APPLICATION OF NEW HOMOTOPY ANALYSIS METHOD AND OPTIMAL HOMOTOPY ASYMPTOTIC METHOD FOR SOLVING FUZZY FRACTIONAL ORDINARY DIFFERENTIAL EQUATIONS

DULFIKAR JAWAD HASHIM

DOCTOR OF PHILOSOPHY
UNIVERSITY UTARA MALAYSIA
2022

Awang Had Salleh
Graduate School
of Arts And Sciences

Universiti Utara Malaysia

PERAKUAN KERJA TESIS / DISERTASI
(Certification of thesis / dissertation)

Kami, yang bertandatangan, memperakukan bahawa
(We, the undersigned, certify that)

DULFIKAR JAWAD HASHIM

calon untuk Ijazah PhD
(candidate for the degree of)

telah mengemukakan tesis / disertasi yang bertajuk:
(has presented his/her thesis / dissertation of the following title):

"APPLICATION OF NEW HOMOTOPY ANALYSIS METHOD AND OPTIMAL HOMOTOPY ASYMPTOTIC METHOD FOR SOLVING FUZZY FRACTIONAL ORDINARY DIFFERENTIAL EQUATIONS"

seperti yang tercatat di muka surat tajuk dan kulit tesis / disertasi.
(as it appears on the title page and front cover of the thesis / dissertation).

Bahawa tesis/disertasi tersebut boleh diterima dari segi bentuk serta kandungan dan meliputi bidang ilmu dengan memuaskan, sebagaimana yang ditunjukkan oleh calon dalam ujian lisan yang diadakan pada : ***24 Mac 2022.***
That the said thesis/dissertation is acceptable in form and content and displays a satisfactory knowledge of the field of study as demonstrated by the candidate through an oral examination held on:
24 March 2022.

Pengerusi Viva: *(Chairman for VIVA)*	Prof. Dr. Haslinda Ibrahim	Tandatangan *(Signature)*
Pemeriksa Luar: *(External Examiner)*	Assoc. Prof. Dr. Farah Aini Abdullah	Tandatangan *(Signature)*
Pemeriksa Dalam: *(Internal Examiner)*	Assoc. Prof. Dr. Nazihah Ahmad	Tandatangan *(Signature)*
Nama Penyelia/Penyelia-penyelia: *(Name of Supervisor/Supervisors)*	Dr. Ali Fareed Jameel	Tandatangan *(Signature)*
Nama Penyelia/Penyelia-penyelia: *(Name of Supervisor/Supervisors)*	Dr. Teh Yuan Ying	Tandatangan *(Signature)*

Tarikh:
(Date) ***24 March 2022***

Permission to Use

In presenting this thesis in fulfillment of the requirements for a postgraduate degree from Universiti Utara Malaysia, I agree that the Universiti Library may make it freely available for inspection. I further agree that permission for the copying of this thesis in any manner, in whole or in part, for scholarly purpose may be granted by my supervisor or, in their absence, by the Dean of Awang Had Salleh Graduate School of Arts and Sciences. It is understood that any copying or publication or use of this thesis or parts thereof for financial gain shall not be allowed without my written permission. It is also understood that due recognition shall be given to me and to Universiti Utara Malaysia for any scholarly use which may be made of any material from my thesis.

Requests for permission to copy or to make other use of materials in this thesis, in whole or in part, should be addressed to :

Dean of Awang Had Salleh Graduate School of Arts and Sciences
UUM College of Arts and Sciences
Universiti Utara Malaysia
06010 UUM Sintok

Abstrak

Fenomena fizikal yang kompleks dengan sifat keturunan serta ketidakpastian diiktiraf untuk dihuraikan dengan baik menggunakan persamaan pembeza biasa pecahan kabur (PPBPK). Pendekatan analitik untuk menyelesaikan PPBPK bertujuan untuk memberikan penyelesaian bentuk tertutup yang dianggap sebagai penyelesaian tepat. Walau bagaimanapun, bagi kebanyakan PPBPK, penyelesaian analitik tidak mudah diperolehi. Selain itu, kebanyakan fenomena fizikal yang kompleks cenderung kepada ketiadaan penyelesaian analitikal. Pendekatan penganggaran boleh menangani kelemahan ini dengan menyediakan penyelesaian bentuk terbuka, dengan beberapa PPBPK dapat diselesaikan menggunakan kaedah-kaedah dalam kelas penganggaran berangka. Walau bagaimanapun, kaedah tersebut kebanyakannya digunakan untuk masalah linear atau yang dilinearkan dan tidak dapat menyelesaikan PPBPK bertertib tinggi secara langsung. Sementara itu, kaedah kelas anggaran-analitik di bawah pendekatan penganggaran bukan sahaja terpakai untuk PPBPK tak linear tanpa memerlukan pelinearan atau pendiskretan tetapi juga mempunyai keupayaan untuk menentukan ketepatan penyelesaian tanpa memerlukan penyelesaian tepat untuk perbandingan. Walau bagaimanapun, kaedah-kaedah anggaran-analitik sedia ada tidak dapat memastikan penumpuan penyelesaian. Namun begitu, untuk menyelesaikan persamaan pembeza biasa pecahan bukan kabur, wujud kaedah berasaskan gangguan: kaedah analisis homotopi pecahan (KAH-P) dan kaedah asimptotik homotopi optimum pecahan (KAHO-P), yang memiliki keupayaan kawalan penumpuan. Oleh itu, penyelidikan ini bertujuan untuk membangunkan kaedah anggaran-analitik baru yang berpenumpuan terkawal: KAH-P kabur (KAH-PK) dan KAHO-P kabur (KAHO-PK), untuk menyelesaikan masalah nilai awal biasa pecahan kabur tertib pertama dan kedua serta masalah nilai sempadan biasa pecahan kabur. Dalam pembangunan teori, pemantapan penumpuan penyelesaian dibangunkan berdasarkan parameter kawalan penumpuan. Dalam kerja eksperimen, penumpuan penyelesaian ditentukan dengan menggunakan sifat nombor kabur. KAH-PK dan KAHO-PK bukan sahaja dapat menyelesaikan masalah tak linear yang sukar bahkan juga mampu menyelesaikan masalah bertertib tinggi secara langsung tanpa menurunkannya ke sistem tertib pertama. Kajian perbandingan menunjukkan prestasi cemerlang bagi kaedah yang dibangunkan berbanding dengan kaedah lain, dengan KAH-PK dan KAHO-PK secara individunya unggul dari segi ketepatan.

Kata kunci: Persamaan pembeza biasa pecahan kabur, Kaedah analisis homotopi (KAH), Kaedah asimptotik homotopi optimum (KAHO), Kaedah penganggaran, Kaedah penganggar-analitik.

Abstract

Physical phenomena that are complex and have hereditary features as well as uncertainty are recognized to be well-described using fuzzy fractional ordinary differential equations (FFODEs). The analytical approach for solving FFODEs aims to give closed-form solutions that are considered exact solutions. However, for most FFODEs, the analytical solutions are not easily derived. Moreover, most complex physical phenomena tend to lack analytical solutions. The approximation approach can handle this drawback by providing open-form solutions where several FFODEs are solvable using the approximate-numerical class of methods. However, those methods are mostly employed for linear or linearized problems, and they cannot directly solve FFODES of high order. Meanwhile, the approximate-analytic class of methods under the approximation approach are not only applicable to nonlinear FFODEs without the need for linearization or discretization, but also can determine solution accuracy without requiring the exact solution for comparison. However, existing approximate-analytical methods cannot ensure convergence of the solution. Nevertheless, to solve non-fuzzy fractional ordinary differential equations, there exist perturbation-based methods: the fractional homotopy analysis method (F-HAM) and the optimal homotopy asymptotic method (F-OHAM), that possess convergence-control ability. Therefore, this research aims to develop new convergence-controlled approximate-analytical methods, fuzzy F-HAM (FF-HAM) and fuzzy F-OHAM (FF-OHAM), for solving first-order and second-order fuzzy fractional ordinary initial value problems and fuzzy fractional ordinary boundary value problems. In the theoretical development, the establishment of the convergence of the solutions is done based on the convergence-control parameters. In the experimental work, the convergence of solutions is determined using properties of fuzzy numbers. FF-HAM and FF-OHAM are not only able to solve difficult nonlinear problems but are also able to solve high-order problems directly without reducing them into first-order systems. The developed methods demonstrate the excellent performance of the developed methods in comparison to other methods, where FF-HAM and FF-OHAM are individually superior in terms of accuracy.

Keywords: Fuzzy fractional ordinary differential equations, Homotopy analysis method (HAM), Optimal homotopy asymptotic method (OHAM), Approximation methods, Approximate-analytical methods.

Acknowledgement

Firstly, and last Alhamdulillah First and foremost, I would like to thank Allah S.W.T for giving me the strength, health, and wellness to finish this dissertation. I would like to express my special appreciation and thanks to some of whom it is possible to give a particular mention here. First and foremost, all praises and thanks to the Almighty Allah SWT for granting me with patience, guidance, and health, as well as giving me the chance to work in an environment such as University Utara Malaysia (UUM) and School of Quantitative sciences particularly. Secondly, I would like to express my sincere and utmost gratitude to my amazing supervisors, Dr. Ali Fareed, and Dr. Teh Yuan Ying for the patience, guidance, encouragement, and advice has provided throughout my time as the student. My gratitude also goes to lecturers, administrative, and technical staff for providing a conducive environment and support during my study. I would like to thank those who are close to my heart; my big family, my darling father, and my mother for their continued support, my dearest brothers, and precious sisters, for making my dream comes true, and keep motivating me throughout this journey, Finally, my sincerest regards to Dr. Abdelkarim Alomari for his moral and scientific support.

Table of Contents

List of Tables

List of Figures

List of Abbreviations

ODE	Ordinary Differential Equation
IVP	Initial Value Problem
BVP	Boundary Value Problem
FODE	Fractional Ordinary Differential Equation
FOIVP	Fractional Ordinary Initial Value Problem
FOBVP	Fractional Ordinary Boundary Value Problem
FFOIVP	Fuzzy Fractional Ordinary Initial Value Problem
FFODE	Fuzzy Fractional Ordinary Differential Equation
FFOBVP	Fuzzy Fractional Ordinary Boundary Value Problem
HAM	Homotopy Analysis Method
F-HAM	Fractional Homotopy Analysis Method
FF-HAM	Fuzzy Fractional Homotopy Analysis Method
OHAM	Optimal Homotopy Asymptotic Method
F-OHAM	Fractional Optimal Homotopy Asymptotic Method
FF-OHAM	Fuzzy Fractional Optimal Homotopy Asymptotic Method
RKHSM	Reproducing Kernel Hilbert Space Method
FRPSM	Fractional Residual Power Series Method
SCM	The spectral collocation method

CHAPTER ONE
INTRODUCTION

1.1 Background of the Study

Classical calculus provides a powerful tool in the modelling of dynamic processes. However, there are many complex systems with anomalous dynamics in nature, possessing hereditary properties of various materials and processes (Cui et al., 2018). For such systems, classical models are often not enough to describe their features. Fractional-order models are more accurate than integer-order models since there are more degrees of freedom in the fractional-order models. The fractional calculus apparently captures some of the hereditary properties in the system (Failla & Zingales, 2020). Fractional calculus is not modern; it is a generalization of the traditional calculus theory which deals with the integer order (Machado et al., 2014). In fractional calculus, the derivative and integral found in classical calculus are generalized to arbitrary real or complex order, that is, to non-integer order (Dalir & Bashour, 2010). The beginning of the theory of fractional calculus dated back to the seventeenth century when Leibniz wrote to L'Hôpital in the year 1695 to tell him about the derivative $\frac{d^{(\beta)}}{dx^{(\beta)}}$ of order $\beta = 0.5$. This letter marked the first appearance of fractional calculus (Dalir & Bashour, 2010).

Whilst classical calculus has unique definitions and clear physical as well as geometrical interpretations for the integer order derivatives and integrals, definitions for the derivative and integral of factional order are not unique where several definitions have been proposed since 1695 (Li & Deng, 2007). The definitions include Riemann-Liouville (Li et al., 2011), Caputo (Li et al., 2011), Riesz (Çelik & Duman, 2012), Erdelyi-Kober (Pagnini, 2012), Jumarie (Atangana & Secer, 2013), Davison and Essex (Atangana & Secer, 2013), Coimbra (Abdelkawy et al., 2015), and Hadamard (Matar, 2018). Among those definitions, although Caputo and Reimann-Liouville are the commonly used, Caputo is deemed more suitable for dealing with the initial as well as boundary value conditions and unlike Reimann-Liouville's, the Caputo definition for constant function is zero (Li et al., 2011).

Although classical calculus is a powerful tool for modelling many complex real phenomena, fractional calculus is considered an indispensable instrument in handling such complex phenomena that are based on long memory terms (Bonyah et al., 2019). Memory refers to the output or results at the present time that are dependent on the history of the variables at the previous time. Problems that depend on the memory of the variables cannot be handled by classical calculus (Picozzi & West, 2002). Fractional ordinary differential equations (FODEs), which are the generalization of the classical ordinary differential equations (ODEs) to a non-integer order, describe dynamically changing systems (Somathilake, 2020). Moreover, FODEs have gradually gained considerable interest from researchers and practitioners over the years due to their capability to model and explain complex phenomena or systems, making the nature of such systems closely understandable (Troparevsky et al., 2019). In addition, FODEs exhibit non-local property, as opposed to ODEs having derivatives of integer order which exhibit local property. Local property signifies that the integer derivative at a certain time is dependent on the current state, whereas non-local property indicates that the fractional derivative at a certain time depends on historical states. Therefore, models built from FODEs behave more like the real process since the aftereffect is included in the inner dynamics (Scalas et al., 2000). In dealing with FODE models, employing Caputo fractional derivatives has unique advantages, as it allows for the inclusion of supplementary conditions in the problem structure, it is suitable for structuring realistic problems. Furthermore, the Caputo derivative for constant function is zero, which means that the rate of change in physical systems can be appropriately modelled under the Caputo sense of differentiability. The Riemann-Liouville fractional derivatives generally cause difficulty in describing and modelling complicated real phenomena. On the other hand, Caputo derivatives, which is a special case of Riemann-Liouville's, tend to be more efficient when dealing with real-world problems (Matlob & Jamali, 2019).

FODE involves fractional derivatives of the form $\frac{d^{(\beta)}}{dx^{(\beta)}}$, which are defined for $\beta > 0$, where β is not necessary an integer (Băleanu & Mustafa, 2010). FODEs can be classified depending on the underlying definition of non-integer derivative used in equation. For example, Eq. (1.1) below (Patrício et al., 2019)

$$D^{(\beta)}y(x) = g\big(x, y(x)\big), x_0 \le x \le X \tag{1.1}$$

is called FODE of the Caputo type when it contains a fractional derivative operator of order β in Caputo sense of differentiability, and defined as

$$D^{(\beta)}y(x) = I^{(m-\beta)}\left(\frac{d^m}{dx^m}y(x)\right) = \frac{1}{\Gamma(m-\beta)}\int_0^x (x-t)^{m-\beta-1}y^{(m)}(t)\,dt, \tag{1.2}$$

where $m-1 < \beta \le m$, $m \in \mathbb{N}$, such that $m-1$ and m refer to the local integer derivative, and $x > 0$ (Garrappa, 2018). Furthermore, if the Eq. (1.1) subjects to the initial condition

$$y(x_0) = y_0, \tag{1.3}$$

Then both Eq. (1.1) and Eq. (1.3) constitute a typical fractional initial-value problem (IVP) involving ordinary differential equation in the Caputo sense of differentiability.

Likewise, imposing boundary conditions to FODE in Eq. (1.1) would generally yield a fractional boundary value problem (BVP) involving ordinary differential equation. IVPs and BVPs have been the focus of many FODE studies (Cui et al., 2018; Demirci & Ozalp, 2012; Li & Zeng, 2013; Pakdaman et al., 2017).

Although FODE is capable of handling hereditary properties in a system, for the aim of understanding the system or phenomena involving vagueness, the structuring of mathematical models under uncertainty is considered sensitive and important. The description of real-world problems usually depends on the solutions of mathematical models of the phenomena (Frolov et al., 2020). Therefore, a major challenge encountered by a model builder would be to describe imprecise concepts of uncertainty in an accurate form to some extent (Mansour et al., 2019). Fuzzy set theory would be the tool required to handle such uncertainty. The development of fuzzy set theory began in 1965 when Lotfi Asker Zadeh introduced the theory of fuzzy sets in a published article (Zadeh, 1978). Chang and Zadeh (1996) proposed the concept of fuzzy numbers and explored the notion of the arithmetic operations on fuzzy numbers (Chang & Zadeh, 1996). In 1972, Chang and Zadeh put forward the notion of the fuzzy derivative for the first time. The extension principle theory was used by Dubois and Prade in their approach in 1982 (Dubois & Prade, 1982). Fuzzy ODEs take into account the information of uncertainty about the behaviour of a dynamical system to obtain a more realistic and flexible model (James et al,. 2001). The idea of the fuzzy ODEs is to form a suitable setting for mathematical modelling of real-world problem (Ngan et al., 2020). Fuzzy ODEs have been broadly studied after the first invention of the notion of fuzzy differential equations (Kandel & Byatt, 1980). However, the earlier discussed problems were strictly not fuzzy differential equations, since the theory of fuzzy set was not yet clear and specific at the time. It was only after the introduction of Hukuhara derivatives that fuzzy ODEs were developed (Dubois & Prade, 1982). FODE was extended to fuzzy fractional ordinary differential equation (FFODE), making its first appearance in 2010 (Agarwal et al., 2010). That study paved the way for studies on FFODEs by the introduction of the fractional calculus theory in a fuzzy setting to deal with FFODEs.

With the presence of uncertainty, the initial value in Eq. (1.3) might not be precisely identified, allowing one to assume any value in the form of "less than," "around," or "more than" (Mosleh & Otadi, 2015). Therefore, it has been suggested to incorporate fuzzy set theory into FODE, and subsequently, FFODE has been introduced as a more realistic model to cope with the presence of uncertainty (Bencsik et al., 2006).

For the purpose of constructing mathematical model of FFODEs, Eq. (1.1) can be formulated from crisp domain to fuzzy environment as follows (Abdul Rahman & Ahmad, 2017):

$$D^{(\beta)}\tilde{y}(x) = g\big(x, \tilde{y}(x)\big), x_0 \leq x \leq X. \quad (1.4)$$

Furthermore, if Eq. (1.4) is subject to the fuzzy initial condition of

$$\tilde{y}(x_0) = \tilde{y}_0. \quad (1.5)$$

If Eq. (1.4) constitutes a typical fuzzy fractional ordinary initial-value problem (FFOIVP), involving an ordinary differential equation in the Caputo sense of differentiability, Likewise, imposing fuzzy boundary conditions on the FFODE in Eq. (1.1) would generally yield a fuzzy fractional ordinary boundary value problem (FFOBVP).

1.2 Problem Statement

FFODE, which is well-known as a powerful mathematical tool, has been utilized in modelling numerous phenomena and systems with uncertainty in application areas including nuclear physics (Das et al., 2013; Salahshour et al., 2015), viscosity (Ahmadian et al., 2017; Sin et al., 2018), liquid kinetic (Ahmadian et al., 2015), robotics (Deng, 2019; Efe, 2008), dynamic system (Najariyan & Zhao, 2018; H. Sun et al., 2010), HIV (Wasques et al., 2020) and Covid-19 (Alderremy et al., 2021). However, FFODEs remain impractical until they are solved, due to where the availability of solutions would allow scientists and engineers to make reasonable predictions about the phenomena being studied. Solving FFODEs is considered difficult due to the complexity of parameters of uncertainty in addition to the fractional operators involving the equations.

In solving FFODEs, there are two main approaches: the analytical approach and the approximation approach. In the analytical approach, the aim is to present a closed-form solution. A closed-form solution is considered the exact solution to the problem (Kudryashov, 2020). The solution may be expressed as the sum of a finite number of elementary functions such as polynomial, exponential, trigonometric, and hyperbolic functions. The advantage of a closed-form solution is that it provides an overall view of the solution to the problem. Moreover, in the analysis of results, using the closed-form solutions generally does not require a huge amount of computation (Bulut et al., 2013).

Within the analytical approach, there are two main classes of methods: analytic-analysis class and analytic-transformation class. Many studies have been devoted in order to find the analytical solution of FFODEs for example, in the study

(Allahviranloo et al., 2012) has extracted the analytical solution of FFODEs, while another study (Vu et al., 2019) has established an analytical method to solve linear FFOIVP and introduced the existence and uniqueness results of FFODEs. Other analytical methods include the studies of (Allahviranloo et al., 2013; Hoa et al., 2019; Salahshour et al., 2012). This class of methods focuses on framing the physical and engineering problems in terms of well-understood mathematical functions where the exact solution of those functions can be calculated for specific instances (Dehghan et al., 2011). However, since it could be challenging to obtain analytic solution through problem analysis, the analytic-transformation class of methods can be used such as Laplace and Sumudu transforms where FFODEs are simplified into a solvable problem (Allahviranloo et al., 2009). Nevertheless, the obtained solutions are generally limited to the linear category of FFODEs (Panahi, 2017). It is unfortunate that most of the complex physical phenomena described using nonlinear FFODEs lack analytical solutions (Hasan et al., 2017). In many instances, analytical solutions cannot be found (Ghanbari & Akgul, 2021; Verma & Kumar, 2020). Nevertheless, the solutions to such equations are always in demand due to practical interests, such as the Riccati differential equation. Therefore, to deal with such instances in a more realistic manner, FFODEs are commonly solved using the approximation approach, which includes the approximate-numerical class of methods (Hasan et al., 2017).

The approximate-numerical class of methods give solutions approximately in numeric value form where estimation of errors utilizes arithmetic procedures (Moore & Ertürk, 2020). The enormous growth of technological speed and memory capacity in computational capability has paved the way towards the advancement and development of methods of approximation approach. Employment of the numerical methods for FFODEs began when the numerical solution of linear FFOIVPs was introduced using the extension of classical Euler method to the fuzzy setting (Ahmad et al., 2013). In the approximation approach, the aim is to obtain an approximate solution, where an open-form solution is sought instead of a closed-form. However, approximate-numerical class of methods cannot solve FFODEs of high order directly, instead they require transformation to system of first-order. Most studies employ approximate-numerical class of methods for linear first-order problems (Ahmadian et al., 2013; Mazandarani & Kamyad, 2013; Prakash et al., 2015).

Another class of methods under the approximation approach is the approximate-analytic class. Unlike the approximate-numerical class of methods, this class of methods is applicable to linear and nonlinear FFODEs without the need for linearization or discretization (Hasan et al., 2017; Khodadadi & elik, 2013). The solution will be given in a polynomial function series, making it easy to present in graphical form and to demonstrate the degree of solution and convergence of the solution. The methods have the ability to determine the accuracy of the obtained solution without requiring the exact solution for comparison. The methods under this approximate-numerical class include the variational iteration method (Panahi, 2017), the spectral collocation method (Esmaeilbeigi et al., 2018), the reproducing kernel Hilbert Space Method (Abdel Aal et al., 2019) and the Adomian decomposition method (Askari et al., 2019). These methods, however, are incapable of simplifying complex nonlinear mathematical problems (He, 2004).

Perturbation-based methods, which contribute through simplification of complex nonlinear problems, have initially been developed for handling weak nonlinear problems (Liao, 2004). A perturbation-based method called the Modified Homotopy Perturbation Method manages to provide a simple function that is asymptotically equivalent to the solution of the given FFODE problem, that is, by seeking an expansion with respect to the asymptotic sequence (Khan et al., 2014). However, this perturbation-based approximate-numerical method can only be used with existing small or large physical or engineering parameters in the equation (Liu et al., 2017). Moreover, all those existing approximate-analytical methods do not provide freedom to choose the auxiliary functions, and it is even more unfortunate that those methods cannot ensure the convergence area of the solution (Bahia et al., 2021).

Fortunately, there exist methods in non-fuzzy FODEs that have been found to be capable of ensuring convergence of solution using the concept of homotopy analysis method (HAM) and optimal homotopy asymptotic method (OHAM). The methods function on a perturbation-based technique with the ability to control the convergence of the solution of the FODEs where these methods have been found to be beneficial in overcoming the difficulty in ensuring convergence. In classical ODEs, HAM and OHAM have been used while fractional HAM (F-HAM) and fractional OHAM (F-OHAM) have been constructed for non-fuzzy FODEs. HAM, OHAM, F-HAM and F-OHAM are independent for small or large physical parameters because of the homotopy formulation. For non-fuzzy FODEs, the utilization of F-HAM and F-OHAM, which are each based on the auxiliary parameter, have been known to provide not only an easy way but also a fast way to control and modify the convergence region to optimize the series solutions. The approximate solution is gained in the form of series which rapidly converges to the exact solution (Veeresha et al., 2019). F-HAM and F-OHAM does not include discretization of variables, making the methods free from round-off errors. Several studies have utilized the concept of HAM (Jena et al., 2019; Kumar et al., 2020) and OHAM (Hamarsheh et al., 2015; Jameel et al., 2018).

Thus, the concept of HAM and OHAM has the potential to be beneficial for FFODEs as well.

Therefore, this study aims to develop new approximate-analytical methods with convergence-control ability in solving linear and nonlinear FFODEs. The concept of HAM and OHAM will be used to develop the new methods where the developed methods would be capable of handling the difficulty of controlling convergence of the approximate-analytical solutions. The study focuses on the development of FF-HAM and FF-OHAM on three types of problems involving FFODEs: first-order FFOIVPs, second-order FFOIVPs and second-order FFOBVPs under Caputo definitions of differentiability.

1.3 Research Questions

1. What are the apropriate methods for solving first-order and second-order FFOIVPs?
2. What are the apropriate methods for solving second-order FFOBVPs?
3. How to establish the convergence of solutions of the constructed methods based on the convergence-control parameters for each method?
4. Do the proposed methods perform better in term of compared to existing methods in solving FFOIVPs and FFOBVPs?

1.4 Objectives

The aim of this research is to develop new convergence-controlled approximate-analytical methods for solving first-order and second-order fuzzy fractional ordinary initial value problems (FFOIVPs) and fuzzy fractional ordinary boundary value problems (FFOBVPs), which can be achieved by accomplishing the following objectives:

1. To formulate fuzzy fractional homotopy analysis method (FF-HAM) and fuzzy fractional optimal homotopy asymptotic method (FF-OHAM) for solving first-order and second-order FFOIVPs
2. To formulate FF-HAM and FF-OHAM for solving second-order FFOBVPs.
3. To establish the convergence of solutions of the constructed methods based on the convergence-control parameters for each method.
4. To conduct comparative studies between the constructed methods and several existing methods in solving FFOIVPs and FFOBVPs in term of accuracy.

1.5 Scope of the Study

The study focuses on the development of FF-HAM and FF-OHAM on three types of problems involving FFODEs: first order FFOIVPs, second order FFOIVPs, and second-order FFOBVPs under the Caputo definitions of differentiability. Experimental work would be carried out on several examples of linear as well as nonlinear physical and engineering problems. The experimental work on examples of FFODEs in this thesis will be conducted by using Mathematica 12 with an HP Laptop (i7-5500U CPU @ 2.40 GHz, 8.00 GB RAM).

1.6 Significance of the Study

The proposed approximate-analytical methods: FF-HAM and FF-OHAM, would give significant contribution towards overcoming obstacles of existing methods such as VIM, ADM and etc for example the proposed methods will help us to simplify the complexity of the uncertain nonlocal derivative when solving FFODEs, furthermore, unlike all existing methods FF-HAM and FF-OHAM providing the convergence parameters, thus ensuring the convergence of the approximate series solutions. The proposed methods would pave the way towards setting reliable mathematical models of nonlinear form or of high-order to describe the complex real problems in a way that makes the model more realistic. The paved way would be useful to scientists and engineers in handling physical and engineering problems in terms of FFODEs that build on the imprecise concepts under the effect of long memory, referring to the hereditary properties of various materials and processes, which is difficult to accurately describe using conventional models. Solving problems with the proposed methods can help engineers or scientists understand a physical problem better, and thus would help them improve future procedures and designs used to lay out their problems that because of the series solutions provided by the proposed methods which provide us the ability to show the graphical designs of the solutions.

1.7 Organization of the Thesis

This thesis contains seven chapters:

Chapter 1 includes the research background, problem statement, objectives, scope of the study, the significance of the study as well as the thesis organization.

Chapter 2 discusses the literature review of FFODEs including the two approaches of solving methods and the relevant mathematical background pertaining to the proposed approximate-analytic methods.

Chapter 3 lays out the research methodology in solving the three types of FFODEs: first-order FFOIVPs, second-order FFOIVPs and second-order FFOBVPs. The methodology involves two main parts: theory and experiment.

Chapter 4 is devoted to the development of the proposed methods: FF-HAM and FF-OHAM, in solving first-order FFOIVPs. The theoretical part is discussed first followed by the experimental part.

Chapter 5 focuses on the development of the proposed methods: FF-HAM and FF-OHAM, in solving second-order FFOIVPs. Similar to Chapter 4, the theoretical part is discussed first followed by the experimental part.

Chapter 6 demonstrates the development of FF-HAM and FF-OHAM for solving second-order FFOBVPs, which is the third type of FFODEs considered in this study. Similarly, the experimental part is discussed after the theoretical part.

Chapter 7 contains the conclusion of the study providing the summary, contribution and limitations of the study. Suggestions for future research are also provided.

CHAPTER TWO
LITERATURE REVIEW

2.1 Introduction

This chapter discusses fuzzy fractional ordinary differential equations (FFODEs) and the two approaches of solving them. The classes of existing methods under each approach for FFODEs are also reviewed. A section is focused on the review of the existing concept for non-fuzzy FODEs which serves as the basis for the proposed methods for solving FFODEs.

2.2 Fuzzy Fractional Ordinary Differential Equations

The general formulation of a fuzzy fractional ordinary initial value problems (FFOIVPs) is as given in Eq. (1.4) and Eq. (1.5). Similarly, the general formulation of fuzzy fractional boundary value problem (FFOBVP) involving ordinary differential equation constitute imposing boundary conditions to the FFODE in Eq. (1.4).

Memory and hereditary properties of materials and processes are widely recognized to be well predicted using fuzzy fractional differential equations (Agarwal et al., 2018). Usually, data for real scientific and technological processes are pervaded under uncertainty, which may arise in the experiment part, data collection, measurement process or in determining the initial values. Such properties have been known to arise in physical sciences (Ahmadian et al., 2013; Sin et al., 2018), engineering (Alshorman et al., 2018; Rashid et al., 2021; Zhu, 2015) and medical (Ullah et al., 2020; Wasques et al., 2020) requiring the solution methods of FFODEs.

2.3 Solution Methods of FFODEs

There are numerous existing methods in solving FFODEs. In general, the methods can be divided into two approaches: analytical approach and approximate approach.

2.3.1 Analytical Approach

The analytical approach aims towards finding an exact solution. There are two classes of methods under the analytical approach to solve FFODEs: analytic-analysis and analytic-transform.

2.3.1.1 Analytic-Analysis Class of Methods

Analytic-analysis methods act on framing the physical and engineering problems in a well-understood mathematical formulation of which the exact solution can be evaluated using the mathematical function for specific instances.

An analytic-analysis method frames the linear FFODEs of order $0 < \beta < 1$ in a mathematical formulation using Mittag-Leffler function (Allahviranloo et al., 2012). The analytic-analysis solution has been obtained where the method utilizes the sense of Riemann-Liouville-H differentiability. Two examples of homogenous and inhomogeneous nuclear decay equations have been used to illustrate the capability of the this analytic-analysis method. There is also a different analytic-analysis method, whose solution is framed on the solution for solving fuzzy differential equations of integer order (Hoa et al., 2018). That method employs derivatives in the sense of Caputo–Katugampola in solving FFOIVPs. Demonstration of the existence and uniqueness of the analytic-analysis solution obtained has also been presented.

Another analytic-analysis method for solving random FFODEs has also acted on framing the fuzzy fractional problem on fuzzy differential equations of integer order (Vu et al., 2019). That method, which utilizes the derivatives in the sense of Riemann-Liouville in solving FFOIVPs, has also shown the existence and uniqueness of the solution of an FFOIVP for random FFODEs. Furthermore, it has been used to analytically solve the model of fish proportional harvesting in two cases of uncertainties: randomness and fuzziness.

There has also been a development of a different analytic-analysis method, whose solution is framed on power series expansion (Shahidi & Khastan, 2018). A new fuzzy fractional derivative has also been developed and utilized in finding the analytic-analysis solution. Experimental work involving first-order exponential function has been presented illustrating the capability of that analytic-analysis method.

2.3.1.2 Analytic-Transform Class of Methods

Analytic-Transform Methods aims on simplifying the process of obtaining an exact solution by transforming the FFODEs into a solvable and straight forward algebraic problem. Furthermore, even if the transformed problem involves complicated algebraic formulation, the inverse transformation tends to provide easier ways to solve FFODEs. Figure 2.1 depicts the transformation procedures for analytic-transform class of methods.

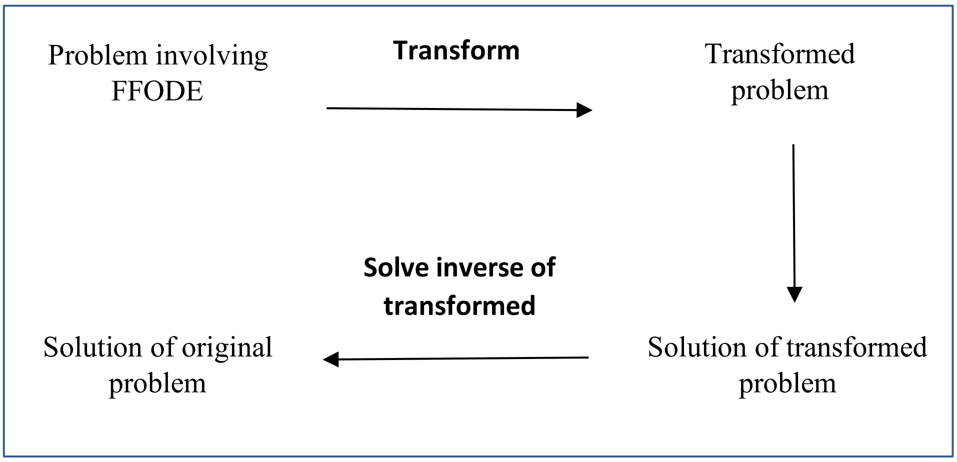

Figure 2.1. Transformation procedures for analytic-transform class of methods

The utilization of fuzzy Laplace transform method has been introduced to find the analytical solutions for FFODEs of order $0 < \beta < 1$ under Riemann–Liouville H-differentiability (Salahshour,. 2012). In that study, the Laplace transform $[L(f(x))]$ and the inverse Laplace transformation $[L^{-1}(f(x))]$ have been used to derive the analytical solution of FFOIVPs. The capability of the method is demonstrated by solving two linear FFOIVPs. The method has been applied to the fuzzy nuclear decay equation of fractional order. In several other studies, the analytic-transform method using fuzzy Laplace transform has been used to solve FFODEs with the employment of fuzzy fractional Caputo differentiability (Allahviranloo et al., 2013), (Takači et al., 2014), and (Das & Roy, 2017). Furthermore, in finding the analytical-transform solutions for several linear FFODEs, another study has also employed the Laplace transform but with the employment of the Caputo generalized Hukuhara derivative (Abdollahi et al., 2019).

Other than Laplace transforms, Sumudu transforms are also considered the best-known methods to simplify FFODEs into a solvable algebra problem. The fuzzy Sumudu transform method has been used to solve linear FFOIVPs under the sense of the fuzzy fractional Caputo derivative (Abdul Rahman & Ahmad, 2017). In addition, new findings regarding the property of fuzzy Sumudu transform method for fuzzy fractional Caputo derivative sense have also been suggested in that study.

An analytic-transform method called Fractional Differential Transform Method has been utilized to obtain analytical solution for FFOBVPs (Mohammed & Ahmed, 2013). Several linear and nonlinear second-order problems have been solved to illustrate the capability of that analytic-transform method, which employs the Caputo fractional derivative. Another analytic-transform method called Fuzzy Generalized Differential Transform Method has been presented to solve fuzzy fractional Basset and Logistic equations using the Caputo fractional derivative (Rivaz et al., 2016). Similar to other studies, analytical solutions of several linear as well as nonlinear examples have also been obtained to demonstrate the simplicity and the capability of the method.

2.3.2 Approximation Approach

The approximation approach aims towards finding an approximate solution, where instead of finding a closed-form solution, it finds an open-form solution. The enormous growth in technological speed and memory capacity in computational capability has paved the way towards advancement and the development of methods of approximation approach. To solve FFODEs, the approximation approach has two groups of methods: approximate-numerical and approximate-analytical**.**

2.3.2.1 Approximate-Numerical Class of Methods

The approximate-numerical class of methods work on the reformulation of the physical, engineering and other real-world problems (Li & Chen, 2018), through which the approximate solution can be determined by means of arithmetic procedures which can be used to estimate the errors (Moore & Ertürk, 2020).

Several extensions of the classical Euler method include fractional Euler method (Ahmad et al,. 2013) and modified fractional Euler method (Mazandarani & Kamyad, 2013) where the methods have been used to solve linear first-order FFOIVPs. Other approximate-numerical methods are operational matrix method (Ahmadian et al., 2013), fractional Jacobi operation matrix method (Sin et al., 2017), Adam-Bash Forth method (Raj & Saradha, 2015) and predictor-corrector method (Prakash et al., 2015). Those methods have been proven for solving first-order FFOIVPs while the Jacobi operation matrix method has also been applied to a physical problem.

2.3.2.2 Approximate-Analytical Class of Methods

The second class of methods under the approximate approach to solving FFODEs are the approximate-analytical methods. For a purely analytical solution, the exact or closed-form solution fulfils the initial or boundary conditions of the FFODEs exactly when it is substituted back. Meanwhile, the approximate-analytical solution approximates the exact solution of an FFODE as an open-form solution with some additional conditions for some level of precision.

There are several methods in the approximate-analytical class of methods, including the Variational Iteration Method (VIM), Reproducing Kernel Hilbert Space Method (RKHSM), Spectral Collocation Method (SCM), Residual Power Series Method (RPSM), Adomian Decomposition Method (ADM), Fractional Residual Power Series Method (FRPSM) and the Modified Homotopy Perturbation Method (MHPM).

VIM has been introduced to solve FFIOVP of order $0 < \beta < 1$, illustrated by solving two examples involving non-fuzzy fractional differential equations with fuzzy initial conditions (Khodadadi & Çelik, 2013). To show the validity and applicability of the method, the approximate findings have been compared with the exact solutions of the given problems where the obtained results from the method using the derivative of fractional Riemann-Liouville have been found to be efficient, accurate, and convenient for dealing with the FFODEs. In another study, VIM has been employed to approximate the analytical solution of nonlinear FFOIVPs (Panahi, 2017). The algorithm was based on a theorem that makes FFODEs equal to the system of the ODEs. The proposed method, whose algorithm is based on a theorem that makes FFODEs equal to the system of the ODEs, has been illustrated by solving two examples with the initial condition as a fuzzy number in order to establish the capability to solve FFODEs.

Another approximate-analytical method is RKHSM. In a study, a new algorithm based on the characterization theorem has been introduced to approximate the analytical solution of the second order FFOIVPs by utilizing the method (Hasan et al., 2017). In that study, the extension of the fuzzy fractional Caputo derivative of order has been introduced to second order by using the concept of generalized Hukuhara differentiability. In a different study, RKHSM has been used to approximate the analytical solution of linear and nonlinear models where the study has also presented a new formulation of multi-fractional order of FFOBVPs under the Caputo sense of differentiability (Abdel Aal et al., 2019).

SCM is a method based on radial base functions as suggested by Esmaeilbeigi et al. (2018) to solve fuzzy fractional Bagley-Torvik equation of order $1 < \beta < 2$ approximately under the fuzzy fractional Caputo differentiability. This approximate-analytical method has been employed to reduce the Bagley-Torvik equation to a different problem that is based on solving two systems of linear differential equations. Among the three examples illustrating the performance of this method, the first and second examples are the fuzzy fractional Bagley-Torvik equations with the inhomogeneous and homogeneous initial conditions, respectively, while the third example is the fuzzy fractional Bagley-Torvik equation with the homogeneous boundary conditions.

FRPSM has been introduced to find the solution of fuzzy fractional Riccati differential equation of order $0 < \beta \leq 1$ approximately under the sense of Caputo fractional differentiability where two examples of first order FFOIVPs have been solved to show the capability of the method (Alshorman et al., 2018).

Adomian decomposition method (ADM) has been adopted to approximate the analytical solution of fuzzy fractional Caputo differential equations (Askari et al., 2019). Furthermore, the uniformity of the convergence of the sequence $\{y_i\ (t)\}$ with several kinds of differentiability to the analytical solution also been demonstrated.

FRPSM is another method under the approximate-analytical class of methods. The method has been used to solve the linear FFOIVP of order $0 < \beta \leq 1$ approximately with 8 terms (Alaroud et al., 2019).

MHPM has been utilized to obtain approximate-analytical solutions of linear and nonlinear FFOIVPs (Khan et al., 2014). To illustrate the capability of MHPM, a comparison study has also been conducted by comparing the obtained approximate solution with the exact solution and also with other known numerical results obtained by fuzzy Euler method.

2.4 Solution Methods with Convergence-Control for non-fuzzy FODEs

There are two methods that have been used to solve non-fuzzy FODEs with the capability of controlling convergence of the solutions: Fractional Homotopy Analysis Method (F-HAM) and Fractional Optimal Homotopy Asymptotic Method (F-OHAM). F-HAM and F-OHAM are methods based on the concept of the original HAM and OHAM that have been used to solve ODEs.

2.4.1 Fractional Homotopy Analysis Method

F-HAM is a method developed from the concept of HAM. The original-HAM is a powerful approximate-analytical method for solving linear and nonlinear differential equations based on the homotopy notion in topology between two continuous functions. Liao (2004) has introduced the homotopy mapping between two continuous functions, such that $\mathcal{H}(x;q) \times [0,1] \rightarrow \mathbb{R}$ to generate a convergent series solution for nonlinear systems.

The concept of HAM has been successfully utilized to solve many nonlinear mathematical models involving ODE, such as original-HAM for classical ODEs (Tan & Abbasbandy, 2008), fuzzy-HAM for fuzzy ODEs (Otadi & Mosleh, 2016) and fractional-HAM for fractional ODEs (Hashim et al., 2009). In a study solving second-order fractional Bagley-Torvik equation, the approximate analytical solution using the concept of HAM has been obtained (Lee et al., 2016). For the experimental part of that study, although the example is stated as FFODEs, due to the crisp form of the initial conditions used and of the coefficients and inhomogeneous terms in the equation, the entire problem is actually non-fuzzy FODE rendering the method in that study as fractional HAM, that is, F-HAM. Furthermore, in the theoretical part of that study, the discussion is about the construction of F-HAM, which has already been done by Hashim et al. (2009).

The concept of HAM has been utilized in various application areas, such as fluid dynamics (Rana & Liao, 2019a), environmental engineering (Jain et al., 2021), disease modelling (Sotonwa & Obabiyi, 2019) and mechanics (Rana & Liao, 2019b). These successful applications have verified the validity, effectiveness, and flexibility of the original-HAM and F-HAM, which are characterized by many features over the other approximate methods, by the avoidance of physically unrealistic assumptions. Also, the original-HAM and F-HAM can yield very rapid convergence of the series solution, and in most cases, only a few iterations would lead to very accurate solutions. The methods allow for the solution of the differential equation to be calculated in the form of an infinite series in which the components can be easily calculated. The HAM and F-HAM utilize a simple method of adjusting and controlling the convergence region of the infinite series solution, that is, by using the convergence-control parameter. Therein lies the importance of these methods over the traditional approximate methods for ODEs and FODEs, especially when the analytic approximations given by the other analytic methods are divergent in the whole domain. Instead, one can gain a convergent series solution simply by choosing the proper auxiliary parameter provided by HAM or F-HAM.

2.4.2 Fractional Optimal Homotopy Asymptotic Method

After the first contribution of Liao (2004), the idea of homotopy in topology become a basic concept and has been widely applied in pure and applied mathematics. One of the limitations of the perturbation methods is to take advantage of the small or large parameter into the nonlinear equation as in the homotopy perturbation method (HPM) (Grover & Tomer, 2011). As a consequence, there was an urgent need to introduce a new approximate perturbation method independent of these parameters where the optimal homotopy asymptotic method (OHAM) was introduced by (Marinca et al., 2008) as a powerful method to allow the simplification of the nonlinear equations. The concept of OHAM provide interesting features that has been found to overcome the obstacles when solving powerful nonlinear differential equation. OHAM has built-in convergence criteria similar to HAM but with a greater degree of flexibility since with OHAM the convergence region can be easily adjusted and controlled. Unlike original-HAM and F-HAM, the original-OHAM and F-OHAM provides many convergence control parameters, making the methods using this concept more flexible for handling the strong nonlinearity with the differential equations. OHAM generally provides better accuracy in the same order of approximation and provide a simple way to ensure the convergence of the solution series. Furthermore, there is great freedom to choose the proper base functions approximating a nonlinear problem. OHAM has an advantage over the traditional methods in that it is independent of any small or large parameters, which in turn makes OHAM widely applicable without limitations.

Original-OHAM has been utilized for solving non-fuzzy ODEs (Mabood et al ,. 2013; Ilie et al,. 2019) and fuzzy-OHAM for solving fuzzy ODEs (Jameel et al., 2018). Practical applications in various sciences have proven the effectiveness of the concept of OHAM, such that for physical application (Shah et al., 2020), engineering (Manafian & Teymuri sindi, 2018) and fluid dynamic (Farooq et al., 2021).

2.5 Chapter Summary

1. Analytical methods:

   a) According to our literature in this chapter, no more than three works were employed the analytical analysis methods for solving linear FFODEs.

   b) Regarding to the analytical transform methods, Laplace transform method and Sumudu transform method considered the most popular methods for solving linear FFODEs.

   c) We conclude that, using the analytical methods for FFODEs requires special skills.

   d) The analytical methods have been unsuccessful for solving the nonlinear FFODEs, while most of the real-world problems govern by the nonlinearity, therefore the analytical methods considered impractical for describing the real-world phenomena.

2. Approximation methods:

   a) According to our literature in this chapter, no more than six works were employed the approximate numerical methods for solving linear FFODEs.

   b) There are several approximate analytical methods including VIM, ADM, RKHSM, FRPSM, SCM, and MHPM were employed for solving FFODEs.

   c) The approximate numerical methods have been unsuccessful for solving the nonlinear FFODEs, on the other hand only VIM was successfully employed for solving nonlinear FFODE.

   d) According to our literature, the exist approximate analytical methods failed to provide a simple way to control and adjust the convergence area.

Therefore, we see it necessary to provide new approximate analytical methods to overcome such obstacles as we will show in the next chapters.

CHAPTER THREE
MATHEMATICAL CONCEPTS AND RESEARCH METHODOLOGY

3.1 Introduction

This chapter describes the mathematical background involving fuzzy set theory, fractional calculus theory, fuzzy fractional derivatives, and the general structure of the Fractional Homotopy Analysis Method (F-HAM) and the Fractional Optimal Homotopy Asymptotic Method (F-OHAM) and introduces the research methodology towards accomplishing the research objectives. The focus problems of this study are three types of FFODEs: first-order FFOIVPs, second-order FFOIVPs, and second-order FFOBVPs. The proposed approximate-analytical methods, FF-HAM and FF-OHAM, are based on the concepts of HAM and OHAM, consecutively, and both methods are constructed for each of the three types of FFODEs.

3.2 Mathematical Background

This section provides the mathematical background of this research. The main definitions, theorems and previous results related to the fuzzy calculus of fractional order are revisited and discussed. The section discusses fuzzy set theory, fractional calculus theory, fuzzy fractional derivatives, (F-HAM) theory and (F-OHAM) theory.

3.2.1 Fuzzy Set Theory

The principle of the fuzzy set theory is an extension of classical set theory consisting of the integer $\{0,1\}$, where this value range has been extended by Zadeh in 1965 to $[0,1]$, such that a fuzzy set A is described throughout a discourse universe X by a function of membership $\mu_A: X \longrightarrow [0,1]$. In Zadeh theory, a changed theoretical approach has been suggested in which the individual entity may have a membership value that is ranged between 0 and 1 and not exactly 0 or 1 (Maier et al., 2020). Zadeh has also demonstrated how basic setting operations like cross-section and union are described on these fuzzy sets; moreover, a coherent framework instrument to address the issues of such operations has been developed. This important framework enables fuzzy sets to be treated intuitively in a consistent and fair manner (Buckley & Yan, 2000).

3.2.1.1 Fuzzy Set

The fuzzy set is a generalized crisp set, in a way that allows the membership function to use any value of the interval [0,1] (Piegat, 2005).

Definition 3.1: (Morales & Mendez, 2012) Given a universal set of objects (x) denoted generally by X. The fuzzy set $\tilde{A}$ in X will be defined through the membership function $\mu_{\tilde{A}}: X \longrightarrow [0,1]$ as a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)): x \in X\}. \tag{3.1}$$

Definition 3.1 shows the relationship between different values of x and the values of the closed interval [0,1]. This relationship, which refers to the degree of truth, is represented by the membership function of the fuzzy set $\tilde{A}$ as shown in Figure 3.1, where the fuzzy set $\tilde{A}$ is entirely described by the membership function. The range of the membership function will be observed by a set of the non-negative real numbers whose supremum is finite and the degree of object membership x to the main set X is defined by the membership function. When the object x is fully embedded in the given set, that means the mapping value equals one exactly; on the other hand, an object does not belong to the specified set if the mapping value is equal to zero. Meanwhile, the values belonging to the open interval (0,1) will characterize the fuzzy objects.

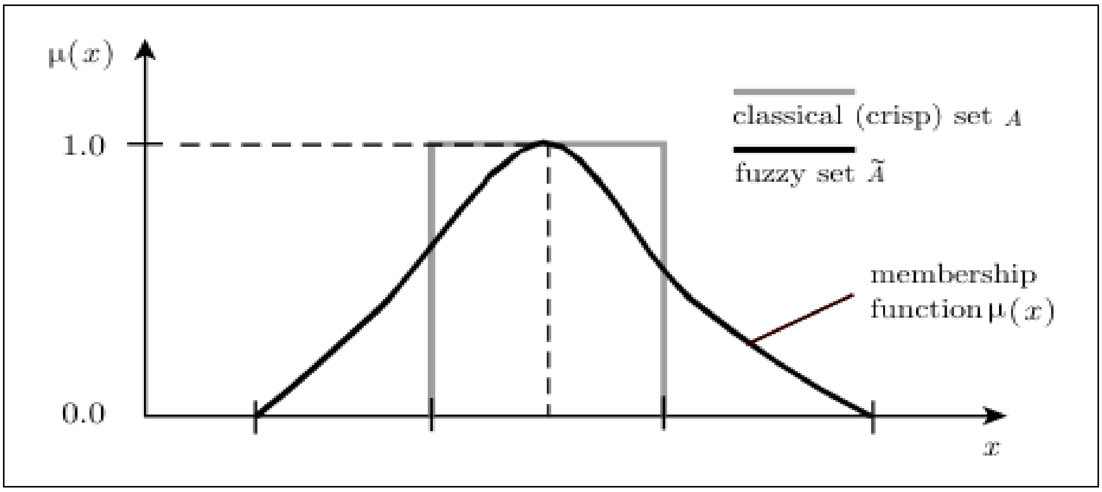

Figure 3.1. Crisp set A and fuzzy set $\tilde{A}$

Definition 3.2: Convex fuzzy set (Kumar & Lata, 2012) Assume $\mathbb{R}^n$ is the n-dimensional Euclidean space. Therefore, any fuzzy set of membership function $\mu_{\tilde{U}}: \mathbb{R}^n \rightarrow [0,1]$ is called convex if

$$\mu\,(\theta a_1 + (1-\theta)a_2) \geq min\{\mu(a_1), \mu(a_2)\} \tag{3.2}$$

for all $a_1, a_2 \in \mathbb{R}^n$ and $\theta \in [0,1]$ where $\tilde{U} = U(\mathbb{R}^n)$ indicate the set of all nonempty fuzzy sets in $\mathbb{R}^n$. In addition, when $\mu(\theta a) = \mu(a), \forall a \in \mathbb{R}^n$ and $\theta > 0$, the fuzzy set defined by the membership function $\mu: \mathbb{R}^n \rightarrow [0,1]$ is called a cone. This cone is a convex fuzzy set as well (Kumar & Lata, 2012).

Definition 3.3: Support of a fuzzy set (Mizukoshi et al., 2007) The crisp set of all members x of the universal set X, which is linked with the nonzero membership function, is called the support of fuzzy set $\tilde{A}$ such that

$$\text{Supp}(\tilde{A}) = \{x \in X | \mu_{\tilde{A}}(x) > 0\} \tag{3.3}$$

3.2.1.2 Extension Principle

One of the major definitions of the fuzzy set theory employed to generalize the classical set to the fuzzy set is called the extension principle theory.

Definition 3.4: (Mizukoshi et al., 2007) Let S be the cartesian product of the universes $S_1, S_2, \dots, S_n$ and $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n$ be n-fuzzy subsets in $S_1, S_2, \dots, S_n$, respectively, with the cartesian product $\tilde{A} = \tilde{A}_1 \times \tilde{A}_2 \times \dots \times \tilde{A}_n$ and g is a mapping from S to a universe Y, with $y = g(s_1, s_2, \dots, s_n)$ for any element $y \in Y$. Then, the extension principle theory allows defining a fuzzy subset $\tilde{B} = g(\tilde{A})$ in Y by

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) : y = g(s_1, s_2, \dots, s_n), s_1, s_2, \dots, s_n \in S\}, \tag{3.4}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{s_1, s_2, \dots, s_n \in g^{-1}(y)} \min\{\mu_{\tilde{A}_1}(s_1), \dots, \mu_{\tilde{A}_n}(s_n)\} & \text{if } y \in \text{range of } (g)\ (g^{-1}(y) \neq \emptyset) \\ 0 & \text{if } y \notin \text{range of } (g)\ (g^{-1}(y) = \emptyset) \end{cases}$$

and g^{-1} represents the inverse function of g. According to (Ahmad et al,. 2013), the extension principle theory for $n = 1$ would be

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) : y = g(s), s \in S\} \tag{3.5}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{s \in g^{-1}(y)} \min\{\mu_{\tilde{A}}(s)\}, & \text{if } y \in \text{range of } (g) \\ 0 & \text{if } y \notin \text{range of } (g) \end{cases}$$

3.2.1.3 The $\boldsymbol{\alpha}$-Cut

The $\boldsymbol{\alpha}$-cut is used to verify that those results or operations, which are already applicable or achieved in crisp sets, are also applicable in fuzzy sets.

Definition 3.5: (Bodjanova, 2006) The α-cut (or α-level set) of the fuzzy set $\tilde{A}$ is indicated by $\tilde{A}_\alpha$ and constitute a crisp set for all $s \in S$ such that $\mu_{\tilde{A}} \geq \alpha$

$$\tilde{A}_\alpha = \{s \in S | \mu_{\tilde{A}} \geq \alpha, \alpha \in [0,1]\}. \tag{3.6}$$

Moreover, the strong α-level sets might also be defined as

$$A_\alpha^+ = \{s \in S | \mu_{\tilde{A}} > \alpha, \alpha \in [0,1]\}. \tag{3.7}$$

According to Definition.3.5, it is clear $\forall\, \alpha_1, \alpha_2 \in [0, 1]$, the following properties hold:

1. $(\tilde{B} \cup \tilde{A})_{\alpha_1} = B_{\alpha_1} \cup \tilde{A}_{\alpha_1}$.
2. $(\tilde{B} \cap \tilde{A})_{\alpha_1} = \tilde{B}_{\alpha_1} \cap \tilde{A}_{\alpha_1}$.
3. If $\tilde{B} \subseteq \tilde{A}$, then $\tilde{B}_{\alpha_1} \subseteq \tilde{A}_{\alpha_1}$.
4. If $\alpha_1 \leq \alpha_2$, then $\tilde{A}_{\alpha_2} \subseteq \tilde{A}_{\alpha_1}$.
5. $\tilde{B} = \tilde{A}$ if and only if $\tilde{B}_{\alpha_1} = \tilde{A}_{\alpha_1}$, $\forall\, \alpha_1 \in [0, 1]$.
6. $\tilde{A}_{\alpha_1} \cap \tilde{A}_{\alpha_2} = \tilde{A}_{\alpha_2}$ and $\tilde{A}_{\alpha_1} \cup \tilde{A}_{\alpha_2} = \tilde{A}_{\alpha_1}$,whenever $\alpha_1 \leq \alpha_2$.

For any fuzzy set $\tilde{A}$, $\forall\, \alpha \in [0, 1]$, the family of subsets of the universal set S is denoted by $\{\tilde{A}_\alpha\}$ such that

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} \alpha\, \tilde{A}_\alpha, \tag{3.8}$$

Eq. (3.8) indicates that all α-levels linked with the fuzzy set $\tilde{A}$ form a family of crisp nested sets, as shown in Figure 3.2.

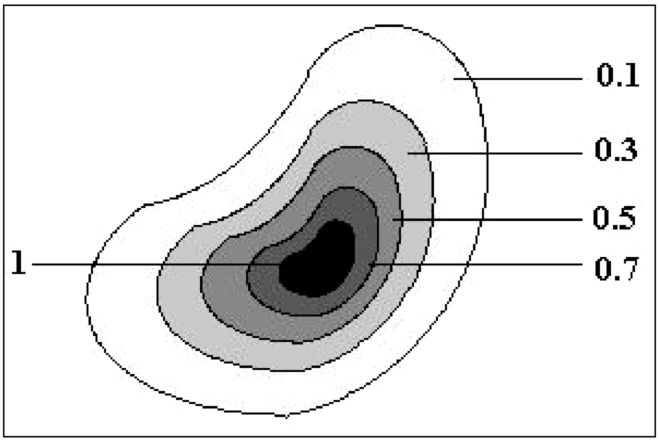

Figure 3.2. Nested α-level sets

Figure 3.2 shows that when the α value is closer to 1, the fuzzy region will be in conformity with the crisp domain.

3.2.1.4 Fuzzy Numbers

Zadeh's fuzzy numbers, which are generally called fuzzy numbers, are vagueness values and are referred by convex and normalized subsets of $\mathbb{R}$. These numbers are related to degrees of membership which show the degrees of truth in telling whether an element exists in the determined set (Roszkowska & Kacprzak, 2016). In addition, the fuzzy numbers can be considered as a fuzzy set defining a fuzzy interval in real numbers, $\mathbb{R}$. Since the interval borders are vague, the interval often constitutes a fuzzy set. In general, there are two end points a_1 and a_3, and a peak point a_2 as $[a_1, a_2, a_3]$ as shown in Figure 3.3. The α-level set also can be applied for fuzzy numbers. Let $[\tilde{A}]_\alpha$ denote the α-cut for the fuzzy number $\tilde{A}$, where the interval gained $[\tilde{A}]_\alpha$ is defined as $[\tilde{A}]_\alpha = [a_1{}^{(\alpha)}, a_3{}^{(\alpha)}]$. As can be observed in Figure 2.4, $[\tilde{A}]_\alpha$ is a crisp interval.

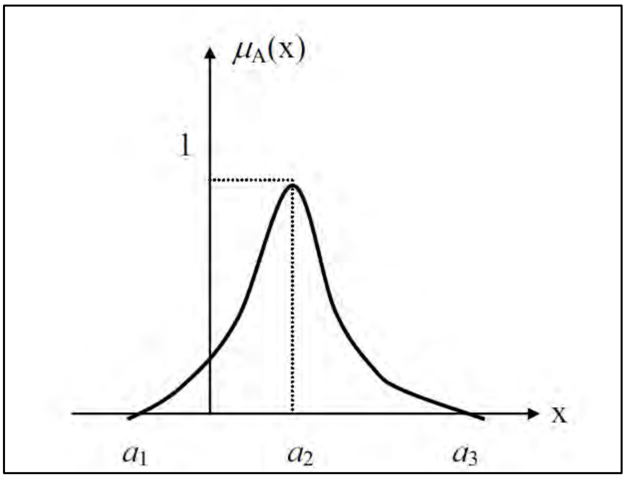

Figure 3.3. Fuzzy numbers $\tilde{A} = [a_1, a_2, a_3]$

Definition 3.6: (Salahshour, 2011) Suppose $\widetilde{U}$ is the set of all semi-continuous upper normal convex fuzzy numbers with α-cuts such that

$$[\mu]_\alpha = \{x \in \mathbb{R}: \mu \geq \alpha\} \tag{3.9}$$

is an ordered pair of membership function defining an arbitrary fuzzy number and given by $[\tilde{\mu}(x)]_\alpha = \left[\underline{\mu}(x), \overline{\mu}(x)\right]_\alpha \ \forall \ \alpha \in [0,1]$, and satisfying the following properties:

1. $\mu(x)$ is normal, that is, there is an $x_0 \in \mathbb{R}$ in which $\mu(x_0) = 1$.
2. $\mu(x)$ is convex fuzzy set, that is, $\mu(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu(x_1)\ , \mu(x_2)\}$ $\forall x_1, x_2 \in \mathbb{R}, \lambda \in [0,1]$.
3. $\forall \mu \in \widetilde{U}$, μ is a semi continuous upper on $\mathbb{R}$; $\{x \in \mathbb{R};\ \mu(x) > 0\}$ is compact.
4. $\underline{\mu}(x)$ is a left continuous non-decreasing bounded function over [0,1].
5. $\overline{\mu}(x)$ is a left continuous non-increasing bounded function over [0,1].
6. $\underline{\mu}(x) \leq \overline{\mu}(x)\ \ \forall \alpha \in [0, 1]$.

Furthermore, α-cuts of any fuzzy number are representing forms of fuzzy sets much more efficient than the given properties or features in Definition 3.6. Also, Uehara and Fujise (1993) show the fuzzy sets can be defined through the families of their α-level sets based on the theorem of the resolution identity theory.

Definition 3.7: (**Triangular Fuzzy Number**) (Salahshour, 2011) Any arbitrary fuzzy number with the following function of membership:

$$\mu\,(x; c, b, a\,) = \begin{cases} 0 & , \; x < c \\ \frac{x-c}{b-c} & , \; c \leq x \leq b \\ \frac{a-x}{a-b} & , \; b \leq x \leq a \\ 0 & , \; x > a \end{cases}, \qquad (3.10)$$

is indicated by the triangular fuzzy number, where a, b and $c \in \mathbb{R}$ with $a > b > c$.

The graph of the triangular membership function $\mu\,(x)$ takes the triangular shape with the base over the interval $[c, b]$ and vertex at $x = b$ as illustrated in Figure 3.4. For the α-cut of μ defined as $[\tilde{\mu}]_{\alpha} = [c + \alpha(b - c), a - \alpha(a - b)]$ for $\alpha \in [0, 1]$, according to Definition 3.3, the support fuzzy set of μ will be defined over the interval $x \in [c, a]$.

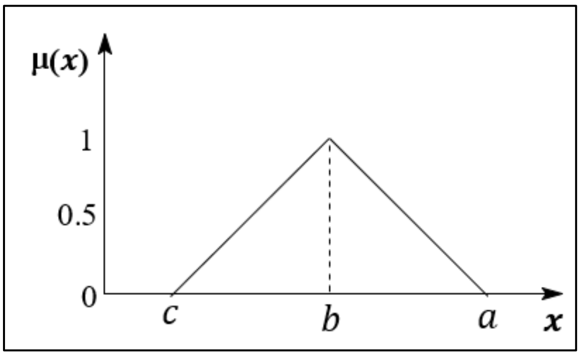

Figure 3.4. Triangular fuzzy number

3.2.1.5 Fuzzy Function

Fuzzy function or fuzzy process refers to the process of images production of the crisp domain in the fuzzy set by fuzzifying a crisp function of a crisp variable (Guang-Quan, 1991).

Definition 3.8: (Fard, 2009) For some interval $X \subseteq \widetilde{U}$, a mapping $\tilde{g}: X \to \widetilde{U}$ is called a fuzzy process (fuzzy function) of crisp variable; accordingly, α-cut of a fuzzy function $\tilde{g}$ will be composed as follows:

$$[\tilde{g}(x)]_\alpha = \left[\underline{g}(x;\alpha), \overline{g}(x;\alpha)\right], x \in X, \alpha \in [0,1] \tag{3.11}$$

Here, $\widetilde{U}$ was predetermined in Definition 3.2. The fuzzy function $\tilde{g}$ is a mapping from the crisp domain to the fuzzy set. In other words, the function in the fuzzy environment and the fuzzy relation coincides with each other mathematically.

3.2.1.6 Fuzzy Differentiation

Chang and Zadeh first presented the notion of the fuzzy derivative (Buckley & Yan, 2000). In its follow-up analysis on fuzzy derivatives, in Dubois and Prade used the theory of extension and suggested two derivative definitions: one centered in Hukuhara and restricted on a convex cone and the other on a whole space in Banach where the second concept was discussed in Chang and Zadeh (1972).

Definition 3.9: (Salahshour, 2011) Given two fuzzy numbers (two fuzzy sets) $\widetilde{\alpha}, \tilde{b} \in \widetilde{U}$, then the Hausdorff distance between $\widetilde{\alpha}$ and $\tilde{b}$ symbolized by $D_H\left(\left[\tilde{a}, \tilde{b}\right]_\alpha\right)$ is such that

$$D([\tilde{a},\tilde{b}]) = \sup\left\{D_H\left([\tilde{a},\tilde{b}]_\alpha\right)\middle|\alpha \in [0,1]\right\}. \quad (3.12)$$

where $(\widetilde{U}, D)$ indicates the Cauchy space (complete metric space).

Suppose that $\widetilde{U}$ is the set of all semi-continuous upper normal convex fuzzy numbers with bounded α-level sets. The α-cuts of fuzzy numbers are always bounded and closed such that the intervals are $[\tilde{\mu}(x)]_\alpha = \left[\underline{\mu}(x), \overline{\mu}(x)\right]_\alpha, x \in \mathbb{R}, \forall\, \alpha \in [0,1]$ (Fard, 2009).

Let $[\tilde{b}(\alpha)] = [\underline{b}(\alpha), \overline{b}(\alpha)], [\tilde{a}(\alpha)] = [\underline{a}(\alpha), \overline{a}(\alpha)]$ represent two fuzzy numbers [Definition 3.7], for $\gamma \geq 0$. We can define the binary operations on fuzzy numbers $\tilde{b}$ and $\tilde{a}$ of addition and scalar multiplication described as follows (Kaleva, 2006):

1. $(\underline{b+a})(\alpha) = (\underline{b}(\alpha) + \underline{a}(\alpha))$
2. $(\overline{b+a})(\alpha) = (\overline{b}(\alpha) + \overline{a}(\alpha))$
3. $(\underline{\gamma b})(\alpha) = \gamma.\overline{b}(\alpha)$.

Assuming $\widetilde{U} \times \widetilde{U} \rightarrow \{0\} \cup \mathbb{R}$, and Hausdorff distance between two fuzzy numbers b, a is $D\left([\tilde{b},\tilde{a}]_\alpha\right) = \sup_{\gamma\in[0,1]}\text{Max}\{|\underline{b}(\alpha) - \underline{a}(\alpha)|, |\overline{b}(\alpha) - \overline{a}(\alpha)|\}$, then the following well-recognized properties hold (Omana, 2009):

1. $D\left([\tilde{d},\tilde{c}]_\alpha\right) = D\left([\tilde{d}+\tilde{b}, \tilde{c}+\tilde{b}]_\alpha\right), \ \forall \tilde{d}, \tilde{c}, \tilde{b} \in \widetilde{U}$.
2. $|r|D\left([\tilde{d},\tilde{c}]_\alpha\right) = D\left([r.\tilde{d}, r.\tilde{c}]_\alpha\right), \ \forall r \in \mathbb{R}, \forall \tilde{d}, \tilde{c} \in \widetilde{U}$.
3. $D([d,c]_\alpha) + D([b,\tilde{a}]_\alpha) \geq D\left([\tilde{d}+\tilde{c}, \tilde{b}+\tilde{a}]_\alpha\right), \ \forall \tilde{d}, \tilde{c}, \tilde{b}, \tilde{a} \in \widetilde{U}$.

Definition 3.10: (Stefanini, 2009) Let $\tilde{z}, \tilde{y} \in \widetilde{U}$; if $\exists\, \tilde{x} \in \widetilde{U}$ such that $\tilde{x} = \tilde{z} - \tilde{y}$, then $\tilde{x}$ is called H-difference or Hukuhara difference such that $\tilde{x} = \tilde{z} \ominus \tilde{y}$.

Definition 3.11: (Mansouri & Ahmady, 2012) Let $\tilde{g}: W \rightarrow \widetilde{U}$ and $y_0 \in W$ where $W \in [x_{0,} X]$, the fuzzy function $\tilde{g}$ is said to be Hukuhara differentiable at y_0, if ∃ an element $[\widetilde{g'}]_\alpha \in \widetilde{U}$, in which $\forall\, \varepsilon > 0$ sufficiently close enough to zero, there exist $\tilde{g}(y_0 + \varepsilon; \alpha) \ominus \tilde{g}(y_0; \alpha)$ and $\tilde{g}(y_0; \alpha) \ominus \tilde{g}(y_0 - \varepsilon; \alpha)$, and the limits in metric space $(\widetilde{U}\,, D)$ are set as follows:

$$\lim_{\varepsilon \to 0+} \frac{\tilde{g}(y_0+\varepsilon;\alpha) \ominus \tilde{g}(y_0;\alpha)}{\varepsilon} = \lim_{\varepsilon \to 0+} \frac{\tilde{g}(y_0;\alpha) \ominus \tilde{g}(y_0-\varepsilon;\alpha)}{\varepsilon}, \tag{3.13}$$

The fuzzy set $[\widetilde{g}'(y_0)]_\alpha$ would represent the Hukuhara derivative or the H-derivative of $[\widetilde{g}'(y)]_\alpha$ at y_0.

Space $(\widetilde{U}\,, D)$ involves the limit given in Definition 3.11, for x_0 or X, then we deem the corresponding derivative on one side. Recall that $[\tilde{z}]_\alpha \ominus [\tilde{y}]_\alpha = [\tilde{x}]_\alpha$, for all $\alpha \in [0,1]$, which means $\tilde{z} \ominus \tilde{y} = \tilde{x} \in \widetilde{U}$ are defined for all α-cut. In consideration of metric D definition, all the α-cuts $[\tilde{g}(0)]_\alpha$ are Hukuhara differentiable at y_0, with Hukuhara-derivatives $[\widetilde{g}'(y_0)]_\alpha$,when $\tilde{g}: W \rightarrow \widetilde{U}$ is the Hukuhara-difference at y_0 with Hukuhara-derivative $[\widetilde{g}'(y_0)]_\alpha$ it guides us to having $[\widetilde{g}'(y)]_\alpha$ as Hukuhara differentiable $\forall\, \alpha \in [0,1]$ which achieve the above limits, that is, if g is differentiable at $x_0 \in [x_0 + b, X]$ then all its α-cuts $[\widetilde{g}'(y)]_\alpha$ are the Hukuhara-difference at x_0.

Theorem 3.1: (Stefanini et al., 2006) Let $\tilde{g}: [x_0 + b, X] \rightarrow \widetilde{U}$ be Hukuhara-differentiable and indicated by

$$[\widetilde{g}'(x)]_\alpha = [\underline{g}'(x), \overline{g}'(x)]_\alpha = [\underline{g}'(x;\alpha), \overline{g}'(x;\alpha)]. \tag{3.14}$$

Then, both lower and upper functions $\underline{g}'(x;\alpha), \overline{g}'(x;\alpha)$ are differentiable $\forall\ \alpha \in [0,1]$, such that

$$[\widetilde{g}'(x)]_\alpha = \left[\left(\underline{g}(x;\alpha)\right)', \left(\overline{g}(x;\alpha)\right)'\right]. \quad (3.15)$$

Theorem 3.2: (Mansouri & Ahmady, 2012) Let $\tilde{g}: [x_0 + b, X] \to \widetilde{U}$ be Hukuhara differentiable and indicated by

$$[\widetilde{g}'(x)]_\alpha = \left[\underline{g}'(x), \overline{g}'(x)\right]_\alpha = \left[\underline{g}'(x;\alpha), \overline{g}'(x;\alpha)\right]. \quad (3.16)$$

Then the upper and lower functions $\underline{g}'(x;\alpha)$ and $\overline{g}'(x;\alpha)$ respectively are differentiable $\forall\ \alpha \in [0,1]$; thus, we can set the k^{th}-order fuzzy derivative as follows:

$$[\tilde{g}^{(k)}(x)]_\alpha = \left[\left(\underline{g}^{(k)}(x;\alpha)\right), \left(\overline{g}^{(k)}(x;\alpha)\right)\right], \forall\ \alpha \in [0,1]. \quad (3.17)$$

According to Definition 3.1, we can set the fuzzy Hukuhara differentiability of order k as in the following Definition 3.12.

Definition 3.12: Let $\tilde{g}: W \to \widetilde{U}$ and $y_0 \in W$, where $W \in [x_{0,}X]$. We can say $\widetilde{g}'$ is Hukuhara differentiable for $x \in \widetilde{U}$, if there exists an element $[\tilde{g}^{(k)}]_\alpha \in \widetilde{U}$ such that $\forall\ \varepsilon > 0$ sufficiently close to zero, there exist $\tilde{g}^{(k-1)}(y_0 + \varepsilon;\alpha) \ominus \tilde{g}^{(k-1)}(y_0;\alpha)$, $\tilde{g}^{(k-1)}(y_0;\alpha) \ominus \tilde{g}^{(k-1)}(y_0 - \varepsilon;\alpha)$. Taking the limits in metric $(\widetilde{U}, D)$, we have

$$\lim_{\varepsilon \to 0+} \frac{\tilde{g}^{(k-1)}(y_0+\varepsilon;\alpha) \ominus \tilde{g}^{(k-1)}(y_0;\alpha)}{\varepsilon} = \lim_{\varepsilon \to 0+} \frac{\tilde{g}^{(k-1)}(y_0;\alpha) \ominus \tilde{g}^{(k-1)}(y_0-\varepsilon;\alpha)}{\varepsilon}. \quad (3.18)$$

3.2.1.7 Fuzzy Integration

Definition 3.13: (Wu, 2000) Assuming the mapping $\tilde{g}: W \to \tilde{U}$ is a closed and bounded fuzzy valued function on the interval $W \in [x_0, X]$ such that $[\tilde{g}(x)]_\alpha = \left[\underline{g}(x;\alpha), \overline{g}(x;\alpha)\right]$. Let $\underline{g}(x;\alpha), \overline{g}(x;\alpha)$ be Riemann integrable on the interval W $\forall\, \alpha \in [0,1]$. Letting

$$\tilde{A}(x;\alpha) = \left[\int_{x_0}^{X} \underline{g}(x;\alpha)dx\,, \int_{x_0}^{X} \overline{g}(x;\alpha)dx\right] \tag{3.19}$$

then $[\tilde{g}(x)]_\alpha$ is fuzzy Riemann-integrable on W and denoted by $[\tilde{g}(x)]_\alpha \in \tilde{U}_{RI}$ on the interval W for $s \in \tilde{A}(x;0)$ with $\int_{x_0}^{X} \tilde{g}(x;\alpha)dx$, such that

$$\mu_{\int_{x_0}^{X} \tilde{g}(x;\alpha)dx}(x) = \sup_{\alpha\in[0.1]} \alpha 1_{\tilde{I}(x;\alpha)}.$$

It is easy to show that

$$\overline{g}(x;\alpha_1) > \overline{g}(x;\alpha_2) > \tilde{g}(x;\alpha) > \underline{g}(x;\alpha_1) > \underline{g}(x;\alpha_2) \tag{3.20}$$

for $\alpha_1 < \alpha_2 < 1$ such that if we assume

$$\int_{x_0}^{X} \underline{g}(x;\alpha)dx = \underline{A}(x;\alpha)\,, \int_{x_0}^{X} \overline{g}(x;\alpha)dx = \overline{A}(x;\alpha) \tag{3.21}$$

then

$$\int_{x_0}^{X} \tilde{g}(x;\alpha)dx = \left[\underline{A}(x;\alpha)\,, \overline{A}(x;\alpha)\,\right] = \tilde{A}(x;\alpha). \tag{3.22}$$

There is also another way to set the fuzzy integration $\left(\int_{x_1}^{x_2} \tilde{g}(x;\alpha)dx\right)$, which is as follows (Wu, 2000):

$$\int_{x_1}^{x_2} \tilde{g}(x;\alpha)dx = \left[\int_{x_1}^{x_2} \underline{g}(x;\alpha)dx\,, \int_{x_1}^{x_2} \overline{g}(x;\alpha)dx\right] \tag{3.23}$$

for $x_1, x_2 \in X$ and $\alpha \in [0,1]$.

3.2.2 Fractional Calculus Theory

The subject of fractional calculus is a notion of traditional calculus formulation to arbitrary real or complex order (Dalir & Bashour, 2010). Unfortunately, there is no physical or even geometrical explanation for the arbitrary order in the works that deals with the calculus so far, and it is considered one of the interesting open issues in the field.

3.2.2.1 Basic principles of fractional derivatives and integrals

As the fractional calculus generalize the principle of classical calculus from integer order into an arbitrary order, an interpolation of an infinite series of classical k-fold integrals and k-fold derivatives can be used for this generalization. This means the principle of fractional derivative and fractional integral are based on the Cauchy formula, where Cauchy has proven that repeated application of ordinary integral can be applied on the continuous function $g(x)$ as:

$$g^{(-k)}(x) = \int_a^x \int_a^{t_1} \dots\dots.. \int_a^{t_{k-1}} g(t_n)\, dt_n dt_{n-1} \dots.\, dt_2 dt_1, \tag{3.24}$$

and given as a single Riemann integration

$$g^{(-k)}(x) = \frac{1}{(k-1)!} \int_a^x (x-t)^{k-1}\, g(x) dx. \tag{3.25}$$

Similar results can be obtained using the k^{th}order derivative by $g^{(k)}(x)$. According to the Cauchy formula in Eq. (3.24) stated earlier, we can obtain the fractional derivative and fractional integral by replacing k with fractional order $\beta > 0$, so that we will get fractional Riemann integral $g^{(-\beta)}(x)$ and fractional Riemann derivative $g^{(\beta)}(x)$. The explanation will be given in detail. Handling the fractional order in the Cauchy formula requires several special functions such as gamma function and beta function.

Definition 3.14: Gamma Function (Kim & Kim, 2020) Gamma function, denoted by Γ, is a straightforward modification of the factorial function $n!$ where $n! = \Gamma(n+1)$ for $n \in \mathbb{N}$, and is defined using Euler's formula as

$$\Gamma(x) = \int_0^\infty z^{x-1} e^{-z}\, dz\,, Re(x) > 0, \text{ for } x \in \mathbb{C} \text{ with real positive part.}$$

Some essential properties of gamma function include the following:

1. $\Gamma(n) = (n-1)!, n \in \mathbb{N}$.
2. $\Gamma(x+1) = x\,\Gamma(x)$.
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
4. $\Gamma(x)\Gamma(1-x) = {}^{\pi}/_{\sin(\pi x)}$.
5. $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!,\ n \in \mathbb{N}$.

