MODIFICATION OF DAL DISTRIBUTION: A COMPARATIVE ANALYSIS OF BURR XII-DAL AND WEIBULL-DAL DISTRIBUTIONS

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Abstrak

Taburan klasik seperti taburan Burr-G dan Weibull-G yang mempunyai kurang dari empat parameter kebiasaanya digunakan untuk mewakili data hayat, tidak memadai untuk menerangkan cerapan sebenar dengan tepat. Taburan ini sering mengalami kesukaran untuk mengenalpasti kerumitan dan gangguan yang terdapat dalam data, yang membawa kepada ketidaktepatan dalam pemodelan dan analisis. Justeru itu, kajian ini memperkenalkan dua model hayat baharu berdasarkan taburan yang dinyatakan sebelum ini untuk menangani isu tersebut. Model yang dinamakan sebagai BurrXII-DAL dan Weibull-DAL, adalah berasaskan sifat taburan DAL, dan mempunyai lima parameter. Analisis statistik dijalankan ke atas model ini, termasuk pemeriksaan terhadap sifat kebolehpercayaan dan pengembangan ketumpatan. Momen biasa dan tidak lengkap, fungsi penjanaan momen, fungsi kuantil, momen berwajaran kebarangkalian, statistik aturan dan entropi Renyi diterbitkan secara eksplisit. Kaedah kebolehjadian maksimum digunakan untuk menghasilkan anggaran bagi parameter untuk model menggunakan kaedah *Broyden-Flether-Goldfrab-Shanno* berulang yang terdapat dalam pakej R. Seterusnya, lima set data sebenar digunakan untuk memperlihatkan keupayaan model BurrXII-DAL dan Weibull-DAL. Pelbagai kriteria kebagusan penyuaian, termasuk AIC, BIC, A*, W* dan nilai-*p* (KS) digunakan. Kedua -dua model, taburan BurrXII-DAL dan Weibull-DAL berprestasi lebih baik kerana ia mempunyai penyuaian fleksibel dengan data sebenar berbanding model kompetitif yang lain. Model tersebut terbukti berkesan kerana ramalan kadar kegagalan adalah lebih tepat dan fungsi kadar bahaya menunjukkan fleksibiliti dengan senario yang rumit. Taburan BurrXII-DAL dan Weibull-DAL dengan lima parameter telah berjaya dibangunkan dalam kajian ini. Kajian ini telah memberikan pandangan yang mendalam mengenai kelebihan dan kelemahan setiap taburan serta mengenal pasti situasi di mana satu taburan mungkin lebih sesuai daripada yang lain.

Katakunci: Taburan keluarga Burr-G dan Weibull-G, Taburan DAL, Taburan BurrXII, Taburan Weibull, Anggaran kebolehjadian maksimum.

Abstract

Classical distributions such as Burr-G and Weibull-G distributions with less than four parameters are commonly used to represent lifetime data, frequently prove insufficient in accurately describing real observations. These distributions often struggle to capture the complexities and nuances present in the data, leading to inaccuracies in modeling and analysis. Hence, this study introduces two new lifetime models based on the aforementioned distributions to address these issues. These models, called BurrXII-DAL and Weibull-DAL, are based on the properties of the DAL distribution, and have five parameters. Statistical analyses are conducted on these models, including examining their reliability properties and density expansion. Their ordinary and incomplete moments, moment generating function, quantile function, probability weighted moments, order statistics, and Renyi entropy have all been derived explicitly. Maximum likelihood technique is used to provide estimates for the model's parameter using Iterative Broyden-Flether-Goldfrab-Shanno method in the R-package. Then, five actual datasets are used to demonstrate the ability of the BurrXII-DAL and Weibull-DAL models. Various measures of goodness-of-fit, including AIC, BIC, A*, W* and *p*-value (KS) are used. Both models, BurrXII-DAL and Weibull-DAL distributions performed better as they are flexible fit to the real data as compared to the other competitive models. The models are proven effective since the prediction of the failure rate is more accurate and the hazard rate function shows flexibility with complicated scenario. BurrXII-DAL and Weibull-DAL distributions with five parameters are successfully developed in this study. This research has provided insights into the strengths and weaknesses of each distribution and has identified situations where one distribution may be more appropriate than the other.

Keywords: Burr-G and Weibull-G family distributions, DAL distribution, BurrXII-Distribution, Weibull-Distribution, Maximum likelihood estimation.

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Finally, this PhD dissertation is dedicated to my Parents.

Table of Contents

List of Tables

List of Figures

List of Abbreviations

BG	Beta generated
GBG	Generalized beta generated
Mc-G	McDonald generalized
Kum	Kumaraswamy
BPF	Beta power function
KPF	Kumaraswamy Power function
LC	Lorenz curve
MWT	Mean waiting time
MLE	Maximum likelihood estimate
MRL	Mean residual life
BC	Bonferroni curve
G	Gini index
Z	Zenga index
A	Atkinson index
GE	Generalized entropy
MML	Method of maximum likelihood
MLSs	Method of least squares
PWMS	Probability weighted moments
DAL	Dimitrakopoulou Adamidis Loukas
W-DAL	Weibull-DAL distribution
BXII-DAL	BurrXII-DAL distribution
AIC	Akaike information criteria

BIC	Bayesian information criteria
KwGPW	Kumaraswamy generalized power Weibull
BENH	Beta Exponentiated Nadarajah Haghighi
EGPW	Exponentiated Generalized Power Weibull
GPW	Generalized Power Weibull
WDa	Weibull Dagum
WPF	Weibull power function
WLx	Weibull Lomax
GPW	Generalized power Weibull
NH	Nadarajah Haghighi
BW	Beta Weibull
KwW	Kumaraswamy Weibull

List of Symbols

pdf	Probability density function
cdf	Cumulative distribution function
x_p	pth quantile
M	Moors' kurtosis
$S(t)$	Survival function
$h(t)$	Hazard rate
$r(t)$	Reversed hazard rate
$H(t)$	Cumulative hazard rate
$m(t)$	Mean residual life
$\bar{\mu}(t)$	Mean waiting time
β_r	Probability weighted moment
$B(a,b)$	Euler beta function
$B_x(a,b)$	Incomplete beta function
$B(a,b,x)$	Complementary incomplete beta function
$I_x(a,b)$	Incomplete beta function ratio

CHAPTER ONE

INTRODUCTION

1.1 Introduction

In our study, we explore the application of probability distributions in reliability analysis, particularly focusing on the Burr XII and DAL and Weibull distributions. These distributions are essential in modeling various real-world scenarios, such as lifetime data analysis and reliability assessment.

The Burr XII distribution is renowned for its flexibility in modeling a wide range of positively skewed data, making it valuable for reliability analysis. Its shape parameter allows it to adapt to various failure rate behaviors over time. Nevertheless, one of the limitations of the Burr XII distribution is its parameter estimation complexity, which can be challenging for small sample sizes. Additionally, it may not always fit well with datasets exhibiting extreme tail behaviors (Smith & Johnson, 2017).

The DAL distribution is an extension of the Weibull distribution, introducing two location parameters, which provide greater flexibility in modeling diverse failure patterns. While this distribution offers significant advantages, such as enhanced model fitting, it can be sensitive to the choice of location parameters and may require more data to estimate accurately. This sensitivity can make parameter estimation more challenging, and the DAL distribution may not always be the best choice for simple failure patterns (Brown, 2008).

The Weibull distribution is a fundamental tool in reliability analysis and is celebrated for its versatility in modeling various failure patterns, such as early-life failures, constant failure rates, and wear-out failures. However, the Weibull distribution may not

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