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HERMITE-HADAMARD, JENSEN'S, FEJÉR'S, AND SIMPSON'S NEW INEQUALITIES FOR A MERGING BETWEEN TWO CONVEX FUNCTIONS

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Abstrak

Ketaksamaan digunakan secara meluas untuk mencari penyelesaian optimum bagi masalah pengoptimuman, kejuruteraan dan pengaturcaraan linear. Kajian terdahulu, hanya satu fungsi cembung tunggal seperti fungsi s-cembung, h-cembung, dan g-cembung digunakan untuk membina ketaksamaan, yang digunakan sebagai ketaksamaan Hermite-Hadamard, Jensen, Fejér, dan Simpson. Ketaksamaan ini menggunakan sifat fungsi cembung tunggal namun terhad kepada sebilangan masalah Sahaja. Oleh itu, penggabungan fungsi cembung ini dapat menyelesaikan lebih banyaj masalah, sekaligus mengembangkan aplikasi dalam pelbagai bidang. Oleh yang demikian, kajian ini bertujuan membina ketaksamaan baharu dengan menggabungkan tiga fungsi cembung tunggal, iaitu fungsi s-cembung, h-cembung, dan g-cembung. Melalui kerangka gabungan kecembungan, kajian ini menyumbang kepada bidang ini secara signifikan dengan menyediakan pelbagai peluasan dalam ketaksamaan Hermite-Hadamard, Jensen, Fejér, dan Simpson bagi fungsi cembung hibrid. Pendekatan menggabungkan dua atau lebih kelas fungsi cembung digunakan untuk membentuk kelas fungsi cembung hibrid bertujuan memperluaskan ketaksamaan bermatematik. Hasil kajian ini membangunkan pelbagai pengembangan ketaksamaan Hermite-Hadamard, Jensen, Fejér, dan Simpson bagi kelas fungsi baharu dengan sifat kecembungan hibrid. Hasil dapatan seterusnya dirumuskan sebagai teorem untuk menerbit anggaran ralat bagi kaedah Simpson, titik tengah, dan petua trapezoid dan juga untuk menentukan nilai purata bagi nombor nyata. Anggaran ralat ini didapati lebih tepat berbanding ketaksamaan sedia ada. Kesimpulannya, kajian ini menambah baik ketaksamaan bermatematik, mengoptimumkan penyelesaian dalam pelbagai bidang di samping mengekalkan sifat kecembungan dan memperluaskan kaedah penghampiran kamiran klasik.

Kata Kunci: Fungsi cembung, ketaksamaan, nilai purata, anggaran ralat.

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Abstract

Inequalities are widely used to find the optimal solution for optimisation, engineering and linear programming problems. In previous works, only a single convex function such as s-convex, h-convex and g-convex functions were used to construct the inequalities namely Hermite-Hadamard, Jensen's, Fejér's, and Simpson's inequalities. These inequalities use the properties of a single convex function but can only address a limited number of problems. By combining these convex functions, more problems can be solved, therefore their applicability to various fields. Thus, this study aims to construct new inequalities by combining three single convex functions namely s-convex, h-convex and g-convex functions. The approach of combining two or more classes of convex functions to form hybrid convex function classes aims to extend mathematical inequalities. The findings of the study establish various extensions of the Hermite-Hadamard, Jensen's, Fejér's, and Simpson's inequalities for new classes of functions with hybrid convexity properties. These results are further formulated as theorems to derive error estimations for the Simpson's, midpoint, and trapezoidal rules and also to determine the mean value of real numbers. These error estimations are found more accurate compared to the existing inequalities. Within the framework of convexity combinations, this study contributes significantly to the field by providing various extensions of the Hermite-Hadamard, Jensen's, Fejér's, and Simpson's inequalities for hybrid convex functions. In conclusion, this study enhanced mathematical inequalities, optimizing solutions in various fields while preserving convexity properties and extending classical integral approximation methods.

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Keywords: Convex function, inequality, mean value, error estimations.

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$\mathbb{R}$	:	The Set of Real Numbers
$\mathbb{R}_+$	:	The Set of All Positive Real Numbers
$[x, y]$	:	The Line Segment Between x and y
$\mathbb{I}$	:	Real Interval $[p, q]$
$\mathbb{I}^*$	:	Interval in $\mathbb{I}$
$\mathbb{I}^{\circ}$	:	Interior of $\mathbb{I}$
$\mathbb{J}$	:	Open Interval of 0 and 1
$\mathbb{J}^*$	:	Closed Interval of 0 and 1
$\mathbb{K}$	:	Positive Real Interval $= [0, \infty)$ or $\mathbb{R}_+$
$L[p, q]$	:	Integrable on $[p, q]$
x, p, q, y	:	Random Points in $\mathbb{R}$
k_1^s	:	s-convex Function in the First Sense
k_2^s	:	s-convex Function in the Second Sense
k_3^s	:	s-convex Function in the Third Sense
k_4^s	:	s-convex Function in the Fourth Sense
$A(p, q)$	:	Arithmetic Mean on Random Points in $\mathbb{I}$
$H(p, q)$	:	Harmonic Mean on Random Points in $\mathbb{I}$
$\bar{L}(p, q)$	:	Logarithmic Mean on Random Points in $\mathbb{I}$
$L_n(p, q)$	:	Generalised Log Mean on Random Points in $\mathbb{I}$
$L_m(p, q)$	:	α-logarithmic mean on Random Points in $\mathbb{I}$
Υ	:	Partition of Real Interval $\mathbb{I}$
$\Upsilon(x, y)$	:	Set of All Possible Partitions in $\mathbb{I}$
$T(f, \Upsilon)$	:	Error Estimation Formulae for Trapezoid Rule
$M(f, g, \Upsilon)$	:	Error Estimation Formulae for Midpoint Rule
$S(f, \Upsilon)$	:	Error Estimation Formulae for Simpson's Rule

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CHAPTER ONE
INTRODUCTION

1.1 Research Background

Since the nineteenth century, inequality has become increasingly important as a fundamental element of analysis (Fink, 2000). Inequality is defined as the difference between two quantities that are not equal and can be less or greater than one another.

Inequalities are used in numerous branches of mathematics such as in optimisation theory (Rashid, Noor, & Noor, 2020), error estimation (Avci, Kavurmaci, Özdemir, & Emin, 2011), and calculus (Anastassiou, 2011), to name a few. In optimisation theory, inequalities can be employed to find an optimum solution to a given problem (Bertsimas & Popescu, 2005). While in error estimation, the estimated error for a function can be obtained by using inequality theory (Chu, Khan, Adil, & Ali, 2017). Meanwhile, in calculus, inequalities can be applied in diverse areas. Different inequalities, such as trapezoid, midpoint, and Simpson inequalities, can be developed using the inequality concept (Dragomir, Cerone, & Sofo, 1998; Chu, Khan, Adil, & Ali, 2017; Du, Li, & Yang, 2017).

Further, convex function and its derivatives are also used for different purposes. For example, the first-order derivative of a convex function is used for testing the continuity of smooth functions of one variable (Roberts, 1993). The second-order derivative of convex function is used to find maximum error estimation for areas under the curve using midpoint and trapezoidal rule (Alomari, Darus, & Dragomir, 2010; Shen, Wang, Wang, & Wise, 2012; Mehrez & Agarwal, 2018), while the third-order derivative of a convex function is specially used for quasi-convex function (Wu,

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Sroysang, Xie, & Chu, 2015), and the fourth-order derivative of convex function is used to find maximum error estimation for areas under the curve for Simpson's rule (Kalsoom, Wu, Hussain, & Latif, 2019). Since higher-order derivatives are complex and require expensive computation, trapezoid, midpoint, and Simpson's methods are preferred to estimate the area under the curve to reduce the number of computations (Dragomir, Agarwal, & Cerone, 1999; Sarıkaya, Budak, & Erden, 2019).

Inequalities also play significant roles in other fields, such as engineering (Cech, 2013; Czasnojc & Grum, 2020), business management (McNeil, 2004; Rouziou, 2019) and decision-making (Attouch & Soubeyran, 2006; Lutz, 2019). In electrical engineering, inequalities are used to transmit the signals with maximum accuracy (Weber, Andrews, Yang, & Veciana, 2007; Aubry, Demaio, Farina, & Wicks, 2013). In business management, cost-profit analysis involving inequalities to optimise profit by reducing the costs incurred and increasing production (Rouziou, 2019; Söker, 2021). While in decision-making, inequalities are often used in linear programming to maximise or minimise the objective function (Anderheggen & Knöpfel, 1972; Schrijver, 1998; Chakraborty, Chandru, Vijay, & Rao, 2020).

According to Hardy and Littlewood (1934), most of mathematical inequalities, such as Hermite-Hadamard (H-H), Jensen's, Fejér's and Simpson's inequalities are the outcomes of a convex function. Moreover, Itrinović and Lacković (1985) mentioned that Jensen's inequality was constructed by using the convex function. As the demand for convex functions is increasing in inequality theory, it is anticipated that new inequalities will be formed to provide a better solution to the problems from the given range of convexity.

Definition 1.1 ***(Boyd & Vandenberghe, 2004)*****:** *Suppose* $x_1 \neq x_2$ *are two points in* $\mathbb{R}^n$*.* *Points of the form*

$$y = ax_1 + (1 - a)x_2,$$

where $a \in \mathbb{R}$*, form a line passing through* x_1 *and* x_2*.*

Based on Definition (1.1), the convex line segment is defined by reducing the interval of a from $\mathbb{R}$ to $[0, 1]$ as defined in Definition (1.2) below:

Definition 1.2 ***(Boyd & Vandenberghe, 2004)*****:** *Suppose* $x_1 \neq x_2$ *are two points in* $\mathbb{R}^n$*.* *Points of the form*

$$y = ax_1 + (1 - a)x_2, \qquad a \in [0, 1].$$

where $a \in [0, 1]$*, form a convex line segment with end points* x_1 *and* x_2 *as follows:*

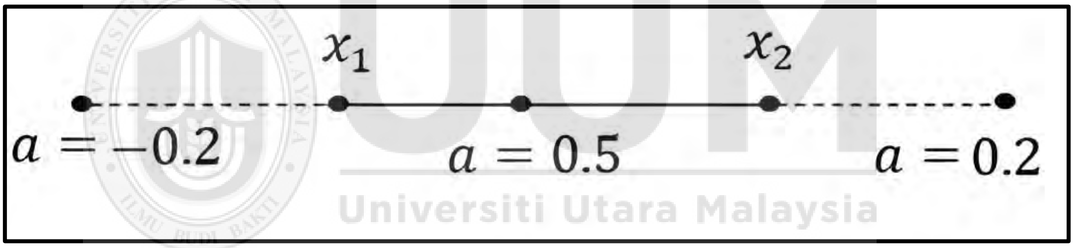

Figure 1. 1. Convex line segment.

1.2 Convex Function

Jensen (1906) introduced the following definition of the convex function for real values at the beginning of the nineteenth century:

Definition 1.3 (***Jensen, 1906*****):** *A function* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$*, is convex if*

$$f(ax_1 + (1 - a)x_2) \leq af(x_1) + (1 - a)f(x_2), \quad x_1, x_2 \in \mathbb{I}, a \in [0, 1]. \tag{1.1}$$

Figure (1.2) shows the geometrical interpretation of a convex function.

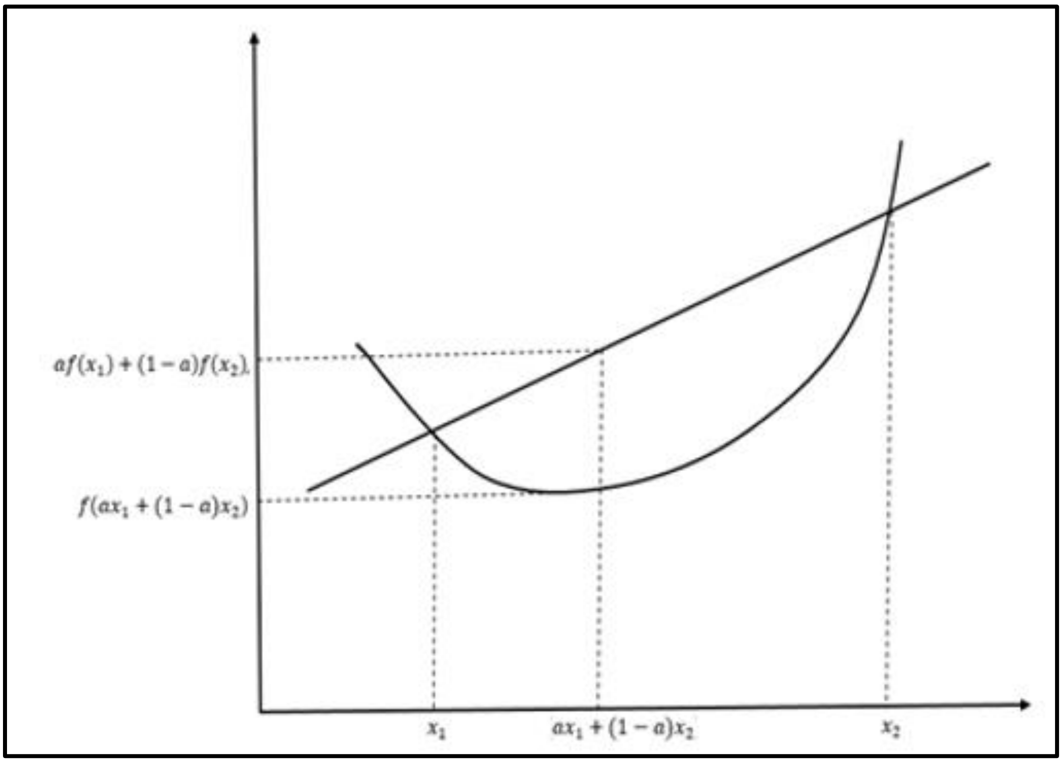

Figure 1. 2. Geometrical interpretation of a convex function.

Remark 1.1: If strict inequality $(<)$ holds on Inequality (1.1), then the function f is called strictly convex whenever $x_1 \neq x_2$ and $a \in (0,1)$.

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Moreover, a function on an n-dimensional real-valued interval is a convex function if the line of this function joining two different points on its graph lies on or above the graph (Peajcariaac & Tong, 1992). Equally, a function is convex if the epigraph (the region above the graph) of this function is a convex set (Duca & Lupşa, 2006). For illustration, refer to Figures 1.3 and 1.4.

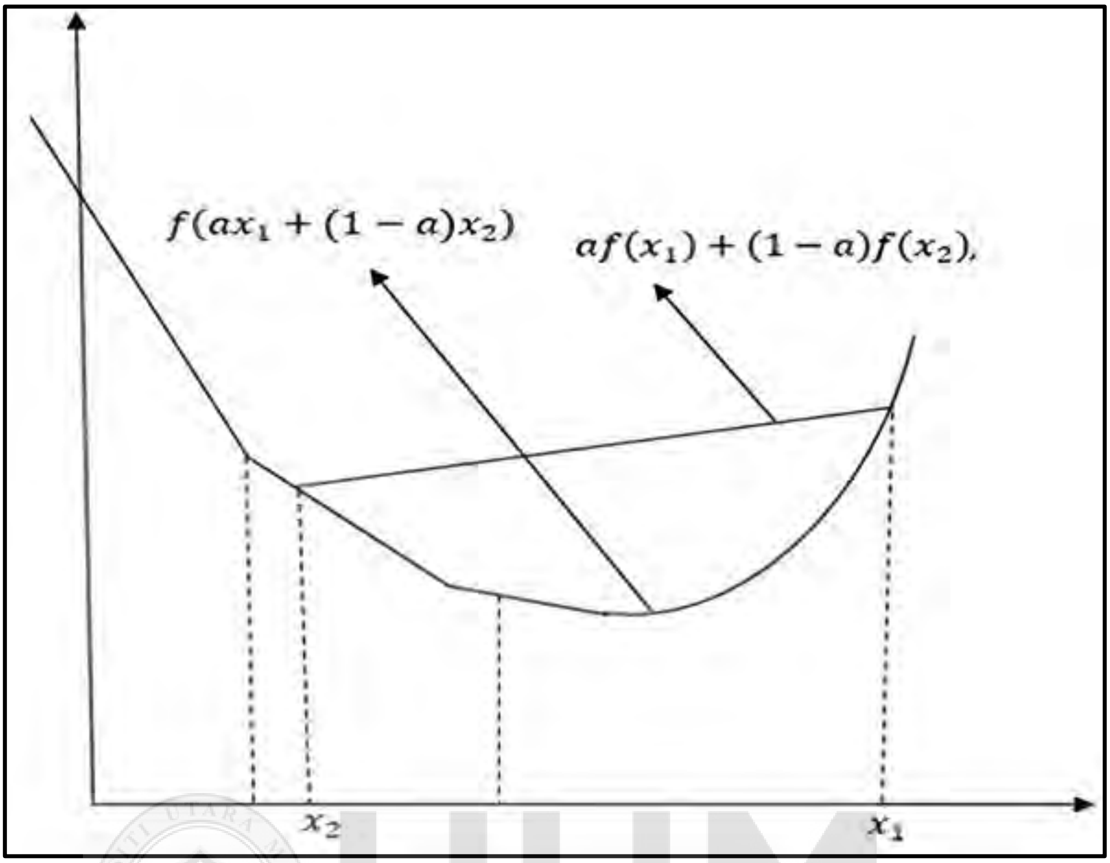

Figure 1.3. The graphical representations of a convex function.

The function f in Figure 1.3 is convex since the line segment for $f(ax_1 + (1-a)x_2)$ joined by x_1 and x_2 (two different points) on the graph lies on or below the segment $af(x_1) + (1-a)f(x_2)$ for all $a \in [0, 1]$.

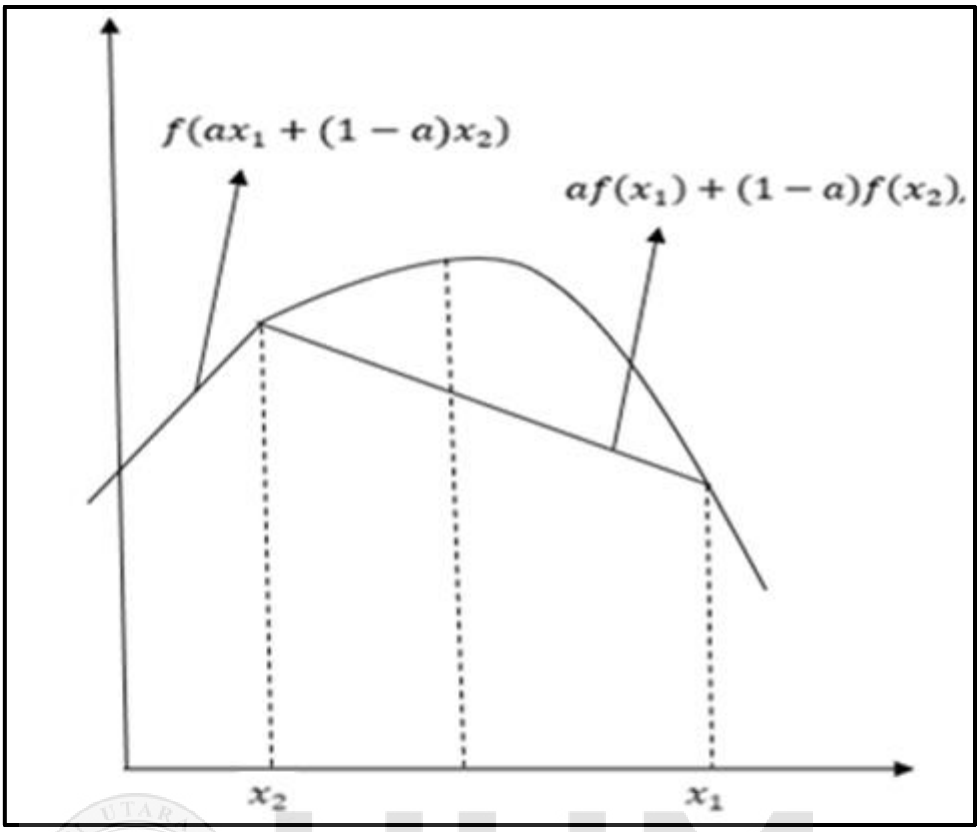

Figure 1.4. The graphical representations of non-convex functions.

On the other hand, the function f in Figure 1.4 is not convex because the line segment for $f(ax_1 + (1-a)x_2)$ joined by x and y (two different points) on the graph lies above the segment $af(x_1) + (1-a)f(x_2)$ for all $a \in [0, 1]$.

An equivalent definition of a convex function is stated as follows:

Definition 1.4 ***(Beckenbach, 1948):*** *A function of real values $f(x_i)$, which is defined on an interval $(q_1 < x_i < q_2)$, is convex if $\forall\ x_1, x_2, q_1, q_2, a, b \in \mathbb{R}$,*

$$f(ax_1 + bx_2) \leq af(x_1) + bf(x_2), \qquad a < x_1 < x_2 < b. \tag{1.2}$$

Referring to Definition (1.4), it is noticed that Beckenbach (1948) ignored three very sensitive but core conditions that must be presented to fulfil the actual nature of the convex function. First, the condition of the interval must be two different points (x_1, x_2), because the line cannot be drawn on a single point and these two points

x_1 and x_2 must be different from each other. Second, the sum of a and b must be equal to one $(a + b = 1)$. Third, the domain of the function must be convex. Many researchers have devoted their efforts to generalise convex function defined in Definition (1.3) and named the generalised functions as s-convex function in the first, second, third, and fourth sense, h-convex function, (h_1, h_2)-convex function, and g-convex function (Matuszewska & Orlicz, 1961; Pinheiro, 2007; Kemali, Sezer, Tınaztepe, & Adilov, 2021; Niculescu, 2000; Varošanec, 2007; Cristescu, Noor, & Awan, 2015; Awan, Noor, & Khalida, 2018). The evolution of convex function is highlighted in the proceeding subsection (1.2.1), as follows:

1.2.1 Evolution of Convex Function

In this subsection, the evolution of convex function is presented. On specific conditions, the generalisations of convex functions are reduced to the convex function defined in Definition (1.3). Matuszewska and Orlicz (1961) proposed the following new function named as s-convex function in the first sense, also known as K_1^s or φ-convex function:

Definition 1.5 ***(Hudzik & Maligranda, 1994):*** *A function $f: \mathbb{K} \subset \mathbb{R}_+ \to \mathbb{R}$ is called s-convex in the first sense,* φ-convex *or K_1^s if $\mathbb{K}$ is a convex set and*

$$f(ax_1 + bx_2) \leq a^s f(x_1) + b^s f(x_2), \qquad \forall a, b \geq 0, \tag{1.3}$$

and $a^s + b^s = 1$ for $s \in (0, 1]$.

The function defined in Definition (1.5) removes the restriction imposed by the standard convex function without sacrificing Definition (1.3). Although Definition (1.5) is taking advantage of the power of the scalar quantities a and b of the convex function, Pinheiro (2007) faced difficulties dealing in graphical presentation of

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Definition (1.5) on condition $a^s + b^s = 1$ for $s \in (0, 1]$. She, therefore, redefined the s-convex function in the first sense that was previously proposed by Matuszewska and Orlicz (1961) and renamed it as s-convex function in the second sense or K_2^s as given below:

Definition 1.6 ***(Pinheiro, 2007):*** *A function $f: \mathbb{K} \subset \mathbb{R}_+ \to \mathbb{R}$ is called s-convex in the second sense or K_2^s if $\mathbb{K}$ is a convex set and*

$$f(ax_1 + bx_2) \leq a^s f(x_1) + b^s f(x_2), \qquad \forall a, b \geq 0, \tag{1.4}$$

and $a + b = 1$ for $s \in (0, 1]$.

As a result, Definition (1.5) and Definition (1.6) are known as s-convex function in the first sense and second sense, respectively. Note that for $s = 1$, s-convex function for both senses will be reduced to the original definition of a convex function given in Definition (1.3). Although Definition (1.5) and Definition (1.6) improved the usability of convex functions in creating new inequalities, it does not deal with the reciprocal in power of the convex function. In response to such flaw, Kemali, Sezer, Tınaztepe, and Adilov (2021) defined the s-convex function in the third sense or K_3^s as follows:

Definition 1.7 ***(Kemali, Sezer, Tınaztepe, & Adilov, 2021):*** *A function $f: \mathbb{K} \subset \mathbb{R}_+ \to \mathbb{R}$ is called s-convex in the third sense or K_3^s if the inequality*

$$f(ax_1 + bx_2) \leq a^{\frac{1}{s}} f(x_1) + b^{\frac{1}{s}} f(x_2) \tag{1.5}$$

is satisfied for all $x_1, x_2 \in \mathbb{K}$ and $a, b \geq 0$ such that $a^s + b^s = 1$ for $s \in (0, 1]$.

Zeynep, Sezer, Tinztepe, and Adilov (2021) in Definition (1.7) of s-convex function in the third sense in graphical representation faced some difficulties in applying the

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condition $a^s + b^s = 1$ for $s \in (0, 1]$ and defined s-convex function in the fourth sense or K_4^s as follows:

Definition 1.8 ***(Zeynep, Sezer, Tinztepe, & Adilov , 2021):*** *A function $f: \mathbb{K} \subset \mathbb{R}_+ \to \mathbb{R}$ is called s-convex in the fourth sense or K_4^s if the inequality*

$$f(ax_1 + bx_2) \leq a^{\frac{1}{s}} f(x_1) + b^{\frac{1}{s}} f(x_2) \tag{1.6}$$

is satisfied for all $x_1, x_2 \in \mathbb{K}$ and $a, b \geq 0$ such that $a + b = 1$ for $s \in (0, 1]$.

Although these generalisations of convex function in Definitions (1.5) and (1.6) can improve the usability of convex functions in producing new inequalities, it does not consider the power of the convex function. In response to such weakness, Niculescu (2000) has constructed g-convex function that is able to calculate the values for the power of a convex function as follows:

Definition 1.9 ***(Niculescu, 2000):*** *Let $f: \mathbb{K} \subset \mathbb{R}_+ \to \mathbb{R}_+$ be a positive function then it is called geometrically convex or g-convex function if*

$$f\left({x_1}^a {x_2}^{(1-a)}\right) \leq [f(x_1)]^a [f(x_2)]^{(1-a)}, \quad a \in [0, 1]. \tag{1.7}$$

Based on Definitions (1.5) and (1.6), although the s-convex function in the first, second, third, and fourth sense is able to calculate the values for the product of a positive scalar quantity of the convex function, it still fails to solve the problems where a positive function is multiplied by the convex function.

In order to improve Definitions (1.5) and (1.6), Varošanec (2007) further defined a new generalisation of the convex function named h-convex function, in which a convex function f is multiplied by a positive real valued function h as follows:

Definition 1.10 ***(Varošanec, 2007):*** *Let* $h: (0,1) \subseteq \mathbb{J} \to \mathbb{R}$ *be non-negative function,* $h \not\equiv 0$. *We say that* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *is a* h*-convex function if* f *is non-negative and for all* $x_1, x_2 \in \mathbb{I}$, $a \in (0,1)$ *we have*

$$f(ax_1 + (1-a)x_2) \leq h(a)f(x_1) + h(1-a)f(x_2). \tag{1.8}$$

It is worth noting that if $h(a) = a^s$ and $h(1-a) = b^s$ for all $s \in [0,1]$, then Definition (1.10) will be reduced to s-convex function in the second sense or K_2^s given in Definition (1.6).

Gabriela, Noor, and Awan (2015) further proposed (h_1, h_2)-convex function to generalise the h-convex function by using two different positive real-valued functions named $h_1$and h_2 as follows:

Definition 1.11 ***(Cristescu, Noor, & Awan, 2015):*** *A function* $f: \mathbb{I} \subset \mathbb{R} \to (0, \infty)$ *is said to be* (h_1, h_2)*-convex function for all* $x_1, x_2 \in \mathbb{I}$ *if*

$$f(ax_1 + (1-a)x_2) \leq h_1(a)f(x_1) + h_2(a)f(x_2), \qquad a \in [0,1]. \tag{1.9}$$

Note that if $h_1(a) = a$ and $h_2(a) = (1-a)$ for all $a \in [0,1]$, then Definition (1.11) is same as Definition (1.3). Similarly, if $h_1(a) = a^s$ and $h_2(a) = (1-a)^s$ for all $a \in [0,1]$ in Definition (1.11), then s-convex function in the second sense stated in Definition (1.6) will be produced. Furthermore, if $h_1(a) = a^{\frac{1}{s}}$ and $h_2(a) = (1-a)^{\frac{1}{s}}$ for all $a \in [0,1]$ and $s \in (0,1]$ in Definition (1.11), then it will be transformed into s-convex function in the third sense defined in Definition (1.7).

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Definition 1.12 ***(Awan, Noor, & Khalida, 2018):*** *Let* $h_1, h_2: (0,1) \subseteq \mathbb{J} \to \mathbb{R}$ *be two real functions such that* $h_1 \not\equiv 0$ *and* $h_2 \not\equiv 0$. *A function* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *for* $a \in (0,1)$ *is called* (h_1, h_2)*-convex function if*

$$f(ax_1 + (1-a)x_2) \leq h_1(a)h_2(1-a)f(x_1) + h_2(a)h_1(1-a)f(x_2). \quad (1.10)$$

The functions presented previously play a crucial role in producing mathematical inequalities (Ekinci & Eroğlu, 2019; Liu & Xu, 2021). Based on these definitions, different types of inequalities such as Hermite-Hadamard (H-H), Jensen's, Fejér's and Simpson's types of inequalities can be formed.

1.3 Common Types of Inequalities

In this section, some basic definitions and applications of H-H, Jensen's, Fejér's, and Simpson's inequality are presented. According to Mitrinovic and Lackovic (1985), H-H inequality was the first in the literature to be established using the convex function. Subsequently, Jensen's, Fejér's, and Simpson's inequalities were constructed.

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1.3.1 Hermite-Hadamard (H-H) Inequality

Hermite-Hadamard (H-H) inequality is used to provide the necessary and sufficient condition for a function to be convex in an interval $\mathbb{I}$ (Dragomir, Cerone, & Sofo, 1998).

Theorem 1.1 ***(Mitrinovic & Lackovic, 1985):*** *Let* $f: \mathbb{I} \to \mathbb{R}_+$ *be a convex function defined on an interval* $\mathbb{I} \in \mathbb{R}$ *and* $p, q \in \mathbb{I}$ *with* $p < q$, *then*

$$(q-p)f\left(\frac{p+q}{2}\right) \leq \int_p^q f(x)dx \leq (q-p)\frac{f(p)+f(q)}{2} \quad (1.11)$$

is called Hermite-Hadamard (H-H) inequality.

This inequality gives an estimation of the mean (average) value of the convex function from both sides to ensure that the convex function is integrable (Khan, Khurshid, & Dragomir, 2018). The estimated value of a convex function is used to obtain an optimal solution from possible solutions when dealing with normally distributed data in statistics (Stein & Charles, 2011).

The following are two main results based on Hermite-Hadamard (H-H) inequality analysis:

1. The mean (average) value of convex function f defined on $\mathbb{I}$ will always lie between the defined interval of f as $x = \left[\frac{p+q}{2}\right]$.
2. The trapezoidal formula estimates the integral from the right side, while the midpoint formula estimates it from the left.

1.3.2 Jensen's Inequality

Another equally important inequality is Jensen's inequality, which is the extension of H-H inequality (Gürbüz, Akdemir, Rashid, & Set, 2020). Jensen's inequality is defined by using only the first and last terms of H-H inequality and, hence, it is also known as the refinement of H-H inequality (Peajcariaac & Tong, 1992).

Theorem 1.2 ***(Jensen, 1906):*** *Let $f\colon \mathbb{I} \to \mathbb{R}$ be a convex function defined on $\mathbb{I} \in \mathbb{R}$, then*

$$f\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i x_i\right) \leq \frac{1}{A_n}\sum_{i=1}^{n} a_i f(x_i) \tag{1.12}$$

is called Jensen's inequality, where $x = (x_1, \dots, x_n) \in \mathbb{I}^n$, $a = (a_1, \dots, a_n) \in [0,1]^n$ *and* $A_n = \sum_{i=1}^{n} a_i$.

Note that the function used in Inequality (1.12) will always be positive and a_i be the positive n-tuples on x.

Jensen's inequality is important as it can be utilised to produce new inequalities, such as Young's inequality, Holder's inequality, Minkowski's inequality, Shannon's inequality, and Levinson's inequality. These inequalities are commonly used in electrical engineering for communication and data storage (Rassias, Matić, Pearce, & Pečarić, 2000; Dragomir, 2006). In statistics, Jensen's inequality is used to define the lower bounds of a random variable (Taleb & Douady, 2013). Further, Jensen's inequality is also used to find the arithmetic mean (average) when dealing with convex functions (Niculescu & Persson, 2004).

1.3.3 Fejér's Inequality

Fejér's inequality was constructed by Fejér (1880-1959) while working on trigonometric functions. Similar to both Hermite-Hadamard (H-H) and Jensen's inequalities, Fejér's inequality, as defined below, can be used for harmonic analysis of convex functions (Latif, Dragomir, & Momoniat, 2017).

Theorem 1.3 (*Pearce & Dragomir, 2002*): *If* $f: \mathbb{I} \to \mathbb{R}$ *is convex and* g *be a non-negative function defined on* $p, q \in \mathbb{I} \subset \mathbb{R}$, *then*

$$f\left(\frac{p+q}{2}\right)\int_{p}^{q} g(x)dx \le \frac{1}{q-p}\int_{p}^{q} f(x)g(x)dx$$

$$\leq \frac{f(p)+f(q)}{2}\int_{p}^{q} g(x)dx \tag{1.13}$$

is called Fejér's inequality.

Note that by substituting $g(x) \equiv 1$ in Inequality (1.13), Hermite-Hadamard (H-H) inequality can be obtained (Farid, 2016). The extension of H-H inequality, when dealing with Fejér's inequality for a convex function produces H-H-Fejér's inequality for that function (Napoles, 2020).

1.3.4 Simpson's Inequality

The last but not least important inequality is Simpson's inequality. Dragomir, Agarwal, and Cerone (1999) derived Simpson's inequality based on the following theorem:

Theorem 1.4 ***(Dragomir, Agarwal, & Cerone, 1999):*** *If* $f: \mathbb{I} \to \mathbb{R}$ *is a differentiable and continuous function on the interval* $(p,q) \subset \mathbb{I} \subset \mathbb{R}$ *and* $\| f^{(4)} \|_{\propto} := sup\left|f^{(4)}(x)\right| < \infty$*, then*

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{q-p}\int_{p}^{q} f(x)dx\right|$$

$$\leq \frac{1}{2880} \| f^{(4)} \|_{\infty} (q-p)^{5} \tag{1.14}$$

is called Simpson's inequality.

On the application side, this inequality is used to calculate the estimation of error in the midpoint formula (Avci, Kavurmaci, & Özdemir, 2011). This inequality can be employed to calculate error estimates in numerical analysis (Dragomir, Agarwal, &

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Cerone, 1999). Recently, the generalisation of mathematical inequalities such as H-H, Jensen’s, Fejér’s, and Simpson’s inequalities are used to find the error estimations for different types of numerical formulas, such as Mid-point (Obeidat, Latif, & Dragomir, 2022), Trapezoid (Özcan & İşcan, 2019), and Simpson’s (Özcan & İşcan, 2019). Finding the error estimations for these formulas is time-consuming because it involves higher derivatives in its calculations (Dragomir, Agarwal, & Cerone, 1999). To avoid higher-order derivatives, researchers devoted their efforts to generalise Simpson’s inequality since this inequality used fourth-order derivative and find the error estimation for Simpson’s formula.

In some studies, researchers have attempted to combine different convex functions to construct new generalised inequalities so that the usage of different convex functions can be extended to wider areas of application (Napoles, 2020). Noor, Khalida, Safdar, and Farhat (2017) and Sun and Liu (2017) have successfully combined α-convex and m-convex functions in which the properties of these combined function are used to develop new definition of (α, m)-convex function and then extended the H-H inequality. Meanwhile, Rehman, Farid, Bibi, Jung, and Kang (2021) successfully merged s-convex and m-convex functions to form new H-H inequality.

1.4 Problem Statement

It is perceived that convex functions and mathematical inequalities are closely related to each other as numerous inequalities have been derived from convex functions (Niculescu & Persson, 2004). Early works on constructing the inequalities, such as Hermite-Hadamard (H-H), Jensen’s, Fejér’s, and Simpson’s inequalities, were based on a single convex function.

It has been revealed in the previous works that the inequalities inherit the properties of the employed convex functions. In order to construct inequalities with broader properties and wider applicability, two approaches have been opted. The first approach is to construct new convex functions with enhanced properties (Matuszewska & Orlicz, 1961; Niculescu, 2000; Pinheiro, 2007; Varošanec, 2007; Gabriela, Noor, & Awan, 2015). However, these functions can only be applied to limited problems. The s-convex function, for example, can be used to solve problems involving variables with positive scalar power, such as distance function with constant acceleration and can also generalised the inequalities involving s-convex function. Meanwhile, the h-convex function, which is continuous and differentiable at all points except zero, can be employed to solve optimisation problems such as minimising the cost and maximising the profit and can generalised the inequality using h-convex function. Further, g-convex function deals with the problem where the power of the convex function is required. This function is used in engineering problems to draw curves of different structures for buildings and can generalised the inequality based on g-convex function. However, the inequalities derived from these functions can only be applied to limited problems because these inequalities deals with s-convex, h-convex, and g-convex functions individually.

In the second approach, the existing convex functions are merged to obtain new inequalities with enhanced properties such as symmetry, differentiability, monotonicity, and continuity. The properties obtained by the second approach were found to be more extensive than those obtained by the former approach. Therefore, several researchers have attempted to construct new inequalities using this approach. Rashid, Noor, Khalida, and Akdemir (2019) combined exponential and s-convex functions to extend H-H inequality. This resulted inequality was then used to approximate the error for mid-point formula and to find the growth or decay rate for large data analysis which could not be performed by the previous H-H inequality. In an attempt to extend H-H inequality, Shi, Ye, Zhao, and Liu (2020) merged the log-convex and h-convex functions. The newly constructed inequality was then used to estimate the errors in the wave propagation system that occurred due to the use of partial or incomplete information. Further, Abdeljawad, Rashid, Hammouch, and Chu (2020) constructed a new extension of Jensen's and H-H inequalities by combining m-convex and s-convex functions. These inequalities were used to calculate the bivariate mean based on the relationship between the two types of data.

Likewise, Set, Sardari, Ozdemir, and Rooin (2012) merged α-convex and m-convex functions to extend H-H inequality. The extended H-H inequality was then used to formulate the optimisation problems. These problems assist in estimating the minimum amount of energy transferred by a mechanical system. For example, these problems help in converting an input force into motion as an output. Later, Liu and Xu (2021) enhanced this work by combining (α, m)-convex and s-convex functions to extend H-H inequality. Besides applying this inequality to optimum solution in mechanical work, it was also used in quadratic programming to derive new mathematical formulations for finding the errors for quadrature rules.

It was discovered that the properties of s-convex, h-convex, and g-convex functions are crucial in the development of mathematical inequalities such as H-H, Jensen's, Fejér's, and Simpson's which then be applied in diverse areas such as to formulate error estimation of numerical analysis and to calculate mathematical mean (average) (Matuszewska & Orlicz, 1961; Niculescu, 2000; Pinheiro, 2007; Varošanec, 2007; Gabriela, Noor, & Awan, 2015). Unfortunately, there haven't been many studies on how these functions might be combined to extend the current inequities in the aforementioned publications.

H-H inequality calculates the optimal solution for optimisation and linear programming problems by estimating the mean value of merged convex functions. This inequality is often used to determine whether a numerical or analytical method is convergent or divergent, as well as to check the function's stability (Mohammed, Sarikaya, & Baleanu, 2020). H-H inequality also refines Jensen’s inequality by combining convex functions. Through the utilisation of the first and last terms of H-H inequality, special means can be calculated. This special means can then be used in Jensen’s inequality to solve optimisation problems. Meanwhile, Jensen's inequality plays an important role in economics and finance because it is used to calculate the average mean value of random variables (Brownlee, 2020). Furthermore, H-H and Jensen’s inequalities are employed to extend many other inequalities such as Fejér’s inequality, Simpson’s inequality, arithmetic-geometric (AG)-mean inequality, harmonic-mean inequality, Minkowski’s inequality, Young's inequality, and Holder's inequality. Moreover, based on Fejér’s and Simpson’s inequalities, approximate solutions for Trapezoid, Mid-point, and Simpson’s rule can be obtained (Niculescu, 2018).

According to researchers, H-H, Jensen’s, Fejér’s, and Simpson’s inequalities based on convex functions are very important and still need a lot of attention to be extended due to the fact that these inequalities are the basis for many other inequalities (Kalsoom, Wu, Hussain, & Latif, 2019; Mohammed, Sarikaya, & Baleanu, 2020; Napoles, 2020; Budak, Erden, & Ali, 2021; Chu, Rashid, Abdeljawad, Khalid, & Kalsoom, 2021; Jung, Saleem, Bilal, Nazeer, & Ghafoor, 2021; Szostok, 2021).

In this study, we attempt to extend H-H, Jensen's, Fejér's, and Simpson's inequalities by using hybrid convex functions resulting from combining s-convex, h-convex, and g-convex functions.

1.5 Objectives of the Study

The main objective of this study is to construct new inequalities by combining three convex functions, i.e., s-convex, h-convex, and g-convex functions. In order to accomplish this objective, the following sub-objectives must be fulfilled:

1. To merge/consolidate between two of s-convex, h-convex, and g-convex functions.
2. To establish new inequalities for H-H, Jensen's, Fejer's, and Simpson's that are based on the merging convex functions.
3. To formulate the error estimations based on trapezoid, mid-point, and Simpson's rule on the newly constructed inequalities.
4. To validate newly constructed inequalities, previously proven H-H, Jensen's, Fejér's, and Simpson's inequalities will be found by applying specific conditions.
5. To apply the newly constructed inequalities in solving special means (averages).

1.6 Significance of the Study

The new inequalities, which are constructed by combining s-convex, h-convex, and g-convex functions, will extend the properties (e.g., symmetry, continuity, monotonicity, and differentiability) of these inequalities and thus broaden their applicability in mathematics and other related fields such as error estimations for numerical rules (Trapezoidal, Mid-point, and Simpson) and to calculate the mean (average) value for real numbers.

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1.7 Scope of the Study

The extensions of H-H, Jensen's, Fejér's, and Simpson's inequalities that are based on s-convex, h-convex, and g-convex functions will be considered in this study.

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CHAPTER TWO
LITERATURE REVIEW

Convex functions are classified based on different properties such as differentiability, monotonicity, symmetry, and continuity (Attouch & Soubeyran, 2006). These properties are the fundamentals of the development for creating convex functions, extending existing convex functions, as well as for employing convex functions to derive new mathematical inequalities (Borwein & Vanderwerff, 2010). Mathematical inequalities such as H-H, Jensen's, Fejér's, and Simpson's inequality are the results of convex functions (Itrinović & Lacković, 1985).

This chapter begins with the introduction of convex functions followed by their historical overview. The essential definitions that are pertinent to this study are also included in this chapter followed by H-H, Jensen's, Fejér's, and Simpson's inequality which were constructed based on convex functions. The summary of the literature review is also provided to highlight the existing works on inequalities. Finally, the research gaps are presented at the end of this chapter to demonstrate the importance of this work to fill those gaps.

2.1 Introduction

Convex functions play an important role in forming new extensions in inequality theory, linear programming, and optimization. Convex functions and inequalities are crucial parts of mathematical applications, functions, and derivation of new mathematical inequalities (Rashid, Noor, & Khalida, 2020). The generalised nature of convexity, which includes symmetric, continuous, and differentiable properties, supports the utilisation of convex functions in the fields of engineering, error estimation, and analysis. The applicability of convexity motivates researchers to explore its concept in depth (Rashid, Noor, & Khalida, 2020). Many integral inequalities, for example, Hermite-Hadamard (H-H), Jensen's, Fejér's, and Simpson's, are derived using convex functions (Dragomir, Agarwal, & Cerone, 1999; Alzer & Qiu, 2003; Bernardis, Nitti, & Scaglia, 2017).

Convex functions provide a significant contribution in inequality theory by extending s-convex in Definitions (1.5-1.8), h-convex in Definitions (1.10-1.12), g-convex in Definition (1.9), harmonically (HA)-convex in Definition (2.4), α-convex in Definition (2.7), and m-convex function in Definition (2.8) (Ekinci & Eroğlu, 2019). Due to their features, convex functions are used in extending mathematical inequalities. The current review of the literature also shows that numerous important applications based on the extensions of inequalities (Chen & Wu, 2016; Gürbüz, Akdemir, Rashid, & Set, 2020).

The long history of convexity was present since ancient Greece in the field of geometry (Papadopoulos, 2019). The evolution of convex functions can be traced back to the 20th century, during which time the study of inequalities attained recognition as a distinct mathematical discipline, mainly due to the contributions of Hermite-Hadamard (H-H), Jensen, Fejér, and Simpson (Dwilewicz, 2009). The following is a brief history of the convex function.

2.2 Preliminaries

In this section, some important basic definitions, lemmas, and theorems related to convex functions and mathematical inequalities are discussed.

2.2.1 Basic Definitions:

Since the definition of a convex function heavily depends on a convex set, this section presents some basic definitions of convex set and function that will be used in the following chapters.

Definition 2.1 ***(Dwilewicz, 2009):*** *A set $\mathbb{U}$ in $\mathbb{R}^n$ is convex if with any two points x_1 and x_2 belonging to space $\mathbb{S}$ the entire segment joining x_1 and x_2 lies in $\mathbb{S}$.*

$$z = ax_1 + (1-a)x_2, \; a \in [0,1], \tag{2.1}$$

Furthermore, Dwilewicz (2009) stated that if the line segment between any two different points of a set $\mathbb{U}$ always lies between the same set $\mathbb{U}$, then it is called convex set. Geometrically, this set has the entire line beyond its two points as shown in Figure (2.1).

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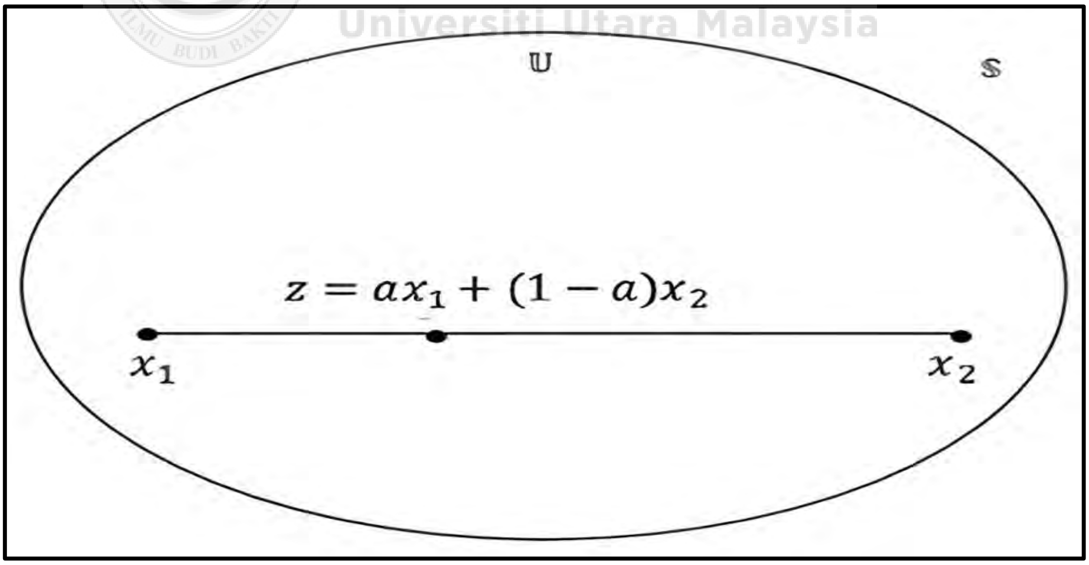

Figure 2.1. Geometrical interpretation of line segment of a convex set $\mathbb{U}$.

For example, consider $x_1, x_2 \in \mathbb{U} \subseteq \mathbb{S}$. Now, to prove that $z \in \mathbb{U}$, select any point of $a \in [0,1]$. Since $x_1, x_2 \in \mathbb{U} \subseteq \mathbb{S}$, then $x_1, x_2 \leq q$. As $a \in [0,1]$, it follows that $p \leq$

$z \leq q$. Because x_1, x_2, and a are chosen arbitrarily, then $z = ax_1 + (1 - a)x_2 \in \mathbb{U}$ for all $x_1, x_2 \in \mathbb{U}$.

Importantly, note that if a set has more than two points and all these points can easily be seen from all the other points of the same set, then this set is known as a convex set. Figure 2.2 represents some graphical forms of different convex and non-convex sets.

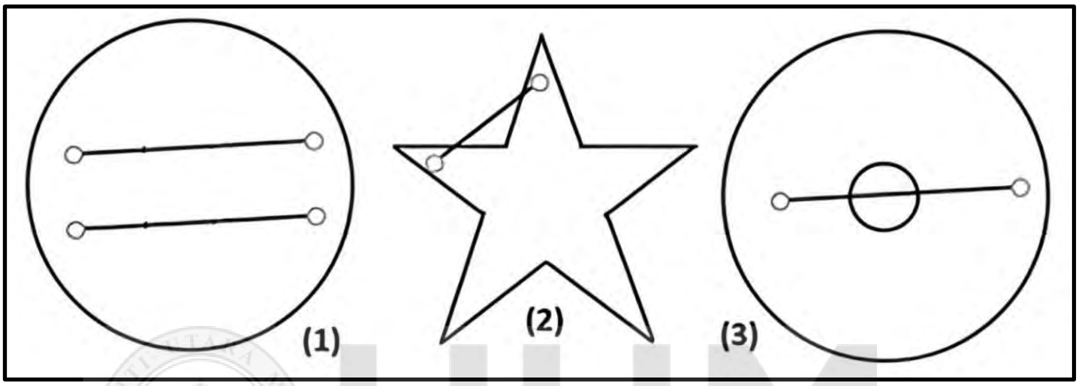

Figure 2.2. Different shapes of convex and non-convex sets.

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Different shapes (circle, star, and disk) in Figure 2.2, described convex and non-convex shapes.