Definition 3.15: (Kaur et al., 2020): Beta function denoted by $B(n,m)$ is defined using the following integration formula:

$$B(n,m) = \int_0^1 z^{n-1}(1-z)^{m-1} dz\,, Re(n) > 0, Re(m) > 0 \tag{3.26}$$

In some cases, $B(n,m)$ is more convenient than Γ especially for handling the power functions. Beta function can be described as follows:

i) The adoption is upon the concept of the Gamma function where beta function can be defined as $B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$.

ii) $B(n,m) = B(m,n)$ (Symmetric property).

Therefore, beta function is considered a special case of the gamma function.

3.2.2.2 Fractional Derivatives in Crisp Environment

Different forms of fractional derivatives in crisp environment have been introduced by a variety of mathematicians. Recently, the fractional derivatives of the Riemann-Liouville and Caputo can be considered the most commonly utilized. We proceed to discuss their specification and practical concepts. Starting with the Cauchy formulation, followed by replacing $k \in \mathbb{N}$ by a positive real number β, we get

$$\int_a^y dy_1 \int_a^{y_1} dy_2 \int_a^{y_2} dy_3 \dots \int_a^{y_{k-1}} g(y_k)\, dy_k = \frac{1}{(k-1)!} \int_a^y \frac{f(y)}{(x-y)^{1-k}} dy. \quad (3.27)$$

In addition, by replacing $(k-1)!$ with generalization of Gamma function, the fractional integral formula will be defined as given in the following Definition 3.16.

Definition 3.16: (Li et al., 2011) The fractional integral operator of Riemann-Liouville sense of the function $g \in L_1[a,b]$ of order β over the interval $[a,b]$ is denoted by $\mathcal{J}_{a^+}^{\beta} f(x)$ such that

$$\mathcal{J}_{a^+}^{\beta} g(x) = \mathcal{J}^{\beta} = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_a^x \frac{g(y)}{(x-y)^{1-\beta}} dy, 0 < y < x, \beta > 0 \\ g(x), \qquad \beta = 0 \end{cases}$$

The following are some interesting features or properties of the operator $\mathcal{J}_{a^+}^{\beta}$:

1. $\mathcal{J}_{a^+}^{0} g(x) = g(x)$, where this means $\mathcal{J}_{a^+}^{0} = I$ (identity operator).
2. $\mathcal{J}_{a^+}^{\beta}(c g_1(x) + g_2(x)) = c\mathcal{J}_{a^+}^{\beta} g_1(x) + \mathcal{J}_{a^+}^{\beta} g_2(x), \beta > 0, c \in \mathbb{C}$, (linear operator).

   If $g(x)$ is a continuous function at $x \geq 0$, we have $\lim_{\beta \to 0} \mathcal{J}_{a^+}^{\beta} g(x) = g(x)$, and

   $\mathcal{J}_{a^+}^{\beta_1}\left(\mathcal{J}_{a^+}^{\beta_2} g(x)\right) = \mathcal{J}_{a^+}^{\beta_1+\beta_2} g(x) = \mathcal{J}_{a^+}^{\beta_2}\left(\mathcal{J}_{a^+}^{\beta_1} g(x)\right), \forall \beta_1, \beta_2 \geq 0.$

3. $\mathcal{J}_{a^+}^{\beta} c = \frac{c}{\Gamma(\beta+1)} x^{\beta}$, $\forall\, c \in \mathbb{R}$, and $\beta > 0$.

4. For $g(x) = (x-a)^r, r > -1$, the fractional integral of $g(x)$ is

   $\mathcal{J}_{a^+}^{\beta} (x-a)^r = \frac{\Gamma(r+1)}{\Gamma(r+1+\beta)} (x-a)^{r+\beta}, \beta > 0, x > a.$

Next, it is necessary to define the operator in the sense of Riemann-Liouville derivative.

Definition 3.17: (Li et al., 2011): The fractional derivative operator in the sense of Riemann-Liouville of order β is denoted by $\ddot{D}_{a^+}^{\beta} g(x)$ such that

$$\ddot{D}_{a^+}^{\beta} g(x) = \ddot{D}^{\beta} = \begin{cases} \frac{1}{\Gamma(m-\beta)} \frac{d^m}{dx^m} \int_a^x \frac{g(y)}{(x-y)^{\beta-m+1}}\, dy, x > a, m-1 < \beta \le m \\ \frac{d^m}{dx^m} g(x), \qquad \beta = m, \end{cases} \tag{3.28}$$

for $\beta > 0$, where $m - 1 < \beta \le m, m \in \mathbb{N}$ and $g \in C^n[a, b]$.

In particular, if $m = 1$, then $\ddot{D}_{a^+}^{\beta} g(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x \frac{g(y)}{(x-y)^{\beta}}\, dy,\ x > a, \beta \in (0,1]$.

It must be noted that when $\beta \in \mathbb{N}$, the fractional operator $\ddot{D}_{a^+}^{\beta}$ becomes the standard differential operator $D^m = \frac{d^m}{dx^m}$.

There are many interesting properties involving Riemann-Liouville fractional derivative operator $\ddot{D}_{a^+}^{\beta}$ as follows:

1. $\ddot{D}_{a^+}^{0} g(x) = g(x)$, where this means $\ddot{D}_{a^+}^{0} = I$ (identity operator).
2. For $g(x) = (x-a)^r, r > -1$, we have $\ddot{D}_{a^+}^{\beta} (x-a)^{\beta} = \frac{\Gamma(r+1)}{\Gamma(r+1-\beta)} (x-a)^{r-\beta}$, $\beta > 0, x > a.$

3. $\ddot{D}^{\beta}_{a^+}g(x) = D^m \mathcal{J}^{m-\beta}_{a^+} g(x)$, for $\beta > 0$, where $m-1 < \beta < m, m \in \mathbb{N}$, which means the fractional operator $\ddot{D}^{\beta}_{a^+}$ of order β is equivalent to the composition of the $(m-\beta)$-fold integration and m-th order differentiation in reverse order.

Remark 3.1: Let $g \in L_r[a,b]$ for $r \in [1,\infty)$. Then, for $\beta_1, \beta_2 > 0$, the following facts hold everywhere over the interval $[a,b]$.

1. $\mathcal{J}^{\beta_1}_{a^+}\mathcal{J}^{\beta_2}_{a^+}g(x) = \mathcal{J}^{\beta_1+\beta_2}_{a^+}g(x)$,
2. $\ddot{D}^{\beta_1}_{a^+}\mathcal{J}^{\beta_1}_{a^+}g(x) = g(x)$,
3. $\ddot{D}^{\beta_1}_{a^+}\mathcal{J}^{\beta_2}_{a^+}g(x) = \mathcal{J}^{\beta_2-\beta_1}_{a^+}g(x)$, for $\beta_2 > \beta_1$ and $\beta_1 \in \mathbb{N}$.

Theorem 3.3: (Diethelm, 2010) The following properties are remarked for the Riemann-Liouville differential operator and Riemann-Liouville integration operator:

1. Let $g(x)$ is a continuous function for $x \geq 0$, and $\beta > 0$, where $m-1 < \beta < m$, $m \in \mathbb{N}$, then $g(x) = \ddot{D}^{\beta}_{a^+}\mathcal{J}^{\beta}_{a^+}f(x)$.
2. For $\beta > 0$,where $m-1 < \beta < m$, $m \in \mathbb{N}$, and $g(x)$ is function of x such that $\mathcal{J}^{\beta}_{a^+}\ddot{D}^{\beta}_{a^+}g(x)$ exists. We have

$$\mathcal{J}^{\beta}_{a^+}\ddot{D}^{\beta}_{a^+}g(x) = g(x) - \left(\sum_{k=0}^{m-1} \frac{1}{\Gamma(\beta-k)}(x-a)^{\beta-(k+1)} \lim_{y\to a^+} D^{m-(k+1)}\mathcal{J}^{m-\beta}_{a^+}g(y)\right).$$

In particular, for $m = 1$, and $0 < \beta < 1$, we have

$$\mathcal{J}^{\beta}_{a^+}\ddot{D}^{\beta}_{a^+}g(x) = g(x) - \left(\frac{1}{\Gamma(\beta)}(x-a)^{\beta-1} \lim_{y\to a^+} \mathcal{J}^{m-\beta}_{a^+}g(y)\right).$$

The improved Riemann differential operator of fractional order was developed based on the viscoelasticity theory by Caputo and Mainardi in 1971 and it has been called

Caputo fractional derivative or Caputo fractional differential operator to deal with the contradictions of Riemann–Liouville derivative while modeling the real-world phenomena. The Caputo fractional derivative is defined as follows:

Definition 3.18: Caputo Fractional Differential Operator (Li et al., 2011) For $\beta \in (m-1, m]$, $\forall\ m \in \mathbb{N}$, the Caputo fractional differential operator of order $\beta > 0$, with $a, x, \beta \in \mathbb{R}$ is denoted by $D_{a^+}^{\beta} g(x)$, such that

$$D_{a^+}^{\beta} g(x) = D^{\beta} = \frac{1}{\Gamma(m-\beta)} \int_a^x \frac{g^{(m)}(y)}{(x-y)^{\beta-m+1}}\, dy, \tag{3.29}$$

for $\beta = m$, we have $D_{a^+}^{\beta} g(x) = \frac{d^m}{dx^m} g(x)$.

The Caputo fractional operator $D_{a^+}^{\beta}$ has remarkable properties as follows:

1. $D_{a^+}^{\beta}(c_1 g_1(x) + c_2 g_2(x)) = c_1 D_{a^+}^{\beta} g_1(x) + c_2\, D_{a^+}^{\beta} g_2(x)$ (Linear operator).

2. $\forall\, \beta > 0$, we have $D_{a^+}^{\beta} c = 0$, for any constant $c \in \mathbb{R}$.

3. For $\beta > 0$, where $m - 1 < \beta \leq m$, with $m, n \in \mathbb{R}$ and assuming $D_{a^+}^{\beta} g(x)$ exists for the function $g(x)$, then we have $D_{a^+}^{\beta} D^m g(x) = D_{a^+}^{\beta+m} g(x) \neq D^n D_{a^+}^{\beta} g(x)$.

4. For $\beta > 0$, where $m - 1 < \beta \leq m$, with $m \in \mathbb{R}$, suppose $D_{a^+}^{\beta} g(x)$ exists for the function $g(x)$, then we have $D_{a^+}^{\beta} g(x) = \mathcal{J}_{a^+}^{m-\beta} D^m g(x)$, where $\mathcal{J}_{a^+}^{\beta} = D_{a^+}^{-\beta}$. This means $D_{a^+}^{\beta}$ is equivalent to the composition of the $(m-\beta)$-fold integration and m-th order differentiation.

Note also that the fractional integral operator $\mathcal{J}_{a^+}^{\beta}$ refers to the left inverse operator of the Caputo fractional derivative $D_{a^+}^{\beta}$, that is, $I = \mathcal{J}_{a^+}^{\beta} D_{a^+}^{\beta}$ (Hashim et al., 2009).

Remark 3.2: In general, the Caputo fractional operator and Riemann- Liouville fractional operator do not coincide, which means $D_{a^+}^{\beta} g(x) \neq \ddot{D}_{a^+}^{\beta} g(x)$. That is true because $\mathcal{J}_{a^+}^{m-\beta} D^m g(x) \neq D^m \mathcal{J}_{a^+}^{m-\beta} g(x)$.

Remark 3.3: $\frac{d^m}{dt^m} g(x) = \lim_{\beta \to m} D_{a^+}^{\beta} g(x) = \lim_{\beta \to m} \ddot{D}_{a^+}^{\beta} g(x)$, for $\beta > 0$,

where $m - 1 < \beta \leq m, m \in \mathbb{N}$, and $\beta \in \mathbb{R}$.

Theorem 3.4: (Diethelm, 2010): For $\beta > 0$, where $m - 1 < \beta \leq m$, the relationship between Caputo fractional differential operator and Riemann-Liouville fractional operator can be seen as follows:

$$D_{a^+}^{\beta} g(x) = \ddot{D}_{a^+}^{\beta} g(x) - \left(\sum_{k=0}^{m-1} \frac{g^{(k)}(0)}{\Gamma(k+1-\beta)} (x-a)^{k-\beta} \right). \tag{3.30}$$

Corollary 3.1: (Diethelm, 2010) The following relationship between the fractional differential operators of Caputo and Riemann-Liouville is achieved

$$D_{a^+}^{\beta} g(x) = \ddot{D}_{a^+}^{\beta} \left(g(x) - \sum_{k=0}^{m-1} \frac{g^{(k)}(0)}{k!} (x-a)^k \right). \tag{3.31}$$

Now, we can use the Taylor series to expand $D_{a^+}^{\beta} g(x)$ such that

$$D_{a^+}^{\beta} g(x) = D_{a^+}^{\beta} \left(\sum_{k=0}^{m-1} \frac{g^{(k)}(0)}{k!} (x-a)^k \right) = \sum_{k=0}^{m-1} \frac{g^{(k)}(0)}{k!} D_{a^+}^{\beta} (x-a)^k.$$

Corollary 3.2: (Diethelm, 2010) For $\beta \in (m-1, m]$, where $m \in \mathbb{N}$, and $r \in \mathbb{R}$, the Caputo fractional derivative of the power function $g(x) = (x-a)^r$ is given by

$$D_{a^+}^{\beta} g(x) = \begin{cases} \frac{\Gamma(r+1)}{\Gamma(r+1-\beta)}(x-a)^{r-\beta}, & m-1 < \beta \le m, r > m-1, m \in \mathbb{N}, r \in \mathbb{R} \\ 0, & \text{otherwise.} \end{cases}$$

3.2.3 Fuzzy Fractional Derivatives

With the existence of uncertainty or vagueness in dynamical systems, the systems of the ordinary fractional differential equations in the crisp environment are not applicable. Thus, FFODEs provide an effective environment for mathematical simulation when modelling the physical or engineering challenges in which uncertainty or ambiguity prevails. Relevant definitions of fuzzy fractional derivatives including the Hukuhara-differentiability are given in the following definitions.

Definition 3.19: (Salahshour et al., 2012) For any continuous fuzzy valued function $\tilde{g} \in C^{\mathcal{F}}[a,b] \cap L^{\mathcal{F}}[a,b]$, the fuzzy fractional Riemann-Liouville integration of $\tilde{g}(x)$ will be defined by the following form:

$$\mathcal{J}_{a+}^{\beta} \tilde{g}(x) = \frac{1}{\Gamma(\beta)} \int_a^x \tilde{g}(y)(x-y)^{\beta-1}\, dy, \text{ for } \beta, x \in \mathbb{R} \text{ and } x > a \tag{3.32}$$

$\forall\, \alpha \in [0,1]$, α-cuts for fuzzy valued function $\tilde{g}$ can be represented by

$$\tilde{g}(x;\alpha) = \left[\underline{g}(x;\alpha), \overline{g}(x;\alpha)\right].$$

Then, according to the lower and upper fuzzy valued functions $\left[\underline{g}(x;\alpha), \overline{g}(x;\alpha)\right]$, we can demonstrate the sense of Riemann-Liouville integration in the fuzzy setting of the fuzzy-valued function $\tilde{g}(x)$, $\forall\, \alpha \in [0,1]$ as follows:

Theorem 3.5: (Salahshour et al,. 2012) The fuzzy fractional Riemann–Liouville integration operator $\mathcal{J}_{a+}^{\beta}$ of the continuous fuzzy valued function $\tilde{g} \in C^{\mathcal{F}}[a,b] \cap L^{\mathcal{F}}[a,b]$ can be given as

$$\left[\mathcal{J}_{a+}^{\beta}\tilde{g}(x;\alpha)\right] = \left[\frac{1}{\Gamma(\beta)}\int_a^x \underline{g}(y;\alpha)(x-y)^{\beta-1}\,dy\,,\,\frac{1}{\Gamma(\beta)}\int_a^t \overline{g}(y;\alpha)(x-y)^{\beta-1}\,dy\right]. \tag{3.33}$$

Next, for the fuzzy-value function $\tilde{g}$, we will define the fuzzy fractional derivatives of Riemann-Liouville sense of order $0 < \beta \leq 1$.

Definition 3.20: (Salahshour et al., 2012) Let $\tilde{g} \in C^{\mathcal{F}}[a,b] \cap L^{\mathcal{F}}[a,b]$, $x_0 \in (a,b)$. The continuous fuzzy fractional valued function $\tilde{g}(x)$ is called a differentiable function at x_0 in the sense of fuzzy Riemann-Liouville and this differential operator is denoted by $\left(\ddot{D}_{a^+}^{\beta}\tilde{g}\right)(x)$, such that $\forall\beta \in (0,1]$, for $\alpha \in [0,1]$ we have $\left(\ddot{D}_{a^+}^{\beta}\tilde{g}\right)(x_0) = \left[\left(\ddot{D}_{a^+}^{\beta}\underline{g}\right)(x_0), \left(\ddot{D}_{a^+}^{\beta}\overline{g}\right)(x_0)\right]$,

where

$$\left(\ddot{D}_{a^+}^{\beta}\underline{g}\right)(x_0,\alpha) = \left[\frac{1}{\Gamma(1-\beta)}.\frac{d}{dx}\int_a^x \frac{\underline{g}(y,\alpha)}{(x-y)^{\beta}}\,dy\right]_{x=x_0},$$

and

$$\left(\ddot{D}_{a^+}^{\beta}\overline{g}\right)(x_0,\alpha) = \left[\frac{1}{\Gamma(1-\beta)}.\frac{d}{dx}\int_a^x \frac{\overline{g}(y,\alpha)}{(x-y)^{\beta}}\,dy\right]_{x=x_0}.$$

Definition 3.21: (Hasan et al., 2017) Suppose $\tilde{g}(x) = \left[\underline{g}(x), \overline{g}(x)\right] \in C^{\mathcal{F}}[0,b] \cap L^{\mathcal{F}}[0,b]$ and $\left(D_{a^+}^{\beta}\tilde{g}\right)(x)$ is the fuzzy fractional Caputo-Hukuhara differentiable function of order $\beta \in (0,1]$, for $\alpha \in [0,1]$, and $x > a$. Then, we have $\left(D_{a^+}^{\beta}\tilde{g}\right)(x) = \left[\left(D_{a^+}^{\beta}\underline{g}\right)(x,\alpha), \left(D_{a^+}^{\beta}\overline{g}\right)(x,\alpha)\right]$,

where

$$\left(D_{a^+}^{\beta}\underline{g}\right)(x,\alpha) = \frac{1}{\Gamma(1-\beta)}\int_a^x \frac{\underline{g}^{(1)}(y,\alpha)}{(x-y)^{\beta}}\,dy,$$

and

$$\left(D_{a^+}^{\beta}\overline{g}\right)(x,\alpha) = \frac{1}{\Gamma(1-\beta)}\int_a^x \frac{\overline{g}^{(1)}(y,\alpha)}{(x-y)^{\beta}}\,dy.$$

Definition 3.22: (Hasan et al., 2017) Let $\beta \in (1,2]$, and $\tilde{g}: [a,b] \to \widetilde{U}$, such that $\tilde{g}$ and $\tilde{g}' \in C^{\mathcal{F}}[0,b] \cap L^{\mathcal{F}}[0,b]$. Then, we can define the fuzzy fractional derivative in the sense of Caputo of the fuzzy function $\tilde{g}$ at $x \in (a,b)$ as follows:

$$\left(D^{\beta}\tilde{g}\right)(x) = \frac{1}{\Gamma(2-\beta)}\int_a^x \frac{\tilde{g}''(x)}{(y-x)^{\beta-1}}\,dx,\; x > a \tag{3.34}$$

3.2.4 General Structure of Fractional Homotopy Analysis Method

To describe the fractional HAM, the nonlinear fractional IVP is considered (Demirci & Ozalp, 2012):

$$\mathcal{N}[y(x)] = 0\,, x_0 \le x \le X, \tag{3.35}$$

where $\mathcal{N}$ represents the nonlinear operator of the fractional differential equation of order $m-1 < \beta \le m$ for integer number $m \ge 1$, x is the crisp independent variable and $y(x)$ is the unknown function of variable x.

Liao (2004) constructed the zeroth-order deformation equation in the crisp domain, followed by Ghazanfari & Veisi (2011) to describe the zeroth-order deformation equation of fractional order $\beta > 0$ as follows:

$$\mathcal{H}(x;q) = (1-q)\mathcal{L}_{\beta}[y(x;q) - y_0(x)] - qhH(x)\mathcal{N}[y(x;q)], \tag{3.36}$$

where $q \in [0,1]$ is the embedding parameter which acts on a change in the homotopy function continuously, $\mathcal{L}_\beta = D^{(\beta)}$ is the Caputo fractional operator, $y_0(x)$ is the initial guess of the function $y(x;q)$, $y(x;q)$ represents the unknown function that must satisfy the initial condition function $y_0(x)$, h is the convergence control parameter, and $H(x) \neq 0$ is the auxiliary function providing a convergence series solution of F-HAM. Here, the auxiliary convergence functions h and $H(x)$ play vital tools to control and modify the convergence of the F-HAM series solution (Ghoreishi et al., 2011) and (Ghoreishi et al., 2012). For $q = 0$, it is easy to note the homotopy function $\mathcal{H}(x;q)$ as follows:

$$\mathcal{H}(x;0) = \mathcal{L}_\beta[y(x;0) - y_0(x)] = 0, \tag{3.37}$$

and for $q = 1$, we have:

$$\mathcal{H}(x;1) = -hH(x)\,\mathcal{N}[y(x;1)] = 0. \tag{3.38}$$

Now, by setting $\mathcal{H}(x;q) = 0$, we can express Eq.(3.36) as follows

$$(1-q)\mathcal{L}_\beta[y(x;q) - y_0(x)] = qhH(x)\,\mathcal{N}[y(x;q)]. \tag{3.39}$$

When the embedding parameter increasing continuously from zero to one, the F-HAM series solution will be deforming from the initial guess $\mathcal{H}(x;0)$ to the exact solution $\mathcal{H}(x;1)$ (Liao, 2009). The next step is to expand the unknown function $y(x;q)$ using the Tylor series with respect to q to obtain the approximate-analytic series solution as follows:

$$y(x;q) = y_0(x) + \sum_{j=1}^{k} y_j(x)\,q^j, \tag{3.40}$$

where

$$y_j(x) = \frac{1}{j!}\frac{\partial^j y(x;q)}{\partial q^j}. \quad (3.41)$$

One has to be knowledgeable in making the suitable selections for the three auxiliary functions: the linear operator $\mathcal{L}_\beta$ which is the fractional derivative, the convergence control parameter h, and the auxiliary function $H(x)$, since the ideal selections would make the F-HAM series solution converge to the exact solution at $q = 1$ as follows:

$$y(x;1) = y_0(x) + \sum_{j=1}^{\infty} y_j(x) = Y(x), \quad (3.42)$$

where Eq. (3.42) must represent one of the solutions of the given problem. It should be noted that we can easily construct another fractional method which is based on HPM, through F-HAM by setting $h = -1$, and $H(x) = 1$ (Turkyilmazoglu, 2011) as follows:

$$(1-q)\mathcal{L}_\beta[y(x;q) - y_0(x)] + q\mathcal{N}[y(x;q)] = 0. \quad (3.43)$$

HPM is a special case of HAM (Turkyilmazoglu, 2011) (Liao, 2005). Similarly, fractional HPM is also considered a special case of F-HAM. Based on Eq. (3.41), we can extract or construct the governing equations based on the zeroth-order deformation equation. Now, define the vectors

$$\vec{y}_j = \{y_0(t), y_1(t), \dots, y_k(t)\}. \quad (3.44)$$

Then, by differentiating the zeroth-order deformation equation described in Eq. (3.36) k times with respect to q, followed by setting $q = 0$, and then dividing them by $k!$, we will extract the k^{th}-order deformation equation as shown in Eq. (3.45) as follows:

$$\mathcal{L}_\beta[y_k(x) - \psi_k y_{k-1}(x)] = h\,\mathcal{R}_k(\vec{y}_{k-1}(x)), \quad (3.45)$$

where

$$\mathcal{R}_k(\vec{y}_{k-1}(x)) = \frac{1}{(k-1)!}\frac{\partial^{k-1}\mathcal{N}[y(x;q)]}{\partial q^{k-1}}\bigg|_{q=0},\ \psi_k = \begin{cases}0, & k \le 1\\ 1, & k > 1\end{cases}. \tag{3.46}$$

3.2.5 General Structure of Fractional Optimal Homotopy Asymptotic Method (F-OHAM)

Consider the following inhomogeneous nonlinear fractional ordinary differential equation of order $\beta \in (m-1, m]$ for integer $m \ge 1$ (Hamarsheh et al., 2015)

$$\mathcal{L}_\beta\left(y(x)\right) - \mathcal{N}\left(y(x)\right) - \mathrm{G}(x) = 0,\ \ x \in [x_0, X], \tag{3.47}$$

subject to the following boundary condition

$$ß\left(y(x), \frac{\partial y}{\partial x}\right) = 0, \tag{3.48}$$

where $\mathcal{N}$ represents the nonlinear operator of the fractional differential equation, $\mathcal{L}_\beta = D^{(\beta)}$ is the linear operator which represents the Caputo fractional derivative [Definition 3.18], x is the crisp independent variable, $\mathrm{G}(x)$ represents the inhomogeneous term of Eq. (3.47), $y(x)$ is the unknown function of variable x and ß is the boundary operator. A family of equations is constructed using F-OHAM which is given by:

$$\begin{cases}(1-q)\left[\mathcal{L}_\beta\big(y(x;q)\big) - \mathrm{G}(x)\right] = \mathcal{H}(q)\left[\mathcal{L}_\beta\big(y(x;q)\big) - \mathrm{G}(x) - \mathcal{N}\big(y(x;q)\big)\right],\\ ß\left(y(x;q), \frac{\partial y(x;q)}{\partial x}\right) = 0,\end{cases} \tag{3.49}$$

where $q \in [0,1]$ is an embedding parameter, $\mathcal{H}(q) \neq 0$ is a nonzero auxiliary function, for $q = 0$ we have $\mathcal{H}(0) = 0$, and $y(x;q)$ is an unknown function.

Obviously, when $q = 0$ and $q = 1$, we get

$$\begin{cases} y(x;0) = y_0(x), \\ y(x;1) = Y(x). \end{cases} \tag{3.50}$$

Meanwhile for $q = 0$ we have

$$\begin{cases} \mathcal{L}_\beta\big(y_0(x)\big) - \mathrm{G}(x) = 0, \\ ß\left(y_0(x), \frac{\partial y_0(x)}{\partial x}\right) = 0. \end{cases} \tag{3.51}$$

Subsequently, when q increases from 0 to 1, the approximate-analytic solution of Eq. (3.47), $y(x;q)$ deforms from $y(x;0)$ to the exact solution $Y(x)$.

We choose the auxiliary function $\mathcal{H}(q)$ in the following form:

$$\mathcal{H}(q) = \sum_{j=1}^{k} S_j q^j = S_1 q + S_2 q^2 + \cdots + S_k q^k \tag{3.52}$$

where $S_1, S_2, \ldots, S_k$ are constants to be determined. Now we expand $y\big(x; q, S_j\big)$ into Taylor's series about q obtaining the approximate-analytic series solution as follows:

$$y\big(x; q, \textstyle\sum_{j=1}^{k} S_j\big) = y_0(x) + \sum_{j=1}^{k} y_j(x, \sum_{j=1}^{k} S_j)\, q^j. \tag{3.53}$$

Now, by substituting Eq. (3.53) into Eq. (3.49) followed by equating the coefficient of like powers of q, we will obtain the following linear equations. The zeroth-order series solution formula is given by Eq. (3.51) while the first-order and second-order series solution formulas are given as follows:

First-order series solution formula:

$$\begin{cases} \mathcal{L}_\beta\big(y_1(x) - y_0(x)\big) + \mathrm{G}(x) = S_1 \mathcal{N}_0\big(y_0(x)\big) \\ ß\left(y_1(x), \frac{\partial y_1}{\partial x}\right) = 0 \end{cases} \tag{3.54}$$

Second-order series solution formula:

$$\begin{cases} \mathcal{L}_\beta(\, y_2(x)) - \mathcal{L}_\beta(\, y_1(x)) = S_2\mathcal{N}_0(\, y_0(x)) + S_1[\mathcal{N}(\, y_0(x),\, y_1(x))] \\ + S_1[\mathcal{L}_\beta(\, y_1(x))] \\ \qquad ß\left(\, y_2(x), \frac{\partial\, y_2}{\partial x}\right) = 0. \end{cases} \tag{3.55}$$

.

.

.

.

.

The general k^{th}-order series solution formula with respect to $y_k(x)$ is given by:

$$\begin{cases} \mathcal{L}_\beta(\, y_k(x)) - \mathcal{L}_\beta(\, y_{k-1}(x)) = S_k\mathcal{N}_0(\, y_0(x)) + \sum_{j=1}^{k-1} S_j\left[\mathcal{L}_\beta\left(\, y_{k-j}(x)\right)\right] \\ + \sum_{j=1}^{k-1} S_j\left[\mathcal{N}_{k-j}(\sum_{i=0}^{k-1}\, y_i(x))\right] \\ \qquad ß\left(\, y_k(x), \frac{\partial\, y_k}{\partial x}\right) = 0, k = 3,4 \ldots \end{cases} \tag{3.56}$$

where $\mathcal{N}_{k-j}\left(\sum_{i=0}^{k-1}\, y_i(x)\right)$ is the coefficient q^k in the expansion of $\mathcal{N}(y(x;q))$ about the embedding parameter q.

$$\mathcal{N}\left(y\left(x; q, \sum_{j=1}^{k} S_j\right)\right) = \mathcal{N}_0(y_0(x)) + \sum_{j=1}^{k}\, \mathcal{N}_j\left(\sum_{j=0}^{k} y_j\right)q^j \tag{3.57}$$

It has been observed that the convergence of the series in Eq. (3.56) depends upon the auxiliary constants $S_1, S_2, \ldots S_k$; then at $q = 1$, we obtain the exact solution:

$$Y\left(x, \sum_{j=1}^{k} S_j\right) = y_0(x) + \sum_{j=1}^{\infty}\, y_j(x, \sum_{j=1}^{\infty} S_j). \tag{3.58}$$

Substituting Eq. (3.53) into Eq. (3.47) yields the following residual:

$$RE\left(x; q, \sum_{j=1}^{k} S_j\right) = \mathcal{L}_\beta\left(y\left(x; q, \sum_{j=1}^{k} S_j\right)\right) - \mathrm{G}(x) - \mathcal{N}\left(y\left(x; q, \sum_{j=1}^{k} S_j\right)\right) \tag{3.59}$$

If $RE = 0, y\left(x; q, \sum_{j=1}^{k} S_j\right)$ then it will yield the exact solution $Y\left(x, \sum_{j=1}^{\infty} S_j\right)$. However, this ideal outcome generally does not happen, especially in nonlinear problems. To find the optimal values of $S_j = 1,2,..,k$, we apply the Least Squares method (Nawaz et al., 2010) as follows:

$$J\left(x, \sum_{j=1}^{k} S_j\right) = \int_{x=x_0}^{X} (RE)^2\left(x; q, \sum_{j=1}^{k} S_j\right)\, dx \tag{3.60}$$

where RE is the residual of Eq. (3.47)

$$RE = \mathcal{L}_\beta(y) - \mathrm{G}(x) - \mathcal{N}(y), \tag{3.61}$$

and

$$\frac{\partial J}{\partial S_1} = \frac{\partial J}{\partial S_2} = \cdots \frac{\partial J}{\partial S_k} = 0, \tag{3.62}$$

where x_0 and X are the endpoints of the given problem to locate the desired S_j for $j = 1,2,..k$. With these constants known, the approximate-analytic solution series (of order k) is well-determined.

3.3 Research Methodology

This section describes the research methodology towards accomplishing the research objectives. The focus problem of this study are three types of FFODEs: first-order FFOIVPs, second-order FFOIVPs and second-order FFOBVPs. The proposed approximate-analytical methods: FF-HAM and FF-OHAM, are based on the concept of HAM and OHAM, consecutively and both methods are constructed for each of the three types of FFODEs.

3.3.1 First-order FFOIVPs

The development of both FF-HAM and FF-OHAM for first-order FFOIVPs is divided into two parts: theoretical development and experimental work. Results of both parts will be given in Chapter 4.

3.3.1.1 Theoretical Development of FF-HAM and FF-OHAM for First-order FFOIVPs

The theoretical development of FF-HAM and FF-OHAM for first-order FFOIVPs consists of three steps as follows:

1. Defuzzification of FFODEs, that is, of first order FFOIVPs.
2. Construction of FF-HAM and FF-OHAM for first order FFOIVPs.
3. Establishment of convergence of FF-HAM and FF-OHAM solution series for first-order FFOIVPs.

3.3.1.2 Experimental Work of FF-HAM and FF-OHAM for First-order FFOIVPs

There are two examples in the experimental work of FF-HAM and FF-OHAM for first-order FFOIVPs. Description of each example is given in Table 3.1.

Table 3.1

Description of Examples of First order FFOIVPs

Example	Description
4.1	- linear - previously solved by FRPSM (Alaroud et al., 2019) - being inhomogeneous, it is considered as a complicated linear problem
4.2	- nonlinear - a non-fuzzy fractional Ricatti problem - previously solved by VIM for non-fuzzy problem (Jafari & Tajadodi, 2010) - has never been solved in fuzzy environment. - fuzzification of the problem is created

The experimental specification for first order FFOIVPs is given in Table 3.2 while Table 3.3 presents the experimental specification for comparative study for first-order FFOIVPs.

Table 3.2

Experimental Specification of First Order FFOIVPs

Example	Solution Series Order	Fractional Order
4.1	fifth-order and eighth-order FF-HAM and FF-OHAM series solution	$\beta = 0.5$
	eighth-order FF-HAM and FF-OHAM series solution	$\beta = 1$ (for comparison with FRPSM)
4.2	sixth-order FF-HAM series solution	$\beta = 0.5$
	sixth-order FF-OHAM series solution	$\beta = 0.9$

Table 3.3

Experimental Specification of Comparative Study of First-Order FFOIVPs

Example	FF-HAM	FF-OHAM
4.1	Direct comparison made with FRPSM Indirect comparison shown with FFHPM	Direct comparison made with FRPSM Indirect comparison shown with FFHPM
4.2	No comparison	No comparison

3.3.2 Second order FFOIVPs

The development of FF-HAM and FF-OHAM for second-order FFOIVPs also involves two parts: theoretical development and experimental work. Chapter 5 consists of results of both parts for second-order FFOIVPs.

3.3.2.1 Theoretical Development of FF-HAM and FF-OHAM for Second-order FFOIVPs

There are three steps in the theoretical development as follows:

1. Defuzzification of FFODEs, that is, of second order FFOIVPs.
2. Construction of FF-HAM and FF-OHAM for second order FFOIVPs.
3. Establishment of convergence of FF-HAM and FF-OHAM solution series for second-order FFOIVPs.

3.3.2.2 Experimental Work of FF-HAM and FF-OHAM for Second-order FFIVPs

There are three examples in the experimental work of FF-HAM and FF-OHAM for second-order FFOIVPs. Description of each example is given in Table 3.4.

Table 3.4
Description of Examples of Second order FFOIVPs

Example	Description
5.1	- linear - previously solved by RKHSM (Hasan et al., 2017) - considered as a complicated linear problem because it is inhomogeneous
5.2	- linear - mixed order (integer and fractional) - previously solved by RKHSM (Hasan et al., 2017) - this problem is inhomogenous and contains the first derivative besides the second-order fractional derivative ($D^{(\beta)}y(x)$ and $y'(x)$)
5.3	- nonlinear - second-order fractional quadratic Ricatti problem - previously solved by ADM for non-fuzzy case (Momani & Shawagfeh, 2006) - has never been solved in fuzzy environment. - fuzzification of the problem is created

The experimental specification for second-order FFOIVPs is given in Table 3.5 while Table 3.6 presents the experimental specification of comparative study for second-order FFOIVPs.

Table 3.5

Experimental Specification for second-order FFOIVPs

Example	Solution Series Order	Fractional Order
5.1	fifth-order FF-HAM series solution	1.9
	fifth-order FF-OHAM series solution	2 (for comparison with RKHSM)
5.2	third-order FF-HAM and FF-OHAM series solution	1.9
	fifth-order FF-HAM and FF-OHAM series solution	2 (for comparison with RKHSM)
5.3	sixth-order FF-HAM series solution	1.5
	sixth-order FF-OHAM series solution	1.9

Table 3.6

Experimental Specification of Comparative Study for Second-Order FFOIVPs

Example	FF-HAM	FF-OHAM
5.1	Comparison with FRPSM at same fractional derivative and same series terms	Comparison with RKHSM
5.2	Comparison study with FRPSM at same fractional derivative and same series terms	Comparison with RKHSM
5.3	No comparison	No comparison

3.3.3 Second Order FFOBVPs

Similar to the first two types of FFODE considered in this study, the development of FF-HAM and FF-OHAM for second-order FFOBVPs involves two parts: theoretical development and experimental work. Results of the two parts will be given in Chapter 6.

3.3.3.1 Theoretical Development of FF-HAM and FF-OHAM for Second Order FFOBVPs

The theoretical development consists of three steps as follows:

1. Defuzzification of FFODEs, that is, of second order FFOBVPs.
2. Construction of FF-HAM and FF-OHAM for second order FFOBVPs.
3. Establishment of convergence of FF-HAM and FF-OHAM solution series for second-order FFOBVPs.

3.3.3.2 Experimental Work of FF-HAM and FF-OHAM for Second-order FFOBVPs

For second order FFOBVPs, there are three examples in the experimental work of FF-HAM and FF-OHAM. Description of each example is given in Table 3.7.

Table 3.7

Description of Examples of Second order FFOBVPs

Example	**Description**
6.1	- linear - Bagley Torvik equation - this application problem occurs quite frequently in various branches of Applied Mathematics and Mechanics - previously solved by SCM (Esmaeilbeigi et al., 2018): - considered as a complicated linear problem because of its inhomogeneous term
6.2	- linear - multi fractional orders - previously solved by RKHSM (Abdel Aal et al., 2019): - this problem is inhomogeneous and contains the fuzzy terms in differential equation and about the boundary conditions - contains multi fuzzy fractional order of first-order and second-order
6.3	- nonlinear fractional Temperature distribution equation - previously solved by green-CAS wavelet method for non-fuzzy case (Ismail et al., 2019) - has never been solved in fuzzy environment. - fuzzification of the problem is created

The experimental specification of second-order FFOBVPs is given in Table 3.8 while Table 3.9 presents the experimental specification of comparative study for second-order FFOBVPs.

Table 3.8

Experimental Specification for Second-Order FFOBVPs

Example	Solution Series Order	Fractional Order
6.1	third-order FF-HAM and FF-OHAM series solution	$\beta_1 = 1.5$
	fifth-order FF-HAM and FF-OHAM series solution	$\beta_1 = 1.5$ and α-cut $= 0.5$ (for comparison with SCM fifth order solution series)
6.2	sixth-order FF-HAM and FF-OHAM series solution	Second-order $\beta_1 = 1.8$ First-order $\beta_2 = 0.9$
6.3	tenth-order FF-HAM and FF-OHAM series solution	$\beta_1 = 1.9$

Table 3.9

Experimental Specification of Comparative Study for Second-Order FFOBVPs

Example	FF-HAM	FF-OHAM
6.1	Comparison with SCM	Comparison with SCM
6.2	No comparison	Comparison with exact solutions
6.3	No comparison	No comparison

CHAPTER FOUR
FF-HAM AND FF-OHAM FOR FIRST-ORDER FUZZY FRACTIONAL ORDINARY INITIAL VALUE PROBLEMS

4.1 Introduction

This chapter discusses the development of the proposed FF-HAM and FF-OHAM on first-order fuzzy fractional ordinary initial value problems (FFOIVPs), which is the first type of FFODEs considered in this study. This chapter first elaborates on the theoretical part of the development of the methods, consisting of three steps. This is followed by the experimental part of the development, which shows the solving process of two first-order FFOIVPs using the developed methods. Towards the end of the chapter, based on the theoretical and experimental parts of the study, a discussion of the developed methods in terms of characteristics and advantages is provided.

4.2 Theoretical Development of FF-HAM and FF-OHAM for First-order FFOIVPs

The theoretical development of FF-HAM and FF-OHAM consists of three steps as follows:

1. Defuzzification of FFODEs

In this step, the first-order FFODEs are defuzzified under the sense of first-order fuzzy fractional Caputo derivative as explained in Section 3.2.3. Furthermore, it includes defuzzification of the fuzzy initial condition for lower and upper bound based on the extension principle theory and properties of the fuzzy calculus such as α-cut definition and fuzzy number definition.

2. Construction of FF-HAM and FF-OHAM for first-order FFOIVPs

The FF-HAM and FF-OHAM are constructed to handle the first-order FFOIVPs based on the extension principle theory and properties of the fuzzy calculus such as α-cut definition and fuzzy number definition.

3. Establishment of convergence of FF-HAM and FF-OHAM solution series

Establishing the convergence analyses of FF-HAM and FF-OHAM on first-order FFOIVPs is done depending on the minimization of the residual form of the given problem in order to select the optimal convergence control parameter of FF-HAM (h) and the optimal convergence control parameters of FF-OHAM ($S_1, S_2, S_3, S_4, \dots S_k$) (here k is the order of the FF-OHAM series).

The theoretical development of each of the two proposed methods follows the three steps. However, Step One needs to be done only once since it is applicable for both methods. In this study, the construction of FF-HAM will be done first, followed by FF-OHAM.

4.3 Theoretical Development of FF-HAM for First Order FFOIVPs

For first order FFOIVPs, the proposed FF-HAM is theoretically developed in three steps.

4.3.1 Defuzzification of first order FFIVPs

Firstly, we will be dealing with the defuzzification of FFOIVP of order $0 < \beta \leq 1$. Consider the following FFOIVP:

$$\begin{cases} \tilde{y}^{(\beta)}(x) = \tilde{g}\big(x, \tilde{y}(x)\big), \ \ x \in [x_0, X], \\ \qquad\qquad \tilde{y}(x_0) = \tilde{y}_0, \end{cases} \tag{4.1}$$

where $\tilde{g}$ is a fuzzy function of the non-fuzzy independent variable x and fuzzy dependent variable $\tilde{y}$, $\tilde{y}^{(\beta)}$ represents the first-order fuzzy fractional Caputo derivative [Definition 3.21] of the fuzzy function $\tilde{y}(x)$ mentioned in Definition 3.8, $\tilde{y}(x_0)$ represents the initial condition and $\tilde{y}_0$ is a fuzzy number which is defined as in Definition 3.7.

Now, based on Definition 3.5, $\forall\alpha \in [0,1]$ we can observe the fuzzy function $\tilde{y}$ by $[\tilde{y}]_\alpha = [\underline{y}, \overline{y}]_\alpha$. That means for $x \in [x_0, X]$, the set of α-cut of $\tilde{y}(x)$ can be denoted by:

$$[\tilde{y}(x)]_\alpha = \left[\underline{y}(x;\alpha), \overline{y}(x;\alpha)\right], \tag{4.2}$$

$$[\tilde{y}(x_0)]_\alpha = \left[\underline{y}(x_0;\alpha), \overline{y}(x_0;\alpha)\right]. \tag{4.3}$$

In addition, $\forall\alpha \in [0,1]$ we can write the fuzzy function $\tilde{g}(x, \tilde{y})$ as follows:

$$[\tilde{g}(x, \tilde{y})]_\alpha = \left[\underline{g}(x, \tilde{y};\alpha), \overline{g}(x, \tilde{y};\alpha)\right], \tag{4.4}$$

where

$$\begin{cases} \underline{g}(x, \tilde{y};\alpha) = g_l\left[x, \underline{y}, \overline{y}\right]_\alpha, \\ \overline{g}(x, \tilde{y};\alpha) = g_u\left[x, \underline{y}, \overline{y}\right]_\alpha. \end{cases} \tag{4.5}$$

Since $\tilde{y}^{(\beta)}(x;\alpha) = \tilde{g}(x, \tilde{y}(x;\alpha);\alpha)$, by employing the extension principle theory [Definition 3.4] and fuzzy number definition [Definition 3.6], we will define the following membership function

$$\begin{cases} \underline{g}(x, \tilde{y}(x;\alpha)) = min\{\tilde{g}(x, \tilde{\mu}(\alpha)) | \tilde{\mu}(\alpha) \in \tilde{y}(x;\alpha)\} \\ \overline{g}(x, \tilde{y}(x;\alpha)) = max\{\tilde{g}(x, \tilde{\mu}(\alpha)) | \tilde{\mu}(\alpha) \in \tilde{y}(x;\alpha)\}' \end{cases} \tag{4.6}$$

where

$$\begin{cases} \underline{g}(x, \tilde{y}(x;\alpha)) = g_l\left(x, \underline{y}(x;\alpha), \overline{y}(x;\alpha)\right) = g_l\big(x, \tilde{y}(x;\alpha)\big) \\ \overline{g}(x, \tilde{y}(x;\alpha)) = g_u\left(x, \underline{y}(x;\alpha), \overline{y}(x;\alpha)\right) = g_u\big(x, \tilde{y}(x;\alpha)\big) \end{cases} \tag{4.7}$$

This procedure of defuzzification of first-order FFOIVP will be used in the fuzzification of HAM and OHAM for solving FFDEs involving ODE of the first order.

Remark 4.1: The defuzzification procedure deals with the first-order FFOIVPs. Meanwhile, in this study, several of the first-order non-fuzzy FOIVPs will be fuzzified in terms of the function or initial condition depending on the nature of the physical or engineering problems. The same procedure will also be used for defuzzification.

Remark 4.2: In this study, the fuzzy function described by $\tilde{g}\big(x, \tilde{y}(x)\big)$ could be homogenous or inhomogeneous and problems could involve fuzzy coefficients numbers such that

$$\tilde{y}^{(\beta)}(x) = \tilde{a}\tilde{g}\big(x, \tilde{y}(x)\big) \ + \tilde{b}\mathrm{G}(x), \tag{4.8}$$

where $\tilde{a}$ and $\tilde{b}$ are the fuzzy numbers.

Remark 4.3: The existence and uniqueness of the solution of first-order FFOIVP have been previously introduced and proven (Salahshour et al., 2012).

4.3.2 Construction of FF-HAM for First-order FFOIVPs

The procedure in the construction of the proposed FF-HAM involves reformulation of the existing F-HAM for solving crisp fractional IVPs followed by the formulation of FF-HAM for first-order FFOIVPs. This section utilizes the definitions and properties of fuzzy set theory provided in Section 3.2 including the fuzzy number, α-cut, and the extension principle theory to formulate a new procedure to approximate the solution of the first-order linear and nonlinear FFOIVPs by FF-HAM, where the process begins with fuzzifying the existing F-HAM followed by its defuzzification to provide a new procedure for solving linear and nonlinear FFOIVPs of order $0 < \beta \leq 1$.

Consider the following first-order FFOIVP:

$$\begin{cases} \tilde{y}^{(\beta)}(x) = \tilde{g}\big(x, \tilde{y}(x)\big) + \tilde{G}(x), \quad x \in [x_0, X], \\ \qquad\qquad \tilde{y}(x_0) = \tilde{y}_0, \end{cases} \tag{4.9}$$

here, $\tilde{y}(x)$ refer the unknown fuzzy function of crisp variable x, $\tilde{g}\big(x, \tilde{y}(x)\big)$ is the fuzzy function [Definition 3.8] of fuzzy variable $\tilde{y}$ and crisp variable x, $\tilde{G}(x)$ represents the inhomogeneous fuzzy term of crisp variable x, and $\tilde{y}_0$ is the fuzzy number [Definition 3.7]. Now, according to the defuzzification procedure of Eq. (4.1), we can generalize Eq. (4.9) $\forall \alpha \in [0,1]$ [Definition 3.5] into the following form:

$$\mathcal{N}[\tilde{y}(x;\alpha)] = 0, \tag{4.10}$$

such that

$$\begin{cases} \mathcal{N}\left[\underline{y}(x;\alpha)\right] = 0, \\ \mathcal{N}[\overline{y}(x;\alpha)] = 0. \end{cases} \tag{4.11}$$

We can rewrite Eq. (4.10) as a lower bound and upper bound function by utilizing the theory of the extension principle [Definition 3.4] as follows:

$$g_l([\tilde{y}(x;q)]) = min\{\, \mathcal{N}[\underline{y}(x;q)]_\alpha, \mathcal{N}[\overline{y}(x;q)]_\alpha \colon \mu | \mu \in [\tilde{y}(x;\alpha)]\} \tag{4.12}$$

$$g_u([\tilde{y}(x;q)]) = max\{\mathcal{N}[\underline{y}(x;q)]_\alpha, \mathcal{N}[\overline{y}(x;q)]_\alpha \colon \mu | \mu \in [\tilde{y}(x;\alpha)]\} \tag{4.13}$$

where μ referred to the membership function of Eq. (4.9) using Definition 3.1. We can construct the zeroth-order deformation equation (Liao, 2004) of Eq.(4.9) as follows:

$$\begin{cases} (1-q)\underline{\mathcal{L}}_\beta\left[[\underline{y}(x;q)]_\alpha - \underline{y}_0(x;\alpha)\right] = q\underline{h}(\alpha)H(x)\, g_l([\tilde{y}(x;q)]_\alpha), \\ (1-q)\overline{\mathcal{L}}_\beta\left[[\overline{y}(x;q)]_\alpha - \overline{y}_0(x;\alpha)\right] = q\overline{h}(\alpha)H(x)\, g_u([\tilde{y}(x;q)]_\alpha). \end{cases} \tag{4.14}$$

As has been mentioned in Section 3.2.4 2, $0 \leq q \leq 1$ represents the embedding parameter, $\tilde{h}(\alpha) = \left[\underline{h}(\alpha), \overline{h}(\alpha)\right]$ refers to the convergence control parameter with the property $\tilde{h}(\alpha) \neq 0$, $H(x)$ is the auxiliary functions while the operators $\underline{\mathcal{L}}_\beta = \frac{\partial^{(\beta)}\left[\underline{y}(x;q)\right]_\alpha}{\partial x^{(\beta)}}$ and $\overline{\mathcal{L}}_\beta = \frac{\partial^{(\beta)}\overline{y}[(x;q)]_\alpha}{\partial x^{(\beta)}}$ [Definition 3.21] are the auxiliary linear operators.

Unlike OHAM, HAM depends on the initial approximation besides the auxiliary functions. Here, we will define the fuzzy initial approximation $\tilde{y}_0(x)$ of the fuzzy function $\tilde{y}(x;\alpha)$ $\forall\alpha \in [0,1]$ as $[\tilde{y}_0(x)]_\alpha = \left[\underline{y}_0(x;\alpha), \overline{y}_0(x;\alpha)\right] = \left[\underline{y}_0, \overline{y}_0\right]$, where $\tilde{y}_0$ are triangular fuzzy numbers [Definition 3.7].

It is possible to conclude the initial approximation $\tilde{y}_0$ and the exact analytical solution $\tilde{Y}(x)$ for Eq.(4.9) by setting $q = 0$ and $q = 1$ respectively as follows:

$$\begin{cases} \left[\underline{y}(x;0)\right]_\alpha = \underline{y}_0(x;\alpha), \\ [\overline{y}(x;0)]_\alpha = \overline{y}_0(x;\alpha). \end{cases} \qquad \begin{cases} \left[\underline{y}(x;1)\right]_\alpha = \underline{Y}(x;\alpha), \\ [\overline{y}(x;1)]_\alpha = \overline{Y}(x;\alpha). \end{cases} \tag{4.15}$$

At the time when the embedding parameter changes from zero to one, the approximate fuzzy solutions $[\tilde{y}(x;q)]_\alpha$, deforms from the initial approximation $[\tilde{y}_0(x)]_\alpha$, to the exact solution $[\tilde{Y}(x)]_\alpha$. Now, by expanding the approximate solution $[\tilde{y}(x;q)]_\alpha$ as a Taylor series with respect to q, $\forall\alpha \in [0,1]$, we obtain

$$\begin{cases} [y(x;q)]_\alpha = \underline{y}_0(x;\alpha) + \sum_{j=1}^{k} \underline{y}_j(x;\alpha)\, q^j \\ [\overline{y}(x;q)]_\alpha = \overline{y}_0(x;\alpha) + \sum_{j=1}^{k} \overline{y}_j(x;\alpha)\, q^j \end{cases} \tag{4.16}$$

where

$$\begin{cases} \underline{y}_j(x;\alpha) = \frac{1}{j!}\frac{\partial^j[\underline{y}(x;q)]_\alpha}{\partial q^j}\Big|_{q=0} \\ \overline{y}_j(x;\alpha) = \frac{1}{j!}\frac{\partial^j[\overline{y}(x;q)]_\alpha}{\partial q^j}\Big|_{q=0} \end{cases} \tag{4.17}$$

The optimal selection for the initial guess $[\tilde{y}_0(x)]_\alpha$ and the auxiliary functions including linear operator $\tilde{\mathcal{L}}_\beta = [\underline{\mathcal{L}}_\beta, \overline{\mathcal{L}}_\beta]$ [Definition 3.21], the auxiliary function $H(x)$, and the convergence control parameter $\tilde{h}(\alpha) = [\underline{h}(\alpha), \overline{h}(\alpha)]$ make the series solution converge to the exact solution at $q = 1$, such that

$$\begin{cases} [\underline{Y}(x)]_\alpha = \underline{y}_0(x;\alpha) + \sum_{j=1}^{\infty} \underline{y}_j(x;\alpha) \\ [\overline{Y}(x)]_\alpha = \overline{y}_0(x;\alpha) + \sum_{j=1}^{\infty} \overline{y}_j(x;\alpha) \end{cases}. \tag{4.18}$$

Clearly, it is impossible to find the analytical solution with FF-HAM as an infinite series of solutions especially for the nonlinear FFODEs, so we will truncate the series after k times, by defining the vectors

$$\begin{cases} \vec{\underline{y}}_j(x;\alpha) = \{\underline{y}_0(x;\alpha), \underline{y}_1(x;\alpha), \dots, \underline{y}_k(x;\alpha)\}, \\ \vec{\overline{y}}_j(x;\alpha) = \{\overline{y}_0(x;\alpha), \overline{y}_1(x;\alpha), \dots, \overline{y}_k(x;\alpha)\}. \end{cases} \tag{4.19}$$

For the purpose of extracting the k^{th}-order deformation equation, we will be differentiating the zeroth-order deformation equation in Eq. (4.14) k times with respect to the embedding parameter q, followed by setting $q = 0$ and after that dividing them by $k!$, obtaining

$$\begin{cases} \underline{\mathcal{L}}_\beta \left[\underline{y}_k(x;\alpha) - \psi_k \underline{y}_{k-1}(x;\alpha)\right] = \underline{h}(\alpha)\mathcal{R}_k(\vec{\underline{y}}_{k-1}(x;\alpha)) \\ \overline{\mathcal{L}}_\beta\left[\overline{y}_k(x;\alpha) - \psi_k \overline{y}_{k-1}(x;\alpha)\right] = \overline{h}(\alpha)\mathcal{R}_k(\vec{\overline{y}}_{k-1}(x;\alpha)) \end{cases} \tag{4.20}$$

where

$$\begin{cases} \mathcal{R}_k\left(\underline{\vec{y}}_{k-1}(x;\alpha)\right) = \frac{1}{(k-1)!}\frac{\partial^{k-1} g_l([\tilde{y}(x;q)]_\alpha)}{\partial q^{k-1}}\Big|_{q=0} \\ \mathcal{R}_k\left(\overline{\vec{y}}_{k-1}(x;\alpha)\right) = \frac{1}{(k-1)!}\frac{\partial^{k-1} g_u([\tilde{y}(x;q)]_\alpha)}{\partial q^{k-1}}\Big|_{q=0} \end{cases} \tag{4.21}$$

for $k \geq 1$ we can construct the solution of the k^{th}-order deformation as follows:

$$\begin{cases} \underline{y}_k(x;\alpha) = \psi_k \underline{y}_{k-1}(x;\alpha) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \mathcal{R}_k(\underline{\vec{y}}_{k-1}(x;\alpha)) \\ \overline{y}_k(x;\alpha) = \psi_k \overline{y}_{k-1}(x;\alpha) + \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)} \mathcal{R}_k(\overline{\vec{y}}_{k-1}(x;\alpha)) \end{cases}, \tag{4.22}$$

such that

$$\psi_k = \begin{cases} 0, & k \leq 1 \\ 1, & k > 1 \end{cases}. \tag{4.23}$$

where $\tilde{\mathcal{L}}_\beta^{\ -1} = \tilde{\mathcal{J}}^{(\beta)} = \left[\underline{\mathcal{J}}^{(\beta)}, \overline{\mathcal{J}}^{(\beta)}\right]$ [Definitions 3.18 and 3.19], such that the approximate solution constructed by FF-HAM can be extracted from Eq. (4.22) as follows:

$$\begin{cases} \underline{y}(x;\alpha;\underline{h}) = \underline{y}_0(x;\alpha) + \underline{y}_1(x;\alpha;\underline{h}) + \cdots + \underline{y}_k(x;\alpha;\underline{h}), \\ \overline{y}(x;\alpha;\overline{h}) = \overline{y}_0(x;\alpha) + \overline{y}_1(x;\alpha;\overline{h}) + \cdots + \overline{y}_k(x;\alpha;\overline{h}). \end{cases} \tag{4.24}$$

It is remarkable that the approximate series solution by FF-HAM in Eq. (4.24) involves the convergence control parameter $\tilde{h}$, and while taking into account the appropriate choice of $\tilde{h}$, we can ensure the convergence series solution of Eq. (4.9). The optimal selection of the convergence control parameter will be discussed in the next section.

The exact analytical solution of Eq. (4.9) is given by

$$\begin{cases} \underline{Y}(x) = \sum_{j=0}^{\infty} \underline{y}_j(x;\alpha;\underline{h}), \\ \overline{Y}(x) = \sum_{j=0}^{\infty} \overline{y}_j(x;\alpha;\overline{h}). \end{cases} \tag{4.25}$$

4.3.3 Establishment of Convergence of FF-HAM Solution Series for First-order FFOIVPs

In this theoretical development step, the convergence of FF-HAM solution series for first-order FFOIVPs is established. From Section 4.3.2, we can say that the convergence of series solution by FF-HAM of Eq. (4.9) would depend largely on the convergence control parameter $\tilde{h}$. Thus, proper selection of the convergence parameter ensures that an accurate convergent solution is obtained. One popular technique to select $\tilde{h}$, is the technique that builds towards obtaining the approximate solution with minimum residual error.

The technique to select $\tilde{h}$ is related to the minimum residual error. Assume that the residual of Eq. (4.9) is denoted by $\widetilde{ER} = \left[\underline{ER}, \overline{ER}\right]$ and defined by the substitution of the approximate series solution by HAM in Eq. (4.24) in the given FFOIVP in Eq.(4.9) as follows :

$$\begin{cases} \underline{ER}(x;\alpha;\underline{h}) = \underline{y}^{(\beta)}(x;\alpha;\underline{h}) - g_l\left(x;\tilde{y}(x;\alpha;\tilde{h})\right) - \underline{\mathrm{G}}(x;\alpha) \\ \overline{ER}(x;\alpha;\overline{h}) = \overline{y}^{(\beta)}(x;\alpha;\overline{h}) - g_u\left(x;\tilde{y}(x;\alpha;\tilde{h})\right) - \overline{\mathrm{G}}(x;\alpha) \end{cases} \tag{4.26}$$

Now $\forall\alpha \in [0,1]$, using the corresponding squared residual error, followed by integrating the residual over the given interval $x \in [x_0, X]$, we obtain the following form:

$$\begin{cases} \underline{SER}(x;\alpha;\underline{h}) = \int_{x_0}^{X}\left[\underline{ER}(x;\alpha;\underline{h})\right]^2 dx \\ \overline{SER}(x;\alpha;\overline{h}) = \int_{x_0}^{X}\left[\overline{ER}(x;\alpha;\overline{h})\right]^2 dx \end{cases}. \tag{4.27}$$

This is followed by setting the values of fractional order $0 < \beta \leq 1$, and then depending on the partial derivatives of $\widetilde{SER}(x;\alpha;\tilde{h})$ with respect to $\tilde{h}(\alpha)$ for all $\alpha \in [0,1]$ [Definition 3.5], we can determine the optimal value of the convergence control parameter $\tilde{h}(\alpha)$ by solving the system of nonlinear equations generated by the partial derivatives of $\widetilde{SER}(x;\alpha;\tilde{h})$ numerically in terms of $\tilde{h}(\alpha)$. Assessment of the optimal value among the best values of $\tilde{h}(\alpha)$ is aimed towards obtaining the most accurate FF-HAM series solution $\tilde{y}(x;\alpha)$, for all $\alpha \in [0,1]$ by plotting the $\tilde{h}$-curves in terms of upper and lower FF-HAM solutions. These curves define the best region of the $\tilde{h}(\alpha)$ values, which are the horizontal line segment with respect to $\tilde{y}(x;\alpha)$ for $x_0 < x < X$. It is familiar in fuzzy environment to find the contract h-curves and to find the optimal value $\tilde{h}(\alpha)$ for each $\alpha \in [0,1]$. We get to choose the best value of $\tilde{h}(\alpha)$ that provides the best accurate solution of the FFOIVP with its corresponding fuzzy level set $\tilde{\alpha} = [\underline{\alpha}, \overline{\alpha}]$, and then applied $\underline{\alpha}$ for each lower-level set to get the best lower approximate solution. A similar step is applied to $\overline{\alpha}$ to determine the best upper solution.

4.4 Theoretical Development of FF-OHAM for First-order FFOIVPs

FF-OHAM can be considered as a modification of FF-HAM based on minimizing the residual error where the control and adjustment of the convergence region S are provided conveniently. Similar to FF-HAM, the theoretical development of FF-OHAM also consists of three steps.

4.4.1 Defuzzification of FFODE

In this study, Step One needs to be done only once since it is applicable for both FF-HAM and FF-OHAM. Step One, which refers to the defuzzification of FFODE has been done in Section 4.3.1.

4.4.2 Construction of FF-OHAM for First-order FFOIVPs

The procedure in the construction of the proposed FF-OHAM involves reformulation of the existing F-OHAM for solving crisp fractional IVPs followed by the formulation of FF-OHAM for first-order FFOIVPs. Similar to the formulation of FF-HAM for first-order FFOIVPs, the formulation of FF-OHAM also utilizes the definitions and properties and definitions of fuzzy set theory given in Section 3.2. The procedure of the formulation begins with the process of fuzzifying the existing F-OHAM followed by its defuzzification. Consider the following FFOIVP of order $0 < \beta \leq 1$ (Khan et al., 2014).

$$\begin{cases} \tilde{\mathcal{L}}_\beta\, \tilde{y}(x) - \tilde{\mathcal{N}}\left(y(x)\right) - \tilde{\mathrm{G}}(x) = 0, \;\; x \in [x_0, X], \\ \tilde{y}(x_0) = \tilde{y}_0. \end{cases} \tag{4.28}$$

From Section 3.2.5, $\tilde{\mathcal{N}}\left(y(x)\right) = \left[g_l\left(\tilde{y}(x)\right), g_u\left(\tilde{y}(x)\right)\right]$ is such that for all $\alpha \in [0,1]$ [Definition 3.5] we rewrite Eq. (4.28)) in the following lower bound based on the extension principle [Definition 3.4]:

$$\begin{cases} \underline{\mathcal{L}}_\beta\left(\underline{y}(x;\alpha)\right) - \underline{\mathrm{G}}(x;\alpha) - g_l\left(x, \tilde{y}(x;\alpha)\right) = 0, \;\; x \in [x_0, X], \\ \underline{y}(x_0) = \underline{y}_0, \end{cases} \tag{4.29}$$

and for the upper bound, we have

$$\begin{cases} \overline{\mathcal{L}}_\beta\left(\overline{y}(x;\alpha)\right) - \overline{\mathrm{G}}(x;\alpha) - g_u\left(x, \tilde{y}(x;\alpha)\right) = 0, \;\; x \in [x_0, X], \\ \overline{y}(x_0) = \overline{y}_0. \end{cases} \tag{4.30}$$

Based on F-OHAM described in Section 3.2.5, Eq. (4.29) and Eq. (4.30) can be written as follows:

$$\begin{cases} (1-q)\left[\underline{\mathcal{L}}_\beta\left(\left[\underline{y}(x;q)\right]_\alpha\right)-\underline{\mathrm{G}}(x;\alpha)\right]=\underline{\mathcal{H}}(q;\alpha)\left[\underline{\mathcal{L}}_\beta\left(\left[\underline{y}(x;q)\right]_\alpha\right)\right] \\ \quad -\underline{\mathcal{H}}(q;\alpha)\left[\underline{\mathrm{G}}(x;\alpha)\right]-\underline{\mathcal{H}}(q;\alpha)\ g_l([\tilde{y}(x;q)]_\alpha) \\ (1-q)\left[\overline{\mathcal{L}}_\beta([\overline{y}(x;q)]_\alpha)-\overline{\mathrm{G}}(x;\alpha)\right]=\overline{\mathcal{H}}(q;\alpha)\left[\overline{\mathcal{L}}_\beta([\overline{y}(x;q)]_\alpha)\right] \\ \quad -\overline{\mathcal{H}}(q;\alpha)\left[\overline{\mathrm{G}}(x;\alpha)\right]-\overline{\mathcal{H}}(q;\alpha)[g_u([\tilde{y}(x;q)]_\alpha)] \\ \tilde{y}(x_0)=\tilde{y}_0, \end{cases} \tag{4.31}$$

where $\tilde{\mathcal{L}}_\beta=\left[\underline{\mathcal{L}}_\beta,\overline{\mathcal{L}}_\beta\right]=\left[\frac{\partial^{(\beta)}\left[\underline{y}(x;q)\right]_\alpha}{\partial x^{(\beta)}},\frac{\partial^{(\beta)}[\overline{y}(x;q)]_\alpha}{\partial x^{(\beta)}}\right]$ [Definition 3.21] are the linear operators and $q\in[0,1]$ is an embedding parameter, $\widetilde{\mathcal{H}}(q;\alpha)=\left[\underline{\mathcal{H}}(q),\overline{\mathcal{H}}(q)\right]_\alpha$ is a nonzero auxiliary fuzzy function for $q\neq 0$, and $[\tilde{y}(x;q)]_\alpha$ is an unknown fuzzy function [Definition 3.8].

Obviously, for $q=0$ and $q=1$, we obtain

$$\begin{cases} \left[\underline{y}(x;0)\right]_\alpha=\underline{y}_0(x;\alpha), & \left[\underline{y}(x;1)\right]_\alpha=\underline{Y}(x;\alpha), \\ [\overline{y}(x;0)]_\alpha=\overline{y}_0(x;\alpha). & [\overline{y}(x;1)]_\alpha=\overline{Y}(x;\alpha). \end{cases} \tag{4.32}$$

Thus, as q increases from 0 to 1, the approximate solution $[\tilde{y}(x;q)]_\alpha$ changes from $\tilde{y}_0(x;\alpha)$ to the exact solution of Eq.(4.29), $\underline{Y}(x;\alpha)$, and Eq.(4.30), $\overline{Y}(x;\alpha)$, where $\tilde{y}_0(x;\alpha)$ is obtained from Eq.(4.31) for $q=0$ as follows:

$$\begin{cases} \underline{\mathcal{L}}_\beta\left(\left[\underline{y}_0(x;0)\right]_\alpha\right)-\underline{\mathrm{G}}(x;\alpha)=0, & \underline{y}\,(x_0;0)=\underline{y}_0, \\ \overline{\mathcal{L}}_\beta\left([\overline{y}_0(x;0)]_\alpha\right)-\overline{\mathrm{G}}(x;\alpha)=0, & \overline{y}\,(x_0;0)=\overline{y}_0, \end{cases} \tag{4.33}$$

and by applying the Riemann-Liouville operator $\tilde{\mathcal{J}}^{(\beta)}$ [Definition 3.19] on the left side of Eq. (4.33), and based on the fractional Caputo derivative [Definition 3.18], we have

$$\begin{cases} \underline{y}_0(x;\alpha)-\underline{\mathcal{J}}^{(\beta)}\,\underline{\mathrm{G}}(x;\alpha)=0, & \underline{y}\,(x_0;0)=\underline{y}_0, \\ \overline{y}_0(x;\alpha)-\overline{\mathcal{J}}^{(\beta)}\,\overline{\mathrm{G}}(x;\alpha)=0, & \overline{y}\,(x_0;0)=\overline{y}_0. \end{cases} \tag{4.34}$$

We choose the auxiliary function $\widetilde{\mathcal{H}}(q;\alpha)$ for Eq.(4.31) in the following form:

$$\begin{cases} \underline{\mathcal{H}}(q;\alpha) = \underline{S}_1(\alpha)q + \underline{S}_2(\alpha)q^2 + \cdots = \sum_{j=1}^{k} \underline{S}_j(\alpha)q^j, \\ \underline{\mathcal{H}}(q;\alpha) = \overline{S}_1(\alpha)q + \overline{S}_2(\alpha)q^2 + \cdots = \sum_{j=1}^{k} \overline{S}_j(\alpha)q^j, \end{cases} \tag{4.35}$$

where $\tilde{S}_1(\alpha) = [\underline{S}_1(\alpha), \overline{S}_1(\alpha)], \tilde{S}_2(\alpha) = [\underline{S}_2(\alpha), \overline{S}_2(\alpha)]$, ... [Definitions 3.4 and 3.5] are the constants to be found for all $\alpha \in [0,1]$. Expanding $\left[\tilde{y}\left(x; q, S_j(\alpha)\right)\right]_\alpha$ into Taylor's series about q, we obtain the approximate series solution

$$\begin{cases} \left[\underline{y}\left(x; q, \underline{S}_j(\alpha)\right)\right]_\alpha = \underline{y}_0(x;\alpha) + \sum_{j=1}^{k} \left[\underline{y}_j\left(x, \underline{S}_j(\alpha)\right)\right]_\alpha q^j, \\ \left[\overline{y}\left(x; q, \overline{S}_j(\alpha)\right)\right]_\alpha = \overline{y}_0(x;\alpha) + \sum_{j=1}^{k} \left[\overline{y}_j\left(x, \overline{S}_j(\alpha)\right)\right]_\alpha q^j. \end{cases} \tag{4.36}$$

Now, as has been mentioned in Section 3.2.5, by substituting Eq. (4.36) into Eq. (4.31) and then by collecting the coefficient of like powers of q, we will obtain the linear equations where the zeroth-order problem is given by Eq. (4.34) while the first-order and second-order problems are given as follows:

First-order Problem:

$$\begin{cases} \underline{\mathcal{L}}_\beta\left(\underline{y}_1(x;\alpha) - \underline{y}_0(x;\alpha)\right) + \underline{\mathrm{G}}(x;\alpha) = \underline{S}_1(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right), \\ \overline{\mathcal{L}}_\beta\left(\overline{y}_1(x;\alpha) - \overline{y}_0(x;\alpha)\right) + \overline{\mathrm{G}}(x;\alpha) = \overline{S}_1(\alpha) g_{u_0}\left(\tilde{y}_0(x;\alpha)\right), \end{cases} \tag{4.37}$$

$$\tilde{y}_1(x_0) = \tilde{0}, \tag{4.38}$$

and by applying the Riemann-Liouville operator $\tilde{\mathcal{J}}^{(\beta)}$ [Definition 3.19] on the left side of Eq. (4.37), we have

$$\begin{cases} \underline{y}_1(x;\alpha) = \underline{y}_0(x;\alpha) + \underline{\mathcal{J}}^{(\beta)}\left(\underline{S}_1(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right) - \underline{\mathrm{G}}(x;\alpha)\right), \\ \overline{y}_1(x;\alpha) = \overline{y}_0(x;\alpha) + \overline{\mathcal{J}}^{(\beta)}\left(\overline{S}_1(\alpha) g_{u_0}\left(\tilde{y}_0(x;\alpha)\right) - \overline{\mathrm{G}}(x;\alpha)\right), \end{cases} \tag{4.39}$$

$$\tilde{y}_1(x_0) = \tilde{0}. \tag{4.40}$$

Second-order Problem:

$$\begin{cases} \underline{\mathcal{L}}_\beta\left(\underline{y}_2(x;\alpha)\right) - \underline{\mathcal{L}}_\beta\left(\underline{y}_1(x;\alpha)\right) = \underline{S}_2(\alpha) g_{l_0}(\tilde{y}_0(x;\alpha)) \\ \quad +\underline{S}_1(\alpha)\left[\underline{\mathcal{L}}_\beta\left(\underline{y}_1(x;\alpha)\right) + g_{l_1}(\tilde{y}_0(x;\alpha),\, \tilde{y}_1(x;\alpha))\right] \\ \overline{\mathcal{L}}_\beta\left(\overline{y}_2(x;\alpha)\right) - \overline{\mathcal{L}}_\beta\left(\overline{y}_1(x;\alpha)\right) = \overline{S}_2(\alpha) g_{u_0}(\tilde{y}_0(x;\alpha)) \\ \quad +\overline{S}_1(\alpha)\left[\overline{\mathcal{L}}_\beta\left(\overline{y}_1(x;\alpha)\right) + g_{u_1}(\tilde{y}_0(x;\alpha),\, \tilde{y}_1(x;\alpha))\right] \end{cases} \tag{4.41}$$

$$\tilde{y}_2(x_0) = \tilde{0} \tag{4.42}$$

And by applying the Riemann-Liouville operator $\tilde{\mathcal{J}}^{(\beta)}$ [Definition 3.19] on the left side of Eq. (4.41), we have

$$\begin{cases} \underline{y}_2(x;\alpha) = \left(1 + \underline{S}_1(\alpha)\right) \underline{y}_1(x;\alpha) + \\ \underline{\mathcal{J}}^{(\beta)}\left(\underline{S}_2(\alpha) g_{l_0}(\tilde{y}_0(x;\alpha)) + \underline{S}_1(\alpha) g_{l_1}(\tilde{y}_0(x;\alpha),\, \tilde{y}_1(x;\alpha))\right) \\ \overline{y}_2(x;\alpha) = \left(1 + \overline{S}_1(\alpha)\right) \overline{y}_1(x;\alpha) + \\ \overline{\mathcal{J}}^{(\beta)}\left(\overline{S}_2(\alpha) g_{u_0}(\tilde{y}_0(x;\alpha)) + \overline{S}_1(\alpha) g_{u_1}(\tilde{y}_0(x;\alpha),\, \tilde{y}_1(x;\alpha))\right) \end{cases} \tag{4.43}$$

$$\tilde{y}_2(x_0) = \tilde{0} \tag{4.44}$$

Then, the general k^{th} -order formula with respect to $\tilde{y}_k(x;\alpha)$ is given by:

$$\begin{cases} \underline{\mathcal{L}}_\beta\left(\underline{y}_k(x;\alpha)\right) - \underline{\mathcal{L}}_\beta\left(\underline{y}_{k-1}(x;\alpha)\right) = \underline{S}_k(\alpha) g_{l_0}(\tilde{y}_0(x;\alpha)) \\ \quad + \sum_{j=1}^{k-1} \underline{S}_j(\alpha)\left[\underline{\mathcal{L}}_\beta\left(\underline{y}_{k-j}(x;\alpha)\right) + g_{l_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)\right] \\ \overline{\mathcal{L}}_\beta\left(\overline{y}_k(x;\alpha)\right) - \overline{\mathcal{L}}_\beta\left(\overline{y}_{k-1}(x;\alpha)\right) = \overline{S}_k(\alpha) g_{u_0}(\tilde{y}_0(x;\alpha)) \\ \quad + \sum_{j=1}^{k-1} \overline{S}_j(\alpha)\left[\overline{\mathcal{L}}_\beta\left(\overline{y}_{k-j}(x;\alpha)\right) + g_{u_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)\right] \end{cases} \tag{4.45}$$

$$\tilde{y}_k(x_0) = \tilde{0} \tag{4.46}$$

And by applying the Riemann-Liouville operator $\tilde{\mathcal{J}}^{(\beta)}$ [Definition 3.19] on the left side of Eq. (4.45), we have

$$\begin{cases} \underline{y}_k(x;\alpha) = \underline{y}_{k-1}(x;\alpha) + \sum_{j=1}^{k-1} \underline{S}_j(\alpha)\ \underline{y}_{k-j}(x;\alpha) + \\ \underline{\mathcal{J}}^{(\beta)} \left(\underline{S}_k(\alpha) g_{l_0}(\ \tilde{y}_0(x;\alpha)) + \sum_{j=1}^{k-1} \underline{S}_j(\alpha) g_{l_{k-j}}\left(\sum_{i=0}^{k-1}\ \tilde{y}_i(x;\alpha)\right)\right) \\ \overline{y}_k(x;\alpha) = \overline{y}_{k-1}(x;\alpha) + \sum_{j=1}^{k-1} \overline{S}_j(\alpha)\ \overline{y}_{k-j}(x;\alpha) + \\ \overline{\mathcal{J}}^{(\beta)} \left(\overline{S}_k(\alpha) g_{u_0}(\ \tilde{y}_0(x;\alpha)) + \sum_{j=1}^{k-1} \overline{S}_j(\alpha) g_{u_{k-j}}\left(\sum_{i=0}^{k-1}\ \tilde{y}_i(x;\alpha)\right)\right) \end{cases} \quad (4.47)$$

$$\tilde{y}_k(x_0) = \tilde{0} \quad (4.48)$$

where $g_{l_{k-j}}\left(\sum_{i=0}^{k-1}\ \tilde{y}_i(x;\alpha)\right)$ and $g_{u_{k-j}}\left(\sum_{i=0}^{k-1}\ \tilde{y}_i(x;\alpha)\right)$ are the coefficients of q^k in the expansion of $g_l[\tilde{y}(x;q)]_\alpha$ and $g_u[\tilde{y}(x;q)]_\alpha$ about the embedding parameter q. Such that we have the lower and upper bound as follows:

$$\begin{cases} g_l\left(\left[\tilde{y}\left(x;q,\sum_{j=1}^{k}\tilde{S}_j(\alpha)\right)\right]_\alpha\right) = g_{l_0}(\tilde{y}_0(x;\alpha)) + \sum_{j=1}^{k}\ g_{l_k}\left(\sum_{j=0}^{k}[\tilde{y}_j]_\alpha\right) q^j, \\ g_u\left(\left[\tilde{y}\left(x;q,\sum_{j=1}^{k}\tilde{S}_j(\alpha)\right)\right]_\alpha\right) = g_{u_0}(\tilde{y}_0(x;\alpha)) + \sum_{j=1}^{k}\ g_{u_k}\left(\sum_{j=0}^{k}[\tilde{y}_j]_\alpha\right) q^j. \end{cases} \quad (4.49)$$

It has been observed that the convergence of the series of Eq. (4.36) largely depends on the convergence parameters $\tilde{S}_j(\alpha)$; then at $q = 1$, we obtain the following fuzzy exact solution

$$\begin{cases} \left[\underline{Y}\left(x,\sum_{j=1}^{\infty}\underline{S}_j(\alpha)\right)\right]_\alpha = \underline{y}_0(x;\alpha) + \sum_{j=1}^{\infty}\left[\underline{y}_j\left(x,\sum_{j=1}^{\infty}\underline{S}_j(\alpha)\right)\right]_\alpha \\ \left[\overline{Y}\left(x,\sum_{j=1}^{\infty}\overline{S}_j(\alpha)\right)\right]_\alpha = \overline{y}_0(x;\alpha) + \sum_{j=1}^{\infty}\left[\overline{y}_j\left(x,\sum_{j=1}^{\infty}\overline{S}_j(\alpha)\right)\right]_\alpha \end{cases}. \quad (4.50)$$

4.4.3 Establishment of convergence of FF-OHAM Solution Series for First-order FFOIVPs

From the observation of the approximate series solution $\left[\tilde{y}\left(x;q,\tilde{S}_j(\alpha)\right)\right]_\alpha$ in Eq.(4.36), we can say that the convergence of series solution by FF-OHAM of Eq.(4.28) would depend largely on the convergence control auxiliary constants $\tilde{S}_1(\alpha), \tilde{S}_2(\alpha), \ldots, \tilde{S}_k(\alpha)$. Thus, proper selection of the convergence parameter would ensure that an accurate convergent solution is possibly obtained. These auxiliary constants must be determined for each $\alpha \in [0,1]$ [Definition 3.5].

