-C_H}{C_w-C_H},$$

So, equation (2) becomes:

$$\frac{-Vx}{H}f'\left(\frac{-V}{H}f'\right)+Vf\left(\frac{-Vx}{H^2}f''\right)= -\frac{1}{\rho_{nf}}\frac{\partial p}{\partial x}+\nu_{nf}\left(\frac{-Vx}{H^3}f'''\right)+\frac{\sigma_{nf}B_\circ^2}{\rho_{nf}}\left(\frac{Vx}{H}\right)f'$$

Simplifying the above equation:

$$\frac{VH}{\nu_f}\left(f'^2 - ff''\right) = \frac{-1}{\rho_{nf}}\frac{\partial p}{\partial x} - f''' + \frac{\sigma_{nf}B_\circ^2 H^2}{\mu_{nf}}f' \tag{12}$$

Similarly, from equation (3), and taking derivative w.r.t x, we have:

$$\frac{\partial^2 p}{\partial x \partial y} = 0 \tag{13}$$

Taking derivative of equation (12), w.r.t y and put equation (13), we get:

$$f^{iv} + RA_1(1-\varphi)^{2.5}(f'f'' - ff''') + B^\circ M^2(1-\varphi)^{2.5}f'' = 0 \tag{14}$$

where $R = \frac{VH}{\nu}$ is Reynolds number ($R > 0$ for suction $R < 0$ for injection), $M^2 = \frac{\sigma B_\circ^2 H^2}{\mu_f}$ is Hartman number, and, the values of A_1, A_2, A_3 are:

$$A_1 = \frac{\rho_{nf}}{\rho_f} = (1-\varphi) + \frac{\rho_s}{\rho_f}\varphi \tag{15}$$

$$A_2 = \frac{\left(\rho C_p\right)_{nf}}{\left(\rho C_p\right)_f} = (1-\varphi) + \frac{\left(\rho C_p\right)_s}{\left(\rho C_p\right)_f}\varphi \tag{16}$$

$$A_3 = \frac{\kappa_{nf}}{\kappa_f} = \frac{\kappa_s + 2\kappa_f - 2\varphi\left(\kappa_f - \kappa_s\right)}{\kappa_s + 2\kappa_f + 2\varphi\left(\kappa_f - \kappa_s\right)} \tag{17}$$

Moreover, boundary conditions becomes

$$f(1) = \frac{1}{2}, f'(1) = 0$$

$$f''(0) = 0, f(0) = 0 \tag{18}$$

Appendix-B

DERIVATION OF THE STABILITY ANALYSIS FOR SECTION 3.3.2

For stability, similarity variables defined as:

$$u = -Vx^* f'(y^*), v = Vf(y^*), \theta(y^*) = \frac{T-T_H}{T_w-T_H} \text{ where } x^* = \frac{x}{H}, y^* = \frac{y}{H} \tag{17}$$

The governing equations of (3.56) – (3.57) for unsteady case $\tau = t$, can be written as:

$$\frac{\partial^4 f}{\partial \eta^4} + RA_1(1-\varphi)^{2.5}\left[\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta^2} - f\frac{\partial^3 f}{\partial \eta^3}\right] + M^2(1-\varphi)^{2.5}\frac{\partial^2 f}{\partial \eta^2} = \frac{\partial^3 f}{\partial \tau \partial \eta^2} \tag{18}$$

$$\frac{\partial^2 \theta}{\partial \eta^2} + Pr\frac{A_2}{A_3} f\frac{\partial \theta}{\partial \eta} = \frac{\partial \theta}{\partial \tau} \tag{19}$$

The stability analysis of the steady flow solution $f(\eta) = f_\circ(\eta)$ and $\theta(\eta) = \theta_\circ(\eta)$.

$$f(\eta) = f_\circ(\eta) + e^{-\lambda t}F(\eta, t) \tag{20}$$

$$\theta(\eta) = \vartheta_\circ(\eta) + e^{-\lambda t}G(\eta, t) \ , \tag{21}$$

Where $0 < F(\eta, t) \ll 1$, $0 < G(\eta, t) \ll 1$ and λ is the unknown eigenvalues, $F(\eta, t)$ and $G(\eta, t)$ are the smallest relative to $f_\circ(\eta)$ and $\theta_\circ(\eta)$ respectively.

Taking derivative of Eqs. (20) and (21)

$$\left.\begin{aligned}
\frac{\partial f}{\partial \eta} &= f_\circ' + e^{-\lambda t}F' \\
\frac{\partial^2 f}{\partial \eta^2} &= f_\circ'' + e^{-\lambda t}F'' \\
\frac{\partial^3 f}{\partial \eta^3} &= f_\circ''' + e^{-\lambda t}F''' \\
\frac{\partial^4 f}{\partial \eta^4} &= f_\circ'''' + e^{-\lambda t}F'''' \\
\frac{\partial^3 f}{\partial \eta^2 \partial \tau} = \frac{\partial}{\partial \tau}\left(\frac{\partial^2 f}{\partial \eta^2}\right) &= \frac{\partial}{\partial \tau}\left(f_\circ'' + e^{-\lambda t}F''\right) = te^{-\lambda t}\left(\frac{\partial F''}{\partial \tau}\right)\frac{\partial t}{\partial \tau} - \lambda e^{-\lambda t}F'' \\
\frac{\partial \theta}{\partial \eta} &= \theta_\circ' + e^{-\lambda t}G' \\
\frac{\partial^2 \theta}{\partial \eta^2} &= \theta_\circ'' + e^{-\lambda t}G''
\end{aligned}\right\} \tag{22}$$

Use the above relation into (18) – (19) and assume $\tau = 0$.