1. The circle involves all the points inside and include all the boundary points of a circle, that is why it is a convex set.
2. The star-shape is not convex because the points on-line do not lie between the set, for this reason, it is not a convex set.
3. The disk misses the internal hole points from all inner boundaries, which is why it is not convex.

The convex set has played a basic role in constructing the convex function. At the beginning of the 19th century, Jensen (1906) introduced the definition of the convex function for real values as follows:

Definition 2.2 ***(Jensen, 1906):*** *A function* $f: \mathbb{I} \to \mathbb{R}$ *is convex on* $a, b \in [0, 1]$*, if*

$$f(ax_1 + bx_2) \le af(x_1) + bf(x_2), \quad x_1, x_2 \in \mathbb{I}, \quad and\; a + b = 1. \tag{2.2}$$

From a geometrical point of view, if the graph of a function f lies below its chord, then f is a convex function (refer to Figure 2.3).

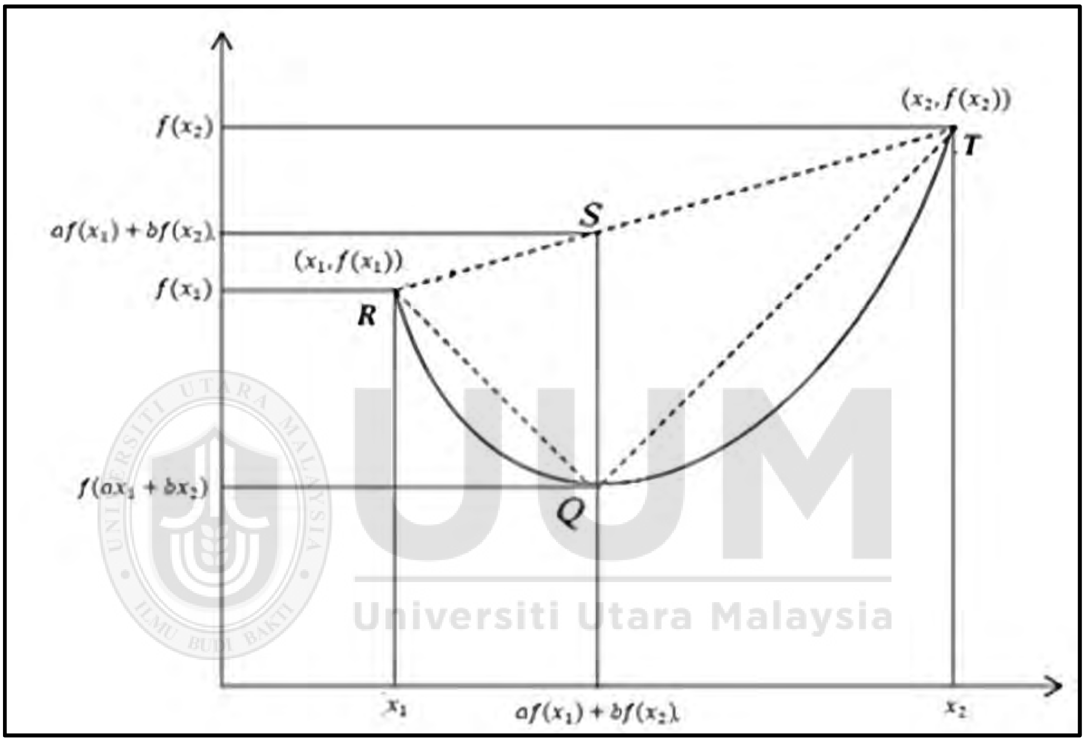

Figure 2. 3. Geometrical interpretation of a convex function.

Based on Figure 2.3, the points of the graph f always lie under or on the chord joining $\big(x_1, f(x_1)\big)$, $\big(x_2, f(x_2)\big)$ for all $x_1, x_2 \in \mathbb{I}$ and $a + b = 1$.

If R, S and T are any three different points on a graph of f such that S lies between R and T, then S lies below or on its chord RS. Unvaryingly, for all distinct values $r, t, u \in [x_1, x_2]$ and $[x_1, x_2] \in \mathbb{I}$ with $r < t < u$, then

$$(u - r)f(t) \le (u - t)f(r) + (t - r)f(u) \tag{2.3}$$

holds (Niculescu & Persson, 2004). Comparatively, Inequality (2.3) can also be represented as

$$f(t) \leq \frac{t-u}{r-u} f(r) + \frac{r-t}{r-u} f(u), \qquad r, t, u \in \mathbb{R}. \tag{2.4}$$

Alternatively, an equivalent definition of a convex function is defined as follows:

Definition 2.3 ***(Peajcariaac & Tong, 1992):*** *A function* $f: \mathbb{I} \to \mathbb{R}$ *is convex for all* $x_1, x_2 \in \mathbb{I} \subseteq \mathbb{R}$, *if*

$$f\left(\frac{\lambda x_1 + \mu x_2}{\lambda + \mu}\right) \leq \frac{\lambda f(x_1) + \mu f(x_2)}{\lambda + \mu}, \qquad \lambda, \mu \in \mathbb{R}_+. \tag{2.5}$$

We noticed that many researches utilised convex functions according to their interests. Since there has not been enough extensions in the field of convex functions to work in other areas of studies such as in statistics, linear programming, and optimisation, it is necessary to extend these functions according to the nature of the problems (Gordji, Rostamian & Delasen, 2016). For example, consider the functions $f(x), g(x)$, and $h(x)$ defined as follows:

$$f(x) = e^{3(x-1)},$$

$$g(x) = \frac{16}{9}\left(x - \frac{1}{4}\right)^2,$$

$$h(x) = \begin{cases} -4x + 1 & \text{if} \quad 0 \leq x < \frac{1}{4} \\ 0 & \text{if} \quad \frac{1}{4} \leq x < \frac{3}{4} \\ 4x - 3 & \text{if} \quad \frac{3}{4} \leq x \leq 1. \end{cases}$$

It is important to note that unlike the first two convex functions $f(x)$, $g(x)$, and the third convex function $h(x)$ does not follow to the asymptotic theory of statistics (Birke & Dette, 2007). Besides, the third function does not estimate the error precisely at the points where the regression function is differentiable. The graphs of $f(x), g(x)$, and $h(x)$ are depicted in Figures 2.4-2.6.

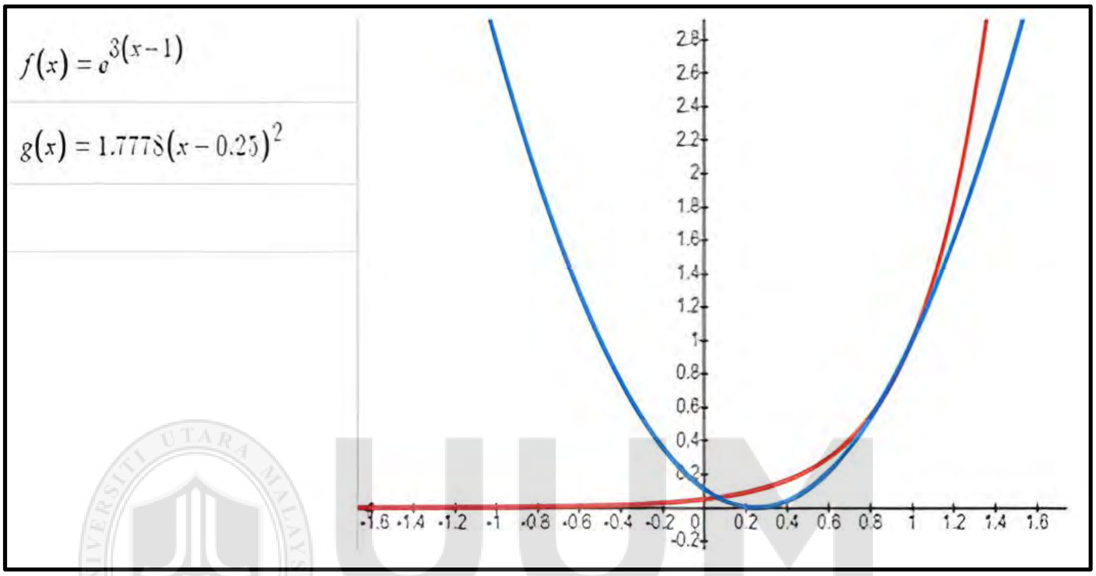

Figure 2. 4. Geometrical interpretation of a $f(x)$ (red line) and $g(x)$ (blue line).

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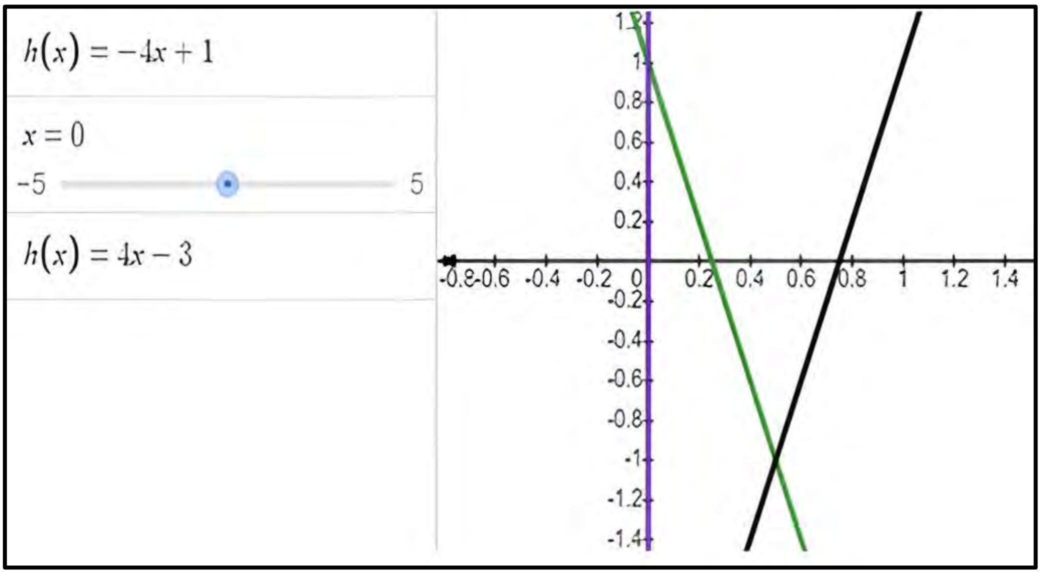

Figure 2.5. Geometrical interpretation of a $h(x)$.

Another example deals with the linear programming problem, specifically focusing on the incorporation of a fixed increment algorithm. This algorithm is used to calculate the estimated error of numerical formula, but it fails to separate the classes of convex functions (Leng, He, Leo, & Qin, 2021). Therefore, extending the convex function and its family is crucial, as under specific conditions, these extensions allow for the separation of classes of convex functions and which may then be used according to the nature of the problem (Kantalo, Wannalookkhee, Nonlaopon, & Budak, 2023; Kórus & Valdés, 2023; Zhang, Ji, & Qi, 2023).

Only recently, numerous extended forms of convex functions and their families have been studied (Farid, Akram, Tawfiq, Ro, Tchier, & Zainab, 2023; Kantalo, Wannalookkhee, Nonlaopon, & Budak, 2023; Kórus & Valdés, 2023; Xie, Ali, Budak, Fečkan, & Sitthiwirattham, 2023). These extended convex functions not only expand the domain of these functions but also extend mathematical inequalities such as H-H and Simpson's. As a result, these extended inequalities can be used in finding the error estimation of numerical formulas and create a link with the mathematical means. This promising development motivates us to explore the extended forms of convex functions, in particular s-convex, h-convex, and g-convex. The primary reason for selecting these convex functions lies in their fundamental properties such as symmetry, differentiability, continuity, and multiplicity. These properties play a crucial role in extending mathematical inequalities such as H-H, Jensen's, Fejér's, and Simpson's.

2.2.2 Related Lemmas, Theorems and Definitions

In this sub-section, some previously defined lemmas, theorems, definitions, and special means are defined to help our study. These lemmas, theorems, definitions, and special means are stated as follows:

Lemma 2.1 ***(Dragomir & Agarwal, 1998):*** *Let* $f: \mathbb{I} \to \mathbb{R}$ *be a differentiable mapping with* $p < q$ *and* $\mathbb{I} \in (p, q)$*. If* $f' \in L[p, q]$*, then*

$$\frac{f(p)+f(q)}{2} - \frac{1}{(q-p)}\int_p^q f(x)dx$$

$$= \left(\frac{q-p}{2}\right)\int_0^1 [1-2a]f'(pa+(1-a)q)da. \qquad (2.6)$$

Lemma 2.2 ***(Sarikaya, Set, Yaldiz, & Basak, 2013):*** *Let* $f: \mathbb{I} \to \mathbb{R}$ *be a differentiable mapping with* $p < q$ *and* $\mathbb{I} \in [p, q]$*. If* $f' \in L[p, q]$*, then*

$$\frac{f(p)+f(q)}{2} - \frac{\Gamma(1+\alpha)}{2(q-p)^{\alpha}}\left[J_{p+}^{\alpha} f(q) + J_{q-}^{\alpha} f(p)\right] = \left(\frac{q-p}{2}\right)$$

$$\times \int_0^1 [(1-a)^{\alpha} - (a)^{\alpha}]\, f'(pa+(1-a)q)da. \qquad (2.7)$$

here

$$J_{p+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_p^x (x-a)^{\alpha-1} f(x)dx\,, \qquad x > p$$

and

$$J_{p-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^q (a-x)^{\alpha-1} f(x)dx\,, \qquad x < q$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function and $J_{p-}^{0} f(x) = J_{q-}^{0} f(x) = f(x)$.

Lemma 2.3 ***(Kirmaci, 2004):*** *Let $f: \mathbb{I} \to \mathbb{R}$ be a differentiable mapping with $p < q$ and $\mathbb{I} \in [p, q]$. If $f' \in L[p, q]$, then*

$$\frac{1}{(q-p)}\int_p^q f(x)dx - f\left(\frac{q+p}{2}\right) = (q-p)\int_0^{\frac{1}{2}} af'(pa+(1-a)q)da$$

$$+ (q-p)\int_{\frac{1}{2}}^1 (1-a)f'(pa+(1-a)q)da. \qquad (2.8)$$

Lemma 2.4 ***(Bai, Qi, & Xi, 2013)*****:** *For $\eta \geq 1$, $0 \leq a \leq 1$ and $\left(\frac{1}{s}\right) \in (0,1)$, we get $\eta^{a^{\left(\frac{1}{s}\right)}} \leq \eta^{\left(\frac{1}{s}\right)a+1-\left(\frac{1}{s}\right)}$*

$$\int_0^1 f^{a^{(s)}}(p) f^{(1-a)^{(s)}}(q)da \leq f^{(s)a+1-(s)}(p) f^{s(1-a)+1-(s)}(q)$$

$$= [f(p)f(q)]^{1-(s)} L\left(f^{(s)}(p), f^{(s)}(q)\right). \qquad (2.9)$$

Lemma 2.5 ***(Özdemir, Yıldız, Akdemir, & Set, 2013):*** *Let a function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{I}^{\circ}$ where $p, q \in \mathbb{I}$ with $p < q$. If $f'' \in L[p, q]$, then*

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) = \frac{(q-p)^2}{16}\left[\int_0^1 a^2 f''\left(a\left(\frac{p+q}{2}\right) + p(1-a)\right)da\right.$$

$$\left. + \int_0^1 (1-a)^2 f''\left(ap + (1-a)\left(\frac{p+q}{2}\right)da\right)\right]. (2.10)$$

Lemma 2.6 ***(Hwang, 2003):*** *suppose $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$, $p, q \in \mathbb{I}$ with $p < q$, $f^{(n)}$ exists on $\mathbb{I}$ and $f^{(n)} \in L(p, q)$ for $n \geq 1$. We have the following identity*

$$\frac{1}{q-p}\int_p^q f(x)dx - \left(\frac{f(p)+f(q)}{2}\right) - \sum_{i=2}^{n-1}\frac{(i-1)(q-p)^i}{2(i+1)!}f^i(p)$$

$$= \frac{(q-p)^i}{2}\left[\int_0^1 a^{(n-1)}(n-2a)f^n(ap+(1-a)q)da\right]. \quad (2.11)$$

Lemma 2.7 ***(Sarikaya, Set, & Ozdamir, 2010):*** *Let $f: \mathbb{I} = [p, q] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function such that $f' \in L[p, q]$ on $t \in [0, 1]$, then*

$$\frac{1}{6}\left[f(p)+f(q)+4f\left(\frac{p+q}{2}\right)\right]-\left(\frac{1}{q-p}\right)\int_p^q f(x)dx = \left(\frac{q-p}{2}\right)$$
$$\times \int_0^1 \left[\left(\frac{a}{2}-\frac{1}{3}\right)f'\left(\frac{1+a}{2}q-\frac{1-a}{2}p\right)+\left(\frac{1}{3}-\frac{a}{2}\right)f'\left(\frac{1+a}{2}p-\frac{1-a}{2}q\right)\right]da. \quad (2.12)$$

Theorem 2.1 ***(Cloud, Drachman & Lebedev, 1998)*****:** *Let $x = x_1, x_2, \ldots x_n$ and $y = y_1, y_2, \ldots y_n$ be two positives n-tuples and p, $q > 1$ be two non-zero numbers such that*

$$\frac{1}{p}+\frac{1}{q}=1.$$

Then

$$\sum_{k=1}^{n} x_k y_k \leq \left(\sum_{k=1}^{n} {x_k}^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n} {y_k}^q\right)^{\frac{1}{q}}, \qquad x_k, y_k \in \mathbb{R}. \quad (2.13)$$

Theorem 2.2 ***(Mitrinovic, Pecaric, & Fink, 2013):*** *Let $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a, b]$, then*

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}, \quad (2.14)$$

with equality hold if and only if $A|f(x)|^p = B|f(x)|^q$ almost everywhere for some constant A and B.

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Lemma 2.8 ***(Obeidat, Latif, & Dragomir, 2022):*** *Let* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be a differential mapping on* $\mathbb{I}^{\circ}$ *and* $p, q \in \mathbb{I}^{\circ}$ *with* $p < q$. *Suppose that* $g: [p, q] \to [0, \infty)$ *is a differentiable mapping. If* $f' \in L[p, q]$, *then for any positive integer* k,

$$g(p)\left(\frac{f(p)+f(q)}{2}\right) - g(p)f\left(\frac{p+q}{2}\right) + \left(\frac{q-p}{4k}\right)$$

$$\times \left\{ \int_0^k \left\{ g'\left(\left(\frac{k+a}{2k}\right)p + \left(\frac{k-a}{2k}\right)q\right) + g'\left(p+q-\left(\left(\frac{k+a}{2k}\right)p + \left(\frac{k-a}{2k}\right)q\right)\right)\right\} da\right.$$

$$\left.\times \int_0^k f\left\{\left(\left(\frac{k+a}{2k}\right)p + \left(\frac{k-a}{2k}\right)q\right) + f\left(p+q-\left(\left(\frac{k+a}{2k}\right)p + \left(\frac{k-a}{2k}\right)q\right)\right)\right\} da \right\}$$

$$= \left(\frac{q-p}{4k}\right)\int_0^k \left\{ g\left(\left(\frac{k+a}{2k}\right)p + \left(\frac{k-a}{2k}\right)q\right)\right.$$

$$\left. - g\left(p+q-\left(\left(\frac{k+a}{2k}\right)p + \left(\frac{k-a}{2k}\right)q\right)\right) + g_q(q)\right\} da$$

$$\times f'\left\{\left(p+q-\left[\left(\frac{k+a}{2k}\right)p + \left(\frac{k-a}{2k}\right)q\right]\right)\right.$$

$$\left. - f'\left(\left(\left(\frac{k+a}{2k}\right)p + \left(\frac{k-a}{2k}\right)q\right)\right)\right\} da. \tag{2.15}$$

Lemma 2.9 ***(Sarikaya, Set, & Ozdemir, 2010)***: *Let* $f: \mathbb{I} \to \mathbb{R}$ *be a differentiable function on* $\mathbb{I} \in [p, q]$ *and* $g: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be symmetric on* $\left(\frac{p+q}{2}\right)$, *then*

$$\frac{1}{6(q-p)}\left[f(p) + 4f\left(\frac{p+q}{2}\right) + f(q)\right]\int_p^q g(x)dx$$

$$- \frac{1}{(q-p)}\int_p^q f(x)g(x)dx \leq \frac{(q-p)}{2}\left[f'\left(\left(\frac{1-a}{2}\right)p + \left(\frac{1+a}{2}\right)q\right)\right] da$$

$$\times \int_0^1 \frac{1}{2}\left[\int_0^a f\left\{\left(\frac{1-a}{2}\right)p + \left(\frac{1+a}{2}\right)q\right\} da\right.$$

$$\left. -\frac{1}{3}\left\{\int_0^1 f\left(\frac{1-a}{2}\right)p + \left(\frac{1+a}{2}\right)q\right\} da\right]$$

$$+\frac{(q-p)}{2}\left[f'\left(\left(\frac{1+a}{2}\right)p + \left(\frac{1-a}{2}\right)q\right)\right] da$$

$$\times \int_0^1 \frac{1}{3}\int_0^a f\left[\left(\frac{1+a}{2}\right)p + \left(\frac{1-a}{2}\right)q\right] da$$

$$\left. -\frac{1}{2}\int_0^1 f\left[\left(\frac{1+a}{2}\right)p + \left(\frac{1-a}{2}\right)q\right] da\right].$$

Lemma 2.10 ***(Alomari, Darus, & Dragomir, 2009)***: *Let $f: \mathbb{I} \to \mathbb{R}$ be a differentiable function on $\mathbb{I} \in [p, q]$ and $g: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be symmetric on $\left(\frac{p+q}{2}\right)$, then*

$$\frac{1}{6}\left[f(p) + 4f\left(\frac{p+q}{2}\right) + f(q)\right] - \frac{1}{(q-p)}\int_p^q f(x)dx$$

$$= (q-p)\int_0^1 t(\alpha)\, f'(pa + (1-a)q)da.$$

Here,

$$t(\alpha) = \begin{cases} a - \frac{1}{6}, & a \in \left[0, \frac{1}{2}\right), \\ a - \frac{5}{6}, & a \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Lemma 2.11 ***(Rostamian, Dragomir, & Sen, 2018)***: *Let $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $|f'|$ be a convex function in the second sense, g be increasing and symmetric function with $p < q$, $\alpha \in [0, 1]$, and $a \in [0, 1]$, then*

$$\left|\left(\frac{f(p)+f(q)}{2}\right)\int_p^q g(x)dx-\int_p^q g(x)f(x)dx\right|$$
$$=\frac{(q-p)^2}{2}\left|\int_0^1 M_g(a)[f'(pa+q(1-a))]da\right|,$$

where

$$M_g(a)=\begin{cases}\int_\alpha^1 g(p\alpha+(1-\alpha)q)d\alpha+\int_0^\alpha g(p\alpha+(1-\alpha)q)d\alpha, & 0\le\alpha\le\frac{1}{2},\\ \int_0^\alpha g(p\alpha+(1-\alpha)q)d\alpha+\int_\alpha^1 g(p\alpha+(1-\alpha)q)d\alpha, & \frac{1}{2}\le\alpha\le 1.\end{cases}$$

Definition 2.4 ***(İşcan, 2014):*** *Let* $\mathbb{I}\subset\mathbb{R}\backslash\{0\}$ *be a real interval. A function* $f:\mathbb{I}\subset\mathbb{R}\to\mathbb{R}$ *is said to be harmonically* (HA)*-convex, if*

$$f\left(\frac{xy}{ax+(1-a)y}\right)\le af(x)+(1-a)f(y), \qquad (2.16)$$

$x,y\in\mathbb{I}$ *and* $a\in[0,1]$.

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Further, the mathematical means to find the relationship with the newly constructed H-H, Jensen's, Fejér's, and Simpson's inequalities are as follows:

Definition 2.5 ***(Dragomir & Pearce, 2003)***: *Let us consider the following means (averages) for real numbers* $\mathbb{R}$ *such as*

Arithmetic Mean

$$A(p,q)=\frac{p+q}{2}, \qquad p,q\in\mathbb{R}.$$

Geometric Mean

$$G(p,q)=\sqrt{pq}, \qquad p,q\in\mathbb{R}_+.$$

Harmonic Mean

$$H(p,q)=\frac{2}{\frac{1}{p}+\frac{1}{q}},\qquad p,q\in\mathbb{R}\backslash\{0\}.$$

Logarithmic Mean

$$\bar{L}(p,q)=\frac{q-p}{ln|q|-ln|p|},\qquad p,q\in\mathbb{R}\backslash\{0\}.$$

Generalised log-Mean

$$L_n(p,q)=\left[\frac{q^{n+1}-p^{n+1}}{(n+1)(q-p)}\right]^{\frac{1}{n}},\ p,q\in\mathbb{R}\backslash\{0\}\text{ and }n\in\mathbb{R}.$$

α-logarithmic mean

$$L_m(p,q)=\left[\frac{q^{\alpha+1}-p^{\alpha+1}}{(\alpha+1)(q-p)}\right]^{\frac{1}{\alpha}},p,q\in\mathbb{R}\backslash\{0\}\text{ and }m\in\mathbb{R}.$$

q-logarithmic mean

$$Q_n(p,q)=\frac{\left(q^{\frac{n+1}{n}}-p^{\frac{n+1}{n}}\right)}{\left(q^{\frac{1}{n}}-p^{\frac{1}{n}}\right)(n+1)},\ p,q\in\mathbb{R}\backslash\{0\}\text{ and }n\in\mathbb{R}.$$

p-logarithmic mean

$$P_n(p,q)=\left(\frac{n}{n+1}\right)^n\left(\frac{\left(q^{\frac{n+1}{n}}-p^{\frac{n+1}{n}}\right)}{(q-p)}\right)^n,p,q\in\mathbb{R}\backslash\{0\}\ \text{ and }\ n\in\mathbb{R}.$$

2.3 Evolution of Mathematical Inequalities Based on Convex Functions

In this section, some evolutions of H-H, Jensen’s, Fejér’s, and Simpson’s inequality based on convex functions will be presented. H-H inequality is a necessary and sufficient condition for convex functions (Mihai, Awan, Noor, Kim, & Noor, 2018).

2.3.1 Historical Overview of Convex Function

Convexity is a simple, useful, and natural concept, which was highlighted at the time of Archimedes (Fink, 2000). According to Niculescu and Persson (2004), Archimedes claimed that a curve on any two different points in the plane is convex if some or all the straight lines between these points connecting these two curve points fall on the same side of the curve. He stated that the plane will be convex if some curved points fall on one end of the same side while others fall on the line itself but not on the opposite side. However, these arguments are only valid for the simple arcs traced on the Euclidean plane. Additionally, in geometry, Archimedes claimed that a set of points is convex if the points in this set contain all line segments between each of its points. This remark is known as the postulate of Archimedes (Dwilewicz, 2009).

The concept of convexity was extended to the convex functions by Fourier in connection with the problem of the solvability of linear inequalities (Varošanec, 2007). Similarly, Cauchy studied convex curves and discovered that if a circle contained a closed convex curve, then the perimeter of this closed convex curve will be smaller than that of this circle (Napoles, 2020).

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These early theories attract researchers to work on convex function as it can be used in various fields of mathematics as well as in other branches of science to build new applications. In order to find the optimum solutions for every problem encountered in various fields, there is an increasing demand for convex functions in inequality theory. As a result, new types of inequalities are constructed that offer the optimum solution from the available range of convex functions (Sarikaya, Saglam, & Yildirim, 2008; Jia & Peng, 2010; Özdemir, Yıldız, Akdemir, & Set, 2013; Mohammed, Sarikaya, & Baleanu, 2020; Napoles, 2020; Budak, Erden, & Ali, 2021; Nasri, Aissaoui, Bouhali, Frioui, Meftah, Zennir, & Radwan, 2023; Breaz, Yildiz, Cotîrlă, Rahman, & Yergöz, 2023). The extended convex functions, such as s-convex, h-convex, geometrically or g-convex, harmonically (HA)-convex, α-convex, and m-convex, are the main motivating factors behind the establishment of new inequalities by researchers. It is noticeably important to note that many forms of H-H inequalities are the results of the convex functions that have been discussed in the previous subsection.

2.3.2 Hermite-Hadamard (H-H) Inequality

In this subsection, some important findings about the H-H inequality will be presented. The H-H inequality was firstly discovered in 1881 by Hermite but it was nowhere mentioned in mathematics literature. Theorem (1.1) in Chapter 1 is a common way to describe the H-H inequality. Without compromising the basic properties in Theorem (1.1), H-H inequality can also be defined as follows:

Theorem 2.3 ***(Mitrinović, Peter, Eckmann, & Waerden, 1970):*** *Let $f: \mathbb{I} \to \mathbb{R}_+$ be a convex function defined on $\mathbb{I} \in \mathbb{R}$ for $p, q \in \mathbb{I}$ and $p < q$, then*

$$[q-p] f\left(\frac{p+q}{2}\right) \leq \int_p^q f(x)dx \leq [q-p]\left(\frac{f(p)+f(q)}{2}\right) \tag{2.17}$$

is called H-H inequality.

Equally, H-H inequality for s-convex function can be described as follows:

Theorem 2.4 ***(Chen & Wu, 2016):*** *Let $f, g: [p, q] \to \mathbb{R},\ p, q \in [0, \infty),\ p < q$ be functions such that $g,\ f, g \in L[p, q]$. If f is convex and nonnegative on $[p, q]$ and g is s- convex function on $[p, q]$ for some fixed $s \in (0, 1]$, then*

$$\left[\frac{1}{q-p}\right]\int_{p}^{q} f(x)g(x)dx \leq \frac{1}{s+2}[f(p)g(p)+f(q)g(q)]$$

$$+\frac{1}{(s+1)(s+2)}[f(p)g(q)+f(q)g(p)] \qquad (2.18)$$

is called H-H inequality for s-convex function.

Chen and Wu (2016) argued that Theorem (2.4) is restricted to only s-convex function, thus the need to introduce another type of H-H inequality that enables us to consider the problem of h-convex function. Similar to Theorem (2.4), Darvish, Dragomir, Nazari, and Taghavi (2017) described the H-H inequality for h-convex function as follows:

Theorem 2.5 ***(Darvish, Dragomir, Nazari, & Taghavi, 2017):*** *Let $f: \mathbb{I} \to \mathbb{R}_+$ be a h-convex function defined on $\mathbb{I} \in \mathbb{R}_+$ for all $p, q \in \mathbb{I}$ with $p < q$, $a \in [0, 1]$, and $f \in L[p, q]$, then*

$$\left(\frac{1}{2h\left(\frac{1}{2}\right)}\right) f\left[\frac{p+q}{2}\right] \leq \frac{1}{q-p}\int_{0}^{1} f(ap+(1-a)q)da$$

$$\leq \big(f(p)+f(q)\big)\int_{0}^{1} f(a)da \qquad (2.19)$$

is called H-H inequality for h-convex function.

Likewise, the description of H-H inequality for g-convex function is given in Theorem (2.6) below:

Theorem 2.6 ***(Taghavi, Darvish, Nazari, & Dragomir, 2016):*** *Let $f: \mathbb{I} \to \mathbb{R}_+$ be a g-convex function defined on $\mathbb{I} \in \mathbb{R}_+$ for all $p, q \in \mathbb{I}$ with $p < q$ and $a \in [0, 1]$, then*

$$f(\sqrt{pq}) \leq \frac{1}{ln(q) - ln(p)} \int_{p}^{q} \frac{1}{a} \sqrt{\left[f(a) f\left(\frac{pq}{a}\right)\right]}\, da \leq \left(\sqrt{f(p) f(q)}\right) \tag{2.20}$$

is called H-H inequality for g-convex function.

Theorems (2.3) - (2.6) shows that the H-H type of inequalities are only applicable when estimating values in functional analysis and related problems. These H-H inequalities are unable to determine a convex function's estimated mean (average) value. To compute the estimated mean value of a convex function, Jensen's inequality was introduced.

2.3.3 Jensen's Inequality

This subsection aims at highlighting some fundamental theorems of Jensen's inequality, in which its construction is mostly founded by convex functions. This type of inequality is also similarly constructed as H-H inequality; thus, it shares many similar advantages that H-H inequality offers.

Theorem 2.7 ***(Jensen, 1906):*** *Let $f: \mathbb{I} \to \mathbb{R}$ be a convex function defined on $\mathbb{I} \in \mathbb{R}$, then*

$$f\left(\frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i}\right) \leq \frac{\sum_{i=1}^{n} a_i f(x_i)}{\sum_{i=1}^{n} a_i} \tag{2.21}$$

is called Jensen's inequality, where $x = (x_1, \dots, x_n) \in \mathbb{I}^n$ and $a = (a_1, \dots, a_n) \in [0, 1]^n$.

Note that the function used in this inequality will always be positive and a_i's are positive n-tuples on x (Becker, 2015). Due to this constraint, this inequality is confined to calculating values for a linear function. For example, if a function $f(x)$ is increasing then, $\overline{f(x)} > f(\bar{x})$ and only depend on the nature of the function.

Jessen (1931) further generalised Jensen's inequality for a convex function using positive linear functions as given in the following Theorem (2.8):

Theorem 2.8 ***(Jessen, 1931):*** *Let* $\mathbb{I}$ *be any convex cone of some real-valued convex function than linear functions* $f, g: \mathbb{I} \to \mathbb{R}$*, satisfies*

$$
\begin{array}{ll}
f(f_1 + g_1) \leq f(f_1) + f(g_1) & \forall\, f_1, g_1 \in \mathbb{I}, \\
f(af_1) \leq af(f_1) & \forall\, a \geq 0, \forall\, f \in \mathbb{I}, \\
f_1 \leq g_1 \Rightarrow f(f_1) \leq f(g_1) & \forall\, f_1, g_1 \in \mathbb{I}, \\
f(\pm 1) \leq \pm f(1) & \forall\, a \geq 0,\, f \in \mathbb{I}.
\end{array}
$$

Some of the generalised Jensen's inequality results are highlighted in Theorems (2.9) – (2.12) as follows:

Theorem 2.9 ***(Peajcariaac & Tong, 1992):*** *Let* $f: \mathbb{I} \to \mathbb{R}$ *be a convex function and* g *be a non-negative function defined on* $\mathbb{I} \in \mathbb{R}$ *for all* $p, q \in \mathbb{I}$ *with* $p < q$*, then*

$$f\left(\frac{\int_p^q g(x)dx}{q-p}\right) \leq \left(\frac{\int_p^q f[g(x)]dx}{q-p}\right). \tag{2.22}$$

According to Pearce and Dragomir (2002), Theorem (2.9) only considers a non-negative convex function. To overcome this constraint, Pearce and Dragomir (2002) extended this work by constructing Jensen's inequality for n-numbers as follows:

Theorem 2.10 ***(Pearce & Dragomir, 2002):*** *Let* $f: \mathbb{I} \to \mathbb{R}$ *be a convex function defined on* $\mathbb{I} \in \mathbb{R}$*, and* $a_i \geq 0,\ p, q \in \mathbb{I}$ *with* $\sum_{i=1}^{n} a_i = 1$*, then*

$$f\left(\sum_{i=1}^{n} a_i x_i\right) \leq \sum_{i=1}^{n} a_i f(x_i),$$

$$\leq \left(\sum_{i=1}^{n} a_i x_i\right)\left[\frac{f(q)-f(p)}{q-p}\right] + \frac{qf(p)-pf(q)}{q-p}. \tag{2.23}$$

Although Theorem (2.10) is able to generalise Jensen's inequality for n-numbers of convex function, it failed to describe mean (average) for h-convex function. The ability to calculate the mean for the h-convex function thus is necessary to generalisation the Jensen's inequality for the h-convex function. Jensen's inequality is now being extended for the h-convex function as follows:

Theorem 2.11 ***(Vivas, , Hernández, & Merentes, 2016):*** *Let* $h: \mathbb{I} \to \mathbb{R}$ *be a* h*-convex function,* $f: \mathbb{I} \to \mathbb{R}$ *is a convex function on* $\mathbb{I} \in \mathbb{R}_+$ *for all* $x_1, x_2, x_3, \ldots x_n \geq 0$ *and* $a_i \geq 0,\ A_n = \sum_{i=1}^{n} a_i$ *with* $\sum_{i=1}^{n} a_i = 1$*, then*

$$f\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i x_i\right) \leq \sum_{i=1}^{n} h\left(\frac{a_i}{A_n}\right) f(x_i). \tag{2.24}$$

Theorem (2.11) extends Jensen's inequality for h-convex function, but it is unable to solve the problems involving s-convex function. For this reason, Chen (2013) developed Jensen's inequality by considering s-convex function as follows:

Theorem 2.12 ***(Chen, 2013):*** *Let* $f: \mathbb{I} \to \mathbb{R}$ *be a* s*-convex function in the second sense for all* $x_1, x_2, x_3, \ldots x_n,\ s \geq 0$ *defined on* $\mathbb{I} \in \mathbb{R}_+$*, and* $a_i \geq 0$ *with* $\sum_{i=1}^{n} a_i = 1$*. Then*

$$f\left(\sum_{i=1}^{n} a_i x_i\right) \leq \sum_{i=1}^{n} (a_i)^s f(x_i). \tag{2.25}$$

Although the inequalities discussed in this subsection are able to calculate the special mean (average), they still fail to multiply two different functions if either one is convex. Thus, to cater this issue, H-H and Jensen's inequalities have been extended. Among the famous extensions is Fejér's inequality.

2.3.4 Fejér's Inequality

Fejér's inequality uses a product of two different functions through which one should be a convex function. This inequality can be used to find the means.

Theorem 2.13 ***(Fejér, 1906):*** *If* $f: \mathbb{I} \to \mathbb{R}$ *is convex and* $g: \mathbb{I} \to \mathbb{R}^+$ *be a non-negative function defined on* $[p, q] \in \mathbb{I} \subseteq \mathbb{R}$ *for all* $x \in \mathbb{I}$, *then*

$$f\left[\frac{p+q}{2}\right]\int_{p}^{q} g(x)dx \leq \int_{p}^{q} f(x)g(x)dx \leq \frac{f(p)+f(q)}{2}\int_{p}^{q} g(x)dx \qquad (2.26)$$

is called Fejér's inequality.

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As has been previously noted that this inequality is the generalisation of H-H inequality. If we substitute $g(x) \equiv 1$, Theorem (2.13) will be converted to the H-H inequality in Theorem (1.1) (Napoles, 2020). Further, Brenner and Alzer (1991) generalised the Fejér's inequality as described in Theorem (2.14):

Theorem 2.14 ***(Brenner & Alzer, 1991):*** *Let* f *be a convex and* g *be an integrable and symmetric function on* $\bar{x} = \left[\frac{ap+bq}{a+b}\right]$ *defined on* $p, q \in \mathbb{I} \in \mathbb{R}$ *such that* $f: \mathbb{I} \to \mathbb{R}$, *then*

$$f\left[\frac{ap+bq}{a+b}\right]\int_{\bar{x}-y}^{\bar{x}+y} g(x)dx \;\; \leq \int_{\bar{x}-y}^{\bar{x}+y} [f(x)g(x)]dx,$$

$$\leq \frac{af(p)+bf(q)}{a+b}\int_{\bar{x}-y}^{\bar{x}-y} g(x)dx\,, \quad 0 \leq y \leq \frac{q-p}{a+b}.$$

Although this developed inequality in Theorem (2.14) is able to avoid the limitation posed by its predecessors, it still fails to solve the problem of differentiation. In Theorem (2.14), the symmetric property of convex function is used whereas a function of n variables is symmetric if its value is the same no matter the order of its arguments. Notice that H-H, Jensen's, and Fejér's inequalities do not employ the derivative of functions in their derivation. Simpson's inequality was then introduced to make use of derivatives of a function in its derivation.

2.3.5 Simpson's Inequality

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We begin this subsection by stating a theorem of Simpson's inequality for a convex function. In addition, Simpson's inequality for differentiable convex function is described as follows:

Theorem 2.15 ***(Dragomir, Agarwal, & Cerone, 1999):*** *If* $f:\mathbb{I}\to\mathbb{R}$ *is a differentiable and continuous function and* $\parallel f^{(4)} \parallel_{\infty} := sup\left|f^{(4)}(x)\right| < \infty$ *is convex on* $p,q \in \mathbb{I} \in \mathbb{R}$*, then*

$$\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\left[\frac{1}{q-p}\right]\int_{p}^{q} f(x)dx$$

$$\leq \frac{1}{2880} \parallel f^{(4)} \parallel_{\propto} (q-p)^4 \qquad (2.27)$$

is called Simpson's inequality.

Since Simpson's inequality is able solve the problems related to differentiable function, it is regarded as a useful inequality, especially when determining mean and estimating errors from Simpson's formula (Mevlüt, Yildiz, & Ekinci, 2013). In Theorem (2.16), the extension of Simpson's inequality is constructed using s-convex function in the second sense as given below.

Theorem 2.16 ***(Alomari, Mohammad, Darus, & Dragomir, 2009):*** *Let $f: \mathbb{I} \to \mathbb{R}$ be a differentiable and continuous function on $\mathbb{I}^{\circ}$ and $|f'|$ is a s-convex function in the second sense for $s \in (0,1)$ on $p, q \in \mathbb{I} \in \mathbb{R}$ and $p < q$, then*

$$\frac{1}{6}\left[f(p) + 4f\left(\frac{p+q}{2}\right) + f(q)\right] - \left[\frac{1}{q-p}\right]\int_{p}^{q} f(x)dx$$
$$\leq (q-p)\frac{6^{-s} - 9(2^{-s}) + 5^{s+2}(6^{-s}) + 3s - 12}{18(s^2 + 3s + 2)}[|f'(p)| + |f'(q)|]. \quad (2.28)$$

Note that Theorem (2.16) is the extension of Simpson's inequality based on differentiable s-convex function. However, it fails to fulfil all the properties of h-convex and g-convex functions.

Recently, there have been efforts to explore novel approaches for merging different types of convex function families to produce new inequalities. This effort is deemed important as the usability of these new functions and their new inequalities may be expanded into relevant fields of study (Dong, Zeb, Farid, & Sidra, 2021). Feng, Ghafoor, Chu, Qureshi, Feng, Yao, and Qiao (2020) have merged p-convex and h-convex functions to extend mathematical inequalities.

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Definition 2.6 ***(Feng*, Gafoor, *Chu, Qureshi, Feng, Yao, & Qiao, 2020):*** *If $f:\mathbb{I}\to\mathbb{R}$ be a p-convex function and h be a non-negative function defined on $p,q\in\mathbb{I}\in\mathbb{R}$, then*

$$f(ax^p+(1-a)y^p)^{\frac{1}{p}}\le h(a)f(x)+h(1-a)f(y),\qquad a\in[0,1] \tag{2.29}$$

is called (p,h)-convex function.

They further constructed extensions of the H-H and Jensen's inequalities and create a link between means and these extended H-H and Jensen's inequalities. Based on the newly constructed hybrid (p,h)-convex function in Definition (2.6), H-H and Jensen's inequalities are extended as described in Theorems (2.17) and (2.18), respectively:

Theorem 2.17 ***(Feng*, Gafoor, *Chu, Qureshi, Feng, Yao, & Qiao, 2020):*** *Let $f:\mathbb{I}\to\mathbb{R}$ be a (p,h)-convex function on $p,q\in\mathbb{I}\in\mathbb{R}$ and $p_1<q_1$, then*

$$\int_0^1 f\left(\frac{{p_1}^p+{q_1}^p}{2}\right)^{\frac{1}{p}}da\le\left(\frac{p}{{q_1}^p-{p_1}^p}\right)\int_{p_1}^{q_1} f(x)dx,$$

$$\le f(p_1)+\{f(q_1)-f(p_1)\}\int_0^1 h(a)da. \tag{2.30}$$

Theorem 2.18 ***(Feng*, Gafoor, *Chu, Qureshi, Feng, Yao, & Qiao, 2020):*** *Let $f:\mathbb{I}\to\mathbb{R}$ be a (p,h)-convex function defined on $\mathbb{I}\in\mathbb{R}$. Then*

$$f\left(\sum_{i=1}^{n}a_i{x_i}^p\right)^{\frac{1}{p}}\le h(A_{n-1})f\left(\sum_{i=1}^{n}\frac{a_i}{A_{n-1}}(x_i)^p\right)^{\frac{1}{p}}+\big(1-h(A_{n-1})\big)f(x_i) \tag{2.31}$$

is called Jensen's inequality, where $x=(x_1,\dots,x_n)\in\mathbb{I}^n$, $a=(a_1,\dots,a_n)\in[0,1]$ and $A_n=\sum_{i=1}^{n}a_i$.

Meanwhile, Dong, Zeb, Farid, and Bibi (2021) has also combined m-convex and α-convex functions to form an (α, m)-convex function as described below:

Definition 2.7 (*Dong, Zeb, Farid, & Bibi, 2021*): *A function* $f: [0, q] \to \mathbb{R}$ *is said to be* (α, m)*-convex function for* $(\alpha, m) \in [0, 1]^2$ *for all* $x, y \in \mathbb{I}$ *and* $a \in [0, 1]$, *if*

$$f(ax + m(1 - a)y) \leq (a^{\alpha})f(x) + m(1 - a^{\alpha})f(y). \quad (2.32)$$

Substituting $m = 1$ and $\alpha = 1$ in Definition (2.5) produces Definition (1.1). Meanwhile, replacing $\alpha = 1$ in Definition (2.5) leads to the following definition:

Definition 2.8 (*Dong, Zeb, Farid, & Bibi, 2021*): *A function* $f: [0, q] \to \mathbb{R}$ *is said to be* m*-convex function, for all* $m \in [0, 1]$, $x, y \in \mathbb{I} \in [0, 1]$, *and* $a \in [0,1]$, *if*

$$f(ax + m(1 - a)y) \leq (a)f(x) + m(1 - a)f(y). \quad (2.33)$$

Furthermore, merging Definition (2.7) with Definition (1.6), Wang, Liu, Yin, and Qi (2023) defined (α, m, s)-convex function as follows:

Definition 2.9 *(Wang, Liu, Yin, & Qi, 2023)*: *A function* $f: [0, q] \to \mathbb{R}$ *is said to be* (α, m, s)*-convex function for* $(\alpha, m) \in [0, 1]^2, s \in [0, 1]$, *for all* $x, y \in \mathbb{I}$ *and* $a \in [0, 1]$, *if*

$$f(ax + m(1 - a)y) \leq (a^{\alpha s})f(x) + m(1 - a^{\alpha s})f(y). \quad (2.34)$$

We end this subsection by illustrating the existing combination of convex functions mentioned in Napoles (2020). Here, in Fig 2.6, some abbreviations are used such as convex function (C), convexity with respect to another function (CO), harmonically convex (HC), strongly harmonically convex (SHC-C), h-convex (h-C), p-convex (p-C), s-convex in first and second sense (SC-1 and SC-2), Godunova-Levin function (G-

L), m-convex (m-C), g-convex (G-C), (m, g)-convex (mG-C), α-convex (), (α, m)-convex (α, m-C), (α, m, g)-convex (α, m),G-C, quasi-convex (cq-C), geometrically-arithmetically convex function (GAC) and etc.

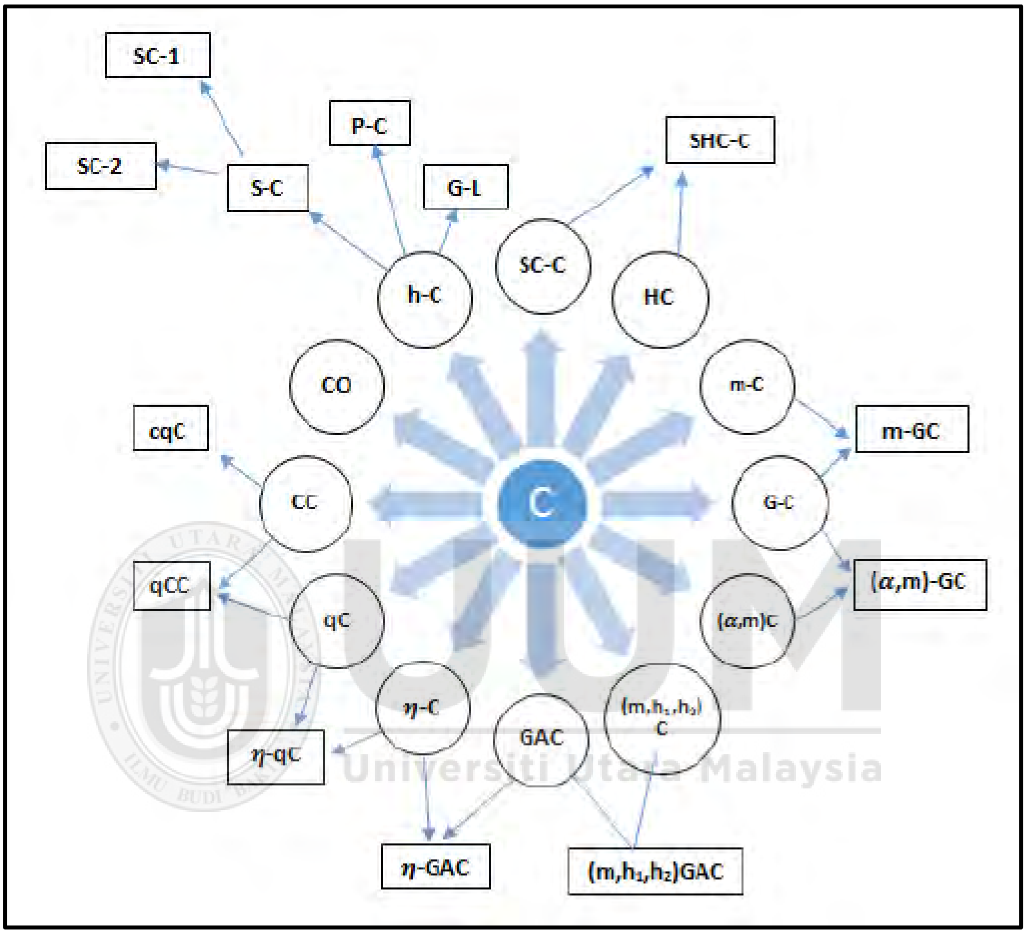

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Figure 2.6. Already merged convex functions.