The procedure for determining the optimal values of the convergence constants for each $\alpha \in [0,1]$ largely depends on the residual error reduction. Firstly, substituting Eq.(4.36) into Eq.(4.29) and Eq.(4.30) would yield the following residual:

$$\begin{cases} \underline{RE}\left(x; q, \sum_{j=1}^{k} \underline{S}_j(\alpha); \alpha\right) = \underline{\mathcal{L}}_\beta\left(\underline{y}\left(x; q, \sum_{j=1}^{k} \underline{S}_j(\alpha); \alpha\right)\right) - \underline{\mathrm{G}}(x;\alpha) \\ \qquad\qquad -g_l\left(\tilde{y}\left(x; q, \sum_{j=1}^{k} \tilde{S}_j(\alpha); \alpha\right)\right), \\ \overline{RE}\left(x; q, \sum_{j=1}^{k} \overline{S}_j(\alpha); \alpha\right) = \bar{\mathcal{L}}_\beta\left(\overline{y}\left(x; q, \sum_{j=1}^{k} \overline{S}_j(\alpha); \alpha\right)\right) - \overline{\mathrm{G}}(x;\alpha) \\ \qquad\qquad -g_u\left(\tilde{y}\left(x; q, \sum_{j=1}^{k} \tilde{S}_j(\alpha); \alpha\right)\right). \end{cases} \tag{4.51}$$

As has been mentioned in Section 3.2.5, if $\widetilde{RE} = 0$, then $\left[\tilde{y}\left(x; q, \tilde{S}_j(\alpha)\right)\right]_\alpha$ yields the exact solution, where generally it does not happen, especially in nonlinear FFOIVPs. To identify the auxiliary constants of $\tilde{S}_j(\alpha)$, $j = 1,2,..k$, we choose x_0 and X such that the optimum values of $\tilde{S}_j(\alpha)$ for the convergent solution of the given problem is obtained. To find the optimal values of $\tilde{S}_j(\alpha)$, we apply the method of Least Squares for each $\alpha \in [0,1]$ [Definition 3.5] as follows:

$$S\widetilde{RE}\left(x; q, \sum_{j=1}^{k} \tilde{S}_j(\alpha)\,; \alpha\right) = \int_{x_0}^{X} \widetilde{RE}^2\left(x; q, \sum_{j=1}^{k} \tilde{S}_j(\alpha)\,; \alpha\right)\, dx \tag{4.52}$$

where $\widetilde{RE}$ is the residual

$$\begin{cases} \left[\underline{RE}\right]_\alpha = \underline{\mathcal{L}}_\beta\left(\left[\underline{y}\right]_\alpha\right) - \underline{\mathrm{G}}(x;\alpha) - g_l([\tilde{y}]_\alpha) \\ \left[\overline{RE}\right]_\alpha = \bar{\mathcal{L}}_\beta([\overline{y}]_\alpha) - \overline{\mathrm{G}}(x;\alpha) - g_u([\tilde{y}]_\alpha) \end{cases}, \tag{4.53}$$

and

$$\frac{\partial\, \widetilde{SRE}}{\partial \tilde{S}_1(\alpha)} = \frac{\partial\, \widetilde{SRE}}{\partial \tilde{S}_2(\alpha)} = \cdots = \frac{\partial\, \widetilde{SRE}}{\partial \tilde{S}_k(\alpha)} = 0 \tag{4.54}$$

The values of the convergence control parameters should be real values, so the complex values will be ignored.

4.5 Experimental Work of FF-HAM and FF-OHAM for First-order FFOIVPs

This section demonstrates the experimental work for the application of the developed methods FF-HAM and FF-OHAM in solving first-order FFOIVPs. There are two experimental examples demonstrated where for each example, the solution method using the developed FF-HAM is demonstrated first and followed by the one using the developed FF-OHAM. The general solving steps in the experimental work involve the defuzzification of the first-order FFOIVPs, implementation of the proposed methods, obtaining optimal parameters, obtaining the approximate-analytical solution and plotting the obtained solution to ensure that the solution satisfies the properties of fuzzy numbers. Comparative study with other available numerical results is also carried out.

4.5.1 Example 4.1

Consider the following inhomogeneous linear FFOIVP of order $\beta \in (0,1]$ (Alaroud et al., 2019)

$$\begin{cases} y^{(\beta)}(x) - y(x) = [1+\alpha, 3-\alpha], 0 \leq x \leq 1, \\ \qquad \tilde{y}(0) = \tilde{y}_0(x;\alpha) = [\tilde{0}]_\alpha. \end{cases} \tag{4.55}$$

subject to the exact solution when $\beta = 1$ (Alaroud et al., 2019)

$$Y(x;\alpha) = 2e^x + [1-\alpha, \alpha-1](1-e^{-x}),\ \forall \alpha \in [0,1], \tag{4.56}$$

where for $\beta \in (0,1]$, $y^{(\beta)}(x) = D^{(\beta)}y(x)$ [Definition 3.21], $[1+\alpha, 3-\alpha]$ and $[1-\alpha, \alpha-1]$ represent the fuzzy numbers [Definition 3.7] and $\tilde{y}(0)$ is the fuzzy initial condition.

We begin the solving steps for Example 4.1 with the defuzzification step. This step is the same for both methods: FF-HAM and FF-OHAM. Based on the fuzzy number definition [Definition 3.6], we have freedom for the defuzzification of the initial condition $\tilde{y}(0)$. We will consider the fuzzy initial condition to be $[\tilde{0}] = [\underline{0}, \overline{0}]$.

Then we will define the linear operator of Eq.(4.55) as $\tilde{\mathcal{L}}_\beta = \widetilde{D}^{(\beta)}$ [Definition 3.21], and the left inverse operator of $D^{(\beta)}$ is denoted by $\tilde{\mathcal{L}}_\beta{}^{-1} = \tilde{\mathcal{J}}^{(\beta)}$ [Definition 3.19] while $D^{(\beta)}$ is the fractional derivative in Caputo sense [Definition 3.21], $\mathcal{J}^{(\beta)}$ refers to the fractional operator of the Riemann-Liouville integral [Definition 3.19] and the initial guess denotes by $[\tilde{0}] = [\underline{0}, \overline{0}]$.

4.5.1.1 Solving Example 4.1 by FF-HAM

In solving Example 4.1 by FF-HAM, based on Section 4.3.2, we will construct the zeroth-order and k^{th}-order deformation equation of Eq. (4.55) as follows:

$$\begin{cases} \underline{y}_k(x;\alpha) = \psi_k \underline{y}_{k-1}(x;\alpha) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \mathcal{R}_k(\vec{\underline{y}}_{k-1}(x;\alpha)) \\ \overline{y}_k(x;\alpha) = \psi_k \overline{y}_{k-1}(x;\alpha) + \overline{h}(\alpha) \overline{\mathcal{J}}^{(\beta)} \mathcal{R}_k(\vec{\overline{y}}_{k-1}(x;\alpha)) \end{cases} \tag{4.57}$$

where $\underline{y}_k(0;\alpha) = 0$ and $\overline{y}_k(0;\alpha) = 0$ such that

$$\begin{cases} \mathcal{R}_k\left(\vec{\underline{y}}_{k-1}(x;\alpha)\right) = \underline{y}_{k-1}{}^{(\beta)}(x;\alpha) - \underline{y}_{k-1}(x;\alpha) - (1+\alpha) \\ \mathcal{R}_k\left(\vec{\overline{y}}_{k-1}(x;\alpha)\right) = \overline{y}_{k-1}{}^{(\beta)}(x;\alpha) - \overline{y}_{k-1}(x;\alpha) - (3-\alpha) \end{cases} \tag{4.58}$$

Here, Eq. (4.58) represents the nonlinear function of FFOIVPs. Then, by expanding Eq. (4.58) for $k = 1,2,3, \ldots$, we obtain the FF-HAM series solution components for the lower and the upper bound [Definition 3.4] for Eq.(4.55) as follows:

For initial approximation:

$$\tilde{y}(0) = \tilde{y}_0(x;\alpha) = [\underline{0}, \overline{0}]. \quad (4.59)$$

For first-order problem:

$$\begin{cases} \underline{y}_1(x;\alpha) = \psi_1 \underline{y}_0(x;\alpha) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \mathcal{R}_1\left(\underline{\vec{y}}_0(x;\alpha)\right), \\ \overline{y}_1(x;\alpha) = \psi_1 \overline{y}_0(x;\alpha) + \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)} \mathcal{R}_1\left(\overline{\vec{y}}_0(x;\alpha)\right), \end{cases} \quad (4.60)$$

and since $\psi_1 = 0$, and $\mathcal{R}_1\left(\widetilde{\vec{y}_0}(x;\alpha)\right) = \tilde{y}_0{}^{(\beta)}(x) - \tilde{y}_0(x) - [(1+\alpha),(3-\alpha)]$ we have

$$\begin{cases} \underline{y}_1(x;\alpha) = \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \left(\underline{y}_0{}^{(\beta)}(x;\alpha) - \underline{y}_0(x;\alpha) - (1+\alpha)\right), \\ \overline{y}_1(x;\alpha) = \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)} \left(\overline{y}_0{}^{(\beta)}(x;\alpha) - \overline{y}_0(x;\alpha) - (3-\alpha)\right), \end{cases} \quad (4.61)$$

and we can rewrite Eq. (4.61) using Definition 3.18 as follows:

$$\begin{cases} \underline{y}_1(x;\alpha) = \underline{h}(\alpha)\ \underline{y}_0(x;\alpha) - \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \left(\underline{y}_0(x;\alpha) + (1+\alpha)\right), \\ \overline{y}_1(x;\alpha) = \overline{h}(\alpha)\, \overline{y}_0(x;\alpha) - \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)} \left(\overline{y}_0(x;\alpha) + (3-\alpha)\right). \end{cases} \quad (4.62)$$

For second-order problem:

$$\begin{cases} \underline{y}_2(x;\alpha) = \psi_2 \underline{y}_1(x;\alpha) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \mathcal{R}_2(\underline{\vec{y}}_1(x;\alpha)) \\ \overline{y}_2(x;\alpha) = \psi_2 \overline{y}_1(x;\alpha) + \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)} \mathcal{R}_2(\overline{\vec{y}}_1(x;\alpha)) \end{cases} \quad (4.63)$$

and since $\psi_2 = 1$, and $\mathcal{R}_2\left(\widetilde{\vec{y}_1}(x;\alpha)\right) = \tilde{y}_1{}^{(\beta)}(x) - \tilde{y}_1(x)$, we have

$$\begin{cases} \underline{y}_2(x;\alpha) = \underline{y}_1(x;\alpha) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \left(\underline{y}_1{}^{(\beta)}(x;\alpha) - \underline{y}_1(x;\alpha)\right) \\ \overline{y}_2(x;\alpha) = \overline{y}_1(x;\alpha) + \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)} \left(\overline{y}_1{}^{(\beta)}(x;\alpha) - \overline{y}_1(x;\alpha)\right) \end{cases} \quad (4.64)$$

and we can rewrite Eq. (4.64) using Definition 3.18 as follows:

$$\begin{cases} \underline{y}_2(x;\alpha) = \underline{y}_1(x;\alpha) + \underline{h}(\alpha)\,\underline{y}_1(x;\alpha) \; - \underline{h}\,(\alpha)\,\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_1(x;\alpha)\right) \\ \overline{y}_2(x;\alpha) = \overline{y}_1(x;\alpha) + \overline{h}(\alpha)\,\overline{y}_1(x;\alpha) \; - \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_1(x;\alpha)\right) \end{cases} \quad (4.65)$$

The general form is as follows:

$$\begin{cases} \underline{y}_{k+1}(x;\alpha) = \underline{y}_k(x;\alpha) + \underline{h}(\alpha)\,\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_k{}^{(\beta)}(x;\alpha) - \underline{y}_k(x;\alpha)\right), \\ \overline{y}_{k+1}(x;\alpha) = \overline{y}_k(x;\alpha) + \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_k{}^{(\beta)}(x;\alpha) - \overline{y}_k(x;\alpha)\right), \end{cases} \quad (4.66)$$

and we can rewrite Eq. (4.66) as follows:

$$\begin{cases} \underline{y}_{k+1}(x;\alpha) = \underline{y}_k(x;\alpha) + \underline{h}(\alpha)\,\underline{y}_k(x;\alpha) - \underline{h}(\alpha)\,\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_k(x;\alpha)\right), \\ \overline{y}_{k+1}(x;\alpha) = \overline{y}_k(x;\alpha) + \overline{h}(\alpha)\overline{y}_k(x;\alpha) - \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_k(x;\alpha)\right). \end{cases} \quad (4.67)$$

The FF-HAM series solution of k^{th}-order for Eq. (4.55) is $\tilde{y}(x;\alpha;\tilde{h}) = \sum_{j=0}^{k} \tilde{y}_j(x;\alpha;\tilde{h})$.

To show the accuracy of FF-HAM for Eq. (4.55), by taking the residual error as mentioned in Section 4.3.3, we have

$$\begin{cases} \underline{ER}(x;\alpha;\underline{h}) = \left(\underline{y}^{(\beta)}(x;\alpha;h) - \underline{y}(x;\alpha;h) - (1+\alpha)\right), \\ \overline{ER}(x;\alpha;\overline{h}) = \left(\overline{y}^{(\beta)}(x;\alpha;h) - \overline{y}(x;\alpha;h) - (3-\alpha)\right). \end{cases} \quad (4.68)$$

It is remarked that the accuracy of FF-HAM corresponding to the residual error in Eq. (4.68) depends on x values and $\tilde{h}$ values. According to Section 4.3.3, the convergence control parameter $\tilde{h}(\alpha) = [\underline{h}(\alpha), \overline{h}(\alpha)]$ plays a pivotal role in controlling and adjusting the convergence region of the homotopy series solution. We will use the properties of FF-HAM convergence in Section 4.3.3 to find the best value of $\tilde{h}(\alpha)[\underline{h}(\alpha), \overline{h}(\alpha)]$ that gives the most accurate solution of Eq. (4.55). For this purpose of finding the best value, firstly, for a fixed value of $0 \le \alpha \le 1$ [Definition 3.5], for example $\alpha = 0.8$, we will plot $\tilde{h}(0.8) = [\underline{h}(0.8), \overline{h}(0.8)]$-curves of lower bound $\underline{y}(0.2; 0.8; \underline{h})$ and upper bound $\overline{y}(0.2; 0.8; \overline{h})$ depending on the integral of the squared residual error described in Eq.(4.69) over the whole region $x \in [0,0.2]$ with the fifth-order and eighth-order FF-HAM solution series of Eq.(4.55) for $\beta = 0.5$, $\alpha = 0.8$ and at $x = 0.2$ where

$$\begin{cases} \underline{SER}(0.2; 0.8; \underline{h}) = \int_{x=0}^{0.2} [\underline{ER}(0.2; 0.8; \underline{h})]^2 \, dx, \\ \overline{SER}(0.2; 0.8; \overline{h}) = \int_{x=0}^{0.2} [\overline{ER}(0.2; 0.8; \overline{h})]^2 \, dx. \end{cases} \tag{4.69}$$

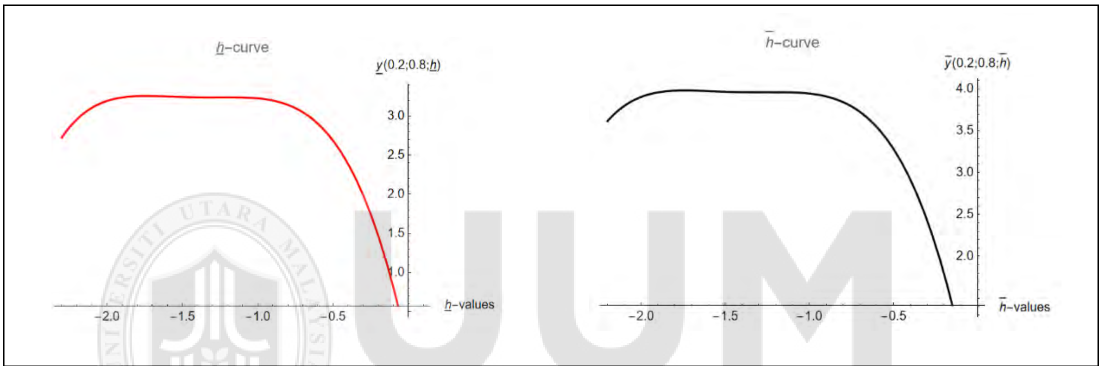

Figure 4.1. The $\tilde{h}(\alpha)$-curves for the fuzzy solution of Eq. (4.55) given by fifth-order FF-HAM for $\beta = 0.5$, $x = 0.2$ and $\alpha = 0.8$ when $H(x) = 1$

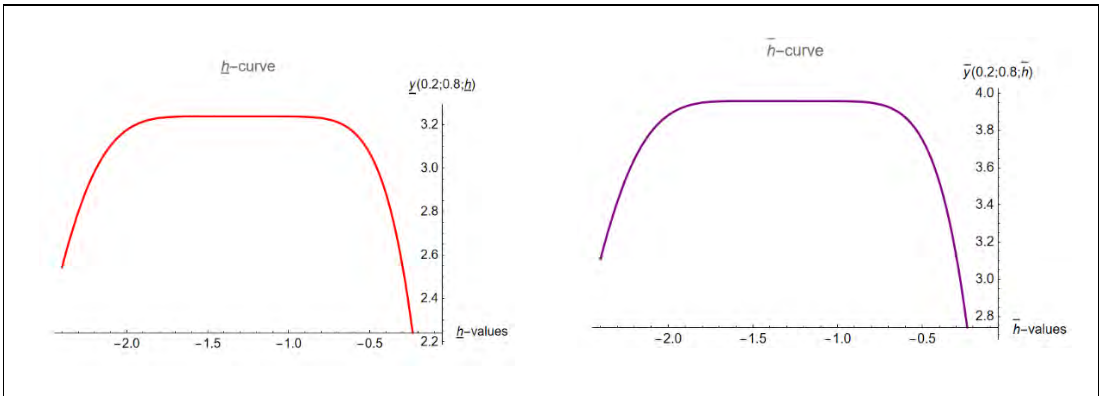

Figure 4.2. The $\tilde{h}(\alpha)$-curve for the fuzzy solution of Eq. (4.55) given by eighth–order FF-HAM for $\beta = 0.5$, $x = 0.2$ and $\alpha = 0.8$ when $H(x) = 1$

Figures 4.1 and 4.2 illustrate the $\tilde{h}(\alpha)$-curves. Secondly, based on Figures 4.1 and 4.2, we can conclude that the valid region of the fuzzy convergence control parameters $\tilde{h}(\alpha)$ for Eq. (4.55) where the FF-HAM series solution corresponds to the line segment nearly parallel to the horizontal axis. In other words, the valid region of $\tilde{h}(\alpha)$ of the fifth-order FF-HAM is bounded by the interval $-1.8 \leq \tilde{h} \leq -0.8$, while the valid region of $\tilde{h}$ of the eighth-order FF-HAM is bounded by the interval $-1.8 \leq \tilde{h} \leq -0.6$. The obtained optimal values of the fifth-order FF-HAM are listed in Table 4.1.

Table 4.1

The optimal values of $\tilde{h}(0.8)$ by fifth-order FF-HAM for lower and upper solutions of Eq. (4.55) for $\beta = 0.5$ and $\alpha = 0.8$

$\underline{\boldsymbol{y}}\left(\boldsymbol{x};\boldsymbol{0.8};\underline{\boldsymbol{h}}(\boldsymbol{0.8})\right)$	$\underline{h}_1 \to -1.7426070281283383$	$\underline{h}_2 \to -1.3231735832323284$
	$\underline{h}_3 \to -1.201823361059829$	$\underline{h}_4 \to -1.1933227160628288$
	$\underline{h}_5 \to -1.1913049993955067$	$\underline{h}_6 \to -1.1547798328037502$
$\overline{\boldsymbol{y}}\left(\boldsymbol{x};\boldsymbol{0.8};\overline{\boldsymbol{h}}(\boldsymbol{0.8})\right)$	$\overline{h}_1 \to -1.742607028128298$	$\overline{h}_2 \to -1.3231735832324185$
	$\overline{h}_3 \to -1.2018249917386437$	$\overline{h}_4 \to -1.1933227160627649$
	$\overline{h}_5 \to -1.191303309529213$	$\overline{h}_6 \to -1.1547800164838038$

Thirdly, we will check the accuracy of fifth-order FF-HAM for Eq. (4.55) by plotting the residual of Eq. (4.55) depending on each fuzzy convergence control parameter such that based on the residual error in Eq. (4.68), for the lower bound we have accuracy illustrated in Figure 4.3.

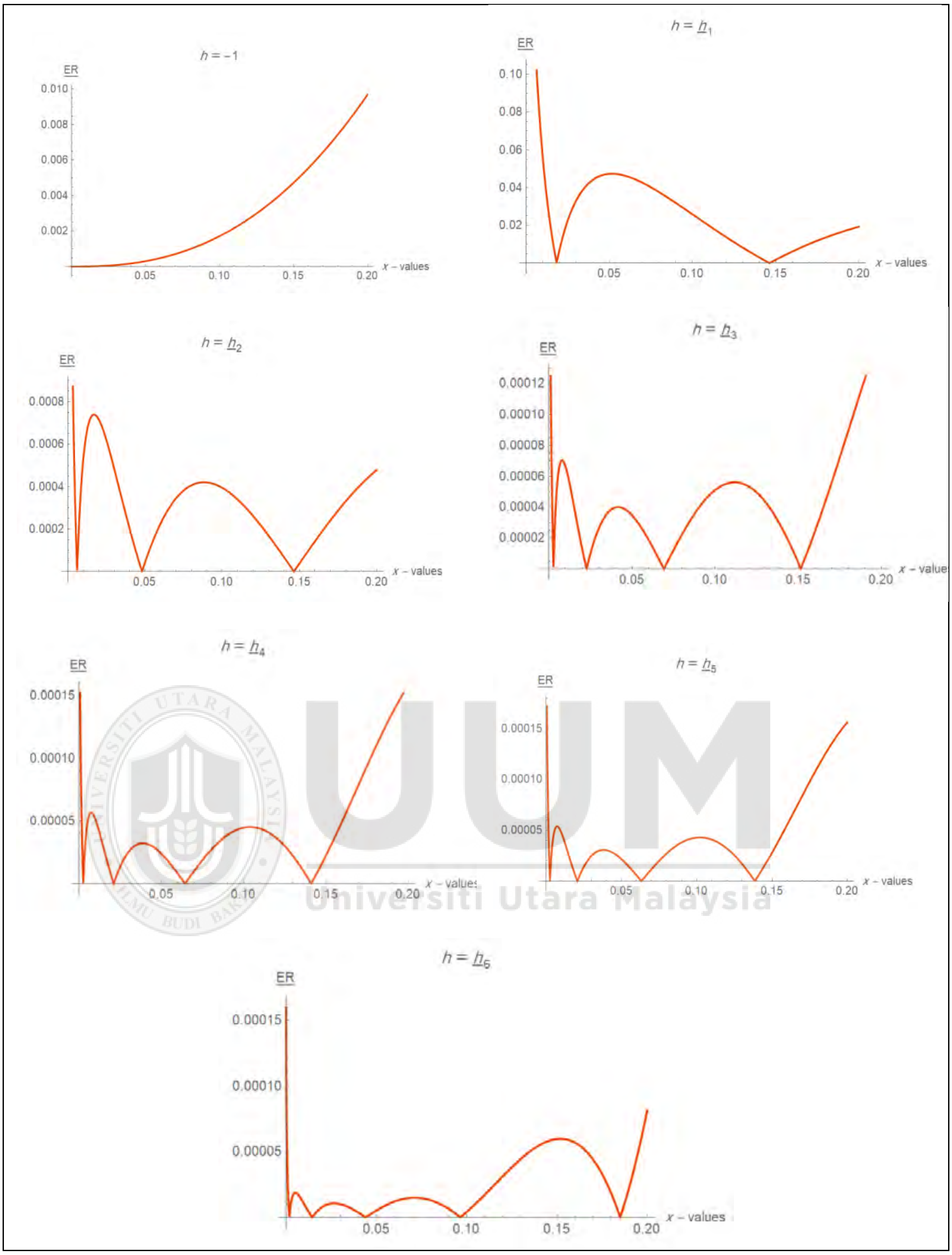

Figure 4.3. The accuracy of the fifth-order FF-HAM linked with the optimal values of the lower convergence control parameters $\underline{h}(0.8)$ for solving Eq. (4.55) at $\beta = 0.5$ for all $x \in [0,0.2]$

Figure 4.3 illustrates the summary of the accuracy of fifth-order FF-HAM solution for Eq. (4.55) based on each lower convergence control parameter, also it is easy to conclude that $\underline{h}_6 = -1.1547798328037502$ represent the best optimal convergence control parameter and provides the more accurate lower solution for Eq. (4.55).

Finally, the best lower optimal convergence control parameter $\underline{h}_6$ will be used to find the lower solution $\underline{y}(0.2; 0.8; \underline{h}_6)$ for Eq. (4.55).

Table 4.2
The lower solution and error of Eq. (4.55) by fifth-order FF-HAM when $\beta = 0.5$ at $x = 0.2$ for $h = -1$, and $h = \underline{h}_6$ $\forall\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\underline{\boldsymbol{ER}}]_{\alpha}, \underline{\boldsymbol{h}} = -\mathbf{1}$	$[\underline{\boldsymbol{ER}}]_{\alpha}, \underline{\boldsymbol{h}}_{\mathbf{6}}$	$[\underline{\boldsymbol{y}}]_{\alpha}, \underline{\boldsymbol{h}} = -\mathbf{1}$	$[\underline{\boldsymbol{y}}]_{\alpha}, \underline{\boldsymbol{h}}_{\mathbf{6}}$
0	-0.00538	-0.00005	0.79730	0.79902
0.2	-0.00646	-0.00005	0.95675	0.95882
0.4	-0.00754	-0.00006	1.11620	1.11863
0.6	-0.00861	-0.00007	1.27567	1.27843
0.8	-0.00969	-0.00008	1.43513	1.43823
1	-0.01077	-0.00009	1.59459	1.59804

Table 4.2 shows the lower solution for Eq.(4.55) by fifth-order FF-HAM based on the $\underline{h} = -1$ which gives the solution by HPM (Turkyilmazoglu, 2011), and the solution by fifth-order FF-HAM based on the best optimal convergence control parameter $\underline{h} = \underline{h}_6$. Simultaneously, Table 4.2 illustrates that the best optimal convergence control parameter $\underline{h} = \underline{h}_6$ provides more accurate approximate solutions compared with the HPM. Next to obtain the upper bound, based on the residual error in Eq. (4.68) we will plot the accuracy of fifth-order FF-HAM for Eq. (4.55) based on each upper convergence control parameter listed in Table 4.1 where the plots are given in Figure 4.4.

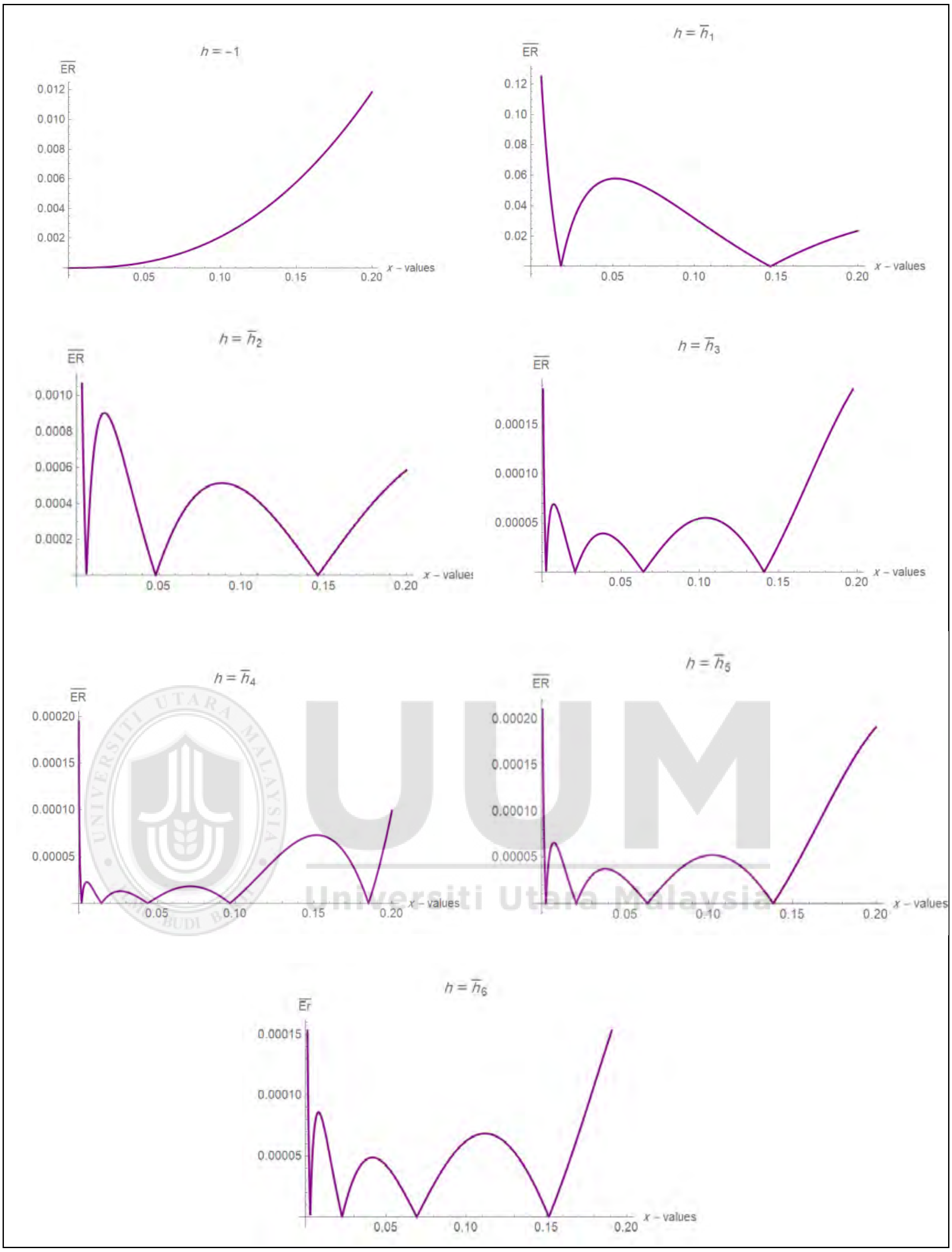

Figure 4.4. The accuracy of the fifth-order FF-HAM linked with the optimal values of the upper convergence control parameters $\bar{h}(0.8)$ for solving Eq. (4.55) at $\beta = 0.5$ for all $x \in [0,0.2]$

Figure 4.4 illustrates the fifth-order FF-HAM accuracy based on each upper convergence control parameter $\bar{h}(0.8)$ for Eq. (4.55). In addition, from Figure 4.4, we can say the upper best convergence control parameter is $\overline{h} = \overline{h}_6$ since $\overline{h}_6$ gives the more accurate upper solution $\overline{y}(0.2; 0.8; \overline{h})$ for Eq. (4.55).

Table 4.3

The upper solution and error of Eq. (4.55) by fifth-order FF-HAM when $\beta = 0.5$ at $x = 0.2$ for $h = -1$ and $h = \overline{h}_6$ $\forall \alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\overline{\boldsymbol{ER}}]_{\alpha}, \overline{\boldsymbol{h}} = -\mathbf{1}$	$[\overline{\boldsymbol{ER}}]_{\alpha}, \overline{\boldsymbol{h}}_{\mathbf{6}}$	$[\overline{y}]_{\alpha}, \overline{\boldsymbol{h}} = -\mathbf{1}$	$[\overline{y}]_{\alpha}, \overline{\boldsymbol{h}}_{\mathbf{6}}$
0	−0.01619	−0.00014	2.39188	2.39706
0.2	−0.01507	−0.00013	2.23242	2.23725
0.4	−0.01400	−0.00012	2.07296	2.07745
0.6	−0.01292	−0.00010	1.91350	1.91765
0.8	−0.01184	−0.00010	1.75404	1.75784
1	−0.01077	−0.00009	1.59459	1.59804

Table 4.3 illustrates the upper solution of Eq. (4.55) using fifth-order FF-HAM based on $\overline{h} = -1$, and the best optimal convergence control parameter $h = \overline{h}_6$. Thus, from Table 4.3, we can conclude that $\overline{h}_6$ gives the optimal upper approximation $\overline{y}(0.2; 0.8; \overline{h})$ for Eq. (4.55) at fractional order $\beta = 0.5$.

Tables 4.2-4.3 display the fifth-order FF-HAM series solution of Eq.(4.55) using the optimal values of $\tilde{h} = [\underline{h}_6, \overline{h}_6]$. In addition, Tables 4.2-4.3 show that the fifth-order FF-HAM approximate solutions at $x = 0.2$ for all $\alpha \in [0,1]$ satisfy the fuzzy numbers [Definition 3.7] and fuzzy differential equations solution (Salahshour, 2011).

For further clarification, Figure 4.5 displays the triangular solutions for Eq.(4.55) satisfying the property of fuzzy numbers.

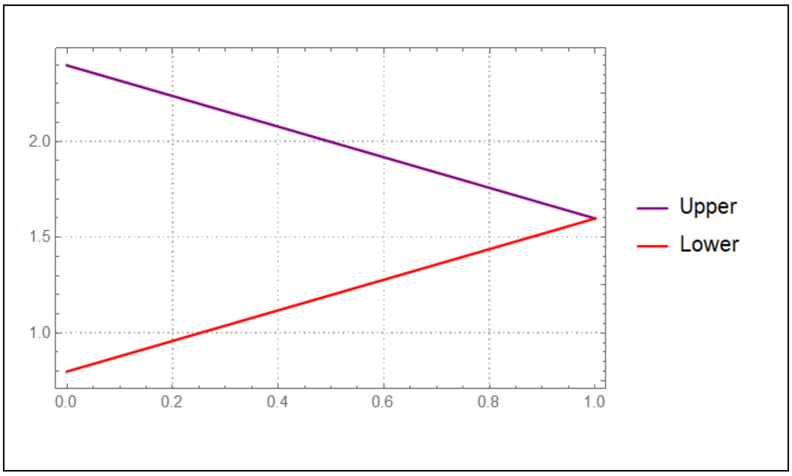

Figure 4.5. The approximate solution of Eq. (4.55) given by fifth-order FF-HAM at $\beta = 0.5$ and $x = 0.2$ for all $\alpha \in [0,1]$

Figure 4.6 summarizes the solutions of fifth-order FF-HAM over all $x \in [0,0.2]$ and $\alpha \in [0,1]$ corresponding to the best optimal convergence control values $\tilde{h} = [\underline{h}_6, \overline{h}_6]$ of Eq. (4.55) in a three-dimensional graph.

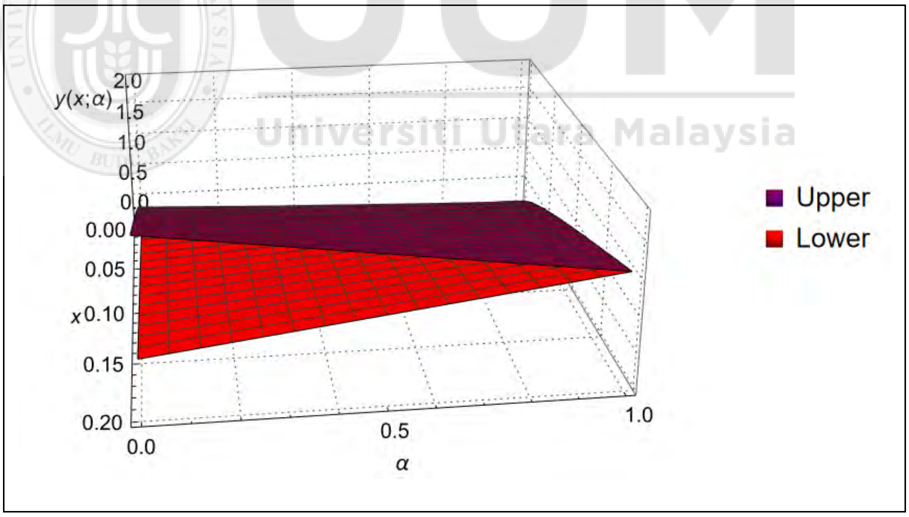

Figure 4.6. The three-dimensional approximate solution of Eq. (4.55) given by fifth-order FF-HAM over all $x \in [0,0.2]$ at $\beta = 0.5$ and for all $\alpha \in [0,1]$

From Tables 4.2-4.3 and Figures 4.5-4.6, it can be observed that the fifth-order FF-HAM solutions for Eq. (4.55) of fractional order $\beta = 0.5$ fulfill the properties of the fuzzy differential equation in the form of triangular fuzzy numbers [Definition 3.7] for all $\alpha \in [0,1]$ [Definition 3.5] and $x \in [0,0.2]$.

To study the behaviour of FF-HAM with different series order for solving first-order FFOIVPs, we proceed to solve Eq. (4.55) by FF-HAM at the same $x \in [0,0.2]$ and $\alpha = 0.8$ but with order eight instead of order five to analyse the convergence dynamic of FF-HAM for different terms of the series approximate solution.

Firstly, we will look at the optimal convergence control parameters $\tilde{h}$ in the valid convergent region of eighth-order FF-HAM. Based on Figure 4.2, we conclude that the valid region of $\tilde{h}$ for the series solution of eighth-order FF-HAM is $-1.8 \leq \tilde{h} \leq -0.6$. The optimal values of $\tilde{h}(0.8)$ are listed in Table 4.4.

Table 4.4
The optimal values of $\tilde{h}(0.8)$ by eighth-order FF-HAM for lower and upper solutions of Eq. (4.55) for $\beta = 0.5$ and $\alpha = 0.8$

$\underline{\boldsymbol{y}}\left(\boldsymbol{x}; \mathbf{0.8}; \underline{\boldsymbol{h}}(\mathbf{0.8})\right)$	$\underline{h}_1 \rightarrow -1.569932526719038$	$\underline{h}_2 \rightarrow -1.1531519585288363$
	$\underline{h}_3 \rightarrow -1.3272504264015281$	$\underline{h}_4 \rightarrow -1.2236986616057783$
	$\underline{h}_5 \rightarrow -1.165494276436343$	$\underline{h}_6 \rightarrow -1.1263472072580956$
$\overline{\boldsymbol{y}}\left(\boldsymbol{x}; \mathbf{0.8}; \overline{\boldsymbol{h}}(\mathbf{0.8})\right)$	$\overline{h}_1 \rightarrow -1.5699325268073594$	$\overline{h}_2 \rightarrow -1.338501484014069$
	$\overline{h}_3 \rightarrow -1.3272504258014728$	$\overline{h}_4 \rightarrow -1.223698663123549$
	$\overline{h}_5 \rightarrow -1.1654942748731745$	$\overline{h}_6 \rightarrow -1.1263472078300167$

Table 4.4 illustrates the new convergence control parameters of FF-HAM solution series after changing the series order from 5 to 8 for solving Eq. (4.55) at the same fractional derivative order $\beta = 0.5$ at $x = 0.2$ and $\alpha = 0.8$. Since FF-HAM provides a flexible way to select the best optimal convergence control parameter, we will

employ $\tilde{h}(0.8) = [\underline{h}_2(0.8), \underline{h}_2(0.8)]$ to find the eighth-order fuzzy series solution of Eq. (4.55). Secondly, in Figures 4.7- 4.8, we plot the accuracy of the eighth-order FF-HAM corresponding to the best optimal convergence control parameters $\tilde{h}(0.8) = [\underline{h}_2(0.8), \underline{h}_2(0.8)]$ in Table 4.4. Based on the residual errors, the eighth-order FF-HAM provide more accurate solutions of Eq. (4.55).

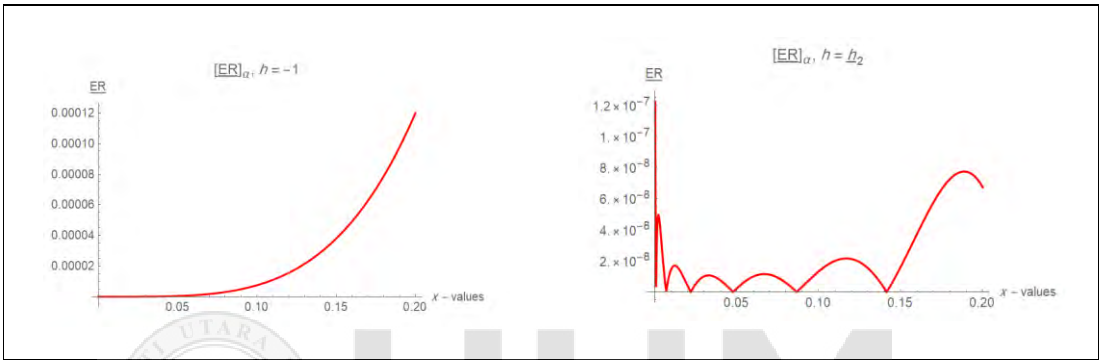

Figure 4.7. The accuracy of eighth-order FF-HAM linked with $h = -1$, and the optimal lower convergence control parameter $\underline{h}_2(0.8)$ for solving Eq. (4.55) at $\beta = 0.5$ and for all $x \in [0,0.2]$

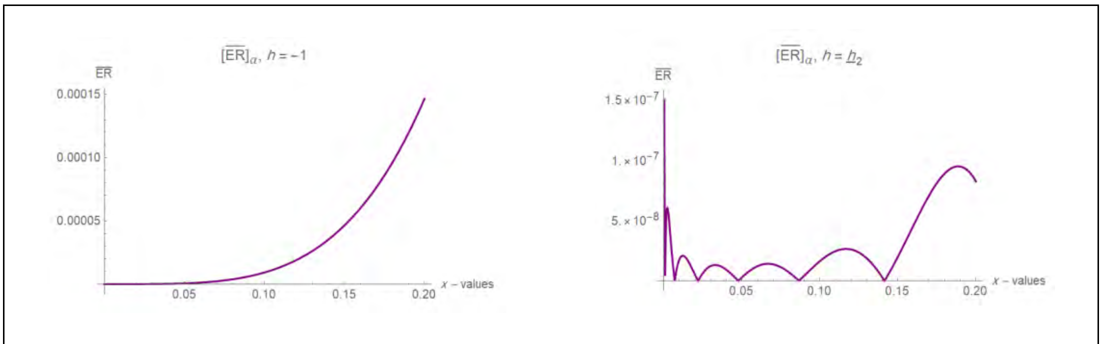

Figure 4.8. The accuracy of eighth-order FF-HAM linked with $h = -1$, and the optimal upper convergence control parameter $\underline{h}_2(0.8)$ for solving Eq. (4.55) at $\beta = 0.5$ and for all $x \in [0,0.2]$

Figures 4.7-4.8 illustrate the accuracy of Eq. (4.55) by eighth-order FF-HAM corresponding to the best optimal convergence control parameter. In Tables 4.5-4.6, we tabulate the residual errors $[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ of Eq.(4.55) of the approximate solutions $\underline{y}(0.2;\alpha;\underline{h}_2)$ and $\overline{y}(0.2;\alpha;\underline{h}_2)$ obtained by using eighth-order FF-HAM according to the best convergence control parameter $\tilde{h}(0.8) = [\underline{h}_2(0.8), \underline{h}_2(0.8)]$.

Table 4.5

The lower solution and error of Eq. (4.55) by eighth-order FF-HAM when $\beta = 0.5$ at $x = 0.2$ for $h = -1$ and $h = \underline{h}_2$ $\forall\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\underline{\boldsymbol{ER}}]_{\boldsymbol{\alpha}}, \boldsymbol{h = -1}$	$[\underline{\boldsymbol{ER}}]_{\boldsymbol{\alpha}}, \underline{\boldsymbol{h}}_{\boldsymbol{2}}$	$[\underline{\boldsymbol{y}}]_{\boldsymbol{\alpha}}, \boldsymbol{h = -1}$	$[\underline{\boldsymbol{y}}]_{\boldsymbol{\alpha}}, \underline{\boldsymbol{h}}_{\boldsymbol{2}}$
0	-0.00007	-3.74050×10^{-8}	0.79900	0.79901
0.2	-0.00008	-4.48859×10^{-8}	0.95880	0.95882
0.4	-0.00009	-5.23669×10^{-8}	1.11860	1.11862
0.6	-0.00010	-5.98480×10^{-8}	1.27840	1.27843
0.8	-0.00012	-6.73290×10^{-8}	1.43820	1.43823
1	-0.00013	-7.48100×10^{-8}	1.59800	1.59803

Table 4.6

The upper solution and error of Eq. (4.55) by eighth-order FF-HAM when $\beta = 0.5$ *at* $x = 0.2$ *for* $h = -1$*, and* $h = \underline{h}_2$ $\forall\alpha \in [0,1]$

α	$[ER]_{\alpha}, h = -1$	$[ER]_{\alpha}, \underline{h}_2$	$[\overline{y}]_{\alpha}, h = -1$	$[\overline{y}]_{\alpha}, \underline{h}_2$
0	-0.00019	-1.12215×10^{-7}	2.39700	2.39705
0.2	-0.00018	-1.04733×10^{-7}	2.23720	2.23725
0.4	-0.00017	-9.72530×10^{-8}	2.07740	2.07744
0.6	-0.00016	-8.97719×10^{-8}	1.91760	1.91764
0.8	-0.00014	-8.22909×10^{-8}	1.75780	1.75784
1	-0.00013	-7.48100×10^{-8}	1.59800	1.59803

Tables 4.5 and 4.6 illustrate the FF-HAM approximate solution of Eq. (4.55) at x =0.2 for all $\alpha \in [0,1]$ where Figure 4.9 and Figure 4.10 provide graphical displays in two-dimensional and three-dimensional respectively.

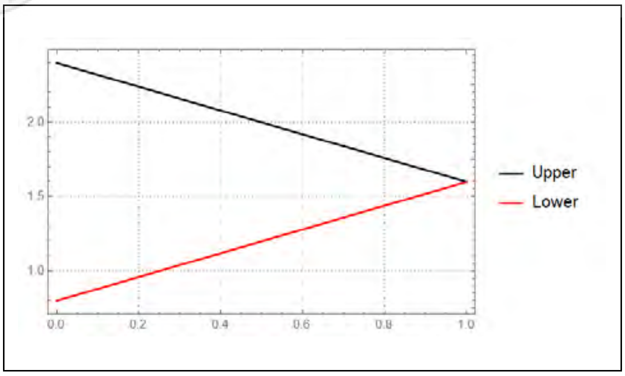

Figure 4.9. The approximate solution of Eq. (4.55) given by eighth-order FF-HAM for $\beta = 0.5$, and $x = 0.2$ for all $\alpha \in [0,1]$

Figure 4.10 illustrates the summary of the fuzzy solutions $[\tilde{y}]_{\alpha}$ for Eq. (4.55) by eighth-order FF-HAM over all $x \in [0,0.2]$ and for all $\alpha \in [0,1]$.

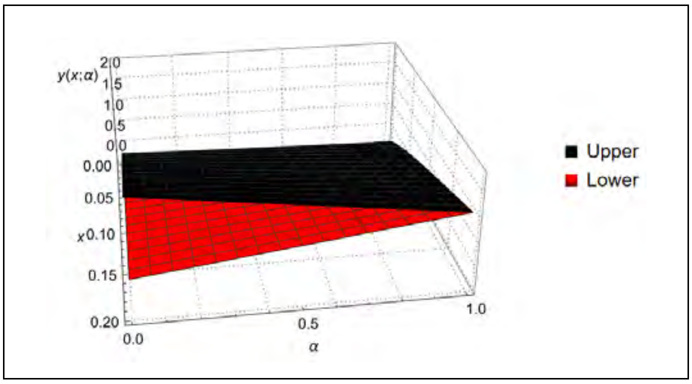

Figure 4.10. The three-dimensional approximate solution of Eq. (4.55) given by eighth-order FF-HAM over all $x \in [0,0.2]$ at $\beta = 0.5$ and for all $\alpha \in [0,1]$

Tables 4.5-4.6 and Figures 4.9-4.10 clarify that the eighth-order FF-HAM satisfy the fuzzy numbers [Definition 3.7] and fuzzy differential equation properties (Salahshour, 2011) of Eq.(4.55), which takes the triangular fuzzy number shape [Definition 3.7].

To reveal the accuracy of the solution obtained based on the fifth-order and eighth-order FF-HAM for Eq. (4.55), the accuracy of the fuzzy solutions based on the residual of Eq. (4.55) in Eq. (4.68) on the interval $x \in [0,0.2]$ for all $\alpha \in [0,0.2]$ corresponding to the best optimal convergence control parameter $\tilde{h}$ is illustrated in Figures 4.11 and 4.12.

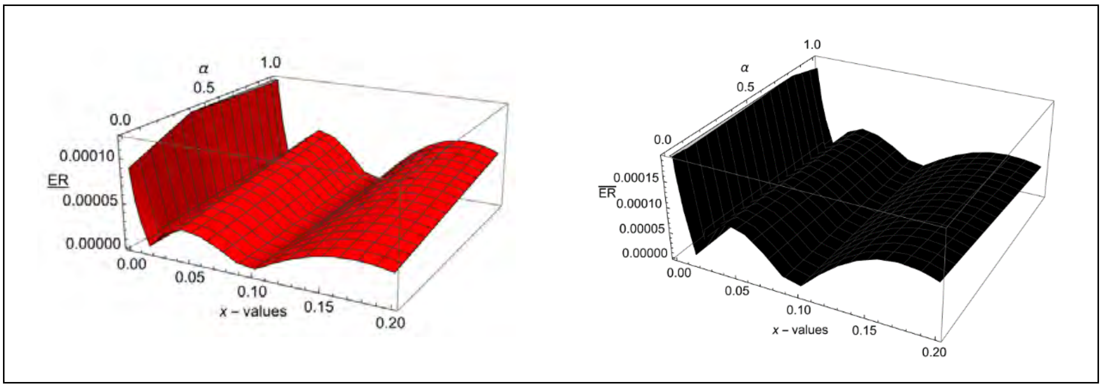

Figure 4.11. The accuracy of fifth-order FF-HAM for solving Eq. (4.55) of order $\beta = 0.5$ for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.2]$

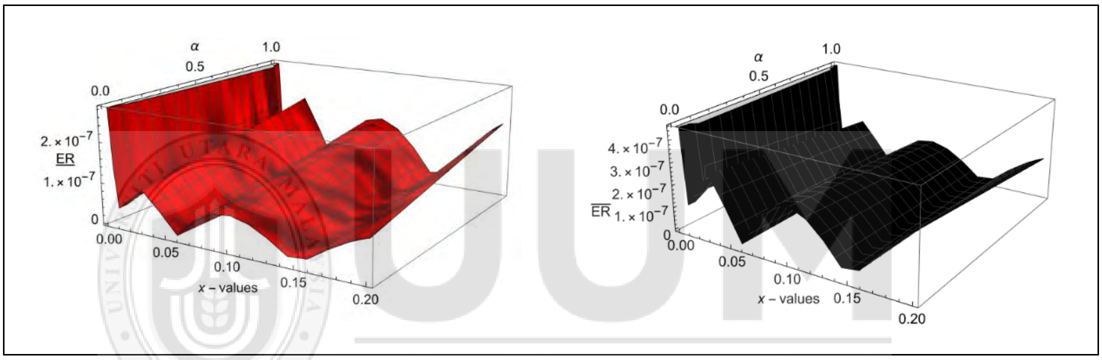

Figure 4.12. The accuracy of eighth-order FF-HAM for solving Eq. (4.55) of order $\beta = 0.5$ for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.2]$

It can be concluded from the Tables 4.2-4.3, Tables 4.5-4.6 and Figures 4.11-4.12 that the FF-HAM provides more accurate solution as the approximate series order increases.

4.5.1.2 Comparative Study of FF-HAM for Example 4.1

For comparative study, we compare the eighth-order FF-HAM and the eighth-order fractional residual power series method (FRPSM) (Alaroud et al., 2019) for solving

Eq.(4.55) under Hukuhara Caputo differentiability [Definition 3.21] at $\beta = 1$ for different values of $x \in [0,1]$, and $\alpha \in [0,1]$ [Definition 3.5].

Firstly, we will check the valid region of convergence control parameters of Eq. (4.55) at $\beta = 1$ of the eighth-order FF-HAM at $\alpha = 1, 0.5,$ and 0, where the regions are illustrated in Figures 4.13-4.15.

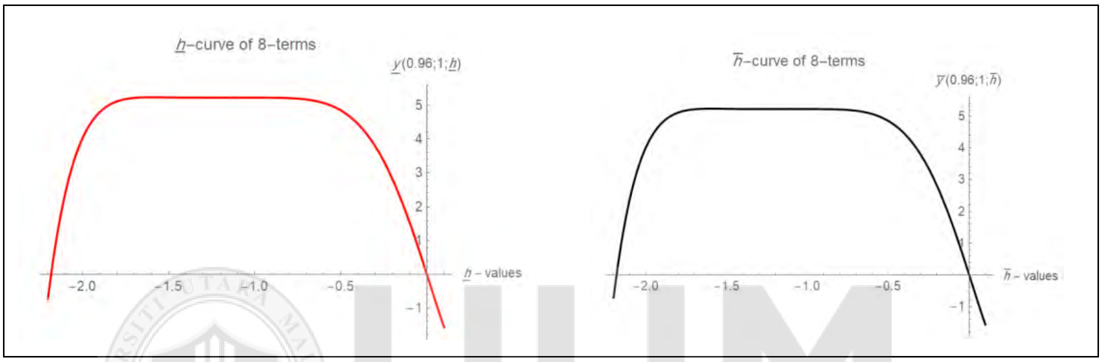

Figure 4.13. The $\tilde{h}$-curve for the fuzzy solution of Eq. (4.55) given by eighth-order FF-HAM for $\beta = 1$, $x = 0.96$ and $\alpha = 1$ when $H(x) = 1$

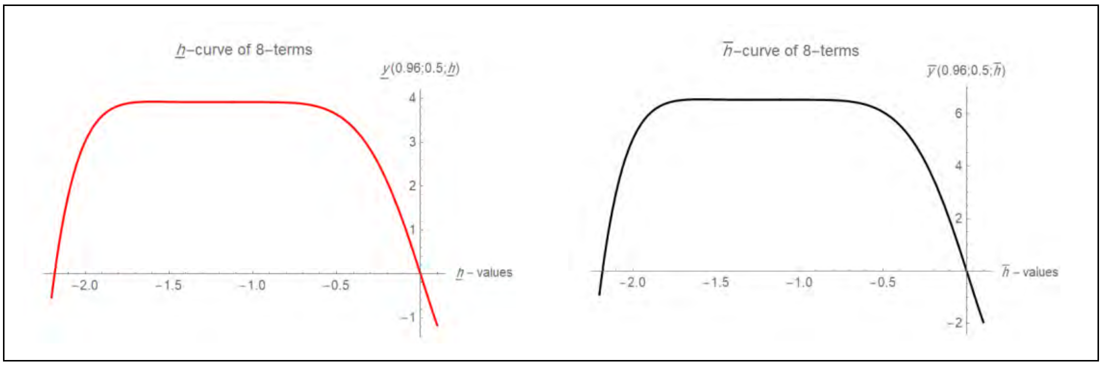

Figure 4.14. The $\tilde{h}$-curve for the fuzzy solution of Eq. (4.55) given by eighth-order FF-HAM for $\beta = 1$, $x = 0.96$ and $\alpha = 0.5$ when $H(x) = 1$

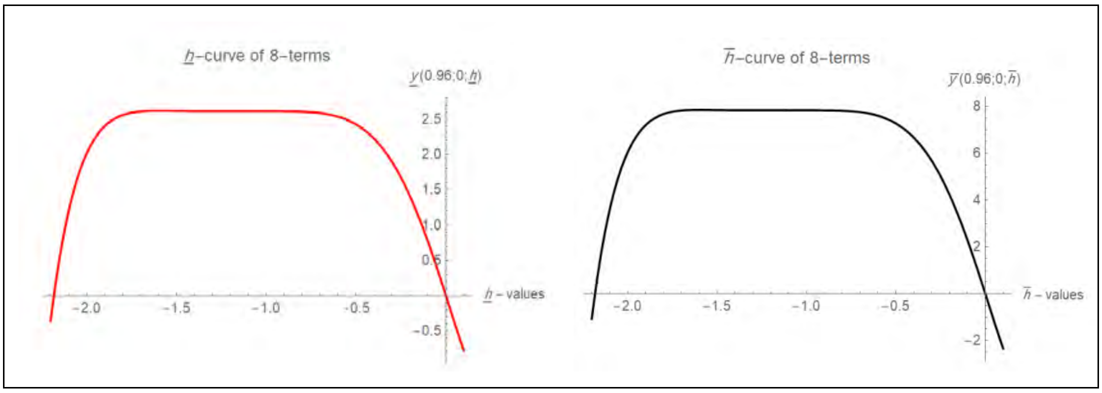

Figure 4.15. The $\tilde{h}$-curve for the fuzzy solution of Eq. (4.55) given by eighth-order FF-HAM for $\beta = 1$, $x = 0.96$ and $\alpha = 0$ when $H(x) = 1$

From Figures 4.13-4.15, it can be observed that the valid region of the optimal convergence control parameters for solving Eq. (4.55) by eighth-order FF-HAM will remain unchanged when the fractional order is changed from $\beta = 0.5$ to $\beta = 1$ and that it will be over the interval $-1.8 \leq \tilde{h} \leq -0.6$ for $\alpha = 0,0.5$ and 1. In other words, we can say that the valid region of the convergence control parameters by FF-HAM will not be affected by the fractional order of the first-order linear FFOIVPs. Furthermore, by using FF-HAM as usual at each value of $x \in [0,1]$ we must find the convergence control parameters, where in this problem, we will find the optimal convergence control parameters over all $x \in [0,1]$ and for all $\alpha \in [0,1]$. Thus, FF-HAM would be more economical in terms of CPU.

Secondly, depending on the residual of Eq. (4.55) of order $\beta = 1$, we will select the best optimal convergence control parameters, then the best value of the convergent control parameter for $\alpha = 0,0.5$, and 1 is $\tilde{h} = -1.0635601570097657$. Now we can tabulate the absolute errors $[\underline{Err}]_\alpha$ and $[\overline{Err}]_\alpha$ of the approximate solutions $\underline{y}(x;\alpha;\tilde{h})$ and $\overline{y}(x;\alpha;\tilde{h})$ that described in Eq.(4.70) obtained by using eighth-order FF-HAM series solution where the absolute errors are given as follows:

$$\begin{cases} \underline{Err}\,(x;\alpha;\underline{h}) = \left|\underline{Y}(x;\alpha) - \underline{y}(x;\alpha;\underline{h})\right|, \\ \overline{Err}\,(x;\alpha;\overline{h}) = \left|\overline{Y}(x;\alpha) - \overline{y}(x;\alpha;\overline{h})\right|. \end{cases} \tag{4.70}$$

Now, based on the absolute error $\widetilde{Err}$ in Eq.(4.70) we will compare the accuracy of eighth-order FF-HAM and eighth-order FRPSM to determine the accuracy of FF-HAM for solving first-order FFOIVPs.

Table 4.7

Numerical comparison of approximate solutions of Eq. (4.55) for different values of x, at $\alpha = 1$ and $\beta = 1$

x	**FRPSM** $\overline{Err}$	**HAM** $\overline{Err}$	**FRPSM** $\underline{Err}$	**HAM** $\underline{Err}$
0.32	3.00×10^{-10}	5.90×10^{-12}	3.00×10^{-10}	5.89×10^{-12}
0.48	8.00×10^{-9}	4.25×10^{-11}	8.00×10^{-9}	4.25×10^{-11}
0.64	1.06×10^{-7}	5.63×10^{-10}	1.06×10^{-7}	5.62×10^{-10}
0.80	8.04×10^{-7}	3.16×10^{-9}	8.04×10^{-7}	3.16×10^{-9}
0.96	4.21×10^{-6}	2.20×10^{-8}	4.21×10^{-6}	2.20×10^{-8}

Table 4.8

Numerical comparison of approximate solutions of Eq. (4.55) for different values of x at $\alpha = 0.5$ and $\beta = 1$

x	**FRPSM** $\overline{Err}$	**HAM** $\overline{Err}$	**FRPSM** $\underline{Err}$	**HAM** $\underline{Err}$
0.32	0	5.82×10^{-13}	0	9.82×10^{-13}
0.48	1.00×10^{-8}	2.72×10^{-11}	6.00×10^{-9}	4.52×10^{-11}
0.64	1.33×10^{-7}	4.63×10^{-10}	7.90×10^{-8}	7.72×10^{-10}
0.80	1.00×10^{-6}	2.86×10^{-8}	6.03×10^{-7}	4.77×10^{-8}
0.96	5.27×10^{-6}	3.30×10^{-7}	3.16×10^{-6}	5.50×10^{-7}

Table 4.9

Numerical comparison of approximate solutions of Eq. (4.55) for different values of x at $\alpha = 0$ and $\beta = 1$

x	**FRPSM** $\overline{Err}$	**HAM** $\overline{Err}$	**FRPSM** $\underline{Err}$	**HAM** $\underline{Err}$
0.32	4.00×10^{-10}	3.94×10^{-13}	1.00×10^{-10}	4.22×10^{-12}
0.48	1.20×10^{-8}	1.81×10^{-11}	4.00×10^{-9}	2.71×10^{-11}
0.64	1.59×10^{-7}	3.09×10^{-10}	5.30×10^{-8}	6.71×10^{-10}
0.80	1.20×10^{-6}	1.91×10^{-8}	4.02×10^{-7}	6.17×10^{-9}
0.96	6.32×10^{-6}	2.20×10^{-7}	2.11×10^{-6}	2.22×10^{-8}

Based on the findings in Tables 4.7-4.9, we concluded that the approximate series solutions by eighth-order FF-HAM linked with the best optimal convergence control parameter value $\tilde{h}$ at different values of x and α are more accurate compared to the results obtained by the eighth-order solution of FRPSM.

4.5.1.3 Solving Example 4.1 by FF-OHAM

To solve Example 4.1 by FF-OHAM, we can construct a fifth-order FF-OHAM with five convergence control parameters $\left(\tilde{S}_1(0.8), \tilde{S}_2(0.8), \ldots, \tilde{S}_5(0.8)\right)$. In this way, we can obtain the same results with five convergence control parameters for each $\alpha \in [0,1]$. Now we can apply the fifth-order FF-OHAM for Eq. (4.55) of order $\beta = 0.5$ for all $\alpha \in [0,1]$ as follows:

For lower bound:

$$\begin{cases} (1-q)\left[\underline{D}^{(\beta)}\left(\underline{y}(x;\alpha)\right) - (1+\alpha)\right] = \sum_{j=1}^{k} \underline{S}_j(\alpha) q^j \left[\underline{D}^{(\beta)}\left(\underline{y}(x;\alpha)\right)\right] \\ \qquad - \sum_{j=1}^{k} \underline{S}_j(\alpha) q^j \, [1+\alpha] - \sum_{j=1}^{k} \underline{S}_j(\alpha) q^j \left[\underline{y}(x;\alpha)\right] \end{cases} \tag{4.71}$$

and for upper bound:

$$\begin{cases}(1-q)\left[\overline{D}^{(\beta)}\left(\overline{y}(x;\alpha)\right)-(3-\alpha)\right]=\sum_{j=1}^{k}\overline{S}_j(\alpha)q^j\left[\overline{D}^{(\beta)}\left(\overline{y}(x;\alpha)\right)\right] \\ \qquad -\sum_{j=1}^{k}\overline{S}_j(\alpha)q^j\,[3-\alpha]-\sum_{j=1}^{k}\overline{S}_j(\alpha)q^j\,[\,\overline{y}(x;\alpha)]\end{cases} \tag{4.72}$$

Then, we will list the zeroth-order problem and the first five order problems as follows:

Zeroth-order problem:

$$\begin{cases}\tilde{y}_0(x;\alpha)=\tilde{J}^{(\beta)}[1+\alpha,3-\alpha], \\ \qquad \tilde{y}_0(0;\alpha)=\left[\tilde{0}\right]_\alpha.\end{cases} \tag{4.73}$$

First-order problem:

$$\begin{cases}\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)=\left(1+\tilde{S}_1(\alpha)\right)\tilde{y}_0(x;\alpha)-\tilde{J}^{(\beta)}[(1+\alpha),(3-\alpha)]- \\ \tilde{S}_1(\alpha)\,\tilde{J}^{(\beta)}\left(\tilde{y}_0(x)+[(1+\alpha),(3-\alpha)]\right), \\ \qquad \tilde{y}_1(0;\alpha)=\left[\tilde{0}\right]_\alpha.\end{cases} \tag{4.74}$$

Second-order problem:

$$\begin{cases}\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)=\left(1+\tilde{S}_1(\alpha)\right)\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)+ \\ \tilde{S}_2(\alpha)\,\tilde{y}_0(x;\alpha)-\tilde{S}_2(\alpha)\tilde{J}^{(\beta)}(\tilde{y}_0(x;\alpha)+[(1+\alpha),(3-\alpha)]) \\ -\tilde{S}_1(\alpha)\,\tilde{J}^{(\beta)}\left(\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)\right), \\ \qquad \tilde{y}_2(0;\alpha)=\left[\tilde{0}\right]_\alpha.\end{cases} \tag{4.75}$$

Third-order problem:

$$\begin{cases}\tilde{y}_3\left(x,\tilde{S}_1(\alpha),\ldots,\tilde{S}_3(\alpha);\alpha\right)=\left(1+\tilde{S}_1(\alpha)\right)\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)+ \\ \tilde{S}_3(\alpha)\tilde{y}_0(x;\alpha)-\tilde{S}_3(\alpha)\tilde{J}^{(\beta)}(\tilde{y}_0(x;\alpha)+[(1+\alpha),(3-\alpha)]) \\ +\tilde{S}_2(\alpha)\,\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)-\tilde{S}_2(\alpha)\tilde{J}^{(\beta)}\left(\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)\right), \\ \qquad \tilde{y}_3(0;\alpha)=\left[\tilde{0}\right]_\alpha.\end{cases} \tag{4.76}$$

Fourth-order problem:

$$\begin{cases} \tilde{y}_4\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_4(\alpha);\alpha\big) = \big(1+\tilde{S}_1(\alpha)\big)\,\tilde{y}_3\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_3(\alpha);\alpha\big) - \\ \tilde{S}_1(\alpha)\tilde{\mathcal{J}}^{(\beta)}\,\tilde{y}_3\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_3(\alpha);\alpha\big) + \tilde{S}_2(\alpha)\Big(\tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big)\Big) \\ -\tilde{S}_2(\alpha)\,\tilde{\mathcal{J}}^{(\beta)}\,\Big(\tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big)\Big) + \tilde{S}_3(\alpha)\,\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big) - \tilde{S}_3(\alpha) \\ \tilde{\mathcal{J}}^{(\beta)}\Big(\,\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big)\Big) + \tilde{S}_4(\alpha)\big(\,\tilde{y}_0(x;\alpha)\big) - \tilde{S}_4(\alpha)\tilde{\mathcal{J}}^{(\beta)}\big(\,\tilde{y}_0(x;\alpha)\big), \\ \qquad +[(1+\alpha),(3-\alpha)], \\ \qquad \tilde{y}_4(0;\alpha) = [\tilde{0}]_\alpha. \end{cases} \tag{4.77}$$

Fifth-order problem:

$$\begin{cases} \tilde{y}_5\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_5(\alpha);\alpha\big) = \big(1+\tilde{S}_1(\alpha)\big)\,\tilde{y}_4\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_4(\alpha);\alpha\big) \\ +\tilde{S}_2(\alpha)\,\tilde{y}_3\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_3(\alpha);\alpha\big) + \tilde{S}_3(\alpha)\,\tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big) + \tilde{S}_4(\alpha) \\ \Big(\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big)\Big) + \tilde{S}_5(\alpha)\Big(\,\tilde{y}_0(x;\alpha)\Big) - \tilde{S}_1(\alpha)\tilde{\mathcal{J}}^{(\beta)}\,\tilde{y}_4\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_4(\alpha);\alpha\big) \\ -\tilde{S}_2(\alpha)\tilde{\mathcal{J}}^{(\beta)}\,\tilde{y}_3\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_3(\alpha);\alpha\big) - \tilde{S}_3(\alpha)\tilde{\mathcal{J}}^{(\beta)}\,\tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big) \\ -\tilde{S}_4(\alpha)\tilde{\mathcal{J}}^{(\beta)}\,\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big) - \tilde{S}_5(\alpha)\tilde{\mathcal{J}}^{(\beta)}\big(\,\tilde{y}_0(x;\alpha) + [(1+\alpha),(3-\alpha)]\big) \\ \qquad \tilde{y}_5(0;\alpha) = [\tilde{0}]_\alpha. \end{cases} \tag{4.78}$$

Based on Section 4.4.2, the solution of Eq. (4.55) by fifth-order FF-OHAM is given as follows:

$$\begin{cases} \underline{y}(x;\alpha) = \underline{y}_0(x;\alpha) + \underline{y}_1\big(x,\underline{S}_1(\alpha);\alpha\big) + \underline{y}_2\big(x,\underline{S}_1(\alpha),\underline{S}_2(\alpha);\alpha\big) + \\ \quad \underline{y}_3\big(x,\underline{S}_1(\alpha),\underline{S}_2(\alpha),\underline{S}_3(\alpha);\alpha\big) + \underline{y}_4\big(x,\underline{S}_1(\alpha)..,\underline{S}_4(\alpha);\alpha\big) + \\ \quad \underline{y}_5\big(x,S_1(\alpha)..,\underline{S}_5(\alpha);\alpha\big) \\ \overline{y}(x;\alpha) = \overline{y}_0(x;\alpha) + \overline{y}_1\big(x,\overline{S}_1(\alpha);\alpha\big) + \overline{y}_2\big(x,\overline{S}_1(\alpha),\overline{S}_2(\alpha);\alpha\big) + \\ \quad \overline{y}_3\big(x,\overline{S}_1(\alpha),\overline{S}_2(\alpha),\overline{S}_3(\alpha);\alpha\big) + \overline{y}_4\big(x,\overline{S}_1(\alpha)..,\overline{S}_4(\alpha);\alpha\big) + \\ \quad \overline{y}_5\big(x,\overline{S}_1(\alpha)..,\overline{S}_5(\alpha);\alpha\big) \end{cases} \tag{4.79}$$

According to Section 4.4.3, we can calculate the optimal convergence control parameters values for the lower and upper bound at $\alpha = 0.8$, where the resulting optimal values are tabulated in Table 4.10.

Table 4.10

The optimal values of the convergence control parameters by fifth-order FF-OHAM for solving Eq. (4.55) for $\beta = 0.5$ *at* $x = 0.2$ $\forall\alpha \in [0,1]$

$\tilde{S}_j(\mathbf{0.8})$	$\tilde{S}_1(0.8) \rightarrow -1.1638339125014454$
	$\tilde{S}_2(0.8) \rightarrow -0.00026513060144121246$
	$\tilde{S}_3(0.8) \rightarrow 0.00029351604664000047$
	$\tilde{S}_4(0.8) \rightarrow -0.00002928410128387554$
	$\tilde{S}_5(0.8) \rightarrow 0$

Table 4.10 illustrates the optimal fuzzy convergence control parameters for the lower and upper bound at $\alpha = 0.8$. Now, based on the optimal convergence control parameters listed in Table 4.10, we calculate the accuracy of the lower and upper solution of Eq. (4.55) and tabulate them in in Table 4.11.

Table 4.11

The approximate solution and error of Eq. (4.55) by fifth-order FF-OHAM for $\beta = 0.5$ *at* $x = 0.2$ $\forall\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\underline{ER}]_\alpha, \tilde{S}_j(\mathbf{0.8})$	$[\overline{ER}]_\alpha, \tilde{S}_j(\mathbf{0.8})$	$[\underline{y}]_\alpha, \tilde{S}_j(\mathbf{0.8})$	$[\overline{y}]_\alpha, \tilde{S}_j(\mathbf{0.8})$
0	-0.00003	-0.00009	0.79902	2.39705
0.2	-0.00004	-0.00008	0.95882	2.23724
0.4	-0.00004	-0.00008	1.11862	2.07744
0.6	-0.00005	-0.00007	1.27842	1.91764
0.8	-0.00005	-0.00007	1.43822	1.75783
1	-0.00006	-0.00006	1.59803	1.59803

Table 4.11 illustrates the accuracy of the approximate solution of Eq. (4.55) with fractional order $\beta = 0.5$ solved by fifth-order FF-OHAM. The fifth-order FF-OHAM satisfies the fuzzy number solution.

We can summarize the fuzzy approximate solutions of Eq. (4.55) for all $\alpha \in [0,1]$ and $x \in [0,0.2]$ for the optimal $\tilde{S}_j(0.8)$ as illustrated in the three-dimensional graph in Figure 4.16.

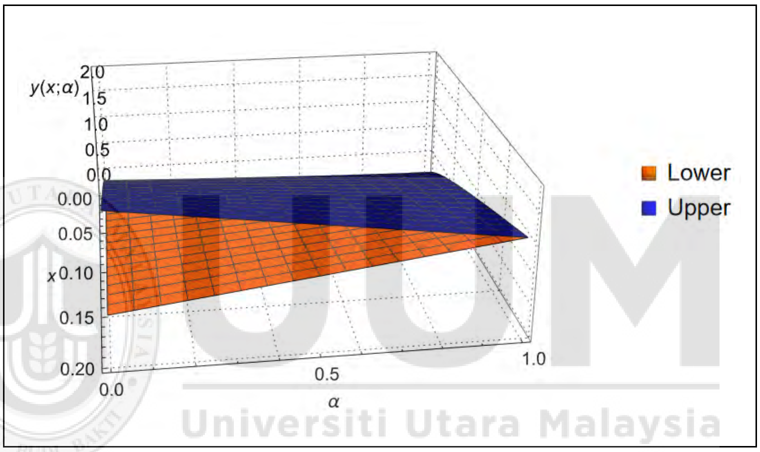

Figure 4.16. The three-dimensional approximate solution given by fifth-order FF-OHAM over all $x \in [0,0.2]$ at $\beta = 0.5$ and for all $\alpha \in [0,1]$

Figure 4.16 and Table 4.11 show that the fifth-order FF-OHAM approximate solutions for Eq. (4.55) at $x = 0.2$ for all $\alpha \in [0,1]$ for fractional order $\beta = 0.5$ satisfy the definition of fuzzy numbers [Definition 3.7] and the solution of the fuzzy differential equation.

As has been indicated previously, FF-HAM gives a more accurate series solution as the series order increases. Now to clarify the accuracy of the FF-OHAM series solutions with different order, we shall proceed to solve Eq. (4.55) by eighth-order FF-

OHAM instead of a fifth-order series solution to probe into the convergence dynamic of FF-OHAM.

Firstly, for $\alpha = 0.4$, we will be looking for the optimal convergence control parameters $\tilde{S}_j(0.4)$, for $j = 1,2,3,\ldots,8$. Table 4.12 lists the optimal fuzzy convergence control parameters for the lower and upper bound at $\alpha = 0.4$ by eighth-order FF-OHAM.

Table 4.12

The optimal values of the convergence control parameters by eighth-order FF-OHAM for solving Eq. (4.55) for $\beta = 0.5$ at $x = 0.2\ \forall\alpha \in [0,1]$

$\tilde{\boldsymbol{S}}_j(\mathbf{0.4})$	$\tilde{S}_1(0.4) \to -1.058066976496025$	$\tilde{S}_2(0.4) \to -0.0036968263587707053$
	$\tilde{S}_3(0.4) \to 0.007130733978260273$	$\tilde{S}_4(0.4) \to -0.004471798756197687$
	$\tilde{S}_5(0.4) \to 0.0012602590704241605$	$\tilde{S}_6(0.4) \to -0.0003003934392337749$
	$\tilde{S}_7(0.4) \to 0.00010387832134684885$	$\tilde{S}_8(0.4) \to 0$

Next, based on the optimal convergence control parameters listed in Table 4.12, we calculate and tabulate the accuracy of the lower and upper solution of Eq. (4.55) in Table 4.13.

Table 4.13

The approximate solution and error of Eq. (4.55) by eighth-order FF-OHAM for $\beta = 0.5$ *at* $x = 0.2\ \forall \alpha \in [0,1]$

α	$[\underline{ER}]_\alpha, \tilde{S}_j(0.4)$	$[\overline{ER}]_\alpha, \tilde{S}_j(0.4)$	$[\underline{y}]_\alpha, \tilde{S}_j(0.4)$	$[\overline{y}]_\alpha, \tilde{S}_j(0.4)$
0	-5.06343×10^{-7}	-1.51903×10^{-6}	0.79902	2.39705
0.2	-6.07612×10^{-7}	-1.41776×10^{-6}	0.95882	2.23725
0.4	-7.08880×10^{-7}	-1.31649×10^{-6}	1.11862	2.07744
0.6	-8.10149×10^{-7}	-1.21522×10^{-6}	1.27843	1.91764
0.8	-9.11418×10^{-7}	-1.11395×10^{-6}	1.43823	1.75784
1	-1.01269×10^{-6}	-1.01269×10^{-6}	1.59803	1.59803

Table 4.13 illustrates the accuracy of Eq. (4.55) by eighth-order FF-OHAM and the results establish that the series solutions by FF-OHAM will approach the exact solutions as the series order increases. Figures 4.17-4.18 illustrate the accuracy of fifth-order and eighth-order FF-OHAM for solving Eq. (4.55) for $\beta = 0.5\ \forall x \in [0,0.2]$ and for all $\alpha \in [0,1]$.