From Eq. (18),

$$\left(f_\circ'''' + e^{-\lambda t}F''''\right) + RA_1(1-\varphi)^{2.5}\left[\left(f_\circ' + e^{-\lambda t}F'\right)\left(f_\circ'' + e^{-\lambda t}F''\right) - \left(f_\circ(\eta) + e^{-\lambda t}F\right)\left(f_\circ'' + e^{-\lambda t}F''\right)\right] + M^2(1-\varphi)^{2.5}\left(f_\circ'' + e^{-\lambda t}F''\right) + \lambda e^{-\lambda t}F'' = 0,$$

Expand the equation,

$$f_\circ'''' + RA_1(1-\varphi)^{2.5}(f_\circ' f_\circ'' - f_\circ f_\circ''') + M^2(1-\varphi)^{2.5} f_\circ'' + e^{-\lambda t}F'''' + RA_1(1-\varphi)^{2.5}\left[\left(f_\circ' F'' e^{-\lambda t} + f_\circ'' F' e^{-\lambda t} + F'F'' e^{-2\lambda t}\right) - \left(f_\circ F''' e^{-\lambda t} + FF''' e^{-\lambda t}\right)\right] + M^2(1-\varphi)^{2.5} e^{-\lambda t}F'' + \lambda e^{-\lambda t}F'' = 0,$$

Since we have assumed that $F(\eta, t)$ is small, therefore the product of their derivatives are also small. So by neglecting the terms and considering the steady state:

$$f_\circ'''' + RA_1(1-\varphi)^{2.5}(f_\circ' f_\circ'' - f_\circ f_\circ''') + M^2(1-\varphi)^{2.5} f_\circ'' = 0$$

The stability equation becomes:

$$F'''' + RA_1(1-\varphi)^{2.5}[(f_\circ' F'' + f_\circ'' F') - (f_\circ F''' + F f_\circ''')]$$

$$+M^2(1-\varphi)^{2.5}F'' + \lambda F'' = 0 \qquad (23)$$

Similarly from (19),

$$\left(\theta_\circ'' + e^{-\lambda t}G''\right) + Pr\frac{A_2}{A_3}\left(f_\circ(\eta) + e^{-\lambda t}F\right)\left(\theta_\circ' + e^{-\lambda t}G'\right) + \lambda e^{-\lambda t}G = 0,$$

Expand and rearrange the equation:

$$\theta_\circ'' + Pr\frac{A_2}{A_3}(f_\circ \theta_\circ') + e^{-\lambda t}G'' + Pr\frac{A_2}{A_3}\left(fG'e^{-\lambda t} + e^{-\lambda t}F\theta_\circ' + e^{-2\lambda t}FG'\right) = 0,$$

Since, we have assumed that $G(\eta, t)$ and $F(\eta, t)$ are small, therefore the product of their derivatives are also small. So by neglecting the terms and considering the steady state:

$$\theta_\circ'' + Pr\frac{A_2}{A_3}(f_\circ \theta_\circ') = 0$$

The stability equation becomes:

$$G'' + Pr\frac{A_2}{A_3}(fG' + F\theta_\circ') + \lambda G = 0. \qquad (24)$$

Boundary equations becomes:

At wall $y = H$

$$u = 0 \Rightarrow= -Vx^* f'(y^*) = 0 \Rightarrow f'\left(\frac{y}{H}\right) = 0 \Rightarrow f'(1) = 0 \qquad (25)$$

$$, v = \frac{V}{2} = Vf(y^*) \Rightarrow f\left(\frac{y}{H}\right) = \left(\frac{1}{2}\right) = f(1) = \frac{1}{2} \qquad (26)$$

$$\theta(y^*) = \frac{T - T_H}{T_w - T_H} => \theta(1) = 1 \text{ as } T = T_w \tag{27}$$

At $y = 0$

$$\frac{\partial u}{\partial y} = 0 => \frac{-V}{H} x^* f''(y^*) => f''(0) = 0 \tag{28}$$

$$v = 0 = Vf(y^*) => f\left(\frac{y}{H}\right) = 0 => f(0) = 0 \tag{29}$$

$$\theta(y^*) = \frac{T - T_H}{T_w - T_H} => \theta(0) = 0 \text{ as } T = T_H \tag{30}$$

Put Eqns. (25) – (30) into Eqns. (20) - (21) and use $\tau = t$, we get the boundary conditions for stability:

$$\left.\begin{array}{c} F(1) = \frac{1}{2}, F'(1) = 1, G(1) = 1 \\ F''(0) = 0, F(0), G(0) = 0 \end{array}\right\} \tag{31}$$

APPENDIX C

MAPLE PROGRAM

This maple program solves the problem of steady laminar incompressible nanofluid in a porous channel with the help of shooting method.