As illustrated in Figure 2.6, the progress in this area is significant and vast because of the convex function properties including symmetry, differentiability, and positivity. These properties of convex functions help in solving the problems of different areas such as inequality theory. Thus, we will focus our attention solely on three main families of convex functions, namely, the s-convex, h-convex, and g-convex, as these functions constitute the primary generalizations of convex functions.

2.4 Summary of Literature Review

Based on an exhaustive literature review, it was discovered that the recent trend in constructing inequalities was obtained by combining different families of convex functions which have significant benefits.

The summary of the literature review is illustrated in the form of a flow chart in Figure 2.7 below:

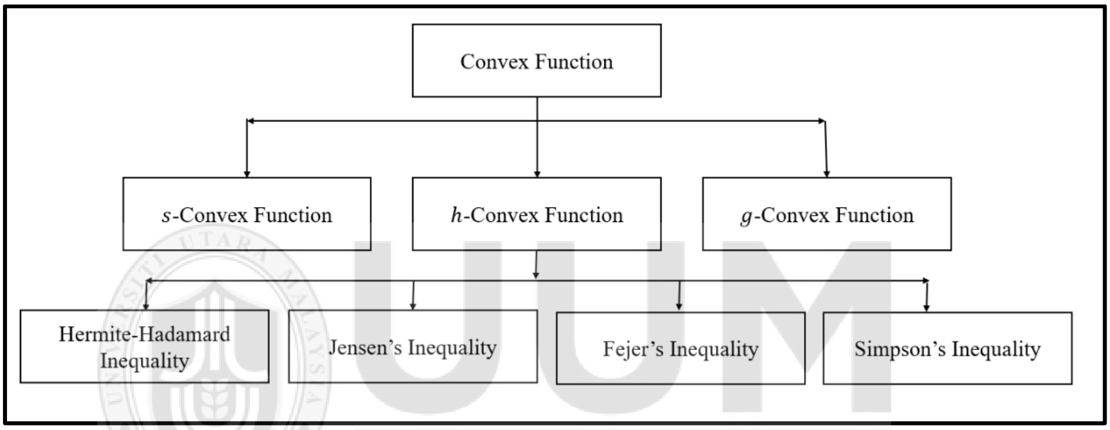

Figure 2.7. Summary of literature review in flow chart

2.5 Research Gap

Based on the flow chart shown in the previous Figure 2.7, limited work has been carried out to extend Hermite-Hadamard (H-H), Jensen's, Fejér's and Simpson's inequality by using s-convex, h-convex, and g-convex function. Therefore, in this study, we attempt to merge s-convex, h-convex, and g-convex functions for constructing new hybrid convex functions. These newly constructed hybrid functions are then employed to extend Hadamard (H-H), Jensen's, Fejér's and Simpson's inequalities as depicted in Figure 2.8.

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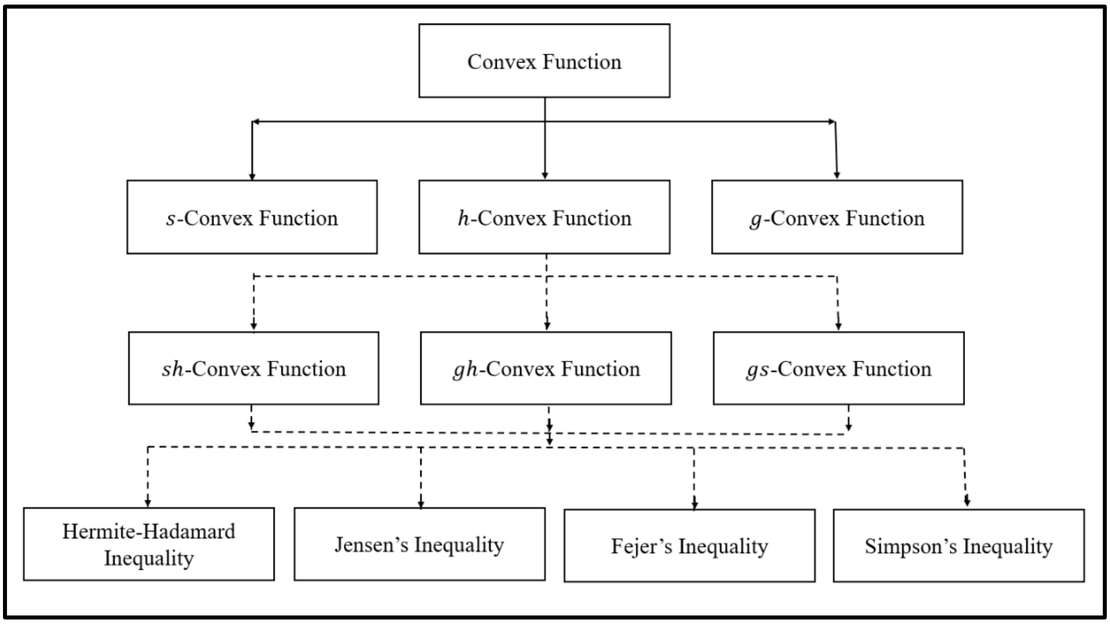

Figure 2.8. Flow chart of research gap

In this study, we only consider the extension of Hermite-Hadamard (H-H), Jensen's, Fejér's, and Simpson's inequalities due to the fact that many other inequalities such as Hermite-Hadamard-Fejer (Latif, 2023), Ostrowski's (Xie, Ali, Budak, Fečkan, & Sitthiwirattham, 2023), Jensen-Mercer (Fahad, Wang, & Butt, 2023), and Quantum-Hermite-Hadamard (Chasreechai, Ali, Ashraf, Sitthiwirattham, Etemad, Sen, & Rezapour, 2023) inequalities are derived from these inequalities.

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CHAPTER THREE
HERMITE-HADAMARD INEQUALITIES BASED ON HYBRID CONVEX FUNCTIONS

This chapter describes three novel integral forms of Hermite-Hadamard (H-H) inequality based on the hybrid of s-convex, h-convex, and g-convex functions. These new inequalities can form new relationship with mathematical means (averages) for real numbers and find error estimation formulae for trapezoidal formula.

3.1 Introduction

In this chapter, Hermite-Hadamard (H-H) inequality is extended based on hybrid convex functions without losing its originality. By applying certain conditions, the formal form, as well as some previous H-H inequality can be obtained.

Moreover, the extended H-H inequality create a link with the mathematical means (averages) and calculate error estimation formulae of the trapezoidal formula.

3.2 Extension of Hermite-Hadamard Inequality by Merging $(\boldsymbol{h_1}, \boldsymbol{h_2})$-convex and $\boldsymbol{s}$-convex Functions

In this section, several new definitions are introduced based on the existing definitions. Definitions (1.5) and (1.11) are merged to form Definition (3.1) of (h_1, h_2, s)-convex function in the first sense as stated below:

Definition 3.1: *Let $h_1, h_2: [0,1] \subset \mathbb{J}^* \to \mathbb{R}$ be two positive functions such that $h_1 \not\equiv 0$ and $h_2 \not\equiv 0$. A function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ for $a \in [0, 1]$, $x, y \in \mathbb{I}$, and $s \in (0, 1]$ is called (h_1, h_2, s)-convex function in the first sense if $h_1{}^s(a) + h_2{}^s(1 - a) = 1$ and*

$$f(ax + (1-a)y) \leq {h_1}^s(a)f(x) + {h_2}^s(1-a)f(y). \quad (3.1)$$

The newly introduced Definition (3.1) can be combined with Definitions (1.6) and (1.11) to construct Definition (3.2) of (h_1, h_2, s)-convex function in the second sense as follows:

Definition 3.2: *Let $h_1, h_2: [0,1] \subset \mathbb{J}^* \to \mathbb{R}$ be two positive functions such that $h_1 \not\equiv 0$ and $h_2 \not\equiv 0$. A function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ for $a \in [0,1]$, $x, y \in \mathbb{I}$, and $s \in (0,1]$ is called (h_1, h_2, s)-convex function in the second sense if $h_1(a) + h_2(1-a) = 1$ and*

$$f(ax + (1-a)y) \leq {h_1}^s(a)f(x) + {h_2}^s(1-a)f(y). \quad (3.2)$$

Meanwhile, combining Definition (1.7) and (1.11) produces new Definition (3.3) of (h_1, h_2, s)-convex function in the third sense:

Definition 3.3: *Let $h_1, h_2: [0,1] \subset \mathbb{J}^* \to \mathbb{R}$ be two positive functions such that $h_1 \not\equiv 0$ and $h_2 \not\equiv 0$. A function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ for $a \in [0,1]$, $x, y \in \mathbb{I}$, and $s \in (0,1]$ is called (h_1, h_2, s)-convex function in the third sense if ${h_1}^s(a) + {h_2}^s(1-a) = 1$ and*

$$f(ax + (1-a)y) \leq {h_1}^{\frac{1}{s}}(a)f(x) + {h_2}^{\frac{1}{s}}(1-a)f(y). \quad (3.3)$$

Finally, Definition (3.4) of (h_1, h_2, s)-convex function in the fourth sense as stated below can be obtained by combining Definitions (1.8) and (1.11):

Definition 3.4: *Let $h_1, h_2: [0,1] \subset \mathbb{J}^* \to \mathbb{R}$ be two positive functions such that $h_1 \not\equiv 0$ and $h_2 \not\equiv 0$. A function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ for $a \in [0,1]$, $x, y \in \mathbb{I}$, and $s \in (0,1]$ is called (h_1, h_2, s)-convex function in the fourth sense if $h_1(a) + h_2(1-a) = 1$ and*

$$f(ax + (1-a)y) \leq {h_1}^{\frac{1}{s}}(a)f(x) + {h_2}^{\frac{1}{s}}(1-a)f(y). \quad (3.4)$$

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Similar as Definition (3.1), (3.2), (3.3), and (3.4), Definition (1.6) and (1.12) are combined to form another hybrid (h_1, h_2, s)-convex function as follows:

Definition 3.5: *Let* $h_1, h_2: (0,1) \subset \mathbb{J} \to \mathbb{R}$ *be two positive functions such that* $h_1 \not\equiv 0$ *and* $h_2 \not\equiv 0$. *A function* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *for* $a \in [0,1]$, $x, y \in \mathbb{I}$, *and* $s \in [0,1]$ *is called* (h_1, h_2, s)*-convex function if*

$$f(ax + (1-a)y) \leq {h_1}^s(a){h_2}^s(1-a)f(x) + {h_2}^s(a){h_1}^s(1-a)f(y). \qquad (3.5)$$

If ${h_1}^s(a){h_2}^s(1-a) = h(a)$ and ${h_2}^s(a){h_1}^s(1-a) = h(1-a)$, then Definition (3.5) will be reduced to Definition (1.10) for h-convex function. While when $s = 1$, Definition (3.5) will be reduced to Definition (1.11) for (h_1, h_2)-convex function.

Now, we extend H-H inequality based on hybrid convex function defined in Definition (3.5).

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Theorem 3.1: *Let a function* $f: \mathbb{I}^* \subset \mathbb{R} \to \mathbb{R}$ *be* (h_1, h_2, s)*-convex, differentiable, positive, and symmetric such that* $x < p < q < y$, $a \in [0,1]$ *and* $s \in [0,1]$ *with* ${h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right) \not\equiv 0$, *then*

$$\frac{1}{2{h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)} f\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q f(x)dx$$

$$\leq [f(p) + f(q)]\int_0^1 {h_1}^s(a)\,{h_2}^s(1-a)da.$$

Proof: Since $x < p < q < y$ and $a \in [0,1]$, by using convex line segment property,

$$x = aq + (1-a)p, \qquad (3.6)$$

and

$$y = ap + (1-a)q. \qquad (3.7)$$

Integrating Inequality (3.5) with respect to a gives

$$\int_0^1 f(aq+(1-a)p)da \le f(q)\int_0^1 {h_1}^s(a){h_2}^s(1-a)\,da$$

$$+f(p)\int_0^1 {h_1}^s(1-a){h_2}^s(a)da,$$

which leads to

$$\frac{1}{q-p}\int_p^q f(x)dx \le f(q)\int_0^1 {h_1}^s(a){h_2}^s(1-a)\,da$$

$$+f(p)\int_0^1 {h_1}^s(1-a){h_2}^s(a)\,da. \tag{3.8}$$

Note that x from Equation (3.6) can also be written as $x = a(q-p)+p$. Taking the derivative of x with respect to a gives, $da = \frac{dx}{(q-p)}$.

Since (h_1, h_2, s)-convex function is positive and symmetric, then Inequality (3.8) becomes

$$\frac{1}{q-p}\int_p^q f(x)dx \le [f(p)+f(q)]\int_0^1 {h_1}^s(a){h_2}^s(1-a)da. \tag{3.9}$$

Without loss of generality, put the values of x and y from Equation (3.6) and Equation (3.7) and then set $a = \frac{1}{2}$ in Inequality (3.5), yields

$$f\left(\frac{p+q}{2}\right) \le {h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)\left[f\left(\frac{p+q}{2}\right)+f\left(\frac{p+q}{2}\right)\right],$$

$$f\left(\frac{p+q}{2}\right) \le 2{h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)f\left(\frac{p+q}{2}\right), \tag{3.10}$$

and $f\left(\frac{x+y}{2}\right)$ can be expressed as

$$f\left(\frac{x+y}{2}\right) = f\left(\frac{ap+(1-a)q+aq+(1-a)p}{2}\right),$$

$$= f\left(\frac{p+q}{2}\right).$$

Therefore, Inequality (3.10) can be written as

$$f\left(\frac{p+q}{2}\right) \leq 2{h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right) f\left(\frac{x+y}{2}\right),$$

or equivalent to

$$f\left(\frac{p+q}{2}\right) \leq 2{h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right) f(aq+(1-a)p). \tag{3.11}$$

Integrating Inequality (3.11) with respect to a where $a \in [0, 1]$, yields

$$\int_0^1 f\left(\frac{p+q}{2}\right) da \leq 2{h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right) \int_0^1 f(aq+(1-a)p)\, da, \tag{3.12}$$

which can be simplified as

$$\int_0^1 f\left(\frac{p+q}{2}\right) da = f\left(\frac{p+q}{2}\right) a\Big|_0^1 = f\left(\frac{p+q}{2}\right)(1-0) = f\left(\frac{p+q}{2}\right),$$

and Inequality (3.12) will be reduced to

$$\frac{1}{2{h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)} f\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q f(x)dx. \tag{3.13}$$

Now, combining Inequality (3.9) and Inequality (3.13) produces

$$\frac{1}{2{h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)} f\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q f(x)dx$$

$$\leq [f(p)+f(q)]\int_0^1 {h_1}^s(a)\,{h_2}^s(1-a)da.$$

This completes the proof.

Remarks 3.1:

1. If $h_1{}^s(a) = h_2{}^s(1-a) = 1$ in Theorem (3.1), then the H-H inequality in Theorem (2.1) will be obtained.

2. By substituting $s = 1$ in Theorem (3.1), the result will be H-H inequality for (h_1, h_2)-convex function.

Note that Theorem (3.1) successfully extended H-H inequality by using hybrid (h_1, h_2, s)-convex function in Definition (3.5).

In the following theorems, we will extend H-H inequality by considering hybrid (h_1, h_2, s)-convex function in third and the fourth sense defined in Definitions (3.3) and (3.4), respectively.

From now, we will use Definition (3.3) of (h_1, h_2, s)-convex function in the third sense in terms of second derivative for $s \in (0, 1]$ and $h_1{}^s(a) + h_2{}^s(1-a) = 1$ with $h_1(a) \not\equiv 0$ and $h_2(1-a) \not\equiv 0$ as follows:

$$f''\left(a\left(\frac{p+q}{2}\right) + (1-a)p\right)$$

$$\leq h_1{}^{\left(\frac{1}{s}\right)}(a) f''\left(\frac{p+q}{2}\right) + h_2{}^{\left(\frac{1}{s}\right)}(1-a) f''(p), \qquad (3.14)$$

and

$$f''\left(aq + (1-a)\left(\frac{p+q}{2}\right)\right)$$

$$\leq h_1{}^{\left(\frac{1}{s}\right)}(a) f''(q) + h_2{}^{\left(\frac{1}{s}\right)}(1-a) f''\left(\frac{p+q}{2}\right). \qquad (3.15)$$

Theorem 3.2: *Let a function* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be differentiable, positive, and symmetric on* $\mathbb{I}^{\circ}$ *where* $p, q \in \mathbb{I}$ *with* $p < q$, $s \in (0,1]$, *and* $a \in [0,1]$. *If* $f'' \in L[p,q]$ *be a* (h_1, h_2, s)*-convex function in the third sense, then*

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[f''\left(\frac{q+p}{2}\right)\int_0^1 a^2 h_1{}^{\left(\frac{1}{s}\right)}(a)da\right.$$

$$\left.+f''(p)\int_0^1 a^2 h_2{}^{\left(\frac{1}{s}\right)}(1-a)da\right] + \frac{(q-p)^2}{16}$$

$$\times\left[f''(q)\int_0^1 (1-a)^2 h_1{}^{\left(\frac{1}{s}\right)}(a)da + f''\left(\frac{p+q}{2}\right)\right.$$

$$\left.\times\int_0^1 (1-a)^2 h_2{}^{\left(\frac{1}{s}\right)}(1-a)da\right].$$

Proof: Since $p < q$, $a \in [0,1]$, and $|f''|$ is a (h_1, h_2, s)-convex function in the third sense, then by substituting Inequalities (3.14) and (3.15) in Lemma (2.5), results

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\int_0^1 a^2\left\{f''\left(h_1{}^{\left(\frac{1}{s}\right)}(a)\left(\frac{p+q}{2}\right)\right)\right.\right.$$

$$\left.\left.+h_2{}^{s}(1-a)f''(p)\right\}da\right]$$

$$+\frac{(q-p)^2}{16}\left[\int_0^1 (1-a)^2\left\{f''\left(h_1{}^{\left(\frac{1}{s}\right)}(a)(q)\right)da\right.\right.$$

$$\left.\left.+h_2{}^{\left(\frac{1}{s}\right)}(1-a)f''\left(\frac{p+q}{2}\right)\right\}da\right],$$

which leads to

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[f''\left(\frac{q+p}{2}\right)\int_0^1 a^2 h_1{}^{\left(\frac{1}{s}\right)}(a)da\right.$$

$$\left.+f''(p)\int_0^1 a^2 h_2{}^{\left(\frac{1}{s}\right)}(1-a)da\right] + \frac{(q-p)^2}{16}$$

$$\times\left[f''(q)\int_0^1(1-a)^2h_1^{\left(\frac{1}{s}\right)}(a)da+f''\left(\frac{p+q}{2}\right)\right.$$

$$\left.\times\int_0^1(1-a)^2h_2^{\left(\frac{1}{s}\right)}(1-a)da\right]. \qquad (3.16)$$

Inequality (3.16) completes the proof.

Remark 3.2:

1. By substituting ${h_1}^s(a)=h^s(a)$ and ${h_2}^s(1-a)=h^s(1-a)$, Theorem (3.2) will produce H-H inequality for (h,s)-convex function.

2. When ${h_1}^s(a)=a$ and ${h_2}^s(1-a)=(1-a)$, Theorem (3.2) will be producing the H-H inequality for a convex function.

Theorem 3.3: *Let a function $f:\mathbb{I}\subset\mathbb{R}\to\mathbb{R}$ be differentiable, positive, and symmetric on $\mathbb{I}^{\circ}$ where $p,q\in\mathbb{I}$ with $p<q$, $s\in(0,1]$, and $a\in[0,1]$. If $f''\in L[p,q]$ be a (h_1,h_2,s)-convex function in the third sense and $\frac{1}{c}+\frac{1}{d}=1$ for $1<c<\infty$ and $d>1$, then*

$$\frac{1}{q-p}\int_p^q f(x)dx-f\left(\frac{p+q}{2}\right)\le\frac{(q-p)^2}{16}\left[\frac{1}{2c+1}\right]^{\frac{1}{c}}\left[\left(h_1^{\left(\frac{1}{s}\right)}(a)\left|f''\left(\frac{p+q}{2}\right)\right|^d\right.\right.$$

$$\left.+h_2^{\left(\frac{1}{s}\right)}(1-a)|f''(p)|^d\right)^{\frac{1}{d}}$$

$$+\left(h_1^{\left(\frac{1}{s}\right)}(a)|f''(q)|^d+h_2^{\left(\frac{1}{s}\right)}(1-a)\right.$$

$$\left.\left.\times\left|f''\left(\frac{p+q}{2}\right)\right|^d\right)^{\frac{1}{d}}\right].$$

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Proof: Since $p < q$ and $a \in [0, 1]$ for $1 < c < \infty$ and $d > 1$, then by utilising Lemma (2.5) in Theorem (2.1) leads to

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\int_0^1 a^{2c}\,da\right]^{\frac{1}{c}}$$

$$\times\left(\int_0^1 f''\left(a\left(\frac{p+q}{2}\right)+(1-a)p\right)^d\right)^{\frac{1}{d}} da$$

$$+\frac{(q-p)^2}{16}\left[\int_0^1 (1-a)^{2c}\,da\right]^{\frac{1}{c}}$$

$$\times\left(\int_0^1 f''\left(aq+(1-a)\left(\frac{p+q}{2}\right)\right)^d\right)^{\frac{1}{d}} da. \quad (3.17)$$

As $|f''|$ is a (h_1, h_2, s)-convex function in the third sense, then by substituting Inequalities (3.14) and (3.15) in Inequality (3.17), results

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\int_0^1 a^{2c}\,da\right]^{\frac{1}{c}}$$

$$\times\left(h_1^{\left(\frac{1}{s}\right)}(a)\left|f''\left(\frac{p+q}{2}\right)\right|^d\right.$$

$$\left.+h_2^{\left(\frac{1}{s}\right)}(1-a)|f''(p)|^d\right)^{\frac{1}{d}}$$

$$+\frac{(q-p)^2}{16}\left[\int_0^1 (1-a)^{2c}\,da\right]^{\frac{1}{c}}$$

$$\times\left(h_1^{\left(\frac{1}{s}\right)}(a)|f''(q)|^d\right.$$

$$\left.+h_2^{\left(\frac{1}{s}\right)}(1-a)\left|f''\left(\frac{p+q}{2}\right)\right|^d\right)^{\frac{1}{d}} da. \qquad (3.18)$$

Since (h_1, h_2, s)-convex function in the third sense is positive and symmetric, then

$$\int_0^1 a^{2c}\, da = \int_0^1 (1-a)^{2c}\, da = \frac{1}{(2c+1)},$$

and Inequality (3.18) will be reduced to

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\frac{1}{2c+1}\right]^{\frac{1}{c}}$$

$$\times\left(\int_0^1 f''\left(a\left(\frac{q+p}{2}\right)+(1-a)p\right)^d da\right)^{\frac{1}{d}}$$

$$+\frac{(q-p)^2}{16}\left[\frac{1}{2c+1}\right]^{\frac{1}{c}}$$

$$\times\left(\int_0^1 f''\left(aq+(1-a)\left(\frac{p+q}{2}\right)\right)^d da\right)^{\frac{1}{d}},$$

which can be written as

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\frac{1}{2c+1}\right]^{\frac{1}{c}}$$

$$\times\left[\left(h_1^{\left(\frac{1}{s}\right)}(a)\left|f''\left(\frac{p+q}{2}\right)\right|^d\right.\right.$$

$$\left.+h_2^{\left(\frac{1}{s}\right)}(1-a)|f''(p)|^d\right)^{\frac{1}{d}}$$

$$+\left(h_1^{\left(\frac{1}{s}\right)}(a)|f''(q)|^d\right.$$

$$\left.\left.+h_2^{\left(\frac{1}{s}\right)}(1-a)\left|f''\left(\frac{p+q}{2}\right)\right|^d\right)^{\frac{1}{d}}\right]. \qquad (3.19)$$

Inequality (3.19) completes the proof.

Remark 3.3:

1. By using ${h_1}^s(a) = h^s(a)$ and ${h_2}^s(1-a) = h^s(1-a)$ in Theorem (3.3), results for (h_1, h_2, s)-convex function will be converted to (h, s)-convex function.
2. Considering ${h_1}^s(a) = a$ and ${h_2}^s(1-a) = (1-a)$ in Theorem (3.3), H-H inequality in (3.19) will be converted to H-H inequality for a convex function.

Theorem 3.4: *Let a function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be differentiable, positive, and symmetric on $\mathbb{I}^\circ$ where $p, q \in \mathbb{I}$ with $p < q$, $s \in (0,1]$, and $a \in [0,1]$. If $f'' \in L[p,q]$ be a (h_1, h_2, s)-convex function and $\frac{1}{c} + \frac{1}{d} = 1$ for $1 < c < \infty$ and $d > 1$, then*

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\frac{1}{3}\right]^{1-\frac{1}{d}}$$

$$\times\left[\left(\left|f''\left(\frac{p+q}{2}\right)\right|^d \int_0^1 (a)^2 {h_1}^{\left(\frac{1}{s}\right)}(a)\, da\right.\right.$$

$$\left.+|f''(p)|^d \int_0^1 (a)^2 {h_2}^{\left(\frac{1}{s}\right)}(1-a)\right)^{\frac{1}{d}} da$$

$$+\left(|f''(q)|^d \int_0^1 (1-a)^2 {h_1}^{\left(\frac{1}{s}\right)}(a)\, da +\right.$$

$$\left.\left.+\left|f''\left(\frac{p+q}{2}\right)\right|^d \int_0^1 (1-a)^2 {h_2}^{\left(\frac{1}{s}\right)}(1-a)\right)^{\frac{1}{d}} da\right].$$

Proof: Since $p < q$ and $a \in [0,1]$, then by applying inequality $pq \leq cp^c + dq^d$ for all $c,\ d > 0$ and $c + d = 1$ on Inequality (3.18) and then using Lemma (2.5), yields

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\left(\int_0^1 a^2 da\right)^{\left(1-\frac{1}{d}\right)}\right.$$

$$\times\left(\int_0^1 a^2\left|f''\left(a\left(\frac{p+q}{2}\right)+(1-a)p\right)\right|^d\right)^{\frac{1}{d}} da$$

$$+\left(\int_0^1 (1-a)^2\, da\right)^{\left(1-\frac{1}{d}\right)}\left(\int_0^1 (1-a)^2\right.$$

$$\left.\times\left|f''\left(ap+(1-a)\left(\frac{p+q}{2}\right)\right)^d\right|\right)^{\frac{1}{d}} da\Bigg]. \quad (3.20)$$

Since $|f''|$ is a (h_1, h_2, s)-convex function in the third sense, then by utilising Inequalities (3.14) and (3.15), right hand side of Inequality (3.20) can be expanded to

$$\int_0^1 (a)^2\left|f''\left(a\left(\frac{p+q}{2}\right)+(1-a)p\right)\right|^d da$$

$$\leq\left|f''\left(\frac{p+q}{2}\right)\right|^d\int_0^1 (a)^2 h_1^{\left(\frac{1}{s}\right)}(a)da$$

$$+|f''(p)|^d\int_0^1 (a)^2 h_2^{\left(\frac{1}{s}\right)}(1-a)da, \quad (3.21)$$

and

$$\int_0^1 (a)^2\left|f''\left(aq+(1-a)\left(\frac{p+q}{2}\right)\right)\right|^d da$$

$$\leq |f''(q)|^d\int_0^1 (1-a)^2\, h_1^{\left(\frac{1}{s}\right)}(a)da + \left|f''\left(\frac{p+q}{2}\right)\right|^d$$

$$\times\int_0^1 (1-a)^2 h_2^{\left(\frac{1}{s}\right)}(1-a)da. \quad (3.22)$$

Since (h_1, h_2, s)-convex function in the third sense is positive and symmetric, then

$$\int_0^1 a^2\,da = \int_0^1 (1-a)^2\,da = \frac{1}{3},$$

and substituting Inequalities (3.21) and (3.22) in Inequality (3.20), yields

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \le \frac{(q-p)^2}{16}\left[\frac{1}{3}\right]^{1-\frac{1}{d}}$$

$$\times\left[\left(\left|f''\left(\frac{p+q}{2}\right)\right|^d \int_0^1 (a)^2 h_1^{\left(\frac{1}{s}\right)}(a)\,da\right.\right.$$

$$\left.+|f''(p)|^d \int_0^1 (a)^2 h_2^{\left(\frac{1}{s}\right)}(1-a)\right)^{\frac{1}{d}} da$$

$$+\left(|f''(q)|^d \int_0^1 (1-a)^2 h_1^{\left(\frac{1}{s}\right)}(a)\,da\right.$$

$$+\left|f''\left(\frac{p+q}{2}\right)\right|^d$$

$$\left.\left.\times\int_0^1 (1-a)^2 h_2^{\left(\frac{1}{s}\right)}(1-a)\right)^{\frac{1}{d}} da\right]. \qquad (3.23)$$

This inequality (3.23) completes the proof.

Remark 3.4:

1. By substituting ${h_1}^s(a) = h^s(a)$ and ${h_2}^s(1-a) = h^s(1-a)$, Theorem (3.4) will be converted to the H-H inequality for (h,s)-convex function.

2. By using ${h_1}^s(a) = a$ and ${h_2}^s(1-a) = 1-a$ in Theorem (3.4), the produced results in Inequality (3.23) will be converted to the H-H inequality of a convex function.

Theorem 3.5: *Let a function* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be differentiable, positive, and symmetric on* $\mathbb{I}^\circ$ *where* $p, q \in \mathbb{I}$ *with* $p < q$, $s \in (0,1]$, *and* $a \in [0,1]$. *If* $f'' \in L[p,q]$ *be a*

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(h_1, h_2, s)*-convex function in the third sense and* $\frac{1}{c} + \frac{1}{d} = 1$ *for* $1 < c < \infty$ *and* $d > 1$*, then*

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \le \frac{(q-p)^2}{16}\left(\frac{1}{c(2c+1)} + \frac{1}{d}\right.$$

$$\times\left(\left|f''\left(\frac{p+q}{2}\right)\right|^d \int_0^1 h_1^{\left(\frac{1}{s}\right)}(a)\, da\right.$$

$$\left.\left. + |f''(p)|^d \int_0^1 h_2^{\left(\frac{1}{s}\right)}(1-a)\, da\right)\right)$$

$$+ \frac{(q-p)^2}{16}\left(\frac{1}{c(2c+1)} + \frac{1}{d}\right.$$

$$\times\left(|f''(q)|^d \int_0^1 h_1^{\left(\frac{1}{s}\right)}(a)\, da\right.$$

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$$\left.\left. + |f''(p)|^d \int_0^1 h_2^{\left(\frac{1}{s}\right)}(1-a)\, da\right)\right).$$

Proof: Since $p < q$ and $a \in [0,1]$, then by utilising Young's inequality for product $pq \le \frac{p^c}{c} + \frac{q^d}{d}$ for all $p, q \ge 0$, $1 < c < \infty$ and $d > 1$ on Inequality (3.18) yields

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \le \frac{(q-p)^2}{16}$$

$$\times \int_0^1 \left(\frac{a^{2c}}{c} + \frac{\left|f''\left(a\left(\frac{p+q}{2}\right) + (1-a)p\right)\right|^d}{d}\right) da$$

$$+ \frac{(q-p)^2}{16}\int_0^1 \left(\frac{(1-a)^{2c}}{c}\right.$$

$$+\frac{\left|f''\left(aq+(1-a)\left(\frac{p+q}{2}\right)\right)\right|^{d}}{d}\Bigg)\,da.$$

As $|f''|$ is a (h_1, h_2, s)-convex function in the third sense, positive, and symmetric, then

$$\int_0^1 a^{2c}\,da = \int_0^1 (1-a)^{2c}\,da = \frac{1}{(2c+1)},$$

and substituting Inequalities (3.14) and (3.15) in Inequality (3.20), yields

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left(\frac{1}{c(2c+1)} + \frac{1}{d}\left(\left|f''\left(\frac{p+q}{2}\right)\right|^{d}\right.\right.$$

$$\left.\left.\times \int_0^1 h_1^{\left(\frac{1}{s}\right)}(a) + |f''(p)|^{d}\int_0^1 h_2^{\left(\frac{1}{s}\right)}(1-a)\,da\right)\right)$$

$$+\frac{(q-p)^2}{16}\left(\frac{1}{c(2c+1)} + \frac{1}{d}\right.$$

$$\times\left(|f''(q)|^{d}\int_0^1 h_1^{\left(\frac{1}{s}\right)}(a)\,da\right.$$

$$\left.\left. + |f''(p)|^{d}\int_0^1 h_2^{\left(\frac{1}{s}\right)}(1-a)da\right)\right). \tag{3.24}$$

Inequality (3.24) completes the proof.

Remark 3.5:

1. By using ${h_1}^{s}(a) = h^{s}(a)$ and ${h_2}^{s}(1-a) = h^{s}(1-a)$, Theorem (3.5) will be converted to (h, s)-convex function.

2. Substituting $h_1{}^s(a) = a$ and $h_2{}^s(1-a) = 1-a$ in Theorem (3.5), the H-H inequality formed in Inequality (3.24) will be converted to the results for convex function.

Note that Theorems (3.2), (3.3), (3.4), and (3.5) will be reduced to extended H-H inequality for (h_1, h_2, s)-convex function in the fourth sense for $s \in (0, 1]$, if we impose the condition $h_1(a) + h_2(1-a) = 1$ with $h_1(a) \not\equiv 0$ and $h_2(1-a) \not\equiv 0$.

Furthermore, in the proceeding subsection, the concept of (h_1, h_2, s)-convex function in the second sense defined in Definition (3.2) and m-convex function from Definition (2.8) are combined to form a new hybrid convex function called (h_1, h_2, s, m)-convex function. Moreover, H-H inequality is also extended by using hybrid (h_1, h_2, s, m)-convex function in the second sense.

3.2.1 Extension of Hermite-Hadamard Inequality by Merging $(\boldsymbol{h_1}, \boldsymbol{h_2}, \boldsymbol{s})$-convex and $\boldsymbol{m}$-convex Functions

In this subsection, another new hybrid convex function called (h_1, h_2, s, m)-convex function in the second sense is formed by combining Definition (2.8) and Definition (3.2). Furthermore, extension of H-H inequality will be considered in accordance with Definition (3.6) as follows:

Definition 3.6: *A function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ and $h_1, h_2: (0, 1) \subseteq \mathbb{J} \to \mathbb{R}$, then f is called (h_1, h_2, s, m)-convex in the second sense for all $a \in [0, 1]$, if $h_1(a) + h_2(1-a) = 1$ and*

$$f(ax + m(1-a)y) \leq h_1{}^s(a) f(x) + m h_2{}^s(1-a) f(y), \qquad x, y \in \mathbb{I}.$$

Here, $m \in (0, 1]$ with $s \in [0, 1]$, $h_1(a) \not\equiv 0$, and $h_2(1-a) \not\equiv 0$ are two positive functions.

From now, to extend H-H inequality, we will use Definitions (3.6) of (h_1, h_2, s, m)-convex function in the second sense in terms of second derivative with $s \in [0, 1]$ and $h_1(a) + h_2(1-a) = 1$, such as

$$f''\left(h_1{}^s(a)\left(\frac{p+q}{2}\right) + mh_2{}^s(1-a)(p)\right) \leq h_1{}^s(a)f''\left(\frac{p+q}{2}\right)$$
$$+mh_2{}^s(1-a)f''(p), \qquad (3.25)$$

and

$$f''\left(h_1{}^s(a)(q) + mh_2{}^s(1-a)\left(\frac{p+q}{2}\right)\right) \leq h_1{}^s(a)f''(q)$$
$$+mh_2{}^s(1-a)f''\left(\frac{p+q}{2}\right). \qquad (3.26)$$

Based on Inequalities (3.25) and (3.26), H-H inequality is extended in Theorems (3.6), (3.8), (3.10), and (3.12) as follows:

Theorem 3.6: *Let a function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be differentiable, positive, and symmetric on $\mathbb{I}^\circ$, where $p, q \in \mathbb{I}$, $p < q$, $a \in [0,1]$, $m \in (0,1]$, and $s \in [0,1]$. If $|f''|$ be a (h_1, h_2, s, m)-convex function in the second sense, then*

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[f''\left(\frac{q+p}{2}\right)\int_0^1 a^2 h_1{}^s(a)da\right.$$
$$\left. +mf''\left(\frac{p}{m}\right)\int_0^1 a^2 h_2{}^s(1-a)da\right]$$
$$+\frac{(q-p)^2}{16}\left[f''(q)\int_0^1 (1-a)^2 h_1{}^s(a)da\right.$$
$$\left. +mf''\left(\frac{p+q}{2m}\right)\int_0^1 (1-a)^2 h_2{}^s(1-a)da\right].$$

Proof: Since $p < q$, $a \in [0,1]$ and $|f''|$ is a (h_1, h_2, s, m)-convex function in the second sense, then by substituting Inequalities (3.25) and (3.26) in Lemma (2.5) results

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\int_0^1 a^2 \left\{f''\left({h_1}^s(a)\left(\frac{p+q}{2}\right)\right)\right.\right.$$

$$\left.\left. +mf''\left(\left({h_2}^s(1-a)\right)\left(\frac{p}{m}\right)\right)\right\} da\right]$$

$$+\frac{(q-p)^2}{16}\left[\int_0^1 (1-a)^2\left\{f''\left({h_1}^s(a)(q)\right)da\right.\right.$$

$$\left.\left. +mf''\left(\left({h_2}^s(1-a)\right)\left(\frac{p+q}{2m}\right)\right)\right\} da\right],$$

which leads to

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[f''\left(\frac{q+p}{2}\right)\int_0^1 a^2 {h_1}^s(a)da\right.$$

$$\left. +mf''\left(\frac{p}{m}\right)\int_0^1 a^2 {h_2}^s(1-a)da\right]$$

$$+\frac{(q-p)^2}{16}\left[f''(q)\int_0^1 (1-a)^2 {h_1}^s(a)da\right.$$

$$+mf''\left(\frac{p+q}{2m}\right)$$

$$\left.\times \int_0^1 (1-a)^2 {h_2}^s(1-a)da\right]. \tag{3.27}$$

Inequality (3.27) completes the proof.

Remark 3.6:

1. By substituting $m = 1$ in Theorem (3.6), then the results will be converted to extended H-H inequality for (h_1, h_2, s)-convex function.

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2. By applying ${h_1}^s(a) = h^s(a)$ and ${h_2}^s(1-a) = h^s(1-a)$ in Theorem (3.6), then results will be converted to extended H-H inequality for (h, s)-convex function.

3. By employing ${h_1}^s(a) = a$ and ${h_2}^s(1-a) = 1-a$ in Theorem (3.6), then the findings will be transformed into extended H-H inequality for convex function.

Theorem 3.7: *Let a function* $f\colon \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be differentiable, positive, and symmetric on* $\mathbb{I}^\circ$ *where* $p, q \in \mathbb{I}$, $p < q$, $a \in [0,1]$, $m \in (0,1]$, *and* $s \in [0,1]$. *If* $f'' \in L[p,q]$ *be a* (h_1, h_2, s, m)*-convex function in the second sense and* $\frac{1}{c} + \frac{1}{d} = 1$ *for* $c > 0$ *and* $d > 1$ *then*

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\frac{1}{2a+1}\right]^{\frac{1}{c}}\left[\left({h_1}^s(a)\left|f''\left(\frac{p+q}{2}\right)\right|^d\right.\right.$$

$$\left.+m{h_2}^s(1-a)\left|f''\left(\frac{p}{m}\right)\right|^d\right)^{\frac{1}{d}}$$

$$+\left({h_1}^s(a)|f''(q)|^d\right.$$

$$\left.\left.+\,m{h_2}^s(1-a)\left|f''\left(\frac{p+q}{2m}\right)\right|^d\right)^{\frac{1}{d}}\right].$$

Proof: Since $p < q$, $c > 0$, $d > 1$ and $a \in [0,1]$, then by applying Lemma (2.5) on Theorem (2.2), produces

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\int_0^1 a^{2c}\right]^{\frac{1}{c}} da$$

$$\times\left(\int_0^1 f''\left(a\left(\frac{p+q}{2}\right) + (1-a)p\right)^d\right)^{\frac{1}{d}} da$$

$$+\frac{(q-p)^2}{16}\left[\int_0^1 (1-a)^{2c}\right]^{\frac{1}{c}} da$$

$$\times\left(\int_0^1 f''(aq)^d\right.$$

$$\left.+(1-a)\left(\frac{p+q}{2}\right)\right)^d\Bigg)^{\frac{1}{d}} da. \tag{3.28}$$

Since (h_1, h_2, s, m)-convex function in the second sense is positive and symmetric, then

$$\int_0^1 a^{2c}\, da = \int_0^1 (1-a)^{2c}\, da = \frac{1}{2c+1}.$$

Substituting Inequalities (3.25) and (3.26) in Inequality (3.28), results

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \le \frac{(q-p)^2}{16}\left[\frac{1}{2c+1}\right]^{\frac{1}{c}}$$

$$\times\left(\int_0^1 f''\left(a\left(\frac{q+p}{2}\right)+(1-a)p\right)^d\right)^{\frac{1}{d}} da$$

$$+\frac{(q-p)^2}{16}\left[\frac{1}{2c+1}\right]^{\frac{1}{c}}$$

$$\times\left(\int_0^1 f''\left(aq+(1-a)\left(\frac{p+q}{2}\right)\right)^d\right)^{\frac{1}{d}} da,$$

which leads to

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \le \frac{(q-p)^2}{16}\left[\frac{1}{2c+1}\right]^{\frac{1}{c}}\left[\left(h_1{}^s(a)\left|f''\left(\frac{p+q}{2}\right)\right|^d\right.\right.$$

$$\left.+mh_2{}^s(1-a)\left|f''\left(\frac{p}{m}\right)\right|^d\right)^{\frac{1}{d}}$$

$$+(h_1{}^s(a)|f''(q)|^d$$

$$+mh_2{}^s(1-a)\left|f''\left(\frac{p+q}{2m}\right)\right|^d\Bigg)^{\frac{1}{d}}\Bigg]. \quad (3.31)$$

Inequality (3.31) completes the proof.

Remark 3.7:

1. By substituting $m = 1$ in Theorem (3.7), then the results will be converted to extended H-H inequality for (h_1, h_2, s)-convex function.
2. By using $h_1{}^s(a) = h^s(a)$ and $h_2{}^s(1-a) = h^s(1-a)$ in Theorem (3.7), then results will be transformed into extended H-H inequality for (h, s)-convex function.
3. By utilising $h_1{}^s(a) = a$ and $h_2{}^s(1-a) = 1-a$ in Theorem (3.7), the results will then produce the extended H-H inequality for a convex function.

Theorem 3.8: *Let a function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be differentiable, positive, and symmetric on $\mathbb{I}^\circ$ where $p, q \in \mathbb{I}$, $p < q$, $a \in [0,1]$, $m \in (0,1]$, and $s \in [0,1]$. If $f'' \in L[p,q]$ be a (h_1, h_2, s, m)-convex function in the second sense and $\frac{1}{c} + \frac{1}{d} = 1$ for $c > 0$, $d > 1$ then*

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\frac{1}{3}\right]^{1-\frac{1}{d}}$$

$$+\Bigg[\Bigg(\left|f''\left(\frac{p+q}{2}\right)\right|^d \int_0^1 (a)^2 h_1{}^s(a)\, da$$

$$+m\left|f''\left(\frac{p}{m}\right)\right|^d \int_0^1 (a)^2 h_2{}^s(1-a)\Bigg)^{\frac{1}{d}} da$$

$$+\left(|f''(q)|^d \int_0^1 (1-a)^2 {h_1}^s(a)da\right.$$

$$+m\left|f''\left(\frac{p+q}{2m}\right)\right|^d$$

$$\left.\times \int_0^1 (1-a)^2 {h_2}^s(1-a)\right)^{\frac{1}{d}} da\Bigg].$$

Proof: Since $p < q$ and $a \in [0,1]$, then by applying inequality $pq \leq cp^c + dq^d$ for all $c, d > 0$ and $c + d = 1$ on Inequality (3.28) and then using Lemma (2.5), yields

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left(\int_0^1 a^2 da\right)^{\left(1-\frac{1}{d}\right)}$$

$$\times\left[\left(\int_0^1 a^2 \left|f''\left(a\left(\frac{p+q}{2}\right)+(1-a)p\right)\right|^d\right)^{\frac{1}{d}} da\right.$$

$$+\left(\int_0^1 (1-a)^2 da\right)^{\left(1-\frac{1}{d}\right)}\left(\int_0^1 (1-a)^2\right.$$

$$\left.\left.\times\left|f''\left(ap+(1-a)\left(\frac{p+q}{2}\right)\right)^d\right|^{\frac{1}{d}}\right)da\right]. \qquad (3.32)$$

As $|f''|$ is a (h_1, h_2, s, m)-convex function in the second sense, then by utilising Inequalities (3.25) and (3.26), results

$$\int_0^1 (a)^2 \left|f''\left(a\left(\frac{p+q}{2}\right)+(1-a)p\right)\right|^d da \leq \left|f''\left(\frac{p+q}{2}\right)\right|^d$$

$$\times\int_0^1 (a)^2 {h_1}^s(a)da + m\left|f''\left(\frac{p}{m}\right)\right|^d$$

$$\times\int_0^1 (a)^2 {h_2}^s(1-a)da, \qquad (3.33)$$

and

$$\int_0^1 (1-a)^2 \left|f''\left(aq+(1-a)\left(\frac{p+q}{2}\right)\right)\right|^d da \leq |f''(q)|^d \int_0^1 (1-a)^2 {h_1}^s(a)da$$

$$+m\left|f''\left(\frac{p+q}{2m}\right)\right|^d$$

$$\times \int_0^1 (1-a)^2 {h_2}^s(1-a)da. \quad (3.34)$$

Since (h_1, h_2, s, m)-convex function in the second sense is positive and symmetric, then

$$\int_0^1 a^2\, da = \int_0^1 (1-a)^2\, da = \frac{1}{3},$$

and substituting Inequalities (3.33) and (3.34) in Inequality (3.32), results

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left[\frac{1}{3}\right]^{1-\frac{1}{d}}$$

$$\times\left[\left(\left|f''\left(\frac{p+q}{2}\right)\right|^d \int_0^1 (a)^2 {h_1}^s(a)da\right.\right.$$

$$\left. +m\left|f''\left(\frac{p}{m}\right)\right|^d \int_0^1 (a)^2 {h_2}^s(1-a)\right)^{\frac{1}{d}} da$$

$$+\left(|f''(q)|^d \int_0^1 (1-a)^2 {h_1}^s(a)da\right.$$

$$+m\left|f''\left(\frac{p+q}{2m}\right)\right|^d$$

$$\left.\times \int_0^1 (1-a)^2 {h_2}^s(1-a)\right)^{\frac{1}{d}} da. \quad (3.35)$$

Inequality (3.35) completes the proof.

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Remark 3.8:

1. By substituting $m = 1$ in Theorem (3.8), the results will be converted to H-H inequality for (h_1, h_2, s)-convex function.

2. By using ${h_1}^s(a) = h^s(a)$ and ${h_2}^s(1-a) = h^s(1-a)$ in Theorem (3.8), then results will be converted into extended H-H inequality for (h, s)-convex function.

3. By applying ${h_1}^s(a) = a$ and ${h_2}^s(1-a) = 1-a$ in Theorem (3.8), the results will be transformed into extended H-H inequality for the convex function.

Theorem 3.9: *Let a function* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be differentiable, positive, and symmetric on* $\mathbb{I}^\circ$ *where* $p, q \in \mathbb{I}$ *with* $p < q$, $a \in [0,1]$, *and* $s \in [0,1]$. *If* $f'' \in L[p,q]$ *be a* (h_1, h_2, s, m)*-convex function in the second sense and* $\frac{1}{c} + \frac{1}{d} = 1$ *for* $c > 0$ *and* $d > 1$ *then*

$$\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \leq \frac{(q-p)^2}{16}\left(\frac{1}{c(2c+1)}\right) + \frac{1}{d}$$

$$\times \left(\left|f''\left(\frac{p+q}{2}\right)\right|^d \int_0^1 {h_1}^s(a)da\right.$$

$$+m\left|f''\left(\frac{p}{m}\right)\right|^d \int_0^1 {h_2}^s(1-a)da + \frac{(q-p)^2}{16}$$

$$\times \frac{1}{c(2c+1)} + \frac{1}{d}\left(|f''(q)|^d \int_0^1 {h_1}^s(a)da\right.$$

$$\left.+m\left|f''\left(\frac{p}{m}\right)\right|^d \int_0^1 {h_2}^s(1-a)\,da\right)$$

Proof: Since $p < q$, $c > 0$, $d > 1$ and $a \in [0, 1]$, then utilising Young's inequality $pq \le \frac{p^c}{c} + \frac{q^d}{d}$ for all $p, q > 0$ on Inequality (3.28) yields

$$\begin{aligned}\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \le\ & \frac{(q-p)^2}{16} \\ & \times \int_0^1 \left(\frac{a^{2c}}{c} + \frac{\left|f''\left(a\left(\frac{p+q}{2}\right)+(1-a)p\right)\right|^d}{d}\right) da \\ & + \frac{(q-p)^2}{16}\int_0^1 \left(\frac{(1-a)^{2c}}{c}\right. \\ & \left. + \frac{\left|f''\left(aq+(1-a)\left(\frac{p+q}{2}\right)\right)\right|^d}{d}\right) da.\end{aligned}$$

As (h_1, h_2, s, m)-convex function in the second sense is positive and symmetric, then

$$\int_0^1 a^{2c}\, da = \int_0^1 (1-a)^{2c}\, da = \frac{1}{2c+1}.$$

Since $|f''|$ is a (h_1, h_2, s, m)-convex function in the second sense, then by using Inequalities (3.25) and (3.26) results

$$\begin{aligned}\frac{1}{q-p}\int_p^q f(x)dx - f\left(\frac{p+q}{2}\right) \le\ & \frac{(q-p)^2}{16}\left(\frac{1}{c(2c+1)}\right. \\ & + \frac{1}{d}\left(\left|f''\left(\frac{p+q}{2}\right)\right|^d \int_0^1 {h_1}^s(a)da\right. \\ & \left.\left. + m\left|f''\left(\frac{p}{m}\right)\right|^d \int_0^1 {h_2}^s(1-a)\, da\right)\right) \\ & + \frac{(q-p)^2}{16}\left(\frac{1}{c(2c+1)}\right.\end{aligned}$$

$$+\frac{1}{d}\left(|f''(q)|^d \int_0^1 {h_1}^s(a)da\right.$$

$$\left.+m\left|f''\left(\frac{p}{m}\right)\right|^d \int_0^1 {h_2}^s(1-a)da\right)\Bigg). \qquad (3.36)$$

Inequality (3.36) completes the proof.