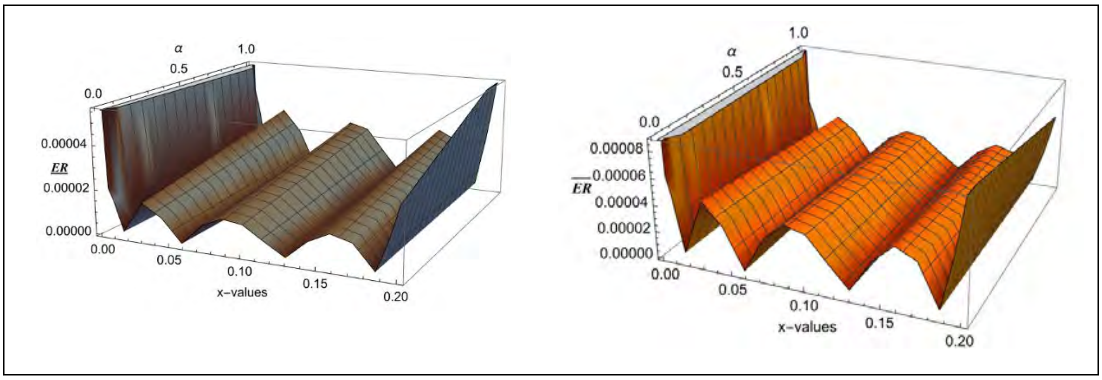

Figure 4.17. The accuracy of fifth-order FF-OHAM for solving Eq. (4.55) for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.2]$

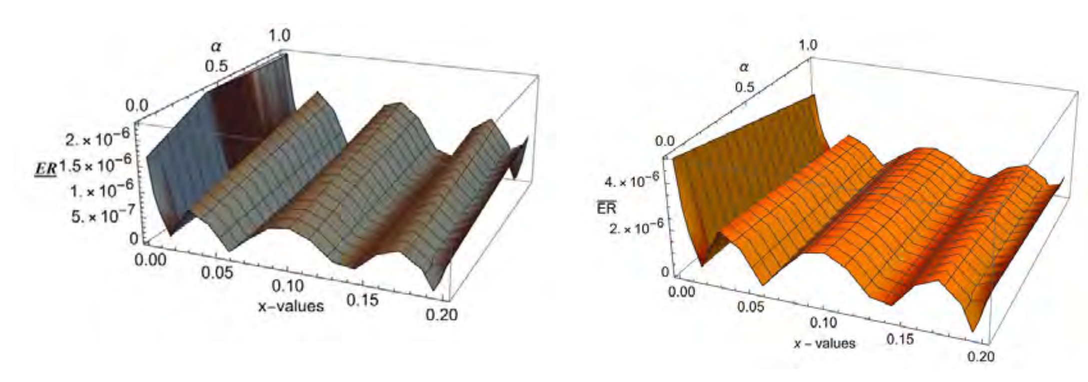

Figure 4.18. The accuracy of eighth-order FF-OHAM for solving Eq. (4.55) for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.2]$

Figures 4.17-4.18 indicate the eight-order FF-OHAM provides more accurate solution compared with fifth-order FF-OHAM. On the other hand, based on the upper and lower solution in Table 4.13, we can conclude that the series solutions for Eq. (4.55) by eighth-order FF-OHAM satisfy the triangular solution of the fuzzy differential equations.

In addition, we can summarize the fuzzy approximate solutions of Eq. (4.55) for all $\alpha \in [0,1]$ [Definition 3.5] and $x \in [0,0.2]$ based on the optimal convergence parameters $\tilde{S}_j(0.8)$ in the three-dimensional graph in Figure 4.19.

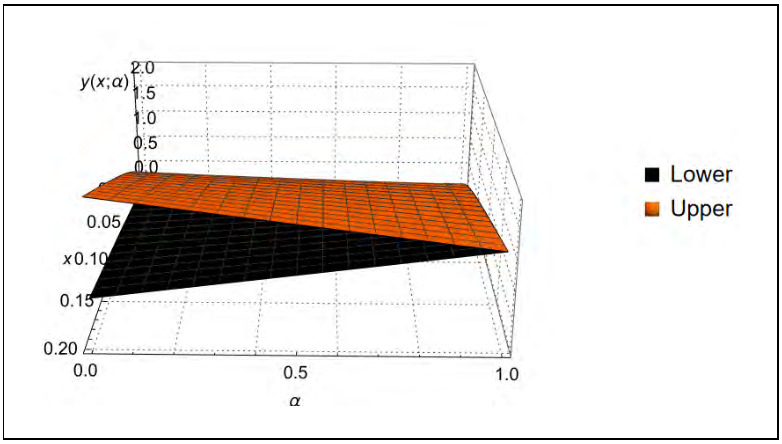

Figure 4.19. The three-dimensional approximate solution of Eq. (4.55) given by eighth-order FF-OHAM over all $x \in [0,0.2]$ at $\beta = 0.5$, and for all $\alpha \in [0,1]$

Figure 4.19 and Table 4.13 show that the eighth-order FF-OHAM approximate solutions at $x = 0.2$ for all $\alpha \in [0,1]$ for fractional order $\beta = 0.5$ satisfy the fuzzy numbers [Definition 3.7] and the solution of fuzzy differential equation.

4.5.1.4 Comparative Study of FF-OHAM for Example 4.1

For comparative study, we compare the eighth-order FF-OHAM and the eighth-order fractional residual power method (FRPSM) (Alaroud et al., 2019) under Hukuhara Caputo differentiability [Definition 3.21] with the fractional derivative order $\beta = 1$ at different values of $\alpha \in [0,1]$ [Definition 3.5] and $x \in [0,1]$.

After the verification process which depends on the residual of Eq. (4.55) of order $\beta = 1$ for $\alpha = 0,0.5$, and 1, it is found that the optimal convergence control parameters $\tilde{S}_j(\alpha)$, for $j = 1,2,3,\dots,8$ will change due to the change of fractional order for Eq. (4.55) from $\beta = 0.5$ to $\beta = 1$.

Now, we find the absolute errors $[\underline{Err}]_\alpha$ and $[\overline{Err}]_\alpha$ of the approximate solutions $\underline{y}(x, \tilde{S}_j; \alpha)$ and $\overline{y}(x, \tilde{S}_j; \alpha)$ obtained using eighth-order FF-OHAM series solution for different values of $x \in [0,1]$ and $\alpha \in [0,1]$ as follows:

$$\begin{cases} \underline{Err}\left(x, \tilde{S}_j; \alpha\right) = \left|\underline{Y}(x; \alpha) - \underline{y}\left(x, \tilde{S}_j; \alpha\right)\right|, \\ \overline{Err}\left(x, \tilde{S}_j; \alpha\right) = \left|\overline{Y}(x; \alpha) - \overline{y}\left(x, \tilde{S}_j; \alpha\right)\right|. \end{cases} \tag{4.80}$$

The absolute errors are tabulated in Table 4.14-4.16.

Table 4.14

Numerical comparison of approximate solutions of Eq. (4.55) for different values of x at $\alpha = 1$ when $\beta = 1$

x	FRPSM $\overline{Err}$	OHAM $\overline{Err}$	FRPSM $\underline{Err}$	OHAM $\underline{Err}$
0.32	3.00×10^{-10}	1.65×10^{-10}	3.00×10^{-10}	1.65×10^{-10}
0.48	8.00×10^{-9}	3.41×10^{-9}	8.00×10^{-9}	3.41×10^{-9}
0.64	1.06×10^{-7}	2.68×10^{-8}	1.06×10^{-7}	2.68×10^{-8}
0.80	8.04×10^{-7}	1.24×10^{-8}	8.04×10^{-7}	1.24×10^{-8}
0.96	4.21×10^{-6}	1.68×10^{-8}	4.21×10^{-6}	1.68×10^{-8}

Table 4.15

Numerical comparison of approximate solutions of Eq. (4.55) for different values of x at $\alpha = 0.5$ when $\beta = 1$

x	FRPSM $\overline{Err}$	OHAM $\overline{Err}$	FRPSM $\underline{Err}$	OHAM $\underline{Err}$
0.32	0	2.51×10^{-10}	0	1.50×10^{-10}
0.48	1.00×10^{-8}	6.09×10^{-10}	6.00×10^{-9}	3.65×10^{-10}
0.64	1.33×10^{-7}	1.79×10^{-8}	7.90×10^{-8}	4.19×10^{-9}
0.80	1.00×10^{-6}	1.29×10^{-8}	6.03×10^{-7}	7.76×10^{-9}
0.96	5.27×10^{-6}	5.76×10^{-9}	3.16×10^{-6}	3.46×10^{-9}

Table 4.16

Numerical comparison of approximate solutions of Eq. (4.55) for different values of x at $\alpha = 0$ when $\beta = 1$

x	FRPSM $\overline{Err}$	OHAM $\overline{Err}$	FRPSM $\underline{Err}$	OHAM $\underline{Err}$
0.32	4.00×10^{-10}	4.50×10^{-11}	1.00×10^{-10}	1.50×10^{-11}
0.48	1.20×10^{-8}	1.51×10^{-10}	4.00×10^{-9}	5.04×10^{-11}
0.64	1.59×10^{-7}	7.53×10^{-9}	5.30×10^{-8}	2.51×10^{-9}
0.80	1.20×10^{-6}	1.80×10^{-8}	4.02×10^{-7}	6.19×10^{-9}
0.96	6.32×10^{-6}	1.47×10^{-8}	2.11×10^{-6}	8.40×10^{-9}

4.5.2 Example 4.2

Consider the following nonlinear fractional Riccati differential equation introduced in (Jafari & Tajadodi, 2010)

$$\begin{cases} D^{(\beta)}y(x) + y^2(x) - 1 = 0, \quad 0 < \beta \leq 1, \\ \qquad\qquad y(0) = 0, \end{cases} \tag{4.81}$$

Since this example is a non-fuzzy FODE, we will first introduce a new fuzzification of the equation. In this study, the fuzzy version of Eq. (4.81) is created.

Consider the new fuzzification of nonlinear first-order fuzzy fractional Riccati differential equation:

$$\begin{cases} D^{(\beta)}\tilde{y}(x) + \tilde{y}^2(x) - 1 = 0, \quad 0 < \beta \leq 1, \\ \qquad\qquad \tilde{y}(0) = [\tilde{0}] = [\underline{0}, \overline{0}]. \end{cases} \tag{4.82}$$

Based on Section 4.3.2, we can make Eq. (4.82) more generalized $\forall\, \alpha \in [0,1]$ [Definition 3.5] such that

$$\tilde{y}(x) = \tilde{y}(x;\alpha) = \left[\underline{y}(x;\alpha), \overline{y}(x;\alpha)\right]. \tag{4.83}$$

From the definitions of the fuzzy numbers [Definition 3.6], we can defuzzify the initial conditions. Let $[\tilde{0}]_{\alpha}$ be triangular fuzzy numbers [Definition 3.7] such that $\forall\ \alpha \in [0,1]$ we have

$$[\tilde{0}]_{\alpha} = [-1,0,1]_{\alpha} = [\alpha - 1, 1 - \alpha]. \tag{4.84}$$

Here, the linear operators of Eq.(4.82) are $\tilde{\mathcal{L}}_{\beta} = \left[\underline{D}^{(\beta)}\left(\underline{y}(x;\alpha)\right), \overline{D}^{(\beta)}\left(\overline{y}(x;\alpha)\right)\right]$ $\forall\ \alpha \in [0,1]$.

4.5.2.1 Solving Example 4.2 by FF-HAM

According to Section 4.3.2, we can construct the zeroth-order and k^{th}-order deformation equations of FF-HAM for Eq. (4.82) as follows:

$$\begin{cases} \underline{y}_k(x;\alpha) = \psi_k \underline{y}_{k-1}(x;\alpha) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \mathcal{R}_k(\underline{\vec{y}}_{k-1}(x;\alpha)) \\ \overline{y}_k(x;\alpha) = \psi_k \overline{y}_{k-1}(x;\alpha) + \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)} \mathcal{R}_k(\overline{\vec{y}}_{k-1}(x;\alpha)) \end{cases} \tag{4.85}$$

with $\underline{y}_k(0;\alpha) = 0, \overline{y}_k(0;\alpha) = 0$ such that

$$\begin{cases} \mathcal{R}_k\left(\underline{\vec{y}}_{k-1}(x;\alpha)\right) = \underline{y}_{k-1}{}^{(\beta)}(x;\alpha) + \sum_{k=0}^{m-1} \underline{y}_k(x;\alpha)\, \underline{y}_{m-1-k}(x;\alpha) - 1, \\ \mathcal{R}_k\left(\overline{\vec{y}}_{k-1}(x;\alpha)\right) = \overline{y}_{k-1}{}^{(\beta)}(x;\alpha) + \sum_{k=0}^{m-1} \overline{y}_k(x;\alpha)\, \overline{y}_{m-1-k}(x;\alpha) - 1. \end{cases} \tag{4.86}$$

For the initial approximation:

$$\tilde{y}_0(x;\alpha) = [\alpha - 1, 1 - \alpha].$$

From Eq. (4.86), we can obtain the FF-HAM series solution components for the lower and the upper bound, for $j = 1,2,3,.,k+1$ as follows:

$$\begin{cases}\tilde{y}_1(x;\alpha) = \tilde{h}\,\tilde{\mathcal{J}}^{(\beta)}\left(\tilde{y}_0{}^{(\beta)}(x;\alpha) - [\tilde{y}_0(x;\alpha)]^2 - 1\right), \\ \qquad\qquad \tilde{y}_1(x;\alpha) = 0\end{cases}$$

$$\begin{cases}\tilde{y}_2(x;\alpha) = \tilde{y}_1(x;\alpha) + \tilde{h}\,\tilde{\mathcal{J}}^{(\beta)}\left(\tilde{y}_1{}^{(\beta)}(x;\alpha) - 2\tilde{y}_1(x;\alpha)\tilde{y}_0(x;\alpha)\right), \\ \qquad\qquad \tilde{y}_2(x;\alpha) = 0\end{cases}$$

.

.

$$\begin{cases}\tilde{y}_{k+1}(x;\alpha) = \tilde{y}_k(x;\alpha) + \tilde{h}\,\tilde{\mathcal{J}}^{(\beta)}\left(\tilde{y}_k{}^{(\beta)}(x;\alpha) - \sum_{m=0}^{n-1}\tilde{y}_k(x;\alpha)\,\tilde{y}_{m-1-k}(x;\alpha)\right), \\ \qquad\qquad \tilde{y}_k(x;\alpha) = 0\end{cases}$$

For the lower bound, we have

$$\begin{cases}\underline{y}_{k+1}(x;\alpha) = \underline{y}_k(x;\alpha) - \underline{h}(\alpha)\,\tilde{\mathcal{J}}^{(\beta)}\left[\sum_{m=0}^{n-1}\underline{y}_k(x;\alpha)\,\underline{y}_{m-1-k}(x;\alpha)\right] \\ +\underline{h}(\alpha)\,\tilde{\mathcal{J}}^{(\beta)}\left[\underline{y}_k{}^{(\beta)}(x;\alpha)\right], \\ \qquad\qquad \underline{y}_k(x;\alpha) = 0\end{cases} \tag{4.87}$$

Here, some of the first lower bound terms of the proposed fuzzy fractional series solution using FF-HAM for Eq. (4.87), starting with the lower initial guess are listed as follows:

$\underline{y}_0 = (\alpha - 1)$,

$\underline{y}_1 = c_0 x^{\beta}$,

$\underline{y}_2 = \left(\underline{h}(\alpha) + 1\right)c_0 x^{\beta} + \underline{h}c_1 x^{2\beta}$,

$\underline{y}_3 = \left(\underline{h}(\alpha) + 1\right)^2 c_0 x^{\beta} + \left(\underline{h}(\alpha)\left(\underline{h}(\alpha) + 1\right)c_1 + c_3\right)x^{2\beta} + (c_2 + c_4)x^{3\beta}$,

$\underline{y}_4 = \left(\underline{h}(\alpha) + 1\right)^3 c_0 x^{\beta} + \left(\underline{h}(\alpha)\left(\underline{h}(\alpha) + 1\right)^2 c_1 + \left(\underline{h}(\alpha) + 1\right)c_3 + c_5\right)x^{2\beta} +$

$\quad \left(\left(\underline{h}(\alpha) + 1\right)(c_2 + c_4) + c_6\right)x^{3\beta} + c_7 x^{4\beta}$,

and so on, and the first few coefficients of c_k are as follows:

$c_0 = \frac{(\alpha^2-2\alpha)}{\Gamma(1.7)}\underline{h}(\alpha),$

$c_1 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2\alpha-2)c_0,$

$c_2 = \frac{\Gamma(2.4)}{\Gamma(3.1)}\underline{h}(\alpha){c_0}^2,$

$c_3 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2\alpha-2)\underline{h}(\alpha)\big(\underline{h}(\alpha)+1\big)c_0,$

$c_4 = \frac{\Gamma(2.4)}{\Gamma(3.1)}(2\alpha-2)\underline{h}^2(\alpha)c_1,$

$c_5 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2\alpha-2)\,\underline{h}(\alpha)\big(\underline{h}(\alpha)+1\big)^2 c_0,$

$c_6 = \frac{\Gamma(2.4)}{\Gamma(3.1)}\Big((2\alpha-2)\Big(\underline{h}(\alpha)\big(\underline{h}(\alpha)\big(\underline{h}(\alpha)+1\big)c_1+c_3\big)\Big)+2\underline{h}(\alpha)\big(\underline{h}(\alpha)+1\big){c_0}^2\Big),$

$c_7 = \frac{\Gamma(3.1)}{\Gamma(3.8)}\Big((2\alpha-2)(c_2+c_4)\underline{h}(\alpha)+2(c_0c_1)\underline{h}(\alpha)\Big).$

For the upper bound, we have

$$\begin{cases} \overline{y}_{k+1}(x;\alpha) = \overline{y}_k(x;\alpha) - \overline{h}(\alpha)\,\tilde{J}^{(\beta)}\left[\sum_{m=0}^{n-1}\overline{y}_k(x;\alpha)\,\overline{y}_{m-1-k}(x;\alpha)\right] \\ +\overline{h}(\alpha)\,\tilde{J}^{(\beta)}\left[\overline{y}^{(\beta)}{}_k(x;\alpha)\right], \\ \qquad\qquad \overline{y}_k(x;\alpha) = 0. \end{cases} \tag{4.88}$$

Several of the first upper bound terms of the FF-HAM series solution of Eq.(4.88) starting with upper initial guess are listed as follows:

$\overline{y}_0 = (1-\alpha),$

$\overline{y}_1 = r_0 x^{\beta},$

$\overline{y}_2 = r_1 x^{\beta} + r_2 x^{2\beta},$

$\overline{y}_3 = \big(r_1+\overline{h}(\alpha)r_1\big)x^{\beta} + (r_2+r_3+r_5)x^{2\beta} + (r_4+r_6)x^{3\beta},$

$\overline{y}_4 = \big(\overline{h}(\alpha)+1\big)^2 x^{\beta} + \Big(\big(\overline{h}(\alpha)+1\big)(r_2+r_3+r_5)+r_7\Big)x^{2\beta} + \Big(\big(\overline{h}(\alpha)+1\big)(r_4 +$

$\qquad r_6) + r_8\Big)x^{3\beta} + r_9 x^{4\beta},$

and so on, and some of the first coefficients for the upper bound are as follows:

$r_0 = c_0,$

$r_1 = r_0 + \frac{(\alpha^2-2\alpha)}{\Gamma(1.7)}\overline{h}^2(\alpha),$

$r_2 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2-2\alpha)\overline{h}(\alpha)r_0,$

$r_3 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2-2\alpha)\overline{h}^2(\alpha)r_0,$

$r_4 = \frac{\Gamma(2.4)}{\Gamma(3.1)}\overline{h}(\alpha){r_0}^2,$

$r_5 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2-2\alpha)\overline{h}r_1,$

$r_6 = \frac{\Gamma(2.4)}{\Gamma(3.1)}(2-2\alpha)\overline{h}(\alpha)r_2,$

$r_7 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2-2\alpha)\overline{h}(\alpha)\big(\overline{h}(\alpha)+1\big)r_1,$

$r_8 = \frac{\Gamma(2.4)}{\Gamma(3.1)}\big((2-2\alpha)(r_2+r_3+r_5)+2r_0r_1\big)\overline{h}(\alpha),$

$r_9 = \frac{\Gamma(3.1)}{\Gamma(3.8)}\big((2-2\alpha)(r_4+r_6)+2r_0r_2\big)\overline{h}(\alpha).$

Based on Section 4.3.3, the accuracy of the sixth-order FF-HAM for Eq. (4.82) corresponding to the residual error described in Eq. (4.26) depends on $\tilde{x}$ values and $\tilde{h}$ values as follows:

$$\begin{cases} \underline{ER}(x;\alpha;\underline{h}) = \left(\underline{y}^{(\beta)}(x;\alpha;\underline{h}) + \left[\underline{y}(x;\alpha;\underline{h})\right]^2 - 1\right), \\ \overline{ER}(x;\alpha;\overline{h}) = \left(\overline{y}^{(\beta)}(x;\alpha;\overline{h}) + \left[\overline{y}(x;\alpha;\overline{h})\right]^2 - 1\right). \end{cases} \quad (4.89)$$

Furthermore, the convergence control parameter $\tilde{h}(\alpha)$ plays a pivotal role in controlling and adjusting the convergence region of the homotopy series solution.

We will use the properties of FF-HAM convergence in Section 4.3.3 to find the best value of $\tilde{h}(\alpha)$. To achieve this, for a fixed value of $0 \le \alpha \le 1$ [Definition 3.5], for example $\alpha = 0.4$, we will plot $\tilde{h}(0.4)$-curves of lower bound $\underline{y}\left(0.1; 0.4; \underline{h}(0.4)\right)$ and upper bound $\overline{y}\left(0.1; 0.4; \overline{h}(0.4)\right)$ depending on the integral of the squared residual error described in Eq.(4.90) over the whole region of $x \in [0,0.1]$ of the sixth-order FF-HAM for Eq. (4.82) for $\beta = 0.5$, $\alpha = 0.4$, and at $x = 0.1$. The integral of the squared residual error is given as follows:

$$\begin{cases} \underline{SER}\left(0.1; 0.4; \underline{h}(0.4)\right) = \int_{x=0}^{0.1} \left[\underline{ER}\left(0.1; 0.4; \underline{h}(0.4)\right)\right]^2 dx, \\ \overline{SER}\left(0.1; 0.4; \overline{h}(0.4)\right) = \int_{x=0}^{0.1} \left[\overline{ER}\left(0.1; 0.4; \overline{h}(0.4)\right)\right]^2 dx. \end{cases} \tag{4.90}$$

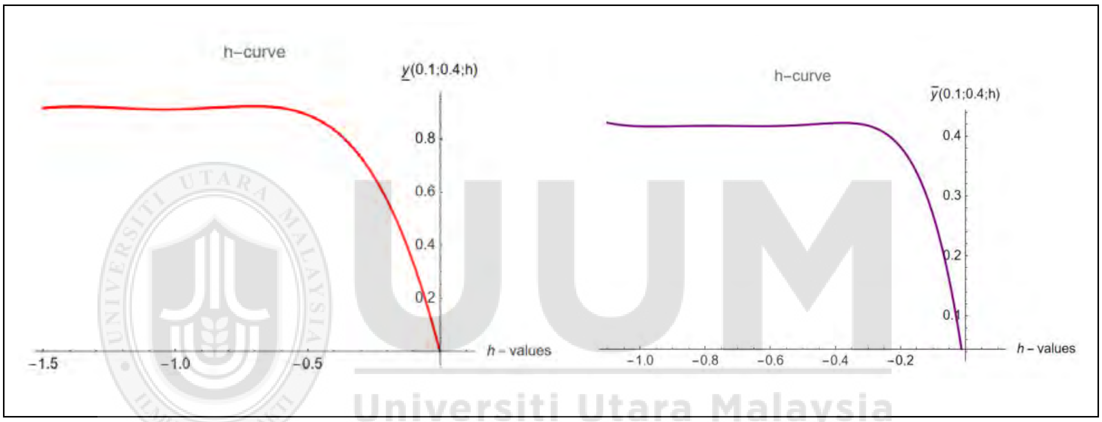

Figure 4.20. The $\tilde{h}(0.4)$-curves for the fuzzy solution of Eq. (4.82) given by sixth-order FF-HAM for $\beta = 0.5$ and $H(x) = 1$

Based on the curves in Figure 4.20, it is easy to discover the valid region of $\tilde{h}(0.4)$, which corresponds to the line segment nearly parallel to the horizontal axis. Based on Section 4.3.3 and Figure 4.20, it is clear that the lower bound of the FF-HAM series solution is convergent when $-1.4 \leq \underline{h} \leq -0.6$ while the upper bound is convergent when $-1 \leq \overline{h} \leq -0.3$.

Table 4.17

The optimal values of $\tilde{h}(0.4)$ by sixth-order FF-HAM for lower and upper solutions of Eq. (4.82) for $\beta = 0.5$ and $\alpha = 0.4$

$\underline{y}\left(x; 0.4; \underline{h}(0.4)\right)$	$\underline{h}_1 \to -0.753336757733535$	$\underline{h}_2 \to -0.7065223661736013$
$\overline{y}\left(x; 0.4; \overline{h}(0.4)\right)$	$\overline{h}_1 \to -0.784072088471834$	$\overline{h}_2 \to -0.6312524183906582$
	$\overline{h}_3 \to -0.3765152033989584$	

Then the best value of the convergent control parameter is $\tilde{h} = \left[\underline{h}_2, \overline{h}_2\right]$ as given in Table 4.17.

Tables 4.18-4.19 give the approximate solutions $\tilde{y}\left(0.1; 0.4; \tilde{h}\right)$, $\tilde{y}(0.1; 0.4; -1)$ obtained by sixth-order FF-HAM for Eq. (4.82) and the residual errors $\widetilde{ER}\left(0.1; 0.4; \tilde{h}\right)$, $\widetilde{ER}(0.1; 0.4; -1)$.

Table 4.18

The approximate lower solution and error of Eq. (4.82) by sixth-order FF-HAM when $\beta = 0.5$ at $x = 0.1$ for $h = -1$ and $h = \underline{h}_2$ $\forall \alpha \in [0,1]$

α	$[\underline{ER}]_\alpha, \underline{h} = -1$	$[\underline{ER}]_\alpha, \underline{h}_2$	$[\underline{y}]_\alpha, \tilde{h} = -1$	$[\underline{y}]_\alpha, \underline{h}_2$
0	0	0	-1	-1
0.2	0.00132	0.00062	-0.59429	-0.59487
0.4	-0.00235	0.00743	-0.29388	-0.29196
0.6	-0.00207	0.00405	-0.05190	-0.05055
0.8	0.00247	-0.00094	0.15283	0.15250
1	0.00362	-0.00252	0.33065	0.32977

Table 4.19

The approximate upper solution and error of Eq. (4.82) by sixth-order FF-HAM when $\beta = 0.5$ at $x = 0.1$ for $h = -1$ and $h = \overline{h}_2$ $\forall\alpha \in [0,1]$

α	$\overline{[ER]}_{\alpha}, \tilde{h} = -1$	$\overline{[ER]}_{\alpha}, \overline{h}_2$	$[\overline{y}]_{\alpha}, \tilde{h} = -1$	$[\overline{y}]_{\alpha}, \overline{h}_2$
0	0	0	1	1
0.2	0.00164	0.00004	0.88600	0.88576
0.4	−0.00080	0.00019	0.76348	0.76360
0.6	0.00330	−0.00016	0.63142	0.63182
0.8	−0.00034	−0.00144	0.48837	0.48817
1	0.00362	−0.00252	0.33065	0.32977

Tables 4.18-4.19 illustrate the accuracy of the approximate solution of Eq. (4.82) with fractional order $\beta = 0.5$ that is solved by sixth-order FF-HAM when $h = -1$ and $h = [\underline{h}_2,. \overline{h}_2]$.

We summarize the fuzzy approximate solutions of Eq. (4.82) for all $\alpha \in [0,1]$ and $x \in [0,0.1]$ when $\tilde{h} = -0.753336757733535$ in the three-dimensional graph in Figure 4.21.

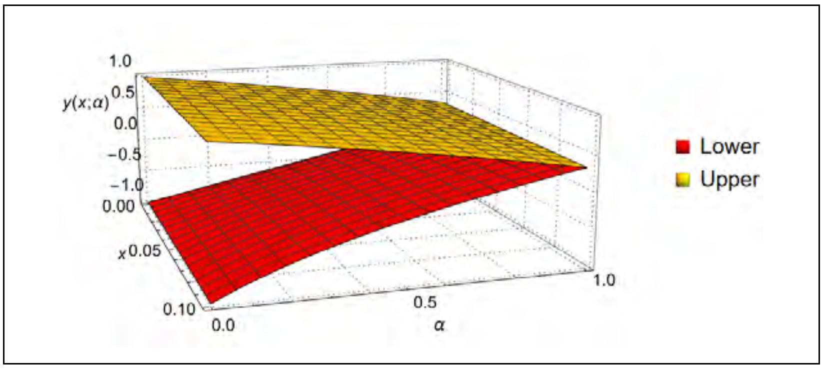

Figure 4.21. The three-dimensional approximate solution of Eq. (4.82) given by sixth-order FF-HAM over all $x \in [0,0.1]$ at $\beta = 0.5$, and for all $\alpha \in [0,1]$

To study the behaviour of different fractional order, we proceed to solve Eq. (4.82) by sixth-order FF-HAM at $\alpha = 0.6$, for $x \in [0,0.1]$ with a different fractional order, $\beta = 0.9$, to analyse the convergence dynamic of FF-HAM for solving first-order nonlinear FFOIVPs at a different fractional order.

Firstly, for this purpose, for a fixed value of $0 \leq \alpha \leq 1$ say $\alpha = 0.6$, we will plot $\tilde{h}(0.6)$ -curves of lower bound $\underline{y}(0.1; 0.6; \underline{h})$ and upper bound $\overline{y}(0.1; 0.6; \overline{h})$ of the sixth-order FF-HAM for Eq.(4.82) for $\beta = 0.9$, $\alpha = 0.6$ and at $x = 0.1$ as in Figure 4.22 to extract the valid region of the convergence control parameters.

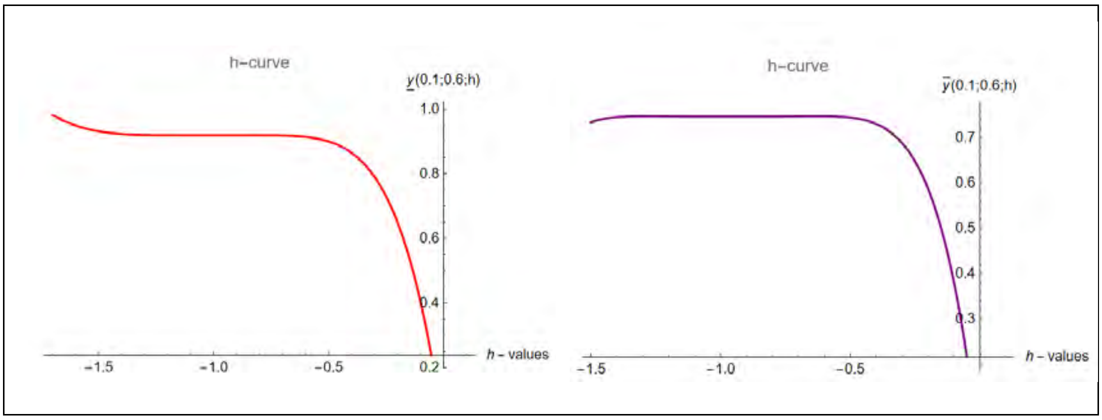

Figure 4.22. The $\tilde{h}(0.6)$-curves for the fuzzy solution of Eq. (4.82) given by sixth-order FF-HAM for $\beta = 0.9$ and $H(x) = 1$

From Figure 4.22, it can be concluded that the valid region of the convergence control parameters, which corresponds to the line segment nearly parallel to the horizontal axis, has changed to $-1.5 \leq \tilde{h} \leq -0.5$ after changing the fractional order of Eq.(4.82) to $\beta = 0.9$ where the best value of the convergent control-parameter is $\tilde{h} = [\underline{h}, \overline{h}] = [-1.0051037230460373, -0.994947838798828]$.

In Table 4.20, we have tabulated the residual errors $\widetilde{ER}\left(0.1; 0.6; \tilde{h}(0.6)\right)$, $\widetilde{ER}(0.1; 0.6; -1)$ obtained using sixth-order FF-HAM.

Table 4.20

Residual errors of Eq. (4.82) given by sixth-order FF-HAM approximate series solution with $\beta = 0.9$ for $x = 0.1$ and for all $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\underline{\boldsymbol{ER}}]_\alpha, \boldsymbol{h} = -\mathbf{1}$	$[\underline{\boldsymbol{ER}}]_\alpha, \underline{\boldsymbol{h}}$	$[\overline{\boldsymbol{ER}}]_\alpha, \boldsymbol{h} = -\mathbf{1}$	$[\overline{\boldsymbol{ER}}]_\alpha, \overline{\boldsymbol{h}}$
0	0	0	0	0
0.2	2.60092×10^{-7}	-1.59331×10^{-7}	1.83968×10^{-7}	-1.59331×10^{-7}
0.4	-1.26138×10^{-6}	6.96395×10^{-7}	-1.37400×10^{-6}	6.96395×10^{-7}
0.6	-1.36804×10^{-6}	2.23860×10^{-7}	-1.06341×10^{-6}	2.23860×10^{-7}
0.8	9.56578×10^{-7}	1.11164×10^{-7}	1.42354×10^{-6}	1.11164×10^{-7}
1	2.65180×10^{-6}	-3.26357×10^{-7}	2.65180×10^{-6}	-3.26357×10^{-7}

In addition, we can summarize the fuzzy approximate solutions of Eq. (4.82) for all $\alpha \in [0,1]$ [Definition 3.5] and $x \in [0,0.1]$ when $\tilde{h} = [-1.0051037230460373, -0.994947838798828]$ in the three-dimensional graph in Figure 4.23.

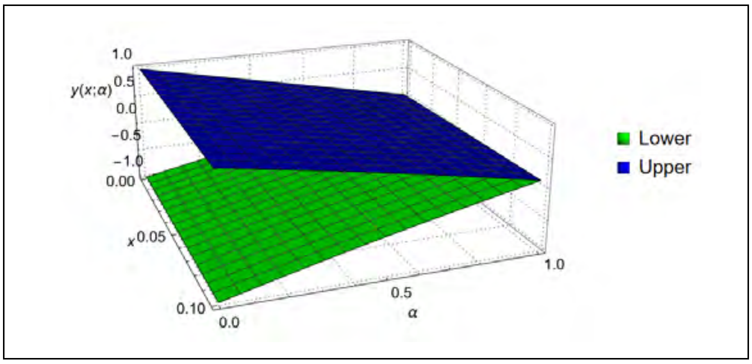

Figure 4.23. The three-dimensional approximate solution of Eq. (4.82) given by sixth-order FF-HAM over all $x \in [0,0.1]$ at $\beta = 0.9$ and for all $\alpha \in [0,1]$

Tables 4.18-4.20, Figure 4.21 and Figure 4.23 indicate that the sixth-order FF-HAM approximate solutions at $x = 0.1$ for all $\alpha \in [0,1]$ for fractional orders $\beta = 0.5$ and $\beta = 0.9$ satisfy the solution of the fuzzy numbers [Definition 3.7].

Figures 4.24-4.25 show the residual errors $\widetilde{ER}(x;h;\alpha)$ for $x \in [0,0.1]$ for the upper and lower solutions $\forall\alpha \in [0,1]$ for fractional orders $\beta = 0.5$ and $\beta = 0.9$ for Eq. (4.82).

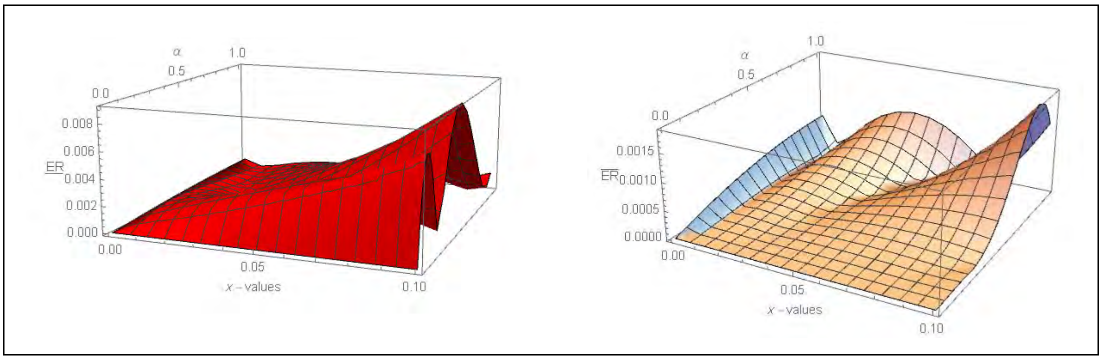

Figure 4.24. Residual errors of the sixth-order FF-HAM for solving Eq. (4.82) with order $\beta = 0.5$ for all $x \in [0,0.1]$ and for all $\alpha \in [0,1]$

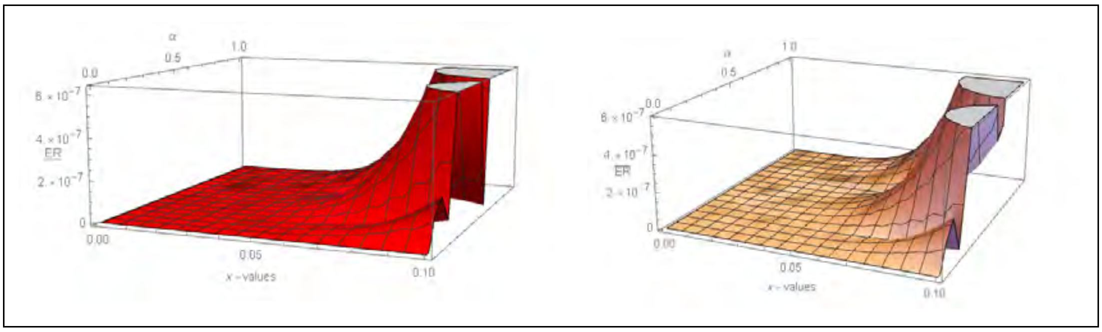

Figure 4.25. Residual errors of the sixth-order FF-HAM for solving Eq. (4.82) with order $\beta = 0.9$ for all $x \in [0,0.1]$ and for all $\alpha \in [0,1]$

From Tables 4.18-4.20 and Figures 4.24-4.25, it can be concluded that the sixth-order FF-HAM for solving Eq. (4.82) give better solution when the fractional order of Eq. (4.82) is large. Moreover, the $\tilde{h}$ curve convergent region and optimal values of $\tilde{h}$ also change according to the value of the fractional derivative order.

4.5.2.2 Solving Example 4.2 by FF-OHAM

In solving Example 4.2 by FF-OHAM, we construct the sixth-order FF-OHAM series solution with six convergence control parameters $\left(\tilde{S}_1(\alpha), \dots, \tilde{S}_6(\alpha)\right)$ for all $\alpha \in [0,1]$. According to Section 4.4.2, we can build the approximate series solution for Eq. (4.82) of order $\beta = 0.5$ for all $\alpha \in [0,1]$ as follows:

For lower and upper bound:

$$\begin{cases} (1-q)\left[\underline{D}^{(\beta)}\left(\underline{y}(x;\alpha)\right) - 1\right] = \\ \sum_{j=1}^{6} \underline{S}_j(\alpha) q^j \left[\underline{D}^{(\beta)}\left(\underline{y}(x;\alpha)\right) - 1 + \left(\underline{y}(x;\alpha)\right)^2\right], \\ (1-q)\left[\overline{D}^{(\beta)}(\overline{y}(x;\alpha)) - 1\right] = \\ \sum_{j=1}^{6} \overline{S}_j(\alpha) q^j \left[\overline{D}^{(\beta)}(\overline{y}(x;\alpha)) - 1 + \left(\overline{y}(x;\alpha)\right)^2\right]. \end{cases} \tag{4.91}$$

Then, we will list the zeroth-order problem and the first six-order problems as follows:

Zeroth-order problem:

$$\begin{cases} \tilde{y}_0(x;\alpha) = \tilde{J}^{(\beta)}[1], \\ \tilde{y}_0(0;\alpha) = [\alpha - 1, 1 - \alpha]. \end{cases} \tag{4.92}$$

First-order problem:

$$\begin{cases} \tilde{y}_1\left(x, \tilde{S}_1(\alpha); \alpha\right) = \left(1 + \tilde{S}_1(\alpha)\right)\tilde{y}_0(x;\alpha) + \tilde{J}^{(\beta)}\left[\tilde{S}_1(\alpha)\left(\tilde{y}_0{}^2(x;\alpha) - 1\right) - 1\right], \\ \tilde{y}_1(0;\alpha) = 0. \end{cases}$$

(4.93)

Second-order problem:

$$\begin{cases} \tilde{y}_2\left(x, \tilde{S}_1(\alpha), \tilde{S}_2(\alpha); \alpha\right) = \left(1 + \tilde{S}_1(\alpha)\right)\left(\tilde{y}_1\left(x, \tilde{S}_1(\alpha); \alpha\right)\right) + \tilde{S}_2(\alpha)\tilde{y}_0(x;\alpha) \\ +\tilde{J}^{(\beta)}\left[2\tilde{S}_1(\alpha)\left(\tilde{y}_0(x;\alpha)\tilde{y}_1\left(x, \tilde{S}_1(\alpha); \alpha\right)\right) + \tilde{S}_2(\alpha)\left(\tilde{y}_0{}^2(x;\alpha) - 1\right)\right], \\ \tilde{y}_1(0;\alpha) = 0. \end{cases} \tag{4.94}$$

Third-order problem:

$$\begin{cases} \tilde{y}_3\left(x, \tilde{S}_1(\alpha), \tilde{S}_2(\alpha), \tilde{S}_3(\alpha); \alpha\right) = \left(1 + \tilde{S}_1(\alpha)\right)\tilde{y}_2\left(x, \tilde{S}_1(\alpha), \tilde{S}_2(\alpha); \alpha\right) \\ +\tilde{S}_3(\alpha)\tilde{y}_0(x;\alpha) + \tilde{S}_2(\alpha)\tilde{y}_1\left(x, \tilde{S}_1(\alpha); \alpha\right) + \tilde{S}_3(\alpha)\tilde{J}^{(\beta)}\left(\left(\tilde{y}_0(x;\alpha)\right)^2 - 1\right) \\ +2\tilde{S}_2(\alpha)\tilde{J}^{(\beta)}\left(\tilde{y}_0(x;\alpha)\tilde{y}_1\left(x, \tilde{S}_1(\alpha); \alpha\right)\right) + \tilde{S}_1(\alpha)\tilde{J}^{(\beta)} \\ \left(2\tilde{y}_0(x;\alpha)\tilde{y}_2\left(x, \tilde{S}_1(\alpha), \tilde{S}_2(\alpha); \alpha\right) + \left(\tilde{y}_1\left(x, \tilde{S}_1(\alpha); \alpha\right)\right)^2\right), \\ \tilde{y}_3(0;\alpha) = 0. \end{cases} \tag{4.95}$$

Fourth-order problem:

$$\begin{cases} \tilde{y}_4\left(x,\tilde{S}_1(\alpha),\ldots,\tilde{S}_4(\alpha);\alpha\right)=\left(1+\tilde{S}_1(\alpha)\right)\tilde{y}_3\left(x,\tilde{S}_1(\alpha),.,\tilde{S}_3(\alpha);\alpha\right)+\tilde{S}_4(\alpha) \\ \tilde{y}_0(x;\alpha)+\tilde{S}_2(\alpha)\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)+\tilde{S}_3(\alpha)\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)+\tilde{S}_4(\alpha) \\ \tilde{J}^{(\beta)}\left(\left(\tilde{y}_0(x;\alpha)\right)^2-1\right)+\tilde{S}_2(\alpha)\tilde{J}^{(\beta)}\left(2\tilde{y}_0(x;\alpha)\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)\right) \\ 2\tilde{S}_1(\alpha)\tilde{J}^{(\beta)}\left(\tilde{y}_0(x;\alpha)\tilde{y}_3\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha),\tilde{S}_3(\alpha);\alpha\right)\right)+\ 2\tilde{S}_1(\alpha)\tilde{J}^{(\beta)} \\ \left(\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)\right)+\tilde{S}_2(\alpha)\tilde{J}^{(\beta)}\left(\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)\right)^2 \\ +2\tilde{S}_3(\alpha)\tilde{J}^{(\beta)}\left(\tilde{y}_0(x;\alpha)\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)\right) \\ , \\ \tilde{y}_4(0;\alpha)=0. \end{cases} \tag{4.96}$$

Fifth-order problem:

$$\begin{cases} \tilde{y}_5\left(x,\tilde{S}_1(\alpha),\ldots,\tilde{S}_5(\alpha);\alpha\right)=\left(1+\tilde{S}_1(\alpha)\right)\tilde{y}_4\left(x,\tilde{S}_1(\alpha),\ldots,\tilde{S}_4(\alpha);\alpha\right)+\tilde{S}_5(\alpha) \\ \tilde{y}_0(x;\alpha)+\tilde{S}_4(\alpha)\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)+\tilde{S}_3(\alpha)\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)+\tilde{S}_2(\alpha) \\ \tilde{y}_3\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha),\tilde{S}_3(\alpha);\alpha\right)+\tilde{S}_5(\alpha)\tilde{J}^{(\beta)}\left(\left(\tilde{y}_0(x)\right)^2-1\right)+\tilde{S}_1(\alpha)\tilde{J}^{(\beta)} \\ \left(\left(\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)\right)^2+2\tilde{y}_0(x;\alpha)\tilde{y}_4\left(x,\tilde{S}_1(\alpha),\ldots,\tilde{S}_4(\alpha);\alpha\right)\right)+ \\ \tilde{S}_1(\alpha)\tilde{J}^{(\beta)}\left(2\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)\tilde{y}_3\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha),\tilde{S}_3(\alpha);\alpha\right)\right)+2\tilde{S}_2(\alpha)\tilde{J}^{(\beta)} \\ \left(\tilde{y}_0(x;\alpha)\tilde{y}_3\left(x,\tilde{S}_1(\alpha),.,\tilde{S}_3(\alpha);\alpha\right)+\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)\right) \\ +\tilde{S}_3(\alpha)\tilde{J}^{(\beta)}\left(\left(\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)\right)^2+2\tilde{y}_0(x;\alpha)\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)\right) \\ +2\tilde{S}_4(\alpha)\tilde{J}^{(\beta)}\left(\tilde{y}_0(x;\alpha)\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)\right), \\ \tilde{y}_5(0;\alpha)=0. \end{cases}$$

(4.97)

Sixth-order problem:

$$\begin{cases} \tilde{y}_6\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_6(\alpha);\alpha\big) = \big(1+\tilde{S}_1(\alpha)\big)\tilde{y}_5\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_5(\alpha);\alpha\big) + \tilde{S}_6(\alpha) \\ \tilde{y}_0(x;\alpha) + \tilde{S}_5(\alpha)\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big) + \tilde{S}_4(\alpha)\tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big) + \tilde{S}_3(\alpha) \\ \tilde{y}_3\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_3(\alpha);\alpha\big) + \tilde{S}_2(\alpha)\tilde{y}_4\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_4(\alpha);\alpha\big) + \tilde{S}_6(\alpha)\tilde{\mathcal{J}}^{(\beta)} \\ \Big(\big(\tilde{y}_0(x;\alpha)\big)^2 - 1\Big) + 2\tilde{S}_1(\alpha)\tilde{\mathcal{J}}^{(\beta)}\Big(\tilde{y}_0(x;\alpha)\tilde{y}_5\big(x,\tilde{S}_1(\alpha),\ldots,\tilde{S}_5(\alpha);\alpha\big)\Big) \\ +2\tilde{S}_1(\alpha)\tilde{\mathcal{J}}^{(\beta)}\begin{pmatrix}\tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big)\tilde{y}_3\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha),\tilde{S}_3(\alpha);\alpha\big) + \\ \tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big)\tilde{y}_4\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_4(\alpha);\alpha\big)\end{pmatrix} \\ +2\tilde{S}_2(\alpha)\tilde{\mathcal{J}}^{(\beta)}\Big(\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big)\tilde{y}_3\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_3(\alpha);\alpha\big)\Big) + \tilde{S}_2(\alpha)\tilde{\mathcal{J}}^{(\beta)} \\ \Big(2\tilde{y}_0(x;\alpha)\tilde{y}_4\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_4(\alpha);\alpha\big) + \Big(\tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big)\Big)^2\Big) + \\ 2\tilde{S}_3(\alpha)\tilde{\mathcal{J}}^{(\beta)}\tilde{y}_0(x;\alpha)\tilde{y}_3\big(x,\tilde{S}_1(\alpha),..,\tilde{S}_3(\alpha);\alpha\big) + 2\tilde{S}_3(\alpha)\tilde{\mathcal{J}}^{(\beta)} \\ \Big(+\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big)\tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big)\Big) + \tilde{S}_4(\alpha)\tilde{\mathcal{J}}^{(\beta)} \\ \Big(\Big(\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big)\Big)^2 + 2\tilde{y}_0(x;\alpha)\tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big)\Big) \\ +2\tilde{S}_5(\alpha)\tilde{\mathcal{J}}^{(\beta)}\Big(\tilde{y}_0(x;\alpha)\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big)\Big), \\ \qquad \tilde{y}_6(0;\alpha) = 0. \end{cases} \tag{4.98}$$

Using Mathematica 12 Package to find the approximate solutions for the lower and upper bounds of Eqs. (4.92)-(4.98) as follows:

$$\tilde{y}(x;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{j=1}^{6}\tilde{y}_j\big(x,\tilde{S}_1(\alpha),\ldots,\tilde{S}_j(\alpha);\alpha\big) \tag{4.99}$$

In the next step, we will set the residual of Eq. (4.82) using the sixth-order FF-OHAM to extract the optimal convergence control parameters as follows:

$$\widetilde{ER} = D^{(\beta)}\tilde{y}(x;\alpha) + \tilde{y}^2(x;\alpha) - 1. \tag{4.100}$$

According to Section 4.4.3, we can extract the optimal convergence control parameters for each $\alpha \in [0,1]$ for lower and upper bound, whose values are tabulated in Tables 4.21-4.22 .

Table 4.21

Lower auxiliary convergence parameters of sixth-order FF-OHAM for solving Eq. (4.82) at $\beta = 0.5$, $x = 0.1$ for all $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$\underline{\boldsymbol{S}}_{\boldsymbol{j}}$		
0	$\underline{S}_1 = -1$	$\underline{S}_2 = 0$	$\underline{S}_3 = 0$
	$\underline{S}_4 = 0$	$\underline{S}_5 = 0$	$\underline{S}_6 = 0$
0.2	$\underline{S}_1 = -1.1080543674241816$	$\underline{S}_2 = 0.025082505544452394$	$\underline{S}_3 = 0.007784390047795941$
	$\underline{S}_4 = 0.001832774758137834$	$\underline{S}_5 = -0.00014058900488874261$	$\underline{S}_6 = -0.00015978709216646174$
0.4	$\underline{S}_1 = -0.8897307268258035$	$\underline{S}_2 = 0.011973364540677519$	$\underline{S}_3 = -0.02079307152658097$
	$\underline{S}_4 = 0.024514513954526477$	$\underline{S}_5 = -0.00664798550909057$	$\underline{S}_6 = -0.0002275368305446968$
0.6	$\underline{S}_1 = -1.0211193966556242$	$\underline{S}_2 = 0.0512538667316031$	$\underline{S}_3 = 0.007689410793817843$
	$\underline{S}_4 = -0.009401243629295222$	$\underline{S}_5 = -0.0019415578005048326$	$\underline{S}_6 = 0.0007320205597871645$
0.8	$\underline{S}_1 = -0.9783536856816559$	$\underline{S}_2 = 0.05540834610388592$	$\underline{S}_3 = -0.00893758766378853$
	$\underline{S}_4 = 0.008344214394395815$	$\underline{S}_5 = 0.006821612920501401$	$\underline{S}_6 = -0.0014260034961302156$
1	$\underline{S}_1 = -1$	$\underline{S}_2 = 0.04402328220089973$	$\underline{S}_3 = -0.0070834247125287665$
	$\underline{S}_4 = -0.0002553367856894602$	$\underline{S}_5 = 0.004149025286753004$	$\underline{S}_6 = -0.000041145750901672 09$

Table 4.22

Upper auxiliary convergence parameters of sixth-order FF-OHAM for solving Eq. (4.82) at $\beta = 0.5$, $x = 0.1$ for all $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$\overline{\boldsymbol{S}}_{\boldsymbol{j}}$		
0	$\overline{S}_1 = -1$	$\overline{S}_2 = 0$	$\overline{S}_3 = 0$
	$\overline{S}_4 = 0$	$\overline{S}_5 = 0$	$\overline{S}_6 = 0$
0.2	$\overline{S}_1 = -0.9166788915163324$	$\overline{S}_2 = 0.018471620611987847$	$\overline{S}_3 = -0.015993950404120678$
	$\overline{S}_4 = 0.0026880408663043235$	$\overline{S}_5 = 0.0010060316441829183$	$\overline{S}_6 = -0.0003933008402067269 7$
0.4	$\overline{S}_1 = -0.9292216911472212$	$\overline{S}_2 = 0.03683670457168856$	$\overline{S}_3 = 0.00705439298393826$
	$\overline{S}_4 = -0.0030670795520950523$	$\overline{S}_5 = -0.00061644891102732 04$	$\overline{S}_6 = 0.0000951583706442795$
0.6	$\overline{S}_1 = -0.9733709487888146$	$\overline{S}_2 = 0.04565794859202862$	$\overline{S}_3 = 0.00875213751857428$
	$\overline{S}_4 = 0.03703962133833942$	$\overline{S}_5 = -0.007221675189437583$	$\overline{S}_6 = -0.0033630089348866726$
0.8	$\overline{S}_1 = -0.9723668781511248$	$\overline{S}_2 = 0.04291221013317356$	$\overline{S}_3 = -0.028756609593720887$
	$\overline{S}_4 = -0.011787263058382479$	$\overline{S}_5 = 0.006676129119448582$	$\overline{S}_6 = -0.000036183702189402 34$
1	$\overline{S}_1 = -1$	$\overline{S}_2 = 0.04402328220089973$	$\overline{S}_3 = -0.0070834247125287665$
	$\overline{S}_4 = -0.0002553367856894602$	$\overline{S}_5 = 0.004149025286753004$	$\overline{S}_6 = -4.114575090167209 \times 10^{-5}$

Tables 4.21 and 4.22 illustrate the lower and upper convergence control parameters $\forall \alpha \in [0,1]$ respectively. Using these parameters, we will find the approximate series solution for Eq. (4.82) by sixth-order FF-OHAM.

Table 4.23

The approximate solution and error of Eq. (4.82) by sixth-order FF-OHAM when $\beta = 0.5$ *at* $x = 0.1$ *for all* $\alpha \in [0,1]$

α	$[\underline{ER}]_{\alpha}, \tilde{S}_j(\alpha)$	$[\overline{ER}]_{\alpha}, \tilde{S}_j(\alpha)$	$[\underline{y}]_{\alpha}, \tilde{S}_j(\alpha)$	$[\overline{y}]_{\alpha}, \tilde{S}_j(\alpha)$
0	0	0	-1	1
0.2	3.64200×10^{-8}	3.24142×10^{-5}	-0.59456	0.88576
0.4	-4.96679×10^{-5}	-3.99757×10^{-6}	-0.29346	0.76359
0.6	-3.56794×10^{-5}	-1.70482×10^{-4}	-0.05155	0.63187
0.8	1.95245×10^{-5}	-9.49933×10^{-6}	0.15245	0.48841
1	6.44286×10^{-5}	6.44286×10^{-5}	0.33011	0.33011

Table 4.23 illustrates the approximate solution of Eq. (4.82) and it can be concluded that the series solutions for Eq. (4.82) by sixth-order FF-OHAM satisfy the triangular solution [Definition 3.7] of the fuzzy differential equations.

Figure 4.26 summarizes the fuzzy approximate solutions of Eq. (4.82) for all $\alpha \in [0,1]$ and $x \in [0,0.1]$ based on the optimal convergence parameters $\tilde{S}_j(\alpha)$ in a three-dimensional graph.

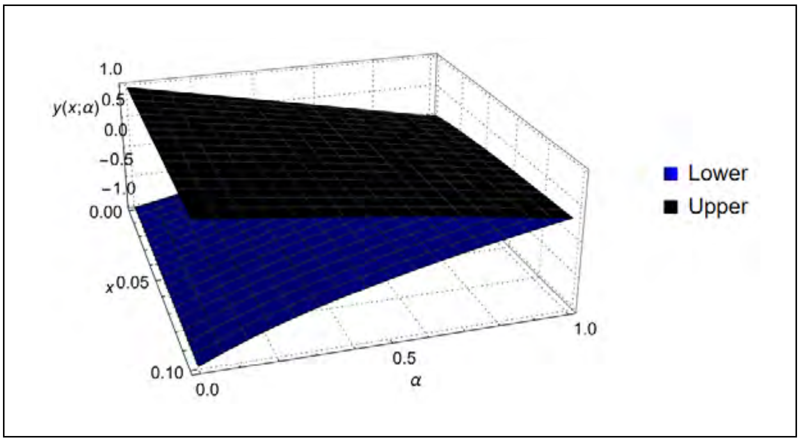

Figure 4.26. The three-dimensional approximate solution of Eq. (4.82) given by sixth-order FF-OHAM over all $x \in [0,0.1]$ at $\beta = 0.5$ and for all $\alpha \in [0,1]$

Next, we will study the behaviour of fractional differentiability. We proceed to solve Eq. (4.82) by sixth-order FF-OHAM for $x \in [0,0.1]$ with a different fractional order, $\beta = 0.9$, to analyse the convergence dynamic of FF-OHAM for solving FFOIVPs at a different fractional order. Thus, the convergence control parameters of sixth-order FF-OHAM for solving Eq. (4.82) at $\beta = 0.9$ will change based on the minimized residual of Eq. (4.82), and the new values are given in Tables 4.24 and 4.25 for lower and upper bound, respectively.

Table 4.24

Lower auxiliary convergence parameters of sixth-order FF-OHAM for solving Eq. (4.82) at $\beta = 0.9$, $x = 0.1$ for all $\alpha \in [0,1]$

α	$\underline{S}_j$		
0	$\underline{S}_1 = -1$ $\underline{S}_4 = 0$	$\underline{S}_2 = 0$ $\underline{S}_5 = 0$	$\underline{S}_3 = 0$ $\underline{S}_6 = 0$
0.2	$\underline{S}_1 = -1.0142950669631587$ $\underline{S}_4 = 3.4122247817517514 \times 10^{-5}$	$\underline{S}_2 = 3.8975299093286465 \times 10^{-3}$ $\underline{S}_5 = -6.171098655268605 \times 10^{-8}$	$\underline{S}_3 = -8.798952175009586 \times 10^{-5}$ $\underline{S}_6 = -3.388024197002148 \times 10^{-7}$
0.4	$\underline{S}_1 = -1.0062110166428124$ $\underline{S}_4 = 6.157923506101239 \times 10^{-5}$	$\underline{S}_2 = 5.023591860436904 \times 10^{-3}$ $\underline{S}_5 = 3.088023288896656 \times 10^{-6}$	$\underline{S}_3 = -1.3190940342858295 \times 10^{-4}$ $\underline{S}_6 = -7.489539083350368 \times 10^{-7}$
0.6	$\underline{S}_1 = -1.0132891322087128$ $\underline{S}_4 = -8.837852907496967 \times 10^{-6}$	$\underline{S}_2 = 4.549356145110974 \times 10^{-3}$ $\underline{S}_5 = 7.967157631214277 \times 10^{-8}$	$\underline{S}_3 = 1.3195758284917703 \times 10^{-4}$ $\underline{S}_6 = 2.079145674496714 \times 10^{-9}$
0.8	$\underline{S}_1 = -1.000957868873054$ $\underline{S}_4 = -2.1944114555294428 \times 10^{-5}$	$\underline{S}_2 = 4.620748229362928 \times 10^{-3}$ $\underline{S}_5 = -2.052584374200022 \times 10^{-6}$	$\underline{S}_3 = 3.8258846015557424 \times 10^{-4}$ $\underline{S}_6 = -4.158061332268335 \times 10^{-8}$
1	$\underline{S}_1 = -1$ $\underline{S}_4 = 2.751163325532934 \times 10^{-6}$	$\underline{S}_2 = 3.4602185795033903 \times 10^{-3}$ $\underline{S}_5 = -2.2769108839977038 \times 10^{-5}$	$\underline{S}_3 = -3.881766446537558 \times 10^{-5}$ $\underline{S}_6 = -2.068300574763506 \times 10^{-9}$

Table 4.25

Upper auxiliary convergence parameters of sixth-order FF-OHAM for solving Eq. (4.82) at $\beta = 0.9$, $x = 0.1$ *for all* $\alpha \in [0,1]$

α	$\overline{S}_j$		
0	$\overline{S}_1 = -1$ $\overline{S}_4 = 0$	$\overline{S}_2 = 0$ $\overline{S}_5 = 0$	$\overline{S}_3 = 0$ $\overline{S}_6 = 0$
0.2	$\overline{S}_1 = -0.9936035232782928$ $\overline{S}_4 = 3.9746883513984584 \times 10^{-5}$	$\overline{S}_2 = 1.5029223368844518 \times 10^{-3}$ $\overline{S}_5 = 2.128848445506741 \times 10^{-6}$	$\overline{S}_3 = 1.1823139172096339 \times 10^{-4}$ $\overline{S}_6 = -3.846095016997509 \times 10^{-8}$
0.4	$\overline{S}_1 = -0.9834930381596945$ $\overline{S}_4 = 3.318853196923199 \times 10^{-5}$	$\overline{S}_2 = 3.8494535367303226 \times 10^{-3}$ $\overline{S}_5 = -2.14149562078394 \times 10^{-6}$	$\overline{S}_3 = 1.1277321426184392 \times 10^{-5}$ $\overline{S}_6 = -2.903679860067431 \times 10^{-7}$
0.6	$\overline{S}_1 = -0.9941009358797868$ $\overline{S}_4 = 9.197540252314956 \times 10^{-5}$	$\overline{S}_2 = 2.690998916726547 \times 10^{-3}$ $\overline{S}_5 = -8.021188602606343 \times 10^{-6}$	$\overline{S}_3 = 5.9572783804273965 \times 10^{-5}$ $\overline{S}_6 = -4.710448682315004 \times 10^{-7}$
0.8	$\overline{S}_1 = -0.9887993739232486$ $\overline{S}_4 = 3.9477057862002565 \times 10^{-5}$	$\overline{S}_2 = 2.604083910457939 \times 10^{-3}$ $\overline{S}_5 = -1.0925943890209795 \times 10^{-5}$	$\overline{S}_3 = 3.728738794076935 \times 10^{-4}$ $\overline{S}_6 = -1.927790490161429 \times 10^{-7}$
1	$\overline{S}_1 = -1$ $\overline{S}_4 = 2.751163655723681 \times 10^{-6}$	$\overline{S}_2 = 3.4602185698092212 \times 10^{-3}$ $\overline{S}_5 = -2.276958381603294 \times 10^{-5}$	$\overline{S}_3 = -3.881758772587082 \times 10^{-5}$ $\overline{S}_6 = -2.068295527428735 \times 10^{-9}$

The convergence control parameters would change according to the fractional order of Eq. (4.82). Based on the new convergence control parameters, we will find the accuracy of Eq. (4.82) by sixth-order FF-OHAM where the values are shown in Table 4.26.

Table 4.26

The approximate solution and error of Eq. (4.82) given by sixth-order FF-OHAM when $\beta = 0.9$ *at* $x = 0.1$ *for all* $\alpha \in [0,1]$

α	$[\underline{ER}]_\alpha, \tilde{S}_j(\alpha)$	$[\overline{ER}]_\alpha, \tilde{S}_j(\alpha)$	$[\underline{y}]_\alpha, \tilde{S}_j(\alpha)$	$[\overline{y}]_\alpha, \tilde{S}_j(\alpha)$
0	0	0	-1	1
0.2	8.71717×10^{-8}	3.51748×10^{-8}	-0.74710	0.84200
0.4	-1.00149×10^{-8}	-7.68927×10^{-9}	-0.50893	0.67664
0.6	-2.00024×10^{-10}	-8.83504×10^{-8}	-0.28411	0.50331
0.8	1.12408×10^{-8}	7.39793×10^{-10}	-0.07148	0.32136
1	3.47663×10^{-8}	3.47663×10^{-8}	0.13004	0.13004

In addition, we can summarize the fuzzy approximate solutions of Eq.(4.82) for all $\alpha \in [0,1]$ and $x \in [0,0.1]$ based on the optimal convergence parameters $\tilde{S}_j(\alpha)$ as illustrated in the three-dimensional graph in Figure 4.27.

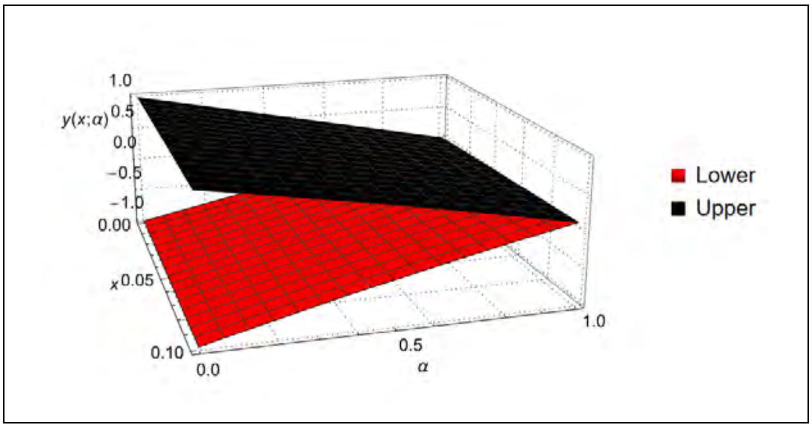

Figure 4.27. The three-dimensional approximate solution of Eq. (4.82) given by sixth-order FF-OHAM over all $x \in [0,0.1]$ at $\beta = 0.9$ and for all $\alpha \in [0,1]$

Table 4.26 and Figure 4.27 illustrate that the sixth-order FF-OHAM approximate solution satisfies the triangular solution [Definition 3.7] of the fuzzy differential equations (Salahshour, 2011). It can be concluded that the sixth-order FF-OHAM approximate solution of Eq. (4.82) will approach the exact solution as the fractional order of Eq. (4.82) increases. Figures 4.28-4.29 illustrate the accuracy of sixth-order FF-OHAM for solving Eq. (4.82) for $\beta = 0.5$ and $\beta = 0.9$ $\forall x \in [0,0.1]$ and for $\alpha = 0.6$.

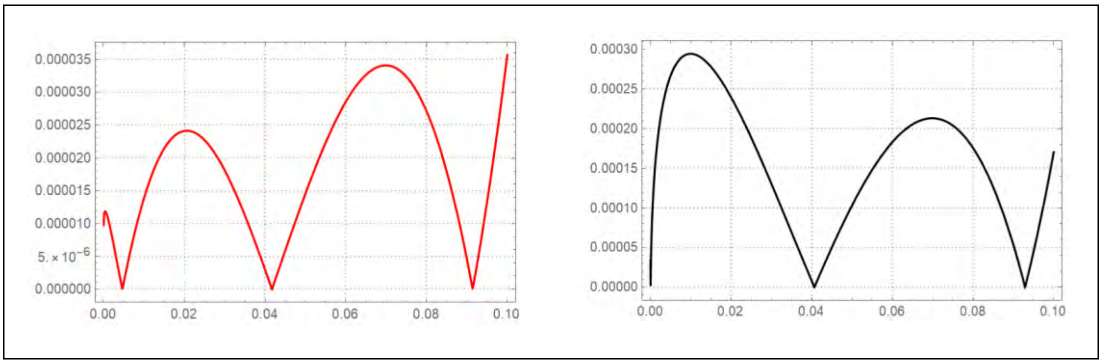

Figure 4.28. Residual errors of Eq. (4.82) by sixth-order FF-OHAM for $\beta = 0.5$ at $\alpha = 0.6$ for all $x \in [0,0.1]$

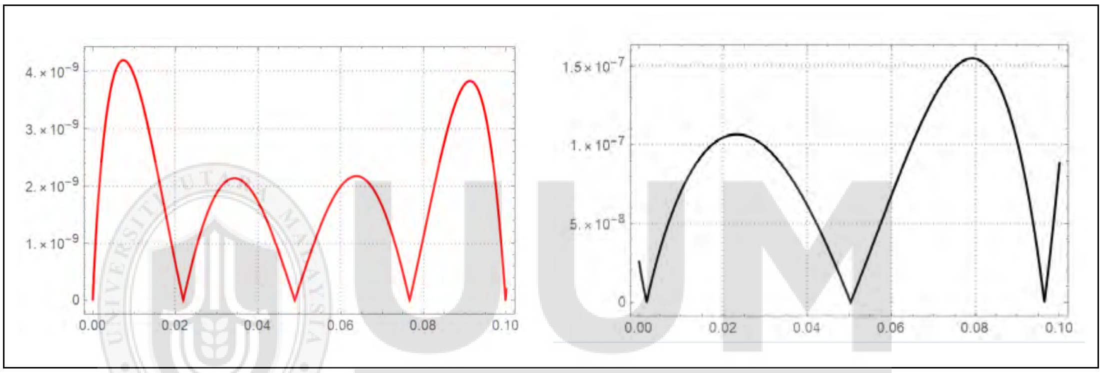

Figure 4.29. Residual errors of Eq. (4.82) by sixth-order FF-OHAM for $\beta = 0.9$ at $\alpha = 0.6$ for all $x \in [0,0.1]$

4.6 Summary of Findings

1. In this chapter, the approximate-analytical methods FF-HAM and FF-OHAM with convergence-control ability have been successfully developed for linear and nonlinear FFOIVPs of order $0 < \beta \leq 1$.

2. The convergence dynamic of each of the developed methods has been introduced to enable selection of the optimal convergence parameter for the purpose of controlling the series solution.

3. The developed FF-HAM and FF-OHAM have been utilized to solve inhomogeneous linear FFOIVPs. The findings are as follows:

   a) FF-HAM and FF-OHAM provide accurate series solution for solving the linear case.

   b) Employing the convergence dynamic of FF-HAM provide us with the optimal convergence parameter h for FF-HAM and we conclude that the optimal convergence parameter provides more accurate solution compared with $h = -1$, which refers to fuzzy fractional HPM.

   c) Employing the convergence dynamic of FF-OHAM provide us with the optimal convergence parameters at any α-cut; will be valid for all a-cut. (Special case for some linear problems)

   d) As the FF-HAM solution series order increases, the accuracy of the solution by FF-HAM also increases and converges to the exact solution. The same is true for FF-OHAM.

   e) Generally, FF-HAM is more accurate and less costly in terms of CPU for solving linear cases.

   f) Based on the absolute error, FF-HAM and FF-OHAM yield more accurate solution compared with FRPSM.

g) All fuzzy fractional solution via FF-HAM and FF-OHAM satisfy the triangular fuzzy solution.

h) Figures and tables provide illustrations of results including optimal parameters, valid region of the convergence parameters, errors and solutions.

4. We introduce a new fuzzy version of fractional Riccati differential equation, followed by the utilization of FF-HAM and FF-OHAM to find the series solution of this nonlinear problem. The findings are as follows:

a) FF-HAM and FF-OHAM provide accurate series solutions for solving non-linear problem.

b) Employing the convergence dynamic of FF-HAM provide us with the optimal convergence parameter h for FF-HAM and we conclude that the optimal convergence parameter provides more accurate solution compared with $h = -1$, which refers to fuzzy Fractional HPM.

c) Unlike the linear case, employing our convergence dynamic of FF-OHAM provide us convergence control parameters at each α-level set separately.

d) For the linear case, we study the convergence dynamic based on the change of the series order while for nonlinear case we study the convergence dynamic based on the change of the fractional order derivative.

e) As the fractional order derivative increases, the accuracy of solution by FF-HAM and FF-OHAM also increases, and the obtained solution converges to the exact solution.

f) Generally, FF-OHAM yields more accurate solution compared with FF-HAM for nonlinear case, but FF-OHAM will be more costly in terms of CPU.

g) All fuzzy fractional solutions by FF-HAM and FF-OHAM satisfy the triangular fuzzy solution for nonlinear case.

h) Figures and tables provide illustrations of results including optimal parameters, valid region of the convergence parameters, errors and solutions.

CHAPTER FIVE

FF-HAM AND FF-OHAM FOR SECOND-ORDER FUZZY FRACTIONAL ORDINARY INITIAL VALUE PROBLEMS

5.1 Introduction

The second type of fuzzy fractional problem being studied is the second-order FFOIVPs. This chapter discusses the development of the proposed FF-HAM and FF-OHAM on this type of problem. Firstly, elaboration on the theoretical part of the development of the method is given. Then, the experimental part of the development that demonstrates the solving process of several second-order fuzzy fractional ordinary initial value problems using the developed methods is presented. Based on the theoretical and experimental parts of the study, discussion of the developed methods in terms of characteristics and advantages are provided towards the end of the chapter.

5.2 Theoretical Development of FF-HAM and FF-OHAM for Second-order FFOIVPs

The theoretical development consists of three steps as follows:

1. Defuzzification of FFODEs

In this step, the FFODEs are defuzzified under the sense of the second-order fuzzy fractional Caputo derivative for lower and upper bound, based on the extension principle theory and properties of the fuzzy calculus such as α-cut definition and fuzzy number definition.

2. Construction of FF-HAM and FF-OHAM for second-order FFOIVPs

In this step, FF-HAM and FF-OHAM are constructed based on the extension principle theory and properties of the fuzzy calculus such as α-cut definition and fuzzy number definition to handle the second-order FFOIVPs.

3. Establishment of convergence of FF-HAM and FF-OHAM solution series

In this step, establishment of convergence analyses of FF-HAM and FF-OHAM solution series for second-order FFOIVPs is made. The analysis depends on the minimization of the residual form of the given problem in order to select the optimal convergence control parameter of FF-HAM (h) and the optimal convergence control parameters of FF-OHAM ($S_1, S_2, S_3, S_4, \dots S_k$) (here k is the order of the FF-OHAM series).

The theoretical development of each of the two proposed methods follows the three steps. However, Step One needs to be done only once since it is applicable for both methods. In this study, the construction of FF-HAM will be done first, followed by FF-OHAM.

5.3 Theoretical Development of FF-HAM for Second-order FFOIVPs

For second-order FFOIVPs, the proposed FF-HAM is theoretically developed in three steps.

5.3.1 Defuzzification of FFODE

This section discusses the defuzzification of FFIVP involving ODE of second-order as follows:

Consider the second-order $1 < \beta \le 2$ FFOIVP:

$$\begin{cases} \tilde{y}^{(\beta)}(x) = \tilde{g}\left(x, \tilde{y}(x), \tilde{y}^{(1)}(x)\right), \;\; x \in [x_0, X], \\ \tilde{y}^{(i)}(x_0) = \tilde{y}_0^{(i)}, i = 0,1, \end{cases} \tag{5.1}$$

where $\tilde{y}^{(k)}(x_0)$ represents the fuzzy numbers [Definition 3.6], $y^{(\beta)}$ represents the fuzzy fractional differential operator in Caputo sense [Definition 3.22], and $y_0^{(\beta)}$ are the fuzzy initial conditions at $x = x_0$.

Now, for $x \in [x_0, X]$, and $\forall\, \alpha \in [0,1]$ [Definition 3.5] the fuzzy function $\tilde{y}$ [Definition 3.8] can be observed by $[\tilde{y}]_\alpha = [\underline{y}, \overline{y}]_\alpha$, such that

$$\begin{cases} [\tilde{y}(x)]_\alpha = \left[\underline{y}(x;\alpha), \overline{y}(x;\alpha)\right], \\ [\tilde{y}'(x)]_\alpha = \left[\underline{y}'(x;\alpha), \overline{y}'(x;\alpha)\right], \end{cases} \tag{5.2}$$

and

$$\begin{cases} [\tilde{y}(x_0)]_\alpha = \left[\underline{y}(x_0;\alpha), \overline{y}(x_0;\alpha)\right], \\ [\tilde{y}'(x_0)]_\alpha = \left[\underline{y}'(x_0;\alpha), \overline{y}'(x_0;\alpha)\right], \end{cases} \tag{5.3}$$

Now, assuming $\tilde{\Upsilon}(x) = \{\tilde{y}(x), \tilde{y}'(x)\}$ and for the defuzzification [Definition 3.4] we have

$$\tilde{\Upsilon}(x;\alpha) = \left[\underline{\Upsilon}(x;\alpha), \overline{\Upsilon}(x;\alpha)\right]$$

$$= \left[\underline{y}(x;\alpha), \underline{y}'(x;\alpha), \overline{y}(x;\alpha), \overline{y}'(x;\alpha)\right].$$

Also, we can write

$$\left[\tilde{g}\left(x, \tilde{Y}\right)\right]_\alpha = \left[\underline{g}\left(x, \tilde{\Upsilon}; \alpha\right), \overline{g}\left(x, \tilde{\Upsilon}; \alpha\right)\right], \tag{5.4}$$

where

$$\begin{cases} \underline{g}(x, \widetilde{\Upsilon}; \alpha) = g_l[x, \underline{\Upsilon}, \overline{\Upsilon}]_\alpha, \\ \overline{g}(x, \widetilde{\Upsilon}; \alpha) = g_u[x, \underline{\Upsilon}, \overline{\Upsilon}]_\alpha. \end{cases} \tag{5.5}$$

Due to $\tilde{y}^{(\beta)}(x) = \tilde{g}\left(x, \widetilde{\Upsilon}(x)\right)$, we define the following membership function [Definition 2.1] by employing the extension principle theory [Definition 3.4]:

$$\begin{cases} \underline{g}(x, \widetilde{\Upsilon}(x;\alpha); \alpha) = \min\{\tilde{g}(x, \tilde{\mu}(\alpha)) | \tilde{\mu}(\alpha) \in \widetilde{\Upsilon}(x;\alpha)\} \\ \overline{g}(x, \widetilde{\Upsilon}(x;\alpha); \alpha) = \max\{\tilde{g}(x, \tilde{\mu}(\alpha)) | \tilde{\mu}(\alpha) \in \widetilde{\Upsilon}(x;\alpha)\} \end{cases} \tag{5.6}$$

where

$$\begin{cases} \underline{g}(x, \widetilde{\Upsilon}(x;\alpha); \alpha) = g_l\left(x, \underline{\Upsilon}(x;\alpha), \overline{\Upsilon}(x;\alpha)\right), \\ \overline{g}(x, \widetilde{\Upsilon}(x;\alpha); \alpha) = g_u\left(x, \underline{\Upsilon}(x;\alpha), \overline{\Upsilon}(x;\alpha)\right). \end{cases} \tag{5.7}$$

5.3.2 Construction of FF-HAM for second-order FFOIVPs

The procedure in the construction of the proposed FF-HAM involves reformulation of the existing F-HAM for solving crisp fractional IVPs followed by the formulation of FF-HAM for second-order FFOIVPs.