> *restart* ;

> *Shootlib* := "D:\\nanofluid/";

$$Shootlib := \text{"D:\nanofluid/"}$$

> *libname* := *Shootlib*, *libname*;

$$libname := \text{"D:\nanofluid/", "C:\Program Files\Maple 18\lib", "."}$$

> *with*(*Shoot*);

$$[shoot]$$

> *with*(*plots*) :

> $M := 1.5; R := 30.0;\ \varphi := 0.03; s := 0.5; \sigma_f := 0.05; \sigma_s := 5980000; S_1 := 0.0; S_2 := 0.0;$

$$M := 1.5$$
$$R := 30.0$$
$$\varphi := 0.03$$
$$s := 0.5$$
$$\sigma_f := 0.05$$
$$\sigma_{0.5} := 5980000$$
$$S_1 := 0.$$
$$S_2 := 0.$$

> $Pr := 6.0;\ \rho_s := 8933; \rho_f := 997.1; C_{ps} := 385; C_{pf} := 4179; K_s := 401; K_f := 0.613;$

$$Pr := 6.0$$
$$\rho_{0.5} := 8933$$
$$\rho_f := 997.1$$
$$C_{ps} := 385$$
$$C_{pf} := 4179$$
$$K_{0.5} := 401$$
$$K_f := 0.613$$

> $A_1 := (1 - \varphi) + \frac{\rho_s}{\rho_f} \cdot \varphi;$

$$A_1 := 1.238769431$$

> $A_2 := (1-\varphi) + \frac{\rho_s \cdot C_{ps}}{\rho_f \cdot C_{pf}} \cdot \varphi;$

$$A_2 := 0.9947610029$$

> $A_3 := \frac{(K_s + 2\cdot K_f - 2\cdot\varphi\cdot(K_f - K_s))}{(K_s + 2\cdot K_f + 2\cdot\varphi\cdot(K_f - K_s))};$

$$A_3 := 1.127038834$$

> $blt1 := 1;\ blt2 := 5;\ blt3 := 6;$

$$blt1 := 1$$
$$blt2 := 5$$
$$blt3 := 6$$

> $B^\circ := \left(1 + \left(\frac{3\cdot\left(\frac{\sigma_s}{\sigma_f} - 1\right)\cdot\varphi}{\left(\frac{\sigma_s}{\sigma_f} + 2\right) - \left(\frac{\sigma_s}{\sigma_f} - 1\right)\cdot\varphi}\right)\right);$

$$B^\circ := 1.092783503$$

> $FNS := \{F(\eta), Fp(\eta), Fpp(\eta), Fppp(\eta), \theta(\eta), \theta p(\eta)\};$

$$FNS := \{F(\eta), Fp(\eta), Fpp(\eta), Fppp(\eta), \theta(\eta), \theta p(\eta)\}$$

>

$ODE := \Big\{ diff(F(\eta),\eta) = Fp(\eta), diff(Fp(\eta),\eta) = Fpp(\eta), diff(Fpp(\eta),\eta) = Fppp(\eta),$
$diff(Fppp(\eta),\eta) = -M^2\cdot(1-\varphi)^{2.5}\cdot Fpp(\eta) - R\cdot A_1\cdot(1-\varphi)^{2.5}\cdot(Fp(\eta)\cdot Fpp(\eta)$
$- F(\eta)\cdot Fppp(\eta)), diff(\theta(\eta),\eta) = \theta p(\eta), \frac{1}{Pr}\cdot diff(\theta p(\eta),\eta) = -\frac{A_2}{A_3}\cdot F(\eta)\cdot\theta p(\eta) \Big\};$

$$ODE := \Big\{0.1666666667\left(\frac{d}{d\eta}\theta p(\eta)\right) = -0.8826324106\,F(\eta)\,\theta p(\eta), \frac{d}{d\eta}F(\eta) = Fp(\eta),$$
$$\frac{d}{d\eta}Fp(\eta) = Fpp(\eta), \frac{d}{d\eta}Fpp(\eta) = Fppp(\eta), \frac{d}{d\eta}Fppp(\eta) = -2.085027819\,Fpp(\eta)$$
$$- 34.43824966\,Fp(\eta)\,Fpp(\eta) + 34.43824966\,F(\eta)\,Fppp(\eta), \frac{d}{d\eta}\theta(\eta) = \theta p(\eta)\Big\}$$

> $IC1 := \{F(1) = s, Fp(1) = S_1\cdot\alpha, Fpp(1) = \alpha, Fppp(1) = \beta, \theta(1) = 0, \theta p(1) = \Omega\};$

$$IC1 := \{F(1) = 0.5, Fp(1) = 0., Fpp(1) = \alpha, Fppp(1) = \beta, \theta(1) = 0, \theta p(1) = \Omega\}$$

> $BC1 := \{F(0) = 0, Fpp(0) = 0, \theta(0) = 1\};$

$$BC1 := \{F(0) = 0, Fpp(0) = 0, \theta(0) = 1\}$$

>

> $infolevel[shoot] := 1:$

> $S1 := shoot(ODE, IC1, BC1, FNS, [\alpha = -27.87843544811985, \beta = -330.38790947783133, \Omega = 0.1]):$

shoot: Step # 1

shoot: Parameter values : alpha = -27.87843544811985 beta = -330.38790947783133 Omega = .1

shoot: Step # 2

shoot: Parameter values : alpha = HFloat(-26.36175378243866) beta = HFloat(-311.6398759119528) Omega = HFloat(-0.8809866159927399)

shoot: Step # 3

shoot: Parameter values : alpha = HFloat(-26.462859744145767) beta = HFloat(-314.26358245169484) Omega = HFloat(-0.9304574939017813)

shoot: Step # 4

shoot: Parameter values : alpha = HFloat(-27.754039617817263) beta = HFloat(-332.54815889671147) Omega = HFloat(-0.9428238639736823)

shoot: Step # 5

shoot: Parameter values : alpha = HFloat(-27.894451024208035) beta = HFloat(-334.82037068238213) Omega = HFloat(-0.9429413671505218)

shoot: Step # 6

shoot: Parameter values : alpha = HFloat(-28.06597704207974) beta = HFloat(-337.31819029760845) Omega = HFloat(-0.9442871983658467)

shoot: Step # 7

shoot: Parameter values : alpha = HFloat(-28.070360342937857) beta = HFloat(-337.38663667071324) Omega = HFloat(-0.9443028164268538)