Remark 3.9:

1. By substituting $m = 1$ in Theorem (3.9), then the results will be converted to H-H inequality for (h_1, h_2, s)-convex function.
2. By using ${h_1}^s(a) = h^s(a)$ and ${h_2}^s(1-a) = h^s(1-a)$ in Theorem (3.9), then results will be converted to H-H inequality for (h, s)-convex function.
3. By applying ${h_1}^s(a) = a$ and ${h_2}^s(1-a) = 1-a$ in Theorem (3.9), the results will be converted to H-H inequality for a convex function.

Note that Theorems (3.6), (3.7), (3.8), and (3.9) will be reduced to extended H-H inequality for (h_1, h_2, s, m)-convex function in the third and fourth sense, if we impose the condition $s = \frac{1}{s}$ for $s \in (0, 1]$ and ${h_1}^s(a) + {h_1}^s(1-a) = 1$, $h_1(a) + h_2(1-a) = 1$ with $h_1(a) \not\equiv 0$, $h_2(1-a) \not\equiv 0$.

Furthermore, in the proceeding subsection, the concept of (h_1, h_2, s, m)-convex function in the second sense defined in Definition (3.6) and harmonically convex function from Definition (2.4) are combined to form a new hybrid (h_1, h_2, s, m)-harmonically convex function in the second sense. Moreover, H-H inequality is also extended by using hybrid (h_1, h_2, s, m)-harmonically convex function in the second sense.

3.2.2 Extension of Hermite-Hadamard Inequality by Merging $(\boldsymbol{h_1}, \boldsymbol{h_2}, \boldsymbol{s}, \boldsymbol{m})$-convex and Harmonically Convex Functions

In this subsection, another new hybrid convex function called (h_1, h_2, s, m)-harmonically convex function in the second sense is formed by combining Definition (2.4) and Definition (3.6). Furthermore, extension of H-H inequality will be considered in accordance with Definition (3.7) as follows:

Definition 3.7: *A function* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *is called* (h_1, h_2, s, m)*-harmonically convex in the second sense for* $a \in [0, 1]$ *on* $\mathbb{I} \subset \mathbb{R}\backslash\{0\}$*, if* $h_1(a) + h_2(1-a) = 1$ *and*

$$f\left(\frac{xy}{ax + m(1-a)y}\right) \leq h_1^s(a)f(x) + mh_2^s(1-a)f(y), \tag{3.37}$$

$x, y \in \mathbb{I}$*,* $s \in [0, 1]$*,* $m \in (0, 1]$ *and* $h_1 \not\equiv 0$*,* $h_2 \not\equiv 0$ *for* $h_1: \mathbb{K} \subset \mathbb{R}_+ \to \mathbb{R}$ *and* $h_2: \mathbb{K} \subset \mathbb{R}_+ \to \mathbb{R}$.

From now, we will extend H-H inequality by using Definition (3.7) of (h_1, h_2, s, m)-harmonically convex in the second sense for all $x_1, y_1, p, q \in \mathbb{I}$ with $\mathbb{I} \subset \mathbb{R}$ and $p < q$. We will use

$$x_1 = \frac{pq}{ap + (1-a)q} \tag{3.38}$$

and

$$y_1 = \frac{pq}{aq + (1-a)p}\ . \tag{3.39}$$

Since, $x_1, y_1, p, q \in \mathbb{I}$, $a \in [0, 1]$ and $p < q$, then Inequality (3.37) of (h_1, h_2, s, m)-harmonically convex in the second sense in terms of x_1 and y_1 can be written as

$$f\left(\frac{x_1y_1}{ax_1+(1-a)y_1}\right)\leq h_1^s(a)f(x_1)+mh_2^s(1-a)f(y_1). \qquad (3.40)$$

Based on Inequality (3.40), H-H inequality can now be extended in Theorems (3.10) and (3.11) as follows:

Theorem 3.10: *Let the functions* $h_1, h_2 : \mathbb{J}^* \subset [0,1] \to \mathbb{R}$ *be differentiable, positive, and symmetric on* $\mathbb{J}^*$ *such that* $h_1 \not\equiv 0$ *and* $h_2 \not\equiv 0$. *If* $f : \mathbb{I} \to \mathbb{R}_+$ *be a* (h_1, h_2, s, m)-*harmonically convex function in the second sense, positive, and symmetric, where* p, $q \in \mathbb{I}$ *with* $p < q$, $s \in [0,1]$, $m \in (0,1]$, *and* $a \in [0,1]$, *then*

$$f\left(\frac{2pq}{p+q}\right)\leq h_1^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q\frac{f(x_1)}{x_1^2}dx_1+mh_2^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q\frac{f(x_1m)}{x_1^2}dx_1.$$

Proof: Since $p < q$ and $a \in [0,1]$, then by substituting Equations (3.38) and (3.39) in Inequality (3.40), results

$$f\left(\frac{2x_1y_1}{x_1+y_1}\right)=f\left(\frac{2\left(\frac{pq}{aq+(1-a)p}\right)\left(\frac{pq}{ap+(1-a)q}\right)}{\frac{pq}{aq+(1-a)p}+\frac{pq}{ap+(1-a)q}}\right),$$

$$=f\left(\frac{2\left(\frac{pq}{aq+(1-a)p}\right)\left(\frac{pq}{ap+(1-a)q}\right)}{\frac{pq\{(ap+(1-a)q)+aq+(1-a)p\}}{(aq+(1-a)p)(ap+(1-a)q)}}\right). \qquad (3.41)$$

Without loss of originality, set $a=\frac{1}{2}$ in the right-hand side of Equation (3.41), yields

$$f\left(\frac{2x_1y_1}{x_1+y_1}\right)=f\left(\frac{2pq}{p+q}\right).$$

Since f is a (h_1, h_2, s, m)-harmonically convex function in the second sense and maintaining $a=\frac{1}{2}$, then by using Inequality (3.40) can be written as

$$f\left(\frac{2pq}{p+q}\right)\leq h_1^s\left(\frac{1}{2}\right)f\left(\frac{pq}{aq+(1-a)p}\right)+mh_2^s\left(\frac{1}{2}\right)f\left(\frac{pq}{ap+(1-a)q}\right). \qquad (3.42)$$

Taking the derivative of x_1 from Equation (3.38) with respect to a gives

$$dx_1 = \frac{pq(q-p)}{(aq+(1-a)p)^2}\,da. \qquad (3.43)$$

Multiply and divide by pq in Equation (3.43), yields

$$dx_1 = \frac{(pq)^2(q-p)}{pq(ap+(1-a)q)^2}\,da,$$

which can be written as

$$dx_1 = \frac{(q-p)}{pq}\left(\frac{pq}{ap+(1-a)q}\right)^2 da. \qquad (3.44)$$

Since, $x_1 = \frac{pq}{ap+(1-a)q}$ from Equation (3.38), then Equation (3.44) can be written as

$$dx_1 = \frac{x_1^2(q-p)}{pq}\,da. \qquad (3.45)$$

Integrating Inequality (3.42) with respect to a, where $a \in [0,1]$, yields

$$\int_0^1 f\left(\frac{2pq}{p+q}\right) da \leq h_1^s\left(\frac{1}{2}\right)\int_0^1 f\left(\frac{pq}{ap+(1-a)q}\right) da$$

$$+mh_2^s\left(\frac{1}{2}\right)\int_0^1 f\left(\frac{pq}{aq+(1-a)p}\right) da. \qquad (3.46)$$

From Inequality (3.46), let $I_1 = \int_0^1 f\left(\frac{pq}{ap+(1-a)q}\right) da$ and $I_2 = \int_0^1 f\left(\frac{pq}{aq+(1-a)p}\right) da$, then by utilising Equations (3.38) and (3.45), results

$$I_1 = \int_0^1 f\left(\frac{pq}{ap+(1-a)q}\right) da = \left(\frac{pq}{q-p}\right)\int_p^q \frac{f(x_1)}{x_1^2}\,dx_1. \qquad (3.47)$$

As f is a (h_1, h_2, s, m)-harmonically convex function in the second sense, positive and symmetric, then similar to Equation (3.47), we have

$$I_2 = \int_0^1 f\left(\frac{pq}{aq + (1-a)p}\right) da = \left(\frac{pq}{q-p}\right) \int_p^q \frac{f(x_1)}{x_1^2} dx_1. \tag{3.48}$$

Multiply with $m \in (0, 1]$ on both sides of Inequality (3.48), gives

$$m \int_0^1 f\left(\frac{pq}{aq + (1-a)p}\right) da = m\left(\frac{pq}{q-p}\right) \int_p^q \frac{f(x_1)}{x_1^2} dx_1. \tag{3.49}$$

As $mf(x_1) \leq f(mx_1)$, then Equation (3.49) can be written as

$$I_2 = \int_0^1 f\left(\frac{pqm}{ap + (1-a)q}\right) da \leq \left(\frac{pq}{q-p}\right) \int_p^q \frac{f(x_1 m)}{x_1^2} dx_1. \tag{3.50}$$

Since $\int_0^1 f\left(\frac{2pq}{p+q}\right) da = f\left(\frac{2pq}{p+q}\right)$ and substituting I_1 and I_2 from Equations (3.47) and (3.50) in Inequality (3.46), results

$$\begin{aligned} f\left(\frac{2pq}{p+q}\right) &\leq h_1^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right) \int_p^q \frac{f(x_1)}{x_1^2} dx \\ &+ m h_2^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right) \int_p^q \frac{f(x_1 m)}{x_1^2} dx_1. \end{aligned} \tag{3.51}$$

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Inequality (3.51) completes the proof.

Theorem 3.11: *Let* $h_1, h_2 \colon \mathbb{J}^* \subset [0,1] \to \mathbb{R}$ *such that* $h_1 \not\equiv 0$, $h_2 \not\equiv 0$. *If* $f \colon \mathbb{I} \to \mathbb{R}_+$ *be a* (h_1, h_2, s, m)*-harmonically convex function in the second sense, positive, and symmetric, where* $p, q \in \mathbb{I}$ *with* $p < q$, $s \in [0,1]$, $m \in (0,1]$, *and* $a \in [0,1]$, *then*

$$\begin{aligned} \left(\frac{pq}{q-p}\right) \int_p^q \frac{f(x_1)}{x_1^2} dx_1 \leq min \Bigg\{ & f(p) \int_0^1 h_1^s(a) da + mf\left(\frac{q}{m}\right) \int_0^1 h_2^s(1-a) da, \\ & f(q) \int_0^1 h_1^s(a) da + mf\left(\frac{p}{m}\right) \int_0^1 h_2^s(1-a) da \Bigg\}. \end{aligned}$$

Proof: Since $p < q$, $a \in [0, 1]$ and f is a (h_1, h_2, s, m)-harmonically convex function in the second sense, then by substituting Equations (3.38) and (3.39) in Inequality (3.40), and then replacing $q = \left(\frac{q}{m}\right)$ for $m \in (0, 1]$, Inequality (3.40) results

$$f\left(\frac{pq}{ap + (1-a)q}\right) \leq h_1^s(a) f(p) + m h_2^s(1-a) f\left(\frac{q}{m}\right). \tag{3.52}$$

Integrate Inequality (3.52) with respect to $a \in [0, 1]$, gives

$$\left(\frac{pq}{q-p}\right) \int_p^q \frac{f(x_1)}{x_1^2} dx_1 \leq f(p) \int_0^1 h_1^s(a) da + m f\left(\frac{q}{m}\right) \int_0^1 h_2^s(1-a) da. \tag{3.53}$$

As f is a (h_1, h_2, s, m)-harmonically convex function in the second sense, positive, and symmetric, then similar to Inequality (3.52), by substituting Equations (3.38) and (3.39) in Inequality (3.40), and then replacing $p = \left(\frac{p}{m}\right)$ for $m \in (0, 1]$, Inequality (3.40) results

$$f\left(\frac{pq}{aq + (1-a)p}\right) \leq h_1^s(a) f(q) + m h_2^s(1-a) f\left(\frac{p}{m}\right). \tag{3.54}$$

Integrate Inequality (3.54) with respect to $a \in [0, 1]$, gives

$$\left(\frac{pq}{q-p}\right) \int_p^q \frac{f(x_1)}{x_1^2} dx_1 \leq f(q) \int_0^1 h_1^s(a) da + m f\left(\frac{p}{m}\right) \int_0^1 h_2^s(1-a) da. \tag{3.55}$$

By combining Inequalities (3.52) and (3.55), results

$$\begin{aligned}\left(\frac{pq}{q-p}\right) \int_p^q \frac{f(x_1)}{x_1^2} dx_1 \quad &\leq min\left\{f(p) \int_0^1 h_1^s(a) da + m f\left(\frac{q}{m}\right) \int_0^1 h_2^s(1-a) da \right. \\ &\left. \times f(q) \int_0^1 h_1^s(a) da + m f\left(\frac{p}{m}\right) \int_0^1 h_2^s(1-a) da \right\}. \end{aligned} \tag{3.56}$$

Inequality (3.56) completes the proof.

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Remarks 3.10:

1. By applying $h_1^s(a) = h_2^s(a) = h(a)$ in Inequality (3.56), we get extended H-H inequality for m-harmonically convex function.

2. By substituting $h_1^s(a) = h_1(a)$ and $h_2^s(a) = h_2(a)$ in Inequality (3.56), then the results will be transformed into H-H inequality for (m, h_1, h_2)-harmonically convex function.

3. By using $h_1^s(a) = a$ and $h_2^s(a) = (1 - a)$ in Inequality (3.56), results for m-*HA* convex function will be obtained.

Note that Theorems (3.10) and (3.11) will be reduced to extended H-H inequality for (h_1, h_2, s)-convex function in the third and fourth sense, if we impose the condition $s = \frac{1}{s}$ for $s \in (0, 1]$ with $h_1^s(a) + h_2^s(1 - a) = 1$ and $h_1(a) + h_2(1 - a) = 1$, such that $h_1(a) \not\equiv 0$ and $h_2(1 - a) \not\equiv 0$.

Furthermore, in the proceeding section, the concept of g-convex function defined in Definition (1.9) and (h_1, h_2)-convex function from Definition (1.11) are combined to form a new hybrid convex function called g-(h_1, h_2) convex function. Moreover, H-H inequality is also extended by using hybrid g-(h_1, h_2) convex function.

3.3 Extension of Hermite-Hadamard Inequality by Merging (h_1, h_2)-convex and g-convex Functions

In this section, another new hybrid convex function called g-(h_1, h_2) convex function is formed by combining Definitions (1.9) and (1.11). Furthermore, the extension of H-H inequality will be considered in accordance with Definition (3.8) as follows:

Definition 3.8: *A function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ is called g-(h_1, h_2)-convex function for $x, y \in \mathbb{I}$ and $a \in [0,1]$ if*

$$f\left(x^{(1-a)}y^{a}\right) \leq f(x)^{h_1(1-a)} f(y)^{h_2(a)}. \tag{3.57}$$

Remarks 3.11:

1. By using $h_1(1-a) = a$ and $h_2(a) = (1-a)$, Definition (3.8) will be reduced to Definition (1.9) of geometrically convex function.

2. If substituting $h_1(1-a) = h(1-a)$ and $h_2(a) = h(a)$ in Definition (3.8), then the Definition for the (g, h)-convex function will be produced.

3. By utilising $h_1(1-a) = (1-a)^s$ and $h_2(a) = a^s$, for all $s \in (0,1)$ in Definition (3.8), then the result will be the converted to the definition for (g, s)-convex function.

From now, we will use Definition (3.8) of g-(h_1, h_2) convex function for all $x_2, y_2, p, q \in \mathbb{I}$ with $\mathbb{I} \subset \mathbb{R}$ and $p < q$. We will use

$$x_2 = p^{(1-a)}q^{a} \tag{3.58}$$

and

$$y_2 = p^{a}q^{(1-a)}. \tag{3.59}$$

Since, $p < q$ and $a \in [0,1]$, then Inequality (3.57) of g-(h_1, h_2) convex function in terms of x_2 and y_2 can be written as

$$f\left(x_2^{(1-a)}y_2^{a}\right) \leq f(x_2)^{h_1(1-a)} f(y_2)^{h_2(a)}. \tag{3.60}$$

Based on Inequality (3.60), H-H inequality can now be extended in Theorems (3.12), (3.13), and (3.14) as follows:

Theorem 3.12: *Let* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}_+$ *be a* g-(h_1, h_2) *convex function,* $h_1, h_2: \mathbb{J} \subset (0,1) \to \mathbb{R}$ *and* $g: \mathbb{J} \subset (0,1) \to \mathbb{R}_+$ *for all* $a \in [0,1]$ *be a* g-(h_1, h_2) *convex function, then* $f \circ g$ *will also be a* g-(h_1, h_2)*-convex function.*

Proof: Since $a \in [0,1]$ and g be a g-(h_1, h_2) convex function, then by utilising Inequality (3.60), results

$$g\left(x_2^{(1-a)} y_2^a\right) \leq g(x_2)^{h_1(1-a)} g(y_2)^{h_2(a)}. \tag{3.61}$$

Since f is also a g-(h_1, h_2) convex function, then Inequality (3.61) can be written as

$$\begin{aligned} f\left[g\left(x_2^{(1-a)} y_2^a\right)\right] &\leq f\left[g(x_2)^{h_1(1-a)} g(y_2)^{h_2(a)}\right], \\ &\leq f[g(x_2)]^{h_1(1-a)} f[g(y_2)]^{h_2(a)}. \end{aligned} \tag{3.62}$$

Inequality (3.62) completes the proof.

Theorem 3.13: *Let* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be a* g-(h_1, h_2) *convex function, positive, and symmetric. If* $h_1, h_2: \mathbb{J} \subset (0,1) \to \mathbb{R}$ *are positive functions such that* $p, q \in \mathbb{I}$ *with* $p < q$, *then*

$$\frac{1}{\ln q - \ln p} \int_p^q \frac{f(x_2)}{x_2} dx_2 \leq L\big(f(p), f(q)\big),$$

for $h_1, h_2 \geq a$ *and* $a \in [0,1]$ *with special mean* $L(p,q)$ *is defined as*

$$L(p,q) = \begin{cases} \dfrac{q-p}{\ln q - \ln p}, & q \neq p, \\ p, & q = p. \end{cases}$$

Furthermore,

$$\frac{1}{\ln q - \ln p} \int_p^q f(x_2) dx_2 \leq L\big(pf(p), qf(q)\big),$$

for $h_1, h_2 < a$ *and* $a \in [0,1]$.

Proof: Since $p < q$, $a \in [0, 1]$, and f be a g-(h_1, h_2) convex function such that $h_1, h_2 \geq a$. By taking ln on both sides of x_2 from Equation (3.58), results

$$ln(x_2) = ln\left(p^{(1-a)}q^a\right). \tag{3.63}$$

By applying product rule and then using power rule on Equation (3.63), implies

$$ln(x_2) = (1-a)ln(p) + (a)ln(q). \tag{3.64}$$

Taking derivative of Equation (3.64) with respect to a, gives

$$\left(\frac{1}{x_2}\right)\frac{dx_2}{da} = (-ln(p) + ln(q))$$

or

$$\frac{dx_2}{x_2} = (ln(q) - \ln(p))da. \tag{3.65}$$

Since f is a g-(h_1, h_2) convex function, then by applying integration for a where $a \in [0, 1]$ on Inequality (3.65), results

$$\int_0^1 f\left(x_2^{(1-a)}y_2^a\right)da \leq \int_0^1 \left\{f(x_2)^{h_1(1-a)}f(y_2)^{h_2(a)}\right\}da. \tag{3.66}$$

Note that $x_2, y_2, p, q \in \mathbb{I}$, then by substituting value of x_2 and y_2 from Equations (3.58), (3.59) and da from Equation (3.65) in left hand side of Inequality (3.66), results

$$\int_0^1 f\left(x_2^{(1-a)}y_2^a\right)da = \int_0^1 f\left(p^{(1-a)}q^a\right)da \tag{3.67}$$

and

$$\int_0^1 f\left(p^{(1-a)}q^a\right) da = \left(\frac{1}{\ln q - \ln p}\right)\int_p^q \left(\frac{f(x_2)}{x_2}\right) dx_2. \tag{3.68}$$

By utilising Equation (3.68), Inequality (3.66) can be written as

$$\left(\frac{1}{\ln q - \ln p}\right)\int_p^q \left(\frac{f(x_2)}{x_2}\right) dx_2 \le \int_0^1 \left\{f(x_2)^{h_1(1-a)} f(y_2)^{h_2(a)}\right\} da. \tag{3.69}$$

Since, $\int_0^1 f\left(x_2^{(1-a)} y_2^a\right) da = \int_0^1 f\left(p^{(1-a)}q^a\right) da$ from Equation (3.67) and f is a g-(h_1, h_2) convex function, positive, and symmetric, then Equation (3.69) can be written as

$$\left(\frac{1}{\ln q - \ln p}\right)\int_p^q \left(\frac{f(x_2)}{x_2}\right) dx_2 \le \int_0^1 \left\{f(p)^{h_1(1-a)} f(q)^{h_2(a)}\right\} da. \tag{3.70}$$

By utilising Equation (3.69) and Inequality (3.70) in Inequality (3.68), yields

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$$\left(\frac{1}{\ln q - \ln p}\right)\int_p^q \left(\frac{f(x_2)}{x_2}\right) dx_2 \le \int_0^1 f(p)^{(1-a)} f(q)^a da. \tag{3.71}$$

By solving right hand side of Inequality (3.71), yields

$$\int_0^1 f(p)^{(1-a)} f(q)^a da = \int_0^1 f(p)^{(-a)} f(p)\, f(q)^a da,$$

which can be written as

$$\int_0^1 f(p)^{(-a)} f(p)\, f(q)^a da = \int_0^1 \left(\frac{f(q)}{f(p)}\right)^a f(p) da. \tag{3.72}$$

By applying integration for a where $a \in [0, 1]$ on Equation (3.72), results

$$\int_0^1 \left(\frac{f(q)}{f(p)}\right)^a f(p)da = \frac{f(p)}{ln\left(\frac{f(q)}{f(p)}\right)}\left|\left(\frac{f(q)}{f(p)}\right)^a\right|_0^1,$$

which will be reduced to

$$\int_0^1 \left(\frac{f(q)}{f(p)}\right)^a f(p)da = \frac{f(p)}{lnf(q) - lnf(p)}\left(\frac{f(q)}{f(p)} - 1\right),$$

$$= \frac{f(p)}{lnf(q) - lnf(p)}\left(\frac{f(q) - f(p)}{f(p)}\right),$$

$$\int_0^1 f(p)^{(1-a)} f(q)^a da = \frac{f(q) - f(p)}{lnf(q) - lnf(p)},$$

$$\int_0^1 f(p)^{(1-a)} f(q)^a da = L\big(f(p), f(q)\big). \tag{3.73}$$

By comparing Inequality (3.71) and Equation (3.73), results

$$\left(\frac{1}{ln\,q - ln\,p}\right)\int_p^q \left(\frac{f(x_2)}{x_2}\right) dx_2 = L\big(f(p), f(q)\big). \tag{3.74}$$

Similarly, for $h_1, h_2 < a$, multiply x_2 from equation (3.58) on both sides of Inequality (3.71), yields

$$\left(\frac{1}{ln\,q - ln\,p}\right)\int_p^q f(x_2)dx_2 \quad = \int_0^1 p^{(1-a)} q^a f\big(p^{(1-a)} q^a\big)\, da,$$

$$\leq \int_0^1 p^{(1-a)} q^a f(p)^{h_1(1-a)} f(q)^{h_2(a)} da,$$

$$\leq p^{(1-a)} q^a \int_0^1 f(p)^{(1-a)} f(q)^a da, \tag{3.75}$$

By using Equation (3.75) in Inequality (3.75), results

$$\left(\frac{1}{\ln q - \ln p}\right)\int_{p}^{q} f(x_2)dx_2 \leq L\big(pf(p), qf(q)\big). \tag{3.76}$$

Inequality (3.74) and Inequality (3.76) completes the proof.

Theorem 3.14: *Let* $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be a* g-(h_1, h_2) *convex function, positive, and symmetric. If* $h_1, h_2: \mathbb{J} \subset (0,1) \to \mathbb{R}$ *is positive function such that* $p, q \in \mathbb{I}$ *with* $p < q$*, and* $a \in [0,1]$*, then*

$$\frac{1}{\ln q - \ln p}\int_{p}^{q} f(a)\frac{f(pq)}{a}\left(\frac{da}{a}\right) \leq \int_{0}^{1} [f(p)f(q)]^{h_1(1-a)+h_2(a)}da.$$

Proof: Since $p < q$, $a \in [0,1]$, and f is a g-(h_1, h_2) convex function, then by utilising Inequality (3.60), results

$$f\left(x_2^{(1-a)}y_2^{a}\right) \leq f(x_2)^{h_1(1-a)}f(y_2)^{h_2(a)}. \tag{3.77}$$

Since, f is a positive and symmetric function then similar to Inequality (3.77), we have

$$f\left(x_2^{a}y_2^{(1-a)}\right) \leq f(x_2)^{h_1(1-a)}f(y_2)^{h_2(a)}. \tag{3.78}$$

By multiplying Inequalities (3.77) and (3.78), yields

$$f\left(x_2^{(1-a)}y_2^{a}\right)f\left(x_2^{a}y_2^{(1-a)}\right) \leq f(x_2)^{h_1(1-a)}f(y_2)^{h_2(a)}$$

$$\times f(x_2)^{h_1(1-a)}f(y_2)^{h_2(a)},$$

$$f\left(x_2^{(1-a)}y_2^{a}\right)f\left(x_2^{a}y_2^{(1-a)}\right) \leq [f(x_2)f(y_2)]^{h_1(1-a)+h_2(a)}. \tag{3.79}$$

Apply integration on Inequality (3.79) with respect to a where $a \in [0,1]$, results

$$\int_{0}^{1} f\left(x_2^{(1-a)}y_2^{a}\right)f\left(x_2^{a}y_2^{(1-a)}\right)da \leq \int_{0}^{1} [f(x_2)f(y_2)]^{h_1(1-a)+h_2(a)}da. \tag{3.80}$$

Taking ln on both sides of Equation (3.58), results

$$\ln(x_2) = \ln\big(p^{(1-a)}q^{a}\big). \tag{3.81}$$

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By applying product rule and then power rule on Equation (3.81), results

$$ln(x_2) = (1-a)ln(p) + (a)ln(q). \tag{3.82}$$

By taking derivative of Equation (3.82) with respect to a, yields

$$\left(\frac{1}{x_2}\right)\frac{dx_2}{da} = (-ln(p) + ln(q))$$

or

$$\frac{dx_2}{x_2} = (ln(q) - ln(p))da. \tag{3.83}$$

By substituting Equations (3.58) and (3.83) in Inequality (3.80), results

$$\frac{1}{ln\,q - ln\,p}\int_p^q f(a)\frac{f(pq)}{a}\left(\frac{da}{a}\right) \leq \int_0^1 [f(p)f(q)]^{h_1(1-a)+h_2(a)}da. \tag{3.84}$$

Inequality (3.84) completes the proof.

Remarks 3.12:

1. By substituting $h_1(1-a) = 1-a$ and $h_2(a) = a$, then Theorem (3.14) will be reduced to H-H inequality for g-convex function.
2. By using $h_1(1-a) = h(1-a)$ and $h_2(a) = h(a)$ in Theorem (3.14), then H-H inequality for h-convex function will be produced.
3. By utilising $h_1(1-a) = (1-a)^s$ and $h_2(a) = (a)^s$ for all $s \in [0,1]$ in Theorem (3.14), then the result will be transformed into H-H inequality for s-convex function in the second sense.
4. If $h_1(1-a) = (1-a)^{\frac{1}{s}}$ and $h_2(a) = (a)^{\frac{1}{s}}$ for all $s \in (0,1]$ in Theorem (3.14), then the result will be converted into H-H inequality for s-convex function in the third sense.

Note that Theorems (3.13), and (3.14) successfully extended H-H inequality for g-(h_1, h_2)-convex function.

Furthermore, in proceeding section, the concept of s-convex function in the third sense defined in Definition (1.7) and g-convex function from Definition (1.9) are combined to form a new hybrid convex function named (g, s)-convex function in the third sense. Moreover, H-H inequality is also extended by using hybrid (g, s)-convex function in the third sense.

3.4 Extension of Hermite-Hadamard Inequality by Merging g-convex and s-convex Functions

In this section, another new hybrid convex function called (g, s)-convex function in the third sense is formed by combining Definition (1.7) and Definition (1.9). Furthermore, extension of H-H inequality will be considered in accordance with Definition (3.9) as follows:

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Definition 3.9: *A function $f: \mathbb{I} \subset \mathbb{R} \rightarrow \mathbb{R}$ is called (g, s)-convex function in the third sense for $x, y \in \mathbb{I}$, $s \in (0, 1]$ and $a \in [0, 1]$ if*

$$f\left(x^a y^{(1-a)}\right) \leq f(x)^{a^{\frac{1}{s}}} f(y)^{(1-a)^{\frac{1}{s}}}. \tag{3.85}$$

Remarks 3.13:

1. If $\frac{1}{s} = 1$, then Definition (3.9) will be reduced to the geometrically or g-convex function.
2. If $\frac{1}{s} = s$ in Definition (3.9) with $s \in (0, 1]$, then the Definition for the second sense of s-convex function will be produced.

From now, we will use Definition (3.9) of (g, s)-convex function in the third sense with $x_2 = p^{(1-a)}q^a$ and $y_2 = p^a q^{(1-a)}$ from Equations (3.58) and (3.59), respectively, for all $x_2, y_2, p, q \in \mathbb{I}$ and $a \in [0, 1]$.

Since $p < q$ and $a \in [0, 1]$, then Definition (3.9) of (g, s)-convex function in the third sense in terms of x_2 and y_2 can be written as

$$f\left(x_2^a y_2^{(1-a)}\right) \leq f(x_2)^{a^{\frac{1}{s}}} f(y_2)^{(1-a)^{\frac{1}{s}}}. \tag{3.86}$$

Based on Inequality (3.86), H-H inequality is considered to extend in Theorems (3.15) and (3.16) as follows:

Theorem 3.15: *Let $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be a (g, s)-convex function in the third sense, symmetric, positive, and integrable on $p, q \in \mathbb{I}$ with $p < q$, $s \in (0, 1]$, and $a \in [0, 1]$, then*

$$f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s}\right)}}\left(\sqrt{pq}\right) \leq \frac{1}{\ln q - \ln p}\int_p^q \frac{f(x_2)}{x_2} dx_2 \leq [f(p)f(q)]^{1-\left(\frac{1}{s}\right)} L\left(f^{\left(\frac{1}{s}\right)}(p), f^{\left(\frac{1}{s}\right)}(q)\right),$$

where the special mean $L(p, q)$ is defined as

$$L(p,q) = \begin{cases} \dfrac{q-p}{\ln q - \ln p}, & q \neq p, \\ p, & q = p. \end{cases}$$

Proof: Since $p < q$, and f be a (g, s)-convex function in the third sense then by applying integration for a where $a \in [0, 1]$ on Inequality (3.86) and then utilising Equation (3.69), results

$$\int_0^1 f\left(x_2^{(1-a)} y_2^a\right) da \leq \int_0^1 \left\{ f(x_2)^{(1-a)^{\frac{1}{s}}} f(y_2)^{(a)^{\frac{1}{s}}} \right\} da,$$

which can be written as

$$\int_{0}^{1} f\left(x_2^{(1-a)} y_2^{a}\right) da \leq \int_{0}^{1} \left\{ f(p)^{(1-a)^{\frac{1}{s}}} f(q)^{(a)^{\frac{1}{s}}} \right\} da. \qquad (3.87)$$

By substituting Equation (3.67) in left hand side of Inequality (3.87), results

$$\left(\frac{1}{\ln q - \ln p}\right) \int_{p}^{q} \left(\frac{f(x_2)}{x_2}\right) dx_2 \leq \int_{0}^{1} f(p)^{(1-a)^{\left(\frac{1}{s}\right)}} f(q)^{(a)^{\left(\frac{1}{s}\right)}} da. \qquad (3.88)$$

Since f is a (g,s)-convex function in the third sense, then by using Lemma (2.4) on Inequality (3.88), yields

$$\int_0^1 f^{a^{\left(\frac{1}{s}\right)}}(p) f^{(1-a)^{\left(\frac{1}{s}\right)}}(q) da \leq \int_0^1 f^{\left(\frac{1}{s}\right)(1-a)+1-\left(\frac{1}{s}\right)}(p) f^{\left(\frac{1}{s}\right)a+1-\left(\frac{1}{s}\right)}(q) da,$$

which can be written as

$$\int_0^1 f^{a^{\left(\frac{1}{s}\right)}}(p) f^{(1-a)^{\left(\frac{1}{s}\right)}}(q) da \leq \int_0^1 f^{\left(\frac{1}{s}\right)(1-a)}(p) f^{1-\left(\frac{1}{s}\right)}(p) f^{\left(\frac{1}{s}\right)a}(q) f^{1-\left(\frac{1}{s}\right)}(q) da$$

or

$$\int_0^1 f^{a^{\left(\frac{1}{s}\right)}}(p) f^{(1-a)^{\left(\frac{1}{s}\right)}}(q) da \leq [f(p)f(q)]^{1-\left(\frac{1}{s}\right)} \int_0^1 f^{\left(\frac{1}{s}\right)(1-a)}(p) f^{\left(\frac{1}{s}\right)a}(q) da. \qquad (3.89)$$

Similar as Equation (3.73), by solving $\int_0^1 f^{\left(\frac{1}{s}\right)(1-a)}(p) f^{\left(\frac{1}{s}\right)a}(q) da$, from right hand side of Inequality (3.89), results

$$\int_0^1 f^{\left(\frac{1}{s}\right)(1-a)}(p) f^{\left(\frac{1}{s}\right)a}(q) da = L\left(f^{\left(\frac{1}{s}\right)}(p), f^{\left(\frac{1}{s}\right)}(q)\right). \qquad (3.90)$$

By substituting Equation (3.90) in Inequality (3.89), yields

$$\int_0^1 f^{a^{\left(\frac{1}{s}\right)}}(p) f^{(1-a)^{\left(\frac{1}{s}\right)}}(q) da \leq [f(p)f(q)]^{1-\left(\frac{1}{s}\right)} L\left(f^{\left(\frac{1}{s}\right)}(p), f^{\left(\frac{1}{s}\right)}(q)\right). \qquad (3.91)$$

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Since $\left(\sqrt{pq}\right) = \left(\sqrt{p^a q^{(1-a)} p^{(1-a)} q^a}\right)$, $\forall\, a \in [0, 1]$ and $p, q \in \mathbb{I}$, then by utilising the definition of (g, s)-convex function in the third sense defined in Inequality (3.86) in terms of p, q and then set $a = \frac{1}{2}$, gives

$$f^{\left(\frac{1}{2}\right)^{\left(\frac{1}{s}\right)}}\left(\sqrt{pq}\right) \leq \left[f\left(p^a q^{(1-a)}\right) f\left(p^{(1-a)} q^a\right)\right]^{\left(\frac{1}{2}\right)}.$$

By using arithmetic and geometric means inequality as $\sqrt{pq} \leq \frac{p+q}{2}$, results

$$\leq \frac{f\left(p^a q^{(1-a)}\right) + f\left(p^{(1-a)} q^a\right)}{2}. \tag{3.92}$$

Since, f is symmetric and positive then by integrating Inequality (3.92) with respect to $a \in [0, 1]$, yields

$$\int_0^1 f^{\left(\frac{1}{2}\right)^{\left(\frac{1}{s}\right)}}\left(\sqrt{pq}\right) da \leq \left(\frac{1}{2}\right) \int_0^1 \left[f\left(p^a q^{(1-a)}\right) + f\left(p^{(1-a)} q^a\right)\right] da.$$

Similar to Inequality (3.88), by solving $\int_0^1 f\left(p^a q^{(1-a)}\right) da$ and $\int_0^1 f\left(p^{(1-a)} q^a\right) da$, results

$$\int_0^1 f^{\left(\frac{1}{2}\right)^{\left(\frac{1}{s}\right)}}\left(\sqrt{pq}\right) da \leq \left(\frac{1}{\ln q - \ln p}\right) \int_p^q \left(\frac{f(x_2)}{x_2}\right) dx_2. \tag{3.93}$$

By combining Inequality (3.91) and Inequality (3.93), results

$$f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s}\right)}}\left(\sqrt{pq}\right) \leq \frac{1}{\ln q - \ln p} \int_p^q \frac{f(x_2)}{x_2} dx_2$$

$$\leq [f(p) f(q)]^{1-\left(\frac{1}{s}\right)} L\left(f^{\left(\frac{1}{s}\right)}(p), f^{\left(\frac{1}{s}\right)}(q)\right). \tag{3.94}$$

Inequality (3.94) completes the proof.

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Theorem 3.16: *Let $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be a (g, s)-convex function in the third sense, symmetric, positive, and integrable on $p, q \in \mathbb{I}$ with $p < q$, $s \in (0,1]$, and $a \in [0,1]$, then*

$$(\sqrt{pq}\,) f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s}\right)}}(\sqrt{pq}\,) \le \frac{1}{\ln q - \ln p} \int_{p}^{q} f(x_2)\,dx_2$$

$$\le [f(p)f(q)]^{1-\left(\frac{1}{s}\right)} L\left(pf^{\left(\frac{1}{s}\right)}(p), qf^{\left(\frac{1}{s}\right)}(q)\right),$$

here the special mean $L(p, q)$ is defined in Theorem (3.15).

Proof: Since $p < q$, $a \in [0,1]$, and f be a (g, s)-convex function in the third sense, then by multiplying x_2 from Equation (3.58) on both sides of Inequality (3.87), results

$$\left(\frac{1}{\ln q - \ln p}\right) \int_{p}^{q} f(x_2)\,dx_2 \le p^{(1-a)} q^{a} \int_{0}^{1} f(p)^{(1-a)^{\left(\frac{1}{s}\right)}} f(q)^{a^{\left(\frac{1}{s}\right)}}\,da, \tag{3.95}$$

$$\le \int_{0}^{1} p^{(1-a)} q^{a} f^{(1-a)\left(\frac{1}{s}\right)+1-\left(\frac{1}{s}\right)}(p)\, f^{a\left(\frac{1}{s}\right)+1-\left(\frac{1}{s}\right)}(q)\,da,$$

Similar as Inequality (3.94), by solving $\int_0^1 f^{\left(\frac{1}{s}\right)(1-a)}(p) f^{\left(\frac{1}{s}\right)a}(q)\,da$, from right hand side of Inequality (3.95), results

$$\left(\frac{1}{\ln q - \ln p}\right) \int_{p}^{q} f(x_2)\,dx_2 \le [f(p)f(q)]^{1-\left(\frac{1}{s}\right)} L\left(pf^{\left(\frac{1}{s}\right)}(p), qf^{\left(\frac{1}{s}\right)}(q)\right). \tag{3.96}$$

By multiplying $(\sqrt{pq}\,) = \left(\sqrt{p^{a} q^{(1-a)} p^{(1-a)} q^{a}}\right)$ on both sides of Inequality (3.95), yields

$$(\sqrt{pq}\,) f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s}\right)}}(\sqrt{pq}\,) \le \int_{0}^{1} \sqrt{p^{a} q^{(1-a)} p^{(1-a)} q^{a}}\, f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s}\right)}}$$

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$$\times\left(\sqrt{p^{a}q^{(1-a)}p^{(1-a)}q^{a}}\right)da,$$

which can be written as

$$\begin{aligned}\left(\sqrt{pq}\,\right)f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s}\right)}}\left(\sqrt{pq}\,\right) &\leq \int_{0}^{1}\sqrt{p^{a}q^{(1-a)}p^{(1-a)}q^{a}}\\ &\times\left[f\left(p^{a}q^{(1-a)}\right)f\left(p^{(1-a)}q^{a}\right)\right]^{\frac{1}{2}}da.\end{aligned}$$

By using arithmetic and geometric means inequality as $\sqrt{pq} \leq \frac{p+q}{2}$ and then substituting values from Inequality (3.95), results

$$\begin{aligned}\left(\sqrt{pq}\,\right)f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s}\right)}}\left(\sqrt{pq}\,\right) &\leq \frac{1}{2}\int_{0}^{1}\left[p^{a}q^{(1-a)}f\left(p^{a}q^{(1-a)}\right)\right.\\ &\left.+p^{(1-a)}q^{a}f\left(p^{(1-a)}q^{a}\right)\right]da,\\ &\leq \left(\frac{1}{\ln q-\ln p}\right)\int_{p}^{q}f(x_2)dx_2. \qquad (3.97)\end{aligned}$$

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By combining Inequality (3.96) and Inequality (3.97), results

$$\begin{aligned}\left(\sqrt{pq}\,\right)f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s}\right)}}\left(\sqrt{pq}\,\right) &\leq \frac{1}{\ln q-\ln p}\int_{p}^{q}f(x_2)dx_2\\ &\leq [f(p)f(q)]^{1-\left(\frac{1}{s}\right)}L\left(pf^{\left(\frac{1}{s}\right)}(p),qf^{\left(\frac{1}{s}\right)}(q)\right),\end{aligned}$$

where the special mean $L(p,q)$ is defined in Theorem (3.15) and completes the proof.

Remarks 3.14:

1. By substituting $\frac{1}{s}=1$ in Theorem (3.15) and Theorem (3.16) the results will be reduced to H-H inequality for g-convex function.

2. By utilising $\frac{1}{s} = s$ in Theorem (3.15) and Theorem (3.16), then H-H inequality for s-convex function in the second sense will be produced.

Note that Theorems (3.15) and (3.16) successfully extended H-H inequality for (g, s)-convex function in the third sense.

3.5 Summary

This chapter established some new hybrid convex functions by combining s-convex, h-convex, and g-convex functions. Further, one the basis of these hybrid convex functions, the Hermite-Hadamard (H-H) type of inequalities are extended or generalised. Additionally, this inequality is a new refinement for many other inequalities such as Fejér's type of inequalities. In Section (3.2), H-H inequality is extended by combining s-convex and (h_1, h_2)-convex functions. Similarly, in Section (3.3), H-H inequality is extended by combining g-convex and (h_1, h_2)-convex functions and in Section (3.4), H-H inequality is extended by combining s-convex and g-convex functions.

Furthermore, to highlight the role of hybrid convex functions, some inequalities are employed to show some relation with the mathematical mean (average).

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CHAPTER FOUR
JENSEN'S TYPE OF INEQUALITIES BASED ON HYBRID CONVEX FUNCTION

This chapter describes three novel forms of Jensen's inequality based on the hybrid of s-convex, h-convex, and g-convex functions.

4.1 Introduction

In this chapter, Jensen's inequality is extended based on hybrid convex functions without losing its originality. By applying certain conditions, the standard form, as well as some previous Jensen's inequality can be obtained.

4.2 Extension of Jensen's Inequality by Merging $(\boldsymbol{h_1}, \boldsymbol{h_2})$-convex and $\boldsymbol{s}$-convex Functions

In this section, newly defined hybrid convex functions named (h_1, h_2, s)-convex function in the third sense in Definition (3.3) is considered to extend Jensen's inequality. Before the extension of Jensen's inequality can be made, some Propositions are illustrated as follows:

Proposition 4.1: *Let* $h_1, h_2 \colon \mathbb{J} \subset (0,1) \to \mathbb{R}$ *be two positive functions defined on* $\mathbb{J}$*. If* $f \colon \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *is a* (h_1, h_2, s)*-convex function in the third sense and* g *is an increasing function for all* $x, y \in \mathbb{I}$*, then* $f \circ g$ *is also* (h_1, h_2, s)*-convex function with* $a \in [0,1]$ *and* $s \in (0,1]$.

Proof: Since $x, y \in \mathbb{I}$, $a \in [0,1]$, and $s \in (0,1]$, then

$$(f \circ g)(ax + (1-a)y) = (a)\big(f \circ g(x)\big) + (1-a)\big(f \circ g(y)\big), \tag{4.1}$$

and

$$(f \circ g)(ax + (1-a)y) \leq (a)f\{g(x)\} + (1-a)f\{g(y)\}. \tag{4.2}$$

By using Definition (3.3) on Inequality (4.2), results

$$(f \circ g)(ax + (1-a)y) \leq h_1^{\frac{1}{s}}(a)f\{g(x)\} + h_2^{\frac{1}{s}}(1-a)f\{g(y)\}. \tag{4.3}$$

Inequality (4.3) completes the proof.

Proposition 4.2: *If f and g are two (h_1, h_2, s)-convex function in third sense such that $f, g: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ then $f + g$ is also a (h_1, h_2, s)-convex function.*

Proof: Since $x_1, x_2 \in \mathbb{I}$, $a_1 \in [0,1]$, $a_2 \in [0,1]$ and $s \in (0,1]$, then

$$(f+g)(a_1x_1 + a_2x_2) = f(a_1x_1 + a_2x_2) + g(a_1x_1 + a_2x_2). \tag{4.4}$$

By applying Definition (3.3) on Inequality (4.4), results

$$\begin{aligned}(f+g)(a_1x_1 + a_2x_2) \leq\ & h_1^{\frac{1}{s}}(a_1)f(x_1) + h_2^{\frac{1}{s}}(a_2)f(x_2) \\ & + h_1^{\frac{1}{s}}(a_1)g(x_1) + h_2^{\frac{1}{s}}(a_2)g(x_2),\end{aligned} \tag{4.5}$$

$$\begin{aligned}(f+g)(a_1x_1 + a_2x_2) \leq\ & h_1^{\frac{1}{s}}(a_1)\big(f(x_1) + g(x_1)\big) \\ & + h_2^{\frac{1}{s}}(a_2)\big(f(x_2) + g(x_2)\big).\end{aligned} \tag{4.6}$$

Inequality (4.6) completes the proof.

Proposition 4.3: *If f be (h_1, h_2, s)-convex function for third sense such that $f, g: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ and $\lambda > 0$ be any arbitrary number, then λf is also a (h_1, h_2, s)-convex function.*

Proof: Since $x_1, x_2 \in \mathbb{I}$, $a_1 \in [0,1]$, $a_2 \in [0,1]$ and $s \in (0,1]$, then

$$f(a_1x_1 + a_2x_2) = a_1f(x_1) + a_2f(x_2). \tag{4.7}$$

By multiplying $\lambda > 0$ on both sides of Inequality (4.7) and then applying Definition (3.3), results

$$\lambda f(a_1x_1 + a_2x_2) \leq h_1^{\frac{1}{s}}(a_1)\lambda f(x_1) + h_2^{\frac{1}{s}}(a_2)\lambda f(x_2). \tag{4.8}$$

Inequality (4.8) completes the proof.

Proposition 4.4: *Let* $h_1, h_2 \colon \mathbb{J} \subset (0,1) \to \mathbb{R}$ *be two positive functions. If* $f \colon \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *is an increasing function and* g *is a* (h_1, h_2, s)*-convex function for third sense such that* $f, g \colon \mathbb{I} \to \mathbb{R}$*, then* $f \circ g$ *is also a* (h_1, h_2, s)*-convex function.*

Proof: Since $x_1, x_2 \in \mathbb{I}$, $a_1 \in [0,1]$, $a_2 \in [0,1]$ and $s \in (0,1]$, then

$$(f \circ g)(a_1 x_1 + a_2 x_2) = f\{g(a_1 x_1 + a_2 x_2)\}. \tag{4.9}$$

As g is a (h_1, h_2, s)-convex function for third sense, then by applying Definition (3.3) on Inequality (4.9), results

$$(f \circ g)(a_1 x_1 + a_2 x_2) \leq f\left\{ {h_1}^{\frac{1}{s}}(a_1) g(x_1) + {h_2}^{\frac{1}{s}}(a_2) g(x_2) \right\}. \tag{4.10}$$

As f is an increasing function, then Inequality (4.10) can be written as

$$(f \circ g)(a_1 x_1 + a_2 x_2) \leq {h_1}^{\frac{1}{s}}(a_1)\big(f \circ g(x_1)\big) + {h_2}^{\frac{1}{s}}(a_2)\big(f \circ g(x_2)\big). \tag{4.11}$$

Inequality (4.11) completes the proof.

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Propositions (4.1 - 4.4) will be used in the following theorems to extend Jensen's type of inequality for hybrid (h_1, h_2, s)-convex function in the third sense defined in Definitions (3.3).

We will use mathematical induction to extend Jensen's inequality in Theorem (4.1):

Theorem 4.1: *Let* $h_1, h_2 \colon \mathbb{J} \subset (0,1) \to \mathbb{R}$ *be two positive functions. If* $f \colon \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be a* (h_1, h_2, s)*-convex function in the third sense on* $\mathbb{I}$ *for all* $x, y \in \mathbb{I}$ *and* $s \in (0,1]$*, then for all* $x_i \in \mathbb{I}$ *with* $\sum_{i=1}^{n} a_i = 1$*, the following inequality holds.*

$$f\left(\sum_{i=1}^{n} a_i x_i\right) \leq f\left(\sum_{i=1}^{n} h^{\frac{1}{s}}(a_i) x_i\right), \quad n \in \mathbb{N}, n \geq 2. \tag{4.12}$$

Proof: $x_i \in \mathbb{I}$, $s \in (0,1]$, and $\sum_{i=1}^{n} a_i = 1$, then by using mathematical induction, we have

Base step: Assume that Inequality (4.12) is true for $n = 2$, then Inequality (4.12) can be written as

$$f\left(\sum_{i=1}^{2} a_i x_i\right) \le f\left(\sum_{i=1}^{2} h(a_i) x_i\right),$$

$$f(a_1x_1 + a_2x_2) \le h(a_1)f(x_1) + h(a_2)f(x_2). \tag{4.13}$$

By replacing $h(a_1) = h_1(a_1)$ and $h(a_2) = h_2(1 - a_1)$ with $h_1(a_1) + h_2(1 - a_1) = 1$, then Inequality (4.5) will be converted to the inequality for (h_1, h_2)-convex function defined in Definition (1.11) as follows

$$f(a_1x_1 + a_2x_2) \le h_1(a_1)f(x_1) + h_2(1 - a_1)f(x_2). \tag{4.14}$$

By applying Definition (3.3) on Inequality (4.14), results

$$f(a_1x_1 + a_2x_2) \le {h_1}^{\frac{1}{s}}(a_1)f(x_1) + {h_2}^{\frac{1}{s}}(1 - a_1)f(x_2). \tag{4.15}$$

Inequality (4.15) is the definition of (h_1, h_2, s)-convex function in third sense with two variables x_1 and x_2.