We will introduce the structure of FF-HAM to approximate the analytical solution of second-order FFOIVPs. For this purpose, we consider the following second-order FFOIVP (Hasan et al., 2017)

$$\begin{cases} \tilde{y}^{(\beta)}(x) = \tilde{g}(x, \tilde{y}(x), \tilde{y}'(x)) + \widetilde{G}(x), \;\; x \in [x_0, X], \; \beta \in (1,2], \\ \tilde{y}(x_0) = \tilde{y}_0, \; \tilde{y}'(x_0) = \tilde{y}_0{}', \end{cases} \tag{5.8}$$

here, $\tilde{y}(x)$ refers to the unknown fuzzy function [Definition 3.8] of crisp variable x, $\tilde{y}^{(\beta)}(x)$ is the fuzzy fractional Caputo derivative of $\tilde{y}(x)$ [Definition 3.22],

$\tilde{g}(x, \tilde{y}(x), \tilde{y}'(x))$ is the fuzzy function [Definition 3.8] of the fuzzy variable $\tilde{y}$, $\tilde{y}'(x)$ is the first derivative of the fuzzy variable $\tilde{y}$ and crisp variable x, $\widetilde{G}(x)$ represents the inhomogeneous fuzzy term of crisp variable x, and $\tilde{y}_0, \tilde{y}'(x_0)$ are the fuzzy numbers [Definition 3.7]. Now, we consider Eq. (5.8) followed by the defuzzification of Eq.(5.1) in Section 5.3.1 such that for all $\alpha \in [0,1]$ [Definition 3.5] and according to the F-HAM in Section 3.2.4, we can construct the following correction function of Eq.(5.8) as follows:

$$\begin{cases} \underline{\mathcal{L}}_\beta [\underline{y}(x;q)]_\alpha = \mathcal{N}[\underline{\Upsilon}(x;\alpha)], \\ \overline{\mathcal{L}}_\beta [\overline{y}(x;q)]_\alpha = \mathcal{N}[\overline{\Upsilon}(x;\alpha)]. \end{cases} \tag{5.9}$$

According to the extension principle theory [Definition 3.4], we have:

$$\begin{cases} g_l([\tilde{y}(x;q;\alpha)]) = min\{ \mathcal{N}[\underline{y}(x;q)]_\alpha, \mathcal{N}[\overline{y}(x;q)]_\alpha : \mu | \mu \in [\widetilde{\Upsilon}(x;\alpha)]\}, \\ g_u([\tilde{y}(x;q;\alpha)]) = max\{\mathcal{N}[\underline{y}(x;q)]_\alpha, \mathcal{N}[\overline{y}(x;q)]_\alpha : \mu | \mu \in [\widetilde{\Upsilon}(x;\alpha)]\}, \end{cases} \tag{5.10}$$

here, μ [Definition 3.1] referred to the membership function of Eq. (5.8). We can define the zeroth-order deformation equation constructed by FF-HAM for Eq. (5.8) as follows:

$$\begin{cases} (1-q)\underline{\mathcal{L}}_\beta \left[[\underline{y}(x;q)]_\alpha - \underline{y}_0(x;\alpha) \right] = q\underline{h}(\alpha)H(x)\ g_l([\tilde{y}(x;q)]_\alpha), \\ (1-q)\overline{\mathcal{L}}_\beta \left[[\overline{y}(x;q)]_\alpha - \overline{y}_0(x;\alpha) \right] = q\overline{h}(\alpha)H(x)\ g_u([\tilde{y}(x;q)]_\alpha). \end{cases} \tag{5.11}$$

As has been discussed in Section 4.3.2, $0 \le q \le 1$ represents the embedding parameter, $\tilde{h}(\alpha) = [\underline{h}(\alpha), \overline{h}(\alpha)]$ is a nonzero convergence control parameter, $H(x)$ is the auxiliary function while the operators $\underline{\mathcal{L}}_\beta = \frac{\partial^{(\beta)}[\underline{y}(x;q)]_\alpha}{\partial x^{(\beta)}}$ and $\overline{\mathcal{L}}_\beta = \frac{\partial^{(\beta)}\overline{y}[(x;q)]_\alpha}{\partial x^{(\beta)}}$ [Definition 3.22] are the auxiliary linear operators. Now, we can define the initial approximation of lower and upper bound $[\tilde{y}_0(x)]_\alpha =$

$\left[\underline{y}_0(x;\alpha), \overline{y}_0(x;\alpha)\right]$ by the rule of solution expression (Liao, 1999). From Eq. (5.11), the approximate solution $\tilde{y}(x;\alpha)$ can be expressed for $j = 0,1,2,.,k$ by a set of base functions x^j, then the approximate solution $\tilde{y}(x;\alpha)$ can be expressed as $\tilde{y}(x;\alpha) = \sum_{j=0}^{k}\left[\tilde{d}^j\right]_\alpha x^j$ where $\left[\tilde{d}^j\right]_\alpha$ are fuzzy coefficients to be determined. According to the rule of solution expression $\sum_{j=0}^{k}\left[\tilde{d}^j\right]_\alpha x^j$, we have the following form of the initial guess:

$$\begin{cases} \underline{y}_0(x;\alpha) = \underline{S}_1(\alpha) + \underline{S}_2(\alpha)x, \\ \overline{y}_0(x;\alpha) = \overline{S}_1(\alpha) + \overline{S}_2(\alpha)x, \end{cases} \tag{5.12}$$

where for all $\alpha \in [0,1]$ [Definition 3.5], $\tilde{S}_1(\alpha)$, and $\tilde{S}_2(\alpha)$ are the constants that can be determined easily from the initial conditions in Eq. (5.8). It can be concluded that for $q = 0$ and $q = 1$, we have

$$\begin{cases} \left[\underline{y}(x;0)\right]_\alpha = \underline{y}(x_0;\alpha), \\ [\overline{y}(x;0)]_\alpha = \overline{y}(x_0;\alpha). \end{cases} \quad \begin{cases} \left[\underline{y}(x;1)\right]_\alpha = \underline{Y}(x;\alpha), \\ [\overline{y}(x;1)]_\alpha = \overline{Y}(x;\alpha). \end{cases} \tag{5.13}$$

At the time when the embedding parameter changes from zero to one, the solution $[\tilde{y}(x;q)]_\alpha$ deforms from the initial approximation $[\tilde{y}(x;0)]_\alpha$ to the exact solution $\tilde{Y}(x;\alpha)$. By expanding the approximate solution $[\tilde{y}(x;q)]_\alpha$ as a Taylor series with respect to q, $\forall\alpha \in [0,1]$, we obtain

$$\begin{cases} \left[\underline{y}(x;q)\right]_\alpha = \underline{y}_0(x;\alpha) + \sum_{j=1}^{k}\underline{y}_j(x;\alpha)\, q^j \\ [\overline{y}(x;q)]_\alpha = \overline{y}_0(x;\alpha) + \sum_{j=1}^{k}\overline{y}_j(x;\alpha)\, q^j \end{cases} \tag{5.14}$$

where

$$\begin{cases} \underline{y}_j(x;\alpha) = \frac{1}{j!}\frac{\partial^j\left[\underline{y}(x;q)\right]_\alpha}{\partial q^j}\bigg|_{q=0}, \\ \overline{y}_j(x;\alpha) = \frac{1}{j!}\frac{\partial^j[\overline{y}(x;q)]_\alpha}{\partial q^j}\bigg|_{q=0}. \end{cases} \tag{5.15}$$

As has been mentioned in Section 4.3.2 if the auxiliary linear operator $\tilde{\mathcal{L}}_\beta$, the initial guess $\tilde{y}_0(x;\alpha)$, the auxiliary function $H(x)$, and the convergence control parameter $\tilde{h}(\alpha)$ are properly chosen, the FF-HAM series in Eq. (5.14) will converge to the exact solution at $q = 1$ such that

$$\begin{cases}[\underline{Y}(x;\alpha)] = \underline{y}_0(x;\alpha) + \sum_{j=1}^{\infty} \underline{y}_j(x;\alpha) \\ [\overline{Y}(x;\alpha)] = \overline{y}_0(x;\alpha) + \sum_{j=1}^{\infty} \overline{y}_j(x;\alpha)\end{cases}. \tag{5.16}$$

As has been mentioned in Section 4.3.2, in most cases it is impossible to find the analytical solution with FF-HAM as an infinite series especially for nonlinear cases. Now, according to Section 4.3.2, we define the vectors

$$\begin{cases}\vec{\underline{y}}_j(x;\alpha) = \{\underline{y}_0(x;\alpha), \underline{y}_1(x;\alpha), \dots, \underline{y}_k(x;\alpha)\} \\ \vec{\overline{y}}_j(x;\alpha) = \{\overline{y}_0(x;\alpha), \overline{y}_1(x;\alpha), \dots, \overline{y}_k(x;\alpha)\}\end{cases}. \tag{5.17}$$

In addition, based on Section 4.3.2 and by differentiating the zeroth-order deformation equation k times with respect to the embedding parameter q, followed by setting $q = 0$ and after that, dividing them by $k!$, we will extract the k^{th} -order deformation equation as follows:

$$\begin{cases}\underline{\mathcal{L}}_\beta\left[\underline{y}_k(x;\alpha) - \psi_k \underline{y}_{k-1}(x;\alpha)\right] = \underline{h}(\alpha)\mathcal{R}_k(\vec{\underline{y}}_{k-1}(x;\alpha)) \\ \overline{\mathcal{L}}_\beta\left[\overline{y}_k(x;\alpha) - \psi_k \overline{y}_{k-1}(x;\alpha)\right] = \overline{h}(\alpha)\mathcal{R}_k(\vec{\overline{y}}_{k-1}(x;\alpha))\end{cases} \tag{5.18}$$

where

$$\begin{cases}\mathcal{R}_k\left(\vec{\underline{y}}_{k-1}(x;\alpha)\right) = \frac{1}{(k-1)!}\frac{\partial^{k-1} g_l([\tilde{y}(x;q)]_\alpha)}{\partial q^{k-1}}\Big|_{q=0} \\ \mathcal{R}_k\left(\vec{\overline{y}}_{k-1}(x;\alpha)\right) = \frac{1}{(k-1)!}\frac{\partial^{k-1} g_u([\tilde{y}(x;q)]_\alpha)}{\partial q^{k-1}}\Big|_{q=0}\end{cases} \tag{5.19}$$

The solution of the k^{th}-order deformation for $k \geq 1$ is

$$\begin{cases} \underline{y}_k(x;\alpha) = \psi_k \underline{y}_{k-1}(x;\alpha) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \mathcal{R}_k(\vec{\underline{y}}_{k-1}(x;\alpha)) \\ \overline{y}_k(x;\alpha) = \psi_k \overline{y}_{k-1}(x;\alpha) + \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)} \mathcal{R}_k(\vec{\overline{y}}_{k-1}(x;\alpha)) \end{cases}, \tag{5.20}$$

such that

$$\psi_k = \begin{cases} 0, & k \leq 1 \\ 1, & k > 1 \end{cases}. \tag{5.21}$$

where $\tilde{\mathcal{L}_\beta}^{-1} = \tilde{\mathcal{J}}^{(\beta)} = \left[\underline{\mathcal{J}}^{(\beta)}, \overline{\mathcal{J}}^{(\beta)}\right]$ are the fuzzy Riemann-Liouville integral of order $\beta \in (1,2]$.

5.3.3 Establishment of Convergence of FF-HAM Solution Series for Second-order FFOIVPs

We can apply the convergence analysis of FF-HAM in Section 4.3.3 to Eq. (5.8) based on the residual formula of second-order FFOIVPs in Eq. (5.8).

5.4 Theoretical Development of FF-OHAM for Second-order FFOIVPs

For second-order FFOIVPs, the proposed FF-OHAM is theoretically developed in three steps

5.4.1 Defuzzification of FFODE

Step One needs to be done only once since it is applicable for both FF-HAM and FF-OHAM. Step One has been done in Section 5.3.1.

5.4.2 Construction of FF-OHAM for Second-order FFOIVPs

The procedure in the construction of the proposed FF-OHAM involves reformulation of the existing F-OHAM for solving crisp fractional IVPs followed by the formulation of FF-OHAM for second-order FFOIVPs.

We will proceed with the process of fuzzifying F-OHAM followed by defuzzification using the properties and definitions of fuzzy set theory [Definition 3.1] including the fuzzy number [Definition 3.6], α-cut [Definition 3.5], and extension principle theory [Definition 3.4], to lay out a new procedure for solving linear and nonlinear FFOIVPs of order $1 < \beta \leq 2$. We consider Eq. (5.8) followed by the defuzzification of Eq. (5.1) such that for all $\alpha \in [0,1]$ [Definition 3.5], we have the following lower bound

$$\begin{cases} \underline{\mathcal{L}}_\beta\left(\underline{y}(x;\alpha)\right) - g_l\left(x, \widetilde{Y}(x;\alpha)\right) - \underline{G}(x;\alpha) = 0, \ x \in [x_0, X], \\ \underline{y}^{(i)}(x_0) = \underline{y}_0^{(i)}, i = 0,1, \end{cases} \tag{5.22}$$

and the following upper bound

$$\begin{cases} \overline{\mathcal{L}}_\beta\left(\overline{y}(x;\alpha)\right) - g_u\left(x, \widetilde{Y}(x;\alpha)\right) - \overline{G}(x;\alpha) = 0, \ x \in [x_0, X], \\ \overline{y}^{(i)}(x_0) = \overline{y}_0^{(i)}, i = 0,1. \end{cases} \tag{5.23}$$

According to F-OHAM described in Section 3.2.5, Eq. (5.22) and Eq. (5.23) can be rewritten as follows:

$$\begin{cases} (1-q)\left[\underline{\mathcal{L}}_\beta\left(\left[\underline{y}(x;q)\right]_\alpha\right) - \underline{G}(x;\alpha)\right] = \underline{\mathcal{H}}(q;\alpha)\left[\underline{\mathcal{L}}_\beta\left(\left[\underline{y}(x;q)\right]_\alpha\right)\right] \\ -\underline{\mathcal{H}}(q;\alpha)\left[\underline{G}(x;\alpha)\right] - \underline{\mathcal{H}}(q;\alpha)\left[g_l([\tilde{y}(x;q)]_\alpha)\right] \\ (1-q)\left[\overline{\mathcal{L}}_\beta([\overline{y}(x;q)]_\alpha) - \overline{G}(x;\alpha)\right] = \overline{\mathcal{H}}(q;\alpha)\left[\overline{\mathcal{L}}_\beta([\overline{y}(x;q)]_\alpha)\right] \\ -\overline{\mathcal{H}}(q;\alpha)\left[\overline{G}(x;\alpha)\right] - \overline{\mathcal{H}}(q;\alpha)\, g_u([\tilde{y}(x;q)]_\alpha) \\ \tilde{y}^{(i)}(x_0) = \tilde{y}_0^{(i)}, i = 0,1, \end{cases} \tag{5.24}$$

where, $\tilde{\mathcal{L}}_\beta = [\underline{\mathcal{L}}_\beta, \overline{\mathcal{L}}_\beta] = \left[\frac{\partial^{(\beta)}[\underline{y}(x;q)]_\alpha}{\partial x^{(\beta)}}, \frac{\partial^{(\beta)}[\overline{y}(x;q)]_\alpha}{\partial x^{(\beta)}}\right]$ [Definition 3.22] are the linear operators and $q \in [0, 1]$ is an embedding parameter while $\tilde{\mathcal{H}}(q;\alpha) = [\underline{\mathcal{H}}(q), \overline{\mathcal{H}}(q)]_\alpha$ is a nonzero auxiliary fuzzy function for $q \neq 0$, and $[\tilde{y}(x;q)]_\alpha$ is an unknown fuzzy function.

Obviously, for $q = 0$, and $q = 1$, we obtain

$$\begin{cases} [\underline{y}(x;0)]_\alpha = \underline{y}_0(x;\alpha), & [\underline{y}(x;1)]_\alpha = \underline{Y}(x;\alpha), \\ [\overline{y}(x;0)]_\alpha = \overline{y}_0(x;\alpha), & [\overline{y}(x;1)]_\alpha = \overline{Y}(x;\alpha). \end{cases} \tag{5.25}$$

Thus, as q increases from 0 to 1, the solution $[\tilde{y}(x;q)]_\alpha$ changes from $\tilde{y}_0(x;\alpha)$ to the exact solution of Eq. (5.22) $\underline{Y}(x;\alpha)$, and Eq. (5.23) $\overline{Y}(x;\alpha)$ where $\tilde{y}_0(x;\alpha)$ is obtained from Eq. (5.24) for $q = 0$ as follows:

For the zeroth-order problem we have

$$\begin{cases} \underline{\mathcal{L}}_\beta\left([\underline{y}_0(x;0)]_\alpha\right) - \underline{\mathrm{G}}(x;\alpha) = 0, \\ \overline{\mathcal{L}}_\beta\left([\overline{y}_0(x;0)]_\alpha\right) - \overline{\mathrm{G}}(x;\alpha) = 0, \end{cases} \tag{5.26}$$

by applying the Riemann-Liouville operator $\tilde{\mathcal{J}}^{(\beta)}$ as in Definition 3.19 and also the property of Definition 3.18 on the left side of Eq. (5.26), we have

$$\begin{cases} \underline{y}_0(x;\alpha) = \tilde{\mathcal{J}}^{(\beta)}\, \underline{\mathrm{G}}(x;\alpha), \\ \overline{y}_0(x;\alpha) = \tilde{\mathcal{J}}^{(\beta)}\, \overline{\mathrm{G}}(x;\alpha), \end{cases} \tag{5.27}$$

subject to the following initial condition

$$[\tilde{y}_0(x_0;0)]_\alpha = \tilde{y}_0(x;\alpha),\ [\tilde{y}'_0(x_0;0)]_\alpha = \tilde{y}'_0(x;\alpha). \tag{5.28}$$

We choose the auxiliary function $\tilde{\mathcal{H}}(q;\alpha)$ for Eq. (5.24) in the following form:

$$\begin{cases} \underline{\mathcal{H}}(q;\alpha) = \underline{S}_1(\alpha)q + \underline{S}_2(\alpha)q^2 + \cdots = \sum_{j=1}^{k} \underline{S}_j(\alpha)q^j, \\ \underline{\mathcal{H}}(q;\alpha) = \overline{S}_1(\alpha)q + \overline{S}_2(\alpha)q^2 + \cdots = \sum_{j=1}^{k} \overline{S}_j(\alpha)q^j, \end{cases} \tag{5.29}$$

where $\tilde{S}_1(\alpha) = [\underline{S}_1(\alpha), \overline{S}_1(\alpha)], \tilde{S}_2(\alpha) = [\underline{S}_2(\alpha), \overline{S}_2(\alpha)]$, ... are the constants to be found for all $\alpha \in [0,1]$. Expanding $\left[\tilde{y}\left(x; q, S_j(\alpha)\right)\right]_\alpha$ into Taylor's series about q, we obtain the following approximate series solution:

$$\begin{cases} \left[\underline{y}\left(x; q, \underline{S}_j(\alpha)\right)\right]_\alpha = \underline{y}_0(x;\alpha) + \sum_{j=1}^{k} \left[\underline{y}_j\left(x, \underline{S}_j(\alpha)\right)\right]_\alpha q^j, \\ \left[\overline{y}\left(x; q, \overline{S}_j(\alpha)\right)\right]_\alpha = \overline{y}_0(x;\alpha) + \sum_{j=1}^{k} \left[\overline{y}_j\left(x, \overline{S}_j(\alpha)\right)\right]_\alpha q^j. \end{cases} \tag{5.30}$$

Now, similar to the procedure in Section 4.4.2, by substituting Eq. (5.30) into Eq. (5.24) and then by collecting the coefficient of like powers of q, we will obtain linear equations where the zeroth-order problem is given by Eq. (5.27) while the first-order and second-order problems are given as follows:

First-order Problem:

$$\begin{cases} \underline{\mathcal{L}}_\beta\left(\underline{y}_1(x;\alpha) - \underline{y}_0(x;\alpha)\right) + \underline{\mathrm{G}}(x;\alpha) = \underline{S}_1(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right), \\ \overline{\mathcal{L}}_\beta\left(\overline{y}_1(x;\alpha) - \overline{y}_0(x;\alpha)\right) + \overline{\mathrm{G}}(x;\alpha) = \overline{S}_1(\alpha) g_{u_0}\left(\tilde{y}_0(x;\alpha)\right), \end{cases} \tag{5.31}$$

$$[\tilde{y}_1(x_0;0)]_\alpha = \tilde{y}_1(x;\alpha), \left[\tilde{y}'_1(x_0;0)\right]_\alpha = \tilde{y}'_1(x;\alpha). \tag{5.32}$$

by applying the Riemann-Liouville operator $\tilde{\mathcal{J}}^{(\beta)}$ as in Definition 3.19 and also the property of Definition 3.18 on the left side of Eq. (5.31), we have

$$\begin{cases} \underline{y}_1(x;\alpha) = \underline{y}_0(x;\alpha) + \underline{\mathcal{J}}^{(\beta)}\left(\underline{S}_1(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right) - \underline{\mathrm{G}}(x;\alpha)\right), \\ \overline{y}_1(x;\alpha) = \overline{y}_0(x;\alpha) + \overline{\mathcal{J}}^{(\beta)}\left(\overline{S}_1(\alpha) g_{u_0}\left(\tilde{y}_0(x;\alpha)\right) - \overline{\mathrm{G}}(x;\alpha)\right), \end{cases} \tag{5.33}$$

$$[\tilde{y}_1(x_0;0)]_\alpha = \tilde{y}_1(x;\alpha), \left[\tilde{y}'_1(x_0;0)\right]_\alpha = \tilde{y}'_1(x;\alpha). \tag{5.34}$$

Second-order Problem:

$$\begin{cases} \underline{\mathcal{L}}_\beta\left(\underline{y}_2(x;\alpha)\right) - \underline{\mathcal{L}}_\beta\left(\underline{y}_1(x;\alpha)\right) = \underline{S}_2(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right) \\ +\underline{S}_1(\alpha)\left[\underline{\mathcal{L}}_\beta\left(\underline{y}_1(x;\alpha)\right) + g_{l_1}\left(\tilde{y}_0(x;\alpha),\ \tilde{y}_1(x;\alpha)\right)\right] \\ \overline{\mathcal{L}}_\beta\left(\overline{y}_2(x;\alpha)\right) - \overline{\mathcal{L}}_\beta\left(\overline{y}_1(x;\alpha)\right) = \overline{S}_2(\alpha) g_{u_0}\left(\tilde{y}_0(x;\alpha)\right) \\ +\overline{S}_1(\alpha)\left[\overline{\mathcal{L}}_\beta\left(\overline{y}_1(x;\alpha)\right) + g_{u_1}\left(\tilde{y}_0(x;\alpha),\ \tilde{y}_1(x;\alpha)\right)\right] \end{cases} \tag{5.35}$$

$$[\tilde{y}_2(x_0;0)]_\alpha = \tilde{y}_2(x;\alpha), \left[\tilde{y}'_2(x_0;0)\right]_\alpha = \tilde{y}'_2(x;\alpha). \tag{5.36}$$

by applying the Riemann-Liouville operator $\tilde{\mathcal{J}}^{(\beta)}$ as in Definition 3.19 and also the property of Definition 3.18 on the left side of Eq. (5.35), we have

$$\begin{cases} \underline{y}_2(x;\alpha) = \left(1 + \underline{S}_1(\alpha)\right) \underline{y}_1(x;\alpha) + \\ \underline{\mathcal{J}}^{(\beta)}\left(\underline{S}_2(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right) + \underline{S}_1(\alpha) g_{l_1}\left(\tilde{y}_0(x;\alpha),\ \tilde{y}_1(x;\alpha)\right)\right) \\ \overline{y}_2(x;\alpha) = \left(1 + \overline{S}_1(\alpha)\right) \overline{y}_1(x;\alpha) + \\ \overline{\mathcal{J}}^{(\beta)}\left(\overline{S}_2(\alpha) g_{u_0}\left(\tilde{y}_0(x;\alpha)\right) + \overline{S}_1(\alpha) g_{u_1}\left(\tilde{y}_0(x;\alpha),\ \tilde{y}_1(x;\alpha)\right)\right) \end{cases} \tag{5.37}$$

$$[\tilde{y}_2(x_0;0)]_\alpha = \tilde{y}_2(x;\alpha), \left[\tilde{y}'_2(x_0;0)\right]_\alpha = \tilde{y}'_2(x;\alpha). \tag{5.38}$$

.

.

.

Then, the general k^{th} -order formula with respect to $\tilde{y}_k(x;\alpha)$ is given by:

$$\begin{cases} \underline{\mathcal{L}}_\beta\left(\underline{y}_k(x;\alpha)\right) - \underline{\mathcal{L}}_\beta\left(\underline{y}_{k-1}(x;\alpha)\right) = \underline{S}_k(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right) \\ + \sum_{j=1}^{k-1} \underline{S}_j(\alpha)\left[\underline{\mathcal{L}}_\beta\left(\underline{y}_{k-j}(x;\alpha)\right) + g_{l_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)\right] \\ \overline{\mathcal{L}}_\beta\left(\overline{y}_k(x;\alpha)\right) - \overline{\mathcal{L}}_\beta\left(\overline{y}_{k-1}(x;\alpha)\right) = \overline{S}_k(\alpha) g_{u_0}\left(\tilde{y}_0(x;\alpha)\right) \\ + \sum_{j=1}^{k-1} \overline{S}_j(\alpha)\left[\overline{\mathcal{L}}_\beta\left(\overline{y}_{k-j}(x;\alpha)\right) + g_{u_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)\right] \end{cases} \tag{5.39}$$

$$[\tilde{y}_k(x_0;0)]_\alpha = \tilde{y}_k(x;\alpha), \left[\tilde{y}'_k(x_0;0)\right]_\alpha = \tilde{y}'_k(x;\alpha). \tag{5.40}$$

by applying the Riemann-Liouville operator $\tilde{J}^{(\beta)}$ as in Definition 3.19 and also the property of Definition 3.18 on the left side of Eq. (5.39), we have

$$\begin{cases} \underline{y}_k(x;\alpha) = \underline{y}_{k-1}(x;\alpha) + \sum_{j=1}^{k-1} \underline{S}_j(\alpha)\ \underline{y}_{k-j}(x;\alpha) + \\ \underline{J}^{(\beta)}\left(\underline{S}_k(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right) + \sum_{j=1}^{k-1} \underline{S}_j(\alpha) g_{l_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)\right) \\ \overline{y}_k(x;\alpha) = \overline{y}_{k-1}(x;\alpha) + \sum_{j=1}^{k-1} \overline{S}_j(\alpha)\ \overline{y}_{k-j}(x;\alpha) + \\ \overline{J}^{(\beta)}\left(\overline{S}_k(\alpha) g_{u_0}\left(\tilde{y}_0(x;\alpha)\right) + \sum_{j=1}^{k-1} \overline{S}_j(\alpha) g_{u_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)\right) \end{cases} \tag{5.41}$$

$$[\tilde{y}_k(x_0;0)]_\alpha = \tilde{y}_k(x;\alpha), \left[\tilde{y}'_k(x_0;0)\right]_\alpha = \tilde{y}'_k(x;\alpha). \tag{5.42}$$

where $g_{l_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)$ and $g_{u_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)$ are the coefficients of q^j in the expansion of $g_l[\tilde{y}(x;q)]_\alpha$ and $g_u[\tilde{y}(x;q)]_\alpha$ about the embedding parameter q, such that we have the lower and upper bound as follows:

$$\begin{cases} g_l\left(\left[\tilde{y}\left(x;q,\sum_{j=1}^{k} \tilde{S}_j(\alpha)\right)\right]_\alpha\right) = g_{l_0}(\tilde{y}_0(x;\alpha)) + \sum_{j=1}^{k} g_{l_j}\left(\sum_{j=0}^{k}\left[\tilde{y}_j\right]_\alpha\right) q^j, \\ g_u\left(\left[\tilde{y}\left(x;q,\sum_{j=1}^{k} \tilde{S}_j(\alpha)\right)\right]_\alpha\right) = g_{u_0}(\tilde{y}_0(x;\alpha)) + \sum_{j=1}^{k} g_{u_j}\left(\sum_{j=0}^{k}\left[\tilde{y}_j\right]_\alpha\right) q^j. \end{cases} \tag{5.43}$$

It has been observed that the convergence of the series in Eq.(5.30) depend upon the auxiliary constants $\tilde{S}_1(\alpha), \tilde{S}_2(\alpha), \ldots$, therefore, the approximate series solution of Eq.(5.8) will be in the following form:

$$\begin{cases} \left[\underline{y}\left(x, \sum_{j=1}^{k} \underline{S}_j(\alpha)\right)\right]_\alpha = \underline{y}_0(x;\alpha) + \sum_{j=1}^{k}\left[\underline{y}_j\left(x, \sum_{j=1}^{k} \underline{S}_j(\alpha)\right)\right]_\alpha, \\ \left[\overline{y}\left(x, \sum_{j=1}^{k} \overline{S}_j(\alpha)\right)\right]_\alpha = \overline{y}_0(x;\alpha) + \sum_{j=1}^{k}\left[\overline{y}_j\left(x, \sum_{j=1}^{k} \overline{S}_j(\alpha)\right)\right]_\alpha. \end{cases} \tag{5.44}$$

5.4.3 Establishment of Convergence of FF-OHAM Solution Series for Second-order FFOIVPs

Similar to Section 4.4.3, we will follow the same procedure to analyse the convergence dynamic of the second-order FFOIVPs by substituting Eq. (5.44) into Eq. (5.22) and Eq. (5.23) resulting in the following residual:

$$
\begin{cases}
\underline{RE}\left(x,\sum_{j=1}^{k}\underline{S}_j(\alpha);\alpha\right)=\underline{\mathcal{L}}_\beta\left(\underline{y}\left(x,\sum_{j=1}^{k}\underline{S}_j(\alpha);\alpha\right)\right)-\underline{\mathrm{G}}(x;\alpha) \\
\qquad\qquad -g_l\left(\tilde{y}\left(x,\sum_{j=1}^{k}\tilde{S}_j(\alpha);\alpha\right)\right), \\
\overline{RE}\left(x,\sum_{j=1}^{k}\overline{S}_j(\alpha);\alpha\right)=\overline{\mathcal{L}}_\beta\left(\overline{y}\left(x,\sum_{j=1}^{k}\overline{S}_j(\alpha);\alpha\right)\right)-\overline{\mathrm{G}}(x;\alpha) \\
\qquad\qquad -g_u\left(\tilde{y}\left(x,\sum_{j=1}^{k}\tilde{S}_j(\alpha);\alpha\right)\right).
\end{cases}
\tag{5.45}
$$

Again, similar to Section 4.4.3, if $\widetilde{RE}\ =0$, then $\left[\tilde{y}\left(x,\sum_{j=1}^{k}\tilde{S}_j(\alpha)\right)\right]_\alpha$ yields the exact solution, which generally does not happen, especially in nonlinear FFOIVPs problems. To identify the auxiliary constants of $\tilde{S}_j(\alpha)$ for $j=1,2,..k$, we choose x_0 and X such that the optimum values of $\tilde{S}_j(\alpha)$ for the convergent solution of the desired problem is obtained. To find the optimal values of $\tilde{S}_j(\alpha)$, we apply the method of Least Squares as follows:

$$
S\widetilde{RE}\left(x,\sum_{j=1}^{k}\tilde{S}_j(\alpha)\,;\alpha\right)=\int_{x_0}^{X}\widetilde{RE}^2\left(x,\sum_{j=1}^{k}\tilde{S}_j(\alpha)\,;\alpha\right)\,dx
\tag{5.46}
$$

where $\widetilde{RE}$ is the residual

$$
\begin{cases}
\left[\underline{RE}\right]_\alpha=\underline{\mathcal{L}}_\beta\left(\left[\underline{y}\right]_\alpha\right)-\underline{\mathrm{G}}(x;\alpha)-g_l([\tilde{y}]_\alpha) \\
\left[\overline{RE}\right]_\alpha=\overline{\mathcal{L}}_\beta([\overline{y}]_\alpha)-\overline{\mathrm{G}}(x;\alpha)-g_u([\tilde{y}]_\alpha)
\end{cases}
\tag{5.47}
$$

and

$$
\frac{\partial\,\widetilde{SRE}}{\partial\tilde{S}_1(\alpha)}=\frac{\partial\,\widetilde{SRE}}{\partial\tilde{S}_2(\alpha)}=\cdots\frac{\partial\,\widetilde{SRE}}{\partial\tilde{S}_k(\alpha)}=0
\tag{5.48}
$$

5.5 Experimental Work of FF-HAM and FF-OHAM for Second-order FFOIVPs

This section demonstrates the experimental work for the application of the developed methods FF-HAM and FF-OHAM in solving second-order FFOIVPs. There are three experimental examples demonstrated where for each example, the solution method using the developed FF-HAM is demonstrated first and followed by the one using the developed FF-OHAM. The general solving steps in the experimental work involve the defuzzification of the second-order FFOIVPs, implementation of the proposed methods, obtaining optimal parameters, obtaining the approximate-analytical solution and plotting the obtained solution to ensure that the solution satisfies the properties of fuzzy numbers. Comparative study with other available numerical results is also carried out.

5.5.1 Example 5.1

The following is a second-order linear fuzzy fractional differential equation (Hasan et al., 2017)

$$\begin{cases} D^{(\beta)}y(x) + y(x) = [\alpha, 2-\alpha], \quad 1 < \beta \leq 2, \\ \qquad \tilde{y}(0) = \tilde{y}'(0) = [\alpha - 1, 1-\alpha], \end{cases} \tag{5.49}$$

5.5.1.1 Solving Example 5.1 by FF-HAM

In solving Example 5.1 by FF-HAM, based on Section 5.3.2, we will construct the zeroth-order and k^{th}–order deformation equations for $k \geq 1$ of Eq. (5.49) as follows:

$$\begin{cases} \underline{y}_k(x;\alpha) = \psi_k \underline{y}_{k-1}(x;\alpha) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \mathcal{R}_k(\vec{\underline{y}}_{k-1}(x;\alpha)) \\ \overline{y}_k(x;\alpha) = \psi_k \overline{y}_{k-1}(x;\alpha) + \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)} \mathcal{R}_k(\vec{\overline{y}}_{k-1}(x;\alpha)) \end{cases} \tag{5.50}$$

$\underline{y}_k(0;\alpha) = 0,\ \overline{y}_k(0;\alpha) = 0,\ \underline{y}_k{}'(0;\alpha) = 0\,, \overline{y}_k{}'(0;\alpha) = 0$, and

$$\begin{cases} \mathcal{R}_k\left(\vec{\underline{y}}_{k-1}(x;\alpha)\right) = \underline{y}_{k-1}{}^{(\beta)}(x;\alpha) + \underline{y}_{k-1}(x;\alpha) - \alpha, \\ \mathcal{R}_k\left(\vec{\overline{y}}_{k-1}(x;\alpha)\right) = \overline{y}_{k-1}{}^{(\beta)}(x;\alpha) + \overline{y}_{k-1}(x;\alpha) - (2-\alpha). \end{cases} \tag{5.51}$$

For the initial approximation

$$\begin{cases}\tilde{y}_0(x;\alpha) = [(1-\alpha) + (\alpha-1)(x)] \\ \tilde{y}_0(x;\alpha) = [(\alpha-1) + (1-\alpha)(x)]\end{cases}$$

Expanding Eq. ((5.51) we have

For first-order problem:

$$\begin{cases}\underline{y}_1(x;\alpha) = \underline{h}(\alpha)\,\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_0{}^{(\beta)}(x;\alpha) + \underline{y}_0(x;\alpha) - \alpha\right), \\ \overline{y}_1(x;\alpha) = \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_0{}^{(\beta)}(x;\alpha) + \overline{y}_0(x;\alpha) - (2-\alpha)\right).\end{cases} \tag{5.52}$$

For second-order problem:

$$\begin{cases}\underline{y}_2(x;\alpha) = \underline{y}_1(x;\alpha) + \underline{h}(\alpha)\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_1{}^{(\beta)}(x;\alpha) + \underline{y}_1(x;\alpha)\right), \\ \overline{y}_2(x;\alpha) = \overline{y}_1(x;\alpha) + \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_1{}^{(\beta)}(x;\alpha) + \overline{y}_1(x;\alpha)\right).\end{cases} \tag{5.53}$$

.

.

.

$$\begin{cases}\underline{y}_k(x;\alpha) = \underline{y}_{k-1}(x;\alpha) + \underline{h}(\alpha)\,\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_{k-1}{}^{(\beta)}(x;\alpha) + \underline{y}_{k-1}(x;\alpha)\right), \\ \overline{y}_k(x;\alpha) = \overline{y}_{k-1}(x;\alpha) + \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_{k-1}{}^{(\beta)}(x;\alpha) - \overline{y}_{k-1}(x;\alpha)\right).\end{cases} \tag{5.54}$$

The fifth-order FF-HAM approximate series solution using Mathematica 12 Dsolve Package to find the approximate solutions for the lower and for the upper bounds, we obtain the following series:

$$\begin{cases}\underline{y}(x;\alpha;\underline{h}) = \underline{y}_0(x;\alpha) + \sum_{j=1}^{5}\underline{y}_j(x;\alpha;\underline{h}) \\ \overline{y}(x;\alpha;\overline{h}) = \overline{y}_0(x;\alpha) + \sum_{j=1}^{5}\overline{y}_j(x;\alpha;\overline{h})\end{cases} \tag{5.55}$$

However, we can show the accuracy of FF-HAM for solving Eq. (5.49) by taking the residual error as mentioned in Section 5.3.3 such that for $\beta \in (1,2]$

$$\begin{cases} \underline{ER}(x;\alpha;\underline{h}) = \left(\underline{y}^{(\beta)}(x;\alpha) + \underline{y}(x;\alpha) - \alpha\right), \\ \overline{ER}(x;\alpha;\overline{h}) = \left(\overline{y}^{(\beta)}(x;\alpha) + \overline{y}(x;\alpha) - (2-\alpha)\right). \end{cases} \tag{5.56}$$

It is to be noted that the series in Eq. (5.56) depends on x, α, and the convergent control-parameter $\tilde{h}$. For a fixed value of $0 \leq \alpha \leq 1$ [Definition 3.5], for example $\alpha = 0.4$, we identify the valid region of $\tilde{h}$-curve with fifth-order FF-HAM series approximation by plotting the $\tilde{h}$-curve for the lower and the upper bound of Eq. (5.49) to select the optimal values of $\tilde{h}(0.4)$ in Figure 5.1.

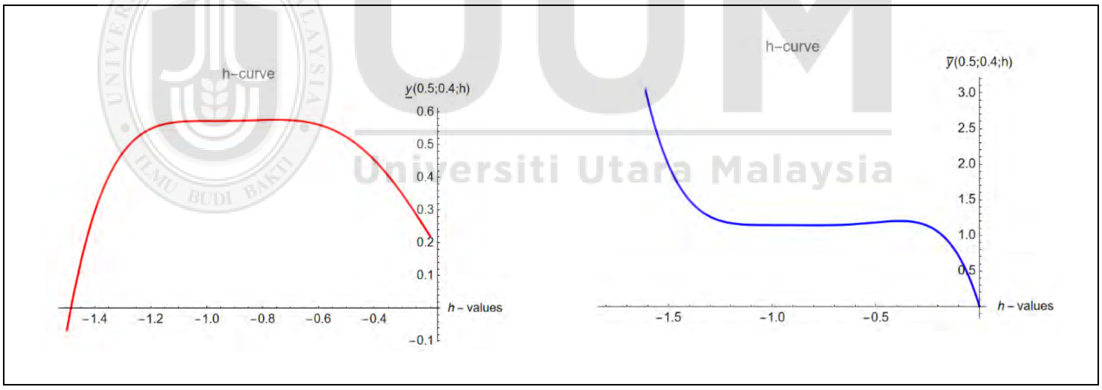

Figure 5.1. The $\tilde{h}(0.4)$-curves for the fuzzy solution of Eq. (5.49) given by fifth-order FF-HAM for $\beta = 1.9$ and $H(x) = 1$

Based on the curves in Figure 5.1, it is easy to reveal the valid region of $\tilde{h}$ when the series solution of FF-HAM corresponds to the line segment nearly parallel to the horizontal axis. In Figure 5.1, the convergent region of $\underline{h}$ is $-1.2 < h < -0.6$ and of $\overline{h}$ is $-1.3 < h < -0.2$. According to Section 5.3.3, the best values of $\tilde{h}$ is $\tilde{h} = -0.9976262061089138$. In Figure 5.2, we illustrate the lower and upper accuracy by fifth-order FF-HAM for all $\alpha \in [0,1]$, and $x \in [0,0.5]$ based on the best optimal convergence control parameter $\tilde{h}$.

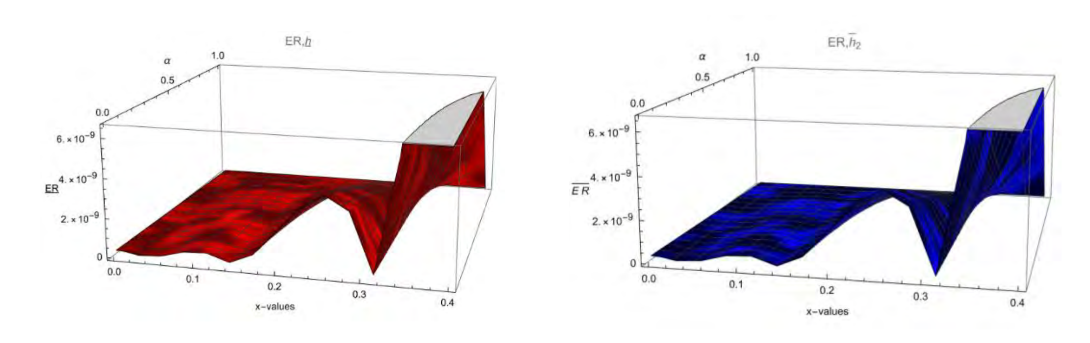

Figure 5.2. The accuracy of fifth-order FF-HAM for solving Eq. (5.49) of order $\beta = 1.9$ for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.5]$

In Table 5.1, the fuzzy solution $\tilde{y}(0.5; 0.4; \tilde{h})$ for Eq.(5.49) and the residual errors$[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ using fifth-order FF-HAM based on the best optimal $\tilde{h} = -0.9976262061089138$ are tabulated.

Table 5.1

The approximate solution and error of Eq. (5.49) by fifth-order FF-HAM when $\beta = 1.9$ *at* $x = 0.5$ *for all* $\alpha \in [0,1]$

α	$[\underline{ER}]_\alpha, \tilde{h}$	$[\overline{ER}]_\alpha, \tilde{h}$	$[\underline{y}]_\alpha, \tilde{h}$	$[\overline{y}]_\alpha, \tilde{h}$
0	8.64608×10^{-8}	-8.68803×10^{-8}	-0.38221	0.66752
0.2	6.91267×10^{-8}	-6.95462×10^{-8}	-0.27724	0.56254
0.4	5.17926×10^{-8}	-5.22121×10^{-8}	-0.17227	0.45760
0.6	3.44585×10^{-8}	-3.48780×10^{-8}	-0.06730	0.35260
0.8	1.71244×10^{-8}	-1.75438×10^{-8}	0.03768	0.24762
1	-2.09746×10^{-10}	-2.09746×10^{-10}	0.14265	0.14265

The summary of the approximate solutions of Eq. (5.49) of order $\beta = 1.9$ for all $\alpha \in [0,1]$ and $x \in [0,0.5]$ based on the optimal convergence parameter $\tilde{h}(0.4)$ is given in the following three-dimensional graph in Figure 5.3.

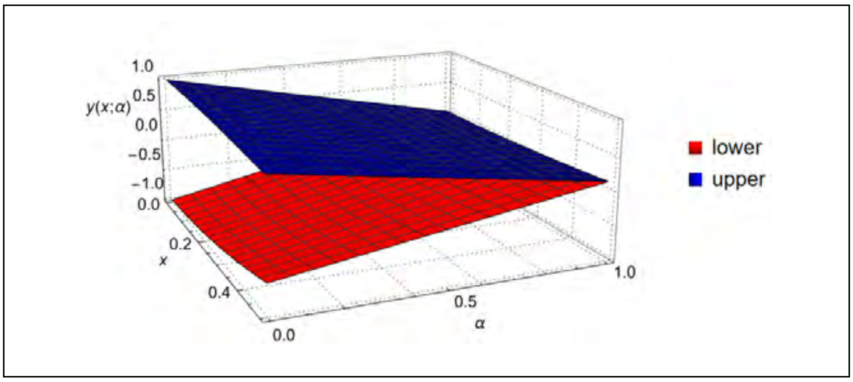

Figure 5.3*.* The three-dimensional approximate solution of Eq. (5.49) given by fifth-order FF-HAM over all $x \in [0,0.5]$ at $\beta = 1.9$, and for all $\alpha \in [0,1]$

5.5.1.2 Comparative Study of FF-HAM for Example 5.1

To show the efficiency of FF-HAM for Example 5.1 in comparison to other approximate methods, we tabulated the residual errors $[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ in Eq.(5.56) of the approximate solutions $\tilde{y}(0.5;\alpha;\tilde{h})$ for Eq.(5.49) of order $\beta = 2$ that obtained by using fifth-order FF-HAM series solution compare with the fifth-order RKHS method (Hasan et al., 2017) for different values of $\alpha \in [0,1]$, yielding the accuracy of FF-HAM for solving second-order linear FFOIVP. Since the optimal convergence control parameter is $\tilde{h} = -0.9976262061089138$, we will utilize this optimal value to find the approximate solution of Eq. (5.49) of order $\beta = 2$.

Table 5.2

Numerical comparison of approximate solutions of Eq. (5.49) for different values of α for $x = 0.5$ and $\beta = 2$

α	RKHS $\overline{ER}$	HAM $\overline{ER}$	RKHS $\underline{ER}$	HAM $\underline{ER}$
0	4.343×10^{-7}	-5.826×10^{-8}	4.343×10^{-7}	5.829×10^{-8}
0.25	3.125×10^{-6}	-4.369×10^{-8}	3.125×10^{-6}	4.372×10^{-8}
0.50	5.817×10^{-6}	-2.912×10^{-8}	5.817×10^{-6}	2.915×10^{-8}
0.75	8.508×10^{-6}	-1.455×10^{-8}	8.508×10^{-6}	1.458×10^{-8}
1.00	1.119×10^{-5}	1.413×10^{-11}	1.119×10^{-5}	1.413×10^{-11}

5.5.1.3 Solving Example 5.1 by FF-OHAM

In this section, we will approximate the analytical solution of Eq. (5.49) by FF-OHAM. We construct fifth-order FF-OHAM series solution with five convergence control parameters $\left(\tilde{S}_1(\alpha), \tilde{S}_2(\alpha), \tilde{S}_3(\alpha), \tilde{S}_4(\alpha), \tilde{S}_5(\alpha)\right)$ for each $\alpha \in [0,1]$. Now, according to Section 5.4.2, the fifth-order FF-OHAM series solution for all $\alpha \in [0,1]$ of Eq. (5.49) can be constructed as follows:

$$\begin{cases}(1-q)\left[\widetilde{D}^{(\beta)}\left(\tilde{y}(x;\alpha)\right) - (\alpha, 2-\alpha)\right] = \sum_{j=1}^{5} \tilde{S}_j(\alpha) q^j \left[\widetilde{D}^{(\beta)}\left(\tilde{y}(x;\alpha)\right)\right] \\ \qquad - \sum_{j=1}^{5} \tilde{S}_j(\alpha) q^j \,[\alpha, 2-\alpha] - \sum_{j=1}^{5} \tilde{S}_j(\alpha) q^j \,[\tilde{y}(x;\alpha)]\end{cases} \tag{5.57}$$

such that

$$\tilde{y}(x;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{j=1}^{5} \tilde{y}_j(x, S_1, .., S_j; \alpha) q^j. \tag{5.58}$$

Zeroth-order problem:

$$\begin{cases}\tilde{y}_0(x;\alpha) = \tilde{J}^{(\beta)}[\alpha, 2-\alpha], \\ \tilde{y}_0(x;\alpha) = \tilde{y}_0{}'(x;\alpha) = [\alpha - 1, 1-\alpha].\end{cases} \tag{5.59}$$

First-order problem:

$$\begin{cases} \tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big) = \Big(1+\tilde{S}_1(\alpha)\Big)\ \tilde{y}_0(x;\alpha) - \tilde{\mathcal{J}}^{(\beta)}(\alpha, 2-\alpha) \\ +\ \tilde{S}_1(\alpha)\,\tilde{\mathcal{J}}^{(\beta)}\ \big(\tilde{y}_0(x;\alpha) - (\alpha, 2-\alpha)\big), \\ \qquad \tilde{y}_1(0;\alpha) = \tilde{y}_1{}'(0;\alpha) = 0. \end{cases} \tag{5.60}$$

Second-order problem:

$$\begin{cases} \tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big) = \Big(1+\tilde{S}_1(\alpha)\Big)\ \tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big) + \tilde{S}_2(\alpha)\tilde{y}_0(x;\alpha) \\ +\tilde{S}_1(\alpha)\tilde{\mathcal{J}}^{(\beta)}\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big) + \tilde{S}_2(\alpha)\tilde{\mathcal{J}}^{(\beta)}\ \tilde{y}_0(x;\alpha) - \tilde{S}_2(\alpha)\tilde{\mathcal{J}}^{(\beta)}(\alpha, 2-\alpha) \\ \qquad \tilde{y}_2(0;\alpha) = \tilde{y}_2{}'(0;\alpha) = 0. \end{cases} \tag{5.61}$$

Third-order problem:

$$\begin{cases} \tilde{y}_3\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha),\tilde{S}_3(\alpha);\alpha\big) = \Big(1+\tilde{S}_1(\alpha)\Big)\ \tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big) + \\ \tilde{S}_2(\alpha)\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big) + \tilde{S}_3(\alpha)\tilde{y}_0(x;\alpha) + \ \tilde{\mathcal{J}}^{(\beta)}\sum_{i=1}^{3}\tilde{S}_i(\alpha) \\ \Big(\tilde{y}_{3-i}\big(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_{3-i}(\alpha);\alpha\big)\Big) - \tilde{S}_3(\alpha)\tilde{\mathcal{J}}^{(\beta)}(\alpha, 2-\alpha) \\ \qquad \tilde{y}_3(0;\alpha) = \tilde{y}_3{}'(0;\alpha) = 0. \end{cases} \tag{5.62}$$

Fourth-order problem:

$$\begin{cases} \tilde{y}_4\big(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_4(\alpha);\alpha\big) = \Big(1+\tilde{S}_1(\alpha)\Big)\ \tilde{y}_3\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha),\tilde{S}_3(\alpha);\alpha\big) + \\ \tilde{S}_2(\alpha)\tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big) + \tilde{S}_3(\alpha)\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big) + \tilde{S}_4(\alpha)\ \tilde{y}_0(x;\alpha) + \\ \tilde{\mathcal{J}}^{(\beta)}\sum_{i=1}^{4}\tilde{S}_i(\alpha)\,\tilde{y}_{4-i}\big(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_{4-i}(\alpha);\alpha\big) - \tilde{S}_4(\alpha)\tilde{\mathcal{J}}^{(\beta)}(\alpha, 2-\alpha) \\ \qquad \tilde{y}_4(0;\alpha) = \tilde{y}_4{}'(0;\alpha) = 0. \end{cases} \tag{5.63}$$

Fifth-order problem:

$$\begin{cases} \tilde{y}_5\big(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_5(\alpha);\alpha\big) = \Big(1+\tilde{S}_1(\alpha)\Big)\ \tilde{y}_4\big(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_4(\alpha);\alpha\big) + \\ \tilde{S}_2(\alpha)\tilde{y}_3\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha),\tilde{S}_3(\alpha);\alpha\big) + \tilde{S}_3(\alpha)\tilde{y}_2\big(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\big) + \\ \tilde{S}_4(\alpha)\tilde{y}_1\big(x,\tilde{S}_1(\alpha);\alpha\big) + \tilde{S}_5(\alpha)\ \tilde{y}_0(x;\alpha) - \tilde{S}_5(\alpha)\tilde{\mathcal{J}}^{(\beta)}(\alpha, 2-\alpha) \\ +\tilde{\mathcal{J}}^{(\beta)}\sum_{i=1}^{5}\tilde{S}_i(\alpha)\,\tilde{y}_{5-i}\big(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_{5-i}(\alpha);\alpha\big) \\ \qquad \tilde{y}_5(0;\alpha) = \tilde{y}_5{}'(0;\alpha) = 0. \end{cases} \tag{5.64}$$

We will use Mathematica 12 DSolve Package to find the solutions for the lower and the upper bounds of Eqs. (5.59)-(5.64), we obtain the following series:

$$\tilde{y}(x;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{j=1}^{5}\tilde{y}_j\big(x,\tilde{S}_1(r),\dots,\tilde{S}_j(\alpha);\alpha\big) \tag{5.65}$$

To compare fifth-order FF-OHAM solution with fifth-order FF-HAM solution, we resolve Eq. (5.49) in several steps. Firstly, we will identify the optimal convergence constants $\tilde{S}_j$ $\forall\alpha \in [0,1]$ in the following Tables 5.3-5.4.

Table 5.3

Lower auxiliary convergence parameters of fifth-order FF-OHAM for solving Eq. (5.49) at $\beta = 1.9$, $x = 0.5$, *for all* $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$\underline{\boldsymbol{S}}_{\boldsymbol{j}}$		
0	$\underline{S}_1 = -0.9978847916495066$ $\underline{S}_4 = 0.00008451713421360928$	$\underline{S}_2 = 0.00033811288345074643$ $\underline{S}_5 = 0.00001391260310964044$7	$\underline{S}_3 = 0.00016916301863427994$
0.2	$\underline{S}_1 = -0.9973247021302214$ $\underline{S}_4 = -5.952352805036373 \times 10^{-7}$	$\underline{S}_2 = -5.226436012331547 \times 10^{-7}$ $\underline{S}_5 = -0.00003477868390061559$	$\underline{S}_3 = -1.923719575086958 \times 10^{-7}$
0.4	$\underline{S}_1 = -0.9973258271316146$ $\underline{S}_4 = -5.302008426025514 \times 10^{-7}$	$\underline{S}_2 = -1.963593898366581 \times 10^{-7}$ $\underline{S}_5 = -0.0000345954250357226$	$\underline{S}_3 = -2.884067072352235 \times 10^{-8}$
0.6	$\underline{S}_1 = -0.9973271500370594$ $\underline{S}_4 = -5.367206209811341 \times 10^{-7}$	$\underline{S}_2 = -8.759807290759088 \times 10^{-8}$ $\underline{S}_5 = -0.00003427258060884567$	$\underline{S}_3 = 2.595474109225254 \times 10^{-8}$
0.8	$\underline{S}_1 = -0.9973301952532944$ $\underline{S}_4 = -6.267344331960221 \times 10^{-7}$	$\underline{S}_2 = -3.321899137298282 \times 10^{-8}$ $\underline{S}_5 = -0.00003313622748667779$	$\underline{S}_3 = 5.42148536233241 \times 10^{-8}$
1	$\underline{S}_1 = -0.9982216419077256$ $\underline{S}_4 = -2.464840588041107 \times 10^{-8}$	$\underline{S}_2 = -1.212750040304593 \times 10^{-13}$ $\underline{S}_5 = -0.000009533021537600279$	$\underline{S}_3 = -9.165596615465692 \times 10^{-11}$

Table 5.4

Upper auxiliary convergence parameters of fifth-order FF-OHAM for solving Eq. (5.49) at $\beta = 1.9$, $x = 0.5$, *for all* $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$\overline{\boldsymbol{S}}_{\boldsymbol{j}}$		
0	$\overline{S}_1 = -0.9995534813377625$ $\overline{S}_4 = 0.0000002444560041648877$	$\overline{S}_2 = -3.30642143303231 \times 10^{-7}$ $\overline{S}_5 = 0.00012300198848974584$	$\overline{S}_3 = -5.02072688741439 \times 10^{-7}$
0.2	$\overline{S}_1 = -0.997321796071799$ $\overline{S}_4 = -3.513094775947534 \times 10^{-7}$	$\overline{S}_2 = 5.741622847012249 \times 10^{-8}$ $\overline{S}_5 = -0.000035391514601860196$	$\overline{S}_3 = 9.702527889855018 \times 10^{-8}$
0.4	$\overline{S}_1 = -0.997321082004045$ $\overline{S}_4 = -3.375043727258685 \times 10^{-7}$	$\overline{S}_2 = 4.835601905232992 \times 10^{-8}$ $\overline{S}_5 = -0.000035480935406938206$	$\overline{S}_3 = 9.232415047717382 \times 10^{-8}$
0.6	$\overline{S}_1 = -0.9973196866623794$ $\overline{S}_4 = -3.085159953029679 \times 10^{-7}$	$\overline{S}_2 = 3.670311781808728 \times 10^{-8}$ $\overline{S}_5 = -0.00003563426371183573$	$\overline{S}_3 = 8.615299524743435 \times 10^{-8}$
0.8	$\overline{S}_1 = -0.997315430369204$ $\overline{S}_4 = -2.188983924468905 \times 10^{-7}$	$\overline{S}_2 = 2.116681568930797 \times 10^{-8}$ $\overline{S}_5 = -0.00003591682931751865$	$\overline{S}_3 = 7.73708153134244 \times 10^{-8}$
1	$\overline{S}_1 = -0.9982221094206769$ $\overline{S}_4 = -2.457293035300185 \times 10^{-8}$	$\overline{S}_2 = -1.203794106131102 \times 10^{-13}$ $\overline{S}_5 = -0.0000095203562610100088$	$\overline{S}_3 = -9.118812778363507 \times 10^{-11}$

Next, we tabulate the residual errors $[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ of the approximate solutions $\underline{y}(0.5;\alpha)$ and $\overline{y}(0.5;\alpha)$ obtained by using fifth-order FF-OHAM series solution based on $\tilde{S}_j(\alpha)$ as follows:

$$\begin{cases} \underline{ER}(x,\underline{S}_j;\alpha) = \left(\underline{y}^{(\beta)}(x;\alpha) + \underline{y}(x;\alpha) - \alpha\right), \\ \overline{ER}(x,\overline{S}_j;\alpha) = \left(\overline{y}^{(\beta)}(x;\alpha) + \overline{y}(x;\alpha) - (2-\alpha)\right). \end{cases} \tag{5.66}$$

Table 5.5

The approximate solution and error of Eq. (5.49) by fifth-order FF-OHAM for $\beta = 1.9$ *at* $x = 0.5\ \forall\alpha \in [0,1]$

α	$[\underline{ER}]_\alpha, \underline{S}_j$	$[\overline{ER}]_\alpha, \overline{S}_j$	$[\underline{y}]_\alpha, \underline{S}_j$	$[\overline{y}]_\alpha, \overline{S}_j$
0	1.14568×10^{-9}	5.87316×10^{-9}	-0.38247	0.66752
0.2	-1.06497×10^{-10}	1.05098×10^{-10}	-0.27724	0.56254
0.4	-8.00920×10^{-11}	7.85572×10^{-11}	-0.17227	0.45757
0.6	-5.36464×10^{-11}	5.20575×10^{-11}	-0.06730	0.35260
0.8	-2.72148×10^{-11}	2.57527×10^{-11}	0.03768	0.24762
1	-8.26006×10^{-14}	-4.88498×10^{-15}	0.14265	0.14265

For all $\alpha \in [0,1]$ and $x \in [0,0.5]$, we summarize the solution of Eq. (5.49) by fifth-order FF-OHAM as illustrated in Figure 5.4.

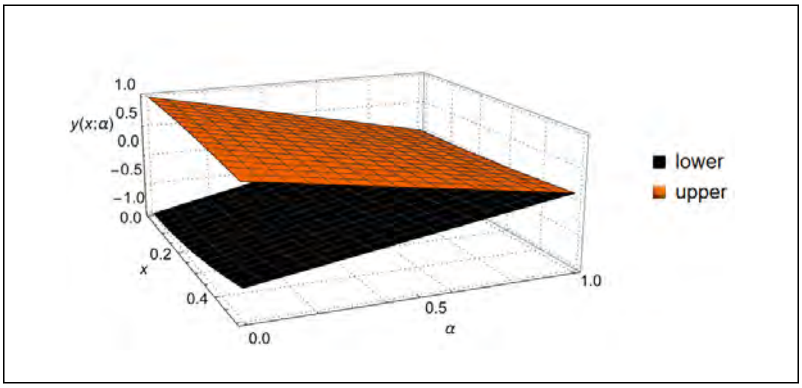

Figure 5.4. The three-dimensional approximate solution of Eq. (5.49) given by fifth-order FF-OHAM over all $x \in [0,0.5]$ at $\beta = 1.9$, and for all $\alpha \in [0,1]$

5.5.1.4 Comparative Study of FF-OHAM for Example 5.1

A comparative study is conducted for Example 5.1. In Table 5.6, we will tabulate the residual errors $[\widetilde{ER}]_\alpha$ for the approximate solutions $\tilde{y}(0.5, \tilde{S}_j; \alpha)$ for Eq.(5.49) of order $\beta = 2$ obtained by using fifth-order FF-OHAM series solution and compare the results with the fifth-order RKHSM (Hasan et al., 2017) for different values of $\alpha \in [0,1]$ at $x = 0.5$.

Table 5.6

Numerical comparison of approximated solutions of Eq. (4.49) for different values of α for $x = 0.5$, and $\beta = 2$

α	RKHS $\overline{ER}$	OHAM $\overline{ER}$	RKHS $\underline{ER}$	OHAM $\underline{ER}$
0	4.343×10^{-7}	-8.205×10^{-9}	4.343×10^{-7}	2.248×10^{-9}
0.25	3.125×10^{-6}	2.668×10^{-11}	3.125×10^{-6}	-2.70×10^{-11}
0.50	5.817×10^{-6}	1.777×10^{-11}	5.817×10^{-6}	-1.807×10^{-11}
0.75	8.508×10^{-6}	4.441×10^{-10}	8.508×10^{-6}	-9.110×10^{-12}
1.00	1.119×10^{-5}	9.672×10^{-12}	1.119×10^{-5}	9.782×10^{-12}

According to Tables 5.1-5.2, Tables 5.5-5.6 and Figures 5.3-5.4 it can be concluded that the fifth-order FF-HAM and fifth-order FF-OHAM satisfy the triangular solution of the fuzzy differential equations (Salahshour, 2011) for solving Eq. (5.49). On the other hand, the proposed methods provide more accurate solution compared with the RKHSM (Hasan et al., 2017), while taking into account that FF-OHAM gives a better solution compared with FF-HAM for all $\alpha \in [0,1]$ at the same order for solving second-order linear FFOIVP.

5.5.2 Example 5.2

Consider the following second-order FFOIVP (Hasan et al., 2017)

$$\begin{cases} D^{(\beta)}y(x) - y'(x) - (x+1) = 0, \quad 1 < \beta \le 2, \\ \qquad y(0) = y'(0) = [\alpha - 2, 1 - 2\alpha], \end{cases} \tag{5.67}$$

5.5.2.1 Solving Example 5.2 by FF-HAM

In this section, we will approximate the analytical solution of Example 5.2 by FF-HAM. The linear operators of Eq. (5.67) are $\underline{\mathcal{L}}_\beta = \underline{D}^{(\beta)}$ and $\overline{\mathcal{L}}_\beta = \overline{D}^{(\beta)}$. Based on Section 5.3.2, we have the following:

$$\begin{cases} \underline{y}_k(x;\alpha) = \psi_k \underline{y}_{k-1}(x;\alpha) + \underline{h}\, \underline{\mathcal{J}}^{(\beta)} \mathcal{R}_k(\vec{\underline{y}}_{k-1}(x;\alpha)) \\ \overline{y}_k(x;\alpha) = \psi_k \overline{y}_{k-1}(x;\alpha) + \overline{h}\, \overline{\mathcal{J}}^{(\beta)} \mathcal{R}_k\left(\vec{\overline{y}}_{k-1}(x;\alpha)\right) \\ \underline{y}_k(0;\alpha) = \underline{y}'_k(0;\alpha) = 0, \overline{y}_k(0;\alpha) = \overline{y}'_k(0;\alpha) = 0. \end{cases} \tag{5.68}$$

such that

$$\psi_k = \begin{cases} 0, & k \le 1 \\ 1, & k > 1 \end{cases} \tag{5.69}$$

$$\begin{cases} \mathcal{R}_k\left(\underline{\vec{y}}_{k-1}(x;\alpha)\right) = \underline{D}^{(\beta)}\underline{y}_{k-1}(x;\alpha) - \underline{y}'_{k-1}(x;\alpha) - (x+1) \\ \mathcal{R}_k\left(\overline{\vec{y}}_{k-1}(x;\alpha)\right) = \overline{D}^{(\beta)}\overline{y}_{k-1}(x;\alpha) - \overline{y}'_{k-1}(x;\alpha) - (x+1) \end{cases} \tag{5.70}$$

For initial approximation

$$\tilde{y}_0(x;\alpha) = \left[\left((\alpha-2)+(\alpha-2)(x)\right), \left((1-2\alpha)+(1-2\alpha)(x)\right)\right].$$

Expanding Eq. (5.70), we have

$$\tilde{y}_1(x;\alpha) = \tilde{h}(\alpha)\tilde{y}_0(x;\alpha) + \tilde{h}(\alpha)\,\tilde{\mathcal{J}}^{(\beta)}\left({y_0}^{(\beta)}(x;\alpha) - \tilde{y}'_0(x;\alpha) - (x+1)\right),$$

$$\tilde{y}_2(x;\alpha) = \tilde{h}(\alpha)\tilde{y}_1(x;\alpha) + \tilde{h}(\alpha)\,\tilde{\mathcal{J}}^{(\beta)}\left({y_1}^{(\beta)}(x;\alpha) - \tilde{y}'_1(x;\alpha)\right),$$

$$\tilde{y}_3(x;\alpha) = \tilde{h}(\alpha)\tilde{y}_2(x;\alpha) + \tilde{h}(\alpha)\,\tilde{\mathcal{J}}^{(\beta)}\left({y_2}^{(\beta)}(x;\alpha) - \tilde{y}'_2(x;\alpha)\right),$$

.

.

.

$$\tilde{y}_k(x;\alpha) = \tilde{h}(\alpha)\tilde{y}_{k-1}(x;\alpha) + \tilde{h}(\alpha)\,\tilde{\mathcal{J}}^{(\beta)}\left({y_{k-1}}^{(\beta)}(x;\alpha) - \tilde{y}'_{k-1}(x;\alpha)\right).$$

with the third-order FF-HAM approximate series solution $\tilde{y}(x;\alpha;\tilde{h}) = \sum_{j=0}^{3}\tilde{y}_j(x;\alpha;\tilde{h})$.

We can show the accuracy of FF-HAM for Eq. (5.67) by taking the residual error as mentioned in Section 5.3.3 such that

$$\begin{cases} \underline{ER}(x;\alpha;\underline{h}) = \left(\underline{y}^{(\beta)}(x;\alpha) - \underline{y}'(x;\alpha) - (x+1)\right), \\ \overline{ER}(x;\alpha;\overline{h}) = \left(\overline{y}^{(\beta)}(x;\alpha) - \overline{y}'(x;\alpha) - (x+1)\right). \end{cases} \tag{5.71}$$

Based on Section 5.3.3, we plot the $\tilde{\hbar}$-curve for $\tilde{y}_k(0.5; 1; \tilde{\hbar})$ of third-order FF-HAM approximate solution. Similar to the previous example, we plot the $\tilde{\hbar}$-curve when $\alpha = 1$ to obtain the optimal value of $\tilde{\hbar}$ of every α-level set for the lower and upper solution of Eq. (5.67).

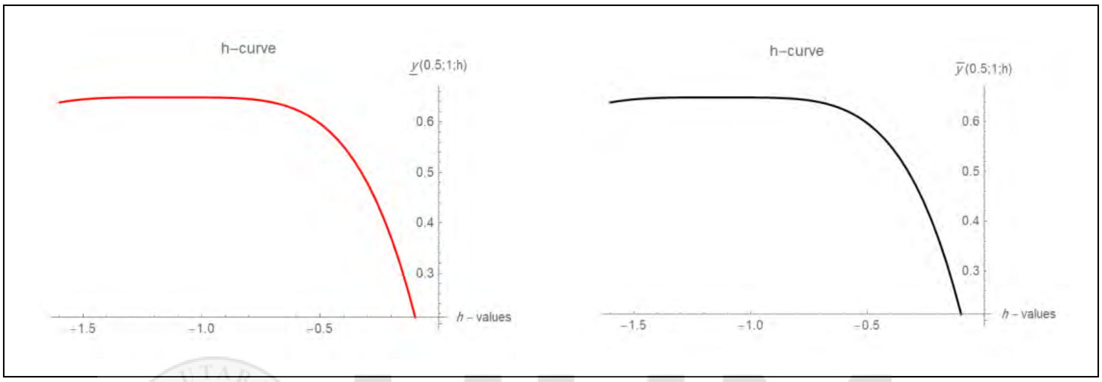

Figure 5.5. The $\tilde{\hbar}(1)$-curves for fuzzy solution of Eq. (5.67) given by third-order FF-HAM for $\beta = 1.9$ and $H(x) = 1$

Observing these curves, it is easy to discover the valid region of $\tilde{\hbar}(1)$, where the series solution of FF-HAM corresponds to the line segment nearly parallel to the horizontal axis. From Figure 5.5 and Section 5.3.3, it is clear that the third-order FF-HAM series solution is convergent when $-1.5 < \tilde{\hbar} < -0.6$. Furthermore, according to Section 5.3.3, the best value of $\tilde{\hbar}(1)$ is $\tilde{\hbar}(\alpha) = -1.1306959383323072$.

In Table 5.7, we tabulate the residual errors$[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ for the approximate solutions $\underline{y}(0.5; \tilde{\hbar}; 1)$ and $\overline{y}(0.5; \tilde{\hbar}; 1)$ of Eq. (5.67) obtained using third-order FF-HAM.

Table 5.7

The approximate solution and error of Eq. (5.67) by third-order FF-HAM for $\beta = 1.9$ *at* $x = 0.5$ $\forall \alpha \in [0,1]$

α	$[\underline{ER}]_{\alpha}, \tilde{h}$	$[\overline{ER}]_{\alpha}, \tilde{h}$	$[\underline{y}]_{\alpha}, \tilde{h}$	$[\overline{y}]_{\alpha}, \tilde{h}$
0	-1.72833×10^{-3}	5.69274×10^{-3}	-3.15314	1.89600
0.25	-1.10990×10^{-3}	4.45590×10^{-3}	-2.73238	1.05448
0.50	-4.91481×10^{-4}	3.21905×10^{-3}	-2.31162	0.21295
0.75	1.26941×10^{-4}	1.98221×10^{-3}	-1.89086	-0.62857
1	7.45363×10^{-4}	7.45363×10^{-4}	-1.47010	-1.47010

The summary of the fuzzy approximate solutions of Eq. (5.67) of order $\beta = 1.9$ for all $\alpha \in [0,1]$ and $x \in [0,0.5]$ based on the optimal convergence parameter $\tilde{h}(1)$ is illustrated in the following three-dimensional graph in Figure 5.6.

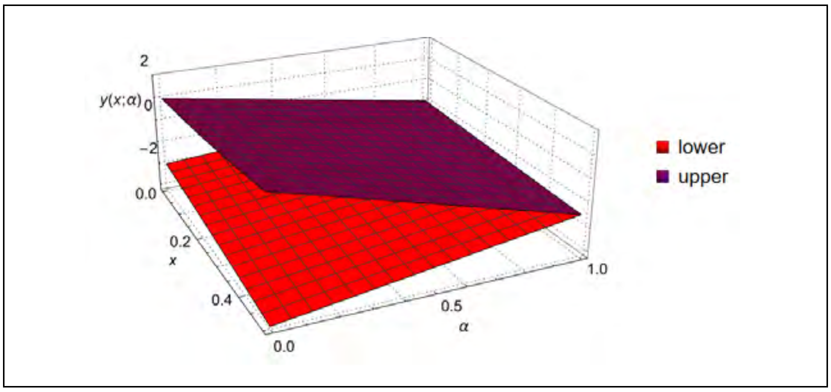

Figure 5.6. The three-dimensional approximate solution of Eq. (5.67) given by third-order FF-HAM over all $x \in [0,0.5]$ at $\beta = 1.9$, and for all $\alpha \in [0,1]$

To study the convergence dynamic of FF-HAM for solving second-order FFOIVPs, we employ the FF-HAM series of order five instead of order three. Figure 5.7 illustrates the valid region of $\tilde{h}$ with fifth-order FF-HAM. The valid region of the convergence control parameters is relatively unchanged after extending the series solution of FF-HAM for solving Eq. (5.67) where the best optimal convergence control parameter is $\tilde{h}(\alpha) = -1.0872665125720007$.

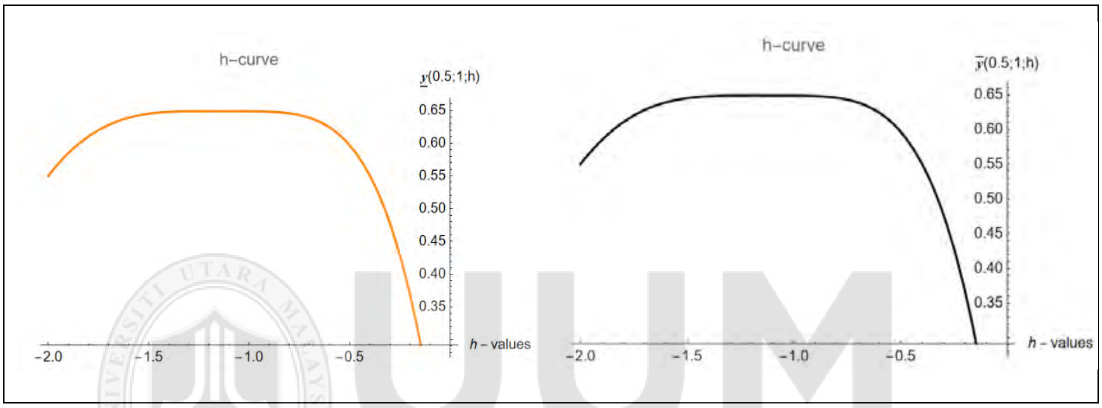

Figure 5.7. The $\tilde{h}(1)$-curves for the fuzzy solution of Eq. (5.67) given by fifth-order FF-HAM for $\beta = 1.9$ and $H(x) = 1$

The new $\tilde{h}(\alpha)$ will be utilized to find the approximate solution of Eq. (5.67) as given in Table 5.8.

Table 5.8

The approximate solution and error of Eq. (5.67) given by fifth-order FF-HAM for $\beta = 1.9$ *at* $x = 0.5$ $\forall \alpha \in [0,1]$

α	$[\underline{ER}]_{\alpha}, \tilde{h}$	$[\overline{ER}]_{\alpha}, \tilde{h}$	$[\underline{y}]_{\alpha}, \tilde{h}$	$[\overline{y}]_{\alpha}, \tilde{h}$
0	-3.80876×10^{-5}	7.73376×10^{-5}	-3.15306	1.89582
0.25	-2.84689×10^{-5}	5.81000×10^{-5}	-2.73232	1.05434
0.50	-1.88501×10^{-5}	3.88625×10^{-5}	-2.31158	0.21286
0.75	-9.23133×10^{-6}	1.96250×10^{-5}	-1.89084	-0.62862
1	3.87438×10^{-7}	3.87438×10^{-7}	-1.47010	-1.47010

We summarize the fuzzy approximate solutions of Eq. (5.67) of order $\beta = 1.9$ for all $\alpha \in [0,1]$ and $x \in [0,0.5]$ based on the optimal convergence parameter $\tilde{h}(1)$ in a three-dimensional graph in Figure 5.8.

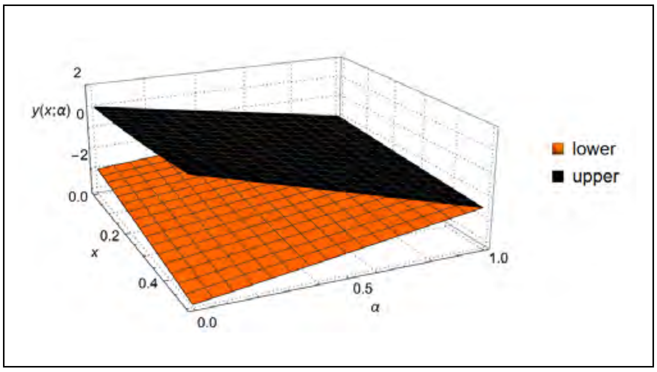

Figure 5.8. The three-dimensional approximate solution of Eq. (5.67) given by fifth-order FF-HAM over all $x \in [0,0.5]$ at $\beta = 1.9$, and for all $\alpha \in [0,1]$

From Tables 5.7-5.8, Figure 5.6 and Figure 5.8, it can be concluded that the third-order and fifth-order FF-HAM satisfies the triangular solution [Definition 3.7] of the fuzzy differential equations (Salahshour, 2011) for solving second-order FFOIVPs. Besides that, Tables 5.7-5.8 indicate that the fifth-order FF-HAM series solution provides more accurate solution compared with the third-order series solution. We can summarize the lower and upper accuracy of third-order and fifth-order FF-HAM based on the best optimal convergence control parameter $\tilde{h}$ for all $\alpha \in [0,1]$ and $x \in [0,0.5]$ as illustrated in the following three-dimensional graphs in Figures 5.9-5.10.

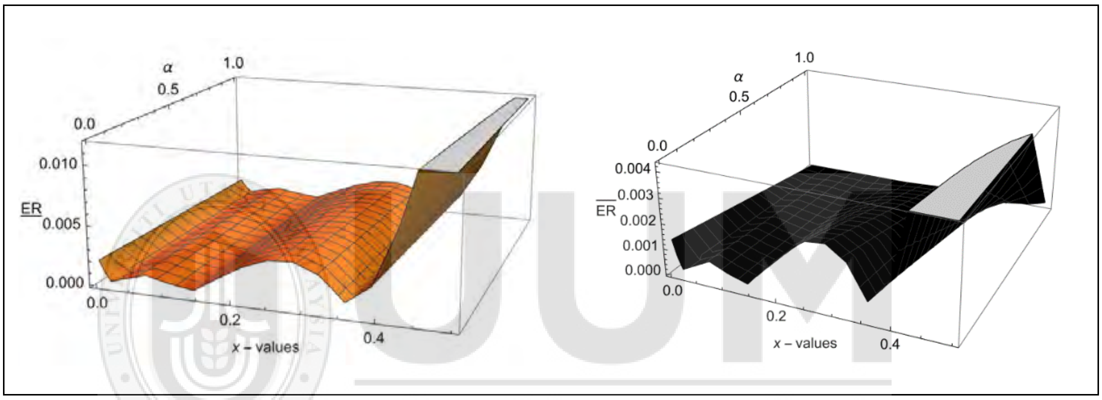

Figure 5.9. The accuracy of third-order FF-HAM for solving Eq. (5.67) for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.5]$

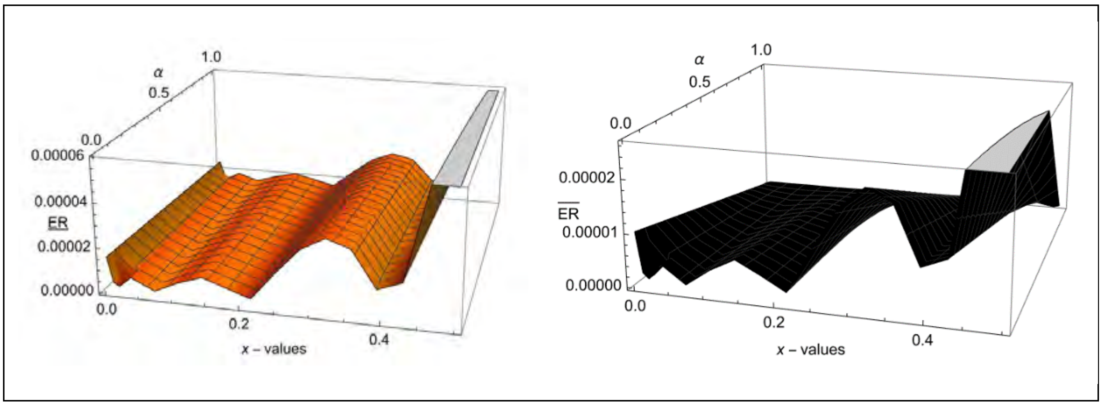

Figure 5.10. The accuracy of fifth-order FF-HAM for solving Eq. (5.67) for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.5]$

Figures 5.9-5.10 clarify that the FF-HAM solution with a higher series order provides a more accurate solution than lower ones.