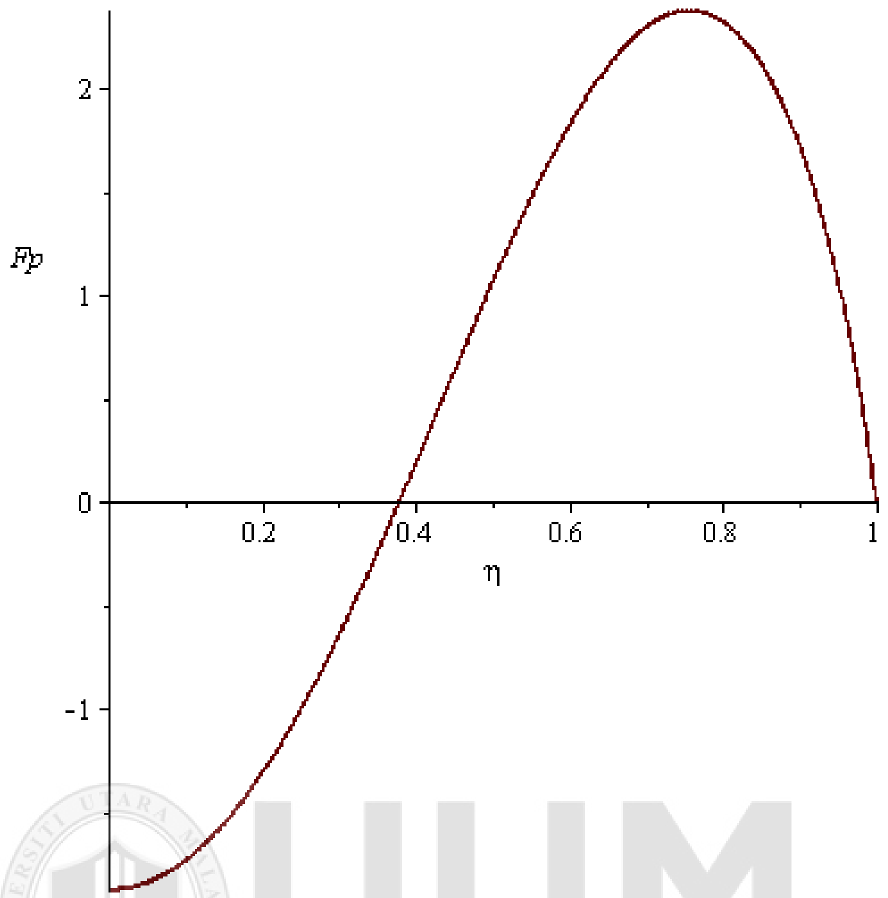
shoot: Step # 8

shoot: Parameter values : alpha = HFloat(-28.07044734864878) beta = HFloat(-337.3879094575454) Omega = HFloat(-0.9443034815316692)

shoot: Step # 9

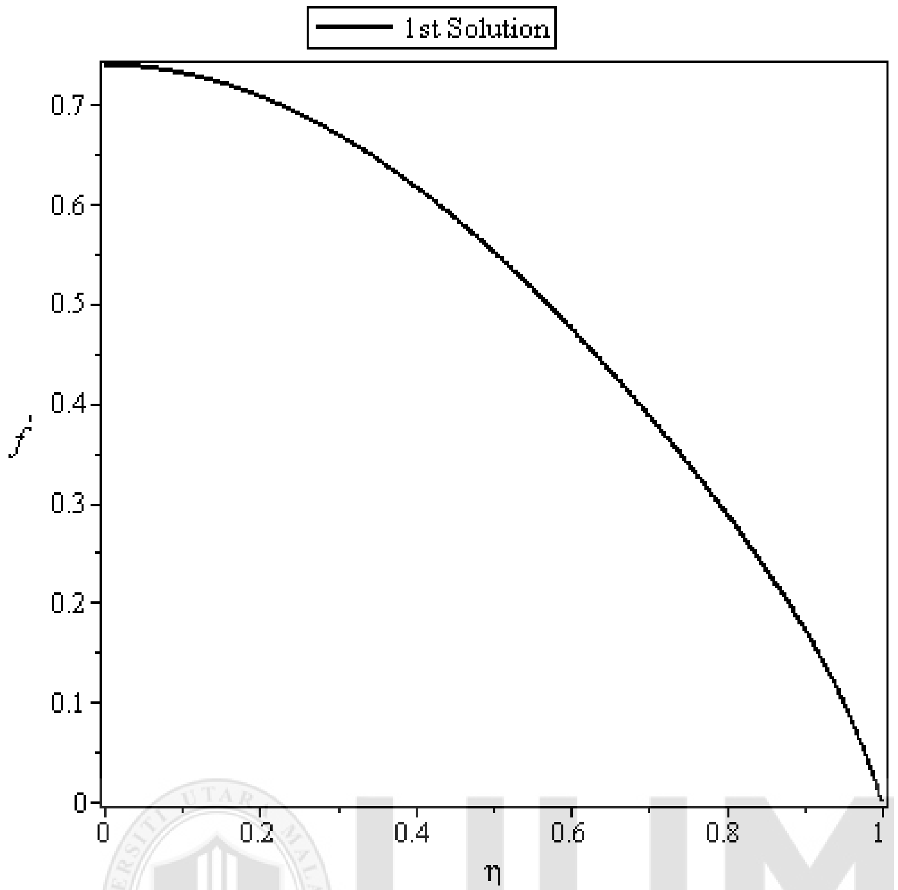
shoot: Parameter values : alpha = HFloat(-28.07044734997453) beta = HFloat(-337.3879094779187) Omega = HFloat(-0.9443034815374989)