Note that such conditions where ${h_1}^{s}(a_1) + {h_2}^{s}(1 - a_1) = 1$ and $h_1(a_1) + h_2(1 - a_1) = 1$, Inequality (4.15) will be transformed into (h_1, h_2, s)-convex function in third sense and fourth sense, respectively.

Hypothesis step: Assume that Inequality (4.12) is true for $m = n - 1$, then

$$f\left(\sum_{i=1}^{n} a_i x_i\right) \le h(a_n)f(x_n) + h(1 - a_i) f\left(\sum_{i=1}^{m} \left(\frac{a_i}{1 - a_m}\right) x_i\right). \tag{4.16}$$

By applying Definition (1.11) of (h_1, h_2)-convex function on Inequality (4.16), then $h(a_n) = h_1(a_n)$ and $\sum_{i=1}^{m}(1 - a_i) = \sum_{i=1}^{m} h_2(1 - a_i)$ with $h_1(a_n) + \sum_{i=1}^{m} h_2(a_i) = 1$. Thus, Inequality (4.16) will be converted to inequality for (h_1, h_2)-convex function such as

$$f\left(\sum_{i=1}^{n} a_i x_i\right) \quad \leq h_1(a_n) f(x_n) + h_2(1-a_i) f\left(\sum_{i=1}^{m}\left(\frac{a_i}{1-a_m}\right) x_i\right). \quad (4.17)$$

Since f is a (h_1, h_2, s)-convex function in third sense, by applying Definition (3.3) on Inequality (4.17) yields

$$f\left(\sum_{i=1}^{n} a_i x_i\right) \leq h_1^{\frac{1}{s}}(a_n) f(x_n) + h_2^{\frac{1}{s}}(1-a_i) f\left(\sum_{i=1}^{m}\left(\frac{a_i}{1-a_m}\right) x_i\right). \quad (4.18)$$

Induction step: Finally, by using the induction hypothesis, we have

$$f\left(\sum_{i=1}^{n} a_i x_i\right) \leq h_1^{\frac{1}{s}}(a_n) f(x_n) + h_2^{\frac{1}{s}}(1-a_i) f\left(\sum_{i=1}^{m}\left(\frac{a_i}{1-a_m}\right) x_i\right). \quad (4.19)$$

Note that the following inequality is also true

$$h_1^{\frac{1}{s}}(a_n) f(x_n) + h_2^{\frac{1}{s}}(1-a_i) f\left(\sum_{i=1}^{m}\left(\frac{a_i}{1-a_m}\right) x_i\right) \leq f\left(\sum_{i=1}^{n} h^{\frac{1}{s}}(a_i) x_i\right). \quad (4.20)$$

By comparing Inequality (4.19) and (4.20), we have

$$f\left(\sum_{i=1}^{n} a_i x_i\right) \leq f\left(\sum_{i=1}^{n} h^{\frac{1}{s}}(a_i) x_i\right). \quad (4.21)$$

Inequality (4.21) completes the proof.

Theorem 4.2: *Let $h_1, h_2 : \mathbb{J} \subset (0,1) \to \mathbb{R}$ be two positive functions and $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be a (h_1, h_2, s)-convex function in third sense on $\mathbb{I}$ and $s \in (0,1]$. If for all $x, y, x_1, x_2, x_3 \in \mathbb{I}$ with $x_1 < x_2 < x_3$ and $x_3 - x_2, x_3 - x_1, x_2 - x_1 \in \mathbb{J}$, then*

$$f(ax + (1-a)y) \leq h_1^{\frac{1}{s}}\left(\frac{x_3 - x_2}{x_3 - x_1}\right) f(x) + h_2^{\frac{1}{s}}\left(\frac{x_2 - x_1}{x_3 - x_1}\right) f(y).$$

Proof: Since $s \in (0,1]$, $a \subset (0,1)$, $x_1, x_2, x_3 \in \mathbb{I}$ with $x_1 < x_2 < x_3$ and $x_3 - x_2$, $x_3 - x_1$, $x_2 - x_1 \in \mathbb{J}$. As h is a positive function then $h(x) = h\left(\frac{x}{y} y\right)$ implies $h(x) \leq$

$h\left(\frac{x}{y}\right)h(y)$ and $\frac{h(x)}{h(y)} \le h\left(\frac{x}{y}\right)$. Similarly, $\left(\frac{x_2-x_1}{x_3-x_1}\right) \in \mathbb{J} \subset (0,1)$, $\left(\frac{x_3-x_2}{x_3-x_1}\right) \in \mathbb{J} \subset (0,1)$ and $\left(\frac{x_2-x_1}{x_3-x_1}\right) + \left(\frac{x_3-x_2}{x_3-x_1}\right) = 1$. Then

$$h(x_3 - x_2) = h\left[\left(\frac{x_3 - x_2}{x_3 - x_1}\right)(x_3 - x_1)\right] \le h\left(\frac{x_3 - x_2}{x_3 - x_1}\right)h(x_3 - x_1), \quad (4.22)$$

and

$$h(x_2 - x_1) = h\left[\left(\frac{x_2 - x_1}{x_3 - x_1}\right)(x_3 - x_1)\right] \le h\left(\frac{x_2 - x_1}{x_3 - x_1}\right)h(x_3 - x_1).$$

Set $a = \left(\frac{x_3-x_2}{x_3-x_1}\right)$, then $(1-a) = \left(\frac{x_2-x_1}{x_3-x_1}\right)$. By utilising Definition (3.3), results

$$f(ax + (1-a)y) \le h_1^{\frac{1}{s}}\left(\frac{x_3 - x_2}{x_3 - x_1}\right)f(x) + h_2^{\frac{1}{s}}\left(\frac{x_2 - x_1}{x_3 - x_1}\right)f(y). \quad (4.23)$$

Inequality (4.23) completes the proof.

We then proceed to Theorem (4.3) by using mathematical induction.

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Theorem 4.3: *Let* $h_1, h_2\colon \mathbb{J} \subset (0,1) \to \mathbb{R}$ *be a positive function and* $f\colon \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be a* (h_1, h_2, s)*-convex function in third sense on interval* $\mathbb{I}$ *and* $s \in (0,1]$. *If* $\forall x_1, x_2, \dots, x_3 \in \mathbb{I}$ *and* $\sum_{i=1}^{n} a_i = 1$ *with* $A_n = \sum_{i=1}^{n} a_i$, *then*

$$f\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i\, x_i\right) \le h_1^{\frac{1}{s}}\left(\frac{a_n}{A_n}\right)f(x_n) + \sum_{i=1}^{n-1} h_2^{\frac{1}{s}}\left(\frac{a_i}{A_{n-1}}\right)f(x_i). \quad (4.24)$$

Proof. Since $s \in (0,1]$, $a \subset (0,1)$, $x_1, x_2, \dots, x_3 \in \mathbb{I}$, then we have

Base step: Assume that Inequality (4.24) is true for $n = 2$, then Inequality (4.24) can be written as

$$f\left(\frac{1}{A_n}\sum_{i=1}^{2} a_i x_i\right) \le \frac{1}{A_n}\left[\sum_{i=1}^{2} h_i^{\frac{1}{s}}(a_i)\, f(x_i)\right],$$

similar to Inequality (3.3) for (h_1, h_2, s)-convex function in third sense with $a = \left(\frac{a_1}{A_1}\right)$ and $(1-a) = \left(\frac{a_2}{A_2}\right)$.

Hypothesis step: Assume that Inequality (4.24) is true for $m = n - 1$, then

$$f\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i\, x_i\right) = \frac{a_n}{A_n} f(x_n) + \frac{1}{A_n}\sum_{i=1}^{n-1}(a_i)\, f(x_i), \tag{4.25}$$

$$f\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i\, x_i\right) = \frac{a_n}{A_n} f(x_n) + \frac{A_m}{A_n}\sum_{i=1}^{m}\left(\frac{a_i}{A_m}\right) f(x_i). \tag{4.26}$$

By applying definition of (h, s)-convex function in the third sense on Inequality (4.26), yields

$$f\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i\, x_i\right) \leq h^{\frac{1}{s}}\left(\frac{a_n}{A_n}\right) f(x_n)$$

$$+h^{\frac{1}{s}}\left(\frac{A_{n-1}}{A_n}\right) f\left(\sum_{i=1}^{n-1}\left(\frac{a_i}{A_{n-1}}\right)(x_i)\right). \tag{4.27}$$

Induction step: Assume that Inequality (4.24) is true for $(n-1)$ and h is a positive function, then similar to Inequality (4.22), $h\left(\frac{a_i}{A_n}\right)$ can be transformed as

$$h\left(\frac{a_i}{A_n}\right) = h\left[\left(\frac{A_{n-1}}{A_n}\right)\left(\frac{a_i}{A_{n-1}}\right)\right] \leq h\left(\frac{A_{n-1}}{A_n}\right) h\left(\frac{a_i}{A_{n-1}}\right). \tag{4.28}$$

By using Inequality (4.27) and Inequality (4.28), yields

$$f\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i\, x_i\right) \leq h^{\frac{1}{s}}\left(\frac{a_n}{A_n}\right)(x_n) + h^{\frac{1}{s}}\left(\frac{A_{n-1}}{A_n}\right) f\left(\sum_{i=1}^{n-1}\left(\frac{a_i}{A_{n-1}}\right)(x_i)\right), \tag{4.29}$$

$$f\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i\, x_i\right) \leq h^{\frac{1}{s}}\left(\frac{a_n}{A_n}\right) f(x_n) + \sum_{i=1}^{n-1} h^{\frac{1}{s}}\left(\frac{a_i}{A_{n-1}}\right) f(x_i). \tag{4.30}$$

By comparing Inequality (3.3) of (h_1, h_2, s)-convex function in third sense with Inequality (4.30), we have $h^{\frac{1}{s}}\left(\frac{a_n}{A_n}\right) = {h_1}^{\frac{1}{s}}(a_1) = {h_1}^{\frac{1}{s}}\left(\frac{a_n}{A_n}\right)$ and $h^{\frac{1}{s}}\left(\frac{a_i}{A_{n-1}}\right) = 1 - {h_1}^{\frac{1}{s}}(a_1) = {h_2}^{\frac{1}{s}}\left(\frac{a_i}{A_{n-1}}\right)$. Then Inequality (4.30) can be written as

$$f\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i\, x_i\right) \leq {h_1}^{\frac{1}{s}}\left(\frac{a_n}{A_n}\right) f(x_n) + \sum_{i=1}^{n-1} {h_2}^{\frac{1}{s}}\left(\frac{a_i}{A_{n-1}}\right) f(x_i). \tag{4.31}$$

Inequality (4.31) completes the proof.

Note that Theorems (4.1), (4.2), and (4.3) successfully extended Jensen's inequality for (h_1, h_2, s)-convex function in third sense.

Remark 4.1:

1. If $h^{\frac{1}{s}}(a_n) = {h_1}^{\frac{1}{s}}(a_1)$ and $h^{\frac{1}{s}}(a_2) = {h_2}^{\frac{1}{s}}(1-a_1)$ with ${h_1}^{\frac{1}{s}}(a_1) + {h_2}^{\frac{1}{s}}(1-a_1) = 1$, then Inequality (4.18) will be reduced to the inequality for (h, s)-convex function for the third sense.
2. By imposing the condition ${h_1}^{s}(a_1) + {h_2}^{s}(a_1) = 1$ and $h_1(a_1) + h_2(a_1) = 1$, then Inequality (4.18) will be converted into (h_1, h_2, s)-convex function in the third sense and fourth sense, respectively.

In the proceeding section, the definition of (h, g)-convex function is considered to extend Jensen's inequality.

4.3 Extension of Jensen's Inequality by Merging *h*-convex and *g*-convex Functions

In this section, another new hybrid convex function called (h, g)-convex function is used to extend Jensen's type of inequality, where h is super multiplicative and g is a positive function.

From now, we will use $n \in \mathbb{N}$ and $\beta_1, \beta_2, \beta_3 \ldots \beta_n$ are positive numbers with $B_n = \beta_1 + \beta_2 + \beta_3 + \cdots + \beta_n$, then $B_n = \sum_{i=1}^{n} \beta_i$ and $X_n = \frac{1}{B_n}\sum_{i=1}^{n} \beta_i x_i$. Set empty product equal to 1 such as $G_{n+1}^{n} \leq \frac{1}{B_n}\prod_{i=n+1}^{n} g(X_i) = 1$.

Theorem 4.4: *Let $f:\mathbb{I}\subset\mathbb{R}\to\mathbb{R}$ be a (h,g)-convex function and h be a positive and multiplicative function on $\mathbb{J}\subset(0,1)$ with $h\not\equiv 0$. Let g be a positive function on $\alpha\in(0,1]$ and $\beta_1,\beta_2,\beta_3\dots\beta_n$ be positive numbers with $B_n=\beta_1+\beta_2+\beta_3+\cdots+\beta_n$, then*

$$f\left(\frac{1}{B_n}\sum_{i=1}^{n}\beta_i x_i\right)\le h\left(\frac{\beta_1}{B_n}\right)f(x_1)G_1^{n-1}+\sum_{i=2}^{n}h\left(\frac{\beta_i}{B_n}\right)f(x_i)g(x_i)G_i^{n-1}. \quad (4.32)$$

Proof: Since $\alpha\in(0,1]$, h is a positive function $\beta_1,\beta_2,\beta_3\dots\beta_n$ be positive numbers with $B_n=\beta_1+\beta_2+\beta_3+\cdots+\beta_n$, then by using mathematical induction, we have

Base Step: Assume that Inequality (4.32) is true for $n=2$, Inequality (4.32) will be converted to the Definition of (h,g)-convex function if $\alpha=\left(\frac{\beta_1}{B_2}\right)$, $(1-\alpha)=\left(\frac{\beta_2}{B_2}\right)$ and $G_1^1=g(X_1)=g(x_1)$ with $G_2^1=1$.

Hypothesis Step: Assume that Inequality (4.32) is true for $m=n-1$, then for all $\beta_1,\beta_2,\beta_3\dots\beta_n$ and $x_1,x_2,x_3\dots x_n$, we have

$$f\left(\frac{1}{B_n}\sum_{i=1}^{n}\beta_i x_i\right)=f\left(\left(\frac{\beta_n}{B_n}\right)x_n+\left(\frac{B_m}{\beta_n}\right)\sum_{i=1}^{m}\left(\frac{\beta_i}{B_m}\right)x_i\right),$$

$$\le h\left(\frac{\beta_n}{B_n}\right)f(x_n)g(x_n)+h\left(\frac{B_m}{\beta_n}\right)$$

$$\times f\left(\sum_{i=1}^{m}\left(\frac{\beta_i}{B_m}\right)x_i\right)g\left(\sum_{i=1}^{m}\left(\frac{\beta_i}{B_m}\right)x_i\right),$$

$$\le h\left(\frac{\beta_n}{B_n}\right)f(x_n)g(x_n)+h\left(\frac{B_m}{\beta_n}\right)$$

$$\times\left[h\left(\frac{\beta_1}{B_m}\right)f(x_1)G_1^{n-2}\right.$$

$$+\sum_{i=2}^{n-1}\left(\frac{\beta_i}{B_m}\right)f(x_n)g(x_n)G_i^{n-2}\Bigg]g(X_{n-1}). \tag{4.33}$$

Inductive step: Since h is a multiplicative function, assume that Inequality (4.33) is true for $m = n - 1$, then

$$f\left(\frac{1}{B_n}\sum_{i=1}^{n}\beta_i x_i\right) \leq h\left(\frac{\beta_n}{B_n}\right)f(x_n)g(x_n)$$

$$+h\left(\frac{\beta_1}{B_n}\right)f(x_1)g(X_{n-1})G_1^{n-2}$$

$$+\sum_{i=2}^{n-1}h\left(\frac{\beta_1}{B_n}\right)f(x_i)g(x_i)g(X_{n-1})G_i^{n-2}. \tag{4.34}$$

Inequality (4.34) completes the proof.

Note that Theorem (4.4) successfully extended Jensen's inequality for (h, g)-convex function. While, if we impose the condition $h\left(\frac{\beta_1}{B_n}\right) = h_1\left(\frac{\beta_1}{B_n}\right)$ and $h\left(\frac{\beta_i}{B_n}\right) = h_2\left(\frac{\beta_i}{B_n}\right)$, then results will be converted into g-(h_1, h_2)-convex function.

Moreover, in upcoming section, Jensen's type of inequality is extended based on newly formed hybrid (g, s)-convex function in third sense.

4.4 Extension of Jensen's Inequality by Merging g-convex and s-convex Functions

In this section, another new hybrid convex function called (g, s)-convex function in third sense in Definition (3.9) is utilized to extend Jensen's inequality as follows:

Theorem 4.5: *Let $f: \mathbb{I} \to \mathbb{R}$ be a (g, s)-convex function defined on $\mathbb{I} \in [p, q] \subset \mathbb{R}$ and differentiable on $s \in (0, 1]$ such that $f' \in L(p, q), p < q$ and $0 < p < u < v < w < q$, then*

$$\left(\frac{\ln f(v) - \ln f(u)}{(\ln v - \ln u)^{\frac{1}{s}}}\right) \leq \left(\frac{\ln f(w) - \ln f(v)}{(\ln w - \ln v)^{\frac{1}{s}}}\right).$$

Proof: Since $s \in (0,1]$, $[p,q] \subset \mathbb{R}$, $0 < p < u < v < w < q$, and f be a (g,s)-convex function with $\left(\frac{w-v}{w-u}\right) \in (0,1)$ and $1 - \left(\frac{w-v}{w-u}\right) \in (0,1)$.

Let $a = \left[\frac{\ln w - \ln v}{\ln w - \ln u}\right]$, then

$$a(\ln w - \ln u) = (\ln w - \ln v)$$

$$a(\ln w) - a(\ln u) = (\ln w - \ln v)$$

$$\ln v - a(\ln u) = \big(\ln w - a(\ln w)\big)$$

By using ln of power rule, we have

$$\ln v - \ln u^a = \ln w - \ln w^a$$

$$\ln v = \ln w - \ln w^a + \ln u^a$$

By applying product rule and quotient rule, results

$$= \ln u^a \frac{w}{w^a}$$

$$\ln v = \ln u^a \, w^{(1-a)},$$

yields

$$v = u^a w^{1-a}. \tag{4.35}$$

By substituting values in Definition (3.9) of (g,s)-convex function in third sense in terms of $u, v,$ and w, results

$$f(v) \leq f(w)^{\left[1-\left(\frac{\ln w - \ln v}{\ln w - \ln u}\right)\right]^{\frac{1}{s}}} f(u)^{\left(\frac{\ln w - \ln v}{\ln w - \ln u}\right)^{\frac{1}{s}}}. \tag{4.36}$$

Taking ln on both sides of Inequality (4.36), and then applying product rule, yields

$$\ln f(v) \leq \ln f(w)^{\left[1-\left(\frac{\ln w - \ln v}{\ln w - \ln u}\right)\right]^{\frac{1}{s}}} + \ln f(u)^{\left(\frac{\ln w - \ln v}{\ln w - \ln u}\right)^{\frac{1}{s}}}. \tag{4.37}$$

Applying power rule on Inequality (4.37) results

$$ln\, f(v) \leq \left(1-\left(\frac{ln\, w - ln\, v}{ln\, w - ln\, u}\right)\right)^{\frac{1}{s}} ln\, f(w)$$

$$+\left(\frac{ln\, w - ln\, v}{ln\, w - ln\, u}\right)^{\frac{1}{s}} ln\, f(u). \qquad (4.38)$$

As $1-\left(\frac{ln\, w-ln\, v}{ln\, w-ln\, u}\right)=\left(\frac{ln\, v-ln\, u}{ln\, w-ln\, u}\right)$, then Inequality (4.38) can be written as

$$ln\, f(v) \leq \left(\frac{ln\, v - ln\, u}{ln\, w - ln\, u}\right)^{\frac{1}{s}} ln\, f(w)$$

$$+\left(\frac{ln\, w - ln\, v}{ln\, w - ln\, u}\right)^{\frac{1}{s}} ln\, f(u), \qquad (4.39)$$

which can be written as

$$\left(\frac{ln\, v - ln\, u}{ln\, w - ln\, u}\right)^{\frac{1}{s}} ln\, f(v) + \left(\frac{ln\, w - ln\, v}{ln\, w - ln\, u}\right)^{\frac{1}{s}} ln\, f(v)$$

$$\leq \left(\frac{ln\, v - ln\, u}{ln\, w - ln\, u}\right)^{\frac{1}{s}} ln\, f(w) + \left(\frac{ln\, w - ln\, v}{ln\, w - ln\, u}\right)^{\frac{1}{s}} ln\, f(u),$$

which can be expanded to

$$\left(\frac{ln\, w - ln\, v}{ln\, w - ln\, u}\right)^{\frac{1}{s}} [ln\, f(v) - ln\, f(u)]$$

$$\leq \left(\frac{ln\, v - ln\, u}{ln\, w - ln\, u}\right)^{\frac{1}{s}} [f(w) - ln\, f(v)], \qquad (4.40)$$

and

$$\frac{(ln\, w - ln\, v)^{\frac{1}{s}}}{(ln\, w - ln\, u)^{\frac{1}{s}}} [ln\, f(v) - ln\, f(u)]$$

$$\leq \frac{(ln\, v - ln\, u)^{\frac{1}{s}}}{(ln\, w - ln\, u)^{\frac{1}{s}}} [f(w) - ln\, f(v)]. \qquad (4.41)$$

Since $u < v < w$ and $s \in (0, 1]$, then Inequality (4.41) can be written as

$$\left(\frac{ln\, f(v) - ln\, f(u)}{(ln\, v - ln\, u)^{\frac{1}{s}}}\right) \leq \left(\frac{ln\, f(w) - ln\, f(v)}{(ln\, w - ln\, v)^{\frac{1}{s}}}\right). \qquad (4.42)$$

Inequality (4.42) completes the proof.

4.5 Summary

This chapter extended Jensen's type of inequalities by considering hybrid convex functions by merging s-convex, h-convex, and g-convex functions. Furthermore, some propositions are proved which will help Jensen's inequality in new refinements for many other inequalities. In Section (4.2), Jensen's inequality is extended by combining s-convex and h-convex functions called hybrid (h_1, h_2, s)-convex function. Similarly, in Section (4.3), Jensen's inequality is extended by combining g-convex and h-convex functions called hybrid (g, h)-convex function. While in Section (4.4), Jensen's inequality is extended by combining s-convex and g-convex functions called hybrid (g, s)-convex function.

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CHAPTER FIVE
FEJÉR'S INEQUALITIES BASED ON HYBRID CONVEX FUNCTIONS

This chapter describes three novel integral forms of Fejér's inequality based on the hybrid of s-convex, h-convex, and g-convex functions. These new inequalities can form new relationship with mathematical means (averages) for real numbers and find error estimation formulae for midpoint formula.

5.1 Introduction

In this chapter, Fejér's inequality is extended based on hybrid convex functions without losing its originality. By applying certain conditions, the formal form, as well as some previous Fejér's inequality can be obtained.

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5.2 Extension of Fejér's Inequality by Merging (h_1, h_2)-convex and s-convex Functions

In this section, several new extensions of Fejér's inequality are formed based on hybrid (h_1, h_2, s)-convex functions in second sense by utilising Definitions (3.2) and (3.5) of hybrid (h_1, h_2, s)-convex functions in second sense.

Firstly, to extend Fejér's inequality for differentiable functions, Lemmas (5.1) and (5.2) are constructed using Equations (3.6) and (3.7) as follows:

Lemma 5.1: *Let* $f, g: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be two differentiable functions on* $\mathbb{I}^{\circ}$ *with* $p, q \in \mathbb{I}$ *and* $p < q$. *If* $f', g' \in L[p, q]$, $\alpha \in [0, 1]$, *and* (h_1, h_2, s)*-convex functions, positive, and symmetric, then*

$$[g(1)-g(0)]\frac{f(p)+f(q)}{2}-\frac{1}{2(q-p)}\int_p^q\left[g'\left(\frac{x-p}{q-p}\right)+g'\left(\frac{q-x}{q-p}\right)\right]dx$$

$$=\left(\frac{q-p}{2}\right)\int_0^1[g(1-a)-g(a)]f'(qa+(1-a)p)da. \qquad (5.1)$$

Proof: Since $p<q$ and $a\in[0,1]$, then by applying integration by parts on right hand side of Equation (5.1), results

$$\int_0^1[g(1-a)-g(a)]f'(qa+(1-a)p)da$$

$$=\left.\frac{[g(1-a)-g(a)]f(qa+(1-a)p)}{q-p}\right|_0^1$$

$$-\int_0^1\frac{[g'(1-a)-g'(a)]f(qa+(1-a)p)}{q-p}da,$$

which can be expanded to

$$=\left[\frac{[g(1-1)-g(1)]f(q(1)+(1-1)p)}{q-p}\right.$$

$$\left.-\frac{[g(1-0)-g(0)]f(q(0)+(1-0)p)}{q-p}\right]$$

$$-\int_0^1\frac{[g'(1-a)-g'(a)]f(qa+(1-a)p)}{q-p}da,$$

which results

$$=\left[\frac{[g(0)-g(1)]f(q)}{q-p}-\frac{[g(1)-g(0)]f(p)}{q-p}\right]$$

$$-\int_0^1\frac{[g'(1-a)-g'(a)]f(qa+(1-a)p)}{q-p}da. \qquad (5.2)$$

Since, $x=aq+(1-a)p$ from Equation (3.6), then when $a=0$ implies $x=p$ and at $a=1$ implies $x=q$.

Without loss of generality, set $a=\frac{q-x}{q-p}$ in Equation (5.2) yields

$$\int_0^1 [g(1-a) - g(a)]f'(qa + (1-a)p)da$$

$$= [g(1) - g(0)]\frac{f(p) + f(q)}{q - p} - \frac{1}{(q-p)^2}$$

$$\times \int_p^q \left[g'\left(\frac{x-p}{q-p}\right) + g'\left(\frac{q-x}{q-p}\right)\right] f(x)dx. \qquad (5.3)$$

By multiplying $\left(\frac{q-p}{2}\right)$ on both sides of Equation (5.3) and using property of symmetry, results

$$[g(1) - g(0)]\frac{f(p) + f(q)}{2} - \frac{1}{2(q-p)}\int_p^q \left[g'\left(\frac{x-p}{q-p}\right) + g'\left(\frac{q-x}{q-p}\right)\right] dx$$

$$= \left(\frac{q-p}{2}\right)\int_0^1 [g(1-a) - g(a)]f'(qa + (1-a)p)da.$$

This completes the proof.

Remarks 5.1:

1. By substituting $g(a) = a$ in Lemma (5.1), Lemma (2.1) will be obtained.

2. When $s = \frac{a^\alpha}{\Gamma(1+\alpha)}$ in Lemma (5.1), Lemma (2.2) will be produced. Here Γ is the Gemma function defined in Lemma (2.2).

Lemma 5.2: *Let* $f, g: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ *be two differentiable functions on* $\mathbb{I}^\circ$ *with* $p, q \in \mathbb{I}$, $p < q$ *and* $a \in [0, 1]$. *If* $f', g' \in L[p, q]$ *and* (h_1, h_2, s)*-convex functions, positive, and symmetric, then*

$$\left| g(0)(f(p) + f(q)) - 2g\left(\frac{1}{2}\right) f\left(\frac{q+p}{2}\right) + \left(\frac{1}{q-p}\right)\left[\int_{\frac{p+q}{2}}^q g'\left(\frac{q-x}{q-p}\right) f(x)dx\right.\right.$$

$$+\int_{p}^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right) f(x)dx\Bigg]\Bigg| = (q-p)\left[\int_{0}^{\frac{1}{2}} g(a)f'(aq+(1-a)p)da\right.$$

$$\left.-\int_{\frac{1}{2}}^{1} g(1-a)f'(aq+(1-a)p)da\right]. \qquad (5.4)$$

Proof: Since $p < q$ and $a \in [0, 1]$, then by applying integration by parts on right hand side of Equation (5.4), results

$$I_3 = \int_{0}^{\frac{1}{2}} g(a)f'(qa+(1-a)p)da = \frac{g(a)f(aq+(1-a)p)}{p-q}\Bigg|_{0}^{\frac{1}{2}}$$

$$-\int_{0}^{\frac{1}{2}} \frac{g'(a)f(aq+(1-a)p)}{p-q}da,$$

which can be expanded to

$$= \left[\frac{g\left(\frac{1}{2}\right) f\left(q\left(\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)p\right)}{p-q}\right.$$

$$\left.-\frac{g(0)f(q(0)+(1-0)p)}{p-q}\right]$$

$$-\int_{0}^{\frac{1}{2}} \frac{g'(a)f(aq+(1-a)p)}{p-q}da,$$

which results to

$$= \left[\frac{g\left(\frac{1}{2}\right) f\left(\frac{p+q}{2}\right)}{(p-q)} - \frac{g(0)f(p)}{p-q}\right]$$

$$-\int_{0}^{\frac{1}{2}} \frac{g'(a)f(aq+(1-a)p)}{p-q}da. \qquad (5.5)$$

Substituting $x = aq + (1-a)p$ from Equation (3.6), taking $a = 0$ implies $x = p$ while $a = \frac{1}{2}$ implies $x = \frac{p+q}{2}$. Set $a = \frac{q-x}{q-p}$ in $\int_0^{\frac{1}{2}} \big(g'(a) f(aq + (1-a)p)\big) da$ from Equation (5.5) results

$$I_3 = \frac{g\left(\frac{1}{2}\right)\left(\frac{p+q}{2}\right)}{q-p} - \frac{g(0)f(p)}{q-p} + \frac{1}{2(q-p)^2} \int_{\frac{p+q}{2}}^{p} g'\left(\frac{q-x}{q-p}\right) f(x)dx. \quad (5.6)$$

Similar to Equation (5.6), by applying integration on $\int_{\frac{1}{2}}^{1} g(1-a) f'(aq + (1-a)p)da$, results

$$I_4 = \int_{\frac{1}{2}}^{1} g(1-a) f'(qa + (1-a)p)da = \frac{g(1-a)f(aq + (1-a)p)}{p-q}\Bigg|_{\frac{1}{2}}^{1}$$

$$- \int_{\frac{1}{2}}^{1} \frac{g'(1-a)f(aq + (1-a)p)}{p-q} da.$$

Since $x = aq + (1-a)p$ form Equation (3.6), then at $a = \frac{1}{2}$ implies $x = \frac{p+q}{2}$ and at $a = 1$ implies $x = q$ and $a = \frac{q-x}{q-p}$, then

$$I_4 = \left[\frac{g\left(\frac{1}{2}\right) f\left(\frac{p+q}{2}\right)}{q-p} - \frac{g(0)f(q)}{q-p}\right] - \frac{1}{2(q-p)^2} \int_{q}^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right) f(x)dx. \quad (5.7)$$

By adding I_3 and I_4 from Equations (5.6) and (5.7) respectively, results

$$= \frac{g\left(\frac{1}{2}\right)\left(\frac{p+q}{2}\right)}{q-p} - \frac{g(0)f(p)}{q-p} + \frac{1}{2(q-p)^2} \int_{\frac{p+q}{2}}^{p} g'\left(\frac{q-x}{q-p}\right) f(x)dx$$

$$+ \frac{g\left(\frac{1}{2}\right) f\left(\frac{p+q}{2}\right)}{q-p} - \frac{g(0)f(q)}{q-p} - \frac{1}{2(q-p)^2} \int_{q}^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right) f(x)dx,$$

which results to

$$[g(0)](f(p)+f(q)) - 2g\left(\frac{1}{2}\right)f\left(\frac{q+p}{2}\right) + \left(\frac{1}{q-p}\right)\left[\int_{\frac{p+q}{2}}^{p} g'\left(\frac{q-x}{q-p}\right)f(x)dx\right.$$

$$\left.+\int_{p}^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right)f(x)dx\right] = \left[\frac{g\left(\frac{1}{2}\right)f(p+q)}{q-p} + \frac{g(0)f(q)}{q-p}\right.$$

$$+\frac{1}{2(q-p)^2}\int_{\frac{p+q}{2}}^{p} g'\left(\frac{q-x}{q-p}\right)f(x)dx$$

$$-\frac{g\left(\frac{1}{2}\right)f(p+q)}{q-p} - \frac{g(0)f(q)}{q-p}$$

$$\left.-\frac{1}{2(q-p)^2}\int_{q}^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right)f(x)dx\right]. \qquad (5.8)$$

As f is (h_1, h_2, s)-convex function, positive, and symmetric then by multiplying $(q-p)$ on both sides of Equation (5.8) and then using modulus and property of symmetry, yields

$$\left|[g(0)](f(p)+f(q)) - 2g\left(\frac{1}{2}\right)f\left(\frac{q+p}{2}\right) + \left(\frac{1}{q-p}\right)\right.$$

$$\left.\times\left[\int_{\frac{p+q}{2}}^{q} g'\left(\frac{q-x}{q-p}\right)f(x)dx + \int_{p}^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right)f(x)dx\right]\right|$$

$$= (q-p)\left[\int_{0}^{\frac{1}{2}} g(a)f'(pa+(1-a)q)da\right.$$

$$\left.-\int_{\frac{1}{2}}^{1} g(1-a)f'(pa+(1-a)q)da\right]. \qquad (5.9)$$

Equation (5.9) completes the proof.

Remarks 5.2:

1. Substituting $g(a) = a$ in Lemma (5.3), then Lemma (2.3) will be produced.

2. By using $s = \frac{a^{\alpha}}{\Gamma(1+\alpha)}$ in Lemma (5.3), then it will be converted to Lemma (2.2).

From now, we will use Definition (3.5) of (h_1, h_2, s)-convex functions in second sense to construct Theorem (5.1). Furthermore, Definition (3.2), Lemma (5.1) and (5.2) will also be utilised to construct Theorems (5.2), (5.3), and (5.4), as follows:

Theorem 5.1: *Let $f, g\colon \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be (h_1, h_2, s)-convex functions in second sense, positive, and symmetric for $p, q \in \mathbb{I}$ with $p < q$, $s \in [0,1]$, and $a \in [0,1]$ with $h_1{}^s\left(\frac{1}{2}\right) h_2{}^s\left(\frac{1}{2}\right) \neq 0$, then*

$$\frac{1}{2h_1{}^{2s}\left(\frac{1}{2}\right) h_2{}^{2s}\left(\frac{1}{2}\right)} f\left(\frac{p+q}{2}\right) g\left(\frac{p+q}{2}\right) - [M(p,q)$$
$$\times \int_0^1 \left[h_1{}^{2s}(a) h_2{}^{2s}(1-a) h_2{}^{2s}(a) h_1{}^{2s}(1-a)\right] da + N(p,q)$$
$$\times \int_0^1 [h_1{}^s(a) h_2{}^s(a) h_1{}^s(1-a) h_2{}^s(1-a)]] \quad \leq \frac{1}{q-p} \int_0^1 f(x) g(x) dx,$$

where $M(p,q) = f(p)g(p) + f(q)g(q)$ and $N(p,q) = f(p)g(q) + f(q)g(p)$.

Proof: Since $p < q$, $s \in [0,1]$, $a \in [0,1]$, and f, g are (h_1, h_2, s)-convex functions in second sense. Similar to Inequality (3.11), set $a = \frac{1}{2}$ and then substituting Equations (3.6) and (3.7) in Inequality (3.5), results

$$f\left(\frac{p+q}{2}\right) \leq h_1{}^s\left(\frac{1}{2}\right) h_2{}^s\left(\frac{1}{2}\right) [f(ap + (1-a)q) + f(aq + (1-a)p)], \quad (5.10)$$

and

$$g\left(\frac{p+q}{2}\right) \leq h_1{}^s\left(\frac{1}{2}\right) h_2{}^s\left(\frac{1}{2}\right) [g(ap + (1-a)q) + g(aq + (1-a)p)]. \quad (5.11)$$

Multiplying Inequalities (5.10) and (5.11), yields

$$f\left(\frac{p+q}{2}\right)g\left(\frac{p+q}{2}\right)$$

$$\leq {h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)[f(ap+(1-a)q)+f(aq+(1-a)p)]$$

$$\times {h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)[g(ap+(1-a)q)+g(aq+(1-a)p)],$$

which can be expanded to

$$f\left(\frac{p+q}{2}\right)g\left(\frac{p+q}{2}\right)$$

$$\leq {h_1}^{2s}\left(\frac{1}{2}\right){h_2}^{2s}\left(\frac{1}{2}\right)[f(ap+(1-a)q)g(ap+(1-a)q)$$

$$+f(aq+(1-a)p)g(aq+(1-a)p)$$

$$+f(ap+(1-a)q)g(aq+(1-a)p)$$

$$+f(aq+(1-a)p)g(ap+(1-a)q)]. \tag{5.12}$$

Since f and g are (h_1, h_2, s)-convex functions in second sense, then by utilising Inequality (3.5), Inequality (5.12) will be expanded to

$$f\left(\frac{p+q}{2}\right)g\left(\frac{p+q}{2}\right) \leq {h_1}^{2s}\left(\frac{1}{2}\right){h_2}^{2s}\left(\frac{1}{2}\right)[f(ap+(1-a)q)g(ap+(1-a)q)$$

$$+f(aq+(1-a)p)g(aq+(1-a)p)$$

$$+\left({h_1}^s(a){h_2}^s(1-a)f(p)+{h_1}^s(1-a){h_2}^s(a)f(q)\right)$$

$$\times\left({h_1}^s(a){h_2}^s(1-a)g(q)+{h_1}^s(1-a){h_2}^s(a)g(p)\right)$$

$$+\left({h_1}^s(a){h_2}^s(1-a)f(q)+{h_1}^s(1-a){h_2}^s(a)f(p)\right)$$

$$\times\left({h_1}^s(a){h_2}^s(1-a)g(p)+{h_1}^s(1-a){h_2}^s(a)g(q)\right)],$$

which results to

$$f\left(\frac{p+q}{2}\right)g\left(\frac{p+q}{2}\right) \leq {h_1}^{2s}\left(\frac{1}{2}\right){h_2}^{2s}\left(\frac{1}{2}\right)[f(ap+(1-a)q)g(ap+(1-a)q)$$

$$+f(aq+(1-a)p)g(aq+(1-a)p)$$

$$+\left(h_1{}^s(a)h_2{}^s(1-a)\right)^2\left(f(p)g(q)\right)+h_1{}^s(a)h_2{}^s(1-a)$$

$$\times\, h_2{}^s(a)h_1{}^s(1-a)\left(f(p)g(p)\right)+h_1{}^s(a)h_2{}^s(1-a)$$

$$\times\, h_2{}^s(a)h_1{}^s(1-a)\left(f(q)g(q)\right)+\left(h_1{}^s(1-a)h_2{}^s(a)\right)^2$$

$$\times\left(f(q)g(p)\right)+\left(h_1{}^s(a)h_2{}^s(1-a)\right)^2\left(f(q)g(p)\right)$$

$$+h_1{}^s(a)h_2{}^s(1-a)h_2{}^s(a)h_1{}^s(1-a)\left(f(q)g(q)\right)$$

$$+h_1{}^s(a)h_2{}^s(1-a)h_2{}^s(a)h_1{}^s(1-a)\left(f(p)g(p)\right)$$

$$+\left(h_1{}^s(1-a)h_2{}^s(a)\right)^2\left(f(p)g(q)\right)\Big].$$

The inequality can now be reduced to

$$\begin{aligned}
f\left(\frac{p+q}{2}\right)g\left(\frac{p+q}{2}\right) \le\ & h_1{}^{2s}\left(\frac{1}{2}\right)h_2{}^{2s}\left(\frac{1}{2}\right)[f(ap+(1-a)q)g(ap+(1-a)q) \\
& +f(aq+(1-a)p)g(aq+(1-a)p) \\
& +2\left[h_1{}^{2s}(a)h_2{}^{2s}(1-a)h_2{}^{2s}(a)h_1{}^{2s}(1-a)\right] \\
& \times[f(p)g(p)+f(q)g(q)] \\
& +2[h_1{}^s(a)h_2{}^s(a)h_1{}^s(1-a)h_2{}^s(1-a)] \\
& \times[f(p)g(q)+f(q)g(p)]\Big]. \qquad (5.13)
\end{aligned}$$

Since $M(p,q)=f(p)g(p)+f(q)g(q)$ and $N(p,q)=f(p)g(q)+f(q)g(p)$, then Inequality (5.13) can be written as

$$\begin{aligned}
f\left(\frac{p+q}{2}\right)g\left(\frac{p+q}{2}\right) \le\ & h_1{}^{2s}\left(\frac{1}{2}\right)h_2{}^{2s}\left(\frac{1}{2}\right)[f(ap+(1-a)q)g(ap+(1-a)q) \\
& +f(aq+(1-a)p)g(aq+(1-a)p) \\
& +2\left[h_1{}^{2s}(a)h_2{}^{2s}(1-a)h_2{}^{2s}(a)h_1{}^{2s}(1-a)\right]M(p,q) \\
& +2[h_1{}^s(a)h_2{}^s(a)h_1{}^s(1-a)h_2{}^s(1-a)]N(p,q)]. \qquad (5.14)
\end{aligned}$$

Similar to Inequality (3.12), apply integration with respect to a where $a\in[0,1]$ on Inequality (5.14), results

$$\int_0^1 f\left(\frac{p+q}{2}\right) g\left(\frac{p+q}{2}\right) da \leq 2{h_1}^{2s}\left(\frac{1}{2}\right){h_2}^{2s}\left(\frac{1}{2}\right)\left[\frac{1}{q-p}\int_0^1 f(x)g(x)dx\right.$$

$$+M(p,q)\int_0^1 \left[{h_1}^{2s}(a){h_2}^{2s}(1-a){h_2}^{2s}(a){h_1}^{2s}(1-a)\right]da$$

$$\left.+N(p,q)\int_0^1 [{h_1}^{s}(a){h_2}^{s}(a){h_1}^{s}(1-a){h_2}^{s}(1-a)]\,da\right]. \quad (5.15)$$

Inequality (5.15) can now be written as

$$\frac{1}{2{h_1}^{2s}\left(\frac{1}{2}\right){h_2}^{2s}\left(\frac{1}{2}\right)} f\left(\frac{p+q}{2}\right) g\left(\frac{p+q}{2}\right) - [M(p,q)$$

$$\times \int_0^1 \left[{h_1}^{2s}(a){h_2}^{2s}(1-a){h_2}^{2s}(a){h_1}^{2s}(1-a)\right]da + N(p,q)$$

$$\left.\times \int_0^1 [{h_1}^{s}(a){h_2}^{s}(a){h_1}^{s}(1-a){h_2}^{s}(1-a)]\right] \leq \frac{1}{q-p}\int_0^1 f(x)g(x)dx. \quad (5.16)$$

Inequality (5.16) completes the proof.

Note that Theorem (5.1) successfully extended Fejér's inequality by using the hybrid of (h_1, h_2, s)-convex function of the second sense in Definition (3.5).

In the following theorems, we will extend Fejér's type of inequality by considering hybrid (h_1, h_2, s)-convex function in second sense as defined in Definition (3.2).

From now, we will use Definition (3.2) of (h_1, h_2, s)-convex function in second sense in terms of first derivative for $s \in [0,1]$ and ${h_1}^{s}(1-a) + {h_2}^{s}(a) = 1$ with $h_1(1-a) \not\equiv 0$ and $h_2(a) \not\equiv 0$ as follows:

$$f'(aq + (1-a)q) \leq {h_1}^{s}(1-a)f'(q) + {h_2}^{s}(a)f'(p). \quad (5.17)$$

Based on Inequality (5.17), Fejér's inequality is considered to extend in Theorems (5.2), (5.3), (5.4), and (5.5), as follows:

Theorem 5.2: *Let $f, g: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be two differentiable, positive, and symmetric functions on $\mathbb{I}^\circ$ where $p, q \in \mathbb{I}$ with $p < q$, $s \in [0,1]$, and $a \in [0,1]$. If $f' \in L[p,q]$ be a (h_1, h_2, s)-convex function in second sense, g be increasing, positive, and symmetric function on $\mathbb{I}$, then*

$$\left|[g(1)-g(0)]\left(\frac{f(p)+f(q)}{2}\right) - \frac{1}{2(q-p)}\int_p^q \left[g'\left(\frac{x-p}{q-p}\right)+g'\left(\frac{q-x}{q-p}\right)\right]f(x)dx\right|$$

$$\leq \left(\frac{q-p}{2}\right)\left\{\int_0^{\frac{1}{2}}[g(1-a)-g(a)][{h_1}^s(a)|f'(p)|+{h_2}^s(1-a)|f'(q)|]da + \int_{\frac{1}{2}}^1 [g(a)-g(1-a)][{h_1}^s(a)|f'(p)|+{h_2}^s(1-a)|f'(q)|]da\right\}.$$

Proof: Since $p < q$, $s \in [0,1]$, $a \in [0,1]$, and $|f'|$ is a (h_1, h_2, s)-convex function in second sense, then by substituting Inequality (5.17) in Lemma (5.1) results

$$[g(1)-g(0)]\frac{f(p)+f(q)}{2} - \frac{1}{2(q-p)}\int_p^q \left[g'\left(\frac{x-p}{q-p}\right)+g'\left(\frac{q-x}{q-p}\right)\right]dx$$

$$\leq \left(\frac{q-p}{2}\right) \times \int_0^1 [g(1-a)-g(a)][{h_1}^s(a)|f'(p)|+{h_2}^s(1-a)|f'(q)|]da. \tag{5.18}$$

By using modulus and then using $\int_0^1 da = \int_0^{\frac{1}{2}} da + \int_{\frac{1}{2}}^1 da$ in right hand side of Inequality (5.18), yields

$$\left|[g(1)-g(0)]\left(\frac{f(p)+f(q)}{2}\right)\right.$$

$$-\frac{1}{2(q-p)}\int_p^q \left[g'\left(\frac{x-p}{q-p}\right)+g'\left(\frac{q-x}{q-p}\right)\right]f(x)dx\Bigg|$$

$$\leq \left(\frac{q-p}{2}\right)\left\{\int_0^{\frac{1}{2}}[g(1-a)-g(a)][h_1{}^s(a)|f'(p)|+h_2{}^s(1-a)|f'(q)|]da\right.$$

$$\left.+\int_{\frac{1}{2}}^1[g(a)-g(1-a)][h_1{}^s(a)|f'(p)|+h_2{}^s(1-a)|f'(q)|]da\right\}. \qquad (5.19)$$

Inequality (5.19) completes the proof.

Corollary 5.1: Suppose Theorem (5.2) with $g(a)=a$, then Inequality (5.19) becomes

$$\left|\left(\frac{f(p)+f(q)}{2}\right)-\frac{1}{(q-p)}\int_p^q f(x)\,dx\right|\leq\left(\frac{q-p}{2}\right)(|f'(p)|+|f'(q)|)$$

$$\times\int_0^{\frac{1}{2}}[1-2a][h_1{}^s(a)+h_2{}^s(1-a)]da.$$

Remarks 5.3:

1. By using $h_1{}^s(a)=a$, and $h_2{}^s(a)=a$ in Corollary (5.1), then it will be reduced to Lemma (2.1).
2. By substituting $s=1$ in Corollary (5.1), results for (h_1,h_2)-convex function will be produced.

Corollary 5.2: In Theorem (5.2), replace $g(a)=\frac{a^\alpha}{\Gamma(1+\alpha)}$, becomes

$$\left|\left(\frac{f(p)+f(q)}{2}\right)-\frac{\Gamma(1+\alpha)}{2(q-p)^\alpha}\left[J_p^\alpha f(q)+J_q^\alpha f(p)\right]\right|\leq\left(\frac{q-p}{2\Gamma(1+\alpha)}\right)$$

$$\times(|f'(p)|+|f'(q)|)\int_0^{\frac{1}{2}}[(1-a)^\alpha-a^\alpha][h_1{}^s(a)+h_2{}^s(1-a)]da.$$

Remarks 5.4:

1. By substituting $h_1{}^s(a) = a$, and $h_2{}^s(a) = a$ in Corollary (5.2), it will be reduced to Inequality for (h_1, h_2, s)-convex function.
2. By using $s = 1$ in Corollary (5.2), then the results for (h_1, h_2)-convex function will be produced.

Theorem 5.3: *Let $f, g: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ be two differentiable, positive, and symmetric functions on $\mathbb{I}^\circ$ where $p, q \in \mathbb{I}$ with $p < q$, $s \in [0,1]$, and $a \in [0,1]$. Let $|f'|$ be a (h_1, h_2, s)-convex function in second sense and g be increasing, positive, and symmetric function with $s \in [0,1]$ and $a \in [0,1]$, then*

$$\left| g(0)\big(f(p)+f(q)\big) - 2g\left(\frac{1}{2}\right) f'\left(\frac{p+q}{2}\right) + \left(\frac{1}{q-p}\right)\left[\int_{\frac{p+q}{2}}^{q} g'\left(\frac{x-p}{q-p}\right) f(x)dx + \int_{p}^{\frac{p+q}{2}} g'\left(\frac{q-x}{q-p}\right) f(x)dx\right]\right| \le (q-p)(|f'(p)| + |f'(q)|) \times \int_0^{\frac{1}{2}} [g(a)][h_1{}^s(a) + h_2{}^s(1-a)]da.$$

Proof: Since $p < q$, $s \in [0,1]$, $a \in [0,1]$, and $|f'|$ as (h_1, h_2, s)-convex function in second sense, then by utilising Inequality (5.17) in Lemma (5.2), results

$$\left| [g(0)]\big(f(p)+f(q)\big) - 2g\left(\frac{1}{2}\right) f\left(\frac{q+p}{2}\right) + \left(\frac{1}{q-p}\right)\left[\int_{\frac{p+q}{2}}^{q} g'\left(\frac{q-x}{q-p}\right) f(x)dx + \int_{p}^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right) f(x)dx\right]\right| \le (q-p)\left[\int_0^{\frac{1}{2}} g(a) \times \left[h_1{}^s(a)|f'(p)| + \big(h_2{}^s(1-a)\big)|f'(q)|\right]da\right.$$

$$+\int_{\frac{1}{2}}^{1} g(1-a)$$

$$\times\left[{h_1}^s(a)|f'(p)|+\left({h_2}^s(1-a)\right)|f'(q)|\right]da\Big],$$

(5.20)

which can be written as

$$=(q-p)[|f'(p)|+|f'(q)|]$$

$$\times\left[\int_0^{\frac{1}{2}} g(a)\left[{h_1}^s(a)+\left({h_2}^s(1-a)\right)\right]da\right]. \quad (5.21)$$

Inequality (5.21) completes the proof.