5.5.2.2 Solving Example 5.2 by FF-OHAM

To solve Example 5.2 by FF-OHAM, we can construct k^{th}-order FF-OHAM series solution with k convergence control parameters $\left(\tilde{S}_1(\alpha), \dots., \tilde{S}_k(\alpha)\right)$ for each $\alpha \in [0,1]$ for all $\alpha \in [0,1]$ for Eq. (5.67) as follows:

$$\begin{cases}(1-q)\left[\widetilde{D}^{(\beta)}\left(\tilde{y}(x;\alpha)\right)-(x+1)\right]=\widetilde{\mathcal{H}}(q;\alpha)\left[\widetilde{D}^{(\beta)}\left(\tilde{y}(x;\alpha)\right)\right] \\ \qquad -\widetilde{\mathcal{H}}(q;\alpha)[x+1]-\widetilde{\mathcal{H}}(q;\alpha)\,[\tilde{y}'(x;\alpha)]\end{cases} \tag{5.72}$$

such that

$$\tilde{y}(x;\alpha)=\tilde{y}_0(x;\alpha)+\textstyle\sum_{j=1}^{k}\tilde{y}_j\left(x,S_1,..,S_j;\alpha\right)q^j, \tag{5.73}$$

$$\tilde{y}'(x;\alpha)=\tilde{y}_0{}'(x;\alpha)+\textstyle\sum_{j=1}^{k}\tilde{y}_j{}'\left(x,S_1,..,S_j;\alpha\right)q^j, \tag{5.74}$$

and

$$\widetilde{\mathcal{H}}(q;\alpha)=\textstyle\sum_{j=1}^{k}\tilde{S}_j(\alpha)q^j. \tag{5.75}$$

Zeroth-order problem:

$$\begin{cases}\tilde{y}_0(x,\alpha)=\tilde{\mathcal{J}}^{(\beta)}(x+1) \\ \tilde{y}_0(x;\alpha)=\tilde{y}_0{}'(x;\alpha)=[\alpha-2,1-2\alpha]\end{cases} \tag{5.76}$$

First-order problem:

$$\begin{cases}\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)=\left(1+\tilde{S}_1(\alpha)\right)\,\tilde{y}_0(x;\alpha)-\left(1+\tilde{S}_1(\alpha)\right) \\ \tilde{\mathcal{J}}^{(\beta)}(x+1)-\tilde{S}_1(\alpha)\,\tilde{\mathcal{J}}^{(\beta)}\left(\,\tilde{y}_0{}'(x;\alpha)\right), \\ \qquad \tilde{y}_1(0;\alpha)=\tilde{y}_1{}'(0;\alpha)=0.\end{cases} \tag{5.77}$$

Second-order problem:

$$\begin{cases}\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right) = \left(1+\tilde{S}_1(\alpha)\right)\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)+\tilde{S}_2(\alpha)\tilde{y}_0(x,\alpha)\\ -\tilde{S}_1(\alpha)\tilde{\mathcal{J}}^{(\beta)}\tilde{y}_1{}'\left(x,\tilde{S}_1(\alpha);\alpha\right)-\tilde{S}_2(\alpha)\tilde{\mathcal{J}}^{(\beta)}\left(\tilde{y}_0{}'(x;\alpha)+(x+1)\right)\\ \qquad \tilde{y}_2(0;\alpha) = \tilde{y}_2{}'(0;\alpha)=0.\end{cases} \tag{5.78}$$

Third-order problem:

$$\begin{cases}\tilde{y}_3\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha),\tilde{S}_3(\alpha);\alpha\right) = \left(1+\tilde{S}_1(\alpha)\right)\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)-\\ \tilde{\mathcal{J}}^{(\beta)}\sum_{i=1}^{3}\tilde{S}_i(\alpha)\,\tilde{y}_{3-i}{}'\left(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_{3-i}(\alpha);\alpha\right)-\tilde{S}_3(\alpha)\tilde{\mathcal{J}}^{(\beta)}(x+1)\\ +\tilde{S}_2(\alpha)\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)+\tilde{S}_3(\alpha)\tilde{y}_0(x;\alpha)\\ \qquad \tilde{y}_3(0;\alpha) = \tilde{y}_3{}'(0;\alpha)=0.\end{cases} \tag{5.79}$$

.

.

.

And k^{th}-order problem

$$\begin{cases}\tilde{y}_k\left(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_k(\alpha);\alpha\right) = \tilde{y}_{k-1}\left(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_{k-1}(\alpha);\alpha\right)-\\ \tilde{\mathcal{J}}^{(\beta)}\sum_{i=1}^{k}\tilde{S}_i(\alpha)\,\tilde{y}_{k-i}{}'\left(x,\tilde{S}_1(\alpha);\dots,\tilde{S}_{k-i}(\alpha);\alpha\right)-\tilde{S}_k(\alpha)\tilde{\mathcal{J}}^{(\beta)}(x+1)\\ +\sum_{i=1}^{k}\tilde{S}_i(\alpha)\tilde{y}_{k-i}\left(x,\tilde{S}_1(\alpha);\dots,\tilde{S}_{k-i}(\alpha);\alpha\right),\\ \qquad \tilde{y}_k(0;\alpha) = \tilde{y}_k{}'(0;\alpha)=0.\end{cases} \tag{5.80}$$

Using Mathematica 12 DSolve Package to find the solutions for the lower and the upper bounds for Eqs.(5.76)-(5.80), we obtain the following form:

$$\tilde{y}(x;\alpha)=\tilde{y}_0(x;\alpha)+\sum_{j=1}^{k}\tilde{y}_j\left(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_j(\alpha);\alpha\right) \tag{5.81}$$

We shall solve Eq. (5.67) by third-order FF-OHAM. Firstly, we will identify the optimal convergence constants $\tilde{S}_j$ for $\alpha=0.5$ then use these constants $\forall\alpha\in[0,1]$ such that

$\tilde{S}_1(0.5) = -1.0896927251803454,$

$\tilde{S}_2(0.5) = -0.0007773511599121089$, and

$\tilde{S}_3(0.5) = -0.002125392638088601 7.$

Table 5.9

The approximate solution and error of Eq. (5.67) by third-order FF-OHAM for $\beta = 1.9$ *at* $x = 0.5$ $\forall \alpha \in [0,1]$

α	$[\underline{ER}]_\alpha, \tilde{S}_j(0.5)$	$[\overline{ER}]_\alpha, \tilde{S}_j(0.5)$	$[\underline{y}]_\alpha, \tilde{S}_j(0.5)$	$[\overline{y}]_\alpha, \tilde{S}_j(0.5)$
0	7.52054×10^{-4}	-1.34252×10^{-3}	-3.15298	1.89557
0.25	5.77506×10^{-4}	-9.93427×10^{-4}	-2.73226	1.05415
0.50	4.02958×10^{-4}	-6.44331×10^{-4}	-2.31155	0.21272
0.75	2.28410×10^{-4}	-2.95235×10^{-4}	-1.89084	-0.62870
1	5.386150×10^{-5}	5.386150×10^{-5}	-1.47013	-1.47013

Figure 5.11 provides a summary of the approximate solutions of Eq.(5.67) of order $\beta = 1.9$ for all $\alpha \in [0,1]$ and $x \in [0,0.5]$ based on the optimal convergence parameters $S_j(0.5)$ in a three-dimensional graph.

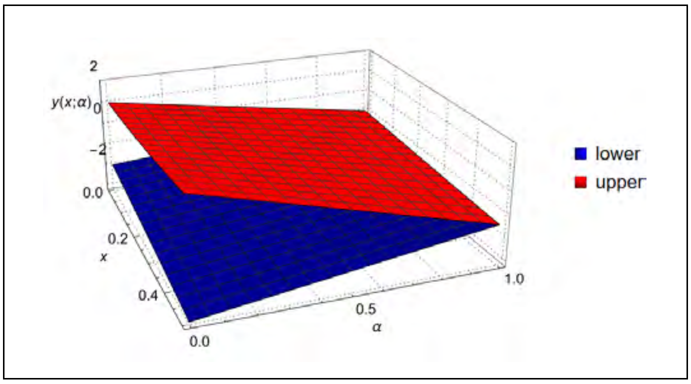

Figure 5.11. The three-dimensional approximate solution of Eq. (5.67) given by fifth-order FF-OHAM over all $x \in [0,0.5]$ at $\beta = 1.9$, and for all $\alpha \in [0,1]$

To analyse the convergence series dynamic for different series order, we employ the fifth-order FF-OHAM to solve Eq. (5.67). First, we identify the optimal convergence constants $\tilde{S}_j$ for $\alpha = 0.5$ and then use these constants $\forall\alpha \in [0,1]$ such that

$\tilde{S}_1(0.5) = -1.0513881401014404,$

$\tilde{S}_2(0.5) = -5.596203690495698 \times 10^{-7},$

$\tilde{S}_3(0.5) = 0.0000038273314625863\,26,$

$\tilde{S}_4(0.5) = 0.00002046681405703\,97$, and

$\tilde{S}_5(0.5) = -0.0025342033689142425.$

Table 5.10

The approximate solution and error of Eq. (5.67) by fifth-order FF-OHAM for $\beta = 1.9$ $at\ x = 0.5\ \forall\alpha \in [0,1]$

α	$[\underline{ER}]_\alpha, \tilde{S}_j(0.5)$	$[\overline{ER}]_\alpha, \tilde{S}_j(0.5)$	$[\underline{y}]_\alpha, \tilde{S}_j(0.5)$	$[\overline{y}]_\alpha, \tilde{S}_j(0.5)$
0	2.08400×10^{-7}	-4.14253×10^{-7}	-3.15306	1.89582
0.25	1.56509×10^{-7}	-3.10478×10^{-7}	-2.73232	1.05434
0.50	1.04622×10^{-7}	-2.06703×10^{-7}	-2.31158	0.21286
0.75	5.27344×10^{-8}	-1.02928×10^{-7}	-1.89084	-0.62862
1	8.46946×10^{-10}	8.46946×10^{-10}	-1.47010	-1.47010

Figure 5.12 gives a three-dimensional graph illustrating the summary of the solutions of Eq. (5.67) of order $\beta = 1.9$ by fifth-order FF-OHAM for all $\alpha \in [0,1]$ and $x \in [0,0.5]$ based on the optimal convergence parameters $\tilde{S}_j(0.5)$.

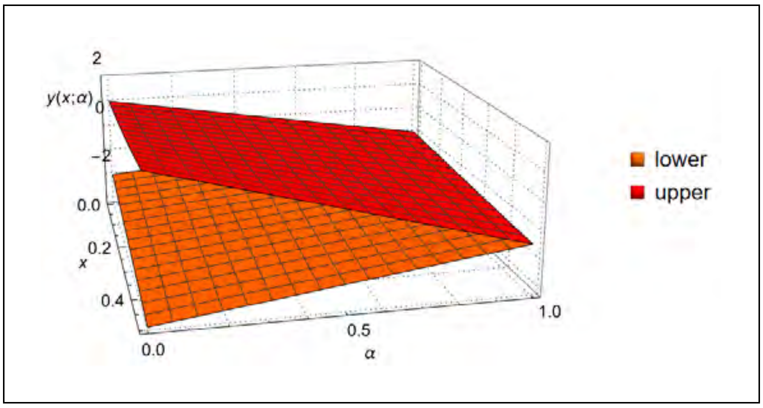

Figure 5.12. The three-dimensional approximate solution of Eq. (5.67) given by fifth-order FF-OHAM over all $x \in [0,0.5]$ at $\beta = 1.9$, and for all $\alpha \in [0,1]$

From Tables 5.9-5.10 and Figures 5.11-5.12, it can be concluded that the third-order as well as the fifth-order FF-OHAM solution series satisfy the triangular property of fuzzy numbers [Definition 3.7] of the fuzzy differential equations (Salahshour, 2011) for solving second-order FFOIVPs.

The lower and upper accuracy by third-order and fifth-order FF-OHAM solution series based on the best optimal convergence control parameter $\tilde{S}_j(0.5)$ are illustrated in the three-dimensional graphs in Figures 5.13-5.14.

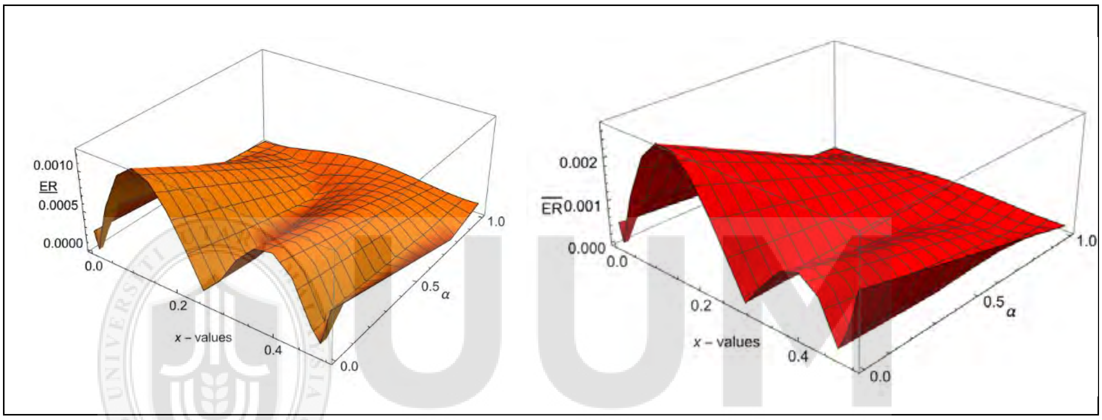

Figure 5.13. The accuracy of third-order FF-OHAM for solving Eq. (5.67) for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.5]$

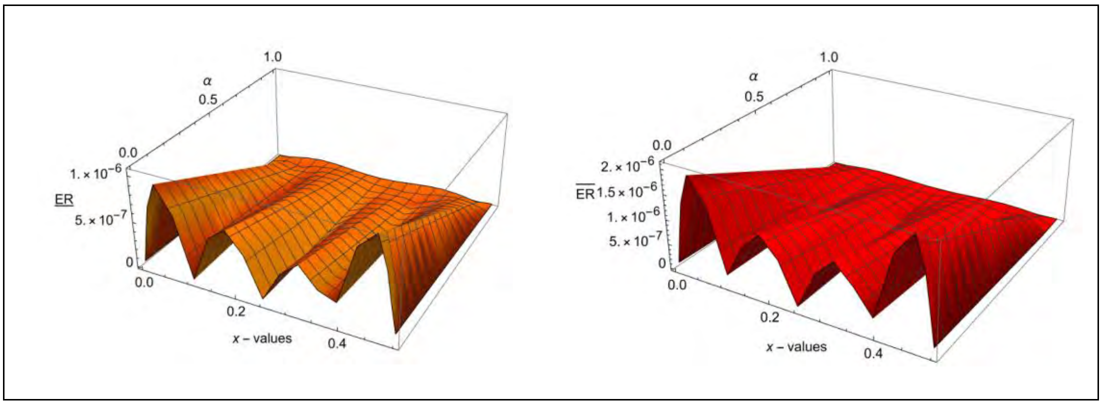

Figure 5.14. The accuracy of fifth-order FF-OHAM for solving Eq. (5.67) for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.5]$

Figures 5.13-5.14 clarify that the FF-OHAM series solution with higher series order provides more accurate solution.

5.5.2.3 Comparative Study of FF-HAM and FF-OHAM for Example 5.2

Comparison study is conducted for Example 5.2 where results are compared among fifth-order RKHSM (Hasan et al., 2017), fifth-order FF-HAM and fifth-order FF-OHAM.

We solve Eq. (5.67) of order $\beta = 2$ by fifth-order FF-HAM based on the optimal convergence control parameter $\tilde{h} = -1.0807134100933249$ and by fifth-order FF-OHAM with five convergence control parameters as tabulated in Table 5.11.

Table 5.11
Fuzzy convergence control parameters by fifth-order FF-OHAM for solving Eq. (5.67) at $\beta = 2, x = 0.5$ *and* $\alpha = 0.1$

$\underline{\boldsymbol{S}}_j(\boldsymbol{\alpha})$	$\overline{\boldsymbol{S}}_j(\boldsymbol{\alpha})$
$\underline{S}_1(0.1) = -1.05173952547119$	$\overline{S}_1(0.1) = -1.0514621170567935$
$\underline{S}_2(0.1) = -6.487940586239462 \times 10^{-7}$	$\overline{S}_2(0.1) = -5.764536242359766 \times 10^{-7}$
$\underline{S}_3(0.1) = 0.0000033007155045371 43$	$\overline{S}_3(0.1) = 0.000003751726288253184$
$\underline{S}_4(0.1) = 0.00002790913682120906$	$\overline{S}_4(0.1) = 0.000022007598326862107$
$\underline{S}_5(0.1) = -0.002640273481124126$	$\overline{S}_5(0.1) = -0.0025563888530125882$

Next, we will compare the numerical outputs by fifth-order FF-HAM and fifth-order FF-OHAM for Eq. (5.67) of order $\beta = 2$ with the fifth-order RKHSM (Hasan et al., 2017) for different values of α as given in Table 5.12.

Table 5.12

Numerical comparison of approximate solutions of Eq. (5.67) for different values of α *for* $x = 0.5$ *and* $\beta = 2$

$\boldsymbol{\alpha}$	$\underline{\boldsymbol{ER}}$ **HAM**	$\underline{\boldsymbol{ER}}$ **OHAM**	$\underline{\boldsymbol{ER}}$ **RKHS**
0.25	3.458934×10^{-6}	3.705105×10^{-8}	3.123591×10^{-5}
0.50	1.712647×10^{-6}	1.308628×10^{-8}	2.474548×10^{-5}

Table 5.13

Numerical comparison of approximate solutions of Eq. (5.67) for different values of α *for* $x = 0.5$ *and* $\beta = 2$

$\boldsymbol{\alpha}$	$\overline{\boldsymbol{ER}}$ **HAM**	$\overline{\boldsymbol{ER}}$ **OHAM**	$\overline{\boldsymbol{ER}}$ **RKHS**
0.25	1.225764×10^{-5}	-7.648175×10^{-8}	3.123591×10^{-5}
0.50	-8.765070×10^{-6}	-4.678749×10^{-8}	2.474548×10^{-5}

Based on Tables 5.12-5.13, it can be concluded that the developed methods: FF-HAM and FF-OHAM gives more accurate approximate solutions compared with RKHSM (Hasan et al., 2017). Comparing the two developed methods, Figures 5.9-5.10 and Figures 5.13-5.14 indicate that in solving second-order linear FFOIVPs, FF-OHAM provides more accurate solutions compared with FF-HAM for different values of the series order and for different values of fractional order β.

5.5.3 Example 5.3

Consider the following second-order nonlinear fractional Riccati differential equation (Momani & Shawagfeh, 2006)

$$\begin{cases} D^{(\beta)}y(x) + y^2(x) - 1 = 0, \quad 1 < \beta \leq 2, \\ \qquad y(0) = \sqrt{2}, \ \ y'(0) = -1, \end{cases} \tag{5.82}$$

Since this example is a non-fuzzy FODE, we will first introduce a new fuzzification of the equation. The fuzzy version of Eq. (5.82) is created as follows:

$$\begin{cases} D^{(\beta)}\tilde{y}(x;\alpha) + \tilde{y}^2(x;\alpha) - 1 = 0, \quad 0 < \beta \leq 2, \\ \qquad \tilde{y}(0) = \sqrt{2}, \ \ \tilde{y}'(0) = -1. \end{cases} \tag{5.83}$$

5.5.3.1 Solving Example 5.3 by FF-HAM

In this section, we will approximate the analytical solution of the fuzzy form of Example 5.3 by FF-HAM. Based on Section 5.3.2, we can construct the zeroth-order and k^{th}-order deformation equations of FF-HAM for Eq. (5.83) as follows:

$$\begin{cases} \underline{y}_k(x;\alpha) = \psi_k \underline{y}_{k-1}(x;\alpha) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)}\mathcal{R}_k(\underline{\vec{y}}_{k-1}(x;\alpha)) \\ \overline{y}_k(x;\alpha) = \psi_k \overline{y}_{k-1}(x;\alpha) + \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)}\mathcal{R}_k(\overline{\vec{y}}_{k-1}(x;\alpha)) \end{cases} \tag{5.84}$$

$$\underline{y}_k(0;\alpha) = \underline{y}'_k(0;\alpha) = 0, \overline{y}_k(0;\alpha) = \overline{y}'_k(0;\alpha) = 0.$$

such that

$$\begin{cases} \mathcal{R}_k\left(\underline{\vec{y}}_{k-1}(x;\alpha)\right) = \underline{y}_{k-1}{}^{(\beta)}(x;\alpha) + \sum_{k=0}^{m-1} \underline{y}_k(x;\alpha)\, \underline{y}_{m-1-k}(x;\alpha) - 1, \\ \mathcal{R}_k\left(\overline{\vec{y}}_{k-1}(x;\alpha)\right) = \overline{y}_{k-1}{}^{(\beta)}(x;\alpha) + \sum_{k=0}^{m-1} \overline{y}_k(x;\alpha)\, \overline{y}_{m-1-k}(x;\alpha) - 1. \end{cases} \tag{5.85}$$

For the initial approximation

$$\tilde{y}_0 = \tilde{y}(0) = \left[(\alpha - 1) + \left(2 - \alpha\left(2 - \sqrt{2}\right)\right)x, (1 - \alpha) + \left(2 - \alpha\left(2 - \sqrt{2}\right)\right)x\right] \quad (5.86)$$

Based on Eq. (5.85), we can obtain the FF-HAM series solution components for the lower and the upper bound, for $k = 1,2,3, \dots$ as follows:

For first-order problem

$$\begin{cases} \underline{y}_1(x;\alpha) = \psi_1 \underline{y}_0(x;\alpha) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \mathcal{R}_1\left(\vec{\underline{y}}_0(x;\alpha)\right), \\ \overline{y}_1(x;\alpha) = \psi_1 \overline{y}_0(x;\alpha) + \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)} \mathcal{R}_1\left(\vec{\overline{y}}_0(x;\alpha)\right), \end{cases} \quad (5.87)$$

Here $\psi_1 = 0$, and $\mathcal{R}_1\left(\widetilde{\vec{y}_0}(x;\alpha)\right) = \tilde{y}_0{}^{(\beta)}(x) + \tilde{y}_0{}^2(x) - 1$ and we can rewrite Eq.(5.87) as follows:

$$\begin{cases} \underline{y}_1(x;\alpha) = \underline{h}(\alpha)\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_0{}^{(\beta)}(x;\alpha) + \underline{y}_0{}^2(x;\alpha) - 1\right), \\ \overline{y}_1(x;\alpha) = \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_0{}^{(\beta)}(x;\alpha) + \overline{y}_0{}^2(x;\alpha) - 1\right), \end{cases} \quad (5.88)$$

Then, we have

$$\begin{cases} \underline{y}_1(x;\alpha) = \underline{h}(\alpha)\left(\underline{y}_0(x;\alpha) + \underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_0{}^2(x;\alpha) - 1\right)\right), \\ \overline{y}_1(x;\alpha) = \overline{h}(\alpha)\left(\overline{y}_0(x;\alpha) + \overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_0{}^2(x;\alpha) - 1\right)\right), \end{cases} \quad (5.89)$$

For second-order problem:

$$\begin{cases} \underline{y}_2(x;\alpha) = \psi_2 \underline{y}_1(x;\alpha) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta)} \mathcal{R}_2(\vec{\underline{y}}_1(x;\alpha)) \\ \overline{y}_2(x;\alpha) = \psi_2 \overline{y}_1(x;\alpha) + \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta)} \mathcal{R}_2(\vec{\overline{y}}_1(x;\alpha)) \end{cases} \quad (5.90)$$

Here, $\psi_2 = 1$, and $\mathcal{R}_2\left(\widetilde{\vec{y}_1}(x;\alpha)\right) = \tilde{y}_1{}^{(\beta)}(x) + \tilde{y}_1{}^2(x)$ and we can rewrite Eq. (5.90) as follows:

$$\begin{cases} \underline{y}_2(x;\alpha) = \underline{y}_1(x;\alpha) + \underline{h}(\alpha)\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_1{}^{(\beta)}(x;\alpha) + \underline{y}_1{}^2(x;\alpha)\right), \\ \overline{y}_2(x;\alpha) = \overline{y}_1(x;\alpha) + \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_1{}^{(\beta)}(x;\alpha) + \overline{y}_1{}^2(x;\alpha)\right), \end{cases} \tag{5.91}$$

Then, we have

$$\begin{cases} \underline{y}_2(x;\alpha) = \left(\underline{h}(\alpha)+1\right)\underline{y}_1(x;\alpha) + \underline{h}(\alpha)\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_1{}^2(x;\alpha)\right), \\ \overline{y}_2(x;\alpha) = \left(\overline{h}(\alpha)+1\right)\overline{y}_1(x;\alpha) + \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_1{}^2(x;\alpha)\right), \end{cases} \tag{5.92}$$

For third-order problem:

$$\begin{cases} \underline{y}_3(x;\alpha) = \left(\underline{h}(\alpha)+1\right)\underline{y}_2(x;\alpha) + \underline{h}(\alpha)\,\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_2{}^2(x;\alpha)\right), \\ \overline{y}_3(x;\alpha) = \left(\overline{h}(\alpha)+1\right)\overline{y}_2(x;\alpha) + \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_2{}^2(x;\alpha)\right). \end{cases} \tag{5.93}$$

For fourth-order problem:

$$\begin{cases} \underline{y}_4(x;\alpha) = \left(\underline{h}(\alpha)+1\right)\underline{y}_3(x;\alpha) + \underline{h}(\alpha)\,\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_3{}^2(x;\alpha)\right), \\ \overline{y}_4(x;\alpha) = \left(\overline{h}(\alpha)+1\right)\overline{y}_3(x;\alpha) + \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_3{}^2(x;\alpha)\right), \end{cases} \tag{5.94}$$

For fifth-order problem

$$\begin{cases} \underline{y}_5(x;\alpha) = \left(\underline{h}(\alpha)+1\right)\underline{y}_4(x;\alpha) + \underline{h}(\alpha)\,\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_4{}^2(x;\alpha)\right), \\ \overline{y}_5(x;\alpha) = \left(\overline{h}(\alpha)+1\right)\overline{y}_4(x;\alpha) + \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_4{}^2(x;\alpha)\right), \end{cases} \tag{5.95}$$

For sixth-order problem

$$\begin{cases} \underline{y}_6(x;\alpha) = \left(\underline{h}(\alpha)+1\right)\underline{y}_5(x;\alpha) + \underline{h}(\alpha)\,\underline{\mathcal{J}}^{(\beta)}\left(\underline{y}_5{}^2(x;\alpha)\right), \\ \overline{y}_6(x;\alpha) = \left(\overline{h}(\alpha)+1\right)\overline{y}_5(x;\alpha) + \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta)}\left(\overline{y}_5{}^2(x;\alpha)\right), \end{cases} \tag{5.96}$$

with sixth-order FF-HAM approximate series solution $\tilde{y}\left(x,\tilde{h};\alpha\right) = \sum_{j=0}^{6}\tilde{y}_j\left(x,\tilde{h};\alpha\right)$

We can demonstrate the accuracy of FF-HAM for Eq. (5.83) by taking the residual error as mentioned in Section 5.3.3, such that for $\beta \in (1,2]$ [Definition 3.22]

$$\begin{cases} \underline{ER}(x;\alpha;\underline{h}) = \left(\underline{y}^{(\beta)}(x;\alpha) + \left[\underline{y}(x;\alpha)\right]^2 - 1\right), \\ \overline{ER}(x;\alpha;\overline{h}) = \left(\overline{y}^{(\beta)}(x;\alpha) + [\overline{y}(x;\alpha)]^2 - 1\right). \end{cases} \tag{5.97}$$

As has been mentioned in previous examples, the series depend on $x, \alpha,$ and the convergent control-parameter $\tilde{h}$. Here, $\tilde{h}$ can be employed to adjust the convergence region of the homotopy analysis solution and we will use the properties of FF-HAM convergence in Section 5.3.3 to find the best optimal convergence control parameters.

The first step is to plot the valid interval of $\tilde{h}$ by a fixed value of $0 \leq \alpha \leq 1$ [Definition 3.5], for example $\alpha = 0.5$, as in Figure 5.15.

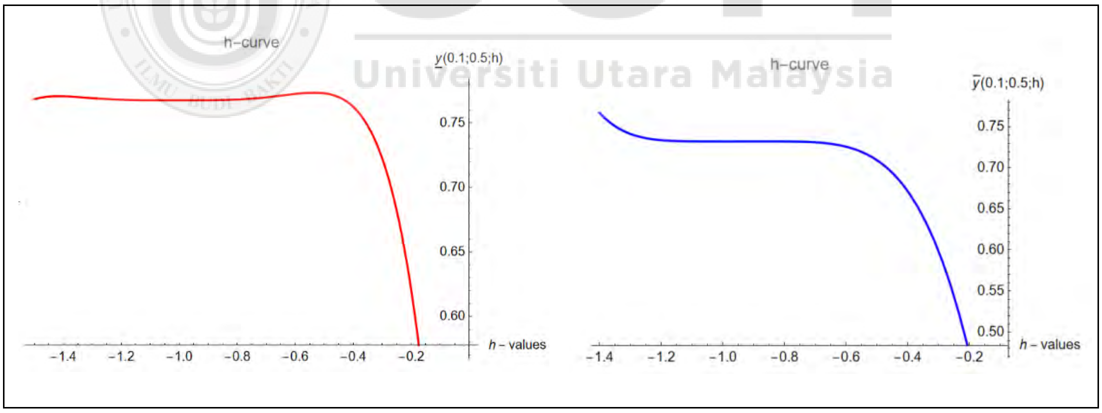

Figure 5.15. The $\tilde{h}(0.5)$-curves for the fuzzy solution of Eq. (5.83) given by the sixth-order FF-HAM for $\beta = 1.5$ and $H(x) = 1$

From the curves in Figure 5.15, it is easy to discover the valid region of $\tilde{h}(0.5)$, which corresponds to the line segment nearly parallel to the horizontal axis. From Figure 5.15 and Section 5.3.3, it is clear that the lower and upper bound of the FF-HAM series solution is convergent when $-1.4 \leq \tilde{h} \leq -0.6$ where the values are listed in Table 5.14.

Table 5.14

The optimal values of $\tilde{h}(0.5)$ by sixth-order FF-HAM for solving Eq. (5.83) for $\beta = 1.5$ and $H(x) = 1$

$\underline{\boldsymbol{y}}(\boldsymbol{x}, \underline{\boldsymbol{h}}(\mathbf{0.5}); \mathbf{0.5})$	$\underline{h}_1 \to -1.4175783308471548$	$\underline{h}_2 \to -1.0042701323832586$
$\overline{\boldsymbol{y}}(\boldsymbol{x}, \overline{\boldsymbol{h}}(\mathbf{0.5}); \mathbf{0.5})$	$\overline{h}_1 \to -0.9937520957278505$	$\overline{h}_2 \to -0.8317791701530487$

In the next step, we will check the accuracy of sixth-order FF-HAM for Eq. (5.83) by plotting the residual depending on the optimal fuzzy convergence control parameters in Table 5.14 where Figures 5.16 and 5.17 give the lower and upper solution accuracy, respectively. Figures 5.16-5.17 illustrate the accuracy of Eq. (5.83) by sixth-order FF-HAM corresponding to the optimal convergence control parameters.

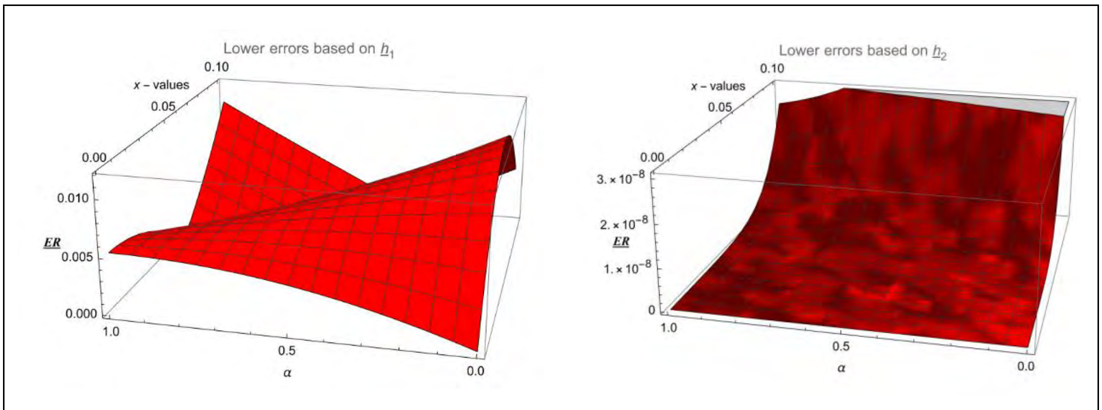

Figure 5.16. The lower solution accuracy of Eq. (5.83) of order $\beta = 1.5$ by sixth-order FF-HAM for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.1]$

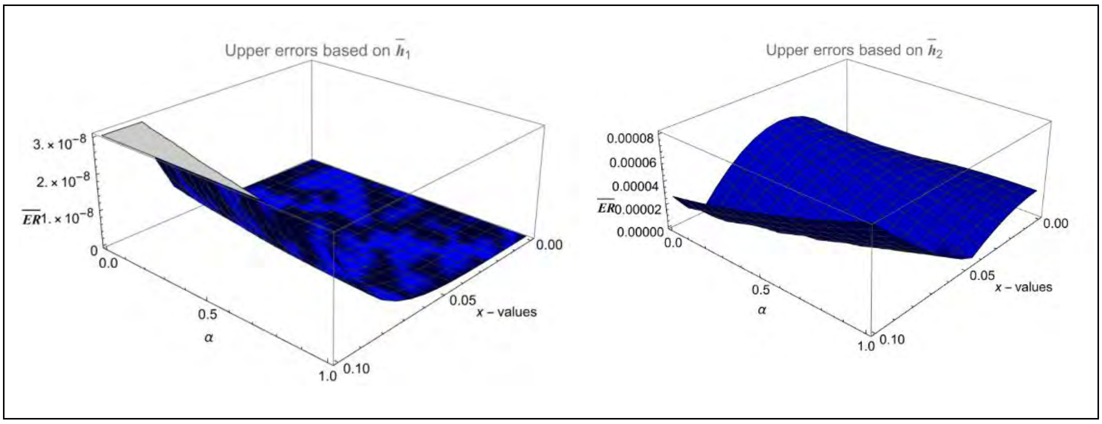

Figure 5.17. The upper solution accuracy of Eq. (5.83) of order $\beta = 1.5$ by sixth-order FF-HAM for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.1]$

In Table 5.15, we tabulate the residual errors $[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ of Eq.(5.83) of the approximate solutions $\underline{y}(0.1; \underline{h}_2; \alpha)$ and $\overline{y}(0.1; \overline{h}_1; \alpha)$ obtained using sixth-order FF-HAM according to the best optimal convergence control parameter $\tilde{h}(0.5) = [\underline{h}_2(0.5), \overline{h}_1(0.5)]$.

Table 5.15

The approximate solution and error of Eq. (5.83) by sixth-order FF-HAM when $\beta = 1.5$ *at* $x = 0.1$ *for all* $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\underline{\boldsymbol{ER}}]_{\alpha}, \underline{\boldsymbol{h}}_2$	$[\overline{\boldsymbol{ER}}]_{\alpha}, \overline{\boldsymbol{h}}_1$	$[\underline{\boldsymbol{y}}]_{\alpha}, \underline{\boldsymbol{h}}_2$	$[\overline{\boldsymbol{y}}]_{\alpha}, \overline{\boldsymbol{h}}_1$
0	1.35730×10^{-7}	1.22253×10^{-7}	-1	1
0.2	8.40226×10^{-8}	7.57969×10^{-8}	-0.79134	0.80847
0.4	5.44195×10^{-8}	4.98075×10^{-8}	-0.58465	0.61510
0.6	3.80973×10^{-8}	3.59015×10^{-8}	-0.37991	0.41987
0.8	2.95211×10^{-8}	2.89109×10^{-8}	-0.17710	0.22277
1	2.53579×10^{-8}	2.57975×10^{-8}	0.02379	0.02379

The summary of the obtained approximate solutions of Eq.(5.83) for all $\alpha \in [0,1]$ and $x \in [0,0.1]$ based on the optimal convergence parameters $\tilde{h}(0.5)$ is illustrated in the three-dimensional graph in Figure 5.18.

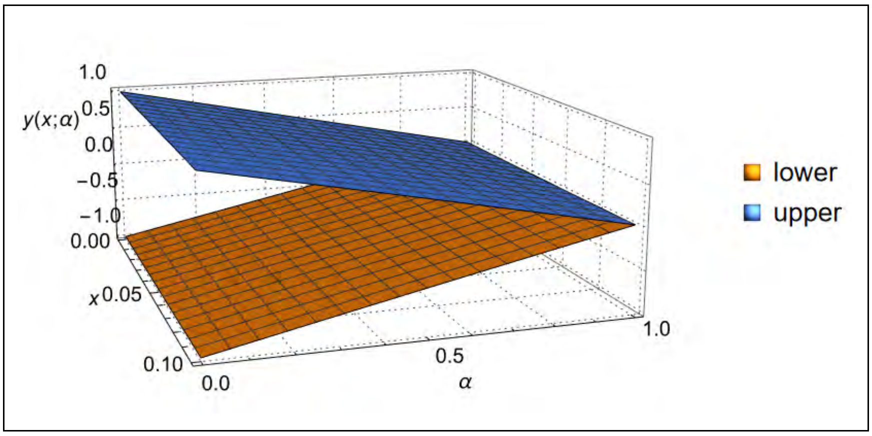

Figure 5.18. The three-dimensional approximate solution of Eq. (5.83) given by sixth-order FF-HAM over all $x \in [0,0.1]$ at $\beta = 1.5$ and for all $\alpha \in [0,1]$

To study the behaviour of second-order fractional differentiability, we shall proceed to solve Eq.(5.83) by sixth-order FF-HAM for $x \in [0,0.1]$ with different fractional order ($\beta = 1.9$) [Definition 3.22] to analyze the convergence dynamic of FF-HAM for solving second-order nonlinear FFOIVPs at a different fractional order.

Firstly, for a fixed value of $0 \leq \alpha \leq 1$ [Definition 3.5], for example $\alpha = 0.5$, we will plot $\tilde{h}(0.5)$ -curves for the lower bound $\underline{y}(0.1; 0.5; \underline{h})$ and upper bound $\overline{y}(0.1; 0.5; \overline{h})$ of the sixth-order FF-HAM for Eq.(5.83) for $\beta = 1.9, \alpha = 0.5$, and at $x = 0.1$ as in Figure 5.19 to extract the valid region of the convergence parameters.

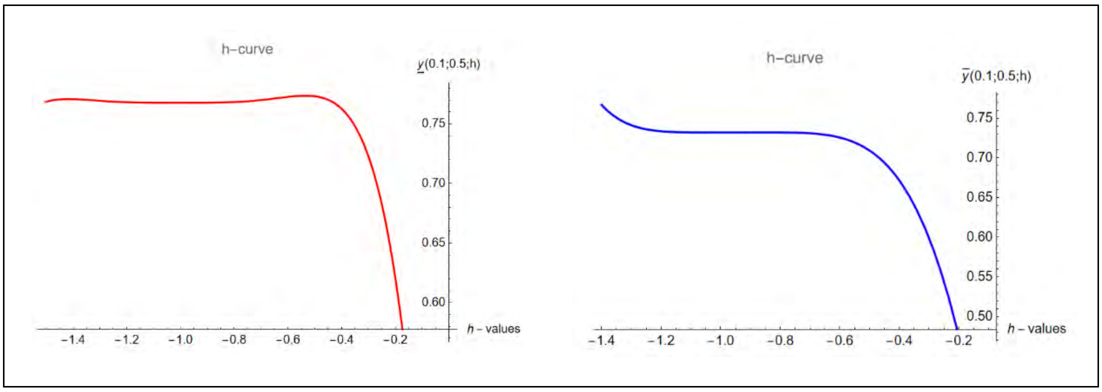

Figure 5.19. The $\tilde{h}(0.5)$-curves for the fuzzy solution of Eq. (5.83) given by sixth-order FF-HAM for $\beta = 1.9$ and $H(x) = 1$

From Figure 5.19, it is easy to conclude the valid region of the convergence control parameters, which corresponds to the line segment nearly parallel to the horizontal axis. The region has changed to $-1.4 \leq \underline{h} \leq -0.6$ for the lower bound and $-1.3 \leq \overline{h} \leq -0.5$ for the upper bound after changing the fractional order of Eq.(5.83) to $\beta = 1.9$.

The best optimal convergence control parameter is $\tilde{h} = [-1.000889336794365, -0.9987375276170942]$ of Eq. (5.83) for all $\alpha \in [0,1]$ is shown in the three-dimensional graph in Figure 5.20.

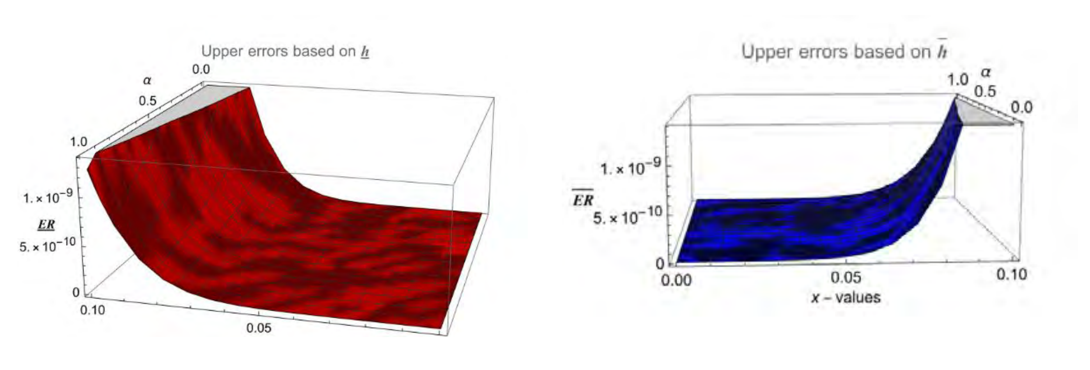

Figure 5.20. The accuracy of Eq. (5.83) of order $\beta = 1.9$ by sixth-order FF-HAM for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,0.1]$

In Table 5.16, the approximate solutions $\underline{y}(0.1; 0.5; \underline{h}), \overline{y}(0.1; 0.5; \overline{h})$ and the residual errors $\widetilde{ER}\left(0.1; 0.5; \tilde{h}(0.5)\right)$ obtained using sixth-order FF-HAM series solution are tabulated.

Table 5.16

The approximate solution and error of Eq. (5.83) by sixth-order FF-HAM when $\beta = 1.9$ at $x = 0.1$ for all $\alpha \in [0,1]$

α	$[\underline{ER}]_\alpha, \underline{h}_2$	$[\overline{ER}]_\alpha, \overline{h}_1$	$[\underline{y}]_\alpha, \underline{h}_2$	$[\overline{y}]_\alpha, \overline{h}_1$
0	5.33302×10^{-9}	5.18915×10^{-9}	-1	1
0.2	3.97659×10^{-9}	3.87722×10^{-9}	-0.79377	0.80622
0.4	2.96698×10^{-9}	2.90544×10^{-9}	-0.58892	0.61107
0.6	2.22023×10^{-9}	2.18954×10^{-9}	-0.38546	0.41453
0.8	1.67018×10^{-9}	1.66344×10^{-9}	-0.18338	0.21661
1	1.26520×10^{-9}	1.27608×10^{-9}	0.01731	0.01731

We summarize the fuzzy approximate solutions of Eq. (5.83) of order $\beta = 1.9$ for all $\alpha \in [0,1]$ and $x \in [0,0.1]$ based on the optimal convergence parameters $\tilde{h}(0.5)$ in the three-dimensional graph in Figure 5.21.

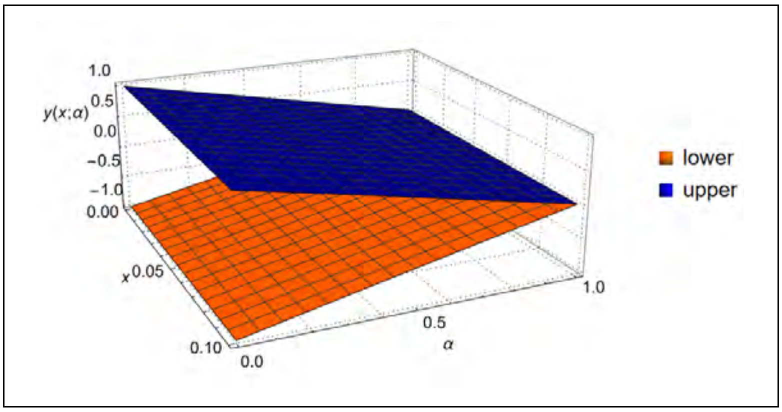

Figure 5.21. The three-dimensional approximate solution of Eq. (5.83) given by sixth-order FF-HAM over all $x \in [0,0.1]$ at $\beta = 1.9$ and for all $\alpha \in [0,1]$

Tables 5.15-5.16 as well as Figures 5.18 and 5.21 indicate that the sixth-order FF-HAM series solutions at $x = 0.1$ for all $\alpha \in [0,1]$ for fractional orders $\beta = 1.5$ and $\beta = 1.9$ satisfy the solution of the fuzzy numbers [Definition 3.7]. Finally, from Figures 5.16, 5.17 and 5.20, we conclude that the accuracy of the solution obtained by the sixth-order FF-HAM for Eq. (5.83) over all $x \in [0,0.1]$ and $\alpha \in [0,1]$ increases as the fractional order increases.

5.5.3.2 Solving Example 5.3 by FF-OHAM

In solving Example 5.3 by FF-OHAM, we will utilize FF-OHAM in Section 5.4.2 to obtain the approximate-analytical solution of Eq. (5.83). We can construct the sixth-order FF-OHAM series solution with six convergence control parameters $\left(\tilde{S}_1(\alpha),\right.$ $\tilde{S}_2(\alpha),\ \tilde{S}_3(\alpha),\ \tilde{S}_4(\alpha),\ \tilde{S}_5(\alpha),\ \left.\tilde{S}_6(\alpha)\right)$ for all $\alpha \in [0,1]$ as follows:

$$\begin{cases}(1-q)\left[\widetilde{D}^{(\beta)}\left(\tilde{y}(x;\alpha)\right)-1\right]=\sum_{j=1}^{k}\underline{S}_j(\alpha)q^j\left[\widetilde{D}^{(\beta)}\left(\tilde{y}(x;\alpha)\right)\right] \\ \sum_{j=1}^{k}\underline{S}_j(\alpha)q^j\left[\left(\tilde{y}(x;\alpha)\right)^2\right]-\sum_{j=1}^{k}\underline{S}_j(\alpha)q^j\end{cases} \tag{5.98}$$

Zeroth-order problem

$$\tilde{y}^{(\beta)}{}_0(x;\alpha)=\tilde{J}^{(\beta)}(1), \tag{5.99}$$

First-order to sixth-order problems

$$\begin{cases}(1-q)\left(\sum_{j=1}^{6}\tilde{y}_j(x;\alpha)\,q^j-1\right)=\sum_{j=1}^{6}\tilde{S}_j(\alpha)q^j \\ \left(\left(\sum_{j=0}^{6}\tilde{y}_j{}^{(\beta)}(x;\alpha)\,q^j\right)+\left(\sum_{j=0}^{6}\tilde{y}_j(x;\alpha)\,q^j\right)\left(\sum_{j=0}^{6}\tilde{y}_j(x;\alpha)\,q^j\right)-1\right) \\ \tilde{y}_j(0;\alpha)=0,\quad \tilde{y}_j{}'(0;\alpha)=0,\end{cases} \tag{5.100}$$

where $\alpha\in[0,1]$ and $j=1,\ldots,6$.

Using Mathematica 12 DSolve package to find the solutions for the lower and the upper bounds of Eq. (5.83), we obtain the following series solution:

$$\tilde{y}(x;\alpha)=\tilde{y}_0(x;\alpha)+\sum_{j=1}^{6}\tilde{y}_j\left(x,\tilde{S}_1(\alpha),\ldots,\tilde{S}_j(\alpha);\alpha\right). \tag{5.101}$$

By using the Least Squares method in Section 5.4.2, we compute the optimal values of $\tilde{S}_1(\alpha)$, $\tilde{S}_2(\alpha)$, $\tilde{S}_3(\alpha)$, $\tilde{S}_4(\alpha)$, $\tilde{S}_5(\alpha)$ and $\tilde{S}_6(\alpha)$ for all $\alpha\in[0,1]$ where the resulting values are tabulated in Tables 5.17-5.18.

Table 5.17

Lower auxiliary convergence parameters of sixth-order FF-OHAM for solving Eq. (5.83) at $\beta=1.5$, $x=0.1$, for all $\alpha\in[0,1]$

$\boldsymbol{\alpha}$	$\underline{\boldsymbol{S}}_{\boldsymbol{j}}$		
0	$\underline{S}_1=-1.0004887138340295$ $\underline{S}_4=-0.00000599452928979768$	$\underline{S}_2=-0.00014207368272515874$ $\underline{S}_5=-0.000004836253337873565$	$\underline{S}_3=-0.000014323031377039$ $\underline{S}_6=-0.00000496445612908855$
0.2	$\underline{S}_1=-1.0001007676640143$ $\underline{S}_4=-2.769871182180139\times10^{-7}$	$\underline{S}_2=-0.00022934368631077467$ $\underline{S}_5=8.806159946984293\times10^{-8}$	$\underline{S}_3=-0.000009002255322928274$ $\underline{S}_6=1.276655184271155\times10^{-8}$
0.4	$\underline{S}_1=-0.9998536558373152$ $\underline{S}_4=-0.000001050182785787753$	$\underline{S}_2=-0.00019019751810859416$ $\underline{S}_5=1.735390064638725\times10^{-7}$	$\underline{S}_3=0.000008866900518437845$ $\underline{S}_6=6.278569450921627\times10^{-9}$
0.6	$\underline{S}_1=-0.9997085958795981$ $\underline{S}_4=-6.625353531260645\times10^{-7}$	$\underline{S}_2=-0.00021008173696003186$ $\underline{S}_5=1.439257751591447\times10^{-7}$	$\underline{S}_3=0.00001401166552130945$ $\underline{S}_6=2.005572615092068\times10^{-9}$
0.8	$\underline{S}_1=-1.0001027813521826$ $\underline{S}_4=-3.865776693947103\times10^{-7}$	$\underline{S}_2=-0.00013211506085237837$ $\underline{S}_5=2.10300103147349\times10^{-7}$	$\underline{S}_3=0.000021718507180436755$ $\underline{S}_6=9.68036322614985\times10^{-10}$
1	$\underline{S}_1=-1.0000283704374622$ $\underline{S}_4=-6.96602423819664\times10^{-7}$	$\underline{S}_2=-0.00006466075683467308$ $\underline{S}_5=-2.682109208985597\times10^{-7}$	$\underline{S}_3=-0.000004617757525836149$ $\underline{S}_6=-1.982097812854571\times10^{-9}$

Table 5.18

Upper auxiliary convergence parameters of sixth-order FF-OHAM for solving Eq. (5.83) at $\beta = 1.5$, $x = 0.1$, for all $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$\overline{\boldsymbol{S}}_{\boldsymbol{j}}$		
0	$\overline{S}_1 = -0.9931349266366354$ $\overline{S}_4 = 5.36347133494925 \times 10^{-7}$	$\overline{S}_2 = -0.0005728248625590717$ $\overline{S}_5 = 8.513448851563934 \times 10^{-7}$	$\overline{S}_3 = -0.000016637334793403948$ $\overline{S}_6 = 8.852409659938568 \times 10^{-7}$
0.2	$\overline{S}_1 = -0.9953381954471$ $\overline{S}_4 = 4.510590100678629 \times 10^{-7}$	$\overline{S}_2 = -0.0001565431926408536$ $\overline{S}_5 = 2.841603533002552 \times 10^{-8}$	$\overline{S}_3 = 0.0000177606674418552 4$ $\overline{S}_6 = 4.240782702485477 \times 10^{-9}$
0.4	$\overline{S}_1 = -0.996176909894881$ $\overline{S}_4 = 7.333084141709041 \times 10^{-7}$	$\overline{S}_2 = -0.00016859020259173555$ $\overline{S}_5 = 6.799006223913968 \times 10^{-8}$	$\overline{S}_3 = 0.000021089481379971113$ $\overline{S}_6 = -1.675081950583952 \times 10^{-9}$
0.6	$\overline{S}_1 = -0.9995178554350718$ $\overline{S}_4 = 6.669523666199332 \times 10^{-7}$	$\overline{S}_2 = -0.00021820608394569673$ $\overline{S}_5 = 6.559944931775348 \times 10^{-8}$	$\overline{S}_3 = 0.000017739340008357773$ $\overline{S}_6 = -5.185550913323237 \times 10^{-12}$
0.8	$\overline{S}_1 = -0.9985120771402487$ $\overline{S}_4 = 4.098463466933682 \times 10^{-7}$	$\overline{S}_2 = -0.0003179368549449074$ $\overline{S}_5 = 8.522093311034238 \times 10^{-8}$	$\overline{S}_3 = 0.000001393024278481928$ $\overline{S}_6 = -1.965389730006275 \times 10^{-10}$
1	$\overline{S}_1 = -1.0000283764316484$ $\overline{S}_4 = -6.966112120957023 \times 10^{-7}$	$\overline{S}_2 = -0.00006465891640968417$ $\overline{S}_5 = -2.682166961013407 \times 10^{-7}$	$\overline{S}_3 = -0.000004617107974551897$ $\overline{S}_6 = -1.982097729252568 \times 10^{-9}$

Eq. (5.83) is solved to obtain sixth-order FF-OHAM series approximate solution at $x = 0.1$ for all $\alpha \in [0,1]$. The residual errors $[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ of the approximate solutions $\underline{y}(0.1;\alpha)$ and $\overline{y}(0.1;\alpha)$ for $\underline{S}_j(\alpha)$ and $\overline{S}_j(\alpha)$ $\forall\alpha \in [0,1]$ are tabulated in Table 5.19.

Table 5.19

The approximate solution and error of Eq. (5.83) by sixth-order FF-OHAM when $\beta = 1.5$ at $x = 0.1$ for all $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\underline{\boldsymbol{ER}}]_\alpha, \underline{\boldsymbol{S}}_j(\boldsymbol{\alpha})$	$[\overline{\boldsymbol{ER}}]_\alpha, \overline{\boldsymbol{S}}_j(\boldsymbol{\alpha})$	$[\underline{y}]_\alpha, \underline{\boldsymbol{S}}_j(\boldsymbol{\alpha})$	$[\overline{y}]_\alpha, \overline{\boldsymbol{S}}_j(\boldsymbol{\alpha})$
0	1.76288×10^{-8}	7.68966×10^{-10}	-1	1
0.2	7.88950×10^{-9}	1.88354×10^{-9}	-0.79134	0.80847
0.4	6.48876×10^{-9}	3.46536×10^{-10}	-0.58465	0.61510
0.6	3.35005×10^{-9}	8.85237×10^{-10}	-0.37991	0.41987
0.8	2.13918×10^{-9}	1.47407×10^{-10}	-0.17710	0.22277
1	3.62353×10^{-9}	3.62348×10^{-9}	0.02379	0.02379

The summary of the obtained approximate solutions of Eq. (5.83) of order $\beta = 1.5$ for all $\alpha \in [0,1]$ and $x \in [0,0.1]$ based on the optimal convergence parameters $\tilde{S}_J(\alpha)$ is illustrated in the three-dimensional graph in Figure 5.22.

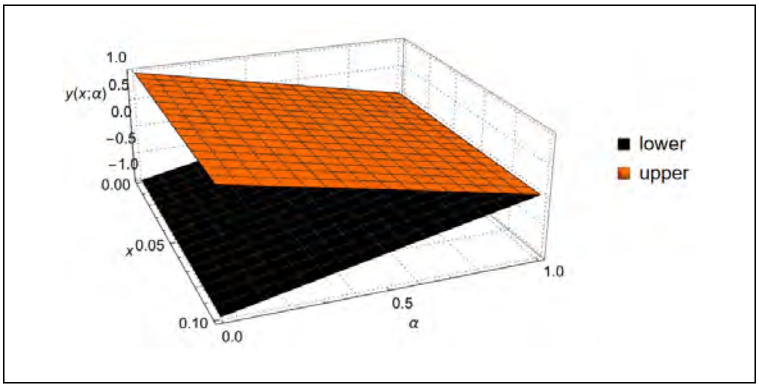

Figure 5.22. The three-dimensional approximate solution of Eq. (5.83) given by sixth-order FF-OHAM over all $x \in [0,0.1]$ at $\beta = 1.5$ and for all $\alpha \in [0,1]$

To study the behaviour of FF-OHAM in terms of fractional order, we proceed to solve Eq. (5.83) by sixth-order FF-OHAM for $x \in [0,0.1]$ with a different fractional order, that is, $\beta = 1.9$, to analyse the convergence dynamic of FF-OHAM for solving second-order nonlinear FFOIVPs. The convergence control parameters of sixth-order FF-OHAM for solving Eq. (5.83) at $\beta = 1.9$ are tabulated in Tables 5.20-5.21 for lower and upper auxiliary convergence parameters of sixth-order FF-OHAM for solving Eq. (5.83) at $\beta = 1.9$, $x = 0.1$ for all $\alpha \in [0,1]$.

Table 5.20

Lower auxiliary convergence parameters of sixth-order FF-OHAM for solving Eq. (5.83) at $\beta = 1.9$, $x = 0.1$ *for all* $\alpha \in [0,1]$

$\boldsymbol{\alpha}$		$\underline{\boldsymbol{S}}_{\boldsymbol{j}}$	
0	$\underline{S}_1 = -0.9999938567283975$	$\underline{S}_2 = -0.000014465219492706975$	$\underline{S}_3 = -2.14819722460408 \times 10^{-7}$
	$\underline{S}_4 = -3.761536183161015 \times 10^{-8}$	$\underline{S}_5 = 2.700197300544636 \times 10^{-8}$	$\underline{S}_6 = 2.121576023938977 \times 10^{-8}$
0.2	$\underline{S}_1 = -1.0000025006750264$	$\underline{S}_2 = -0.000009119188273547559$	$\underline{S}_3 = -2.717774220190315 \times 10^{-7}$
	$\underline{S}_4 = -5.817612504964532 \times 10^{-8}$	$\underline{S}_5 = 8.931839196476066 \times 10^{-9}$	$\underline{S}_6 = 8.82507231850276 \times 10^{-10}$
0.4	$\underline{S}_1 = -1.000082848924036$	$\underline{S}_2 = -0.000028909083409395104$	$\underline{S}_3 = 0.000001591008748323679$
	$\underline{S}_4 = -4.621328489849156 \times 10^{-8}$	$\underline{S}_5 = 7.931348270591485 \times 10^{-9}$	$\underline{S}_6 = 3.143362860026618 \times 10^{-10}$
0.6	$\underline{S}_1 = -1.0000986732596662$	$\underline{S}_2 = -0.00006989310426590048$	$\underline{S}_3 = -3.594276440624393 \times 10^{-7}$
	$\underline{S}_4 = 1.48452142144658 \times 10^{-8}$	$\underline{S}_5 = -3.34320031245028 \times 10^{-9}$	$\underline{S}_6 = -2.644094908098222 \times 10^{-11}$
0.8	$\underline{S}_1 = -0.9997940583768458$	$\underline{S}_2 = -0.0000592101463172233$	$\underline{S}_3 = -0.000001379170317937283$
	$\underline{S}_4 = 9.369322795465534 \times 10^{-9}$	$\underline{S}_5 = -3.578044877335095 \times 10^{-9}$	$\underline{S}_6 = -3.409710142995377 \times 10^{-12}$
1	$\underline{S}_1 = -0.9999828432467399$	$\underline{S}_2 = -0.00009853711010003435$	$\underline{S}_3 = -0.000004162784202806852$
	$\underline{S}_4 = -9.828328135360575 \times 10^{-7}$	$\underline{S}_5 = -7.101344525252669 \times 10^{-7}$	$\underline{S}_6 = -1.954300427855333 \times 10^{-10}$

Table 5.21

Upper auxiliary convergence parameters of sixth-order FF-OHAM for solving Eq. (5.83) at $\beta = 1.9$, $x = 0.1$ *for all* $\alpha \in [0,1]$

$\boldsymbol{\alpha}$		$\overline{\boldsymbol{S}}_{\boldsymbol{j}}$	
0	$\overline{S}_1 = -1.000323172929518$	$\overline{S}_2 = -0.000053517316749387566$	$\overline{S}_3 = -0.0000001682857459177062$
	$\overline{S}_4 = -1.44919062774774 \times 10^{-7}$	$\overline{S}_5 = -7.544042363434265 \times 10^{-8}$	$\overline{S}_6 = -7.000185299174572 \times 10^{-8}$
0.2	$\overline{S}_1 = -1.000373681900666$	$\overline{S}_2 = -0.00005673276413094351$	$\overline{S}_3 = -7.91781823192615 \times 10^{-7}$
	$\overline{S}_4 = -3.883986537909889 \times 10^{-8}$	$\overline{S}_5 = -3.475716058260577 \times 10^{-9}$	$\overline{S}_6 = 4.749018369284861 \times 10^{-10}$
0.4	$\overline{S}_1 = -0.9999851150965483$	$\overline{S}_2 = -0.000008878992505018405$	$\overline{S}_3 = 9.319742535655644 \times 10^{-7}$
	$\overline{S}_4 = -5.311849181123145 \times 10^{-8}$	$\overline{S}_5 = -6.494577094950955 \times 10^{-9}$	$\overline{S}_6 = 3.092036354931575 \times 10^{-10}$
0.6	$\overline{S}_1 = -1.000111371940197$	$\overline{S}_2 = -0.00001806358329426273$	$\overline{S}_3 = 8.664875249879387 \times 10^{-7}$
	$\overline{S}_4 = -3.778009417708393 \times 10^{-8}$	$\overline{S}_5 = -6.323509709197772 \times 10^{-9}$	$\overline{S}_6 = 1.403527301508976 \times 10^{-10}$
0.8	$\overline{S}_1 = -1.0000518821998297$	$\overline{S}_2 = -0.00006902538703088462$	$\overline{S}_3 = -0.000001913173239228596$
	$\overline{S}_4 = 3.868536880090515 \times 10^{-9}$	$\overline{S}_5 = 1.142924386530773 \times 10^{-9}$	$\overline{S}_6 = 2.513010309051114 \times 10^{-12}$
1	$\overline{S}_1 = -0.9999836324038701$	$\overline{S}_2 = -0.00009823694350052632$	$\overline{S}_3 = -0.000004166787186161021$
	$\overline{S}_4 = -9.75822443328472 \times 10^{-7}$	$\overline{S}_5 = -7.050868990007115 \times 10^{-7}$	$\overline{S}_6 = -1.946301994960142 \times 10^{-10}$

This is followed by the calculation of the residual errors $[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ of the approximate solutions $\underline{y}(0.1;\alpha)$ and $\overline{y}(0.1;\alpha)$ obtained using sixth-order FF-OHAM series solution for $\beta = 1.9$ by utilizing the convergence control parameters $\tilde{S}_j$ listed in Tables 5.20-5.21. The calculated residuals are tabulated in Table 5.22.

Table 5.22

The approximate solution and error of Eq. (5.83) by sixth-order FF-OHAM when $\beta = 1.9$ *at* $x = 0.1$ *for all* $\alpha \in [0,1]$

α	$\lfloor ER \rfloor_{\alpha}, \underline{S}_j$	$\lceil ER \rceil_{\alpha}, \bar{S}_j$	$\lfloor y \rfloor_{\alpha}, \underline{S}_j$	$\lceil \bar{y} \rceil_{\alpha}, \bar{S}_j$
0	6.95721×10^{-10}	6.39249×10^{-10}	-1	1
0.2	4.91577×10^{-10}	3.15250×10^{-10}	-0.79751	0.80248
0.4	3.19145×10^{-10}	3.22608×10^{-10}	-0.59558	0.60440
0.6	1.12294×10^{-11}	2.05593×10^{-10}	-0.39421	0.40578
0.8	2.75586×10^{-11}	3.66438×10^{-11}	-0.19338	0.20661
1	4.35295×10^{-12}	4.83494×10^{-12}	0.00689	0.00689

We summarize the solution of Eq. (5.83) of order $\beta = 1.9$ by sixth-order FF-OHAM for all $\alpha \in [0,1]$ and $x \in [0,0.1]$, and based on the optimal convergence parameters $\tilde{S}_k(\alpha)$, in the three-dimensional graph in Figure 5.23.

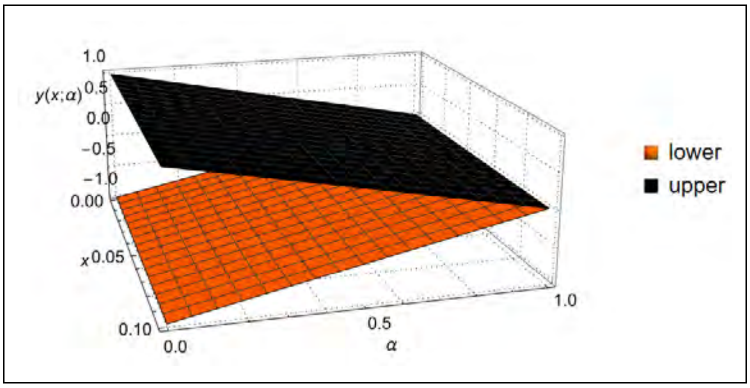

Figure 5. 23. The three-dimensional approximate solution of Eq. (5.83) given by sixth-order FF-HAM over all $x \in [0,0.1]$ at $\beta = 1.9$ and for all $\alpha \in [0,1]$

Figures 5.22-5.23, Table 5.19 and Table 5.22 indicate that the sixth-order FF-OHAM series solutions at $x = 0.1$ for all $\alpha \in [0,1]$ for fractional orders $\beta = 1.5$ and $\beta = 1.9$ satisfy the solution of the fuzzy numbers [Definition 3.7]. In addition, we conclude that the accuracy of the solution obtained by the sixth-order FF-OHAM for Eq. (5.83) over all $x \in [0,0.1]$ and $\alpha \in [0,1]$ increases as the fractional order increases. Finally, from the results of the approximate solution of the nonlinear FFOIVP in Eq. (5.83), it is concluded that FF-OHAM provides more accurate solution compared with FF-HAM.

5.6 Summary of Findings

1. In this chapter, the approximate-analytical methods FF-HAM and FF-OHAM with convergence-control ability have been successfully developed for linear and nonlinear FFOIVPs of order $1 < \beta \leq 2$.

2. The convergence dynamic of the proposed methods that has been introduced in Chapter 4 is valid for FFOIVPs of order $1 < \beta \leq 2$, where it is used to select the optimal convergence parameter for each method for the purpose of controlling the series solution.

3. FF-HAM and FF-OHAM have been successfully utilized to solve the inhomogeneous linear second-order FFIOVPs (without first order derivative). The findings are as follows:

   a) FF-HAM and FF-OHAM provide accurate series solution for solving linear problem.

b) Employing the convergence dynamic of FF-HAM and FF-OHAM has successfully provide us optimal convergence parameters for FF-HAM and FF-OHAM.

c) Generally, FF-OHAM yields more accurate solution than FF-HAM.

d) FF-HAM and FF-OHAM yield more accurate solutions compared with RKHSM.

e) All fuzzy fractional solutions by FF-HAM and FF-OHAM satisfy the triangular fuzzy solution.

f) Figures and tables provide illustrations of results including optimal parameters, valid region of the convergence parameters, errors and solutions.

4. FF-HAM and FF-OHAM have been successfully applied to solve the inhomogeneous linear second-order FFIOVPs containing first-order derivative. The findings are as follows:

a) FF-HAM and FF-OHAM provide an accurate series solution for solving linear case contain first order derivative.

b) Employing our convergence dynamic of FF-HAM and FF-OHAM were successfully to provide us optimal convergence parameters for FF-HAM and FF-OHAM

c) Generally, FF-OHAM more accurate than FF-HAM

d) As the FF-HAM and FF-OHAM solution series order increases, the accuracy of solution also increases and the obtained solution converges to the exact solution.

e) FF-HAM and FF-OHAM yield more accurate solution compared with RKHSM.

f) All fuzzy fractional solutions by FF-HAM and FF-OHAM satisfy the triangular fuzzy solution.

g) Figures and tables provide illustrations of results including optimal parameters, valid region of the convergence parameters, errors and solutions.

5. We introduce a new fuzzy version of fractional Riccati differential equation, followed by the utilization of FF-HAM and FF-OHAM to find the series solution of this nonlinear problem. The findings are as follows:

a) FF-HAM and FF-OHAM provide accurate series solution for solving non-linear problem.

b) Employing the convergence dynamic of FF-HAM and FF-OHAM has successfully provide us the optimal convergence parameters for FF-HAM and FF-OHAM.

c) For the linear case, we study the convergence dynamic based on the change the series order, for nonlinear case we study the convergence dynamic based on the change of the fractional order derivative.

d) As the fractional order derivative increases, the accuracy of the solution by FF-HAM and FF-OHAM increases, and the obtained solution converges to the exact solution.

e) Generally, FF-OHAM yields more accurate solution compared with FF-HAM for nonlinear case, but it will be more costly in terms of CPU.

f) All fuzzy fractional solution by FF-HAM and FF-OHAM satisfy the triangular fuzzy solution for nonlinear case.

g) Figures and tables provide illustrations of results including optimal parameters, valid region of the convergence parameters, errors and solutions.

6. FF-HAM and FF-OHAM provide more accurate solutions for solving second-order FFODEs compared with the first-order FFODEs, on the other hand, FF-OHAM clearly demonstrate its huge ability to control the convergence area for solving second-order FFOIVPs unlike the first-order FFOIVPs.

CHAPTER SIX
FF-HAM AND FF-OHAM FOR SECOND-ORDER FUZZY FRACTIONAL ORDINARY BOUNDARY VALUE PROBLEMS

6.1 Introduction

This chapter discusses the development of the proposed FF-HAM and FF-OHAM on second-order fuzzy fractional ordinary boundary value problems (FFOBVPs), which is the third-type of FFODEs considered in this study. This chapter firstly elaborates on the theoretical part of the development of the methods consisting of three steps. This is followed by the experimental part of the development which shows the solving process of several second-order FFOBVPs using the developed methods. Discussion of the developed methods in terms of characteristics and advantages based on the theoretical and experimental parts of the study are also provided towards the end of the chapter.

6.2 Theoretical Development of FF-HAM and FF-OHAM for Second-order FFOBVPs

The theoretical development of the proposed methods consists of three steps as follows:

1. Defuzzification of FFODEs

In this step, defuzzification the fuzzy fractional ordinary differential equation is done under the sense of the second-order fuzzy fractional Caputo derivative. Furthermore, it includes defuzzification of the fuzzy boundary conditions for lower and upper bound based on the extension principle theory and properties of the fuzzy calculus such as α-cut definition and fuzzy number definition.

2. Construction of FF-HAM and FF-OHAM for second-order FFOBVPs

In this step, FF-HAM and FF-OHAM are constructed based on the extension principle theory and properties of the fuzzy calculus such as α-cut definition and fuzzy number definition to handle the multi second-order FFOBVPs.

3. Establishment of the convergence of FF-HAM and FF-OHAM solution series for second-order FFOBVPs

Establishing the convergence analyses of FFHAM and FFOHAM on second-order FFOBVPs depends on the minimization of the residual form of the given problem in order to select the optimal convergence control parameter of FF-HAM (h) and the optimal convergence control parameters of FF-OHAM ($S_1, S_2, S_3, S_4, \ldots S_k$) (here k is the order of the FF-OHAM series)

The theoretical development of each of the two proposed methods follows the three steps. However, Step One needs to be done only once since it is applicable for both methods. In this study, the construction of FF-HAM will be done first, followed by FF-OHAM.

6.3 Theoretical Development of FF-HAM for Second-order FFOBVPs

For second-order FFOBVPs, the proposed FF-HAM is theoretically developed as follows:

6.3.1 Defuzzification of FFODE

The first step of the development of the proposed methods for solving second-order FFOBVPs is the defuzzification step. This is a general step which applies to both methods.

Consider the second-order FFOBVP as follows:

$$\begin{cases} \tilde{y}^{(\beta_1)}(x) = \tilde{g}\left(x, \tilde{y}(x), \tilde{y}^{(\beta_2)}(x)\right), \ \ x \in [x_0, X], \\ \qquad\qquad 1 < \beta_1 \leq 2, \ \ 0 < \beta_2 \leq 1, \end{cases} \tag{6.1}$$

subject to the following boundary conditions:

$$\begin{cases} \tilde{y}(x_0) = \tilde{a}_0, \ \tilde{y}^{(1)}(x_0) = \tilde{a}_1, \\ \tilde{y}(X) = \tilde{b}_0, \ \tilde{y}^{(1)}(X) = \tilde{b}_1, \end{cases} \tag{6.2}$$

where $\tilde{g}$ is the fuzzy function as in Definition 3.8 while $\tilde{y}^{(\beta_1)}(x)$ and $\tilde{y}^{(\beta_2)}(x)$ are as given in Definitions 3.22 and 3.21 respectively, giving fuzzy fractional derivatives in Caputo sense; and the boundary conditions at the points x_0 and X are fuzzy numbers as defined in Definition 3.7.

For $x \in [x_0, X]$ and $\forall\ \alpha \in [0,1]$ by Definition 3.5, the fuzzy function will be defined by $[\tilde{y}]_\alpha = [\underline{y}, \overline{y}]_\alpha$ based on Definition 3.8 as follows:

$[\tilde{y}(x)]_\alpha = \left[\underline{y}(x;\alpha), \overline{y}(x;\alpha)\right]$, $\left[\tilde{y}^{(\beta_2)}(x)\right]_\alpha = \left[\underline{y}^{(\beta_2)}(x;\alpha), \overline{y}^{(\beta_2)}(x;\alpha)\right]$,

$[\tilde{y}(x_0)]_\alpha = \left[\underline{y}(x_0;\alpha), \overline{y}(x_0;\alpha)\right]$, $\left[\tilde{y}^{(\beta_2)}(x_0)\right]_\alpha = \left[\underline{y}^{(\beta_2)}(x_0;\alpha), \overline{y}^{(\beta_2)}(x_0;\alpha)\right]$, and

$[\tilde{y}(X)]_\alpha = \left[\underline{y}(X;\alpha), \overline{y}(X;\alpha)\right]$, $\left[\tilde{y}^{(\beta_2)}(X)\right]_\alpha = \left[\underline{y}^{(\beta_2)}(X;\alpha), \overline{y}^{(\beta_2)}(X;\alpha)\right]$.

where $\left[\tilde{y}^{(\beta_2)}(x)\right]_\alpha = \left[\underline{y}^{(\beta_2)}(x;\alpha), \overline{y}^{(\beta_2)}(x;\alpha)\right]$, $\left[\tilde{y}^{(\beta_2)}(x_0)\right]_\alpha = \left[\underline{y}^{(\beta_2)}(x_0;\alpha), \overline{y}^{(\beta_2)}(x_0;\alpha)\right]$ and $\left[\tilde{y}^{(\beta_2)}(X)\right]_\alpha = \left[\underline{y}^{(\beta_2)}(X;\alpha), \overline{y}^{(\beta_2)}(X;\alpha)\right]$ are obtained based on Theorem 3.1.

Now, by assuming $\tilde{\hat{Y}}(x) = \{\tilde{y}(x), \tilde{y}^{(\beta_2)}(x)\}$, for defuzzification we have

$$\tilde{\hat{Y}}(x,\alpha) = \left[\underline{\hat{Y}}(x,\alpha), \overline{\hat{Y}}(x,\alpha)\right] = [\underline{y}(x,\alpha), \underline{y}^{(\beta_2)}(x,\alpha), \overline{y}(x;\alpha), \overline{y}^{(\beta_2)}(x;\alpha)].$$

In addition, we can write the α −cut of the fuzzy function as

$$\left[\tilde{g}\left(x, \tilde{\hat{Y}}\right)\right]_\alpha = \left[\underline{g}\left(x, \tilde{\hat{Y}};\alpha\right), \overline{g}\left(x, \tilde{\hat{Y}};\alpha\right)\right] \tag{6.3}$$

Now, by utilizing the concepts of the extension principle theory given in Definition 3.4, we get

$$\tilde{g}\left(x, \tilde{\hat{Y}}(x;\alpha)\right) = \left[\underline{g}\left(x, \tilde{\hat{Y}}(x;\alpha)\right), \overline{g}\left(x, \tilde{\hat{Y}}(x;\alpha)\right)\right], \tag{6.4}$$

where

$$\begin{aligned} \underline{g}\left(x, \tilde{\hat{Y}}(x;\alpha)\right) &= g_l\left(x, \underline{\hat{Y}}(t;\alpha), \overline{\hat{Y}}(x;\alpha)\right) = g_l\left(x, \overline{\hat{Y}}(x;\alpha)\right), \\ \overline{g}\left(x, \tilde{\hat{Y}}(x;\alpha)\right) &= g_u\left(x, \underline{\hat{Y}}(x;\alpha), \overline{\hat{Y}}(x;\alpha)\right) = g_u\left(x, \overline{\hat{Y}}(x;\alpha)\right). \end{aligned} \tag{6.5}$$

which means that $\forall\ \alpha \in [0,1]$, we have

$$\underline{y}^{(\beta_1)}(x;\alpha) = g_l\left(x, \tilde{\hat{Y}}(x;\alpha)\right), \tag{6.6}$$

$$\overline{y}^{(\beta_1)}(x;\alpha) = g_u\left(x, \tilde{\hat{Y}}(x;\alpha)\right), \tag{6.7}$$

where

$$g_l\left(x, \widetilde{\Upsilon}(x;\alpha)\right) = \min\left\{\tilde{y}^{(\beta_1)}(x, \tilde{\mu}(\alpha))\middle|\tilde{\mu}(\alpha) \in \left[\widetilde{\Upsilon}(x;\alpha)\right]_\alpha\right\}, \text{ and}$$

$$g_u\left(x, \widetilde{\Upsilon}(x;\alpha)\right) = \max\left\{\tilde{y}^{(\beta_1)}(x, \tilde{\mu}(\alpha))\middle|\tilde{\mu}(\alpha) \in \left[\widetilde{\Upsilon}(x;\alpha)\right]_\alpha\right\}$$

based on Definition 3.4.