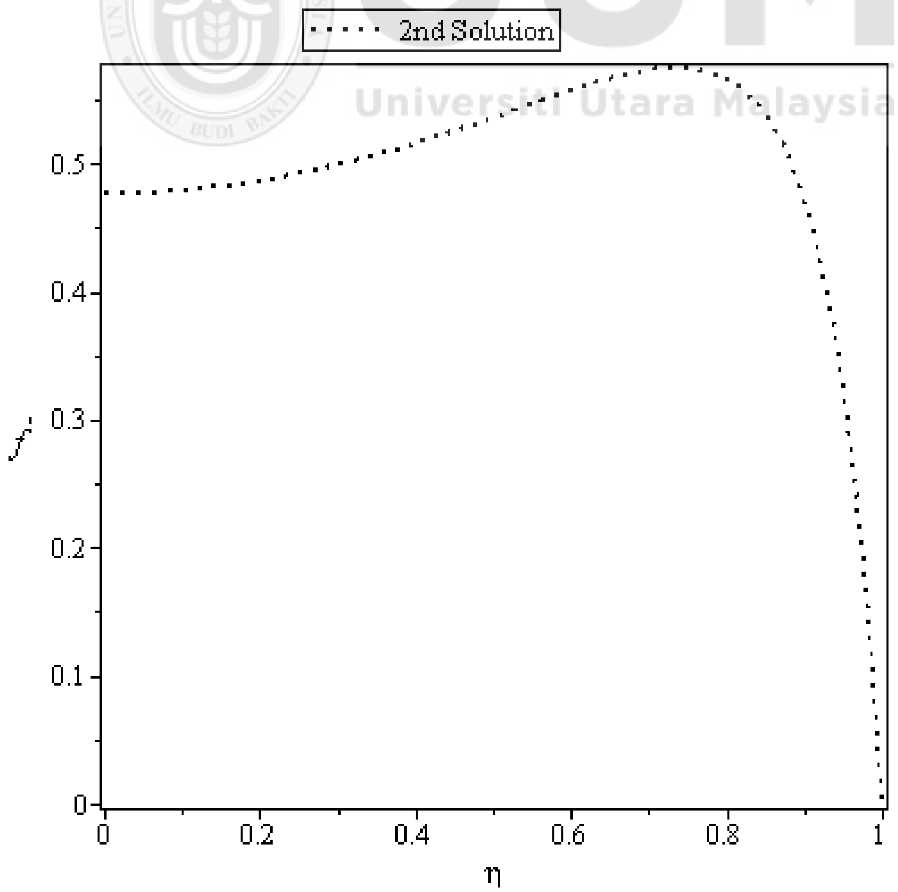
```
> p1 := odeplot(S1, [η, Fp(η)], 0 .. blt1, numpoints = 500) :
> p2 := odeplot(S1, [η, F(η)], 0 ..blt1, numpoints = 500) :
> p3 := odeplot(S1, [η, θ(η)], 0 ..blt1, numpoints = 500) :
>
> p4 := odeplot(S1, [η, Fppp(η)], 0 ..blt1, numpoints = 500) :
>
> display(p1);
```