Corollary 5.3: Suppose Theorem (5.3) with $g(a)=a$, then Inequality (5.21) becomes

$$\left|\left(\frac{1}{q-p}\right)\left[\int_p^q f(x)dx-f\left(\frac{p+q}{2}\right)\right]\right|\le(q-p)(|f'(p)|+|f'(q)|)$$

$$\times\int_0^{\frac{1}{2}} a[{h_1}^s(a)+{h_2}^s(1-a)]da.$$

Remark 5.5:

By substituting ${h_i}^s(a)=a$ for all $i=1,2$ in Corollary (5.3), then Lemma (1) defined in Kirmaci (2004) will be obtained.

Theorem 5.4: *Let* $f:\mathbb{I}\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable, positive, and symmetric function on* $\mathbb{I}^\circ$ *where* $p,q\in\mathbb{I}$ *with* $p<q$, $s\in[0,1]$, *and* $a\in[0,1]$. *If* $|f'|$ *be* (h_1,h_2,s)*-convex function in second sense,* g *be increasing, positive, and symmetric function on* $\mathbb{I}$ *for all* $p,q\in\mathbb{I}$ *with* $p<q$, $s\in[0,1]$, *and* $a\in[0,1]$, *then*

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$$\left|\left(\frac{f(p)+f(q)}{2}\right)\int_p^q g(x)dx - \int_p^q g(x)f(x)dx \leq (q-p)(|f'(p)|+|f'(q)|)\right.$$

$$\times \int_p^{\frac{p+q}{2}} \int_0^{\frac{x-p}{q-p}} g(x)[{h_1}^s(a)+{h_2}^s(1-a)]dxda.$$

Proof: Since $p < q$, $s \in [0,1]$, $a \in [0,1]$, and $|f'|$ is (h_1, h_2, s)-convex function in the second sense. As g is an increasing and symmetric function, then by utilising $\int_0^1 da = \int_0^{\frac{1}{2}} da + \int_{\frac{1}{2}}^1 da$ in Lemma (2.11), results

$$\left|\left(\frac{f(p)+f(q)}{2}\right)\int_p^q g(x)dx - \int_p^q g(x)f(x)dx\right|$$

$$\leq \frac{(q-p)^2}{2}\left\{\int_0^{\frac{1}{2}} |M_g(a)||f'(pa+q(1-a))|da\right.$$

$$\left. + \int_{\frac{1}{2}}^1 |M_g(a)||f'(pa+q(1-a))|da\right\}, \qquad (5.22)$$

where $M_g(a)$ is defined in Lemma (2.11).

As $|f'|$ is (h_1, h_2, s)-convex function in the second sense, then by substituting values for $M_g(a)$ from Lemma (2.11) and then applying Inequality (5.17) on Inequality (5.22), results

$$\left|\left(\frac{f(p)+f(q)}{2}\right)\int_p^q g(x)dx - \int_p^q g(x)f(x)dx\right|$$

$$\leq \frac{(q-p)^2}{2}\left\{\int_0^{\frac{1}{2}} \left|\int_a^1 g(\alpha p+(1-\alpha)q)\right| d\alpha\right.$$

$$\times \left|h_1{}^s(a)f'(p) + h_2{}^s(1-a)f'(q)\right|da$$

$$+\int_{\frac{1}{2}}^{1}\left|\int_{0}^{\alpha} g(\alpha p + (1-\alpha)q)\right|d\alpha$$

$$\times \left|h_1{}^s(a)f'(p) + h_2{}^s(1-a)f'(q)\right|da\}. \qquad (5.23)$$

Similar to Inequality (5.23), using interval $\frac{1}{2} \leq \alpha \leq 1$ for $M_g(a)$ from Lemma (2.11) on Inequality (5.22), yields

$$\left|\left(\frac{f(p)+f(q)}{2}\right)\int_p^q g(x)dx \quad -\int_p^q g(x)f(x)dx\right|$$

$$= \frac{(q-p)^2}{2}\left|\int_0^1 M_g(a)[f'(pa+q(1-a))]da\right|,$$

$$\leq (q-p)^2\left\{2\int_0^{\frac{1}{2}}\left|\int_0^{\alpha} g(\alpha p + (1-\alpha)q)\right|d\alpha\right.$$

$$\times \left|h_1{}^s(a)f'(p) + h_1{}^s(1-a)f'(q)\right|da$$

$$+\int_{\frac{1}{2}}^{1}\left|\int_{\alpha}^{1} g(\alpha p + (1-\alpha)q)\right|d\alpha$$

$$\times \left|h_1{}^s(a)f'(p) + h_1{}^s(1-a)f'(q)\right|da\}. \quad (5.24)$$

Since $x = ap + (1-a)q$ and by applying Inequality (5.17) on Inequality (5.24), results

$$\left|\left(\frac{f(p)+f(q)}{2}\right)\int_p^q g(x)dx - \int_p^q g(x)f(x)dx\right|$$

$$= \frac{(q-p)^2}{2}\left|\int_0^1 M_g(a)[f'(pa+q(1-a))]da\right|,$$

$$\leq (q-p)^2 \left\{ \int_{\frac{p+q}{2}}^{q} \left| \int_{0}^{\frac{q-x}{q-p}} g(x) \right| |h_1{}^s(a) f'(p) \right.$$

$$+h_2{}^s(1-a)f'(q)|dadx$$

$$+ \int_{0}^{\frac{p+q}{2}} \left| \int_{\frac{q-x}{q-p}}^{1} g(x) \right| |h_1{}^s(a) f'(p)$$

$$\left. +h_2{}^s(1-a)f'(q)| \right\} dadx. \tag{5.25}$$

Since g is a symmetric and positive function for $\frac{p+q}{2}$ and $h_1 = h_2$, then right hand side of Inequality (5.25) can be written as

$$\int_{\frac{p+q}{2}}^{q} \int_{0}^{\frac{q-x}{q-p}} g(x) \, |h_1{}^s(a) f'(p) + h_2{}^s(1-a) f'(q)| dadx$$

$$= \int_{\frac{p+q}{2}}^{p} \int_{0}^{\frac{p-x}{p-q}} g(x) \, |h_1{}^s(a) f'(q) + h_2{}^s(1-a) f'(p)| dadx, \tag{5.26}$$

and

$$\int_{0}^{\frac{q-x}{q-p}} |h_1{}^s(1-a)| da = \int_{\frac{q-x}{q-p}}^{1} |h_1{}^s(1-a)| da, \tag{5.27}$$

$$\int_{0}^{\frac{q-x}{q-p}} |h_2{}^s(a)| da = \int_{\frac{q-x}{q-p}}^{1} |h_2{}^s(1-a)| da. \tag{5.28}$$

By substituting values from Equations (5.26), (5.27), and (5.28) in Inequality (5.25), results

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$$\left(\frac{f(p)+f(q)}{2}\right)\int_p^q g(x)dx - \int_p^q g(x)f(x)dx \le (q-p)$$

$$\times\left(|f'(p)|+|f'(q)|\right)\int_p^{\frac{p+q}{2}}\int_0^{\frac{x-p}{q-p}} g(x)\left[{h_1}^s(a)+{h_2}^s(1-a)\right]dxda. \quad (5.29)$$

Inequality (5.29) completes the proof.

Remark 5.6:

1. By using ${h_1}^s(a) = h(a)$ and ${h_2}^s(a) = h(a)$ in Theorem (5.4), Fejér's inequality for (h, s)-convex function will be obtained.
2. By substituting $s = 1$ in Theorem (5.4), results for (h_1, h_2)-convex function will be produced.

Theorem 5.5: *Let $f: \mathbb{I} = [p, q] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable, positive, and symmetric function. Let g be increasing, positive, and symmetric function on $\mathbb{I}$ for all $p, q \in \mathbb{I}$ with $p < q$, $s \in [0,1]$, and $a \in [0,1]$. If $|f'|$ be the (h_1, h_2, s)-convex function in the second sense for any positive integer k, then*

$$\left|\int_p^q g(x)f(x)dx - \left(\frac{f(p)+f(q)}{2}\right)\int_p^q g(x)dx\right|$$

$$\le \left(\frac{q-p}{k}\right)\left(\frac{|f'(p)|+|f'(q)|}{2}\right)\int_0^k \int_p^{\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q} g(x)$$

$$\times\left[{h_1}^s\left(\frac{k+a}{2k}\right)+{h_2}^s\left(\frac{k-a}{2k}\right)\right]dxda.$$

Proof: Since $p < q$, $s \in [0,1]$, $a \in [0,1]$, and $|f'|$ is (h_1, h_2, s)-convex function in second sense with g is increasing and symmetric function, then by utilising $g_q(x) = \int_p^q g(x)dx$ and $g_p(x) = 0$ in Lemma (2.8), results

$$g(p)\left(\frac{f(p)+f(q)}{2}\right)-g(p)f\left(\frac{p+q}{2}\right)$$

$$+\left(\frac{q-p}{4k}\right)\left[\int_0^k\left\{g'\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right.\right.$$

$$\left.+g'\left(p+q-\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right)\right]da$$

$$\times\left\{\int_0^k f\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right.$$

$$\left.+f\left(p+q-\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right)da\right\}$$

$$=\left(\frac{q-p}{4k}\right)\int_0^k\left\{g\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right.$$

$$\left.+g\left(p+q-\left[\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right]\right)\right\}da$$

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$$\times f\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)$$

$$+f\left(p+q-\left[\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right]\right)da,$$

which can be expanded to

$$=\left(\frac{q-p}{4k}\right)\int_0^k\left\{g\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right.$$

$$\left.-g\left(p+q-\left[\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right]\right)+g_q(q)\right\}da$$

$$\times f'\left\{\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right.$$

$$-f'\left(p+q-\left[\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right]\right)\Bigg\}da. \tag{5.30}$$

Since g is a symmetric function for $\left(\frac{p+q}{2}\right)$, then Equation (5.30) can be written as

$$g(p)\left(\frac{f(p)+f(q)}{2}\right)-g(p)f\left(\frac{p+q}{2}\right)$$

$$+\left(\frac{q-p}{4k}\right)\left\{\int_0^k\left\{g'\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right.\right.$$

$$\left.+g'\left(p+q-\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right)\right\}da$$

$$\times\int_0^k f\left\{\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right.$$

$$\left.\left.+f\left(p+q-\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right)\right\}da\right\}$$

$$=\left(\frac{q-p}{2k}\right)\int_0^k\left\{g\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)f\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right\}da$$

$$+\left(\frac{q-p}{2k}\right)\int_0^k\left\{g\left(p+q-\left[\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right]\right)\right.$$

$$\left.\times f\left(p+q-\left[\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right]\right)\right\}da. \tag{5.31}$$

Since, $\int_u^v\{g(x)f(x)\}=\int_u^c\{g(x)f(x)\}+\int_c^v\{g(x)f(x)\}$, then for each $a\in[0,1]$, we have

$$\int_p^{\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q} g(x)dx=\int_{\left(p+q-\left[\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right]\right)}^{q} g(x)dx. \tag{5.32}$$

By using Equation (5.32) in Equation (5.31), results

$$\left(\frac{q-p}{4k}\right)\int_0^k \left\{g\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right.$$

$$+g\left(p+q-\left[\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right]\right)+g_q(x)\}$$

$$\times\left\{f'\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)+f'\left(p+q-\left[\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right]\right)\right\}\,da$$

$$=\left(\frac{q-p}{2k}\right)\int_0^k \left\{\int_p^{\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q} g(x)\right\}dx$$

$$\times\left\{f'\left(p+q-\left[\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right]\right)\right.$$

$$\left.-f'\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right\}da, \tag{5.33}$$

and

$$\int_p^q g(x)f(x)dx$$

$$-\left(\frac{f(p)+f(q)}{2}\right)\int_p^q g(x)dx \quad =\left(\frac{q-p}{k}\right)\int_0^k g(x)dx\,\{f'(p+q)$$

$$-\left[\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right]$$

$$\left.+f'\left(\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q\right)\right\}da. \tag{5.34}$$

Since $|f'(p)+f'(q)|\leq|f'(p)|+|f'(q)|$, then using Equation (5.33) and then using Inequality (5.17) in Equation (5.34), results

$$\left|\int_p^q g(x)f(x)dx-\left(\frac{f(p)+f(q)}{2}\right)\int_p^q g(x)dx\right|\leq\left(\frac{q-p}{2k}\right)\left(\frac{|f'(p)|+|f'(q)|}{2}\right)$$

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$$\times \int_0^k \int_p^{\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q} g(x)\left[{h_1}^s\left(\frac{k+a}{2k}\right)+{h_2}^s\left(\frac{k-a}{2k}\right)\right]dxda. \qquad (5.35)$$

Inequality (5.35) completes the proof.

Note that Theorems (5.1), (5.2), (5.3), (5. 4), and (5.5) successfully extended Fejér's type of inequality by using hybrid (h_1, h_2, s)-convex function in the second sense. While, if we impose the condition $s = \frac{1}{s}$ for $s \in (0, 1]$ with $h_1^s(a) + h_2^s(1 - a) = 1$ and $h_1(a) + h_2(1 - a) = 1$, then Fejér's type of inequalities for (h_1, h_2, s)-convex function in the third and fourth sense can be obtained.

Furthermore, in proceeding subsection, the concept of (h_1, h_2, s, m)-harmonically convex function in the second sense defined in Definition (3.8) is used. Furthermore, Fejér's inequality is considered to extend by using Definition (3.8).

5.2.1 Extension of Fejér's Inequality by Merging (h_1, h_2, s, m)-convex and Harmonically Convex Functions

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In this subsection, another new extension of Fejér's inequality is formed based on hybrid of (h_1, h_2, s, m)-harmonically convex function in the second sense defined in Definition (3.7) is used.

From now, we will use Equations (3.38), (3.39) and Inequality (3.40) to extend Fejér's inequality in the following Theorem (5.6):

Theorem 5.6: *Let* $h_1, h_2: [0, 1] \to \mathbb{R}$ *such that* $h_1, h_2 \not\equiv 0$. *Let* $f, g: \mathbb{I} \to \mathbb{R}_+$ *be* (h_1, h_2, s, m)*-harmonically convex functions in the second sense, positive, and symmetric for all* $x_1, p, q \in \mathbb{I}$ *with* $p < q$, $s \in [0, 1]$, $m \in [0, 1]$, *and* $a \in [0, 1]$, *then*

$$\left(\frac{p-q}{pq}\right) f\left(\frac{2pq}{p+q}\right) g\left(\frac{2pq}{p+q}\right) \leq \left[h_1^s\left(\frac{1}{2}\right)\right]^2 \int_p^q \frac{f(x_1)g(x_1)}{x_1^2} dx_1$$

$$+mh_1^s\left(\frac{1}{2}\right) h_2^s\left(\frac{1}{2}\right) \int_p^q \frac{g(x_1 m)f(x_1)}{x_1^2} dx_1$$

$$+mh_1^s\left(\frac{1}{2}\right) h_2^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right) \int_p^q \frac{f(x_1 m)g(x_1)}{x_1^2} dx_1$$

$$+\left[h_2^s\left(\frac{1}{2}\right)\right]^2 \int_p^q \frac{g(x_1 m)f(x_1 m)}{x_1^2} dx_1. \qquad (5.36)$$

Proof: Since $p < q$, $s \in [0,1]$, $m \in [0,1]$, $a \in [0,1]$, and f, g be (h_1, h_2, s, m)-harmonically convex functions in the second sense, then similar to Inequality (3.42), by utilising $a = \frac{1}{2}$ and then substituting Equations (3.38) and (3.39) in Inequality (3.40), yields

$$f\left(\frac{2pq}{p+q}\right) g\left(\frac{2pq}{p+q}\right) \leq \left[h_1^s\left(\frac{1}{2}\right) f\left(\frac{pqm}{ap+(1-a)q}\right)\right.$$

$$\left.+mh_2^s\left(\frac{1}{2}\right) f\left(\frac{pqm}{ap+(1-a)q}\right)\right]$$

$$\times\left[h_1^s\left(\frac{1}{2}\right) g\left(\frac{pq}{ap+(1-a)q}\right)\right.$$

$$\left.+mh_2^s\left(\frac{1}{2}\right) g\left(\frac{pq}{ap+(1-a)q}\right)\right],$$

Which can be expanded to

$$\leq \left[\left[h_1^s\left(\frac{1}{2}\right)\right]^2 f\left(\frac{pqm}{ap+(1-a)q}\right) g\left(\frac{pq}{ap+(1-a)q}\right)\right.$$

$$+mh_1^s\left(\frac{1}{2}\right) h_2^s\left(\frac{1}{2}\right) f\left(\frac{pq}{ap+(1-a)q}\right) g\left(\frac{pqm}{ap+(1-a)q}\right)$$

$$+mh_1^s\left(\frac{1}{2}\right) h_2^s\left(\frac{1}{2}\right) f\left(\frac{pqm}{ap+(1-a)q}\right) g\left(\frac{pq}{ap+(1-a)q}\right)$$

$$+\left[mh_2^s\left(\frac{1}{2}\right)\right]^2$$

$$\times f\left(\frac{pqm}{ap+(1-a)q}\right)g\left(\frac{pqm}{ap+(1-a)q}\right)\Bigg]. \tag{5.37}$$

Similar to Equations (3.47) and (3.50), integrating Inequality (5.37) on both sides with respect to a where $a \in [0,1]$, results

$$\left(\frac{p-q}{pq}\right)f\left(\frac{2pq}{p+q}\right)g\left(\frac{2pq}{p+q}\right) \leq \left[h_1^s\left(\frac{1}{2}\right)\right]^2 \int_p^q \frac{f(x_1)g(x_1)}{x_1^2}dx_1 + mh_1^s\left(\frac{1}{2}\right)h_2^s\left(\frac{1}{2}\right)$$

$$\times \int_p^q \frac{g(x_1m)f(x_1)}{x_1^2}dx_1 + mh_1^s\left(\frac{1}{2}\right)h_2^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)$$

$$\times \int_p^q \frac{f(x_1m)g(x_1)}{x_1^2}dx_1$$

$$+\left[h_2^s\left(\frac{1}{2}\right)\right]^2 \int_p^q \frac{g(x_1m)f(x_1m)}{x_1^2}dx. \tag{5.38}$$

Inequality (5.28) completes the proof.

Note that Theorem (5.6) successfully extended Fejér's type of inequality by using hybrid (h_1, h_2, s, m)-harmonically convex function in the second sense. While, if we impose the condition $s = \frac{1}{s}$ for $s \in (0,1]$ with $h_1^s(a) + h_2^s(1-a) = 1$ and $h_1(a) + h_2(1-a) = 1$, then Fejér's type of inequality for (h_1, h_2, s, m)-harmonically convex function in the third and fourth sense can be obtained.

Furthermore, in the proceeding section, Definition (3.9) of (g, s)-convex function in the third sense is utilised to extend Fejér's inequality. In addition, we have created a link between mathematical means (averages) with Fejér's inequality.

5.3 Extension of Fejér's Inequality by Merging g-convex and s-convex Functions

In this section, another hybrid convex function called (g, s)-convex function in the third sense defined in Definition (3.9) is used to extend Fejér's inequality.

From now, we will use Equations (3.58), (3.59), and Inequality (3.86) to extend Fejér's type of inequality in Theorems (5.7) and (5.8), as follows:

Theorem 5.7: *Let $f, g: \mathbb{I} \rightarrow \mathbb{R}$ be two (g, s_1)-convex function and (g, s_2)-convex functions in the third sense, positive and symmetric, with $x_2, p, q \in \mathbb{I}$, $p < q$, $a \in [0, 1]$, and integrable on $s_1, s_2 \in (0, 1]$, then*

$$\left(\sqrt{pq}\right) f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_1}\right)}}\left(\sqrt{pq}\right)\left(\sqrt{pq}\right) g^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_2}\right)}}\left(\sqrt{pq}\right)$$
$$\leq \frac{1}{\ln q - \ln p} \int_p^q f(x_2) g(x_2) dx_2$$
$$\leq [f(p)f(q)]^{1-\left(\frac{1}{s_1}\right)} [f(p)f(q)]^{1-\left(\frac{1}{s_2}\right)}$$
$$L\left(pf^{\left(\frac{1}{s_1}\right)}(p) g^{\left(\frac{1}{s_2}\right)}(p), qf^{\left(\frac{1}{s_1}\right)}(q) g^{\left(\frac{1}{s_2}\right)}(q)\right),$$

where the special mean $L(p, q)$ is defined as

$$L(p,q) = \begin{cases} \dfrac{q-p}{\ln q - \ln p}, & q \neq p, \\ p, & q = p. \end{cases}$$

Proof: Since $x_2, p, q \in \mathbb{I}$, $p < q$, $a \in [0, 1]$, and f be a (g, s)-convex function in the third sense. Taking derivative of x_2 from Equation (3.58) with respect to a, we have dx_2 in Equation (3.65).

By multiplying $g(x_2) \not\equiv 0$ on both sides of Inequality (3.75) with $x_2 = p^{(1-a)} q^a$ from Equation (3.58) and then apply Lemma (2.4) on Inequality (3.75), yields

$$\left(\frac{1}{\ln q - \ln p}\right)\int_p^q f(x_2)g(x_2)dx_2 \quad = p^{(1-a)}q^a \int_0^1 f\left(p^{(1-a)}q^a\right)g\left(p^{(1-a)}q^a\right) da,$$

$$\leq p^{(1-a)}q^a \int_0^1 f^{(1-a)^{\left(\frac{1}{s_1}\right)}}(p)\, f^{a^{\left(\frac{1}{s_1}\right)}}(q)$$

$$\times\, g^{(1-a)^{\left(\frac{1}{s_2}\right)}}(p) g^{a^{\left(\frac{1}{s_2}\right)}}(q)da \qquad (5.39)$$

Similar as Equation (3.91), by solving $\int_0^1 f^{\left(\frac{1}{s_1}\right)(1-a)}(p)f^{\left(\frac{1}{s_1}\right)a}(q)da$ and $\int_0^1 g^{\left(\frac{1}{s_2}\right)(1-a)}(p)g^{\left(\frac{1}{s_2}\right)a}(q)da$ from right hand side of Inequality (5.39), results

$$\left(\frac{1}{\ln q - \ln p}\right)\int_p^q f(x_2)g(x_2)dx_2 \leq [f(p)f(q)]^{1-\left(\frac{1}{s_1}\right)}[g(p)g(q)]^{1-\left(\frac{1}{s_2}\right)}$$

$$\times L\left(pf^{\left(\frac{1}{s_1}\right)}(p)g^{\left(\frac{1}{s_2}\right)}(p), qf^{\left(\frac{1}{s_1}\right)}(q)g^{\left(\frac{1}{s_2}\right)}(q)\right). \qquad (5.40)$$

Since, $\left(\sqrt{pq}\right) = \left(\sqrt{p^a q^{(1-a)}p^{(1-a)}q^a}\right)$ with f and g are (g, s_1)-convex function and (g, s_2)-convex functions in the third sense, positive and symmetric in terms of p, q, then by substituting $a = \frac{1}{2}$ and then utilising Inequality (3.86) for f, g in Inequality (3.86), results

$$\left(\sqrt{pq}\right)f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_1}\right)}}\left(\sqrt{pq}\right)\left(\sqrt{pq}\right)g^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_2}\right)}}\left(\sqrt{pq}\right)$$

$$\leq \int_0^1 \left(p^a q^{(1-a)}p^{(1-a)}q^a\right)^{\frac{1}{2}}$$

$$\times \left[f\left(p^a q^{(1-a)}\right)f\left(p^{(1-a)}q^a\right)\right.$$

$$\left.\times\, g\left(p^a q^{(1-a)}\right) g\left(p^{(1-a)}q^a\right)\right)\Big]^{\frac{1}{2}}. \qquad (5.41)$$

Similar to Inequality (3.92), by using arithmetic and geometric mean inequality as $\sqrt{pq} \leq \frac{p+q}{2}$, Inequality (5.41) becomes

$$\left(\sqrt{pq}\right)f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_1}\right)}}\left(\sqrt{pq}\right)\left(\sqrt{pq}\right)g^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_2}\right)}}\left(\sqrt{pq}\right)$$

$$\leq \frac{1}{2}\int_0^1 \left[p^a q^{(1-a)} f\left(p^a q^{(1-a)}\right) g\left(p^a q^{(1-a)}\right)\right.$$

$$\left. + p^{(1-a)} q^a f\left(p^{(1-a)} q^a\right) g\left(p^{(1-a)} q^a\right)\right] da.$$

Similar to Inequality (3.88), by solving $\int_0^1 f\left(p^{(1-a)}q^a\right)g\left(p^{(1-a)}q^a\right)da$, results

$$\int_0^1 f\left(p^{(1-a)}q^a\right)g\left(p^{(1-a)}q^a\right)da = \left(\frac{1}{\ln q - \ln p}\right)\int_p^q f(x_2)g(x_2)dx_2. \tag{5.42}$$

By comparing Inequality (5.40) and Inequality (5.42), results

$$\left(\sqrt{pq}\right)f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_1}\right)}}\left(\sqrt{pq}\right)\left(\sqrt{pq}\right)g^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_2}\right)}}\left(\sqrt{pq}\right)$$

$$\leq \frac{1}{\ln q - \ln p}\int_p^q f(x_2)g(x_2)dx_2$$

$$\leq [f(p)f(q)]^{1-\left(\frac{1}{s_1}\right)}[f(p)f(q)]^{1-\left(\frac{1}{s_2}\right)}$$

$$\times L\left(pf^{\left(\frac{1}{s_1}\right)}(p)g^{\left(\frac{1}{s_2}\right)}(p), qf^{\left(\frac{1}{s_1}\right)}(q)g^{\left(\frac{1}{s_2}\right)}(q)\right),$$

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where the special mean $L(p, q)$ is defined in Theorem (5.7).

Theorem 5.8: *Let* $f, g: \mathbb{I} \to \mathbb{R}$ *be a* (g, s_1)*-convex function and* (g, s_2)*-convex functions in the third sense, positive, and symmetric with* $x_2, p, q \in \mathbb{I}$*,* $p < q$*,* $a \in [0, 1]$*, and integrable on* $s_1, s_2 \in (0, 1]$*, then*

$$f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_1}\right)}}\left(\sqrt{pq}\right)g^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_2}\right)}}\left(\sqrt{pq}\right) \leq \frac{1}{\ln q - \ln p}\int_p^q \frac{f(x_2)g(x_2)}{x_2}dx_2$$

$$\leq [f(p)f(q)]^{1-\left(\frac{1}{s_1}\right)}[f(p)f(q)]^{1-\left(\frac{1}{s_2}\right)}$$

$$\times L\left(f^{\left(\frac{1}{s_1}\right)}(p)g^{\left(\frac{1}{s_2}\right)}(p), f^{\left(\frac{1}{s_1}\right)}(q)g^{\left(\frac{1}{s_2}\right)}(q)\right),$$

where the special mean $L(p,q)$ is defined in Theorem (5.7).

Proof: Since $x_2, p, q \in \mathbb{I}$, $p < q$, and f, g be two (g,s)-convex functions in the third sense.

Note that $x_2 = p^{(1-a)}q^a$ from Equation (3.58), then multiply $g(x_2) \not\equiv 0$ on both sides of Inequality (3.71) and then utilising Lemma (2.4) on Inequality (3.75), yields

$$\left(\frac{1}{\ln q - \ln p}\right)\int_p^q \frac{f(x_2)g(x_2)}{x_2}dx_2 \le \int_0^1 f\left(p^{(1-a)}q^a\right)g\left(p^{(1-a)}q^a\right)da,$$

which can be written as

$$\left(\frac{1}{\ln q - \ln p}\right)\int_p^q \frac{f(x_2)g(x_2)}{x_2}dx_2 \le \int_0^1 \left[f^{(1-a)^{\left(\frac{1}{s_1}\right)}}(p)\, f^{a^{\left(\frac{1}{s_1}\right)}}(q)\right.$$

$$\left.\times\, g^{(1-a)^{\left(\frac{1}{s_2}\right)}}(p)\, g^{a^{\left(\frac{1}{s_2}\right)}}(q)\right]da,$$

which can be expanded to

$$\le \int_0^1 f^{a\left(\frac{1}{s_1}\right)+1-\left(\frac{1}{s_1}\right)}(p) f^{a\left(\frac{1}{s_2}\right)+1-\left(\frac{1}{s_2}\right)}(p)$$

$$\times\, g^{(1-a)\left(\frac{1}{s_1}\right)+1-\left(\frac{1}{s_1}\right)}(q) g^{(1-a)\left(\frac{1}{s_2}\right)+1-\left(\frac{1}{s_2}\right)}(q)da. \tag{5.43}$$

Similar as Equation (3.91), by solving $\int_0^1 f^{\left(\frac{1}{s_1}\right)(1-a)}(p)f^{\left(\frac{1}{s_1}\right)a}(q)da$ and $\int_0^1 g^{\left(\frac{1}{s_2}\right)(1-a)}(p)g^{\left(\frac{1}{s_2}\right)a}(q)da$ from right hand side of Inequality (5.43), results

$$= [f(p)f(q)]^{1-\left(\frac{1}{s_1}\right)}[g(p)g(q)]^{1-\left(\frac{1}{s_2}\right)}$$

$$\times\, L\left(f^{\left(\frac{1}{s_1}\right)}(p)g^{\left(\frac{1}{s_2}\right)}(p), f^{\left(\frac{1}{s_1}\right)}(q)g^{\left(\frac{1}{s_2}\right)}(q)\right). \tag{5.44}$$

Since, $\sqrt{pq} = \sqrt{p^a q^{(1-a)} p^{(1-a)} q^a}$ and f, g are positive and symmetric in terms of p, q, then by substituting $a = \frac{1}{2}$ and then utilising Inequality (3.86) for f, g in Inequality (3.86), results

$$f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_1}\right)}}\left(\sqrt{pq}\right) g^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_2}\right)}}\left(\sqrt{pq}\right) \le \int_0^1 \left(f^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_1}\right)}} \left(\sqrt{p^a q^{(1-a)} p^{(1-a)} q^a}\right) da \right.$$

$$\left. \times\, g^{\left(\frac{1}{2}\right)^{\left(1-\frac{1}{s_1}\right)}} \left(\sqrt{p^a q^{(1-a)} p^{(1-a)} q^a}\right)\right),$$

$$\le \frac{1}{2}\int_0^1 \left[f\left(p^a q^{(1-a)}\right) g\left(p^a q^{(1-a)}\right)\right.$$

$$\left. + f\left(p^{(1-a)} q^a\right) g\left(p^{(1-a)} q^a\right)\right] da$$

$$= \left(\frac{1}{\ln q - \ln p}\right) \int_p^q \frac{f(x_2) g(x_2)}{x_2} dx_2. \qquad (5.45)$$

Now by combining Inequality (5.44) and Inequality (5.45), results

$$f^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_1}\right)}}\left(\sqrt{pq}\right) g^{\left(\frac{1}{2}\right)^{1-\left(\frac{1}{s_2}\right)}}\left(\sqrt{pq}\right) \le \frac{1}{\ln q - \ln p} \int_p^q \frac{f(x_2) g(x_2)}{x_2} dx_2$$

$$\le [f(p) f(q)]^{1-\left(\frac{1}{s_1}\right)} [f(p) f(q)]^{1-\left(\frac{1}{s_2}\right)}$$

$$\times\, L\left(f^{\left(\frac{1}{s_1}\right)}(p) g^{\left(\frac{1}{s_2}\right)}(p), f^{\left(\frac{1}{s_1}\right)}(q) g^{\left(\frac{1}{s_2}\right)}(q)\right),$$

where the special mean $L(p, q)$ is defined in Theorem (5.7).

Note that Theorem (5.7) and (5.8) successfully extended Fejér's type of inequality by using hybrid (g, s)-convex function in the third sense defined in Definition (3.9).

Meanwhile in proceeding section, we merge Definition (3.2) with Definition (2.7) to form a new hybrid (h, s, α, m)-convex function in the second sense. Later, Fejér's type of inequality is also extended by using hybrid (h, s, α, m)-convex function in the second sense.

5.4 Extension of Fejér's Inequality by Merging h-convex and s-convex Functions

In this section, another new hybrid convex function called (h, s, α, m)-convex function in the second sense is formed by merging Definition (2.7) and Definition (3.2) as follows:

Definition 5.1: *Let $h: (0,1) \to \mathbb{R}$ such that $h \not\equiv 0$. If function $f: \mathbb{I} \to \mathbb{R}_+$ is a (h, s, α, m)-convex function in the second sense, positive, and symmetric with $s \in [0,1]$, $\alpha \in [0,1]$, $m \in [0,1]$, and $a \in [0,1]$, then*

$$f(ax + m(1-a)y) \leq h^s(a^\alpha)f(x) + mh^s(1-a^\alpha)f(y).$$

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Note that for $\alpha = 1$ and $m = 1$ in Definition (5.1), then (h, s)-convex function defined in Definition (1.10) will be formed. From now, to extend Fejér's type of inequality, we will use

$$p_1 = ax + m(1-a)y, \tag{5.46}$$

and

$$q_1 = ay + m(1-a)x. \tag{5.47}$$

Based on Equations (5.46) and (5.47), Theorems (5.9) and (5.10) extended Fejér's type of inequality as follows:

Theorem 5.9: *Let $f, g: \mathbb{I} \to \mathbb{R}$ be two (h, s, α, m)-convex functions in the second sense, positive, differentiable, and symmetric on $\mathbb{I}$, $s \in [0,1]$, $a \in [0,1]$, $m \in [0,1]$, $\alpha \in [0,1]$, and $p_1, q_1 \in \mathbb{I}$ with $p_1 < q_1$, then the following inequalities holds.*

$$\frac{1}{mx-y}\int_{x}^{my} f(x)g(x)dx \leq h^{s}(x,x)L + h^{s}(y,y)M + h^{s}(x,y)M + h^{s}(x,y)N.$$

(5.48)

Here,

$$h^{s}(x,x) = p_{1}(x)q_{1}(x), \tag{5.49}$$

$$h^{s}(y,y) = p_{1}(y)q_{1}(y), \tag{5.50}$$

$$h^{s}(x,y) = p_{1}(x)q_{1}(y) + p_{1}(x)q_{1}(y), \tag{5.51}$$

and

$$L = \int_{0}^{1} [h^{s}(a^{\alpha})]^{2} da, \tag{5.52}$$

$$M = \int_{0}^{1} m^{2}[h^{s}(1-a^{\alpha})]^{2} da, \tag{5.53}$$

$$N = \int_{0}^{1} m h^{s}(a^{\alpha}) h^{s}(1-a^{\alpha}) da. \tag{5.54}$$

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Proof: Since $p_1 < q_1$, $a \in [0,1]$, and f, g be (h,s,α,m)-convex functions in the second sense, then by using Definition (5.1), results

$$\begin{aligned} f(ax+m(1-a)y)\, g(ax+m(1-a)y) &\leq [h^{s}(a^{\alpha})f(x) + mh^{s}(1-a^{\alpha})f(y)] \\ &\quad \times [h^{s}(a^{\alpha})g(x) + mh^{s}(1-a^{\alpha})g(y)], \end{aligned}$$

which can be expanded to

$$\begin{aligned} &\leq [h^{s}(a^{\alpha})]^{2} f(x)g(x) \\ &+ m^{2}[h^{s}(1-a^{\alpha})]^{2} f(y)g(y) \\ &+ [mh^{s}(a^{\alpha})][h^{s}(1-a^{\alpha})] \\ &\times [f(x)g(y) + f(y)g(x)]. \end{aligned} \tag{5.55}$$

Integrate Inequality (5.55) on both sides with respect to a where $a \in [0, 1]$, yields

$$\int_0^1 f[(ax + m(1-a)y)g(ax + m(1-a)y)]da$$

$$\leq \int_0^1 [h^s(a^\alpha)]^2 f(x)g(x)da$$

$$+ \int_0^1 m^2[h^s(1-a^\alpha)]^2 f(y)g(y)da$$

$$+ \int_0^1 [mh^s(a^\alpha)][h^s(1-a^\alpha)]$$

$$\times [f(x)g(y) + f(y)g(x)]da. \quad (5.56)$$

By using Equations (5.49)-(5.54) in Inequality (5.56), results

$$\int_0^1 [f(ax + m(1-a)y)g(ax + m(1-a)y)]da$$

$$\leq h^s\,(x,x)L + h^s(y,y)M + h^s(x,y)N. \quad (5.57)$$

Note that $p_1 = ax + m(1-a)y$, then at $a = 0$, $p_1 = my$ and at $a = 1$, $q_1 = x$.

Differentiate p_1 with respect to a gives $(da)x + m(da)y = dp_1$, $da = \frac{dp_1}{x-my}$.

Substituting these values in Inequality (5.57), results

$$\int_{my}^{x} f(p_1)\, g(p_1) \frac{dp_1}{x-my} \leq h^s\,(x,x)L + h^s(y,y)M + h^s(x,y)N,$$

$$\frac{1}{x-my}\int_x^{my} f(p_1)\, g(p_1)dp_1 \leq h^s\,(x,x)L + h^s(y,y)M + h^s(x,y)N. \quad (5.58)$$

Inequality (5.58) completes the proof.

Corollary 5.4: If we put $h^s(a) = a$ and $h^s(1-a^\alpha) = 1-a$ then

$$\frac{1}{y-x}\int_a^b f(p_1)g(p_1)dp_1 \leq \frac{1}{2\alpha+1}h^s(x,x) + \frac{2m^2\alpha^2}{(2\alpha+1)(\alpha+1)}h^s(y,y)$$

$$+\frac{m\alpha}{(2\alpha+1)(\alpha+1)}h^s(x,y).$$

Remark 5.7:

1. If we put $h^s(a) = h(a)$ and $h^s(1-a^{\alpha}) = h(1-a)$ then it will be converted h-convex function.
2. Substituting $m = 1$ and $\alpha = 1$ in Theorem (5.9) and Corollary (5.4), then it will be converted to (h,s)-convex function in the second sense as follows:

$$\frac{1}{mx-y}\int_x^{my} f(x)g(x)dx \leq h^s(x,x)L + h^s(y,y)M + h^s(x,y)M + h^s(x,y)N.$$

Here, $h^s(x,x)$, $h^s(y,y)$, $h^s(x,y)$, L, M, and N are defined in Equations (5.45)-(5.54).

Theorem 5.10: *Let $f, g: \mathbb{I} \to \mathbb{R}$ be two (h,s,α,m)-convex functions in the second sense, positive, differentiable, and symmetric on $\mathbb{I}$, $s \in [0,1]$, $a \in [0,1]$, $\alpha \in [0,1]$, $m \in [0,1]$, and $p_1, q_1 \in \mathbb{I}$ with $p_1 < q_1$, then the following inequalities hold.*

$$2f\left(\frac{x+my}{2}\right)g\left(\frac{x+my}{2}\right) \leq \int_0^1 f(ax+m(1-a)y)g(ax+m(1-a)y)da$$

$$+[h^s(x,x)+n^2h^s(y,y)]S + \frac{1}{2}[h^s(x,y)]T. \qquad (5.59)$$

with

$$T = \left(\int_0^1 [h^s(a^{\alpha})h^s(1-a^{\alpha})][f(x)f(y)+g(x)g(y)]da\right).$$

Here, $h^s(x,x)$, $h^s(y,y)$, $h^s(x,y)$, L, M, and N are defined in Theorem (5.9).

Proof: Since $p < q$, $a \in [0, 1]$, and f, g be (h, s, α, m)-convex functions in the second sense. By setting $a = \frac{1}{2}$ and then substituting values of p_1 and q_1 from Equations (5.46) and (5.47) in Definition (5.1), results

$$f\left(\frac{x+my}{2}\right) g\left(\frac{x+my}{2}\right) \leq \left[f\left(\frac{ax+m(1-a)y}{2}+\frac{(1-a)x+may}{2}\right)\right]$$

$$\times\left[g\left(\frac{ax+m(1-a)y}{2}+\frac{(1-a)x+may}{2}\right)\right],$$

which can be expanded to

$$\leq \frac{1}{4}\left[f(ax+m(1-a)y)+f\big((1-a)x+may\big)\right]$$

$$\times\left[g(ax+m(1-a)y)+g\big((1-a)x+may\big)\right],$$

$$\leq \frac{1}{4}[f(ax+m(1-a)y)g(ax+m(1-a)y)$$

$$+f\big((1-a)x+may\big)g(1-a)x+may)]$$

$$\times \frac{1}{4}[f(ax+m(1-a)y)g\big((1-a)x+may\big)$$

$$+f\big((1-a)x+may\big)g\big((1-a)x+may\big)\big]. \quad (5.60)$$

By applying Definition (5.1) of (h, s, α, m)-convex function in the second sense on Inequality (5.60), yields

$$f\left(\frac{x+my}{2}\right) g\left(\frac{x+my}{2}\right) \leq \frac{1}{4}[f(ax+m(1-a)y)g(ax+m(1-a)y)]$$

$$+f\big((1-a)x+may\big)g\big((1-a)x+may\big)$$

$$+\frac{1}{4}[h^s(a^\alpha)f(x)+mh^s(1-1-a^\alpha)f(y)]$$

$$\times h^s(1-a^\alpha)f(x) + mh^s(a^\alpha)f(y).$$

$$\leq \frac{1}{4} f(ax+m(1-a)y)g(ax+m(1-a)y)$$

$$+f\big((1-a)x+may\big)g\big((1-a)x+may\big)$$

$$+\frac{1}{2}[h^s(a^\alpha)h^s(1-a^\alpha)][f(x)f(y)+g(x)g(y)]$$

$$+\frac{1}{4}[m[h^s(a^\alpha)]^2$$

$$+[h^s(1-a^\alpha)]^2[f(x)g(y)+f(y)g(x)]]. \tag{5.61}$$

Integrate Inequality (5.61) with respect to a where $a \in [0, 1]$, results

$$f\left(\frac{x+my}{2}\right)g\left(\frac{x+my}{2}\right) \leq \frac{1}{4}\left[\left(\int_0^1 f(ax+m(1-a)y)g(ax+m(1-a)y)\right.\right.$$

$$+f\big((1-a)x+may\big)g\big((1-a)x+may\big)\Big]da$$

$$+\frac{1}{2}\left(\int_0^1 [h^s(a^\alpha)h^s(1-a^\alpha)][f(x)f(y)+g(x)g(y)]da\right)$$

$$+\frac{1}{4}\left[\int_0^1 [m[h^s(a^\alpha)]^2+[h^s(1-a^\alpha)]^2]\,da\right]$$

$$\times [f(x)g(y)+f(y)g(x)]. \tag{5.62}$$

By simplifying Inequality (5.62), results

$$f\left(\frac{x+my}{2}\right)g\left(\frac{x+my}{2}\right) \leq \frac{1}{2}\int_0^1 [f(ax+m(1-a)y)g(ax+m(1-a)y)]da$$

$$+\frac{1}{2}[h^s(x,x)+mh^s(y,y)]S+\frac{1}{4}[h^s(x,y)]T.$$

$$2f\left(\frac{x+my}{2}\right)g\left(\frac{x+my}{2}\right) \leq \int_0^1 [f(ax+m(1-a)y)g(ax+m(1-a)y)]da$$

$$+[h^s(x,x)+mh^s(y,y)]S+\frac{1}{2}[h^s(x,y)]T. \tag{5.63}$$

Inequality (5.63) completes the proof.

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Corollary 5. 5: If $h^s(a) = a$ and $h^s(1 - a^\alpha) = 1 - a$, then

$$2f\left(\frac{x+my}{2}\right)g\left(\frac{x+my}{2}\right) \leq \frac{1}{y-x}\int_a^b f(p_1)g(p_1)dp_1 + \frac{(\alpha)}{(2\alpha+1)(\alpha+1)}$$

$$\times\left[h^s(x,x) + mh^s(y,y)\right] + \frac{m(\alpha^2+\alpha+1)}{2(2\alpha+1)(\alpha+1)}$$

$$+\frac{m(\alpha^2+\alpha+1)}{2(2\alpha+1)(\alpha+1)}h^s(x,y).$$

Remark 5.8:

1. If using $h^s(a) = h(a)$ and $h^s(1 - a^\alpha) = h(1 - a)$, then extended Fejér's inequality will be converted to the results for h-convex function.
2. By substituting $m = 1$ and $\alpha = 1$ in Theorem (5.10) and Corollary (5.5), then it will be converted to Fejér's inequality for (h, s)-convex function in the second sense.

$$2f\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right) \leq \int_0^1 f(ax+(1-a)y)g(ax+(1-a)y)da$$

$$+\left[h^s(x,x) + n^2h^s(y,y)\right]S + \frac{1}{2}\left[h^s(x,y)\right]T.$$

Here, $h^s(x,x)$, $h^s(y,y)$, $h^s(x,y)$, L, M, and N are defined in Theorem (5.9).

Note that Theorems (5.9) and (5.10) successfully extended Fejér's type of inequality by using hybrid (h, s, α, m)-convex function in the second sense defined in Definition (5.1). While, if we impose condition $s = \frac{1}{s}$ for $s \in (0, 1]$ with $h^s(a) + h^s(1 - a) = 1$ and $h(a) + h(1 - a) = 1$, then results for (h, s, α, m)-convex function in the second sense will be converted to (h, s, α, m)-convex function in the third and fourth sense, respectively.

In addition, if $\alpha = 1$ and $m = 1$, then results will be converted to Fejér's inequality for (h, s)-convex function in the second sense.

5.5 Summary

This chapter established some new hybrid convex functions by combining s-convex, h-convex, and g-convex functions. Further, on the basis of these hybrid convex functions, the Fejér's type of inequalities are extended or generalised. In Section (5.2) Fejér's inequality is extended by combining s-convex and h-convex functions. Similarly, in Section (5.3) Fejér's inequality is extended by combining g-convex and h-convex functions and in Section (5.4) Fejér's inequality is extended by combining s-convex and g-convex functions.

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CHAPTER SIX
SIMPSON'S TYPE OF INEQUALITIES BASED ON HYBRID CONVEX FUNCTION

This chapter describes three novel integral forms of Simpson's inequality based on the hybrid of s-convex, h-convex, and g-convex functions. These new inequalities can form new relationship with mathematical means (averages) for real numbers and find error estimation formula for Simpson's inequality.

6.1 Introduction

In this chapter, Simpson's inequality is extended based on hybrid convex functions without losing its originality. By applying certain conditions, the formal form, as well as some previous Simpson's inequality can be obtained.

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6.2 Extending Simpson's Inequality by Merging (h_1, h_2)-convex and s-convex Functions

In this section, several new extensions of Simpson's inequalities are formed based on hybrid (h_1, h_2, s)-convex function in third sense previously defined in Definition (3.3). First (h_1, h_2, s)-convex function in third sense in terms of first derivative for $p, q \in \mathbb{I}$ can be written as

$$f'(pa + (1-a)q) \leq h_1^{\frac{1}{s}}(a)f'(p) + h_2^{\frac{1}{s}}(1-a)f'(q). \tag{6.1}$$

Based on Inequality (6.1), Simpson's type of inequality can be extended in Theorems (6.1), (6.2), and (6.3), as follows:

Theorem 6.1: *Let $h_1, h_2: \mathbb{I} \to \mathbb{R}$ be two positive functions and $f: \mathbb{I} \to \mathbb{R}$ be a differentiable function, positive, and symmetric on $\mathbb{I}$ such that $p, q \in \mathbb{I}$ with $p < q$, $f' \in L[p,q]$, and $a \in [0,1]$. If $|f'|$ is (h_1, h_2, s)-convex function in the third sense with $s \in (0,1]$ and $\frac{1}{c} + \frac{1}{d} = 1$ for $1 < c < \infty$, $d > 1$ then*

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2} + 2f\left(\frac{p+q}{2}\right)\right] - \frac{1}{(q-p)}\int_p^q f(x)dx\right| \leq (q-p)|f'(p)|$$

$$\times\left\{\left(\int_0^{\frac{1}{2}} \left|a - \frac{1}{6}\right|^c\right)^{\frac{1}{c}}\left(\int_0^{\frac{1}{2}} h_1^{\frac{d}{s}}(a)\right)^{\frac{1}{d}} + \left(\int_{\frac{1}{2}}^1 \left|a - \frac{5}{6}\right|^c\right)^{\frac{1}{c}}\left(\int_{\frac{1}{2}}^1 h_1^{\frac{d}{s}}(a)\right)^{\frac{1}{d}}\right\} da$$

$$+(q-p)|f'(q)|\left\{\left(\int_0^{\frac{1}{2}} \left|a - \frac{1}{6}\right|^c\right)^{\frac{1}{c}}\left(\int_0^{\frac{1}{2}} h_2^{\frac{d}{s}}(1-a)\right)^{\frac{1}{d}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^1 \left|a - \frac{5}{6}\right|^c\right)^{\frac{1}{c}}\left(\int_{\frac{1}{2}}^1 h_2^{\frac{d}{s}}(1-a)\right)^{\frac{1}{d}}\right\} da.$$

Proof: Since $p < q$, $a \in [0,1]$, $1 < c < \infty$, $d > 1$ and $|f'|$ is (h_1, h_2, s)-convex function in the third sense and positive, by utilising Inequality (6.1) in Lemma (2.10), results

$$\left|\frac{1}{6(q-p)}\left[f(p) + 4f\left(\frac{p+q}{2}\right) + f(q)\right] - \frac{1}{(q-p)}\int_p^q f(x)dx\right|$$

$$= (q-p)\left|\int_0^1 t(\alpha)\, f'(pa + (1-a)q)da\right|,$$

which can be written as

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_{p}^{q} f(x)dx\right|$$

$$=(q-p)\left|\int_{0}^{1} t(\alpha)\, f'(pa+(1-a)q)da\right|. \qquad (6.2)$$

By applying limit values for $t(\alpha)$ from Lemma (2.10) in Equation (6.2), results

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_{p}^{q} f(x)dx\right|$$

$$=(q-p)\left|\int_{0}^{\frac{1}{2}}\left(a-\frac{1}{6}\right)f'(pa+(1-a)q)da\right|$$

$$+\left|\int_{\frac{1}{2}}^{1}\left(a-\frac{5}{6}\right)f'(pa+(1-a)q)da\right|. \qquad (6.3)$$

By using Inequality (6.1) for $|f'|$ of (h_1,h_2,s)-convex function in the third sense in terms of first derivative in Equation (6.3), together with $\int_0^1 da=\int_0^{\frac{1}{2}} da+\int_{\frac{1}{2}}^1 da$, yields

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_{p}^{q} f(x)dx\right|$$

$$\leq (q-p)\left|\int_{0}^{\frac{1}{2}}\left|a-\frac{1}{6}\right| h_1^{\frac{1}{s}}(a)f'(p)+h_2^{\frac{1}{s}}(1-a)f'(q)da\right|$$

$$+(q-p)\left|\int_{\frac{1}{2}}^{1}\left|a-\frac{5}{6}\right| h_1^{\frac{1}{s}}(a)f'(p)+h_2^{\frac{1}{s}}(1-a)f'(q)da\right|,$$

which can be expanded to

$$\leq (q-p)|f'(p)|\left\{\int_0^{\frac{1}{2}}\left|a-\frac{1}{6}\right|h_1^{\frac{1}{s}}(a)+\int_{\frac{1}{2}}^{1}\left|a-\frac{5}{6}\right|h_1^{\frac{1}{s}}(a)\right\}da$$

$$+(q-p)|f'(q)|$$

$$\times\left|\int_0^{\frac{1}{2}}\left|a-\frac{1}{6}\right|h_2^{\frac{1}{s}}(1-a)+\int_{\frac{1}{2}}^{1}\left|a-\frac{5}{6}\right|h_2^{\frac{1}{s}}(1-a)\right|da. \quad (6.4)$$

By utilising Theorem (2.18) for $\frac{1}{c}+\frac{1}{d}=1$ for $1<c<\infty$, $d>1$ and then expanding Inequality (6.4), yields

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_p^q f(x)dx\right|\leq (q-p)|f'(p)|$$

$$\times\left\{\left(\int_0^{\frac{1}{2}}\left|a-\frac{1}{6}\right|^c\right)^{\frac{1}{c}}\left(\int_0^{\frac{1}{2}}h_1^{\frac{d}{s}}(a)\right)^{\frac{1}{d}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}\left|a-\frac{5}{6}\right|^c\right)^{\frac{1}{c}}\left(\int_{\frac{1}{2}}^{1}h_1^{\frac{d}{s}}(a)\right)^{\frac{1}{d}}\right\}da+(q-p)|f'(q)|$$

$$\times\left\{\left(\int_0^{\frac{1}{2}}\left|a-\frac{1}{6}\right|^c\right)^{\frac{1}{c}}\left(\int_0^{\frac{1}{2}}h_2^{\frac{d}{s}}(1-a)\right)^{\frac{1}{d}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}\left|a-\frac{5}{6}\right|^c\right)^{\frac{1}{c}}\left(\int_{\frac{1}{2}}^{1}h_2^{\frac{d}{s}}(1-a)\right)^{\frac{1}{d}}\right\}da. \quad (6.5)$$

Inequality (6.5) completes the proof.