6.3.2 Construction of FF-HAM for Second-order FFOBVPs

The procedure in the construction of the proposed FF-HAM involves reformulation of the existing F-HAM for solving crisp second-order fractional BVPs followed by the formulation of FF-HAM for second-order FFOBVPs.

This section presents the steps towards obtaining the structure of the proposed FF-HAM to approximate the analytical solution of second-order FFOBVPs. Consider the following nonlinear FFOBVP:

$$\begin{cases} \tilde{y}^{(\beta_1)}(x) = \tilde{g}\left(x, \tilde{y}(x), \tilde{y}^{(\beta_2)}(x)\right) + \widetilde{\mathrm{G}}(x)\ x \in [x_0, X], \\ \tilde{y}(x_0) = \tilde{a}_0,\ \tilde{y}'(x_0) = \tilde{a}_1, \\ \tilde{y}(X) = \tilde{b}_0,\ \tilde{y}'(X) = \tilde{b}_1, \\ \beta_1 \in (1,2], \beta_2 \in (0,1], \end{cases} \tag{6.8}$$

here, $\tilde{y}(x)$ refers to the unknown fuzzy function [Definition 3.8] of crisp variable x, $\tilde{y}^{(\beta_1)}(x)$ [Definition 3.22], and $\tilde{y}^{(\beta_2)}(x)$ [Definition 3.21] are the fuzzy fractional Caputo derivative of $\tilde{y}(x)$ for $\beta_1 \in (1,2]$ and $\beta_2 \in (0,1]$, $\tilde{g}$ is the fuzzy function [Definition 3.8] of the fuzzy variable $\tilde{y}$, first fractional derivative of the fuzzy variable $\tilde{y}$, and crisp variable x, $\widetilde{\mathrm{G}}(x)$ represents the inhomogeneous fuzzy term of crisp variable x, and $\tilde{a}_0$, $\tilde{a}_1$, $\tilde{b}_0$, and $\tilde{b}_1$ are fuzzy numbers [Definition 3.7]. Now, considering Eq.(6.8) followed by the defuzzification of Eq.(6.1) such that for all $\alpha \in$

[0,1] [Definition 3.5] and according to the FF-HAM in Section 4.3.2, we can construct the following correction functional as follows:

$$\begin{cases} \underline{\mathcal{L}}_{\beta_1}[\underline{y}(x;q)]_\alpha = \mathcal{N}\left[\underline{\hat{Y}}(x;\alpha)\right], \\ \overline{\mathcal{L}}_{\beta_1}[\underline{y}(x;q)]_\alpha = \mathcal{N}\left[\overline{\hat{Y}}(x;\alpha)\right]. \end{cases} \tag{6.9}$$

According to the extension principle theory [Definition 3.4], we have:

$$\begin{cases} g_l([\tilde{y}(x;q;\alpha)]) = min\{\mathcal{N}[\underline{y}(x;q)]_\alpha, \mathcal{N}[\overline{y}(x;q)]_\alpha : \mu|\mu \in [\tilde{\hat{Y}}(x;\alpha)]\}, \\ g_u([\tilde{y}(x;q;\alpha)]) = max\{\mathcal{N}[\underline{y}(x;q)]_\alpha, \mathcal{N}[\overline{y}(x;q)]_\alpha : \mu|\mu \in [\tilde{\hat{Y}}(x;\alpha)]\}, \end{cases} \tag{6.10}$$

here, μ referred to the membership function of Eq. (6.8) where the definition of this membership function is given in Definition 3.1. We can define the zeroth-order deformation equation constructed by HAM (Liao, 2004) as follows:

$$\begin{cases} (1-q)\underline{\mathcal{L}}_{\beta_1}\left[[\underline{y}(x;q)]_\alpha - \underline{y}_0(x;\alpha)\right] = q\underline{h}(\alpha)H(x)\, g_l([\tilde{y}(x;q)]_\alpha) \\ (1-q)\overline{\mathcal{L}}_{\beta_1}\left[[\overline{y}(x;q)]_\alpha - \overline{y}_0(x;\alpha)\right] = q\overline{h}(\alpha)H(x)\, g_u([\tilde{y}(x;q)]_\alpha) \end{cases}. \tag{6.11}$$

As has been discussed in Section 4.3.2 and Section 5.3.2, $0 \le q \le 1$ represents the embedding parameter, $\tilde{h}(\alpha) = \left[\underline{h}(\alpha), \overline{h}(\alpha)\right]$ is a nonzero convergence control parameter, $H(x)$ is the auxiliary function while the operators $\underline{\mathcal{L}}_{\beta_1} = \frac{\partial^{(\beta_1)}\left[\underline{y}(x;q)\right]_\alpha}{\partial x^{(\beta_1)}}$ and $\overline{\mathcal{L}}_{\beta_1} = \frac{\partial^{(\beta_1)}\overline{y}[(x;q)]_\alpha}{\partial x^{(\beta_1)}}$ [Definition 3.22] are the auxiliary linear operators. Now, we can define the initial approximation of the lower and upper bound $[\tilde{y}_0(x)]_\alpha = \left[\underline{y}_0(x;\alpha), \overline{y}_0(x;\alpha)\right]$ by the rule of solution expression (Liao, 1999). From Eq.(6.11), the approximate solution $\tilde{y}(x;\alpha)$ can be expressed for $k = 0,1,2,..$ by a set of base functions x^k, then the approximate solution $\tilde{y}(x;\alpha)$ can be expressed as $\tilde{y}(x;\alpha) = \sum_{j=0}^{k}\left[\tilde{d}^j\right]_\alpha x^j$. where $\left[\tilde{d}^j\right]_\alpha$, are fuzzy coefficients to be determined. According to the rule of solution expression $\sum_{j=0}^{k}\left[\tilde{d}^{j}\right]_{\alpha}x^{j}$, we have the following form of the initial guess:

$$\begin{cases}\underline{y}_0(x;\alpha)=\underline{S}_1(\alpha)+\underline{S}_2(\alpha)x,\\ \overline{y}_0(x;\alpha)\ =\overline{S}_1(\alpha)+\overline{S}_2(\alpha)x,\end{cases} \tag{6.12}$$

where for all $\alpha\in[0,1]$, $\tilde{S}_1(\alpha)$, and $\tilde{S}_2(\alpha)$ are the constants that can be determined easily from the boundary conditions in Eq. (6.8). The two-point fuzzy boundary conditions of Eq.(6.8) are

$$\begin{cases}[\tilde{y}(x_0;q)]_{\alpha}=\tilde{a}_0, & \frac{\partial}{\partial x}\tilde{a}_0=\tilde{a}_1,\\ [\tilde{y}(X;q)]_{\alpha}=\tilde{b}_0, & \frac{\partial}{\partial x}\tilde{b}_0=\tilde{b}_1.\end{cases} \tag{6.13}$$

It can be concluded that for $q=0$ and $q=1$, we have:

$$\begin{cases}\left[\underline{y}(x;0)\right]_{\alpha}=\underline{y}(x_0;\alpha),\\ [\overline{y}(x;0)]_{\alpha}=\overline{y}(x_0;\alpha).\end{cases}\quad \begin{cases}\left[\underline{y}(x;1)\right]_{\alpha}=\underline{Y}(x;\alpha),\\ [\overline{y}(x;1)]_{\alpha}=\overline{Y}(x;\alpha).\end{cases} \tag{6.14}$$

At a time when the embedding parameter changes from zero to one, the solution $\left[\underline{y}(x;q)\right]_{\alpha}, [\overline{y}(x;q)]_{\alpha}$ deforms from the initial approximation $\left[\underline{y}(x;0)\right]_{\alpha}, [\overline{y}(x;0)]_{\alpha}$ to the exact solution $\underline{Y}(x;\alpha), \overline{Y}(x;\alpha)$. Now by expanding the approximate solution $\left[\underline{y}(x;q)\right]_{\alpha}, [\overline{y}(x;q)]_{\alpha}$ as a Taylor series with respect to q, $\forall\alpha\in[0,1]$ we have

$$\begin{cases}\left[\underline{y}(x;q)\right]_{\alpha}=\underline{y}_0(x;\alpha)+\sum_{j=1}^{k}\underline{y}_j(x;\alpha)\,q^{j}\\ [\overline{y}(x;q)]_{\alpha}=\overline{y}_0(x;\alpha)+\sum_{j=1}^{k}\overline{y}_j(x;\alpha)\,q^{j}\end{cases} \tag{6.15}$$

where

$$\begin{cases}\underline{y}_j(x;\alpha)=\frac{1}{j!}\frac{\partial^{j}\left[\underline{y}(x;q)\right]_{\alpha}}{\partial q^{j}}\Big|_{q=0}\\ \overline{y}_j(x;\alpha)=\frac{1}{j!}\frac{\partial^{j}[\overline{y}(x;q)]_{\alpha}}{\partial q^{j}}\Big|_{q=0}\end{cases}. \tag{6.16}$$

As has been mentioned in Section 4.3.2, and Section 5.3.2 if the auxiliary linear operator $\tilde{\mathcal{L}}_{\beta_1}$ [Definition 3.22], the initial guess $\tilde{y}_0(x;\alpha)$, the auxiliary function $H(x)$, and the convergence control parameter $\tilde{h}(\alpha)$, are all properly chosen, the HAM series solution will converge to the exact solution at $q = 1$ such that

$$\begin{cases} \underline{Y}(x;\alpha) = \underline{y}_0(x;\alpha) + \sum_{j=1}^{\infty} \underline{y}_j(x;\alpha) \\ \overline{Y}(x;\alpha) = \overline{y}_0(x;\alpha) + \sum_{j=1}^{\infty} \overline{y}_j(x;\alpha) \end{cases}. \tag{6.17}$$

As we mentioned in Section 5.3.2, in most cases it is impossible to find the analytical solution with HAM as an infinite series. Now, we define the vectors

$$\begin{cases} \underline{\vec{y}}_j(x;\alpha) = \{\underline{y}_0(x;\alpha), \underline{y}_1(x;\alpha), \dots, \underline{y}_k(x;\alpha)\} \\ \overline{\vec{y}}_j(x;\alpha) = \{\overline{y}_0(x;\alpha), \overline{y}_1(x;\alpha), \dots, \overline{y}_k(x;\alpha)\} \end{cases}. \tag{6.18}$$

Now, by differentiating the zeroth-order deformation equation k times with respect to the embedding parameter q, followed by setting $q = 0$ and next dividing them by $k!$ we can extract the k^{th}-order deformation equation as follows:

$$\begin{cases} \underline{\mathcal{L}}_{\beta_1}\left[\underline{y}_k(x;\alpha) - \psi_k \underline{y}_{k-1}(x;\alpha)\right] = \underline{h}(\alpha)\mathcal{R}_k(\underline{\vec{y}}_{k-1}(x;\alpha)) \\ \overline{\mathcal{L}}_{\beta_1}\left[\overline{y}_k(x;\alpha) - \psi_k \overline{y}_{k-1}(x;\alpha)\right] = \overline{h}(\alpha)\mathcal{R}_k(\overline{\vec{y}}_{k-1}(x;\alpha)) \end{cases} \tag{6.19}$$

where

$$\begin{cases} \mathcal{R}_k\left(\underline{\vec{y}}_{k-1}(x;\alpha)\right) = \frac{1}{(k-1)!}\frac{\partial^{k-1} g_l([\tilde{y}(x;q)]_\alpha)}{\partial q^{k-1}}\Big|_{q=0} \\ \mathcal{R}_k\left(\overline{\vec{y}}_{k-1}(x;\alpha)\right) = \frac{1}{(k-1)!}\frac{\partial^{k-1} g_u([\tilde{y}(x;q)]_\alpha)}{\partial q^{k-1}}\Big|_{q=0} \end{cases} \tag{6.20}$$

The solution of the k^{th}-order deformation for $k \geq 1$ is:

$$\begin{cases} \underline{y}_k(x;\alpha) = \\ \psi_k\left(\underline{y}_{k-1}(x;\alpha)\right) + \underline{h}(\alpha)\,\underline{\mathcal{J}}^{(\beta_1)}\mathcal{R}_k(\vec{\underline{y}}_{k-1}(x;\alpha)) \\ \overline{y}_k(x;\alpha) = \\ \psi_k\left(\overline{y}_{k-1}(x;\alpha)\right) + \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta_1)}\mathcal{R}_k(\vec{\overline{y}}_{k-1}(x;\alpha)) \end{cases}, \tag{6.21}$$

such that

$$\psi_k = \begin{cases} 0, & k \leq 1 \\ 1, & k > 1 \end{cases},$$

where $\tilde{\mathcal{J}}^{(\beta_1)} = \left[\underline{\mathcal{J}}^{(\beta_1)}, \overline{\mathcal{J}}^{(\beta_1)}\right] = \tilde{\mathcal{L}}_{\beta_1}^{\,-1}$ are the fuzzy Riemann-Liouville integral of order $\beta_1 \in (1,2]$.

6.3.3 Establishment of Convergence of FF-HAM Solution Series for Second-order FFOBVPs

We can apply the convergence analysis of FF-HAM in Section 4.3.3 to Eq. (6.8).

6.4 Theoretical Development of FF-OHAM for Second-order FFOBVPs

For second-order FFOBVPs, the proposed FF-HAM is theoretically developed in three steps.

6.4.1 Defuzzification of FFODE

The first step of the development of the proposed methods for solving second-order FFOBVPs is the defuzzification step. Step One needs to be done only once since it is applicable for both FF-HAM and FF-OHAM. Step One has been done in Section 6.3.1.

6.4.2 Construction of FF-OHAM for Second-order FFOBVPs

The procedure in the construction of the proposed FF-OHAM involves reformulation of the existing F-OHAM for solving crisp fractional BVPs followed by the formulation of FF-OHAM for second-order FFOBVPs.

In this section, we fuzzify F-OHAM and then defuzzify it to obtain an approximate solution of linear and nonlinear second-order FFOBVPs using the properties and definitions of fuzzy set theory [Definition 3.1] to provide a new procedure for solving linear and nonlinear second-order FFOBVPs. Considering Eq. (6.8) followed by the defuzzification of Eq.(6.1) such that for all $\alpha \in [0,1]$ [as in Definition 3.5] we have the following lower bound:

$$\begin{cases} \underline{\mathcal{L}}_{\beta_1}\left(\tilde{y}(x;\alpha)\right) - g_l\left(x, \tilde{\hat{\mathrm{Y}}}(x)\right) - \underline{\mathrm{G}}(x) = 0, \;\; x \in [x_0, X], \\ ß\left(\underline{y}(x;\alpha), \frac{\partial[\underline{y}]_\alpha}{\partial x}\right) = 0. \end{cases} \tag{6.22}$$

and the following upper bound:

$$\begin{cases} \overline{\mathcal{L}}_{\beta_1}\left(\tilde{y}(x;\alpha)\right) - g_u\left(x, \tilde{\hat{\mathrm{Y}}}(x)\right) - \overline{\mathrm{G}}(x) = 0, \;\; x \in [x_0, X], \\ ß\left(\overline{y}(x;\alpha), \frac{\partial[\overline{y}]_\alpha}{\partial x}\right) = 0. \end{cases} \tag{6.23}$$

According to F-OHAM in Section 3.2.5, Eqs. (6.22)-(6.23) can be written as follows:

$$\begin{cases} (1-q)\left[\underline{\mathcal{L}}_{\beta_1}\left(\left[\underline{y}(x;q)\right]_\alpha\right) - \underline{\mathrm{G}}(x;\alpha)\right] = \underline{\mathcal{H}}(q;\alpha)\left[\underline{\mathcal{L}}_{\beta_1}\left(\left[\underline{y}(x;q)\right]_\alpha\right)\right] \\ \quad -\underline{\mathcal{H}}(q;\alpha)\left[\underline{\mathrm{G}}(x;\alpha)\right] - \underline{\mathcal{H}}(q;\alpha)[g_l([\tilde{y}(x;q)]_\alpha)] \\ (1-q)\left[\overline{\mathcal{L}}_{\beta_1}([\overline{y}(x;q)]_\alpha) - \overline{\mathrm{G}}(x;\alpha)\right] = \overline{\mathcal{H}}(q;\alpha)\left[\overline{\mathcal{L}}_{\beta_1}([\overline{y}(x;q)]_\alpha)\right] \\ \quad -\overline{\mathcal{H}}(q;\alpha)\left[\overline{\mathrm{G}}(x;\alpha)\right] - \overline{\mathcal{H}}(q;\alpha)[g_l([\tilde{y}(x;q)]_\alpha)] \end{cases} \tag{6.24}$$

subject to the following boundary condition

$$ß\left([\tilde{y}(x;q)]_\alpha, \frac{\partial[\tilde{y}(x;q)]_\alpha}{\partial x}\right) = 0, \tag{6.25}$$

where $\tilde{\mathcal{L}}_{\beta_1} = [\underline{\mathcal{L}}_{\beta_1}, \overline{\mathcal{L}}_{\beta_1}] = \left[\frac{\partial^{(\beta_1)}[\underline{y}(x;q)]_\alpha}{\partial x^{(\beta_1)}}, \frac{\partial^{(\beta_1)}[\overline{y}(x;q)]_\alpha}{\partial x^{(\beta_1)}}\right]$ are the linear operators and $q \in [0,1]$ is an embedding parameter. Here $\widetilde{\mathcal{H}}(q;\alpha) = [\underline{\mathcal{H}}(q), \overline{\mathcal{H}}(q)]_\alpha$ is a nonzero auxiliary fuzzy function for $q \neq 0$, and $[\tilde{y}(x;q)]_\alpha$ is an unknown fuzzy function.

Obviously, for $q = 0$ and $q = 1$, we get

$$\begin{cases} [\underline{y}(x;0)]_\alpha = \underline{y}_0(x;\alpha), & [\underline{y}(x;1)]_\alpha = \underline{Y}(x;\alpha), \\ [\overline{y}(x;0)]_\alpha = \overline{y}_0(t;\alpha). & [\overline{y}(x;1)]_\alpha = \overline{Y}(x;\alpha). \end{cases} \tag{6.26}$$

Thus, as q increases from 0 to 1, the solution $[\tilde{y}(x;q)]_\alpha$changes from $\tilde{y}_0(x;\alpha)$ to the solution of Eqs. (6.22)-(6.23) $\tilde{Y}(x;\alpha)$, where $\tilde{y}_0(x;\alpha)$ is obtained from Eq. (6.24) for $q = 0$ as follows:

$$\begin{cases} \underline{y}_0(x;\alpha) = \tilde{\mathcal{J}}^{(\beta_1)}\, \underline{\mathrm{G}}(x;\alpha), \\ \overline{y}_0(x;\alpha) = \tilde{\mathcal{J}}^{(\beta_1)}\, \overline{\mathrm{G}}(x;\alpha), \end{cases} \tag{6.27}$$

subject to the following boundary condition

$$ß\left(\tilde{y}_0(x;\alpha), \frac{\partial[\tilde{y}_0]_\alpha}{\partial x}\right) = 0. \tag{6.28}$$

We choose the auxiliary function $\widetilde{\mathcal{H}}(q;\alpha)$ for Eq. (6.24) in the following form:

$$\begin{cases} \underline{\mathcal{H}}(q;\alpha) = \underline{S}_1(\alpha)q + \underline{S}_2(\alpha)q^2 + \cdots = \sum_{j=1}^{k} \underline{S}_j(\alpha)q^j, \\ \underline{\mathcal{H}}(q;\alpha) = \overline{S}_1(\alpha)q + \overline{S}_2(\alpha)q^2 + \cdots = \sum_{j=1}^{k} \overline{S}_j(\alpha)q^j, \end{cases} \tag{6.29}$$

where $\tilde{S}_1(\alpha) = [\underline{S}_1(\alpha), \overline{S}_1(\alpha)], \tilde{S}_2(\alpha) = [\underline{S}_2(\alpha), \overline{S}_2(\alpha)], \ldots$ are the constants to be found for all $\alpha \in [0,1]$ [Definition 3.5]. Expanding $\left[\tilde{y}\left(x; q, S_j(\alpha)\right)\right]_\alpha$ into Taylor's series about q, we obtain the following approximate series solution:

$$\begin{cases} \left[\underline{y}\left(x;q,\underline{S}_j(\alpha)\right)\right]_\alpha = \underline{y}_0(x;\alpha) + \sum_{j=1}^{k}\left[\underline{y}_j\left(x,\underline{S}_j(\alpha)\right)\right]_\alpha q^j, \\ \left[\overline{y}\left(x;q,\overline{S}_j(\alpha)\right)\right]_\alpha = \overline{y}_0(x;\alpha) + \sum_{j=1}^{k}\left[\overline{y}_j\left(x,\overline{S}_j(\alpha)\right)\right]_\alpha q^j. \end{cases} \tag{6.30}$$

Now, as in Section 5.4.2, substitute Eq. (6.30) into Eq. (6.24) then by collecting the coefficient of like powers of q, we will obtain the following linear equations where the zeroth-order problem is given by Eq. (6.27), while the first and second-order problems are given as follows:

First-order Problem:

$$\begin{cases} \underline{y}_1(x;\alpha) = \underline{y}_0(x;\alpha) + \underline{\mathcal{J}}^{(\beta_1)}\left(\underline{S}_1(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right) + \underline{G}(x;\alpha)\right), \\ \overline{y}_1(x;\alpha) = \overline{y}_0(x;\alpha) + \overline{\mathcal{J}}^{(\beta_1)}\left(\overline{S}_1(\alpha) g_{u_0}\left(\tilde{y}_0(x;\alpha)\right) + \overline{G}(x;\alpha)\right), \end{cases} \tag{6.31}$$

$$ß\left(\tilde{y}_1(x;\alpha), \frac{\partial[\tilde{y}_1]_\alpha}{\partial x}\right) = 0. \tag{6.32}$$

Second-order Problem

$$\begin{cases} \underline{y}_2(x;\alpha) = \left(1 + \underline{S}_1(\alpha)\right) \underline{y}_1(x;\alpha) + \\ \underline{\mathcal{J}}^{(\beta_1)}\left(\underline{S}_2(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right) + \underline{S}_1(\alpha) g_{l_1}\left(\tilde{y}_0(x;\alpha), \tilde{y}_1(x;\alpha)\right)\right) \\ \overline{y}_2(x;\alpha) = \left(1 + \overline{S}_1(\alpha)\right) \overline{y}_1(x;\alpha) + \\ \overline{\mathcal{J}}^{(\beta_1)}\left(\overline{S}_2(\alpha) g_{u_0}\left(\tilde{y}_0(x;\alpha)\right) + \overline{S}_1(\alpha) g_{u_1}\left(\tilde{y}_0(x;\alpha), \tilde{y}_1(x;\alpha)\right)\right) \end{cases} \tag{6.33}$$

$$ß\left(\tilde{y}_2(x;\alpha), \frac{\partial[\tilde{y}_2]_\alpha}{\partial x}\right) = 0. \tag{6.34}$$

.

.

.

.

Then, the general k^{th} -order formula with respect to $\tilde{y}_k(x;\alpha)$ is given by:

$$\begin{cases} \underline{y}_k(x;\alpha) = \underline{y}_{k-1}(x;\alpha) + \sum_{j=1}^{k-1} \underline{S}_j(\alpha)\left(\underline{y}_{k-j}(x;\alpha)\right) \\ \underline{\mathcal{J}}^{(\beta_1)}\left(\underline{S}_k(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right) + \sum_{j=1}^{k-1} \underline{S}_j(\alpha) g_{l_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)\right) \\ \overline{y}_k(x;\alpha) = \overline{y}_{k-1}(x;\alpha) + \sum_{j=1}^{k-1} \overline{S}_j(\alpha)\left(\overline{y}_{k-j}(x;\alpha)\right) \\ \overline{\mathcal{J}}^{(\beta_1)}\left(\overline{S}_k(\alpha) g_{l_0}\left(\tilde{y}_0(x;\alpha)\right) + \sum_{j=1}^{k-1} \overline{S}_j(\alpha) g_{u_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)\right) \end{cases} \quad (6.35)$$

$$ß\left(\tilde{y}_k(x;\alpha), \frac{\partial[\tilde{y}_k]_\alpha}{\partial x}\right) = 0. \quad (6.36)$$

where $g_{l_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)$ and $g_{u_{k-j}}\left(\sum_{i=0}^{k-1} \tilde{y}_i(x;\alpha)\right)$ are the coefficients of q^j in the expansion of $g_l[\tilde{y}(x;q)]_\alpha$ and $g_u[\tilde{y}(x;q)]_\alpha$ about the embedding parameter q. We have the lower and upper bound as follows:

$$\begin{cases} g_l\left(\left[\tilde{y}\left(x;q,\sum_{j=1}^{k} \tilde{S}_j(\alpha)\right)\right]_\alpha\right) = g_{l_0}(\tilde{y}_0(x;\alpha)) + \sum_{j=1}^{k} g_{l_j}\left(\sum_{j=0}^{k}[\tilde{y}_j]_\alpha\right) q^j, \\ g_u\left(\left[\tilde{y}\left(x;q,\sum_{j=1}^{k} \tilde{S}_j(\alpha)\right)\right]_\alpha\right) = g_{u_0}(\tilde{y}_0(x;\alpha)) + \sum_{j=1}^{k} g_{u_j}\left(\sum_{j=0}^{k}[\tilde{y}_j]_\alpha\right) q^j. \end{cases} \quad (6.37)$$

It has been observed that the convergence of the series in Eq. (6.30) depends upon the auxiliary constants $\tilde{S}_1(\alpha), \tilde{S}_2(\alpha), \ldots$, then at $q = 1$, we obtain the exact solution:

$$\begin{cases} \left[\underline{Y}\left(x, \sum_{j=1}^{\infty} \underline{S}_j(\alpha)\right)\right]_\alpha = \underline{y}_0(x;\alpha) + \sum_{j=1}^{\infty}\left[\underline{y}_j\left(x; \sum_{j=1}^{\infty} \underline{S}_j(\alpha)\right)\right]_\alpha \\ \left[\overline{Y}\left(x, \sum_{j=1}^{\infty} \overline{S}_j(\alpha)\right)\right]_\alpha = \overline{y}_0(x;\alpha) + \sum_{j=1}^{\infty}\left[\overline{y}_j\left(x; \sum_{j=1}^{\infty} \overline{S}_j(\alpha)\right)\right]_\alpha \end{cases} \quad (6.38)$$

Substituting Eq. (6.30) into Eq. (6.22) and Eq. (6.23) yields the following residual:

$$\begin{cases} \underline{RE}\left(x, \sum_{j=1}^{k} \underline{S}_j(\alpha); \alpha\right) = \underline{\mathcal{L}}_{\beta_1}\left(\underline{y}\left(x, \sum_{j=1}^{k} \underline{S}_j(\alpha); \alpha\right)\right) - \underline{\mathrm{G}}(x;\alpha) \\ \qquad - g_l\left(\tilde{y}\left(x, \sum_{j=1}^{k} \tilde{S}_j(\alpha); \alpha\right)\right), \\ \overline{RE}\left(x, \sum_{j=1}^{k} \overline{S}_j(\alpha); \alpha\right) = \overline{\mathcal{L}}_{\beta_1}\left(\overline{y}\left(x, \sum_{j=1}^{k} \overline{S}_j(\alpha); \alpha\right)\right) - \overline{\mathrm{G}}(x;\alpha) \\ \qquad - g_u\left(\tilde{y}\left(x, \sum_{j=1}^{k} \tilde{S}_j(\alpha); \alpha\right)\right). \end{cases} \quad (6.39)$$

Similar to previous encounters with respect to OHAM, if $\widetilde{RE} = 0$, then $\tilde{y}$ yields the exact solution although generally it does not happen, especially in nonlinear FFOBVPs. To identify the auxiliary constants of $\tilde{S}_j(\alpha), j = 1,2,..k$, we choose x_0 and X such that the optimum values of $\tilde{S}_j(\alpha)$ for the convergent solution of the desired problem is obtained. To find the optimal values of $\tilde{S}_j(\alpha)$ here, we apply the method of Least Squares as follows:

$$\widetilde{SRE}\left(x, \sum_{j=1}^{k} \tilde{S}_j(\alpha)\,; \alpha\right) = \int_{x_0}^{X} \widetilde{RE}^2\left(x, \sum_{j=1}^{k} \tilde{S}_j(\alpha)\,; \alpha\right)\, dx, \tag{6.40}$$

where $\widetilde{RE}$ is the residual,

$$\begin{cases} \left[\underline{RE}\right]_\alpha = \underline{\mathcal{L}}_{\beta_1}\left(\left[\underline{y}\right]_\alpha\right) - \underline{\mathrm{G}}(x;\alpha) - g_l([\tilde{y}]_\alpha) \\ \left[\overline{RE}\right]_\alpha = \bar{\mathcal{L}}_{\beta_1}([\overline{y}]_\alpha) - \overline{\mathrm{G}}(x;\alpha) - g_u([\tilde{y}]_\alpha) \end{cases} \tag{6.41}$$

and

$$\frac{\partial\, \widetilde{SRE}}{\partial \tilde{S}_1(\alpha)} = \frac{\partial\, \widetilde{SRE}}{\partial \tilde{S}_2(\alpha)} = \cdots \frac{\partial\, \widetilde{SRE}}{\partial \tilde{S}_k(\alpha)} = 0. \tag{6.42}$$

6.4.3 Establishment of Convergence of FF-OHAM Solution Series for Second-order FFOBVPs

For step-3 of the theoretical part of the development of FF-OHAM for solving second-order FFOBVPs, the same algorithm that has been presented in Chapter 4 and Chapter 5 to establish convergence, is used here.

6.5 Experimental Work of FF-HAM and FF-OHAM for Second-order FFOBVP

This section demonstrates the experimental work for the application of the developed methods FF-HAM and FF-OHAM in solving second-order FFOBVPs. The developed methods are applied to linear and nonlinear FFOBVPs of order $\beta_1 \in (1,2]$ as stated in Definition 3.22. There are three experimental examples demonstrated where for each example, the solution method using the developed FF-HAM is demonstrated first and followed by the one using the developed FF-OHAM. The general solving steps in the experimental work involve the defuzzification of the second-order FFOBVPs, implementation of the proposed methods, obtaining optimal parameters, obtaining the approximate-analytical solution and plotting the obtained solution to ensure that the solution satisfies the properties of fuzzy numbers. Comparative study with other available numerical results is also carried out.

6.5.1 Example 6.1

Consider fuzzy fractional Bagley-Torvik equation (Esmaeilbeigi et al., 2018):

$$D^{(1.5)}\tilde{y}(x) + \tilde{y}(x) = \tilde{F}(x;\alpha),\ x \in [0,1], \tag{6.43}$$

such that

$$\tilde{F}(x;\alpha) = \begin{cases} \underline{F}(x;\alpha) \\ \overline{F}(x;\alpha) \end{cases} = \begin{cases} \alpha(x^2 - x) + 4\alpha\frac{\sqrt{x}}{\sqrt{\pi}}, \\ (2-\alpha)(x^2 - x) + 4(2-\alpha)\frac{\sqrt{x}}{\sqrt{\pi}}. \end{cases} \tag{6.44}$$

subject to the following fuzzy boundary condition

$$\begin{cases} \underline{y}(0;\alpha) = \underline{y}(1;\alpha) = (\alpha - 1), \\ \overline{y}(0;\alpha) = \overline{y}(1;\alpha) = (1-\alpha). \end{cases} \tag{6.45}$$

with the following fuzzy exact solution

$$\begin{cases} \underline{Y}(x;\alpha) = \alpha(x^2 - x), \\ \overline{Y}(x;\alpha) = (2-\alpha)(x^2 - x). \end{cases} \tag{6.46}$$

6.5.1.1 Solving Example 6.1 by FF-HAM

To solve Example 6.1 by FF-HAM, based on Section 6.3.2, we construct the zeroth-order and k^{th}-order deformation equations for $k \geq 1$ of Eq. (6.43) as follows:

$$\begin{cases} \underline{y}_k(x;\alpha) = \\ \psi_k\left(\underline{y}_{k-1}(x;\alpha)\right) + \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(\beta_1)}\mathcal{R}_k(\vec{\underline{y}}_{k-1}(x;\alpha)) \\ \overline{y}_k(x;\alpha) = \\ \psi_k\left(\overline{y}_{k-1}(x;\alpha)\right) + \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(\beta_1)}\mathcal{R}_k(\vec{\overline{y}}_{k-1}(x;\alpha)) \end{cases}, \tag{6.47}$$

$\underline{y}_k(0;\alpha) = 0, \overline{y}_k(0;\alpha) = 0, \underline{y}_k(1;\alpha) = 0, \overline{y}_k(1;\alpha) = 0$, such that

$$\psi_k = \begin{cases} 0, & k \leq 1 \\ 1, & k > 1 \end{cases}, \text{ and}$$

$$\begin{cases} \mathcal{R}_k\left(\vec{\underline{y}}_{k-1}(x;\alpha)\right) = \underline{y}_{k-1}{}^{(1.5)}(x;\alpha) + \underline{y}_k(x;\alpha) - \underline{F}(x;\alpha), \\ \mathcal{R}_k\left(\vec{\overline{y}}_{k-1}(x;\alpha)\right) = \overline{y}_{k-1}{}^{(1.5)}(x;\alpha) + \overline{y}_k(x;\alpha) - \overline{F}(x;\alpha). \end{cases} \tag{6.48}$$

The initial approximation

$$\tilde{y}_0(x;\alpha) = [1-\alpha, \alpha-1]. \tag{6.49}$$

Expanding Eq. (6.48), we have the following:

The first-order series approximation

$$\begin{cases} \underline{y}_1(x;\alpha) = \underline{h}(\alpha)\, \underline{\mathcal{J}}^{(1.5)}\left(\underline{y}_0{}^{(1.5)}(x;\alpha) + \underline{y}_0(x;\alpha) - \underline{F}(x;\alpha)\right), \\ \overline{y}_1(x;\alpha) = \overline{h}(\alpha)\, \overline{\mathcal{J}}^{(1.5)}\left(\overline{y}_0{}^{(1.5)}(x;\alpha) + \overline{y}_0(x;\alpha) - \overline{F}(x;\alpha)\right). \end{cases} \tag{6.50}$$

The second-order series approximation

$$\begin{cases} \underline{y}_2(x;\alpha) = \left(\underline{y}_1(x;\alpha)\right) + \\ \underline{h}(\alpha)\underline{\mathcal{J}}^{(1.5)}\left(\underline{y}_1{}^{(1.5)}(x;\alpha) + \underline{y}_1(x;\alpha)\right), \\ \overline{y}_2(x;\alpha) = \left(\overline{y}_1(x;\alpha)\right) + \\ \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(1.5)}\left(\overline{y}_1{}^{(1.5)}(x;\alpha) + \overline{y}_1(x;\alpha)\right). \end{cases} \tag{6.51}$$

The third-order series approximation

$$\begin{cases} \underline{y}_3(x;\alpha) = \left(\underline{y}_2(x;\alpha)\right) + \\ \underline{h}(\alpha)\underline{\mathcal{J}}^{(1.5)}\left(\underline{y}_2{}^{(1.5)}(x;\alpha) + \underline{y}_2(x;\alpha)\right), \\ \overline{y}_3(x;\alpha) = \left(\overline{y}_2(x;\alpha)\right) + \\ \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(1.5)}\left(\overline{y}_2{}^{(1.5)}(x;\alpha) + \overline{y}_2(x;\alpha)\right). \end{cases} \tag{6.52}$$

.

.

$$\begin{cases} \underline{y}_k(x;\alpha) = \left(\underline{y}_{k-1}(x;\alpha)\right) \\ +\underline{h}(\alpha)\,\underline{\mathcal{J}}^{(1.5)}\left(\underline{y}_{k-1}{}^{(1.5)}(x;\alpha) + \underline{y}_{k-1}(x;\alpha)\right), \\ \overline{y}_k(x;\alpha) = \left(\overline{y}_{k-1}(x;\alpha)\right) \\ +\overline{h}(\alpha)\,\overline{\mathcal{J}}^{(1.5)}\left(\overline{y}_{k-1}{}^{(1.5)}(x;\alpha) + \overline{y}_{k-1}(x;\alpha)\right). \end{cases} \tag{6.53}$$

The third-order series FF-HAM approximate-analytic solution has been obtained using Mathematica 12 Dsolve Package where the lower and the upper bounds have the following form:

$$\begin{cases} \underline{y}(x;\alpha;\underline{h}) = \underline{y}_0(x;\alpha) + \sum_{j=1}^{3} \underline{y}_j(x;\alpha;\underline{h}) \\ \overline{y}(x;\alpha;\overline{h}) = \overline{y}_0(x;\alpha) + \sum_{j=1}^{3} \overline{y}_j(x;\alpha;\overline{h}) \end{cases} \tag{6.54}$$

We can obtain the accuracy of FF-HAM for solving Eq.(6.43) by taking the residual error as mentioned in Section 6.3.3 such that

$$\begin{cases} \underline{ER}(x;\alpha;\underline{h}) = \left(\underline{y}^{(1.5)}(x;\alpha) + \underline{y}(x;\alpha) - \underline{F}(x;\alpha)\right), \\ \overline{ER}(x;\alpha;\overline{h}) = \left(\overline{y}^{(1.5)}(x;\alpha) + \overline{y}(x;\alpha) - \overline{F}(x;\alpha)\right). \end{cases} \tag{6.55}$$

As has been mentioned in Chapter 4 and Chapter 5, these series depend upon x and α as stated in Definition 3.5 and also upon the convergent-control parameter $\tilde{h}$. According to Section 6.3.3, the convergent control parameter $\tilde{h}$ can be employed to adjust the convergence region of the FF-HAM and we use the properties of FF-HAM convergence to find the best value of $\tilde{h}$. Toward this end, for fixed value of $0 \le \alpha \le 1$ say $\alpha = 0.5$ we plot the $\tilde{h}$ -curve of the lower bound and upper bound using the solution of third-order series FF-HAM for Eq. (6.43).

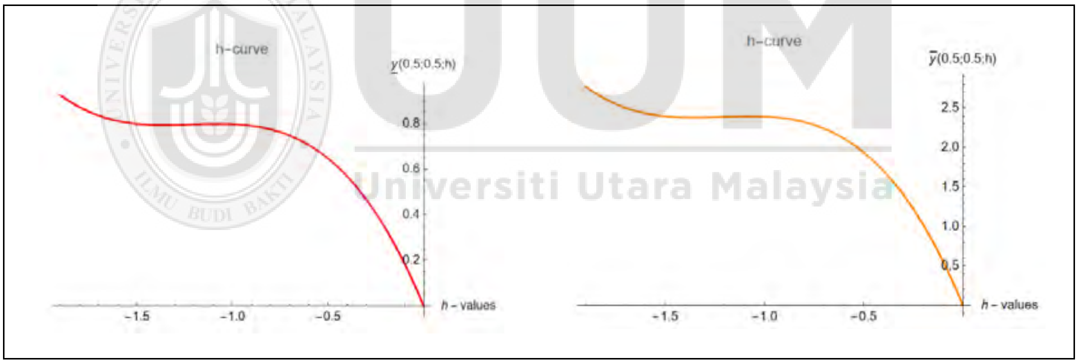

Figure 6.1. The $\tilde{h}$-curve for the fuzzy solution of Eq. (6.43) given by third-order series FF-HAM when $H(x) = 1$

Based on the curves in Figure 6.1, it is easy to discover the valid region of $\tilde{h}(0.5)$ which corresponds to the line segment that is almost parallel to the horizontal axis. Therefore, in this particular case, the FF-HAM series solution is convergent when $-1.5 \le \tilde{h} \le -0.8$, and the best value of h is $\tilde{h} = -1.0488155053478476$.

The accuracy of FF-HAM approximate solution can be measured using residual errors. The residual errors $[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ of the solutions $\underline{y}(0.5;\tilde{h};0.5)$ and $\overline{y}(0.5;\tilde{h};0.5)$ obtained using third-order series FF-HAM are tabulated in Table 6.1.

Table 6.1
The approximate solution and error of Eq. (6.43) by third-order series FF-HAM when $\beta_1 = 1.5$ at $x = 0.5$ for all $\alpha \in [0,1]$

α	$[\underline{ER}]_\alpha, \tilde{h}$	$[\overline{ER}]_\alpha, \tilde{h}$	$[\underline{y}]_\alpha, \tilde{h}$	$[\overline{y}]_\alpha, \tilde{h}$
0	2.02950×10^{-4}	0	-0.50005	0
0.2	1.82655×10^{-4}	2.02950×10^{-5}	-0.45004	-0.05000
0.4	1.62360×10^{-4}	4.05901×10^{-5}	-0.40004	-0.10001
0.6	1.42065×10^{-4}	6.08852×10^{-5}	-0.35003	-0.15001
0.8	1.21770×10^{-4}	8.11802×10^{-5}	-0.30003	-0.20002
1	1.01475×10^{-4}	1.01475×10^{-4}	-0.25002	-0.25002

Table 6.1 illustrates the lower and upper solutions of Eq. (6.43) using third-order series FF-HAM based on the best optimal convergence control parameter $\tilde{h} = -1.0488155053478476$.

The solutions over all $x \in [0,0.5]$ and $\alpha \in [0,1]$ corresponding to the best optimal convergence control value of $\tilde{h}$ for Eq. (6.43) can be summarized in the three-dimensional figure as illustrated in Figure 6.2.

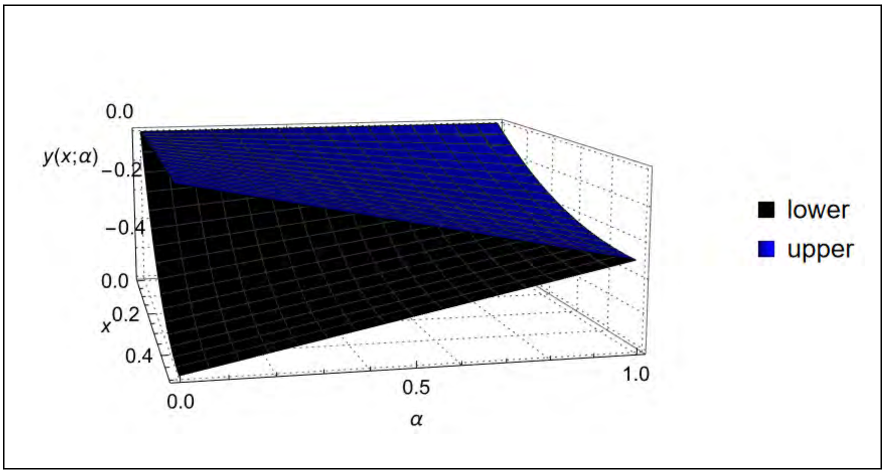

Figure 6.2. The three-dimensional approximate solution of Eq. (6.43) given by third-order FF-HAM over all $x \in [0,0.5]$ at $\beta_1 = 1.5$, and for all $\alpha \in [0,1]$.

In studying the behaviour of FF-HAM for solving second-order FFOBVPs, we shall proceed to solve Eq. (6.43) by FF-HAM at the same $x \in [0,0.5]$ and $\beta_1 = 1.5$ of order five instead of order three to analyse the convergence dynamic of FF-HAM for different terms of the series approximate solution. We begin with the identification of the valid region of the convergence control parameters of fifth-order FF-HAM.

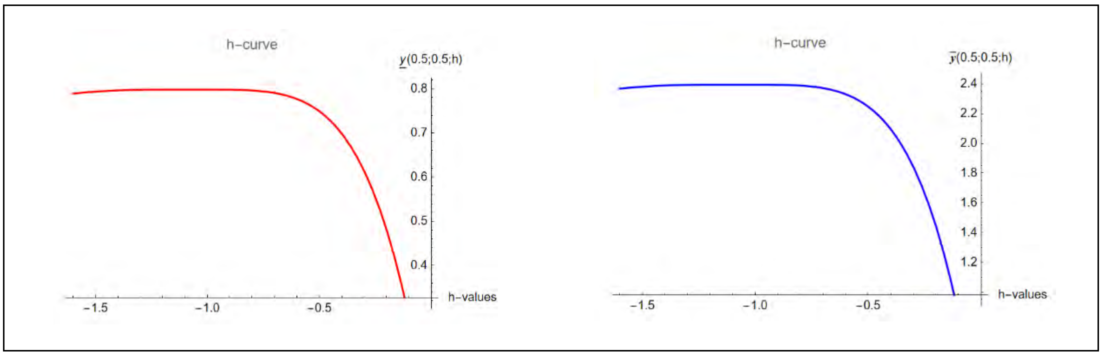

Figure 6.3. The $\tilde{h}$-curve for the fuzzy solution of Eq. (6.43) given by fifth-order FF-HAM when $H(x) = 1$

From Figure 6.3, it is easy to conclude that, the valid region of the convergence control parameters that corresponds to the line segment nearly parallel to the horizontal axis has changed to $-1.5 \leq \tilde{h} \leq -0.5$ after changing the order of FF-HAM, while the best value of the convergent control-parameter is $\tilde{h} = -1.0549896423027272$.

In the following Table 6.2, the residual errors $[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ of the approximate solutions $\underline{y}(0.5;\tilde{h};\alpha)$ and $\overline{y}(0.5;\tilde{h};\alpha)$ obtained by using fifth-order FF-HAM are tabulated.

Table 6.2

The approximate solution and error of Eq. (6.43) by fifth-order FF-HAM when $\beta_1 = 1.5$ *at* $x = 0.5$ *for all* $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\underline{ER}]_\alpha, \tilde{h}$	$[\overline{ER}]_\alpha, \tilde{h}$	$[\underline{y}]_\alpha, \tilde{h}$	$[\overline{y}]_\alpha, \tilde{h}$
0	4.20230×10^{-7}	0	-0.49999	0
0.2	3.78206×10^{-7}	4.20229×10^{-8}	-0.44999	-0.04999
0.4	3.36183×10^{-7}	8.40458×10^{-8}	-0.39999	-0.09999
0.6	2.94160×10^{-7}	1.26069×10^{-7}	-0.34999	-0.14999
0.8	2.52138×10^{-7}	1.68092×10^{-7}	-0.29999	-0.19999
1	2.10115×10^{-7}	2.10115×10^{-7}	-0.24999	-0.24999

Using three-dimensional graph, the solutions by fifth-order FF-HAM over all $x \in [0,0.5]$ and $\alpha \in [0,1]$ corresponding with the best optimal convergence control values $\tilde{h}$ of Eq. (6.43) are illustrated in Figure 6.4.

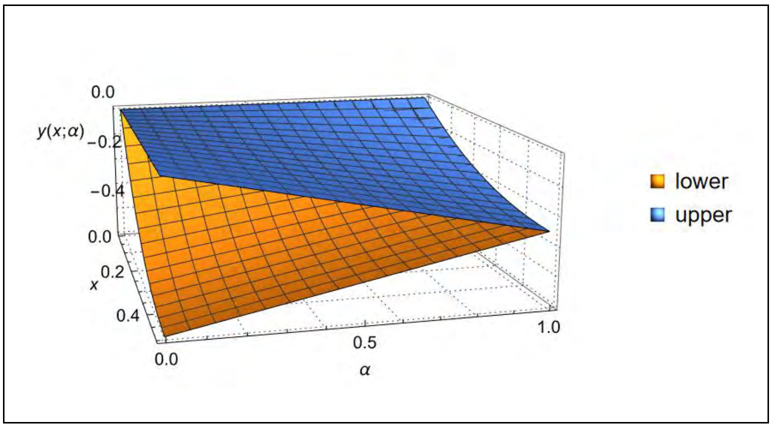

Figure 6.4. The three-dimensional approximate solution of Eq. (6.43) given by fifth-order FF-HAM over all $x \in [0,0.5]$ at $\beta_1 = 1.5$, and for all $\alpha \in [0,1]$

Tables 6.1-6.2 and Figures 6.2-6.4 illustrate that the third-order and fifth-order FF-HAM satisfies the triangular solution of the fuzzy differential equations [Definition 3.7] for Eq. (6.43).

On the other hand, we can conclude that the solution accuracy of Eq.(6.43) by FF-HAM will approach the exact solutions whenever the order of FF-HAM series increases.

6.5.1.2 Comparative Study of FF-HAM for Example 6.1

For comparative study on Example 6.1, Figures 6.5 and 6.6 illustrate the accuracy of the developed fifth-order FF-HAM in comparison to the fifth-order SCM (Esmaeilbeigi et al., 2018) for solving Eq.(6.43) $\forall x \in [0,1]$ for $\alpha = 0.5$ based on the absolute error of Eq.(6.43) as follows:

$$\begin{cases} \underline{ERR}(x;\alpha;\underline{h}) = \left|\underline{y}(x;\alpha;\underline{h}) - \underline{Y}(x)\right|, \\ \overline{ERR}(x;\alpha;\overline{h}) = \left|\overline{y}(x;\alpha;\overline{h}) - \overline{Y}(x)\right|. \end{cases} \tag{6.56}$$

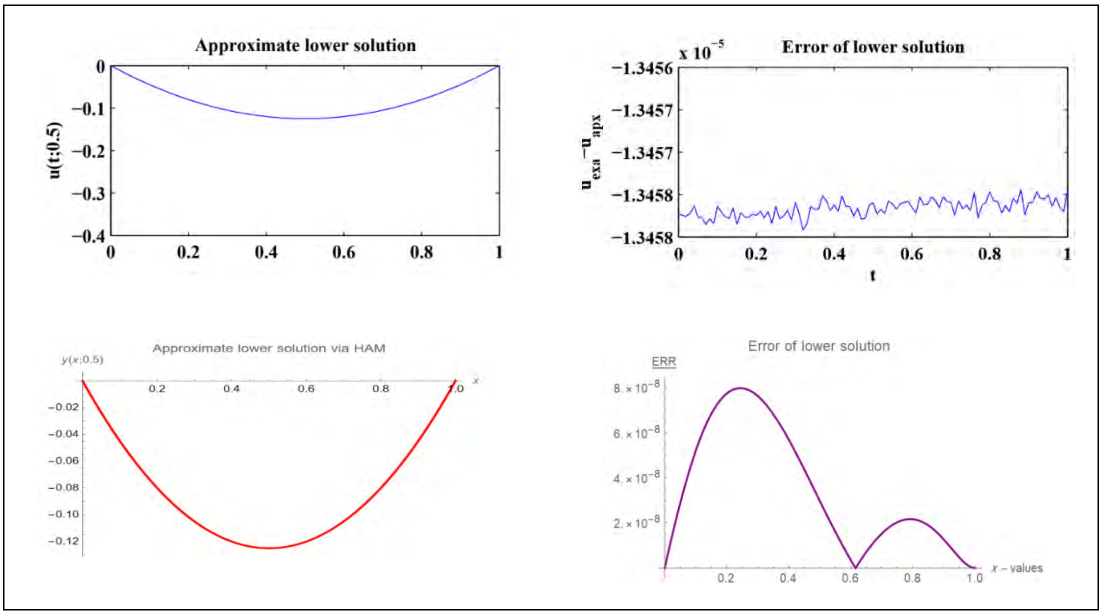

Figure 6.5. Comparison of the lower approximate solution of Eq. (6.43) by fifth-order FF-HAM and fifth-order SCM for $\alpha = 0.5$ and $x \in [0,1]$

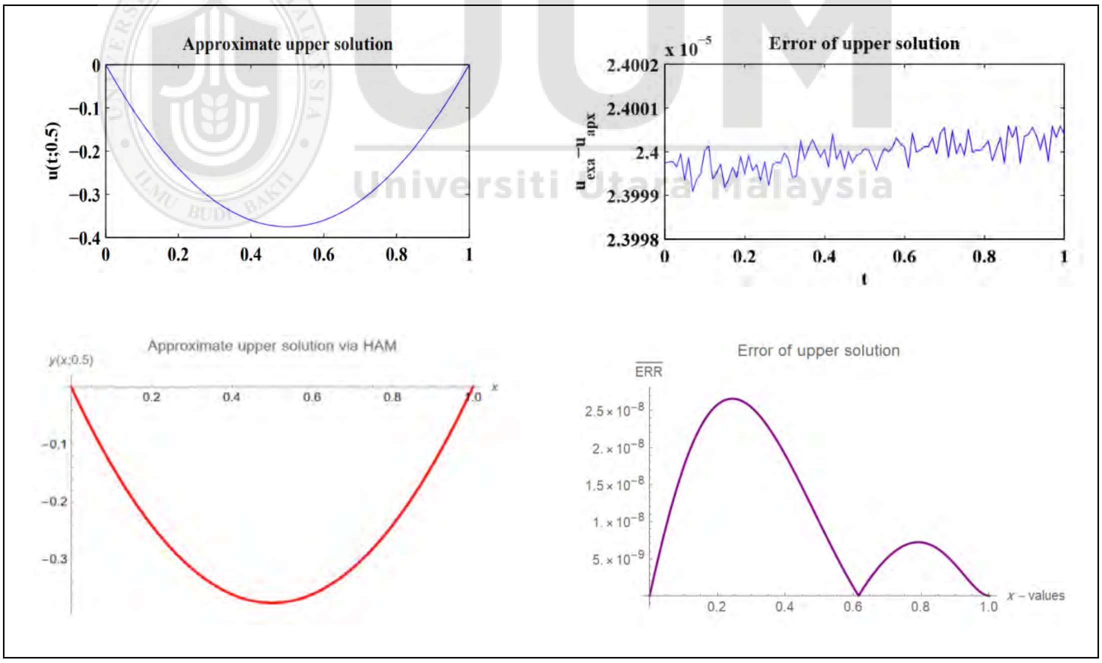

Figure 6.6. Comparison of the upper approximate solution of Eq. (6.43) by fifth-order FF-HAM and fifth-order SCM for $\alpha = 0.5$ and $x \in [0,1]$

We can conclude from Figures 6.5 and 6.6 that in terms of accuracy, the approximate solution solved by the fifth-order FF-HAM series gives a better approximation compared to SCM for all $x \in [0,1]$.

6.5.1.3 Solving Example 6.1 by FF-OHAM

This section demonstrates solving Example 6.1 by FF-OHAM. We obtain the approximate analytical solution by constructing the third-order and fifth-order FF-OHAM series solution with convergence control parameters $\tilde{S}_k(0.5)$ at $\alpha = 0.5\ \forall k$. This way, we can obtain the same results based on the control parameters $\tilde{S}_k(\alpha)$ for each $\alpha \in [0,1]$. In addition, it requires less time for the CPU to run especially in dealing with FFOBVPs.

Now, we can contract the FF-OHAM series solution for all $\alpha \in [0,1]$ of Eq. (6.43) as follows:

$$\begin{cases} (1-q)\left[\widetilde{D}^{(1.5)}\big(\tilde{y}(x;\alpha)\big) - \tilde{F}(x;\alpha)\right] = \sum_{j=1}^{5} \tilde{S}_j(\alpha) q^j \\ \tilde{S}_j(\alpha) q^j \left[\widetilde{D}^{(1.5)}\big(\tilde{y}(x;\alpha)\big) + \tilde{y}(x;\alpha) - \Big(\tilde{F}(x;\alpha)\Big)\right] \end{cases} \tag{6.57}$$

such that

$$\tilde{y}(x;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{j=1}^{k} \tilde{y}_j(x, S_1, .., S_j; \alpha) q^j \tag{6.58}$$

Zeroth-order problem:

$$\begin{cases} \tilde{y}_0(x,\alpha) = \tilde{J}^{(1.5)}\left[\alpha\left((x^2 - x) + 4\frac{\sqrt{x}}{\sqrt{\pi}}\right), (2-\alpha)\left((x^2 - x) + 4\frac{\sqrt{x}}{\sqrt{\pi}}\right)\right] \\ ß\left(\tilde{y}_0(x;\alpha), \frac{\partial[\tilde{y}_0]_\alpha}{\partial x}\right) = 0 \end{cases} \tag{6.59}$$

First-order problem:

$$\begin{cases} \tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right) = \left(1+\tilde{S}_1(\alpha)\right)\,\tilde{y}_0(x;\alpha)+\tilde{S}_1(\alpha)\,\tilde{\mathcal{J}}^{(1.5)}\,\tilde{y}_0(x;\alpha) \\ \qquad -\left(1+\tilde{S}_1(\alpha)\right)\tilde{\mathcal{J}}^{(1.5)}\,\left(\tilde{F}(x;\alpha)\right), \\ \qquad ß\left(\tilde{y}_1(x;\alpha),\frac{\partial[\tilde{y}_1]_\alpha}{\partial x}\right)=0. \end{cases} \tag{6.60}$$

Second-order problem:

$$\begin{cases} \tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right) = \left(1+\tilde{S}_1(\alpha)\right)\,\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)+\tilde{S}_2(\alpha)\tilde{y}_0(x;\alpha)+ \\ \tilde{S}_1(\alpha)\tilde{\mathcal{J}}^{(1.5)}\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)+\tilde{S}_2(\alpha)\tilde{\mathcal{J}}^{(1.5)}\,\tilde{y}_0(x;\alpha)-\tilde{S}_2(\alpha)\tilde{\mathcal{J}}^{(1.5)}\left(\tilde{F}(x;\alpha)\right) \\ \qquad ß\left(\tilde{y}_2(x;\alpha),\dfrac{\partial[\tilde{y}_2]_\alpha}{\partial x}\right)=0. \end{cases} \tag{6.61}$$

Third-order problem:

$$\begin{cases} \tilde{y}_3\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha),\tilde{S}_3(\alpha);\alpha\right) = \left(1+\tilde{S}_1(\alpha)\right)\,\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)+ \\ \tilde{S}_2(\alpha)\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)+\tilde{\mathcal{J}}^{(1.5)}\sum_{j=1}^{3}\tilde{S}_j(\alpha)\,\tilde{y}_{3-j}\left(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_{3-j}(\alpha);\alpha\right) \\ \qquad -\tilde{S}_3(\alpha)\tilde{\mathcal{J}}^{(1.5)}\left(\tilde{F}(x;\alpha)\right)+\tilde{S}_3(\alpha)\tilde{y}_0(x;\alpha) \\ \qquad ß\left(\tilde{y}_3(x;\alpha),\frac{\partial[\tilde{y}_3]_\alpha}{\partial x}\right)=0. \end{cases} \tag{6.62}$$

Fourth-order problem:

$$\begin{cases} \tilde{y}_4\left(x,\tilde{S}_1(\alpha),..,\tilde{S}_4(\alpha);\alpha\right) = \left(1+\tilde{S}_1(\alpha)\right)\,\tilde{y}_3\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha),\tilde{S}_3(\alpha);\alpha\right)+ \\ \tilde{S}_2(\alpha)\tilde{y}_2\left(x,\tilde{S}_1(\alpha),\tilde{S}_2(\alpha);\alpha\right)+\tilde{S}_3(\alpha)\tilde{y}_1\left(x,\tilde{S}_1(\alpha);\alpha\right)+\tilde{S}_4(\alpha)\,\tilde{y}_0(x;\alpha)+ \\ \tilde{\mathcal{J}}^{(1.5)}\sum_{j=1}^{4}\tilde{S}_j(\alpha)\,\tilde{y}_{4-j}\left(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_{4-i}(\alpha);\alpha\right)-\tilde{S}_4(\alpha)\tilde{\mathcal{J}}^{(1.5)}\left(\tilde{F}(x;\alpha)\right) \\ \qquad ß\left(\tilde{y}_4(x;\alpha),\frac{\partial[\tilde{y}_4]_\alpha}{\partial x}\right)=0 \end{cases} \tag{6.63}$$

Fifth-order problem:

$$\begin{cases} \tilde{y}_5\left(x, \tilde{S}_1(\alpha), .., \tilde{S}_5(\alpha); \alpha\right) = \left(1 + \tilde{S}_1(\alpha)\right) \tilde{y}_4\left(x, \tilde{S}_1(\alpha), \ldots .., \tilde{S}_4(\alpha); \alpha\right) + \\ \tilde{S}_2(\alpha)\tilde{y}_3\left(x, \tilde{S}_1(\alpha), \tilde{S}_2(\alpha), \tilde{S}_3(\alpha); \alpha\right) + \tilde{S}_3(\alpha)\tilde{y}_2\left(x, \tilde{S}_1(\alpha), \tilde{S}_2(\alpha); \alpha\right) + \\ \tilde{S}_4(\alpha)\tilde{y}_1\left(x, \tilde{S}_1(\alpha); \alpha\right) + \tilde{S}_5(\alpha)\, \tilde{y}_0(x; \alpha) - \tilde{S}_5(\alpha)\tilde{\mathcal{J}}^{(1.5)}\left(\tilde{F}(x; \alpha)\right) + \\ \tilde{\mathcal{J}}^{(1.5)} \sum_{j=1}^{5} \tilde{S}_j(\alpha)\, \tilde{y}_{5-j}\left(x, \tilde{S}_1(\alpha), \ldots, \tilde{S}_{5-i}(\alpha); \alpha\right) \\ ß\left(\tilde{y}_5(x; \alpha), \frac{\partial [\tilde{y}_5]_\alpha}{\partial x}\right) = 0. \end{cases} \quad (6.64)$$

Next, we will solve Eq. (6.43) with third-order series FF-OHAM using Mathematica 12 Dsolve Package

$$\tilde{y}(x; \alpha) = \tilde{y}_0(x; \alpha) + \sum_{j=1}^{3} \tilde{y}_j\left(x, \tilde{S}_1(0.5), .., \tilde{S}_j(0.5); 0.5\right). \quad (6.65)$$

such that

$\tilde{S}_1(0.5) = -1.065291064957493$,

$\tilde{S}_2(0.5) = -0.0000433898550990572 4$, and

$\tilde{S}_3(0.5) = -0.002073926908232552 3$.

Table 6.3

The approximate solution and error of Eq. (6.43) by third-order FF-OHAM at $x = 0.5$ for all $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\underline{\boldsymbol{ER}}]_\alpha, \tilde{\boldsymbol{S}}_j$	$[\overline{\boldsymbol{ER}}]_\alpha, \tilde{\boldsymbol{S}}_j$	$[\underline{\boldsymbol{y}}]_\alpha, \tilde{\boldsymbol{S}}_j$	$[\overline{y}]_\alpha, \tilde{\boldsymbol{S}}_j$
0	-1.36061×10^{-5}	0	-0.49998	0
0.2	-1.22455×10^{-5}	-1.36061×10^{-6}	-0.44998	-0.04999
0.4	-1.08849×10^{-5}	-2.72123×10^{-6}	-0.39999	-0.09999
0.6	-9.52429×10^{-6}	-4.08184×10^{-6}	-0.34999	-0.14999
0.8	-8.16368×10^{-6}	-5.44245×10^{-6}	-0.29999	-0.19999
1	-6.80306×10^{-6}	-6.80306×10^{-6}	-0.24999	-0.24999

Using three-dimensional graph, we summarize the solutions by third-order FF-OHAM over all $x \in [0,0.5]$ and $\alpha \in [0,1]$ corresponding with the best optimal convergence control values $\tilde{S}_j$ of Eq. (6.43) in Figure 6.7.

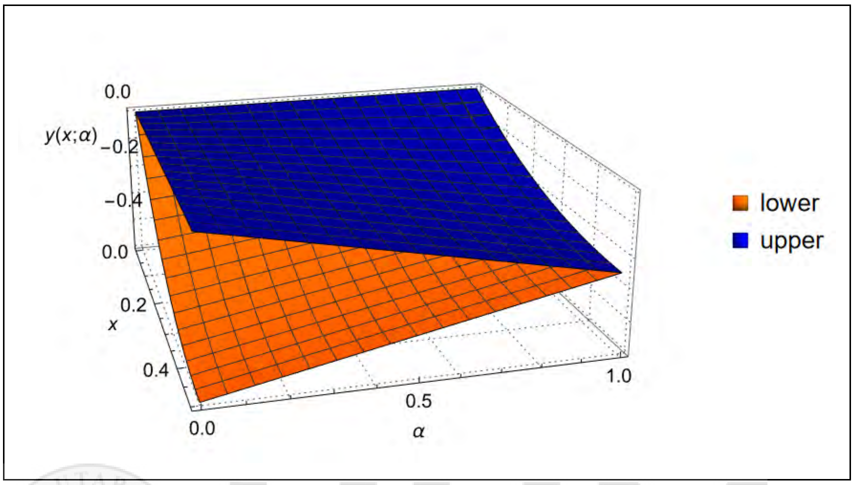

Figure 6.7. The three-dimensional approximate solution of Eq. (6.43) given by third-order FF-OHAM over all $x \in [0,0.5]$, and for all $\alpha \in [0,1]$

To analyse the behaviour of FF-OHAM for solving second-order FFOBVPs, we shall proceed to solve Eq. (6.43) by FF-OHAM at the same $x \in [0,0.5]$ and $\beta_1 = 1.5$ of order five instead of order three to illustrate the convergence dynamic of FF-OHAM for different terms of the approximate series solution.

$$\tilde{y}(x;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{j=1}^{5} \tilde{y}_j\left(x, \tilde{S}_1(0.5), \ldots, \tilde{S}_j(0.5); 0.5\right). \tag{6.66}$$

such that the optimal convergence control parameters are

$\tilde{S}_1(0.5) = -1.0270653590282228,$

$\tilde{S}_2(0.5) = 6.734609572815013 \times 10^{-7},$

$\tilde{S}_3(0.5) = -0.00002508307222870676 7,$

$\tilde{S}_4(0.5) = 0.000131089732250741$, and

$\tilde{S}_5(0.5) = 0.0001858737255737089$.

Table 6.4

The approximate solution and error of Eq. (6.43) by fifth-order FF-OHAM at $x = 0.5$ *for all* $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\underline{\boldsymbol{ER}}]_\alpha, \tilde{\boldsymbol{S}}_j(\mathbf{0.5})$	$[\overline{\boldsymbol{ER}}]_\alpha, \tilde{\boldsymbol{S}}_j(\mathbf{0.5})$	$[\underline{\boldsymbol{y}}]_\alpha, \tilde{\boldsymbol{S}}_j(\mathbf{0.5})$	$[\overline{\boldsymbol{y}}]_\alpha, \tilde{\boldsymbol{S}}_j(\mathbf{0.5})$
0	-2.79607×10^{-8}	0	-0.50000	0
0.2	-2.51647×10^{-8}	-2.79607×10^{-9}	-0.45000	-0.05000
0.4	-2.23686×10^{-8}	-5.59214×10^{-9}	-0.40000	-0.10000
0.6	-1.95725×10^{-8}	-8.38822×10^{-9}	-0.35000	-0.15000
0.8	-1.67764×10^{-8}	-1.11843×10^{-8}	-0.30000	-0.20000
1	-1.39804×10^{-8}	-1.39804×10^{-8}	-0.25000	-0.25000

Figure 6.8 illustrates the summary of the solutions by fifth-order FF-OHAM over all $x \in [0,0.5]$ and $\alpha \in [0,1]$ corresponding with the best optimal convergence control values $\tilde{S}_k$ of Eq. (6.43) in a three-dimensional figure.

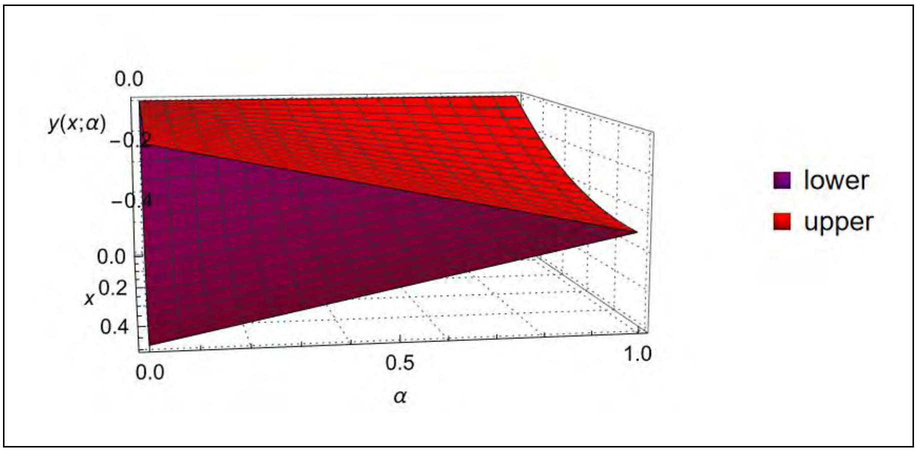

Figure 6.8. The three-dimensional approximate solution of Eq. (6.43) given by fifth-order FF-OHAM over all $x \in [0,0.5]$, and for all $\alpha \in [0,1]$

Tables 6.3-6.4 and Figures 6.7-6.8 illustrate that the third and fifth-order FF-OHAM satisfy the triangular solution [Definition 3.7] of the fuzzy differential equations for Eq. (6.43). On the other hand, we can conclude that the obtained solution of Eq. (6.43) by FF-OHAM will approach the exact solutions whenever the order of OHAM increasing.

6.5.1.4 Comparative Study of FF-OHAM for Example 6.1

The developed FF-OHAM is compared with SCM for Example 6.1. Figures 6.9 and 6.10 illustrate the accuracy of fifth-order FF-OHAM in comparison to the fifth-order SCM for solving Eq. (6.43) $\forall x \in [0,1]$ at $\alpha = 0.5$ based on the absolute error as defined in Eq. (6.56).

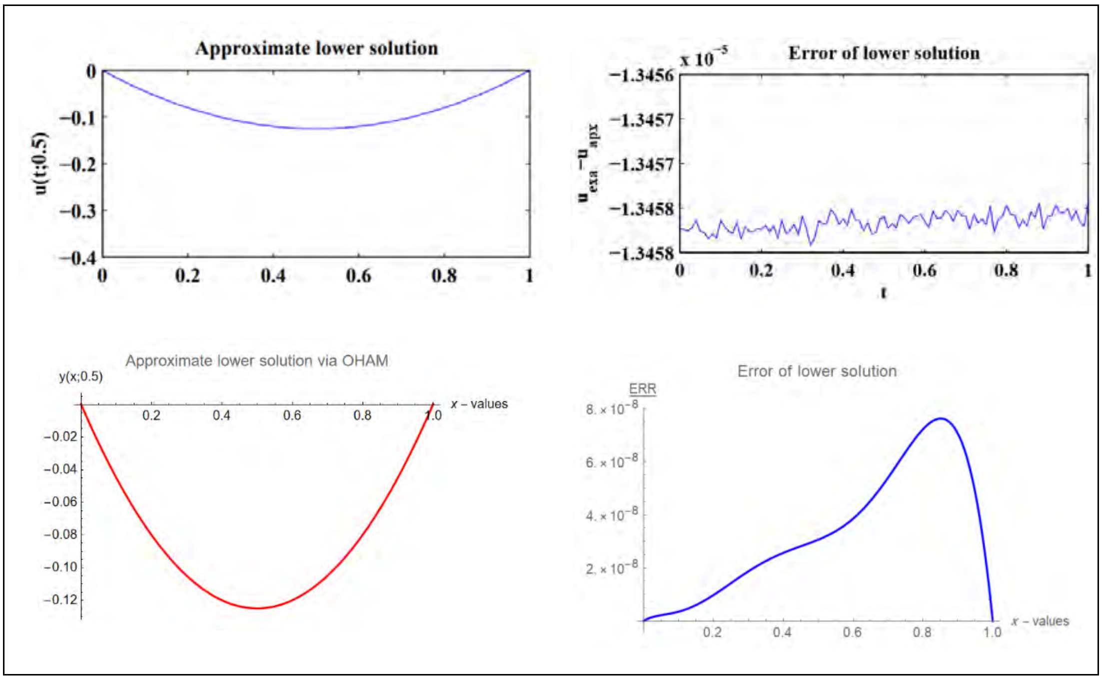

Figure 6.9. Comparison of the lower approximate solution of Eq. (6.43) by fifth-order FF-OHAM and fifth-order SCM for $\alpha = 0.5$ and $x \in [0,1]$

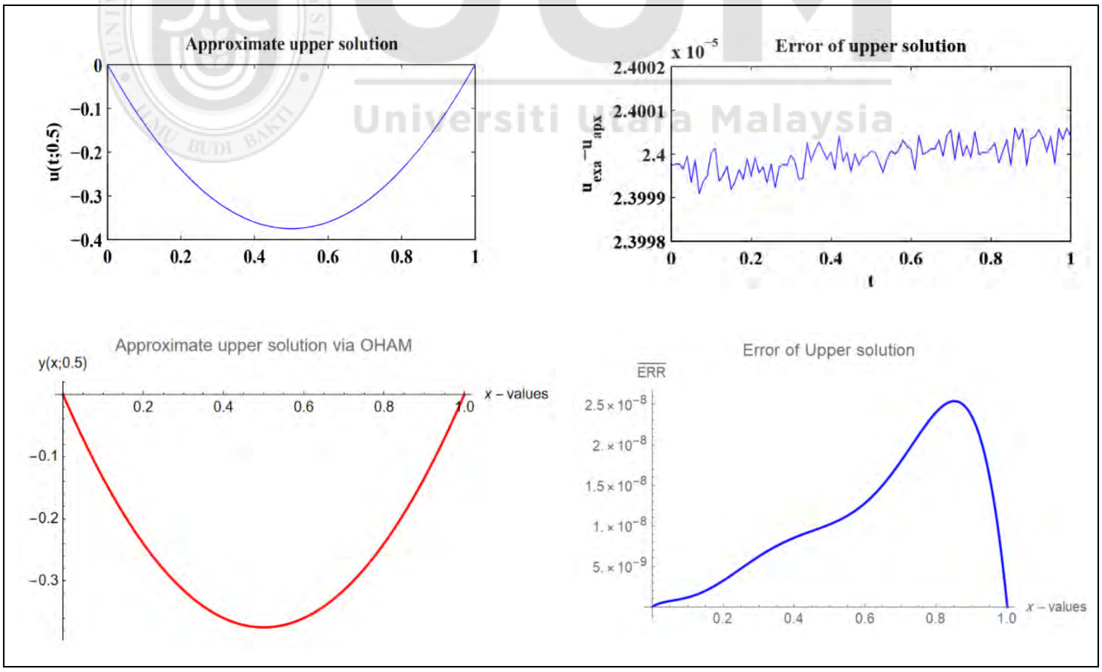

Figure 6.10. Comparison of the upper approximate solution of Eq. (6.43) by fifth-order FF-OHAM and fifth-order SCM for $\alpha = 0.5$ and $x \in [0,1]$

We can conclude from Figures 6.9 and 6.10 that the accuracy of the approximate solution solved by the fifth-order FF-OHAM series gives a better accuracy compared to SCM for all $\alpha \in [0,1]$.

6.5.2 Example 6.2

Consider the following linear FFOBVP (Abdel Aal et al., 2019):

$$\begin{cases} D^{(1.8)}\tilde{y}(x) - D^{(0.9)}\tilde{y}(x) = [\alpha - 1, 1 - \alpha] + 1, \ \ x \in [0,1] \\ \tilde{y}(x_0) = [0,0], \ \tilde{y}(X) = [\alpha - 1, 1 - \alpha], \end{cases} \tag{6.67}$$

here, the linear operator of Eq. (6.67) is $\tilde{\mathcal{L}}_{\beta_1} = \tilde{y}^{(1.8)}(x)$ with the inverse operator $\tilde{\mathcal{L}}^{-1}{}_{\beta_1} = \tilde{\mathcal{J}}^{(1.8)}$.

6.5.2.1 Solving Example 6.2 by FF-HAM

First, we solve this example by FF-HAM. According to Section 6.3.2, we construct the zeroth-order and k^{th}-order deformation equations for $k \geq 1$ of Eq. (6.67) as follows:

$$\begin{cases} \underline{y}_k(x;\alpha) = \\ \psi_k\left(\underline{y}_{k-1}(x;\alpha)\right) + \underline{h}(\alpha)\,\underline{\mathcal{J}}^{(\beta_1)}\mathcal{R}_k(\vec{\underline{y}}_{k-1}(x;\alpha)) \\ \overline{y}_k(x;\alpha) = \\ \psi_k\left(\overline{y}_{k-1}(x;\alpha)\right) + \overline{h}(\alpha)\,\overline{\mathcal{J}}^{(\beta_1)}\mathcal{R}_k(\vec{\overline{y}}_{k-1}(x;\alpha)) \end{cases} \tag{6.68}$$

$\underline{y}_k(0;\alpha) = 0, \ \overline{y}_k(0;\alpha) = 0, \underline{y}_k(1;\alpha) = 0, \ \overline{y}_k(1;\alpha) = 0$, such that

$$\psi_k = \begin{cases} 0, \ k \leq 1 \\ 1, \ k > 1 \end{cases}, \text{ and}$$

$$\begin{cases} \mathcal{R}_k\left(\vec{\underline{y}}_{k-1}(x;\alpha)\right) = \underline{y}_{k-1}{}^{(1.8)}(x;\alpha) - \underline{y}_{k-1}{}^{(0.9)}(x;\alpha) - [\alpha - 1] - 1, \\ \mathcal{R}_k\left(\vec{\overline{y}}_{k-1}(x;\alpha)\right) = \overline{y}_{k-1}{}^{(1.8)}(x;\alpha) - \overline{y}_{k-1}{}^{(0.9)}(x;\alpha) - [1 - \alpha] - 1. \end{cases} \tag{6.69}$$

The initial approximation guess for all $\alpha \in [0,1]$ are given by:

$\tilde{y}_0(x;\alpha) = [(\alpha - 1)\, x\,, (1-\alpha)\, x].$

Expanding Eq. (6.69) we have the following:

The first-order problem

$$\tilde{y}_1(x;\alpha) = \tilde{h}(\alpha)\tilde{y}_0(x;\alpha) - \tilde{h}(\alpha)\, \tilde{\mathcal{J}}^{(1.8)}\big(\tilde{y}_0{}^{(0.9)}(x;\alpha) + [(\alpha-1),(1-\alpha)] + 1\big).$$

For general k^{th}-order problem (for $k \geq 2$)

$$\tilde{y}_k(x;\alpha) = \left(\, \tilde{y}_{k-1}(x;\alpha)\right)$$

$$+\tilde{h}(\alpha)\left(\, \tilde{y}_{k-1}(x;\alpha) - \, \tilde{\mathcal{J}}^{(1.8)}\left(\tilde{y}_{k-1}{}^{(0.9)}(x;\alpha)\right)\right).$$

with k^{th}-order FF-HAM approximate series solution $\tilde{y}(x;\alpha) = \sum_{j=0}^{k} \tilde{y}_j\big(x;\alpha;\tilde{h}\big)$.

We can demonstrate the accuracy of FF-HAM solution for Eq. (6.67) by taking the residual error as mentioned in Section 6.3.3 such that

$$\widetilde{ER}\big(x;\alpha;\tilde{h}\big) = \big(D^{(1.8)}\tilde{y}(x) - D^{(0.9)}\tilde{y}(x) - [\alpha-1,1-\alpha] - 1\big) \qquad (6.70)$$

It is to be noted that the series in Eq. (6.70) depend upon x, α and the convergent control-parameter $\tilde{h}$. We plotted the h-curve for the $\tilde{y}(1;\alpha;\tilde{h})$ by sixth-order FF-HAM. As in previous examples, we plotted the $\tilde{h}$-curve when $\alpha = 0.5$ to obtain the optimal value of $\tilde{h}$.