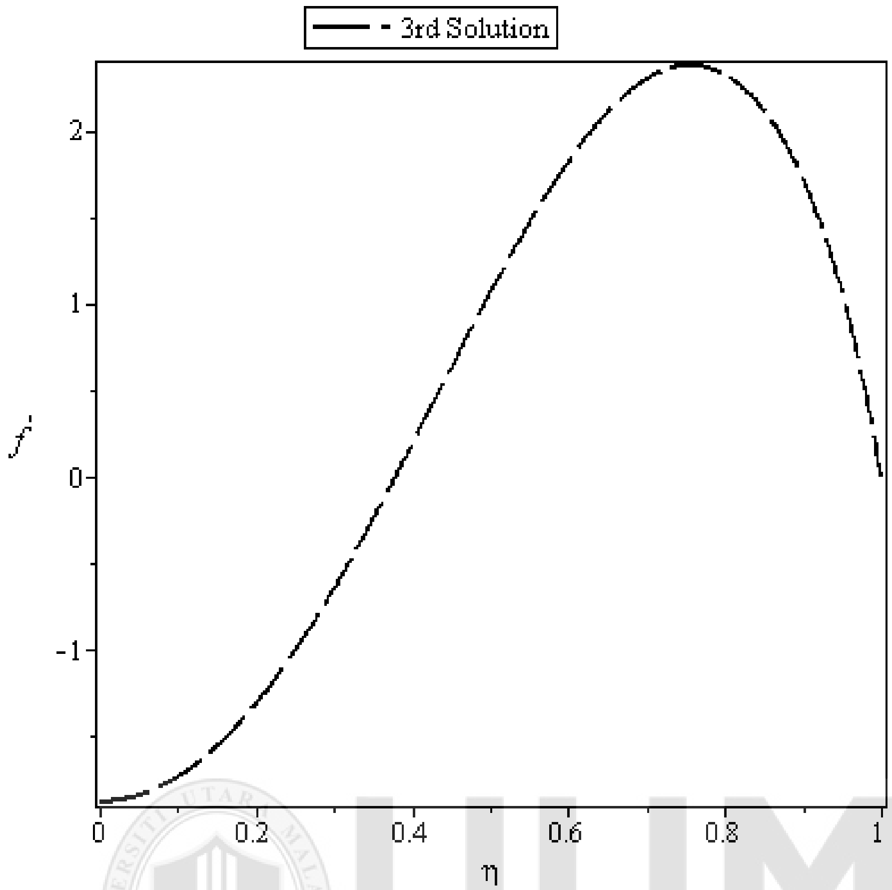


Fp
2
1
0
-1
0.2
0.4
0.6
0.8
1
η

>


1st Solution
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
f'
0
0.2
0.4
0.6
0.8
1
η