Remark 6.1:

1. By substituting $s = 1$ and $h_1 = h_2 = h$ in Theorem (6.1), results will be converted to Simpson's inequality for the h-convex function.
2. By using ${h_1}^s(a) = h^s(a)$ and ${h_2}^s(1-a) = h^s(1-a)$ in Theorem (6.1), then the result will be converted to Simpson's inequality for (h, s)-convex function.
3. By utilising ${h_1}^s(a) = a$ and ${h_2}^s(1-a) = 1-a$ in Theorem (6.1), the result will be transformed to Simpson's inequality for a convex function.

Theorem 6.2: *Let $h_1, h_2 \colon \mathbb{I} \to \mathbb{R}$ be two positive functions and $f \colon \mathbb{I} \to \mathbb{R}$ be a differentiable, positive, and symmetric function on $\mathbb{I}$ such that $p, q \in \mathbb{I}$ with $p < q$, $f' \in L[p, q]$, and $a \in [0, 1]$. If $|f'|$ is (h_1, h_2, s)-convex in the third sense with $s \in (0, 1]$, then*

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_p^q f(x)dx\right|$$

$$\leq (q-p)|f'(p)|\left\{\left(\int_0^{\frac{1}{6}} {h_1}^{\frac{1}{s}}\left(a\left(\frac{1}{6}-a\right)\right)\right)+\left(\int_{\frac{1}{6}}^{\frac{1}{2}} {h_1}^{\frac{1}{s}}\left(a\left(a-\frac{1}{6}\right)\right)\right)\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{6}} {h_1}^{\frac{1}{s}}\left(a\left(\frac{5}{6}-a\right)\right)\right)+\left(\int_{\frac{5}{6}}^{1} {h_1}^{\frac{1}{s}}\left(a\left(a-\frac{5}{6}\right)\right)\right)\right\} da$$

$$+(q-p)|f'(q)|\left(\int_0^{\frac{1}{6}} {h_1}^{\frac{1}{s}}\left((1-a)\left(\frac{1}{6}-a\right)\right)\right)+\left(\int_{\frac{1}{6}}^{\frac{1}{2}} {h_1}^{\frac{1}{s}}\left((1-a)\left(a-\frac{1}{6}\right)\right)\right)$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}} h_1{}^{\frac{1}{s}}\left((1-a)\left(\frac{5}{6}-a\right)\right)\right)+\left(\int_{\frac{5}{6}}^{1} h_1{}^{\frac{1}{s}}\left((1-a)\left(a-\frac{5}{6}\right)\right)\right)\Bigg\} da.$$

Proof: Since $p < q$, $a \in [0, 1]$, then by using Inequality (6.1) in Lemma (2.10) and apply limit values for $t(\alpha)$ from Lemma (2.10) yields

$$\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_p^q f(x)dx$$

$$\leq (q-p)\left(\int_0^{\frac{1}{2}}\left(a-\frac{1}{6}\right)f'(pa+(1-a)q)da\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}\left(a-\frac{5}{6}\right)f'(pa+(1-a)q)da\right). \qquad (6.6)$$

Since $|f'|$ is a (h_1, h_2, s)-convex function in the third sense, then by using absolute value on Inequality (6.6) and using Inequality (6.1) in Inequality (6.6) results

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_p^q f(x)dx\right|$$

$$\leq (q-p)\left|\int_0^{\frac{1}{2}}\left|a-\frac{1}{6}\right|\left(h_1{}^{\frac{1}{s}}(a)f'(p)+h_2{}^{\frac{1}{s}}(1-a)f'(q)\right)da\right|$$

$$+(q-p)\left|\int_{\frac{1}{2}}^{1}\left|a-\frac{5}{6}\right|\left(h_1{}^{\frac{1}{s}}(a)f'(p)+h_2{}^{\frac{1}{s}}(1-a)f'(q)\right)da\right|,$$

which can be written as

$$\leq (q-p)|f'(p)|\left\{\int_{0}^{\frac{1}{2}}\left|a-\frac{1}{6}\right| h_1^{\frac{1}{s}}(a)+\int_{\frac{1}{2}}^{1}\left|a-\frac{5}{6}\right| h_1^{\frac{1}{s}}(a)\right\} da$$

$$+(q-p)\left|f'^{(q)}\right|\left|\int_{0}^{\frac{1}{2}}\left|a-\frac{1}{6}\right| h_2^{\frac{1}{s}}(1-a)da+\int_{\frac{1}{2}}^{1}\left|a-\frac{5}{6}\right| h_2^{\frac{1}{s}}(1-a)da\right|, \quad (6.7)$$

$$\leq (q-p)\left|\int_{0}^{\frac{1}{6}}\left|\frac{1}{6}-a\right|\left(h_1^{\frac{1}{s}}(a)f'(p)+h_2^{\frac{1}{s}}(1-a)f'(q)\right)da\right|$$

$$+(q-p)\left|\int_{\frac{1}{6}}^{\frac{1}{2}}\left|a-\frac{1}{6}\right|\left(h_1^{\frac{1}{s}}(a)f'(p)+h_2^{\frac{1}{s}}(1-a)f'(q)\right)da\right|$$

$$+(q-p)\left|\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\frac{5}{6}-a\right|\left(h_1^{\frac{1}{s}}(a)f'(p)+h_2^{\frac{1}{s}}(1-a)f'(q)\right)da\right|$$

$$+(q-p)\left|\int_{\frac{5}{6}}^{1}\left|a-\frac{5}{6}\right|\left(h_1^{\frac{1}{s}}(a)f'(p)+h_2^{\frac{1}{s}}(1-a)f'(q)\right)da\right|. \quad (6.8)$$

By taking values of $f'(p)$ and $f'(q)$ common from Inequality (6.8), results

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_{p}^{q} f(x)dx\right| \leq (q-p)|f'(p)|$$

$$\times\left\{\left(\int_{0}^{\frac{1}{6}} h_1^{\frac{1}{s}}\left(a\left(\frac{1}{6}-a\right)\right)\right)+\left(\int_{\frac{1}{6}}^{\frac{1}{2}} h_1^{\frac{1}{s}}\left(a\left(a-\frac{1}{6}\right)\right)\right)\right.$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}} h_1^{\frac{1}{s}}\left(a\left(\frac{5}{6}-a\right)\right)\right)+\left(\int_{\frac{5}{6}}^{1} h_1^{\frac{1}{s}}\left(a\left(a-\frac{5}{6}\right)\right)\right)\Bigg\} da$$

$$+(q-p)|f'(q)|\left\{\left(\int_0^{\frac{1}{6}} h_1^{\frac{1}{s}}\left((1-a)\left(\frac{1}{6}-a\right)\right)\right)\right.$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}} h_1^{\frac{1}{s}}\left((1-a)\left(a-\frac{1}{6}\right)\right)\right)+\left(\int_{\frac{1}{2}}^{\frac{5}{6}} h_1^{\frac{1}{s}}\left((1-a)\left(\frac{5}{6}-a\right)\right)\right)$$

$$\left.+\left(\int_{\frac{5}{6}}^{1} h_1^{\frac{1}{s}}\left((1-a)\left(a-\frac{5}{6}\right)\right)\right)\right\} da. \tag{6.9}$$

Inequality (6.9) completes the proof.

Remark 6.2:

1. By substituting $s=1$ and $h_1=h_2=h$ in Theorem (6.2), the result will be converted to Simpson's inequality for h-convex function.
2. By using ${h_1}^s(a)=h^s(a)$ and ${h_2}^s(1-a)=h^s(1-a)$ in Theorem (6.2), then result will be converted to Simpson's inequality for (h,s)-convex function.
3. By utilising ${h_1}^s(a)=a$ and ${h_2}^s(1-a)=1-a$ in Theorem (6.2), the result will be transformed to Simpson's inequality for a convex function.

Theorem 6.3: *Let $h_1,h_2\colon \mathbb{I}\to\mathbb{R}$ be two positive functions and $f\colon\mathbb{I}\to\mathbb{R}$ be a differentiable, positive, and symmetric function on $\mathbb{I}$ such that $p,q\in\mathbb{I}$ with $p<q$, $f'\in L[p,q]$, and $a\in[0,1]$. If $|f'|$ is (h_1,h_2,s)-convex in the third sense with $s\in(0,1]$ and $\frac{1}{c}+\frac{1}{d}=1$ with $c>0$, $d>1$ then*

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_{p}^{q} f(x)dx\right|$$

$$\leq \frac{(q-p)}{12}\left(\frac{2+2^{c+2}}{c+1}\right)^{\frac{1}{c}}\left\{\left(\frac{1}{h_1^{\frac{1}{s}}\left(\frac{1}{2}\right)h_2^{\frac{1}{s}}\left(\frac{1}{2}\right)}\right)^{\frac{1}{c}}+f'\left(\frac{p+q}{2}\right)\right\}da.$$

Proof: Since $p < q$ and $a \in [0,1]$, then by using Inequality (6.1) in Lemma (2.10), results

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_{p}^{q} f(x)dx\right|$$

$$\leq (q-p)\left|\int_{0}^{1} t(\alpha)\, f'(pa+(1-a)q)da\right|. \tag{6.10}$$

By utilising Theorem (2.2) for $\frac{1}{c}+\frac{1}{d}=1$ on Inequality (6.10), yields

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_{p}^{q} f(x)dx\right|$$

$$\leq (q-p)\left|\int_{0}^{1}[t^c(\alpha)]^{\frac{1}{c}}[f'(pa+(1-a)q)^d]^{\frac{1}{d}}da\right|. \tag{6.11}$$

Since $|f'|$ is a (h_1,h_2,s)-convex in the third sense, then by setting $a=\frac{1}{2}$ in right hand side of Inequality (6.1), results

$$\int_{0}^{1} f'(pa+(1-a)q)^d\, da \leq h_1^{\frac{1}{s}}\left(\frac{1}{2}\right)f'(p)+h_2^{\frac{1}{s}}\left(\frac{1}{2}\right)f'(q),$$

$$\int_{0}^{1} f'(pa+(1-a)q)^d\, da \leq \left(\frac{1}{h_1^{\frac{1}{s}}\left(\frac{1}{2}\right)h_2^{\frac{1}{s}}\left(\frac{1}{2}\right)}\right)f'\left(\frac{p+q}{2}\right)^d,$$

and then substituting limit values of $t(\alpha)$ from Lemma (2.10) in Inequality (6.11), yields

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_{p}^{q}f(x)dx\right|\leq(q-p)$$

$$\times\left|\left\{\int_{0}^{\frac{1}{2}}\left|a-\frac{1}{6}\right|^{c}+\int_{\frac{1}{2}}^{1}\left|a-\frac{5}{6}\right|^{c}\right\}^{\frac{1}{c}}\left[\left(\frac{1}{h_1^{\frac{1}{s}}\left(\frac{1}{2}\right)h_2^{\frac{1}{s}}\left(\frac{1}{2}\right)}\right)f'\left(\frac{p+q}{2}\right)^{d}\right]^{\frac{1}{d}}da\right|. \quad (6.12)$$

Since, f is positive and symmetric then we have

$$\int_{0}^{\frac{1}{2}}\left|a-\frac{1}{6}\right|^{c}da=\int_{0}^{\frac{1}{6}}\left|\frac{1}{6}-a\right|^{c}da+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|a-\frac{1}{6}\right|^{c}da=\frac{1}{c+1}\left(\frac{1}{6^{c+1}}+\frac{1}{3^{c+1}}\right),$$

and

$$\int_{\frac{1}{2}}^{1}\left|a-\frac{5}{6}\right|^{c}da=\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\frac{5}{6}-a\right|^{c}da+\int_{\frac{5}{6}}^{1}\left|a-\frac{5}{6}\right|^{c}da=\frac{1}{c+1}\left(\frac{1}{6^{c+1}}+\frac{1}{3^{c+1}}\right).$$

By substituting these values in Inequality (6.12), results

$$\left|\frac{1}{3}\left[\frac{f(p)+f(q)}{2}+2f\left(\frac{p+q}{2}\right)\right]-\frac{1}{(q-p)}\int_{p}^{q}f(x)dx\right|$$

$$\leq\frac{(q-p)}{12}\left(\frac{2+2^{c+2}}{c+1}\right)^{\frac{1}{c}}\left\{\left(\frac{1}{h_1^{\frac{1}{s}}\left(\frac{1}{2}\right)h_2^{\frac{1}{s}}\left(\frac{1}{2}\right)}\right)^{\frac{1}{c}}+f'\left(\frac{p+q}{2}\right)\right\}da. \quad (6.13)$$

Inequality (6.13) completes the proof.

Remark 6.3:

1. By substituting $s=1$ and $h_1=h_2=h$ in Theorem (6.3), results will be converted to Simpson's inequality for h-convex function.

2. By using ${h_1}^s(a) = h^s(a)$ and ${h_2}^s(1-a) = h^s(1-a)$ in Theorem (6.3), then results will be converted to (h, s)-convex function.

3. By utilising ${h_1}^s(a) = a$ and ${h_2}^s(1-a) = 1-a$ in Theorem (6.3), the results will be transformed to Simpson's inequality for a convex function.

Note that Theorems (6.1), (6.2), and (6.3) successfully extended Simpson's inequality for (h_1, h_2, s)-convex function in the third sense. While, if we impose the condition $h_1(a) + h_2(1-a) = 1$ with $h_1(a) \not\equiv 0$ and $h_2(1-a) \not\equiv 0$, then the results will be converted to (h_1, h_2, s)-convex function in fourth sense.

In proceeding section, g-(h_1, h_2)-convex function defined in Definition (3.8) is used to extend Simpson's type of inequality as follows:

6.3 Extending Simpson's Inequality by Merging (h_1, h_2)-convex and g-convex Functions

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In this section, new extension of Simpson's inequality is constructed by considering hybrid g-(h_1, h_2) convex function as follows:

Theorem 6.4: *Let* $f: \mathbb{I} \to \mathbb{R}$ *be a* g-(h_1, h_2) *convex function and n-times differentiable on* $\mathbb{I} \subset \mathbb{R}$ *for all* $p, q \in \mathbb{I}$ *with* $p < q$, $a \in [0, 1]$, *and* $\left|f^{(n)}\right| \leq M$. *Then*

$$\left|\frac{f(p) + f(q)}{2} - \left(\frac{1}{q-p}\right)\int_p^q f(x)dx - \sum_{i=1}^{n-1}\left(\frac{(i-1)(q-p)^i}{2(i-1)!} f^i(p)\right)\right|$$

$$\leq \frac{(q-p)^i}{2(i)!}\left(\frac{i-1}{i+1}\right)^{\left(1-\frac{1}{q}\right)}\left(\int_0^1 a^{(i-1)}(i-2a) f^n(p)^{h_1(a)} f^n(q)^{h_2(1-a)} da\right)^{\frac{1}{q}}.$$

Proof: Since $p < q$, $a \in [0, 1]$, by using Lemma (2.6), we have

$$\left|\frac{f(p)+f(q)}{2}-\left(\frac{1}{q-p}\right)\int_p^q f(x)dx-\sum_{i=1}^{n-1}\left(\frac{(i-1)(q-p)^i}{2(i-1)!}f^i(p)\right)\right|$$

$$=\frac{(q-p)^i}{2(i)!}\int_0^1 {a_1}^{(i-1)}(i-2a)f^n(ap+(1-a)q)da.$$

Next Theorem (2.18) is utilised to get

$$\left|\frac{f(p)+f(q)}{2}-\left(\frac{1}{q-p}\right)\int_p^q f(x)dx-\sum_{i=1}^{n-1}\left(\frac{(i-1)(q-p)^i}{2(i-1)!}f^i(p)\right)\right|$$

$$\leq\frac{(q-p)^i}{2(i)!}\left(\int_0^1 {a_1}^{(i-1)}(i-2a)da\right)^{\left(1-\frac{1}{q}\right)}$$

$$\times\left(\int_0^1 a^{(i-1)}(i-2a)f^n(ap+(1-a)q)da\right)^{\frac{1}{q}},$$

$$\leq\frac{(q-p)^i}{2(i)!}\left(\frac{i-1}{i+1}\right)^{\left(1-\frac{1}{q}\right)}$$

$$\times\left(\int_0^1 a^{(i-1)}(i-2a)f^n(ap+(1-a)q)da\right)^{\frac{1}{q}}.$$

Note that $f^{(n)}$ is a g-(h_1,h_2) convex function, implies

$$\left|\frac{f(p)+f(q)}{2}-\left(\frac{1}{q-p}\right)\int_p^q f(x)dx-\sum_{i=1}^{n-1}\left(\frac{(i-1)(q-p)^i}{2(i-1)!}f^i(p)\right)\right|$$

$$\leq\frac{(q-p)^i}{2(i)!}\left(\frac{i-1}{i+1}\right)^{\left(1-\frac{1}{q}\right)}$$

$$\times\left(\int_0^1 a^{(i-1)}(i-2a)f^n(p)^{h_1(a)}f^n(q)^{h_2(1-a)}da\right)^{\frac{1}{q}}. \tag{6.14}$$

Inequality (6.14) completes the proof.

Note that Theorem (6.4) successfully extended Simpson's inequality for g-(h_1,h_2) convex function defined in Definition (3.8).

Furthermore, (g, s)-convex function in third sense in Definition (3.9) is utilised to extend Simpson's type of inequality.

6.4 Extending Simpson's Inequality by Merging g-convex and s-convex Functions

In this section, another new hybrid convex function, (g, s)-convex function in the third sense defined in Definition (3.9) is considered to extend Simpson's inequality.

From now, to extend Simpson's inequality, Definition (3.9) of (g, s)-convex function in the third sense in terms of first derivative for $p, q \in \mathbb{I}$ will be used such as

$$f'\left(q^{\left(\frac{1+a}{2}\right)}p^{\left(\frac{1-a}{2}\right)}\right) \leq f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}, \tag{6.15}$$

for $a = \left(\frac{1+a}{2}\right)$ and $1 - a = \left(\frac{1-a}{2}\right)$.

Theorem 6.5: *Let $f: \mathbb{I} \to \mathbb{R}$ be a (g, s)-convex function in third sense, monotonically decreasing, and differentiable on $s \in (0, 1]$ such that $p, q \in \mathbb{I}$ with $p < q$, $f' \in L(p, q)$, and $a \in [0, 1]$, then*

$$\left|\frac{1}{6} f(p) + 4f\left(\frac{p+q}{2}\right) + f(q) - \left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right| \leq \left(\frac{q-p}{2}\right)$$

$$\times \begin{cases} \left|[f'(p)f'(q)]^{\left(\frac{1}{2s}\right)}\right| \left[\Phi_1\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right) + \Phi_2\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right)\right], \\ \qquad f'(p) \leq 1; \\ \left|f'(p)^{\left(1-\frac{1}{2s}\right)}\right| \left|f'(q)^{\left(\frac{1}{2s}\right)}\right| \left[\Phi_1\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right) + \Phi_2\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right)\right], \\ \qquad f'(q) \leq 1 \leq f'(p); \\ \left|[f'(p)f'(q)]^{\left(1-\frac{1}{2s}\right)}\right| \left[\Phi_1\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right) + \Phi_2\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right)\right], \\ \qquad 1 \leq f'(q). \end{cases} \tag{6.16}$$

Here, $\Phi_1(\beta)$ and $\Phi_2(\beta)$ are

$$\Phi_1(\beta) = \begin{cases} \dfrac{5}{36}, & \beta = 1 \\ \dfrac{6\beta^{\frac{2}{3}} + (\beta - 2)ln\beta - 3\beta - 3}{6ln^2\beta}, & \beta \neq 1 \end{cases}$$

and

$$\Phi_2(\beta) = \begin{cases} \dfrac{5}{36}, & \beta = 1 \\ \dfrac{6\beta^{-\frac{2}{3}} + (\beta^{-1} - 2)ln\beta^{-1} - 3\beta^{-1} - 3}{6ln^2\beta^{-1}}, & \beta \neq 1 \end{cases}$$

where $\beta(p,q) = \frac{|f'(q)|^c}{|f'(p)|^d}$ *with* $c, d > 0$.

Proof: Since $p < q$ and $a \in [0,1]$, by using modulus property on Lemma (2.7) yields

$$\left|\frac{1}{6}\left[f(p) + f(q) + 4f\left(\frac{p+q}{2}\right)\right] - \left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right| = \left|\frac{q-p}{2}\right.$$

$$\left.\times \int_0^1 \left[\left(\frac{a}{2} - \frac{1}{3}\right)f'\left(\frac{1+a}{2}q - \frac{1-a}{2}p\right) + \left(\frac{1}{3} - \frac{a}{2}\right)f'\left(\frac{1+a}{2}p - \frac{1-a}{2}q\right)\right]da\right|.$$

As $|f'(x)|$ is a (g,s)-convex function and monotonically decreasing, then by using Inequality (6.15) in Lemma (2.7), yields

$$\left|\frac{1}{6}\left[f(p) + f(q) + 4f\left(\frac{p+q}{2}\right)\right] - \left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right|$$

$$\leq \left(\frac{q-p}{2}\right)\int_0^1 \left[\left|\frac{a}{2} - \frac{1}{3}\right| f'\left(|q|^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} |p|^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}\right)\right.$$

$$\left. + \left|\frac{1}{3} - \frac{a}{2}\right| f'\left(|p|^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} |q|^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}\right)\right] da. \tag{6.17}$$

By utilising Lemma (2.4) for $0 < |f'(p)| \leq 1$ and then by taking integration with respect to a where $a \in [0,1]$ on Inequality (6.17), results

$$I_5 = \int_0^1 \left| f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} \right| da \leq \int_0^1 \left[|f'(q)|^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} |f'(p)|^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} \right] da\,, \quad (6.18)$$

which can be written as

$$\leq \int_0^1 \left[|f'(q)|^{\left(\frac{1+a}{2}\right)\frac{1}{s}} |f'(p)|^{\left(\frac{1-a}{2}\right)\frac{1}{s}} \right] da,$$

$$\leq \int_0^1 \left[|f'(q)|^{\frac{1}{2s}} |f'(q)|^{\frac{a}{2s}} \left(\frac{|f'(q)|^{\frac{1}{2s}}}{|f'(p)|^{\frac{a}{2s}}} \right) \right] da,$$

$$\leq |f'(p)f'(q)|^{\frac{1}{2s}} \int_0^1 \left[\left(\frac{|f'(q)|^{\frac{1}{2s}}}{|f'(p)|^{\frac{1}{2s}}} \right)^a \right] da,$$

$$\leq |f'(p)f'(q)|^{\frac{1}{2s}} \Phi_1 \left(\beta \left(\frac{1}{2s}, \frac{1}{2s} \right) \right), \quad (6.19)$$

where Φ_1 and β are defined in Theorem (6.5).

Further, using piecewise integration for a mod function, we have

$$\int_0^1 \left| \frac{a}{2} - \frac{1}{3} \right| da = \int_0^1 \frac{|3a-2|}{6} da. \quad (6.20)$$

By using $t = 3a - 2$ and differentiating t with respect to a, results $da = \frac{dt}{3}$. Substituting these values in Equation (6.20), yields

$$\int_0^1 \left| \frac{a}{2} - \frac{1}{3} \right| da = \frac{1}{18} \int_0^1 |t| dt. \quad (6.21)$$

By applying integration by parts with $f_1 = |t|$ and $g_1{}' = 1$ then $f_1{}' = \frac{t}{|t|}$ and $g_1 = t$, Equation (6.21) will be expended to

$$\int_0^1 \left| \frac{a}{2} - \frac{1}{3} \right| da = t|t| - \int_0^1 |t| dt.$$

By applying integration on $\int |t| dt = \frac{t|t|}{2}$.

By substituting all these values in Equation (6.21), yields

$$\int_0^1 \left|\frac{a}{2} - \frac{1}{3}\right| da = \frac{t|t|}{36}\bigg|_0^1. \tag{6.22}$$

Since, $t = 3a - 2$ then Equation (6.22) will be expended to

$$\int_0^1 \left|\frac{a}{2} - \frac{1}{3}\right| da = \frac{(3x-2)|3x-2|}{36}\bigg|_0^1,$$

$$= \frac{1}{36}\big((3(1)-2)|3(1)-2| - (3(0)-2)|3(0)-2|\big),$$

$$\int_0^1 \left|\frac{a}{2} - \frac{1}{3}\right| da = \frac{5}{36}. \tag{6.23}$$

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By substituting Inequality (6.19) and Equation (6.23) in Inequality (6.18), yields

$$I_5 = \int_0^1 \left| f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} \right| da$$

$$\leq \frac{5}{36} |f'(p)f'(q)|^{\frac{1}{2s}} \left(\Phi_1 \left(\beta \left(\frac{1}{2s}, \frac{1}{2s} \right) \right) \right). \tag{6.24}$$

Similar to Inequality (6.24), we have

$$I_6 = \int_0^1 \left[\left|\frac{1}{3} - \frac{a}{2}\right| \left| f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} da \right| \right.$$

$$\leq \int_0^1 \left[\left[\left|\frac{1}{3} - \frac{a}{2}\right| |f'(q)|^{\left(\frac{1+a}{2}\right)\frac{1}{s}} |f'(p)|^{\left(\frac{1-a}{2}\right)\frac{1}{s}} da, \right. \right.$$

$$\leq \frac{5}{36}|f'(p)f'(q)|^{\frac{1}{2s}}\left(\Phi_2\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)\right). \tag{6.25}$$

By substituting values of I_5 and I_6 from Inequalities (6.24) and (6.25) in right hand side of Inequality (6.17), results

$$\left|\frac{1}{6}f(p)+4f\left(\frac{p+q}{2}\right)+f(q)-\left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right| \leq \left(\frac{q-p}{2}\right)\left(\frac{5}{36}\right)$$

$$\times\left|[f'(p)f'(q)]^{\left(\frac{1}{2s}\right)}\right|\left[\Phi_1\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)+\Phi_2\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)\right]$$

with Φ_1 and Φ_2 is defined in Theorem (6.5) and the first condition is satisfied.

Again, by using Lemma (2.4) for $|f'(q)| \leq 1 \leq |f'(p)|$ and by applying integration with respect to a where $a \in [0, 1]$ on Equation (6.17), then similar to Inequality (6.24), we have

$$I_7 = \int_0^1\left[\left|\frac{a}{2}-\frac{1}{3}\right|\left|[f'(q)]^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}[f'(p)]^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}da\right|\right]$$

$$\leq \int_0^1\left[\left|\frac{1}{3}-\frac{a}{2}\right||f'(q)|^{\left(\frac{1+a}{2}\right)\frac{1}{s}}|f'(p)|^{\left(\frac{1-a}{2}\right)\frac{1}{s}+1-\frac{1}{s}}da\right.,$$

$$= \frac{5}{36}|f'(p)|^{1-\frac{1}{2s}}|f'(q)|^{\frac{1}{2s}}\left(\Phi_1\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)\right). \tag{6.26}$$

Similar to Inequality (6.26), we have

$$I_8 = \int_0^1\left[\left|\frac{1}{3}-\frac{a}{2}\right|\left|f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}da\right|\right.$$

$$\leq \int_0^1\left[\left|\frac{1}{3}-\frac{a}{2}\right||f'(q)|^{\left(\frac{1+a}{2}\right)\frac{1}{s}+1-\frac{1}{s}}|f'(p)|^{\left(\frac{1-a}{2}\right)\frac{1}{s}}da\right],$$

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$$= \frac{5}{36}|f'(p)|^{\frac{1}{2s}}|f'(q)|^{1-\frac{1}{2s}}\left(\Phi_2\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)\right). \tag{6.27}$$

By substituting the values of I_7 and I_8 from Inequalities (6.26) and (6.27) in equation (6.17), results

$$\left|\frac{1}{6}f(p) + 4f\left(\frac{p+q}{2}\right) + f(q) - \left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right| \leq \left(\frac{q-p}{2}\right)\left(\frac{5}{36}\right)$$

$$\times \left|f'(p)^{\left(1-\frac{1}{2s}\right)}\right|\left|f'(q)^{\left(\frac{1}{2s}\right)}\right|\left[\Phi_1\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right) + \Phi_2\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)\right]$$

with Φ_1 and Φ_2 is defined in Theorem (6.5) and the second condition of Theorem (6.6) satisfies.

While, for the third condition using Lemma (2.4) for $1 \leq |f'(q)|$ and by applying integration with respect to a where $a \in [0,1]$ on Equation (6.17), then similar to Inequality (6.24), we have

$$I_9 = \int_0^1 \left[\left|\frac{a}{2}-\frac{1}{3}\right|\left|f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}da\right|\right]$$

$$\leq \int_0^1 \left[\left|\frac{1}{3}-\frac{a}{2}\right||f'(q)|^{\left(\frac{1+a}{2}\right)\frac{1}{s}+1-\frac{1}{s}}|f'(p)|^{\left(\frac{1-a}{2}\right)\frac{1}{s}+1-\frac{1}{s}}da\right],$$

$$= \left(\frac{5}{36}\right)|f'(p)f'(q)|^{1-\frac{1}{2s}}\left(\Phi_1\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)\right). \tag{6.28}$$

Similar to Inequality (6.24), we have

$$I_{10} = \int_0^1 \left[\left|\frac{1}{3}-\frac{a}{2}\right|\left|f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}da\right|\right.$$

$$\leq \int_0^1 \left[\left|\frac{1}{3}-\frac{a}{2}\right| |f'(q)|^{\left(\frac{1+a}{2}\right)\frac{1}{s}+1-\frac{1}{s}}|f'(p)|^{\left(\frac{1-a}{2}\right)\frac{1}{s}+1-\frac{1}{s}}da\right],$$

$$= \left(\frac{5}{36}\right)|f'(p)f'(q)|^{1-\frac{1}{2s}}\left(\Phi_2\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)\right). \qquad (6.29)$$

By substituting the values of I_9 and I_{10} from Inequalities (6.28) and (6.29) in equation (6.17), results

$$\left|\frac{1}{6}f(p)+4f\left(\frac{p+q}{2}\right)+f(q)-\left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right| \leq \left(\frac{q-p}{2}\right)\left(\frac{5}{36}\right)$$

$$\times\left|[f'(p)f'(q)]^{\left(1-\frac{1}{2s}\right)}\right|\left[\Phi_1\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)+\Phi_2\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)\right]$$

with Φ_1 and Φ_2 is defined in Theorem (6.5). The third condition of Theorem (6.6) is also satisfies and this completes the proof.

Theorem 6.6: *Let* $f:\mathbb{I}\to\mathbb{R}$ *be a* (g,s)*-convex function in third sense and differentiable on* $s\in(0,1]$ *such that* $p,q\in\mathbb{I}$, $p<q$, $f'\in L(p,q)$, $a\in[0,1]$, *and monotonically decreasing. If* $|f'(x)|^d$ *with* $d>1$ *is also* (g,s)*-convex function in the third sense on* $\mathbb{I}$, *then*

$$\left|\frac{1}{6}\left[f(p)+f(q)+4f\left(\frac{p+q}{2}\right)\right]-\left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right| \leq \left(\frac{q-p}{2}\right)\left(\frac{2(2^{d+1}+1)}{6^{d+1}(d+1)}\right)^{\frac{1}{d}}$$

$$\times\begin{cases} \left|[f'(p)f'(q)]^{\left(\frac{1}{2s}\right)}\right|\left[\Phi_3\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}+\Phi_4\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}\right], \\ |f'(p)|\leq 1; \\ \left|f'(p)^{\left(1-\frac{1}{2s}\right)}\right|\left|f'(q)^{\left(\frac{1}{2s}\right)}\right|\left[\Phi_3\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}+\Phi_4\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}\right], \\ |f'(q)|\leq 1\leq |f'(p)|; \\ \left|[f'(p)f'(q)]^{\left(1-\frac{1}{2s}\right)}\right|\left[\Phi_3\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}+\Phi_4\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}\right], \\ 1\leq |f'(q)|. \end{cases}$$

Here, $\Phi_3(\beta)$ and $\Phi_4(\beta)$ are

$$\Phi_3(\beta)=\begin{cases} 1, & \beta=1 \\ \dfrac{\beta-1}{ln\,\beta}, & \beta\neq 1, \end{cases}$$

and

$$\Phi_4(\beta)=\begin{cases} 1, & \beta=1 \\ \dfrac{\beta^{-1}-1}{ln\,\beta^{-1}}, & \beta\neq 1. \end{cases}$$

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Proof: Since $p<q$ and $a\in[0,1]$, then by using Lemma (2.7) for $\frac{1}{c}+\frac{1}{d}=1$ with $1<c<\infty, d>1$ on Theorem (2.2), results

$$\left|\frac{1}{6}\left[f(p)+f(q)+4f\left(\frac{p+q}{2}\right)\right]-\left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right|$$

$$=\left(\frac{q-p}{2}\right)\left|\int_0^1\left|\frac{a}{2}-\frac{1}{3}\right|^d da\right|^{\frac{1}{d}}\left|\int_0^1\left[\left|f'\left(\frac{1+a}{2}q-\frac{1-a}{2}p\right)da\right|^c\right|^{\frac{1}{c}}$$

$$+\left|\int_0^1 \left|\frac{1}{3}-\frac{a}{2}\right|^d dt\right|^{\frac{1}{d}} \left|\int_0^1 \left|f'\left(\frac{1+a}{2}p-\frac{1-a}{2}q\right)\right|^c da\right|^{\frac{1}{c}}. \quad (6.30)$$

As $|f'(x)|^c$ is monotonically decreasing and (g,s)-convex function in third sense, then by using Inequality (6.1) in Equation (6.30), results

$$\left|\frac{1}{6}\left[f(p)+f(q)+4f\left(\frac{p+q}{2}\right)\right]-\left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right| \leq \left(\frac{q-p}{2}\right)\left(\frac{2(2^{d+1}+1)}{6^{d+1}(d+1)}\right)^{\frac{1}{d}}$$

$$\times\left[\left|\int_0^1 \left[\left|f'\left(q^{\left(\frac{1+a}{2}\right)}p^{\left(\frac{1-a}{2}\right)}\right)\right|^c\right] dt\right|^{\frac{1}{c}} + \left|\int_0^1 \left|f'\left(p^{\left(\frac{1+a}{2}\right)}q^{\left(\frac{1-a}{2}\right)}\right)\right|^c da\right|^{\frac{1}{c}}\right],$$

$$\leq \left(\frac{q-p}{2}\right)\left(\frac{2(2^{d+1}+1)}{6^{c+1}(d+1)}\right)^{\frac{1}{d}}\left[\left|\int_0^1 f'(q)^{c\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} f'(p)^{c\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} da\right|^{\frac{1}{c}}\right.$$

$$\left.+\left|\int_0^1 f'(p)^{c\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} f'(q)^{c\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} da\right|^{\frac{1}{c}}\right]. \quad (6.31)$$

By using Lemma (2.4) for $0<|f'(p)|\leq 1$ and by applying integration with respect to a where $a\in[0,1]$ on Inequality (6.31), results

$$I_{11}=\int_0^1 \left|f'(q)^{c\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}\right|\left|f'(p)^{c\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}\right| da \leq \int_0^1 \left|f'(q)^{c\left(\frac{1+a}{2s}\right)}\right|\left|f'(p)^{c\left(\frac{1-a}{2s}\right)} da\right|,$$

$$=|f'(p)f'(q)|^{\frac{c}{2s}}\left[\int_0^1 \left(\frac{f'(q)}{f'(p)}\right)^{\frac{c}{2s}}\right]^a da,$$

$$=|f'(p)f'(q)|^{\frac{c}{2s}}\Phi_3\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}. (6.32)$$

Similar to Inequality (6.32), we have

$$I_{12}=\int_0^1\left|f'(p)^{c\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}\right|\left|f'(q)^{c\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}\right|da\ \leq\int_0^1\left|f'(p)^{c\left(\frac{1+a}{2s}\right)}\right|\left|f'(q)^{c\left(\frac{1-a}{2s}\right)}da\right|,$$

$$=|f'(p)f'(q)|^{\frac{c}{2s}}\left[\int_0^1\left(\frac{f'(p)}{f'(q)}\right)^{\frac{c}{2s}}\right]^{c}da,$$

$$=|f'(p)f'(q)|^{\frac{c}{2s}}\Phi_4\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}. \qquad (6.33)$$

By substituting the values of I_{11} and I_{12} from Inequalities (6.32) and (6.33) in Equation (6.31), results

$$\left|\frac{1}{6}\left[f(p)+f(q)+4f\left(\frac{p+q}{2}\right)\right]-\left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right|=\left(\frac{q-p}{2}\right)\left(\frac{2(2^{d+1}+1)}{6^{d+1}(d+1)}\right)^{\frac{1}{d}}$$

$$\times\left|[f'(p)f'(q)]^{\left(\frac{1}{2s}\right)}\right|\left[\Phi_3\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}+\Phi_4\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}\right]$$

with Φ_3 and Φ_4 is defined in Theorem (6.6) and the first condition of Theorem (6.6) is satisfied.

For the second condition, again using Lemma (2.4) for $|f'(q)|\leq 1\leq|f'(p)|$ and by applying integration with respect to a where $a\in[0,1]$ on Inequality (6.31), gives

$$I_{13}=\int_0^1\left|f'(q)^{c\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}\right|\left|f'(p)^{c\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}da\right|$$

$$\leq\int_0^1\left|f'(q)^{c\left(\frac{1+a}{2s}\right)}\right|\left|f'(p)^{c\left(\frac{1-a}{2s}+1-\frac{1}{s}\right)}da\right|,$$

$$=|f'(p)|^{\frac{c}{2s}}|f'(q)|^{c-\frac{c}{2s}}\left[\int_0^1\left(\frac{f'(q)}{f'(p)}\right)^{\frac{c}{2s}}\right]^{a}da,$$

$$= |f'(p)|^{\frac{c}{2s}} |f'(q)|^{c-\frac{c}{2s}} \Phi_3 \left(\beta\left(\frac{c}{2s}, \frac{c}{2s}\right)\right)^{\frac{1}{c}}. \tag{6.34}$$

Similar to Inequality (6.34), we have

$$I_{14} = \int_0^1 \left| f'(p)^{c\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} \right| \left| f'(q)^{c\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} da \right|$$

$$\leq \int_0^1 \left| f'(p)^{c\left(\frac{1+a}{2s}\right)} \right| \left| f'(q)^{c\left(\frac{1-a}{2s}+1-\frac{1}{s}\right)} da \right|,$$

$$= |f'(q)|^{\frac{c}{2s}} |f'(p)|^{c-\frac{c}{2s}} \left[\int_0^1 \left(\frac{f'(p)}{f'(q)} \right)^{\frac{c}{2s}} \right]^a da,$$

$$= |f'(p)|^{\frac{c}{2s}} |f'(q)|^{c-\frac{c}{2s}} \Phi_4 \left(\beta\left(\frac{c}{2s}, \frac{c}{2s}\right)\right)^{\frac{1}{c}}. \tag{6.35}$$

By substituting the values of I_{14} and I_{15} from Inequalities (6.34) and (6.35) in Equation (6.31), results

$$\left| \frac{1}{6}\left[f(p) + f(q) + 4f\left(\frac{p+q}{2}\right) \right] - \left(\frac{1}{q-p}\right) \int_p^q f(x)dx \right| = \left(\frac{q-p}{2}\right) \left(\frac{2(2^{d+1}+1)}{6^{d+1}(d+1)} \right)^{\frac{1}{d}}$$

$$\times \left| f'(p)^{\left(1-\frac{1}{2s}\right)} \right| \left| f'(q)^{\left(\frac{1}{2s}\right)} \right| \left[\Phi_3 \left(\beta\left(\frac{c}{2s}, \frac{c}{2s}\right)\right)^{\frac{1}{c}} + \Phi_4 \left(\beta\left(\frac{c}{2s}, \frac{c}{2s}\right)\right)^{\frac{1}{c}} \right]$$

with Φ_3 and Φ_4 is defined in Theorem (6.6) and the second condition of Theorem (6.6) satisfied.

While, for the third condition, using Lemma (2.4) for $1 \leq |f'(q)|$ and by applying integration with respect to a where $a \in [0, 1]$ on Inequality (6.31) results

$$I_{15}=\int_0^1\left|f'(q)^{c\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}\right|\left|f'(p)^{c\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}da\right|$$

$$\leq\int_0^1\left|f'(q)^{c\left(\frac{1+a}{2s}+1-\frac{1}{s}\right)}\right|\left|f'(p)^{c\left(\frac{1-a}{2s}+1-\frac{1}{s}\right)}da\right|,$$

$$=|f'(p)|^{c-\frac{c}{2s}}|f'(q)|^{c-\frac{c}{2s}}\left[\int_0^1\left(\frac{f'(q)}{f'(p)}\right)^{\frac{c}{2s}}da\right]^a,$$

$$=|f'(p)f'(q)|^{c-\frac{c}{2s}}\Phi_3\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}. \tag{6.36}$$

Similar to Inequality (6.36), we have

$$I_{16}=\int_0^1\left|f'(p)^{c\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}\right|\left|f'(q)^{c\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}da\right|$$

$$\leq\int_0^1\left|f'(p)^{c\left(\frac{1+a}{2s}+1-\frac{1}{s}\right)}\right|\left|f'(q)^{c\left(\frac{1-a}{2s}+1-\frac{1}{s}\right)}da\right|,$$

$$=|f'(q)|^{c-\frac{c}{2s}}|f'(p)|^{c-\frac{c}{2s}}\left[\int_0^1\left(\frac{f'(p)}{f'(q)}\right)^{\frac{c}{2s}}\right]^a da,$$

$$=|f'(p)f'(q)|^{c-\frac{c}{2s}}\Phi_4\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}. \tag{6.37}$$

By substituting the values of I_{15} and I_{16} from Inequalities (6.36) and (6.37) in Equation (6.31), results

$$\left|\frac{1}{6}\left[f(p)+f(q)+4f\left(\frac{p+q}{2}\right)\right]-\left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right|=\left(\frac{q-p}{2}\right)\left(\frac{2(2^{d+1}+1)}{6^{d+1}(d+1)}\right)^{\frac{1}{d}}$$

$$\times\left|[f'(p)f'(q)]^{\left(1-\frac{1}{2s}\right)}\right|\left[\Phi_3\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}+\Phi_4\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}\right]$$

with Φ_3 and Φ_4 is defined in Theorem (6.6) and the third condition of Theorem (6.6) satisfied and completes the proof.

Theorem 6.7: *Let $f: \mathbb{I} \to \mathbb{R}$ be a (g, s)-convex function in third sense, monotonically decreasing and differentiable on $s \in (0, 1]$ such that $f' \in L(p, q),\ p < q$. If $|f'(x)|^d$ with $d > 1$ is also (g, s)-convex function in the third sense on $\mathbb{I}$, then*

$$\left|\frac{1}{6}\left[f(p) + f(q) + 4f\left(\frac{p+q}{2}\right)\right] - \left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right| \leq \left(\frac{q-p}{2}\right)\left(\frac{5}{36}\right)^{\frac{1}{c}}$$

$$\times \begin{cases} \left|[f'(p)f'(q)]^{\left(\frac{1}{2s}\right)}\right|\left[\Phi_1\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}} + \Phi_2\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}\right], \\ \qquad |f'(p)| \leq 1; \\ \left|f'(p)^{\left(1-\frac{1}{2s}\right)}\right|\left|f'(q)^{\left(\frac{1}{2s}\right)}\right|\left[\Phi_1\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}} + \Phi_2\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}\right], \\ \qquad |f'(q)| \leq 1 \leq |f'(p)|; \\ \left|[f'(p)f'(q)]^{\left(1-\frac{1}{2s}\right)}\right|\left[\Phi_1\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}} + \Phi_2\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}\right], \\ \qquad 1 \leq |f'(q)|. \end{cases} \tag{6.38}$$

Here, Φ_1, Φ_2 and β in Inequality (6.38) are similarly defined in Theorem (6.5).

Proof: Since $p < q$ and $a \in [0, 1]$, then by using Lemma (2.7), yields

$$\left|\frac{1}{6}\left[f(p) + f(q) + 4f\left(\frac{p+q}{2}\right)\right] - \left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right|$$

$$\leq \left(\frac{q-p}{2}\right)\left|\int_0^1 \left|\frac{a}{2} - \frac{1}{3}\right| da\right|\left|\int_0^1 \left[\left|f'\left(\frac{1+a}{2}q - \frac{1-a}{2}p\right)\right|\right] da\right|$$

$$+\left|\int_0^1 \left|\frac{1}{3} - \frac{a}{2}\right| da\right|\left|\int_0^1 \left|f'\left(\frac{1+a}{2}p - \frac{1-a}{2}q\right)\right| da\right|. \tag{6.39}$$

As $|f'(x)|^c$ with $c > 1$ is also (g, s)-convex function in the third sense and monotonically decreasing for $\mathbb{I}$, then by using Theorem (2.18) and then utilising Inequality (6.1), Inequality (6.39) will be expanded to

$$\left|\frac{1}{6}\left[f(p) + f(q) + 4f\left(\frac{p+q}{2}\right)\right] - \left(\frac{1}{q-p}\right)\int_p^q f(x)dx\right|$$

$$\leq \left(\frac{q-p}{2}\right)\left(\frac{5}{36}\right)^{1-\frac{1}{c}}\left|\int_0^1 \left|\frac{a}{2} - \frac{1}{3}\right|\left|f'\left(q^{\left(\frac{1+a}{2}\right)}p^{\left(\frac{1-a}{2}\right)}\right)\right|^c da\right|^{\frac{1}{c}}$$

$$+\left|\int_0^1 \left|\frac{1}{3} - \frac{a}{2}\right|\left|f'\left(p^{\left(\frac{1+a}{2}\right)}q^{\left(\frac{1-a}{2}\right)}\right)\right|^c da\right|^{\frac{1}{c}}\Bigg],$$

$$\leq \left(\frac{q-p}{2}\right)\left(\frac{5}{36}\right)^{1-\frac{1}{c}}\left|\int_0^1 \left|\frac{a}{2} - \frac{1}{3}\right|\left|f'\left(q^{c\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}p^{c\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}\right)\right|da\right|^{\frac{1}{c}}$$

$$+\left|\int_0^1 \left|\frac{1}{3} - \frac{a}{2}\right|\left|f'\left(p^{c\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}q^{c\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}\right)\right|da\right|^{\frac{1}{c}}\Bigg]. \tag{6.40}$$

By using Lemma (2.4) for $0 < |f'(p)| \leq 1$ and by applying integration with respect to a where $a \in [0, 1]$ on Inequality (6.40), results

$$I_{17} = \int_0^1 \left[\left|\frac{a}{2} - \frac{1}{3}\right|\left|f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}da\right|\right]$$

$$\leq \int_0^1 \left[\left|\frac{a}{2} - \frac{1}{3}\right||f'(q)|^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}}|f'(p)|^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}}da\right],$$

$$= |f'(p)f'(q)|^{\frac{1}{2s}}\left(\Phi_1\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right)\right). \tag{6.41}$$

Similar to Inequality (6.41), we have

$$I_{18} = \int_0^1 \left[\left|\frac{1}{3} - \frac{a}{2}\right| \left| f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} da \right|\right.$$

$$\leq \int_0^1 \left[\left|\frac{1}{3} - \frac{a}{2}\right| |f'(q)|^{\left(\frac{1+a}{2}\right)\frac{1}{s}} |f'(p)|^{\left(\frac{1-a}{2}\right)\frac{1}{s}} da \right],$$

$$= |f'(p)f'(q)|^{\frac{1}{2s}} \left(\Phi_2 \left(\beta \left(\frac{1}{2s}, \frac{1}{2s} \right) \right) \right). \tag{6.42}$$

By substituting the values of I_{17} and I_{18} from Inequalities (6.41) and (6.42) in Inequality (6.40), results

$$\left| \frac{1}{6} \left[f(p) + f(q) + 4f\left(\frac{p+q}{2}\right) \right] - \left(\frac{1}{q-p}\right) \int_p^q f(x)dx \right| \leq \left(\frac{q-p}{2}\right) \left(\frac{5}{36}\right)^{\frac{1}{c}}$$

$$\times \left| [f'(p)f'(q)]^{\left(\frac{1}{2s}\right)} \right| \left[\Phi_1 \left(\beta \left(\frac{c}{2s}, \frac{c}{2s} \right) \right)^{\frac{1}{c}} + \Phi_2 \left(\beta \left(\frac{c}{2s}, \frac{c}{2s} \right) \right)^{\frac{1}{c}} \right]$$

with Φ_1 and Φ_2 is defined in Theorem (6.5) and the first condition of Theorem (6.7) satisfies.