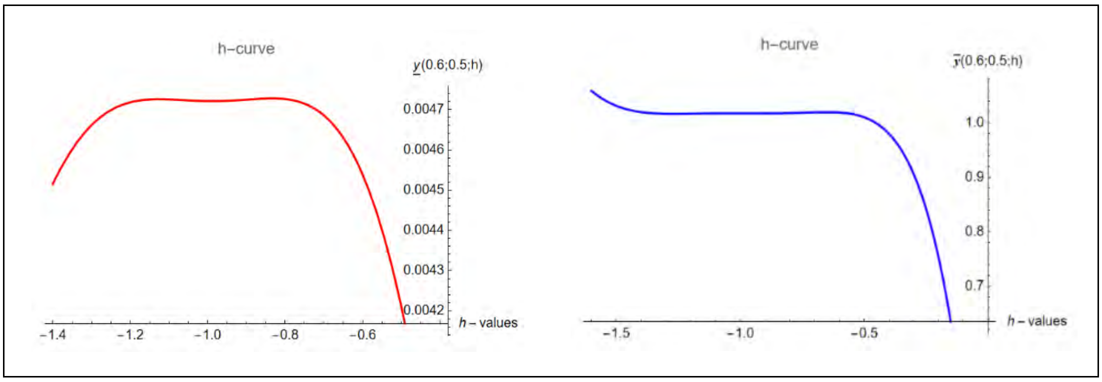

Figure 6.11. The $\tilde{h}$-curve for the fuzzy solution of Eq. (6.67) given by sixth-order FF-HAM when $H(x) = 1$

Based on these curves, it is easy to discover the valid region of $\tilde{h}$ which corresponds to the line segment that is nearly parallel to the horizontal axis. From Figure 6.11 and according to Section 6.3.3, it is clear that the FF-HAM series solution is convergent when $-1.25 \leq \underline{h} \leq -0.8$ and $-1.45 \leq \overline{h} \leq -0.5$ where the best value of the convergent control-parameter is $\tilde{h} = -0.99357382880107 89$.

In Table 6.5, we tabulated the residual errors $[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ and the fuzzy approximate solutions $\tilde{y}(x;\alpha;\tilde{h})$ of Eq. (6.67) by sixth-order FF-HAM.

Table 6.5

The approximate solution and error of Eq. (6.67) by sixth-order FF-HAM at $x = 0.6$ *for all* $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\underline{ER}]_\alpha, \tilde{h}$	$[\overline{ER}]_\alpha, \tilde{h}$	$[\underline{y}]_\alpha, \tilde{h}$	$[\overline{y}]_\alpha, \tilde{h}$
0	2.20636×10^{-6}	-1.19263×10^{-5}	-0.47133	0.21436
0.2	7.93089×10^{-7}	-1.05130×10^{-5}	-0.40276	0.14579
0.4	-6.20177×10^{-7}	-9.09978×10^{-6}	-0.33419	0.07722
0.6	-2.03344×10^{-6}	-7.68651×10^{-6}	-0.26562	0.00866
0.8	-3.44671×10^{-6}	-6.27324×10^{-6}	-0.19705	-0.05991
1	-4.85998×10^{-6}	-4.85997×10^{-6}	-0.12848	-0.12848

The summary of the exact solution and the approximate solutions by sixth-order FF-HAM over all $x \in [0,1]$ and $\alpha \in [0,1]$ corresponding to the best optimal convergence control values $\tilde{h}$ of Eq. (6.67) is illustrated in the following three-dimensional graph in Figure 6.12.

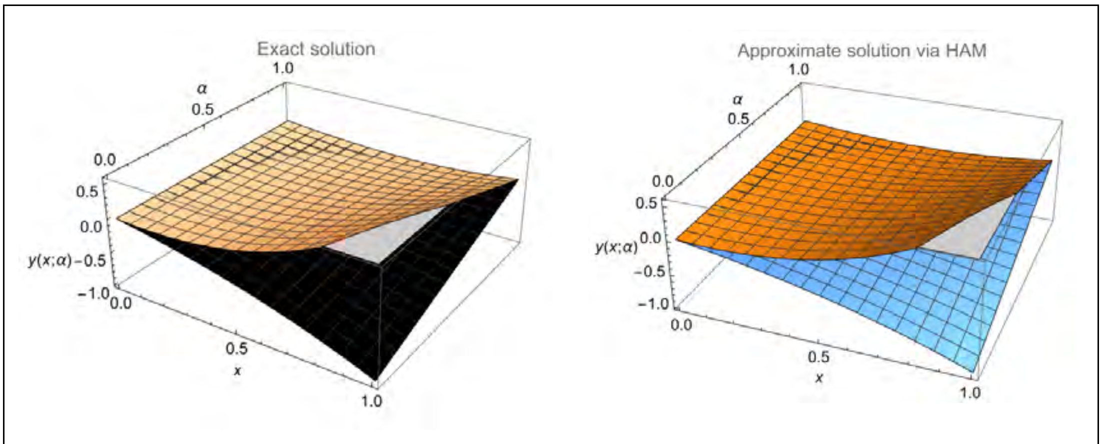

Figure 6.12. The three-dimensional exact solution and approximate solution of Eq. (6.67) given by sixth-order FF-HAM over all $x \in [0,1]$, and for all $\alpha \in [0,1]$

6.5.2.2 Solving Example 6.2 by FF-OHAM

We demonstrate the solving process of Example 6.2 using FF-OHAM. In this example, we use the FF-OHAM as described in Example 6.1 to obtain results based on the control parameters $\tilde{S}_j(0.8)$ for each $\alpha \in [0,1]$.

Next, we can contract the FF-OHAM series solution for all $\alpha \in [0,1]$ of Eq. (6.67) as follows:

Zeroth-order problem

$$\begin{cases} \tilde{y}_0(x,\alpha) = \tilde{\mathcal{J}}^{(1.8)}\big[[\alpha-1,1-\alpha]-1\big] \\ \tilde{y}_0(0;\alpha) = \tilde{0},\ \tilde{y}_0(1;\alpha) = [\alpha-1,1-\alpha] \end{cases} \tag{6.71}$$

First to sixth-order problems

$$\begin{cases} (1-q)\big[\tilde{D}^{(1.8)}\big(\tilde{y}(x;\alpha)\big) - [\alpha-1,1-\alpha]-1\big] = \sum_{j=1}^{6} \tilde{S}_j(\alpha)q^j \\ \tilde{S}_j(\alpha)q^j\big[\tilde{D}^{(1.8)}\big(\tilde{y}(x;\alpha)\big) - \tilde{D}^{(0.9)}\big(\tilde{y}(x;\alpha)\big) - ([\alpha-1,1-\alpha]-1)\big] \\ \tilde{y}_k(0;\alpha) = \tilde{y}_k(1;\alpha) = [\tilde{0}] \end{cases} \tag{6.72}$$

such that

$$\tilde{y}(x;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{j=1}^{6} \tilde{y}_j(x,S_1,..,S_j;\alpha)q^j \tag{6.73}$$

For the next step, using Mathematica 12 DSolve Package to find the solutions for the lower and the upper bounds of Eq. (6.67), for $j = 1,2,..,6$, we obtain the following:

$$\tilde{y}(x;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{j=1}^{k} \tilde{y}_j\big(x,\tilde{S}_1(\alpha),\dots,\tilde{S}_j(\alpha);\alpha\big) \tag{6.74}$$

$\tilde{S}_1(0.8) = -0.987186126120874,$

$\tilde{S}_2(0.8) = 1.271577914752466 \times 10^{-7},$

$\tilde{S}_3(0.8) = -0.0000041097310285705l6,$

$\tilde{S}_4(0.8) = -0.0004885210186436286,$

$\tilde{S}_5(0.8) = 0.0012351064866801835,$

$\tilde{S}_6(0.8) = -0.051631745579960385.$

Table 6.6

The approximate solution and error of Eq. (6.67) by sixth-order FF-OHAM at $x = 0.6$ *for all* $\alpha \in [0,1]$

α	$[\underline{ER}]_\alpha, \tilde{S}_j(0.8)$	$[\overline{ER}]_\alpha, \tilde{S}_j(0.8)$	$[\underline{y}]_\alpha, \tilde{S}_j(0.8)$	$[\overline{y}]_\alpha, \tilde{S}_j(0.8)$
0	-5.58293×10^{-8}	2.75062×10^{-7}	-0.47133	0.21436
0.2	-2.27401×10^{-8}	2.41973×10^{-7}	-0.40276	0.14579
0.4	1.03491×10^{-8}	2.08884×10^{-7}	-0.33419	0.07722
0.6	4.34384×10^{-8}	1.75795×10^{-7}	-0.26562	0.00865
0.8	7.65274×10^{-8}	1.42706×10^{-7}	-0.19705	-0.05991
1	1.09616×10^{-7}	1.09616×10^{-7}	-0.12848	-0.12848

Summarization of the exact solution and the approximate solution by sixth-order FF-OHAM over all $x \in [0,1]$ and $\alpha \in [0,1]$ corresponding to the best optimal convergence control values $\tilde{S}_j(0.8)$ of Eq. (6.67) is illustrated in the following three-dimensional graph in Figure 6.13.

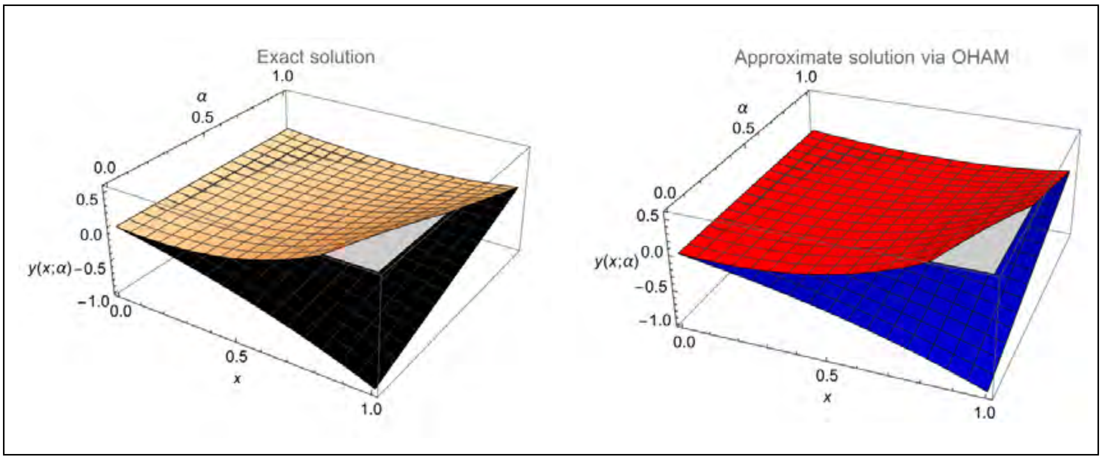

Figure 6.13. The three-dimensional graph of exact solution and approximate solution of Eq. (6.67) given by sixth-order FF-OHAM over all $x \in [0,1]$ and for all $\alpha \in [0,1]$

Tables 6.5-6.6 and Figures 6.12-6.13 clarify that the sixth-order FF-HAM and sixth-order FF-OHAM satisfy the property fuzzy numbers [Definition 3.7] and fuzzy differential equation properties (Salahshour, 2011) of Eq.(6.67) by the formation of the triangular shape of fuzzy numbers. Finally, in revealing the efficiency of the proposed methods for solving Eq. (6.67), the accuracy of the solutions on the interval $x \in [0,1]$ for all $\alpha \in [0,1]$ corresponding to the optimal convergence control parameter is demonstrated in the following Figures 6.14-6.15.

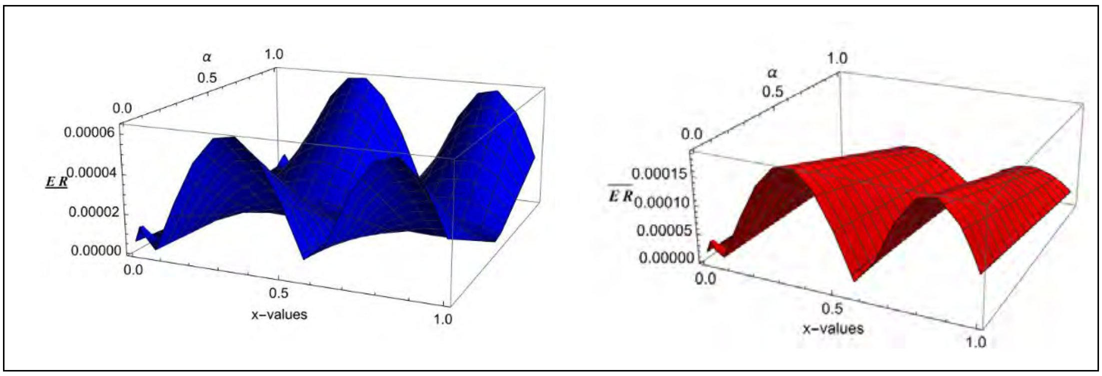

Figure 6.14. The accuracy of Eq. (6.67) by sixth-order FF-HAM for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,1]$

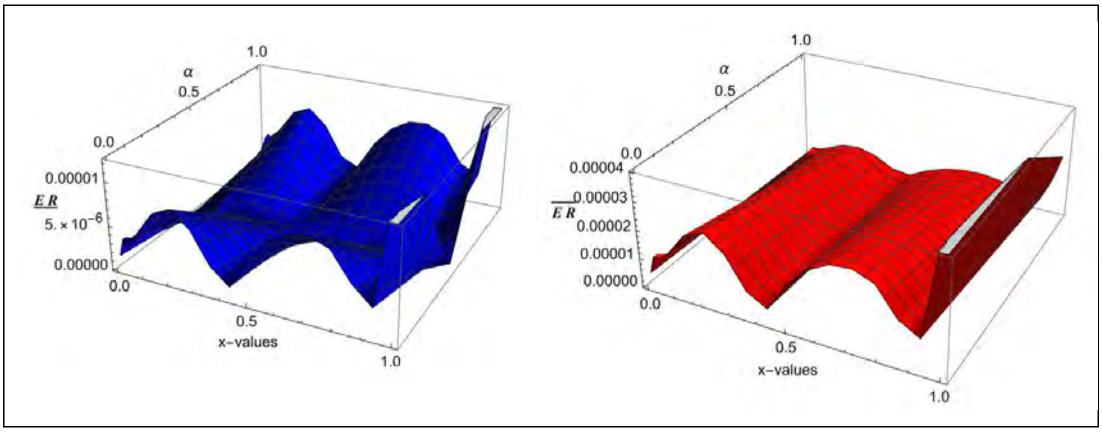

Figure 6.15. The accuracy of Eq. (6.67) by sixth-order FF-OHAM for all three dimensions for $\alpha \in [0,1]$ and $x \in [0,1]$.

6.5.3 Example 6.3

Consider the mathematical model, a nonlinear fractional temperature distribution equation of order $\beta_1 \in (1,2]$ which explains the distribution of the temperature in the lumped convection system in a layer comprised of materials with varying thermal conductivity (Ismail et al., 2019):

$$\begin{cases} D^{(\beta_1)}y(x) - \eta\big(y(x)\big)^4 = 0,\ x \in [0,1] \\ \qquad y'(0) = 0,\ y(1) = 1. \end{cases} \tag{6.75}$$

where x is the time independent variable, $y(x)$ is the dimensionless temperature.

The following is the fuzzy version of Eq. (6.75):

$$\begin{cases} D^{(\beta_1)}\tilde{y}(x;\alpha) - \eta\big(y(x;\alpha)\big)^4 = 0,\ x \in [0,1] \\ \underline{y}'(0;\alpha) = (0.1\alpha - 0.1), \quad \underline{y}(1;\alpha) = (0.1\alpha + 0.9), \\ \overline{y}'(0;\alpha) = (0.1 - 0.1\alpha), \quad \overline{y}(1;\alpha) = (1.1 - 0.1\alpha). \end{cases} \tag{6.76}$$

6.5.3.1 Solving Example 6.3 by FF-HAM

To solve this example by FF-HAM, based on Section 6.3.2, we construct the zeroth-order and k^{th}-order deformation equations for $k \geq 1$ of Eq. (6.76) as follows:

$$\begin{cases} \tilde{y}_k(x;\alpha) = [1-(0.1-0.1\alpha)x, 1+(0.1-0.1\alpha)x] + \\ \qquad \psi_k\big(\tilde{y}_{k-1}(x;\alpha)\big) \\ \qquad +\tilde{h}(\alpha)\, \tilde{\mathcal{J}}^{(\beta_1)}\mathcal{R}_k(\overrightarrow{\tilde{y}_{k-1}}(x;\alpha)) \\ \underline{y}'(0;\alpha) = (0.1\alpha - 0.1), \quad \underline{y}(1;\alpha) = (0.1\alpha + 0.9), \\ \overline{y}'(0;\alpha) = (0.1 - 0.1\alpha), \quad \overline{y}(1;\alpha) = (1.1 - 0.1\alpha) \end{cases} \tag{6.77}$$

such that

$$\psi_k = \begin{cases} 0, & k \leq 1 \\ 1, & k > 1 \end{cases}, \text{ and}$$

$$\begin{cases} \mathcal{R}_k\left(\overrightarrow{\tilde{y}_{k-1}}(x;\alpha)\right) = \tilde{y}_{k-1}{}^{(\beta_1)}(x;\alpha) - \eta \\ \left(\sum_{i=0}^{k-1} \tilde{y}_{k-1-i}(x;\alpha) \sum_{j=0}^{i} \tilde{y}_{i-j}(x;\alpha) \sum_{s=0}^{j} \tilde{y}_s(x;\alpha)\tilde{y}_{j-s}(x;\alpha)\right) \end{cases} \tag{6.78}$$

For the initial approximation guess:

$$\tilde{y}_0(x;\alpha) = [1-(0.1-0.1\alpha)x, 1+(0.1-0.1\alpha)x].$$

Expanding Eq. (6.78) we have the following:

For first-order problem

$$\tilde{y}_1(x;\alpha) = [1-(0.1-0.1\alpha)x, 1+(0.1-0.1\alpha)x] + \tilde{h}(\alpha)\, \tilde{y}_0(x;\alpha) - \eta\, \tilde{h}(\alpha)\, \tilde{\mathcal{J}}^{(\beta_1)}([\,\tilde{y}_0(x;\alpha)]^4)$$

For $k > 1$

$$\tilde{y}_k(x;\alpha) = [1-(0.1-0.1\alpha)x, 1+(0.1-0.1\alpha)x] + \big(\tilde{y}_{k-1}(x;\alpha)\big) + \tilde{h}(\alpha)\, \tilde{y}_{k-1}(x;\alpha) - \eta\, \tilde{h}(\alpha)\, \tilde{\mathcal{J}}^{(\beta_1)}([\,\tilde{y}_{k-1}(x;\alpha)]^4)$$

with tenth-order FF-HAM approximate series solution $\tilde{y}(x;\tilde{h};\alpha) = \sum_{j=0}^{10} \tilde{y}_j(x,\tilde{h};\alpha)$. Since this equation is without exact analytical solution, we can demonstrate the accuracy of the FF-HAM for Eq. (6.76) by taking the residual error as mentioned in Section 6.3.3, such that for $\eta = 0.6$ we have

$$\widetilde{ER}(x;\alpha;\tilde{h}) = \tilde{y}^{(\beta_1)}(x;\alpha;\tilde{h}) - 0.6\,[\tilde{y}(x;\alpha;\tilde{h})]^4. \qquad (6.79)$$

It is to be noted that the series in Eq. (6.79) depend upon x, α and the convergent control-parameter $\tilde{h}$. We have plotted the $\tilde{h}$-curve for $\tilde{y}(0.1;1;\tilde{h})$ by tenth-order approximation of the FF-HAM. As in previous examples we plotted the $\tilde{h}$-curve when $\alpha = 1$ to obtain the optimal value of $\tilde{h}$.

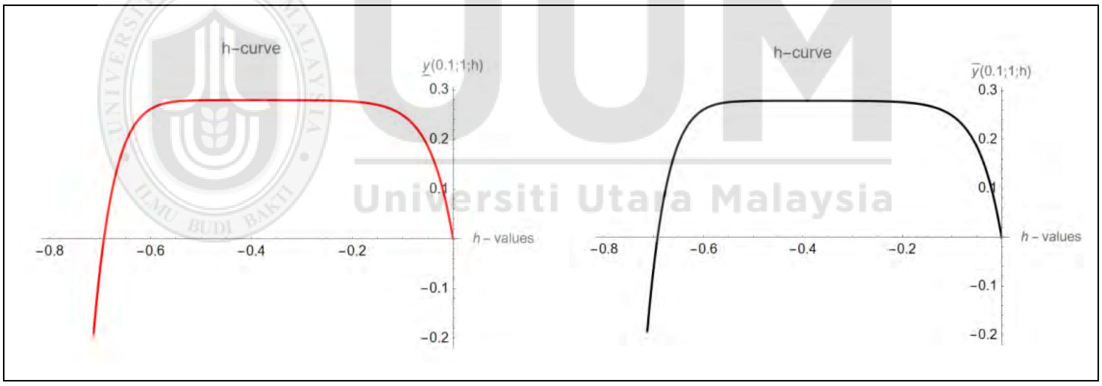

Figure 6.16. The $\tilde{h}$-curve for the fuzzy solution of Eq. (6.76) given by tenth-order FF-HAM when $H(x) = 1$

According to these curves, it is easy to discover the valid region of $\tilde{h}$ when the series solution of FF-HAM corresponds to the line segment nearly parallel to the horizontal axis, such that the valid region of $\tilde{h}$ is bounded by $-0.7 \le \tilde{h} < 0$. According to Section 6.3.3, the best values of $\tilde{h}$ we can obtain in that interval is $\tilde{h} = [-0.45259559205862615, -0.42754247593277295]$.

In Table 6.7, we have tabulated the residual errors$[\underline{ER}]_\alpha$ and $[\overline{ER}]_\alpha$ from Eq. (6.79) of the approximate solutions $\underline{y}(0.1; 0.1;\ \tilde{h})$ and $\overline{y}(0.1; 0.1;\ \tilde{h})$ obtained by using third-order FF-HAM.

Table 6.7

The approximate solution and error of Eq. (6.76) by tenth-order FF-HAM when $\beta_1 = 1.9$ *at* $x = 0.1$ *for all* $\alpha \in [0,1]$

$\boldsymbol{\alpha}$	$[\underline{ER}]_\alpha, \tilde{h}$	$[\overline{ER}]_\alpha, \tilde{h}$	$[\underline{y}]_\alpha, \tilde{h}$	$[\overline{y}]_\alpha, \tilde{h}$
0	-9.04873×10^{-5}	-4.87830×10^{-5}	0.81737	0.83220
0.5	-6.94694×10^{-5}	-4.97735×10^{-5}	0.82173	0.82913
1	-6.81546×10^{-5}	-6.35808×10^{-5}	0.82565	0.82564

The following three-dimensional graph in Figure 6.17 summarizes the solutions by tenth-order FF-HAM over all $x \in [0,0.1]$ and $\alpha \in [0,1]$ corresponding to the best optimal convergence control values $\tilde{h}$ of Eq. (6.76).

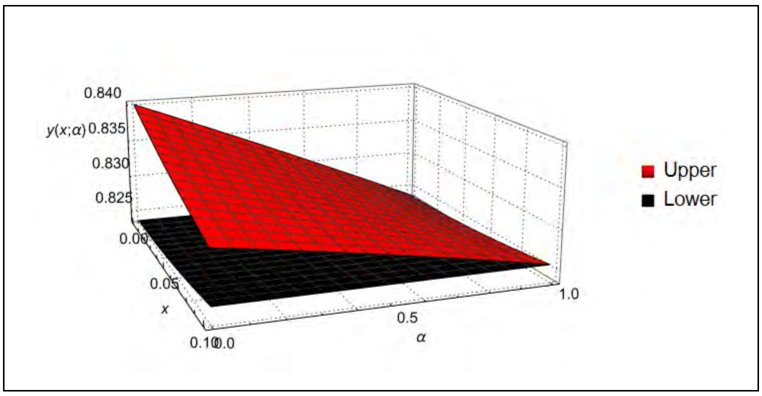

Figure 6.17. The three-dimensional approximate solution of Eq. (6.76) given by tenth-order FF-HAM over all $x \in [0,0.1]$ at $\beta_1 = 1.9$, and for $\eta = 0.6$, and for all $\alpha \in [0,1]$

6.5.3.2 Solving Example 6.3 by FF-OHAM

To solve Example 6.3 by FF-OHAM, based on Section 6.4.2, we can build the approximate series solution for Eq. (6.76) of order $\beta_1 \in (1,2]$ [Definition 3.22] for all $\alpha \in [0,1]$ [Definition 3.5] as follows:

$$(1-q)\left[\widetilde{D}^{(\beta_1)}\left(\tilde{y}(x;\alpha)\right)\right] = \sum_{j=1}^{k} \underline{S}_j(\alpha) q^j \left[\widetilde{D}^{(\beta_1)}\left(\tilde{y}(x;\alpha)\right) - \eta\left(\tilde{y}(x;\alpha)\right)^4\right], \quad (6.80)$$

where

$$\tilde{y}(x;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{j=1}^{k} \tilde{y}_j(x, S_1, .., S_j; \alpha) q^j \quad (6.81)$$

For zeroth-order problem

$$\tilde{y}_0(x;\alpha) = [\tilde{0}]. \quad (6.82)$$

For first to tenth-order problems

$$\begin{cases} (1-q)\left[\widetilde{D}^{(\beta_1)}\left(\tilde{y}(x;\alpha)\right)\right] = \sum_{j=1}^{10} \tilde{S}_j(\alpha) q^j [\widetilde{D}^{(\beta_1)}\left(\tilde{y}(x;\alpha)\right) - \\ \eta\left(\sum_{i=0}^{9} \tilde{y}_{9-i}(x;\alpha) \sum_{j=0}^{i} \tilde{y}_{i-j}(x;\alpha) \sum_{s=0}^{j} \tilde{y}_s(x;\alpha) \tilde{y}_{j-s}(x;\alpha)\right)] \\ \tilde{y}_k{}'(0;\alpha) = \tilde{y}_k(1;\alpha) = \tilde{0} \end{cases} \quad (6.83)$$

Next, using Mathematica 12 DSolve Package to find the solutions for the lower and the upper bounds of Eq. (6.76), for $j = 1,2,..,10$, we obtain

$$\tilde{y}(x;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{j=1}^{10} \tilde{y}_j(x, S_1, .., S_j; \alpha)\, q^j. \quad (6.84)$$

For this example, Tables 6.7 and 6.10 and Figures 6.17-6.18 clarify that the tenth-order FF-HAM and tenth-order FF-OHAM solutions satisfy the fuzzy numbers [Definition 3.7] and fuzzy differential equation properties (Salahshour, 2011) of Eq.(6.76) by the formation of the triangular shape of fuzzy numbers [Definition 3.7] as follows:

Table 6.8

Lower auxiliary convergence parameters of tenth-order FF-OHAM for solving Eq. (6.76) at $\beta_1 = 1.9$, $x = 0.1$, *for all* $\alpha \in [0,1]$

$\boldsymbol{\alpha}$		$\underline{\boldsymbol{S}}_{\boldsymbol{j}}$	
0	$\underline{S}_1 = -0.42754243116098856$ $\underline{S}_4 = 0.057027658640051423$ $\underline{S}_7 = 0.14476539458261192$	$\underline{S}_2 = 0.10029988098228566$ $\underline{S}_5 = -0.026736473824494192$ $\underline{S}_8 = 2.6782513149178886$ $\underline{S}_{10} = 0$	$\underline{S}_3 = -0.13497958972180282$ $\underline{S}_6 = -0.09334881995323582$ $\underline{S}_9 = -3.1547021023298236$
0.5	$\underline{S}_1 = -0.4275424758687012$ $\underline{S}_4 = 0.07545735284696302$ $\underline{S}_7 = 0.33623100353843177$	$\underline{S}_2 = 0.10581949295995502$ $\underline{S}_5 = -0.049825680474098175$ $\underline{S}_8 = -1.780319356305217$ $\underline{S}_{10} = 0$	$\underline{S}_3 = -0.1375566084081214$ $\underline{S}_6 = -0.21083114235974942$ $\underline{S}_9 = 1.8279961729200187$
1	$\underline{S}_1 = -0.4275424759052609$ $\underline{S}_4 = 0.0819534530845722$ $\underline{S}_7 = 0.12107016205095011$	$\underline{S}_2 = 0.10838719398757574$ $\underline{S}_5 = -0.05404354434171497$ $\underline{S}_8 = -0.9412597479377718$ $\underline{S}_{10} = 0$	$\underline{S}_3 = -0.1393691078519383$ $\underline{S}_6 = -0.06431130239382181$ $\underline{S}_9 = 1.0018880129871244$

Table 6.9

Upper auxiliary convergence parameters of tenth-order FF-OHAM for solving Eq. (6.76) at $\beta_1 = 1.9$, $x = 0.1$, *for all* $\alpha \in [0,1]$

$\boldsymbol{\alpha}$		$\overline{\boldsymbol{S}}_{\boldsymbol{j}}$	
0	$\overline{S}_1 = -0.4499999715464292$ $\overline{S}_4 = 0.02200367504397761$ $\overline{S}_7 = 0.20247748961630552$	$\overline{S}_2 = 0.07033774093204971$ $\overline{S}_5 = -0.013738000249538687$ $\overline{S}_8 = -1.167757698543094$ $\overline{S}_{10} = 0$	$\overline{S}_3 = -0.08588077852784994$ $\overline{S}_6 = -0.13177749373901043$ $\overline{S}_9 = 1.1715997608744646$
0.5	$\overline{S}_1 = -0.4500000000118868$ $\overline{S}_4 = 0.053720614964490584$ $\overline{S}_7 = 0.1700603758070842$	$\overline{S}_2 = 0.11085869384502318$ $\overline{S}_5 = -0.019748829952898384$ $\overline{S}_8 = -0.7863871032395353$ $\overline{S}_{10} = 0$	$\overline{S}_3 = -0.1322901831399701$ $\overline{S}_6 = -0.12141795332965101$ $\overline{S}_9 = 0.7686499426055967$
1	$\overline{S}_1 = -0.4499999999893346$ $\overline{S}_4 = 0.0473415579627094$ $\overline{S}_7 = 0.2076338115632854$	$\overline{S}_2 = 0.1095056263140715$ $\overline{S}_5 = -0.0185338348210904$ $\overline{S}_8 = -1.0720018208884141$ $\overline{S}_{10} = 0$	$\overline{S}_3 = -0.1302513333365527$ $\overline{S}_6 = -0.14589007991813402$ $\overline{S}_9 = 1.0605759385659808$

Table 6.10

The approximate solution and error of Eq. (6.76) by tenth-order FF-OHAM when $\beta_1 = 1.9$ *at* $x = 0.1$ *for all* $\alpha \in [0,1]$

α	$[\underline{ER}]_{\alpha}, \underline{S}_j$	$[\overline{ER}]_{\alpha}, \overline{S}_j$	$[\underline{y}]_{\alpha}, \underline{S}_j$	$[\overline{y}]_{\alpha}, \overline{S}_j$
0	-7.73663×10^{-5}	-1.78658×10^{-5}	0.81730	0.83216
0.5	-6.65123×10^{-6}	1.68215×10^{-6}	0.82167	0.82909
1	6.42966×10^{-5}	-2.62733×10^{-5}	0.82560	0.82560

Figure 6.18 illustrates the summary of the solutions by tenth-order FF-OHAM over all $x \in [0,0.1]$ and $\alpha \in [0,1]$ corresponding with the best optimal convergence control values $\tilde{S}_j$ of Eq.(6.76) in three-dimensional graph.

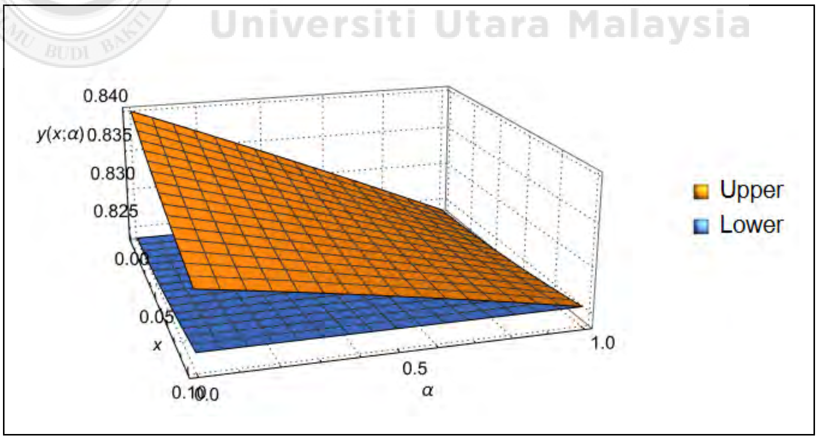

Figure 6.18. The three-dimensional approximate solution of Eq. (6.76) given by tenth-order FF-OHAM over all $x \in [0,0.1]$ at $\beta_1 = 1.9$ and for $\eta = 0.6$, and for all $\alpha \in [0,1]$

6.6 Summary of Findings

1. In this chapter, the approximate-analytical methods FF-HAM and FF-OHAM with convergence-control ability have been successfully developed for linear and nonlinear multi fractional order FFOBVPs of order $1 < \beta_1 \leq 2$ and $0 < \beta_2 \leq 1$.

2. The convergence dynamic of the developed methods that has been introduced in Chapter 4 is valid for multi fractional order FFOBVPs of order $1 < \beta_1 \leq 2$ and $0 < \beta_2 \leq 1$, in selecting the optimal convergence parameter for each method for the purpose of controlling the series solution.

3. FF-HAM and FF-OHAM have been utilized to solve the Bagley-Torvik equation, which is an inhomogeneous linear FFOBVP. The findings are as follows:

    a) FF-HAM and FF-OHAM provide accurate series solution for solving linear case.

    b) Employing the convergence dynamic of FF-HAM and FF-OHAM has successfully provide optimal convergence parameters for FF-HAM and FF-OHAM.

    c) Generally, FF-OHAM yields more accurate solutions than FF-HAM.

    d) As the series order increases, the accuracy of solution by FF-HAM and FF-OHAM also increases and the obtained solution converges to the exact solution.

e) FF-HAM and FF-OHAM yield more accurate solutions compared with SCM.

f) All fuzzy fractional solution by FF-HAM and FF-OHAM satisfy the triangular fuzzy solution.

g) Figures and tables provide illustrations of results including optimal parameters, valid region of the convergence parameters, errors and solutions.

4. FF-HAM and FF-OHAM have been utilized to solve the inhomogeneous linear FFOBVPs containing first and second fractional derivative. The findings are as follows:

a) FF-HAM and FF-OHAM provide accurate series solution for solving linear case for multi fractional derivative.

b) Employing the convergence dynamic of FF-HAM and FF-OHAM has successfully provide the optimal convergence parameters for FF-HAM and FF-OHAM.

c) Generally, FF-OHAM yields more accurate solutions than FF-HAM.

d) Comparing the the series solutions via FF-HAM and FF-OHAM with the exact solution

e) All fuzzy fractional solution by FF-HAM and FF-OHAM satisfy the triangular fuzzy solution.

f) Figures and tables provide illustrations of results including optimal parameters, valid region of the convergence parameters, errors and solutions.

5. We introduce a new fuzzy version of fractional temperature distribution equation, followed by the utilization of FF-HAM and FF-OHAM to find the series solution of this nonlinear problem. The findings are as follows:

a) FF-HAM and FF-OHAM provide accurate series solution for solving non-linear case but for high series order due to the strong nonlinearity with this equation.

b) Employing the convergence dynamic of FF-HAM and FF-OHAM has successfully provide the optimal convergence parameters for FF-HAM and FF-OHAM.

c) Generally, FF-OHAM yields more accurate solution compared with FF-HAM for nonlinear case, but it will be more costly in terms of CPU.

d) All fuzzy fractional solution by FF-HAM and FF-OHAM satisfy the triangular fuzzy solution for nonlinear case.

g) Figures and tables provide illustrations of results including optimal parameters, valid region of the convergence parameters, errors and solutions.

6. FF-HAM and FF-OHAM provide more accurate solutions for solving second-order FFOBVPs compared with the second order FFOIVPs especially for nonlinear case.

CHAPTER SEVEN
CONCLUSION

7.1 Introduction

This chapter provides the conclusion of the study on the development of new approximate-analytic methods with convergence-control capability. This chapter attempts to shed light on the concepts, observations and results which has been discussed in the previous chapters**.** The summary of the study is given followed by contribution of the study, limitations and future research.

7.2 Summary of the Study

FFODEs, with their capability to model and explain complex phenomena including uncertainty, have gained considerable degree of importance among researchers and practitioners. However, the existing approximate-analytic methods lack the ability of controlling the convergence of solutions. Therefore, the aim of the study is to develop new approximate-analytical methods with convergence-control for solving first-order and second-order FFOIVPs and FFOBVPs. Three types of FFODEs have been the focus of this study: first-order FFOIVPs, second-order FFOIVPs and second-order FFOBVPs. The developed approximate-analytical methods are based on the concept of HAM where two methods: FF-HAM and FF-OHAM, are constructed for each of the three types of FFODEs. The convergence of the constructed methods have also been established based on the convergence-control parameters of each method. In general, the study involves two main parts: theoretical part and experimental part. The theoretical part utilizes relevant existing mathematical definitions and theorems of fuzzy set theory, fractional calculus theory and fuzzy fractional derivatives. This part also utilizes the general structure of the existing F-HAM and F-OHAM where this structure is the basis of the development of the proposed methods in this study: FF-HAM and FF-OHAM. Previous studies have utilized the concept of HAM and OHAM to solve crisp linear and nonlinear problems involving FODEs yielding F-HAM and F-OHAM; in this study, the concept of HAM and OHAM has been successfully introduced to develop the proposed FF-HAM and FF-OHAM, by employing fuzzy set theory properties, to solve fuzzy linear and nonlinear problems involving FFODEs. The definitions of differentiability used in the methods are the Caputo fractional derivatives.

The experimental part of this study involves obtaining approximate-analytical solutions of several linear and nonlinear FFODEs with respect to different order of solution series, determining convergence of solutions and also comparing results of the constructed methods against existing methods.

The results of experimentation on the examples using the developed approximate-analytical methods FF-HAM and FF-OHAM indicated that these methods are capable and powerful in finding solutions of linear and nonlinear FFODEs models. The convergence of FF-HAM for FFOIVPs models has been discussed. The solutions obtained by using FF-OHAM with FF-HAM at the same order of terms has been compared and FF-OHAM has been found to give an approximate solution of FFODEs with a higher degree of accuracy. The solution contains the convergence control parameter h which provides a simple way to adjust and control the convergence region of the resulting infinite approximate-analytical solution series. The selection of the convergence-control parameter h is important to guarantee the convergence of homotopy series solutions. It has been shown that choosing any value of convergence-control parameter h from the h-curve convergence region would result in the region giving the optimal value of h. Towards to end, it is concluded that only one value of convergence-control parameter h is needed for any $\alpha \in [0,1]$ and this value of h is the same value for each α-level set. Furthermore, we conclude that the selection of values for the auxiliary constants of FF-OHAM, which are the convergence-control parameters, are very important in adjusting the convergence of the FF-OHAM solution. These auxiliary constants must be determined for each α-level set for all $\alpha \in [0,1]$. Moreover, in certain problems for some α-level set, the optimal convergence-control parameters have been found to be missing as a consequence of the large output. To avoid this problem, we have suggested to extract FF-OHAM solution with convergence-control parameters at a certain α-level making it less computationally demanding with reduction of time-consumption and also making it easier to find the optimal value of these parameters. Finally, we have shown that approximate-analytical methods are very accurate, capable, and powerful in solving linear and nonlinear fuzzy models involving fractional DEs. We have shown the absolute error between our approximate solution and the exact solution in figures and tables. The numerical results show the efficiency of these methods to deal with linear and non-linear models where numerical results that satisfies the properties of fuzzy numbers are illustrated by the triangular fuzzy numbers shape. In addition, a comparative study has been conducted between the proposed methods with other approximate methods, depending on the residual or absolute error, where results indicate the efficiency of the developed methods.

The advantages of the developed methods are as follows:

1. FF-HAM and FF-OHAM are able to solve difficult nonlinear problems

2. FF-HAM and FF-OHAM are able to solve high-order problems directly without reducing it into first-order system.

3. FF-HAM and FF-OHAM are capable of determining the accuracy of the obtained approximate-analytical solution without needing the exact or actual solution. The accuracy can be determined using the residual error. This capability is important especially for nonlinear equations.

4. Equipped with convergence-control ability, FF-HAM and FF-OHAM provide a simple way to ensure the convergence of solution series. In addition, the methods allow great freedom to choose proper base function in approximating a nonlinear problem.

5. FF-HAM and FF-OHAM are capable of yielding approximate-analytical solutions to nonlinear problems without the need for linearization.

6. FF-HAM and FF-OHAM are capable of yielding approximate-analytical solutions to both linear and nonlinear problems without the need for discretization.

7. The approximate solutions obtained using FF-HAM and FF-OHAM are in the form of series which rapidly converges to the exact solution.

8. FF-HAM provides an efficient solution with high accuracy, minimal calculation without having to deal with physically unrealistic assumptions.

9. FF-OHAM has similar built-in convergence criteria as in FF-HAM but with a greater degree of flexibility, rendering it better than FF-HAM.

7.3 Contribution of the study

The study has contributed to mathematics area in terms of the theoretical formulation of the constructed methods and the establishment of convergence of the approximate-analytic solutions. The developed methods are capable of dealing not only with linear and nonlinear FFODEs but also with first-order and second-order FFODEs.

The methods developed in this study have made the contribution to the approximate-analytical class of methods towards overcoming obstacles of existing methods in that class for solving FFODEs by providing the convergence parameters, thus ensuring the convergence of the series solutions. In other words, FF-HAM and FF-OHAM can be employed without having to worry about solution being divergent.

For second-order FFODEs, the developed methods can handle them directly. This study has contributed the convergence-dynamic algorithms of FF-HAM and FF-OHAM whose mechanism is used to control and adjust the convergence of the approximate-analytical series solutions. In addition, the FF-HAM and FF-OHAM have been constructed for a new model of fuzzy fractional boundary value problems of multi-fractional order, which contain first fractional order with $\beta_1 \in (1,2]$ and $\beta_2 \in (0,1]$.

Among the examples in the experimental part of the study, new fuzzy versions of several important fractional differential equations, such as fractional quadratic Riccati differential equation and heat transfer differential equation, have been created in the study to provide new reliable models which would contribute in assisting researchers to handle potential uncertainty cases with respect to these important equations.

7.4 Limitation of the Study

The capability of the developed method FF-HAM and FF-OHAM has only been tested on several physical and engineering applications including Fuzzy fractional Riccati differential equations, Fuzzy fractional Temperature distribution equation and Bagley-Torvik equation. The developed FF-OHAM generally yields an approximate-analytic solution of the FFODE with a high degree of accuracy. However, this method has several requirements. In FF-OHAM, a set of nonlinear algebraic equations needs to be solved at each order of solution series approximation. FF-OHAM includes many unknown convergence-control parameters, where these parameters need to be determined at each fuzzy level set for the lower and the upper solution of FFODEs making it time-consuming and computationally demanding.

7.5 Recommendations for Future Study

This study has developed approximate-analytical methods: FF-HAM and FF-OHAM to solve first-order and second-order FFOIVPs as well as second-order FFOBVPs. For future studies, researcher can focus on several aspects.

The choice of initial function plays important role towards finding the convergent series solution in this study. It is suggested to improve the developed FF-HAM and FF-OHAM to obtain an approximate solution equivalent to the exact solution through a minimal number of iterations.

The capability of the developed method FF-HAM and FF-OHAM has been tested by solving several physical and engineering applications including fuzzy fractional Riccati differential equations, fuzzy fractional Temperature distribution equation and Bagley-Torvik equation. It is suggested to apply the developed FF-HAM and FF-OHAM for solving other complicated physical and engineering problems based on FFODEs such as HIV dynamic and fuzzy fractional kinetic equations in modelling acid hydrolysis reaction.

The developed FF-OHAM gives an approximate solution of FFODEs with a high degree of accuracy. However, since the developed FF-OHAM is considered to be computationally demanding, it is suggested to explore techniques to improve the computational efficiency of the developed FF-OHAM.

In this study, the construction of FF-HAM and FF-OHAM is based on Hukuhara Caputo derivative. For future study, it is suggested to base the construction on other definition of derivatives such as the generalized Hukuhara Caputo derivative.

In this study, the concept of HAM and OHAM has been used in the construction of the proposed FF-HAM and FF-OHAM for solving FFODEs. It is suggested as future research to employ the concept of HAM and OHAM to solve fuzzy fractional PDEs and fuzzy fractional integro differential equations.

REFERENCES

Abdel Aal, M., Abu-Darwish, N., Abu Arqub, O., Al-Smadi, M., & Momani, S. (2019). Analytical solutions of fuzzy fractional boundary value problem of order 2α by using RKHS algorithm. *Applied Mathematics and Information Sciences*, *13*(4), 523–533. https://doi.org/10.18576/amis/130402

Abdelkawy, M. A., Zaky, M. A., Bhrawy, A. H., & Baleanu, D. (2015). Numerical simulation of time variable fractional order mobile-immobile advection-dispersion model. *Romanian Reports in Physics*, *67*(3), 773–791.

Abdollahi, R., Farshbaf Moghimi, M., Khastan, A., & Hooshmandasl, M. R. (2019). Linear fractional fuzzy differential equations with Caputo derivative. *Computational Methods for Differential Equations*, *7*(2), 252–265.

Abdul Rahman, N. A., & Ahmad, M. Z. (2017). Solving fuzzy fractional differential equations using fuzzy Sumudu transform. *The Journal of Nonlinear Sciences and Applications*, *10*(05), 2620–2632. https://doi.org/10.22436/jnsa.010.05.28

Agarwal, R. P., Baleanu, D., Nieto, J. J., Torres, D. F. M., & Zhou, Y. (2018). A survey on fuzzy fractional differential and optimal control nonlocal evolution equations. *Journal of Computational and Applied Mathematics*, *339*, 3–29. https://doi.org/10.1016/j.cam.2017.09.039

Agarwal, R. P., Lakshmikantham, V., & Nieto, J. J. (2010). On the concept of solution for fractional differential equations with uncertainty. *Nonlinear Analysis: Theory, Methods & Applications*, *72*(6), 2859–2862. https://doi.org/10.1016/j.na.2009.11.029

Ahmad, M. Z., Hasan, M. K., & Abbasbandy, S. (2013). Solving fuzzy fractional differential equations using Zadeh's extension principle. *The Scientific World Journal*, *2013*(1), 1–11.

Ahmad, M. Z., Hasan, M. K., & De Baets, B. (2013). Analytical and numerical solutions of fuzzy differential equations. *Information Sciences*, *236*, 156–167. https://doi.org/10.1016/j.ins.2013.02.026

Ahmadian, A., Ismail, F., Salahshour, S., Baleanu, D., & Ghaemi, F. (2017). Uncertain viscoelastic models with fractional order: A new spectral tau method to study the numerical simulations of the solution. *Communications in Nonlinear Science and Numerical Simulation*, *53*, 44–64. https://doi.org/10.1016/j.cnsns.2017.03.012

Ahmadian, A., Salahshour, S., Baleanu, D., Amirkhani, H., & Yunus, R. (2015). Tau method for the numerical solution of a fuzzy fractional kinetic model and its application to the oil palm frond as a promising source of xylose. *Journal of Computational Physics*, *294*, 562–584. https://doi.org/10.1016/j.jcp.2015.03.011

Ahmadian, Ali, Suleiman, M., Salahshour, S., & Baleanu, D. (2013). A Jacobi operational matrix for solving a fuzzy linear fractional differential equation. *Advances in Difference Equations*, *2013*(1), 1–29. https://doi.org/10.1186/1687-1847-2013-104

Alaroud, M., Saadeh, R., Al-smadi, M., Ahmad, R. R., Din, ummul K. S., & Arqub, O. abu. (2019). Solving nonlinear fuzzy fractional IVPs using fractional residual power series algorithm. *IACM*, 170–175.

Alderremy, A. A., Gómez-Aguilar, J. F., Aly, S., & Saad, K. M. (2021). A fuzzy fractional model of coronavirus (COVID-19) and its study with Legendre spectral method. *Results in Physics*, *21*, 103773. https://doi.org/10.1016/j.rinp.2020.103773

Allahviranloo, T., Salahshour, S., & Abbasbandy, S. (2012). Explicit solutions of fractional differential equations with uncertainty. *Soft Computing*, *16*(2), 297–302. https://doi.org/10.1007/s00500-011-0743-y

Allahviranloo, T, Abbasbandy, S., Shahryari, M. R. B., Salahshour, S., & Baleanu, D. (2013). On solutions of linear fractional differential equations with uncertainty. *Abstract and Applied Analysis*, *2013*, 1–13. https://doi.org/10.1155/2013/178378

Allahviranloo, Tofigh, Kiani, N. A., & Motamedi, N. (2009). Solving fuzzy differential equations by differential transformation method. *Information Sciences*, *179*(7), 956–966.

Alshorman, M. A., Zamri, N., Ali, M., & Albzeirat, A. K. (2018). New implementation of residual power series for solving fuzzy fractional Riccati equation. *Journal of Modeling and Optimization*, *10*(2), 81–87.

Askari, S., Allahviranloo, T., & Abbasbandy, S. (2019). Solving fuzzy fractional differential equations by adomian decomposition method used in optimal control theory. *Nternational Transaction Journal of Engineering, Management, & Applied Sciences & Technologies*, *10*(12), 1–10. https://doi.org/10.6084/m9.figshare.11110514

Atangana, A., & Secer, A. (2013). A note on fractional order derivatives and table of fractional derivatives of some special functions. *Abstract and Applied Analysis*, *2013*(1), 1–8. https://doi.org/10.1155/2013/279681

Bahia, G., Ouannas, A., Batiha, I. M., & Odibat, Z. (2021). The optimal homotopy analysis method applied on nonlinear time-fractional hyperbolic partial differential equation s. *Numerical Methods for Partial Differential Equations*, *37*(3), 2008–2022.

Băleanu, D., & Mustafa, O. G. (2010). On the global existence of solutions to a class of fractional differential equations. *Computers & Mathematics with Applications*, *59*(5), 1835–1841. https://doi.org/10.1016/j.camwa.2009.08.028

Bencsik, A. L., Bede, B., Tar, ózsef K., & Fodor, J. (2006). Fuzzy differential equations in modeling of hydraulic differential servo cylinders. *In Third Romanian-Hungarian Joint Symposium on Applied Computational Intelligence (SACI)*.

Bodjanova, S. (2006). Median alpha-levels of a fuzzy number. *Fuzzy Sets and Systems*, *157*(7), 879–891. https://doi.org/10.1016/j.fss.2005.10.015

Bonyah, E., Atangana, A., & Chand, M. (2019). Analysis of 3D IS-LM macroeconomic system model within the scope of fractional calculus. *Chaos, Solitons & Fractals: X*, *2*. https://doi.org/10.1016/j.csfx.2019.100007

Buckley, J.J., & Yan, A. (2000). Fuzzy functional analysis (I): Basic concepts. *Fuzzy Sets and Systems*, *115*(3), 393–402. https://doi.org/10.1016/S0165-0114(98)00161-4

Buckley, James J., & Feuring, T. (2001). Fuzzy initial value problem for Nth-order linear differential equations. *Fuzzy Sets and Systems*, *121*(2), 247–255. https://doi.org/10.1016/S0165-0114(00)00028-2

Bulut, H., Baskonus, H. M., & Belgacem, F. B. M. (2013). The analytical solution of some fractional ordinary differential equations by the Sumudu transform method.

Abstract and Applied Analysis, *2013*.

Çelik, C., & Duman, M. (2012). Crank–Nicolson method for the fractional diffusion equation with the Riesz fractional derivative. *Journal of Computational Physics*, *231*(4), 1743–1750. https://doi.org/10.1016/j.jcp.2011.11.008

Chang, S. S. L., & Zadeh, L. A. (1972). On fuzzy mapping and control. *IEEE Transactions on Systems, Man, and Cybernetics*, *SMC-2*(1), 30–34. https://doi.org/10.1109/TSMC.1972.5408553

Chang, S. S. L., & Zadeh, L. A. (1996). On fuzzy mapping and control. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh* (pp. 180–184). World Scientific.

Cui, Y., Ma, W., Sun, Q., & Su, X. (2018). New uniqueness results for boundary value problem of fractional differential equation. *Nonlinear Analysis: Modelling and Control*, *23*(1), 31–39. https://doi.org/10.15388/NA.2018.1.3

Dalir, M., & Bashour, M. (2010). Applications of fractional calculus. *Applied Mathematical Sciences*, *4*(21), 1021–1032.

Das, A. K., & Roy, T. K. (2017). Exact solution of some linear fuzzy fractional differential equation using Laplace transform method. *Global Journal of Pure and Applied Mathematics*, *13*(9), 5427–5435.

Das, S., Pan, I., & Das, S. (2013). Fractional order fuzzy control of nuclear reactor power with thermal-hydraulic effects in the presence of random network induced delay and sensor noise having long range dependence. *Energy Conversion and Management*, *68*, 200–218. https://doi.org/10.1016/j.enconman.2013.01.003

Dehghan, M., Manafian, J., & Saadatmandi, A. (2011). Analytical treatment of some partial differential equations arising in mathematical physics by using the Exp-function method. *International Journal of Modern Physics B*, *25*(22), 2965–2981.

Demirci, E., & Ozalp, N. (2012). A method for solving differential equations of fractional order. *Journal of Computational and Applied Mathematics*, *236*(11), 2754–2762. https://doi.org/10.1016/j.cam.2012.01.005

Deng, Y. (2019). Fractional-order fuzzy adaptive controller design for uncertain robotic manipulators. *International Journal of Advanced Robotic Systems*, *16*(2), 172988141984022. https://doi.org/10.1177/1729881419840223

Diethelm, K. (2010). *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type. Springer Science & Business Media*. https://doi.org/10.1007/978-3-642-14574-2

Dubois, D., & Prade, H. (1982). Towards fuzzy differential calculus part3: Differentation. *Fuzzy Sets and Systems*, *8*(1982), 225–233.

Efe, M. O. (2008). Fractional Fuzzy Adaptive Sliding-Mode Control of a 2-DOF Direct-Drive Robot Arm. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, *38*(6), 1561–1570. https://doi.org/10.1109/TSMCB.2008.928227

Esmaeilbeigi, M., Paripour, M., & Garmanjani, G. (2018). Approximate solution of the fuzzy fractional Bagley-Torvik equation by the RBF collocation method. *Computational Methods for Differential Equations*, *6*(2), 186–214.

Failla, G., & Zingales, M. (2020). Advanced materials modelling via fractional calculus: Challenges and perspectives. *Philosophical Transactions of the Royal*

Society A: Mathematical, Physical and Engineering Sciences, *378*, 1–13. https://doi.org/10.1098/rsta.2020.0050

Fard, O. S. (2009). An iterative scheme for the solution of generalized system of linear fuzzy differential equations. *World Applied Sciences Journal*, *7*(12), 1597–1604.

Farooq, M., Khan, A., Nawaz, R., Islam, S., Ayaz, M., & Chu, Y. M. (2021). Comparative study of generalized couette flow of couple stress fluid using optimal homotopy asymptotic method and new iterative method. *Scientific Reports*, *11*(1), 1–20. https://doi.org/10.1038/s41598-021-82746-8

Frolov, A. L., Frolova, O. A., Sumina, R. S., & Sviridova, E. N. (2020). Mathematical modeling of axisymmetric flow of granular materials. *Journal of Physics: Conference Series*, *1479*(1). https://doi.org/10.1088/1742-6596/1479/1/012115

Garrappa, R. (2018). Numerical solution of fractional differential equations: A survey and a software tutorial. *Mathematics*, *6*(2), 1–23. https://doi.org/10.3390/math6020016

Ghanbari, B., & Akgul, A. (2021). Abundant new analytical and approximate solutions to the generalized schamel equation. *Materials and Design*, *11*(20), 1–32.

Ghazanfari, B., & Veisi, F. (2011). Homotopy analysis method for the fractional nonlinear equations. *Journal of King Saud University - Science*, *23*(4), 389–393. https://doi.org/10.1016/j.jksus.2010.07.019

Ghoreishi, M., Ismail, A. I. B. M., & Alomari, A. K. (2011). Application of the homotopy analysis method for solving a model for HIV infection of CD4+ T-cells. *Mathematical and Computer Modelling*, *54*(11–12), 3007–3015. https://doi.org/10.1016/j.mcm.2011.07.029

Ghoreishi, M., Ismail, A. I. B. M., Alomari, A. K., & Sami Bataineh, A. (2012). The comparison between homotopy analysis method and optimal homotopy asymptotic method for nonlinear age-structured population models. *Communications in Nonlinear Science and Numerical Simulation*, *17*(3), 1163–1177. https://doi.org/10.1016/j.cnsns.2011.08.003

Grover, M., & Tomer, A. (2011). Comparison of optimal homotopy asymptotic method with homotopy perturbation method of twelfth order boundary value problems. *International Journal on Computer Science and Engineering*, *3*(7), 2739–2747.

Guang-Quan, Z. (1991). Fuzzy continuous function and its properties. *Fuzzy Sets and Systems*, *43*(2), 159–171. https://doi.org/10.1016/0165-0114(91)90074-Z

Hamarsheh, M., Ismail, A., & Odibat, Z. (2015). Optimal homotopy asymptotic method for solving fractional relaxation-oscillation equation. *Journal of Interpolation and Approximation in Scientific Computing*, *2015*(2), 98–111. https://doi.org/10.5899/2015/jiasc-00081

Hasan, S., Alawneh, A., Al-Momani, M., & Momani, S. (2017). Second order fuzzy fractional differential equations under Caputo's H-differentiability. *Applied Mathematics & Information Sciences*, *11*(6), 1–12. https://doi.org/10.18576/amis/110606

Hashim, I., Abdulaziz, O., & Momani, S. (2009). Homotopy analysis method for fractional IVPs. *Communications in Nonlinear Science and Numerical Simulation*, *14*(3), 674–684. https://doi.org/10.1016/j.cnsns.2007.09.014

He, J.-H. (2004). Comparison of homotopy perturbation method and homotopy analysis method. *Applied Mathematics and Computation*, *156*(2), 527–539.

Hoa, N. Van, Vu, H., & Duc, T. M. (2019). Fuzzy fractional differential equations under Caputo–Katugampola fractional derivative approach. *Fuzzy Sets and Systems*, *375*, 70–99. https://doi.org/10.1016/j.fss.2018.08.001

Ilie, M., Biazar, J., & Ayati, Z. (2019). Analytic solution for second-order fractional differential equations via OHAM. *Journal of Fractional Calculus and Applications*, *10*(1), 105–119.

Ismail, M., Saeed, U., Alzabut, J., & ur Rehman, M. (2019). Approximate solutions for fractional boundary value problems via green-CAS wavelet method. *Mathematics*, *7*(12), 1–20. https://doi.org/10.3390/MATH7121164

Jafari, H., & Tajadodi, H. (2010). He's variational iteration method for solving fractional Riccati differential equation. . *International Journal of Differential Equations*, 1–8. https://doi.org/10.1155/2010/764738

Jain, P., Kumbhakar, M., & Ghoshal, K. (2021). Application of homotopy analysis method to the determination of vertical sediment concentration distribution with shear-induced diffusivity. *Engineering with Computers*, 1–20.

Jameel, A. F., Saaban, A., Altaie, S. A., Anakira, N. R., Alomari, A. K., & Ahmad, N. (2018). Solving first order nonlinear fuzzy differential equations using optimal homotopy asymptotic method. *International Journal of Pure and Applied Mathematics*, *118*(1), 49–64. https://doi.org/10.12732/ijpam.v118i1.5

Jena, R. M., Chakraverty, S., & Jena, S. K. (2019). Dynamic response analysis of fractionally damped beams subjected to external loads using homotopy analysis method. *Journal of Applied and Computational Mechanics*, *5*(2), 355–366.

https://doi.org/10.22055/jacm.2019.27592.1419

Kaleva, O. (2006). A note on fuzzy differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, *64*(5), 895–900. https://doi.org/10.1016/j.na.2005.01.003

Kandel, A., & Byatt, W. J. (1980). Fuzzy processes. *Fuzzy Sets and Systems*, *4*(2), 117–152. https://doi.org/10.1016/0165-0114(80)90032-9

Kaur, D., Agarwal, P., Rakshit, M., & Chand, M. (2020). Fractional calculus involving (p, q)-mathieu type series. . *Applied Mathematics and Nonlinear Sciences*, *5*(2), 15–34. https://doi.org/10.2478/amns.2020.2.00011

Khan, N. A., Riaz, F., & Razzaq, O. A. (2014). A comparison between numerical methods for solving fuzzy fractional differential equations. *Nonlinear Engineering*, *3*(3), 155–162. https://doi.org/10.1515/nleng-2013-0029

Khodadadi, E., & Çelik, E. (2013). The variational iteration method for fuzzy fractional differential equations with uncertainty. *Fixed Point Theory and Applications*, *2013*(1), 1–7. https://doi.org/10.1186/1687-1812-2013-13

Kim, T., & Kim, D. S. (2020). Note on the degenerate gamma function. *Russian Journal of Mathematical Physics*, *27*(3), 352–358. https://doi.org/10.1134/S1061920820030061

Kudryashov, N. A. (2020). Method for finding highly dispersive optical solitons of nonlinear differential equations. *Optik*, *206*, 163550.

Kumar, A., & Lata, S. (2012). A note on fuzzy initial value problem for Nth-order fuzzy linear differential equations. *Journal of Fuzzy Set Valued Analysis*, *2012*, 1–3. https://doi.org/10.5899/2012/jfsva-00103

Kumar, S., Kumar, R., Singh, J., Nisar, K. S., & Kumar, D. (2020). An efficient numerical scheme for fractional model of HIV-1 infection of CD4+ T-cells with the effect of antiviral drug therapy. *Alexandria Engineering Journal*, *59*(4), 2053–2064.

Lee, M. O., Kumaresan, N., & Ratnavelu, K. (2016). Solution of fuzzy fractional differential equations using homotopy analysis method. *Applied Mathematical Modelling*, *32*(2), 113–119. https://doi.org/10.1016/j.apm.2009.09.011

Li, C., & Chen, A. (2018). Numerical methods for fractional partial differential equations. *International Journal of Computer Mathematics*, *95*(6–7), 1048–1099. https://doi.org/10.1080/00207160.2017.1343941

Li, C., & Deng, W. (2007). Remarks on fractional derivatives. *Applied Mathematics and Computation*, *187*(2), 777–784. https://doi.org/10.1016/j.amc.2006.08.163

Li, C., Qian, D., & Chen, Y. (2011). On Riemann-Liouville and Caputo derivatives. *Discrete Dynamics in Nature and Society*, *2011*, 1–15. https://doi.org/10.1155/2011/562494

Li, C., & Zeng, F. (2013). The finite difference methods for fractional ordinary differential equations. *Numerical Functional Analysis and Optimization*, *34*(2), 149–179. https://doi.org/10.1080/01630563.2012.706673

Liao, S. (2004). On the homotopy analysis method for nonlinear problems. *Applied Mathematics and Computation*, *147*, 499–513. https://doi.org/10.1016/S0096-3003(02)00790-7

Liao, S. (2005). Comparison between the homotopy analysis method and homotopy perturbation method. *Applied Mathematics and Computation*, *169*(2), 1186–1194. https://doi.org/10.1016/j.amc.2004.10.058

Liao, S. (2009). Notes on the homotopy analysis method: Some definitions and theorems. *Communications in Nonlinear Science and Numerical Simulation*, *14*(4), 983–997. https://doi.org/10.1016/j.cnsns.2008.04.013

Liao, S. J. (1999). An explicit, totally analytic approximate solution for blasius' viscous flow problems. *International Journal of Non-Linear Mechanics*, *34*(4), 759–778. https://doi.org/10.1016/S0020-7462(98)00056-0

Liu, Z.-J., Adamu, M. Y., Suleiman, E., & He, J.-H. (2017). Hybridization of homotopy perturbation method and Laplace transformation for the partial differential equations. *Thermal Science*, *21*(4), 1843–1846.

Mabood, F., Ismail, A. I. M., & Hashim, I. (2013). Application of optimal homotopy asymptotic method for the approximate solution of Riccati equation. *Sains Malaysiana*, *42*(6), 863–867.

Machado, J. A. T., Galhano, A. M. S. F., & Trujillo, J. J. (2014). On development of fractional calculus during the last fifty years. *Scientometrics*, *98*(1), 577–582. https://doi.org/10.1007/s11192-013-1032-6

Maier, C., Mattke, J., Pflügner, K., & Weitzel, T. (2020). Smartphone use while driving: A fuzzy-set qualitative comparative analysis of personality profiles influencing frequent high-risk smartphone use while driving in Germany. *International Journal of Information Management*, *55*, 1–12. https://doi.org/10.1016/j.ijinfomgt.2020.102207

Manafian, J., & Teymuri sindi, C. (2018). An optimal homotopy asymptotic method applied to the nonlinear thin film flow problems. *International Journal of Numerical Methods for Heat and Fluid Flow*, *28*(12), 2816–2841. https://doi.org/10.1108/HFF-08-2017-0300

Mansour, N., Cherif, M. S., & Abdelfattah, W. (2019). Multi-objective imprecise programming for financial portfolio selection with fuzzy returns. *Expert Systems with Applications*, *138*. https://doi.org/10.1016/j.eswa.2019.07.027

Mansouri, S. S., & Ahmady, N. (2012). A numerical method for solving Nth-order fuzzy differential equation by using characterization theorem. *Communications in Numerical Analysis*, *2012*, 1–12. https://doi.org/10.5899/2012/cna-00054

Marinca, V., Herişanu, N., & Nemeş, I. (2008). Optimal homotopy asymptotic method with application to thin film flow. *Open Physics*, *6*(3), 648–653. https://doi.org/10.2478/s11534-008-0061-x

Matar, M. M. (2018). Solution of sequential Hadamard fractional differential equations by variation of parameter technique. *Abstract and Applied Analysis*, *2018*(3), 1–7. https://doi.org/10.1155/2018/9605353

Matlob, M. A., & Jamali, Y. (2019). The concepts and applications of fractional order differential calculus in modeling of viscoelastic systems: A primer. *Critical Reviews in Biomedical Engineering*, *47*(4), 249–276. https://doi.org/10.1615/CritRevBiomedEng.2018028368

Mazandarani, M., & Kamyad, A. V. (2013). Modified fractional Euler method for solving fuzzy fractional initial value problem. *Communications in Nonlinear Science and Numerical Simulation*, *18*(1), 12–21.

https://doi.org/10.1016/j.cnsns.2012.06.008

Mizukoshi, M. T., Barros, L. C., Chalco-Cano, Y., Román-Flores, H., & Bassanezi, R. C. (2007). Fuzzy differential equations and the extension principle. *Information Sciences*, *177*(17), 3627–3635. https://doi.org/10.1016/j.ins.2007.02.039

Mohammed, O. H., & Ahmed, S. A. (2013). Solving fuzzy fractional boundary value problems using fractional differential transform method. *Journal of Al-Nahrain University Science*, *16*(4), 225–232. https://doi.org/10.22401/JNUS.16.4.28

Momani, S., & Shawagfeh, N. (2006). Decomposition method for solving fractional Riccati differential equations. *Applied Mathematics and Computation*, *182*(2), 1083–1092. https://doi.org/10.1016/j.amc.2006.05.008

Moore, T. J., & Ertürk, V. S. (2020). Comparison of the method of variation of parameters to semi-analytical methods for solving nonlinear boundary value problems in engineering. *Nonlinear Engineering*, *9*(1), 1–13. https://doi.org/10.1515/nleng-2018-0148

Morales, O. S., & Mendez, J. J. S. (2012). Partition of a nonempty fuzzy set in nonempty convex fuzzy subsets. *Applied Mathematical Sciences*, *6*(59), 2917–2921.

Mosleh, M., & Otadi, M. (2015). Approximate solution of fuzzy differential equations under generalized differentiability. *Applied Mathematical Modelling*, *39*(10–11), 3003–3015. https://doi.org/10.1016/j.apm.2014.11.035

Najariyan, M., & Zhao, Y. (2018). Fuzzy Fractional Quadratic Regulator Problem Under Granular Fuzzy Fractional Derivatives. *IEEE Transactions on Fuzzy Systems*, *26*(4), 2273–2288. https://doi.org/10.1109/TFUZZ.2017.2783895

Nawaz, R., Islam, S., & Yasin, S. (2010). Solution of tenth order boundary value problems using optimal homotopy asymptotic method (OHAM). *Canadian Journal on Computing in Mathematics, Natural Sciences, Engineering & Medicin*, *1*(2), 37–54.

Ngan, R. T., Son, L. H., Ali, M., Tamir, D. E., Rishe, N. D., & Kandel, A. (2020). Representing complex intuitionistic fuzzy set by quaternion numbers and applications to decision making. *Applied Soft Computing Journal*, *87*, 105961. https://doi.org/10.1016/j.asoc.2019.105961

Omana, R. W. (2009). Lower and upper solutions and existence of W 1,1 -solutions of fuzzy differential equations. *Southern Africa Journal of Pure and Applied Mathematics*, *4*(2009), 29–42.

Otadi, M., & Mosleh, M. (2016). Solution of fuzzy differential equations. *International Journal of Industrial Mathematics*, *8*(1), 73–80.

Pagnini, G. (2012). Erdélyi-Kober fractional diffusion. *Fractional Calculus and Applied Analysis*, *15*(1), 117–127. https://doi.org/10.2478/s13540-012-0008-1

Pakdaman, M., Ahmadian, A., Effati, S., Salahshour, S., & Baleanu, D. (2017). Solving differential equations of fractional order using an optimization technique based on training artificial neural network. *Applied Mathematics and Computation*, *293*(2017), 81–95. https://doi.org/10.1016/j.amc.2016.07.021

Panahi, A. (2017). Approximate solution of fuzzy fractional differential equations. *International Journal of Industrial Mathematics*, *9*(2), 111–118.

Patrício, M. F. S., Ramos, H., & Patrício, M. (2019). Solving initial and boundary value problems of fractional ordinary differential equations by using collocation and fractional powers. *Journal of Computational and Applied Mathematics*, *354*, 348–359. https://doi.org/10.1016/j.cam.2018.07.034

Picozzi, S., & West, B. J. (2002). Fractional langevin model of memory in financial markets. *Physical Review E - 66*, *66*(4), 12. https://doi.org/10.1103/PhysRevE.66.046118

PIEGAT, A. (2005). A new definition of the fuzzy set. *Int. J. Appl. Math. Comput. Sci*, *15*(1), 125–140.

Prakash, P., Nieto, J. J., Senthilvelavan, S., & Sudha Priya, G. (2015). Fuzzy fractional initial value problem. *Journal of Intelligent and Fuzzy Systems*, *28*(6), 2691–2704. https://doi.org/10.3233/IFS-151547

Raj, S. R., & Saradha, M. (2015). Solving hybrid fuzzy fractional differential equations by Adam-Bash forth method. *Applied Mathematical Sciences*, *9*(29), 1429–1432. https://doi.org/10.12988/ams.2015.4121047

Rana, J., & Liao, S. (2019a). A general analytical approach to study solute dispersion in non-Newtonian fluid flow. *European Journal of Mechanics, B/Fluids*, *77*, 183–200. https://doi.org/10.1016/j.euromechflu.2019.04.013

Rana, J., & Liao, S. (2019b). On time independent Schrödinger equations in quantum mechanics by the homotopy analysis method. *Theoretical and Applied Mechanics Letters*, *9*(6), 376–381. https://doi.org/10.1016/j.taml.2019.05.006

Rashid, S., Ashraf, R., & Bayones, F. S. (2021). A Novel Treatment of Fuzzy Fractional Swift–Hohenberg Equation for a Hybrid Transform within the Fractional Derivative Operator. *Fractal and Fractional*, *5*(4), 209. https://doi.org/10.3390/fractalfract5040209

Rivaz, A., Fard, O. S., & Bidgoli, T. A. (2016). Solving fuzzy fractional differential equations by a generalized differential transform method. *SeMA Journal*, *73*(2), 149–170. https://doi.org/10.1007/s40324-015-0061-x

Roszkowska, E., & Kacprzak, D. (2016). The fuzzy saw and fuzzy topsis procedures based on ordered fuzzy numbers. *Information Sciences*, *369*, 564–584. https://doi.org/10.1016/j.ins.2016.07.044

Salahshour, S. (2011). Nth-order fuzzy differential equations under generalized differentiability. *Journal of Fuzzy Set Valued Analysis*, *2011*, 1–14. https://doi.org/10.5899/2011/jfsva-00043

Salahshour, S, Allahviranloo, T., & Abbasbandy, S. (2012). Solving fuzzy fractional differential equations by fuzzy Laplace transforms. *Communications in Nonlinear Science and Numerical Simulation*, *17*(3), 1372–1381. https://doi.org/10.1016/j.cnsns.2011.07.005

Salahshour, S, Allahviranloo, T., Abbasbandy, S., & Baleanu, D. (2012). Existence and uniqueness results for fractional differential equations with uncertainty. *Advanced in Difference Equations*, *2012*(1), 1–12.

Salahshour, Soheil, Ahmadian, A., Senu, N., Baleanu, D., & Agarwal, P. (2015). On Analytical Solutions of the Fractional Differential Equation with Uncertainty: Application to the Basset Problem. *Entropy*, *17*(2), 885–902. https://doi.org/10.3390/e17020885

Scalas, E., Gorenflo, R., & Mainardi, F. (2000). Fractional calculus and continuous-time finance. *Physica A: Statistical Mechanics and Its Applications*, *284*(2000), 376–384. https://doi.org/10.1016/S0378-4371(00)00255-7

Shah, N. A., Ahmad, I., Bazighifan, O., Abouelregal, A. E., & Ahmad, H. (2020). Multistage optimal homotopy asymptotic method for the nonlinear Riccati ordinary differential equation in nonlinear physics. *Applied Mathematics & Information Sciences*, *14*(6), 1009–1016. https://doi.org/10.18576/amis/140608

Shahidi, M., & Khastan, A. (2018). Solving fuzzy fractional differential equations by power series expansion method. *2018 6th Iranian Joint Congress on Fuzzy and Intelligent Systems (CFIS)*, *2018*, 37–39. https://doi.org/10.1109/CFIS.2018.8336621

Sin, K., Chen, M., Choi, H., & Ri, K. (2017). Fractional Jacobi operational matrix for solving fuzzy fractional differential equation1. *Journal of Intelligent & Fuzzy Systems*, *33*(2), 1041–1052. https://doi.org/10.3233/JIFS-162374

Sin, K., Chen, M., Wu, C., Ri, K., & Choi, H. (2018). Application of a spectral method to Fractional Differential Equations under uncertainty1. *Journal of Intelligent & Fuzzy Systems*, *35*(4), 4821–4835. https://doi.org/10.3233/JIFS-18732

Somathilake, L. W. (2020). An efficient numerical method for fractional ordinary differential equations - based on exponentially decreasing random memory on uniform meshes. *Journal of the National Science Foundation of Sri Lanka*, *48*(2), 163–174. https://doi.org/10.4038/jnsfsr.v48i2.9026

Sotonwa, O., & Obabiyi, O. (2019). The convergence of homotopy analysis method for solving Onchocerciasis (Riverblindness). *IOSR Journal of Mathematics*, *15*(5), 79–93. https://doi.org/10.9790/5728-1505047993

Stefanini, L. (2009). A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. *Fuzzy Sets and Systems*, *161*, 1564–1584.

https://doi.org/10.1016/j.fss.2009.06.009

Stefanini, L., Sorini, L., & Guerra, M. L. (2006). Parametric representation of fuzzy numbers and application to fuzzy calculus. *Fuzzy Sets and Systems*, *157*(18), 2423–2455. https://doi.org/10.1016/j.fss.2006.02.002

Sun, H. G., Zhang, Y., Baleanu, D., Chen, W., & Chen, Y. Q. (2018). A new collection of real world applications of fractional calculus in science and engineering. *Communications in Nonlinear Science and Numerical Simulation*, *64*, 213–231. https://doi.org/10.1016/j.cnsns.2018.04.019

Sun, H., Song, X., & Chen, Y. (2010). A class of fractional dynamic systems with fuzzy order. *2010 8th World Congress on Intelligent Control and Automation*, 197–201.

Takači, D., Takači, A., & Takači, A. (2014). On the solutions of fuzzy fractional differential equations. *Fractional Calculus and Applied Analysis*, *4*(1), 98–103.

Tan, Y., & Abbasbandy, S. (2008). Homotopy analysis method for quadratic Riccati differential equation. *Communications in Nonlinear Science and Numerical Simulation*, *13*(3), 539–546. https://doi.org/10.1016/j.cnsns.2006.06.006

Troparevsky, M. I., Seminara, S. A., & Fabio, M. A. (2019). A review on fractional differential equations and a numerical method to solve some boundary value problems. In *Nonlinear Systems -Theoretical Aspects and Recent Applications* (pp. 1–19). IntechOpen. https://doi.org/10.5772/intechopen.86273

Turkyilmazoglu, M. (2011). Some issues on HPM and HAM methods: A convergence scheme. *Mathematical and Computer Modelling*, *53*(2011), 1929–1936. https://doi.org/10.1016/j.mcm.2011.01.022

Uehara, K., & Fujise, M. (1993). Fuzzy inference based on families of a-level sets. *IEEE Transactions on Fuzzy Systems*, *1*(2), 111–124.

Ullah, A., Ullah, A., Ahmad, S., Ahmad, I., & Akgül, A. (2020). On solutions of fuzzy fractional order complex population dynamical model. *Numerical Methods for Partial Differential Equations*, num.22654. https://doi.org/10.1002/num.22654

Veeresha, P., Prakasha, D. G., & Baskonus, H. M. (2019). Solving smoking epidemic model of fractional order using a modified homotopy analysis transform method. *Mathematical Sciences*, *13*(2), 115–128.

Verma, P., & Kumar, M. (2020). An analytical solution of linear/nonlinear fractional-order partial differential equations and with new existence and uniqueness conditions. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, 1–9.

Vu, H., An, T. V., & Van Hoa, N. (2019). On the initial value problem for random fuzzy differential equations with Riemann-Liouville fractional derivative: Existence theory and analytical solution. *Journal of Intelligent and Fuzzy Systems*, *36*(6), 6503–6520. https://doi.org/10.3233/JIFS-182876

Wasques, V., Laiate, B., Santo Pedro, F., Esmi, E., & de Barros, L. C. (2020). *Interactive Fuzzy Fractional Differential Equation: Application on HIV Dynamics* (pp. 198–211). https://doi.org/10.1007/978-3-030-50153-2_15

Wu, H.-C. (2000). The fuzzy Riemann integral and its numerical integration. *Fuzzy Sets and Systems*, *110*, 1–25. https://doi.org/10.1016/S0020-0255(98)00016-4

Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, *1*(1), 3–28.

Zhu, Y. (2015). Existence and uniqueness of the solution to uncertain fractional differential equation. *Journal of Uncertainty Analysis and Applications*, *3*(1), 1–11. https://doi.org/10.1186/s40467-015-0028-6