2nd Solution
0.5
0.4
0.3
0.2
0.1
0
f'
0
0.2
0.4
0.6
0.8
1
η



3rd Solution
2
1
0
-1
f'
0
0.2
0.4
0.6
0.8
1
η

APPENDIX D

Stability Program for Section 3.3.2

This MATLAB program solves the problem of steady laminar incompressible nanofluid in a porous channel with the help of 3-Stage Lobatto III-A Formula:

```
function first_solution
clear all;
clc;
global R phi M a b Pr ros rof cps cpf ks kf A1 A2 A3
ros = 8933; rof = 991.1; cps = 385; cpf = 4179; kf = 0.613; ks =
401;Pr=6.2;
phi = 0.03; R = 33.0; M=0.4;
A1 = (1-phi)+(ros/rof)*phi;
A2 = (1-phi)+ ((ros*cps)/(rof*cpf))*phi;
A3 = (ks+2*kf-2*phi*(kf-ks))/(ks+2*kf+phi*(kf-ks));


a = 0;
b = 1;

solinit = bvpinit(linspace(a,b,5),@guess);
options = bvpset('stats','on','RelTol',1e-7);
sol = bvp4c(@nano_ode,@nano_bc,solinit,options);
figure(1)
plot(sol.x,sol.y(2,:),'b')
xlabel('\eta')
ylabel ('f''(\eta)')
hold on

figure(2)
plot(sol.x,sol.y(5,:),'r')
xlabel('\eta')
ylabel('\theta(\eta)')
hold on

descris=[sol.x; sol.y];
save 'first_sol_casson.txt' descris -ascii
%save first_sol_casson.mat '-struct' 'sol';

fprintf('f(1) = %7.3f.\n', sol.y(1,end));
fprintf('f''(1) = %7.3f.\n', sol.y(2,end));
fprintf('f"(1) = %7.3f.\n', sol.y(3,end));
fprintf('f""(1) = %7.3f.\n', sol.y(4,end));
fprintf('thetha(1) = %7.3f.\n', sol.y(5,end));
fprintf('thetha''(1) = %7.3f.\n', sol.y(6,end));



%-------------------------------------------------------------------
-----
```

```
function dydx = nano_ode(x,y,R,M,phi,A1,A2,A3,Pr)
global R phi A1 A2 A3 M Pr


dydx = [y(2)
    y(3)
    y(4)
    -(M^2)*((1-phi)^2.5)*y(3)-R*A1*((1-phi)^2.5)*(y(2)*y(3)-
y(1)*y(4))
    y(6)
    -(Pr)*(A2/A3)*y(1)*y(6)];
%-------------------------------------------------------------------
-----
function BC = nano_bc(ya,yb)

BC = [ya(1)
    ya(3)
    ya(5)-1
    yb(1)-0.5
    yb(2)
    yb(5)];
%-------------------------------------------------------------------
------

function v = guess(x)

%v = [0 0 0 2 0 1];
v = [-1*exp(-x) sin(x) sin(-x) 2*cos(x) sin(x) cos(x)];



function second_solution
clear all;
clc;
global R phi M a b Pr ros rof cps cpf ks kf A1 A2 A3
ros = 8933; rof = 991.1; cps = 385; cpf = 4179; kf = 0.613; ks =
401;Pr=6.2;
phi = 0.03; R = 33.0; M=0.4;
A1 = (1-phi)+(ros/rof)*phi;
A2 = (1-phi)+ ((ros*cps)/(rof*cpf))*phi;
A3 = (ks+2*kf-2*phi*(kf-ks))/(ks+2*kf+phi*(kf-ks));



a = 0;
b = 1;

solinit = bvpinit(linspace(a,b,5),@guess);
options = bvpset('stats','on','RelTol',1e-7);
sol = bvp4c(@nano_ode,@nano_bc,solinit,options);
figure(1)
plot(sol.x,sol.y(2,:),'b')
xlabel('\eta')
ylabel ('f''(\eta)')
hold on

figure(2)
plot(sol.x,sol.y(5,:),'r')
xlabel('\eta')
```

```
ylabel('\theta(\eta)')
hold on

descris=[sol.x; sol.y];
save 'second_sol_casson.txt' descris -ascii
%save 'second_sol_casson.mat' '-struct' 'sol';

fprintf('f(1) = %7.3f.\n', sol.y(1,end));
fprintf('f''(1) = %7.3f.\n', sol.y(2,end));
fprintf('f"(1) = %7.3f.\n', sol.y(3,end));
fprintf('f""(1) = %7.3f.\n', sol.y(4,end));
fprintf('thetha(1) = %7.3f.\n', sol.y(5,end));
fprintf('thetha''(1) = %7.3f.\n', sol.y(6,end));



%---------------------------------------------------------------------
-----

function dydx = nano_ode(x,y,R,M,phi,A1,A2,A3,Pr)
global R phi A1 A2 A3 M Pr


dydx = [y(2)
    y(3)
    y(4)
    -(M^2)*((1-phi)^2.5)*y(3)-R*A1*((1-phi)^2.5)*(y(2)*y(3)-
y(1)*y(4))
    y(6)
    -(Pr)*(A2/A3)*y(1)*y(6)];
%---------------------------------------------------------------------
-----
function BC = nano_bc(ya,yb)

BC = [ya(1)
    ya(3)
    ya(5)-1
    yb(1)-0.5
    yb(2)
    yb(5)];
%---------------------------------------------------------------------
------

function v = guess(x)

%v = [1 0.5 -4 204 0 0]; %2nd solution
v = [-2*exp(x) sin(x) 4*sin(-x) cos(x) sin(x) cos(-4*x)];



function third_solution
clear all;
clc;
global R phi M a b Pr ros rof cps cpf ks kf A1 A2 A3
ros = 8933; rof = 991.1; cps = 385; cpf = 4179; kf = 0.613; ks =
401;Pr=6.2;
phi = 0.03; R = 33.0; M=0.4;
A1 = (1-phi)+(ros/rof)*phi;
```

```
A2 = (1-phi)+ ((ros*cps)/(rof*cpf))*phi;
A3 = (ks+2*kf-2*phi*(kf-ks))/(ks+2*kf+phi*(kf-ks));


a = 0;
b = 1;

solinit = bvpinit(linspace(a,b,5),@guess);
options = bvpset('stats','on','RelTol',1e-7);
sol = bvp4c(@nano_ode,@nano_bc,solinit,options);
figure(1)
plot(sol.x,sol.y(2,:),'b')
xlabel('\eta')
ylabel ('f''(\eta)')
hold on

figure(2)
plot(sol.x,sol.y(5,:),'r')
xlabel('\eta')
ylabel('\theta(\eta)')
hold on

descris=[sol.x; sol.y];
save 'third_sol_casson.txt' descris -ascii
%save 'third_sol_casson.mat' '-struct' 'sol';

fprintf('f(1) = %7.3f.\n', sol.y(1,end));
fprintf('f''(1) = %7.3f.\n', sol.y(2,end));
fprintf('f"(1) = %7.3f.\n', sol.y(3,end));
fprintf('f""(1) = %7.3f.\n', sol.y(4,end));
fprintf('thetha(1) = %7.3f.\n', sol.y(5,end));
fprintf('thetha''(1) = %7.3f.\n', sol.y(6,end));



%----------------------------------------------------------------------
-----

function dydx = nano_ode(x,y,R,M,phi,A1,A2,A3,Pr)
global R phi A1 A2 A3 M Pr


dydx = [y(2)
    y(3)
    y(4)
    -(M^2)*((1-phi)^2.5)*y(3)-R*A1*((1-phi)^2.5)*(y(2)*y(3)-
y(1)*y(4))
    y(6)
    -(Pr)*(A2/A3)*y(1)*y(6)];
%----------------------------------------------------------------------
-----
function BC = nano_bc(ya,yb)

BC = [ya(1)
    ya(3)
    ya(5)-1
    yb(1)-0.5
    yb(2)
    yb(5)];
```

```
%-----------------------------------------------------------------------
------

function v = guess(x)

%v= [0 0.5 -8 204 0 0]; % 3rd solution
%v = [exp(x) -1*exp(x) -4*exp(x) 204+exp(x) -20+cos(x) -20*sin(-x)];
v = [-4*exp(-2*x) 2*exp(-x) 4*sin(-x) cos(x) sin(x) cos(-4*x)];


function velocity_nano
clear all;
clc;
global R phi M a b Pr ros rof cps cpf ks kf A1 A2 A3
ros = 8933; rof = 991.1; cps = 385; cpf = 4179; kf = 0.613; ks =
401;Pr=6.2;
phi = 0.03; R = 33.0; M=0.4;
A1 = (1-phi)+(ros/rof)*phi;
A2 = (1-phi)+ ((ros*cps)/(rof*cpf))*phi;
A3 = (ks+2*kf-2*phi*(kf-ks))/(ks+2*kf+phi*(kf-ks));


a = 0;
b = 1;
for R = 33.0:0.1:34.0
    if R == 33.0
        lo = load ('first_sol_casson.txt');
        solinit.x=lo(1,:);solinit.y=lo(2:7,:);
        else
      solinit.x = sol.x; solinit.y=sol.y;
end

options = bvpset('stats','off','RelTol',1e-10);
sol = bvp4c(@nano_ode,@nano_bc,solinit,options);


end
figure(1)
 plot(sol.x,sol.y(2,:),'k')
 hold on
figure(2)
 plot(sol.x,sol.y(4,:),'k')
 hold on
save ('first_sol_casson.mat', '-struct', 'sol');

%-----------------------------------------------------------------------
-----

for R = 33.0:0.1:34.0
    if R == 33.0
        lo = load ('second_sol_casson.txt');
        solinit.x=lo(1,:);solinit.y=lo(2:7,:);
        else
      solinit.x = sol.x; solinit.y=sol.y;
end

options = bvpset('stats','off','RelTol',1e-10);
sol = bvp4c(@nano_ode,@nano_bc,solinit,options);
```

```
end
figure(1)
 plot(sol.x,sol.y(2,:),'k')
 hold on
figure(2)
 plot(sol.x,sol.y(4,:),'k')
 hold on
save ('second_sol_casson.mat', '-struct', 'sol');


%-----------------------------------------------------------------------
-----

for R = 33.0:0.1:34.0
    if R == 33.0
        lo = load ('third_sol_casson.txt');
        solinit.x=lo(1,:);solinit.y=lo(2:7,:);
        else
      solinit.x = sol.x; solinit.y=sol.y;
end

options = bvpset('stats','off','RelTol',1e-10);
sol = bvp4c(@nano_ode,@nano_bc,solinit,options);


end
figure(1)
 plot(sol.x,sol.y(2,:),'k')
 hold on
figure(2)
 plot(sol.x,sol.y(4,:),'k')
 hold on
save ('third_sol_casson.mat', '-struct', 'sol');



%-----------------------------------------------------------------------
-----
function dydx = nano_ode(x,y,R,M,phi,A1,A2,A3,Pr)
global R phi A1 A2 A3 M Pr


dydx = [y(2)
    y(3)
    y(4)
    -(M^2)*((1-phi)^2.5)*y(3)-R*A1*((1-phi)^2.5)*(y(2)*y(3)-
y(1)*y(4))
    y(6)
    -(Pr)*(A2/A3)*y(1)*y(6)];
%-----------------------------------------------------------------------
-----
function BC = nano_bc(ya,yb)

BC = [ya(1)
    ya(3)
    ya(5)-1
    yb(1)-0.5
    yb(2)
```

```
    yb(5)];
%-----------------------------------------------------------------------
------

function v = guess(x)

%v = [0 0 0 2 0 1];
%v = [exp(x) sin(x) sin(-x) 2*cos(x) sin(x) cos(x)];
%v = [-1*exp(x) exp(-x) 4*sin(-x) cos(x) sin(x) cos(-4*x)];
%v = [exp(x) -1*exp(x) -4*exp(x) 204+exp(x) -20+cos(x) -20*sin(-x)];
v = [-4*exp(-2*x) 2*exp(-x) 4*sin(-x) cos(x) sin(x) cos(-4*x)];


function stability_nano
format long g

clear all;
clc;
global R phi M a b Pr ros rof cps cpf ks kf A1 A2 A3 D gamma
ros = 8933; rof = 991.1; cps = 385; cpf = 4179; kf = 0.613; ks =
401;Pr=6.2;
phi = 0.03; R = 33.0; M=0.4;
A1 = (1-phi)+(ros/rof)*phi;
A2 = (1-phi)+ ((ros*cps)/(rof*cpf))*phi;
A3 = (ks+2*kf-2*phi*(kf-ks))/(ks+2*kf+phi*(kf-ks));


a = 0;
b = 1;

D = load('first_sol_casson.mat');
%D=load('first_sol_casson.mat');
%D=load('second_sol_casson.mat');%gamma = -5.1979:-0.0001:-5.1985
%D=load('third_sol_casson.mat');
%D=descris;

err = [];
gam = [];

for gamma = 12.2:0.0001:12.3
    solinit = bvpinit(linspace(a,b,5),@guess);

sol = bvp4c(@nano_ode,@nano_bc,solinit);
figure(1)
plot(sol.x,sol.y(2,:),'b')
hold on
plot(sol.x,sol.y(1,:),'r')
hold on
sol.y;
disp([gamma,abs(sol.y(5,end))]);
err = [err,abs(sol.y(5,end))];
gam = [gam,gamma];


end
figure(2)
plot(gam,err, 'LineWidth', 1.5);
hold on
```

```
min(err)
%fprinitf ('eigen value = %7.3f.\n',sol.parameters);

%--------------------------------------------------------------------
---

function dydx = nano_ode(x,y,R,M,phi,A1,A2,A3,Pr,gamma)
global R phi A1 A2 A3 M Pr s D gamma
[s,sp] = deval(D,x);

dydx = [y(2)
    y(3)
    y(4)
    -(M^2)*((1-phi)^2.5)*s(3)-R*A1*((1-
phi)^2.5)*(y(2)*s(3)+s(2)*y(3)-y(1)*s(4)-s(1)*y(4))-gamma*s(3)
    y(6)
    -(Pr)*(A2/A3)*(y(1)*s(6)+s(1)*y(6))-gamma*s(5)];
%--------------------------------------------------------------------
-----

function BC = nano_bc(ya,yb)

BC = [ya(1)
    ya(3)
    ya(5)-1
    ya(6)-1
    yb(1)-0.5
    yb(2)
    ];
%--------------------------------------------------------------------
------

function v = guess(x,gamma)

v = [-1*exp(-x) sin(x) sin(-x) 2*cos(x) sin(x) cos(x)]; %first
solution
%v = [exp(x) exp(x) -4*exp(x) 208*exp(x) sin(x) sin(-x)]; %2nd
solution
%v = [cos(x) exp(x) -8*exp(x) 204+exp(x) sin(x) sin(-x)]; %3rd
solution
```