Further, by using Lemma (2.4) for $|f'(q)| \leq 1 \leq |f'(p)|$ and by applying integration with respect to a where $a \in [0, 1]$ on Inequality (6.40), yields

$$I_{19} = \int_0^1 \left[\left|\frac{a}{2} - \frac{1}{3}\right| \left| f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} da \right| \right]$$

$$\leq \int_0^1 \left[\left|\frac{1}{3} - \frac{a}{2}\right| |f'(q)|^{\left(\frac{1+a}{2}\right)\frac{1}{s}} |f'(p)|^{\left(\frac{1-a}{2}\right)\frac{1}{s}+1-\frac{1}{s}} da \right],$$

$$= |f'(p)|^{1-\frac{1}{2s}} |f'(q)|^{\frac{1}{2s}} \left(\Phi_1 \left(\beta \left(\frac{1}{2s}, \frac{1}{2s} \right) \right) \right). \tag{6.43}$$

Similar to Inequality (6.43), we have

$$I_{20} = \int_0^1 \left[\left| \frac{1}{3} - \frac{a}{2} \right| \left| f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} da \right|$$

$$\leq \int_0^1 \left[\left| \frac{1}{3} - \frac{a}{2} \right| |f'(q)|^{\left(\frac{1+a}{2}\right)\frac{1}{s}+1-\frac{1}{s}} |f'(p)|^{\left(\frac{1-a}{2}\right)\frac{1}{s}} da \right],$$

$$= |f'(p)|^{\frac{1}{2s}} |f'(q)|^{1-\frac{1}{2s}} \left(\Phi_2 \left(\beta \left(\frac{1}{2s}, \frac{1}{2s} \right) \right) \right). \tag{6.44}$$

By substituting the values of I_{19} and I_{20} from Inequalities (6.43) and (6.44) in Inequality (6.40), results

$$\left| \frac{1}{6} \left[f(p) + f(q) + 4f\left(\frac{p+q}{2} \right) \right] - \left(\frac{1}{q-p} \right) \int_p^q f(x)dx \right| \leq \left(\frac{q-p}{2} \right) \left(\frac{5}{36} \right)^{\frac{1}{c}}$$

$$\times \left| f'(p)^{\left(1-\frac{1}{2s}\right)} \right| \left| f'(q)^{\left(\frac{1}{2s}\right)} \right| \left[\Phi_1 \left(\beta \left(\frac{c}{2s}, \frac{c}{2s} \right) \right)^{\frac{1}{c}} + \Phi_2 \left(\beta \left(\frac{c}{2s}, \frac{c}{2s} \right) \right)^{\frac{1}{c}} \right]$$

with Φ_1 and Φ_2 is defined in Theorem (6.5) and the second condition of Theorem (6.7) satisfies.

While, for the third condition using Lemma (2.4) for $1 \leq |f'(q)|$ and by applying integration with respect to a where $a \in [0, 1]$ on Inequality (6.40), gives

$$I_{21} = \int_0^1 \left[\left| \frac{a}{2} - \frac{1}{3} \right| \left| f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} da \right| \right]$$

$$\leq \int_0^1 \left[\left| \frac{1}{3} - \frac{a}{2} \right| |f'(q)|^{\left(\frac{1+a}{2}\right)\frac{1}{s}+1-\frac{1}{s}} |f'(p)|^{\left(\frac{1-a}{2}\right)\frac{1}{s}+1-\frac{1}{s}} da \right],$$

$$= |f'(p)f'(q)|^{1-\frac{1}{2s}} \left(\Phi_1 \left(\beta \left(\frac{1}{2s}, \frac{1}{2s} \right) \right) \right). \tag{6.45}$$

Similar to Inequality (6.45), we have

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$$I_{22} = \int_0^1 \left[\left|\frac{1}{3} - \frac{a}{2}\right| \left| f'(q)^{\left(\frac{1+a}{2}\right)^{\frac{1}{s}}} f'(p)^{\left(\frac{1-a}{2}\right)^{\frac{1}{s}}} da \right| \right.$$

$$\le \int_0^1 \left[\left|\frac{1}{3} - \frac{a}{2}\right| |f'(q)|^{\left(\frac{1+a}{2}\right)\frac{1}{s}+1-\frac{1}{s}} |f'(p)|^{\left(\frac{1-a}{2}\right)\frac{1}{s}+1-\frac{1}{s}} da \right],$$

$$= |f'(p)f'(q)|^{1-\frac{1}{2s}} \left(\Phi_2 \left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right)\right), \tag{6.46}$$

By substituting the values of I_{21} and I_{22} from Inequalities (6.45) and (6.46) in Inequality (6.40), results

$$\left| \frac{1}{6}\left[f(p) + f(q) + 4f\left(\frac{p+q}{2}\right)\right] - \left(\frac{1}{q-p}\right) \int_p^q f(x)dx \right| \le \left(\frac{q-p}{2}\right)\left(\frac{5}{36}\right)^{\frac{1}{c}}$$

$$\times \left| [f'(p)f'(q)]^{\left(1-\frac{1}{2s}\right)} \right| \left[\Phi_1 \left(\beta\left(\frac{c}{2s}, \frac{c}{2s}\right)\right)^{\frac{1}{c}} + \Phi_2 \left(\beta\left(\frac{c}{2s}, \frac{c}{2s}\right)\right)^{\frac{1}{c}} \right]$$

with Φ_1 and Φ_2 is defined in Theorem (6.5) and the third condition of Theorem (6.7) satisfies and completes the proof.

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Remarks 6. 5:

1. Substituting $\frac{1}{s} = 1$ in Theorem (6.5), (6.6) and (6.7), results will be reduced to Simpson's inequality for geometrically convex function.
2. If $\frac{1}{s} = s$ in Theorem (6.5), (6.6) and (6.7), then Simpson's inequality for s-convex function in second sense will be produced.

6.5 Summary

This chapter established some new hybrid convex functions by combining s-convex, h-convex, and g-convex functions. Further, on the basis of these hybrid convex functions, Simpson's type of inequalities is extended or generalised. In Section (6.2) Simpson's inequality is extended by combining s-convex and h-convex functions. Similarly, in Section (6.3) Simpson's inequality is extended by combining g-convex and h-convex functions and in Section (6.4), Simpson's inequality is extended by combining s-convex and g-convex functions.

Furthermore, to highlight the role of hybrid convex functions, some inequalities are employed to show some relation with the mathematical mean (average).

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CHAPTER SEVEN
APPLICATIONS FOR NEWLY EXTENDED MATHEMATICAL INEQUALITIES

This chapter discusses the applications of mathematical inequalities formed by hybrid convex functions. Furthermore, error estimation formulae in term of Trapezoidal, Mid-point, and Simpson's formula and mathematical mean (average) for newly extended H-H, Jensen's, Fejér's, and Simpson's inequalities will be described.

7.1 Introduction

In this chapter, mathematical inequalities, such as H-H, Jensen's, Fejer's, and Simpson's, which were derived in Chapters Three, Four, Five, and Six, are utilised. Through the application of specific conditions, the formal form, as well as some previously established results of H-H, Jensen's, Fejer's, and Simpson's type of inequalities, can be obtained. This approach validates new inequalities and validates the correct extension of these inequalities based on hybrid convex functions.

Furthermore, the extended mathematical inequality creates a link with the mathematical mean (average) for real numbers and derives the error estimation formulae for the numerical formulas such as Trapezoidal, Mid-point, and Simpson's.

7.2 Validation of Mathematical Inequalities

In this section, newly constructed inequalities, such as H-H, Jensen's, Fejér's, and Simpson's, which were introduced in Chapters Three, Four, Five, and Six, are examined. Under specific conditions, these newly developed inequalities are transformed into well-known results that have been previously proven.

7.2.1 Validation of Hermite-Hadamard Inequality

In this subsection, newly constructed H-H type of inequalities, obtained in Chapter Three are considered to be validated on some specific conditions as follows:

In section 3.1, if we impose the condition on Theorem (3.1) as $h_1^s(a) = a$ and $h_2^s(1-a) = (1-a)$, then Theorem (3.1) will be converted to H-H inequality for a convex function stated in Theorem (2.3). Furthermore, by substituting $h_1^s(a)h_2^s(1-a) = h(a)$ for $a = \frac{1}{2}$ in Theorem (3.1), H-H inequality in Theorem (2.5) will be obtained. While, if $h_1^s(a) = a^{s_1}$ and $h_2^s(1-a) = a^{s_2}$ is considered in Theorem (3.1), then H-H inequality for Brecker type of (s_1, s_2)-convex function is found which is stated in Corollary (1) by Awan, Noor, and Khalida (2018). Moreover, in Theorems (3.2), (3.3), (3.4), and (3.5), if substituting $s = 1$ then H-H inequalities for (h_1, h_2)-convex function will be obtained.

Furthermore, in subsection 3.2.1, if considering $m = 1$ in Theorems (3.10) and (3.11), then H-H inequalities for (h, s) convex function will be obtained. In addition, if trying $h_1^s(a) = a$ and $h_2^s(1-a) = (1-a)$ with $m = 1$, then Theorems (3.6), (3.7), (3.8), and (3.9) will be converted to H-H inequality for a convex function.

Moreover, in subsection 3.2.2, if we impose the condition $h_1^s(a) = h^s(a)$ and $h_2^s(1-a) = h^s(1-a)$ with $m = 1$, then Theorems (3.10) and (3.11) will be converted to H-H inequality for (h, s)-harmonically convex function stated in Theorem (3.1) and (3.2) by Noor, Khalida, and Iftikhar (2016).

In section 3.3, if we impose the condition on Theorems (3.12), (3.13), and (3.14) as $h_1^s(a) = h(a)$ and $h_2^s(1-a) = h(1-a)$, then Theorem (3.12), (3.13), and (3.14) will be converted to H-H inequality for geometrically h-convex function derived by Ozdemir, Tunc, and Gurbuz (2012).

Moreover, in section 3.4, if trying $\frac{1}{s} = s$ in Theorems (3.15) and (3.16) then results will be converted to H-H inequality for geometrically s-convex function in second sense obtained by Yin and Wang (2018).

Since, newly constructed H-H inequalities in Chapter Three, successfully forming previously defined H-H inequalities by different researchers.

Note that H-H inequalities in Chapter Three are successfully validated on some specific conditions. Furthermore, in the proceeding subsection, certain conditions are considered to validate Jensen's type of inequalities obtained in Chapter Four.

7.2.2 Validation of Jensen's Type of Inequality

In this subsection, newly constructed Jensen's type of inequalities, obtained in Chapter Four are considered to be validated on some specific conditions as follows:

In section 4.2, if we impose the condition on Theorem (4.1) as $s = 1$, then Theorem (4.1) will be converted to Jensen's inequality for h-convex function obtained by Varošanec (2007). While, if $h_1^s(a) = a$ and $h_2^s(1-a) = (1-a)$ is considered in Theorem (4.2) and (4.3) then Jensen's inequality for a convex function is found stated in Theorem (1.2).

Furthermore, in section 4.3, if considering $G_1^n = 1$ for all $n \in \mathbb{N}$, in Theorems (4.4), then Jensen's inequality for h-convex function will be obtained. In addition, if trying $h(a) = h_1^s(a)$ and $h(1-a) = h_2^s(1-a)$, then Theorems (4.4) will be converted to Jensen's inequality for geometrically (h_1, h_2)-convex function.

Moreover, in section 4.4, if we impose the condition on Theorem (4.5) as $\frac{1}{s} = s$, then Theorem (4.5) will be converted to Jensen's inequality for geometrically s-convex function in second sense.

Note that Jensen's type of inequalities in Chapter Four are successfully validated on some specific conditions. Furthermore, in proceeding subsection, certain conditions are considered to validate Fejér's type of inequalities obtained in Chapter Five.

7.2.3 Validation of Fejér's Type of Inequality

In this subsection, newly constructed Fejér's type of inequalities, obtained in Chapter Five are considered to be validated on some specific conditions as follows:

In chapter five, if we impose a condition $g(x) \equiv 1$, then Fejér's type of inequalities will be converted into H-H inequalities. Furthermore, in section 5.2, if $s = 1$ is considered in Theorems (5.1), (5.2), (5.3), (5.4), and (5.5), then Fejér's type of inequalities for (h_1, h_2)-harmonically convex function will be obtained. In addition, if substituting $m = 1$, in subsection 5.2.1, then results will be converted to Fejér's type of inequalities for (h_1, h_2, s)-harmonically convex function in second sense.

Furthermore, in section 5.3, if considering $\frac{1}{s} = s$, in Theorems (5.7) and (5.8), then Fejér's type of inequalities for geometrically s-convex function in second sense will be obtained. While in section 5.4, if $s = m = \alpha = 1$, then Theorems (5.9) and (5.10) will be converted into Fejér's type of inequalities for h-convex function.

Note that Fejér's type of inequalities in Chapter Five are successfully validated on some specific conditions. Furthermore, in proceeding subsection, certain conditions are considered to validate Simpson's type of inequalities obtained in Chapter Six.

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7.2.4 Validation of Simpson's Type of Inequality

In this subsection, newly constructed Simpson's type of inequalities, obtained in Chapter Six are considered to be validated on some specific conditions as follows:

In section 5.2, if considering $\frac{1}{s} = s$, in Theorems (6.1), (6.2), (6.3), and (6.4), then Simpson's type of inequalities for (h_1, h_2, s)-convex function in second sense will be obtained. While, if choosing $s = 1$, then results will be converted into Simpson's type of inequality for (h_1, h_2)-convex function.

Furthermore, if we impose the condition on Theorem (6.5) as $h_1^s(a) = h(a)$ and $h_2^s(1-a) = h(1-a)$, then Simpson's type of inequality for geometrically h-convex function will be found.

Moreover, in section 6.4, if considering $\frac{1}{s} = s$, in Theorems (6.6), (6.7), and (6.8), then results will be transformed into Simpson's type of inequalities for geometrically s-convex function in second sense.

Since, all the mathematical inequalities such as H-H, Jensen's, Fejér's, and Simpson's derived in Chapters Three, Four, Five, and Six respectively, are successfully validated in section 7.2. While, in section 7.3, we have established a link of mathematical inequalities such as H-H, Jensen's, Fejér's, and Simpson's inequalities with the mathematical means (averages) defined in (2.9).

7.3 Mathematical Means (Average)

In this section, the relationship between newly extended mathematical inequalities and mathematical means (averages) is established.

Note that Definition (2.5) of Mathematical means (averages) (Arithmetic, Geometric, Harmonic, Logarithmic, Generalised Log mean, α-Logarithmic, q-Logarithmic, q-Logarithmic) are used in Propositions (7.1), (7.2), (7.3), and (7.4) to link H-H, Jensen's, Fejér's, and Simpson's type of inequalities with the special means (averages) as follows:

Proposition 7.1: *Let* $0 < p < q$*. Then for* $f(x_1) = \frac{1}{x_1^\alpha}$ *for all* $\alpha \geq 1$*, Theorem (3.10) can be written as*

$$2^{(\alpha_1+\alpha_2)} \mathrm{G}^2(p^\alpha, q^\alpha) H^{-\alpha}(p,q) \leq [2^{\alpha_2} m^{\alpha-1} + 2^{\alpha_1}] L_\alpha^\alpha(p,q). \tag{7.1}$$

Proof: Since $p, q \in \mathbb{R}_+$ and f is a (h_1, h_2, s, m)-harmonically convex function in second sense, then by using $f(x_1) = \frac{1}{x_1^\alpha}$ for all $\alpha \geq 1$ and $h_1^s(a) = a^{\alpha_1}$, $h_2^s(a) = a^{\alpha_2}$, for all $a \in [0, 1]$ and $0 < \alpha_1, \alpha_2 \leq 1$.

Inequality (3.51) can be written as

$$f\left(\frac{2pq}{p+q}\right) \leq \frac{pq h_1^s\left(\frac{1}{2}\right)}{q-p} \int_p^q \frac{f(x_1)}{x_1^2} dx_1 + \frac{pqm h_2^s\left(\frac{1}{2}\right)}{q-p} \int_p^q \frac{f(x_1 m)}{x_1^2} dx_1.$$

After solving Inequality (3.51) using $f\left(\frac{2pq}{p+q}\right) = \frac{1}{\left(\frac{2pq}{p+q}\right)^\alpha}$ and the Harmonic mean, Geometric mean, and generalised log mean from Definition (2.5), yields

$$f\left(\frac{2pq}{p+q}\right) = \left(\frac{2pq}{p+q}\right)^{-\alpha} = H^{-\alpha}(p,q).$$

$$h_1^s\left(\frac{1}{2}\right) = \frac{1}{2^{\alpha_1}} \qquad = 2^{-\alpha_1}. \tag{7.2}$$

Similarly, as Equation (7.2), we can write

$$h_2^s\left(\frac{1}{2}\right) = \frac{1}{2^{\alpha_2}} = 2^{-\alpha_2}.$$

By solving right hand side of Inequality (3.51), results

$$\begin{aligned}
\left(\frac{pq}{q-p}\right)\int_p^q \frac{f(x_1)}{x_1^2}dx_1 &= \left(\frac{pq}{q-p}\right)\int_p^q \frac{1}{{x_1}^{\alpha+2}}dx_1,\\
&= \left(\frac{pq}{-(q-p)(\alpha+1)}\right)\left[q^{-(\alpha+1)} - p^{-(\alpha+1)}\right], \qquad (7.3)\\
&= \left(\frac{pq}{(q-p)(\alpha+1)}\right)\left[\frac{1}{p^{(\alpha+1)}} - \frac{1}{q^{(\alpha+1)}}\right],\\
&= \left(\frac{pq}{(q-p)(\alpha+1)}\right)\left(\frac{q^{(\alpha+1)} - p^{(\alpha+1)}}{q^{(\alpha+1)}p^{(\alpha+1)}}\right),\\
&= \left(\frac{1}{q^\alpha p^\alpha}\right)\left(\frac{q^{(\alpha+1)} - p^{(\alpha+1)}}{(\alpha+1)(q-p)}\right),\\
&= \left(\frac{1}{G^2(q^\alpha, p^\alpha)}\right)\left\{\left(\frac{q^{(\alpha+1)} - p^{(\alpha+1)}}{(\alpha+1)(q-p)}\right)^{\frac{1}{\alpha}}\right\},\\
&= \left(\frac{1}{G^2(q^\alpha, p^\alpha)}\right)L_\alpha^\alpha(p,q). \qquad (7.4)
\end{aligned}$$

Similarly, as Equation (7.4), we obtain

$$\begin{aligned}
\left(\frac{pqm}{q-p}\right)\int_p^q \frac{f(x_1 m)}{x_1^2}dx_1 &= \left(\frac{pq}{q-p}\right)\int_p^q \frac{1}{m^\alpha {x_1}^{\alpha+2}}dx_1,\\
&= \left(\frac{pq}{-(q-p)(\alpha+1)m^{(\alpha-1)}}\right)\left[q^{-(\alpha+1)} - p^{-(\alpha+1)}\right],\\
&= \left(\frac{pq}{(q-p)(\alpha+1)m^{(\alpha-1)}}\right)\left[\frac{1}{p^{(\alpha+1)}} - \frac{1}{q^{(\alpha+1)}}\right],\\
&= \left(\frac{pq}{(q-p)(\alpha+1)m^{(\alpha-1)}}\right)\left(\frac{q^{(\alpha+1)} - p^{(\alpha+1)}}{q^{(\alpha+1)}p^{(\alpha+1)}}\right),\\
&= \left(\frac{1}{q^\alpha p^\alpha m^{(\alpha-1)}}\right)\left(\frac{q^{(\alpha+1)} - p^{(\alpha+1)}}{(\alpha+1)(q-p)}\right),
\end{aligned}$$

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$$= \left(\frac{1}{G^2(q^\alpha, p^\alpha)m^{(\alpha-1)}}\right)\left\{\left(\frac{q^{(\alpha+1)} - p^{(\alpha+1)}}{(\alpha+1)(q-p)}\right)^{\frac{1}{\alpha}}\right\}^{\alpha},$$

$$= \left(\frac{1}{G^2(q^\alpha, p^\alpha)m^{(\alpha-1)}}\right) L_\alpha^\alpha(p,q). \tag{7.5}$$

Substituting values from Equations (7.2)-(7.5) in Inequality (3.40), yields

$$H^{-\alpha}(p,q) \le \left(\frac{1}{2^{\alpha_1}}\right)\left(\frac{1}{G^2(q^\alpha, p^\alpha)}\right) L_\alpha^\alpha(p,q)$$

$$+ \left(\frac{1}{2^{\alpha_2}}\right)\left(\frac{1}{G^2(q^\alpha, p^\alpha)m^{(\alpha-1)}}\right) L_\alpha^\alpha(p,q),$$

$$\le \frac{2^{\alpha_2}m^{(\alpha-1)} + 2^{\alpha_1}}{2^{\alpha_1+\alpha_2}m^{(\alpha-1)}}\left(\frac{1}{G^2(q^\alpha, p^\alpha)}\right) L_\alpha^\alpha(p,q). \tag{7.6}$$

Inequality (7.6) can be written as

$$2^{(\alpha_1+\alpha_2)}\mathrm{G}^2(p^\alpha, q^\alpha)H^{-\alpha}(p,q) \le [2^{\alpha_2}m^{\alpha-1} + 2^{\alpha_1}]L_\alpha^\alpha(p,q). \tag{7.7}$$

Inequality (7.7) completes the proof.

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Note that Proposition (7.1) successfully established a link between H-H inequality for (h_1, h_2, s, m)-harmonically convex function with special means (averages) in Definition (2.5). While, if we use $\frac{1}{q-p}\int_p^q f(x)dx = L_r^r(p,q)$ and $f\left(\frac{p+q}{2}\right) = A(p,q)$, then the whole expressions of theorems can be transformed into the form of special means (averages).

Furthermore, in Proposition (7.2), extended Jensen's inequality is successfully linked with the special means (averages) defined in Definition (2.5) as follows:

Proposition 7.2: *Let $q > p > 0$, $h^{\frac{1}{s}}(a_n) = a_n$ and $n \in \mathbb{N}$, then Theorem (4.1) becomes $Q_n(p,q) \le P_n(p,q)$.*

Proof: Since $q > p > 0$ and $h^{\frac{1}{s}}(a_n) = a_n$, then Inequality (4.12) can be converted to integral form of Jensen's inequality.

Set $f(x) = x$ in integral Jansen's inequality, then

$$\left(\int_p^q x^{n-1}dx\right)^n \leq (q-p)\left(\int_p^q x^n dx\right)^{n-1}, \tag{7.8}$$

$$\left(\frac{(q^n - p^n)}{n}\right)^n \leq (q-p)\left(\frac{(q^{n+1} - p^{n+1})}{n+1}\right)^{n-1}, \tag{7.9}$$

$$\frac{1}{(q-p)(n+1)} \leq \frac{n^n(q^{n+1} - p^{n+1})^{n-1}}{(n+1)^n(q^n - p^n)^n}. \tag{7.10}$$

Multiplying $(q^{n+1} - p^{n+1})$ on both sides of Inequality (7.10) and set $p = p^{\frac{1}{n}}$, $q = q^{\frac{1}{n}}$, yields

$$\frac{\left(q^{\frac{n+1}{n}} - p^{\frac{n+1}{n}}\right)}{\left(q^{\frac{1}{n}} - p^{\frac{1}{n}}\right)(n+1)} \leq \left(\frac{n}{n+1}\right)^n \left(\frac{\left(q^{\frac{n+1}{n}} - p^{\frac{n+1}{n}}\right)}{(q-p)}\right)^n. \tag{7.11}$$

This is equal to $Q_n(p,q) \leq P_n(p,q)$ and completes the proof.

Note that Proposition (7.2) successfully established a link between Jensen's inequality with special means (averages). In Proposition (7.2), q-logarithmic mean $(Q_n(p,q))$ and p-logarithmic mean $(L_\alpha^\alpha(p,q))$ defined in Definition (2.5) of Mathematical means (averages) are used in Jensen's inequality.

In addition, Proposition (7.3) creates a link between Fejér's inequality and mathematical means (averages) defined in Definition (2.5) as follows:

Proposition 7.3: *Let* $p, q \in \mathbb{R}$, $f(x) = x^n, x > 0$ *and* $g(x) = 1$ *with* ${h_i}^s(a) = a^k$, *where* $k \in (-\infty, -2) \cup (-1, 1]$ *in Theorem (5.4), then Inequality (5.25) can be written as*

$$|A(p^n, q^n) - L_n^n(p,q)| \leq \frac{n(q-p)}{(k+1)(k+2)} L(|(p)^{n-1}| + |(q)^{n-1}|)\left(\frac{1+k2^{k+1}}{2^{k+1}}\right).$$

Proof: Since $p, q \in \mathbb{R}$, $f(x) = x^n, x > 0$, and $g(x) = 1$ with ${h_i}^s(a) = a^k$, then Inequality (5.25) can be written as

$$\begin{aligned}
\left|\frac{(p)^n + (q)^n}{2}(q-p)\right| &- \frac{1}{n+1}\{(p)^n + (q)^n\} \\
&\leq n(q-p)(|(p)^{n-1}| + |(q)^{n-1}|) \\
&\times \int_p^{\frac{p+q}{2}} \int_0^{\frac{x-p}{q-p}} [(a)^k + (1-a)^k] dx da, \\
&= \frac{n(q-p)}{k+1}(|(p)^{n-1}| + |(q)^{n-1}|) \\
&\int_p^{\frac{p+q}{2}} \left[\left(\frac{x-p}{q-p}\right)^{k+1} + \left(\frac{q-x}{q-p}\right)^{k+1}\right] dx da, \\
&= \frac{n}{(k+1)(k+2)(q-p)^k} \\
&\times (|(p)^{n-1}| + |(q)^{n-1}|)|(x-p)^{k+2} \\
&+(q-p)^{k+1}(k+2)x - (q-x)^{k+2}|_p^{\frac{p+q}{2}} \\
&\times \left|\frac{(p)^n + (q)^n}{2} - \frac{(q)^{n+1} - (p)^{n+1}}{(n+1)(q-p)}\right. \\
&\leq \frac{n(q-p)}{(k+1)(k+2)}\{(p)^{n-1} + (q)^{n-1}\} \\
&\times \left(\frac{1+k2^{k+1}}{2^{k+1}}\right).
\end{aligned} \tag{7.12}$$

By using Arithmetic mean, generalised log mean, logarithmic mean from Definition (2.5) of mathematical means (averages), Inequality (7.12) will be converted to

$$|A(p^n, q^n) - L_n^n(p,q)| \leq \frac{n(q-p)}{(k+1)(k+2)} L(|(p)^{n-1}| + |(q)^{n-1}|)\left(\frac{1+k2^{k+1}}{2^{k+1}}\right).$$

Note that Proposition (7.3) successfully established a link between Fejér's inequality for (h_1, h_2, s)-convex function with special means (averages). Arithmetic Mean $(A(p^n, q^n))$ and generalised Logarithmic Mean $(L_n^n(p,q))$ defined in Definition (2.5) of Mathematical means (averages), are used in Fejér's inequality.

In Proposition (7.4), Fejér's inequality is successfully linked with the special means (averages) in Definition (2.5) as follows:

Proposition 7.4: *Let* $p, q \in \mathbb{R}$, $f(x) = \frac{1}{x}$, $x > 0$ *and* $g(x) = x$ *with* $h_i{}^s(a) = a^k$, *where* $k \in (-\infty, -2) \cup (-1, 1]$ *in Theorem (5.5), then*

$$|H^{-1}(p,q) - \bar{L}(p,q)| \leq \frac{(q-p)A(|p|^{-2}, |q|^{-2})}{(k+1)(k+2)}\left(k + \frac{1}{2^k}\right).$$

Proof: Since $p, q \in \mathbb{R}$ for $p < q$ and $s \in [0, 1]$, then similar to Propositions (7.1), (7.2), and (7.3), by substituting the values of Harmonic mean, logarithmic mean, Arithmetic mean from Definition (2.5) in Theorem (5.5) for $f(x) = \frac{1}{x}$ completes the proof of proposition (7.4).

Corollary 7.1: Suppose $k = 1$ in Proposition (7.3), we get Proposition (3.3) of Dragomir and Agarwal (1998).

In proceeding Propositions (7.5), (7.6), and (7.7), we established a link between Simpson's inequality and mathematical means (averages) as follows:

Proposition 7.5: *Let* $\mathbb{I} \in [p, q] \subset \mathbb{R}$, *and* $f(x) = \frac{x^{\frac{1}{s}}}{\frac{1}{s}}$ *is applied to the function in Theorem (6.5) with* $s \in (0, 1]$, *then*

$$\left|\frac{1}{6}\left(\frac{2A\left(p^{\frac{1}{s}},q^{\frac{1}{s}}\right)+4A^{\frac{1}{s}}(p,q)}{s}\right)-\frac{L_s^s(p,q)}{s}\right|\leq\left(\frac{q-p}{2}\right)$$

$$\times\begin{cases}\left|G\left[p^{\frac{1}{s}\left(\frac{1}{s}-1\right)},q^{\frac{1}{s}\left(\frac{1}{s}-1\right)}\right]^{\left(\frac{1}{2s}\right)}\right|\left[\Phi_1\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)+\Phi_2\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)\right], & \\ \qquad p^{\left(\frac{1}{s}-1\right)}\leq 1; & \\ \left|G\left[p^{\left(2-\frac{1}{s}\right)\left(\frac{1}{s}-1\right)},q^{\frac{1}{s}\left(\frac{1}{s}-1\right)}\right]\right|\left[\Phi_1\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)+\Phi_2\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)\right], & \\ \qquad q^{\left(\frac{1}{s}-1\right)}\leq 1\leq p^{\left(\frac{1}{s}-1\right)}; & \\ \left|G\left[p^{\left(2-\frac{1}{s}\right)\left(\frac{1}{s}-1\right)},q^{\left(2-\frac{1}{s}\right)\left(\frac{1}{s}-1\right)}\right]\right|\left[\Phi_1\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)+\Phi_2\left(\beta\left(\frac{1}{2s},\frac{1}{2s}\right)\right)\right], & \\ \qquad 1\leq q^{\left(\frac{1}{s}-1\right)}. & \end{cases}$$

Here $\beta(c,d)=\left(p^{\left(\frac{1}{s}-1\right)}\right)^{-c}\left(q^{\left(\frac{1}{s}-1\right)}\right)^{d}$ *for* $c,d>0$ *with* $\Phi_1(\beta)$ *and* $\Phi_2(\beta)$ *are*

$$\Phi_1(\beta)=\begin{cases}\frac{5}{36}, & \beta=1\\ \frac{6\beta^{\frac{2}{3}}+(\beta-2)ln\beta-3\beta-3}{6ln^2\beta}, & \beta\neq 1,\end{cases}$$

and

$$\Phi_2(\beta)=\begin{cases}\frac{5}{36}, & \beta=1\\ \frac{6\beta^{-\frac{2}{3}}+(\beta^{-1}-2)ln\beta^{-1}-3\beta^{-1}-3}{6ln^2\beta^{-1}}, & \beta\neq 1.\end{cases}$$

Proof: Since $p,q\in\mathbb{R}$ for $p<q$ and $s\in(0,1]$ then similar to Propositions (7.1), (7.2), and (7.3), by substituting the values of generalised log mean and geometric mean from Definition (2.5) in Theorem (6.5) for $f(x)=\frac{x^{\frac{1}{s}}}{\frac{1}{s}}$ and $|f'(x)|=x^{\frac{1}{s}-1}$.

By substituting the values in Inequality (6.17), completes the proof.

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Proposition 7.6: *Let* $\mathbb{I} = [p, q] \in \mathbb{R}$, *and* $f(x) = \frac{x^{\frac{1}{s}}}{\frac{1}{s}}$ *is applied to the function in Theorem (6.6) with* $s \in (0, 1]$, *then*

$$\left|\frac{1}{6}\left(\frac{2A\left(p^{\frac{1}{s}}, q^{\frac{1}{s}}\right) + 4A^{\frac{1}{s}}(p,q)}{s}\right) - \frac{L_s^s(p,q)}{s}\right| = \left(\frac{q-p}{2}\right)\left(\frac{2(2^{c+1}+1)}{6^{c+1}(c+1)}\right)^{\frac{1}{c}}$$

$$\times \begin{cases} \left|G\left[p^{\frac{1}{s}\left(\frac{1}{s}-1\right)}, q^{\frac{1}{s}\left(\frac{1}{s}-1\right)}\right]\right|\left[\Phi_3\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}} + \Phi_4\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}\right], \\ \qquad p^{\left(\frac{1}{s}-1\right)} \leq 1; \\ \left|G\left[p^{\left(2-\frac{1}{s}\right)\left(\frac{1}{s}-1\right)}, q^{\frac{1}{s}\left(\frac{1}{s}-1\right)}\right]\right|\left[\Phi_3\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}} + \Phi_4\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}\right], \\ \qquad q^{\left(\frac{1}{s}-1\right)} \leq 1 \leq p^{\left(\frac{1}{s}-1\right)}; \\ \left|G\left[p^{\left(2-\frac{1}{s}\right)\left(\frac{1}{s}-1\right)}, q^{\left(2-\frac{1}{s}\right)\left(\frac{1}{s}-1\right)}\right]\right|\left[\Phi_3\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}} + \Phi_4\left(\beta\left(\frac{c}{2s},\frac{c}{2s}\right)\right)^{\frac{1}{c}}\right], \\ \qquad 1 \leq q^{\left(\frac{1}{s}-1\right)}. \end{cases}$$

Here, $\Phi_3(\beta)$ *and* $\Phi_4(\beta)$ *are*

$$\Phi_3(\beta) = \begin{cases} 1, & \beta = 1 \\ \dfrac{\beta - 1}{ln\beta}, & \beta \neq 1, \end{cases}$$

and

$$\Phi_4(\beta) = \begin{cases} 1, & \beta = 1 \\ \dfrac{\beta^{-1} - 1}{ln\beta^{-1}}, & \beta \neq 1. \end{cases}$$

Proof: Since $p, q \in \mathbb{R}$ for $p < q$ and $s \in (0, 1]$ then similar to Propositions (7.1), (7.2), and (7.3), by substituting the values of generalised log mean and geometric mean from Definition (2.5) in Theorem (6.6) for $f(x) = \frac{x^{\frac{1}{s}}}{\frac{1}{s}}$ completes the proof.

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Proposition 7.7: *Let* $\mathbb{I} = [p, q] \in \mathbb{R}$, *and* $f(x) = \frac{x^{\frac{1}{s}}}{\frac{1}{s}}$ *is applied to the function in Theorem (6.7) with* $s \in (0, 1]$, *then*

$$\left|\frac{1}{6}\left(\frac{2A\left(p^{\frac{1}{s}}, q^{\frac{1}{s}}\right) + 4A^{\frac{1}{s}}(p, q)}{s}\right) - \frac{L_s^s(p, q)}{s}\right| = \left(\frac{q-p}{2}\right)\left(\frac{5}{36}\right)^{\frac{1}{c}}$$

$$\times \begin{cases} \left|G\left[p^{\frac{1}{s}\left(\frac{1}{s}-1\right)}, q^{\frac{1}{s}\left(\frac{1}{s}-1\right)}\right]^{\left(\frac{1}{2s}\right)}\right|\left[\Phi_1\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right) + \Phi_2\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right)\right], \\ \qquad p^{\left(\frac{1}{s}-1\right)} \leq 1; \\ \left|G\left[p^{\left(2-\frac{1}{s}\right)\left(\frac{1}{s}-1\right)}, q^{\frac{1}{s}\left(\frac{1}{s}-1\right)}\right]\right|\left[\Phi_1\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right) + \Phi_2\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right)\right], \\ \qquad q^{\left(\frac{1}{s}-1\right)} \leq 1 \leq p^{\left(\frac{1}{s}-1\right)}; \\ \left|G\left[p^{\left(2-\frac{1}{s}\right)\left(\frac{1}{s}-1\right)}, q^{\left(2-\frac{1}{s}\right)\left(\frac{1}{s}-1\right)}\right]\right|\left[\Phi_1\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right) + \Phi_2\left(\beta\left(\frac{1}{2s}, \frac{1}{2s}\right)\right)\right], \\ \qquad 1 \leq q^{\left(\frac{1}{s}-1\right)}. \end{cases}$$

Here, Φ_1, Φ_2 *and* β *are the same as defined in previous Proposition (6.6).*

Proof: Since $p, q \in \mathbb{R}$ for $p < q$ and $s \in (0, 1]$ then similar to Propositions (7.1), (7.2), and (7.3), by substituting the values of generalised log mean and geometric mean from Definition (2.5) in Theorem (6.7) for $f(x) = \frac{x^{\frac{1}{s}}}{\frac{1}{s}}$ completes the proof.

Proposition 7.8: *Suppose* $p, q \in \mathbb{I}$ *and* $f: \mathbb{I} \to \mathbb{R}$ *be a convex function and n-times differentiable with* $p < q, a \in [0, 1]$, *and* $\left|f^{(n)}\right| \leq M$, *then Theorem (6.4) can be written as*

$$\left|A\big(f(p), f(q)\big) - \left(\frac{1}{q-p}\right)\int_p^q f(x)dx - \sum_{i=1}^{n-1}\left(\frac{(q-p)^2}{2!}f(A(p, q))\right)\right|$$

$$\leq \frac{(q-p)^2}{4}\left(\frac{1}{3}\right)^{\left(1-\frac{1}{q}\right)}\left(\int_0^1 2a(1-a)f^n\left(G(p,q)^{h_k(a)}\right)da\right)^{\frac{1}{q}}.$$

Proof: Since $p < q$ and $a \in [0,1]$, then similar to Propositions (7.1), (7.2), and (7.3), by utilising arithmetic mean and geometric mean from Definition (2.5) and set $i = 2$ in Theorem (6.4) completes the proof.

Note that Propositions (7.5), (7.6), (7.7) and (7.8) successfully established a link between Simpson's inequality for hybrid convex functions with special means (averages).

While in proceeding section, error estimation formulae for numerical formulas such as Trapezoidal, Mid-point, and Simpson's are obtained by using extended mathematical inequalities obtained in Chapter Three, Four, Five, and Six.

7.4 Error Estimation Formula Using Extended Inequalities

In this section, error estimates for Trapezoidal, Mid-point, and Simpson's formulas are considered to calculate by utilising H-H, Jensen's, Fejér's, and Simpson's inequalities developed in Chapters Three, Four, Five, and Six, respectively.

Let Υ be the partition for the interval $[p,q]$ such that $p = x_0, x_1, x_2, \dots, x_n = q$, then the Trapezoidal, Mid-point, and Simpson's formulas are defined as follows:

$$\int_p^q f(x)dx \qquad = T(f,\Upsilon) + E(f,\Upsilon), \tag{7.13}$$

$$\int_p^q f(x)g(x)dx = M(f,g,\Upsilon) + E(f,g,\Upsilon), \tag{7.14}$$

$$\int_p^q f(x)dx \qquad = S(f,\Upsilon) + E(f,\Upsilon). \tag{7.15}$$

By utilising inequalities from Chapters Three, Four, Five, and Six, error estimation formulas in the form of propositions are presented in Propositions (7.9), (7.10), and (7.11) as follows:

Proposition 7.9: *Let* $f : \mathbb{I} \to \mathbb{R}$ *be a differentiable function. Let* f *be the* (h_1, h_2, s)*-convex function in second sense and* $g(x) \equiv 1$ *with* $p < q$*,* $s \in [0, 1]$*, and* $a \in [0, 1]$ *on a partition* Υ*, then Trapezoidal formula (7.13) is satisfying, and Inequality (5.29) will be reduced to*

$$\left|\left(\frac{f(p) + f(q)}{2}\right) - \int_p^q f(x)dx\right| \leq (q - p)(|f'(p)| + |f'(q)|)$$

$$\times \int_p^{\frac{p+q}{2}} \int_0^{\frac{x-p}{q-p}} [{h_1}^s(a) + {h_2}^s(1 - a)]dxda. \qquad (7.16)$$

Proof: In equation (7.13),

$$T(f, \Upsilon) = \sum_{i=0}^{n-1} \left[\frac{f(x_i + x_{i+1})}{2}\right] h_i^s,$$

is the trapezoidal form for $i = 0,1,2, \dots, n - 1$ and $E(f, \Upsilon)$ is the approximate error for this formula. For Proposition (7.5), let all the assumptions discussed in Equation (7.13) be true, then Inequality (5.29) can be written as

$$\left|\left(\frac{f(x_i) + f(x_{i+1})}{2}\right) - \int_{x_i}^{x_{i+1}} f(x)dx\right| \leq (x_{i+1} - x_i)(|f'(x_i)| + |f'(x_{i+1})|)$$

$$\times \int_{x_i}^{\frac{x_i+x_{i+1}}{2}} \int_0^{\frac{x-x_i}{x_{i+1}-x_i}} [{h_1}^s(a) + {h_2}^s(1 - a)]\, dxda. \qquad (7.17)$$

By using triangle inequality $|f(x_i + x_{i+1})| \leq |f(x_i)| + |f(x_{i+1})|$, results

$$\left|T(f,\Upsilon) - \int_p^q f(x)dx\right| \leq \sum_{i=0}^{n-1}\left[\frac{f(x_i + x_{i+1})}{2}\right] - \left|\int_{x_i}^{x_{i+1}} f(x)dx\right|,$$

$$\leq \left|\sum_{i=0}^{n-1}[(x_{i+1} - x_i)(|f'(x_i)| + |f'(x_{i+1})|)]\right.$$

$$\left.\times \int_{x_i}^{\frac{x_i+x_{i+1}}{2}} \int_0^{\frac{x-x_i}{x_{i+1}-x_i}} [h_1{}^s(a) + h_2{}^s(1-a)]dxda\right|.$$

Now the approximate error becomes

$$E(f,\Upsilon) \leq \sum_{i=0}^{n-1}[(x_{i+1} - x_i)(|f'(x_i)| + |f'(x_{i+1})|)]$$

$$\times \int_{x_i}^{\frac{x_i+x_{i+1}}{2}} \int_0^{\frac{x-x_i}{x_{i+1}-x_i}} [h_1{}^s(a) + h_2{}^s(1-a)]dxda\,.$$

This is the error estimation formulae for (h_1, h_2, s)-convex function in second sense of the trapezoidal formula by using H-H inequality.

Note that if we impose a condition $h_1{}^s(a) = a^k$ for $k \geq 1$, then error estimation formulae of trapezoidal formula will be reduced to

$$E(f,\Upsilon) \leq \frac{1}{k+1}\sum_{i=0}^{n-1}[(x_{i+1} - x_i)(|f'(x_i)| + |f'(x_{i+1})|)]$$

$$\times \int_{x_i}^{\frac{x_i+x_{i+1}}{2}} \int_0^{\frac{x-x_i}{x_{i+1}-x_i}} \left[\left(\frac{x_i + x_{i+1} - x}{x_{i+1} - x_i}\right)^{k+1} - \left(\frac{x - x_i}{x_{i+1} - x_i}\right)^{k+1} + 1\right]dx\,.$$

In addition, if $k = 1$, then error estimation formula will be reduced to the formal form of error estimation for a convex function as follows:

$$E(f,\Upsilon) \leq \frac{1}{8}\sum_{i=0}^{n-1}[(x_{i+1}-x_i)^2(|f'(x_i)|+|f'(x_{i+1})|)].$$

Note that Proposition (7.9) successfully established error estimation formulae for hybrid (h_1, h_2, s)-convex function in second sense.

Meanwhile, in Proposition (7.10), error estimation for Midpoint formula defined in equation (7.14) is utilised to find error estimation from Fejér's inequality.

Proposition 7.10: *Let $f, g: \mathbb{I} \to \mathbb{R}$ be differentiable functions. Let f be the (h_1, h_2, s)-convex function in second sense with $p < q$, $s \in [0,1]$, and $a \in [0,1]$ on a partition Υ, then Midpoint formula (7.14) is satisfying, and Inequality (5.35) will give the error estimations for this formula as follows:*

In equation (7.14),

$$M(f,g,\Upsilon) = \sum_{i=0}^{n-1} f\left[\frac{(x_i+x_{i+1})}{2}\right] g_{(x_i,\ x_{i+1})}$$

with

$$g_{(x_i,\ x_{i+1})} = \int_{x_i}^{x_{i+1}} g(x)dx,$$

is the midpoint form and $E(f,g,\Upsilon)$ is the approximate error for this formula such as

$$E(f,g,\Upsilon) \leq \sum_{i=0}^{n-1}\left[\frac{(x_i+x_{i+1})}{k}\right]\left[\frac{|f'(x_i)|+|f'(x_{i+1})|}{2}\right]$$

$$\times \int_0^k \int_p^{\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q} g(x)\left[{h_1}^s\left(\frac{k+a}{2k}\right)+{h_2}^s\left(\frac{k-a}{2k}\right)\right]dxda. \quad (7.18)$$

Proof: Since $i = 0, 1, 2, \dots, n-1$, then applying interval $[x_i,\ x_{i+1}]$ on Theorem (5.5), results

$$\left|\left(\frac{f(x_{i+1})+f(x_i)}{2}\right)\int_{x_i}^{x_{i+1}} g(x)dx - \int_{x_i}^{x_{i+1}} g(x)f(x)dx\right| \leq \left(\frac{x_{i+1}-x_i}{k}\right)$$

$$\times\left(\frac{|f'(x_{i+1})|+|f'(x_i)|}{2}\right)$$

$$\times\int_0^k \int_p^{\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q} g(x)\left[{h_1}^s\left(\frac{k+a}{2k}\right)+{h_2}^s\left(\frac{k-a}{2k}\right)\right]dxda. \tag{7.19}$$

Equation (7.14) can be written as

$$E(f,g,\Upsilon) = \int_p^q f(x)g(x)dx - M(f,g,\Upsilon), \tag{7.20}$$

which will be reduced to

$$= \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} g(x)f(x)dx - \sum_{i=0}^{n-1} f\left[\frac{(x_i+x_{i+1})}{2}\right] g_{(x_i,\ x_{i+1})},$$

$$= \sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}} g(x)f(x)dx - f\left[\frac{(x_i+x_{i+1})}{2}\right]\int_{x_i}^{x_{i+1}} g(x)dx\right].$$

By comparing Equations (7.19) and (7.20), yields

$$E(f,g,\Upsilon) = \sum_{i=0}^{n-1}\left[\frac{(x_i+x_{i+1})}{k}\right]\left[\frac{|f'(x_i)|+|f'(x_{i+1})|}{2}\right]$$

$$\times\int_0^k \int_p^{\left(\frac{k+a}{2k}\right)p+\left(\frac{k-a}{2k}\right)q} g(x)\left[{h_1}^s\left(\frac{k+a}{2k}\right)+{h_2}^s\left(\frac{k-a}{2k}\right)\right]dxda.$$

This is the error estimation formulae of the mid-point formula for Fejér’s inequality.

Note that Proposition (7.10) successfully established error estimation formulae for hybrid (h_1, h_2, s)-convex function in second sense for mid-point formula using Fejér’s inequality.

Meanwhile, in Proposition (7.11), error estimation for Simpson's formula defined in equation (7.15) is utilised to find error estimation from Simpson's inequality.

Proposition 7.11: *Let* $f\colon \mathbb{I} \to \mathbb{R}$ *be a differentiable function. Let* f *be the* (h_1, h_2, s)*-convex function in second sense with* $p < q$*,* $s \in [0,1]$*, and* $a \in [0,1]$ *on a partition* Υ*, then Simpson's formula (7.15) is satisfying, and Theorem (6.1) will give the error estimations for this formula as follows:*

$$E(f, \Upsilon) = \frac{2(x_i + x_{i+1})}{3}\left(\frac{1 + 2^{c+1}}{6(c+1)}\right)^{\frac{1}{c}} \sum_{i=0}^{n-1} (x_i + x_{i+1})^2$$

$$\times \left\{ |f'(x_i)|^d + \left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|^d + \left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|^d + |f'(x_i)|^d \right\}^{\frac{1}{d}}$$

Proof: Since $i = 0, 1, 2, \dots, n-1$, then applying interval $[x_i, x_{i+1}]$ on Theorem (6.1), results

$$\left| \frac{(x_i + x_{i+1})}{3}\left[f(x_i) + f(x_{i+1}) + 4f\left(\frac{x_i + x_{i+1}}{2}\right)\right] - \int_{x_i}^{x_{i+1}} f(x)dx \right|$$

$$\leq \frac{2(x_i + x_{i+1})^2}{3}\left(\frac{1 + 2^{c+1}}{6(c+1)}\right)^{\frac{1}{c}}$$

$$\times \left\{ |f'(x_i)|^d + \left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|^d + \left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|^d + |f'(x_i)|^d \right\}^{\frac{1}{d}}. \quad (7.21)$$

Equation (7.15) can be written as

$$E(f, \Upsilon) = S(f, \Upsilon) - \int_p^q f(x)dx. \quad (7.22)$$

By comparing Equation (7.21) and (7.22), we get

$$S(f, \Upsilon) = \frac{(x_i + x_{i+1})}{3}\left[f(x_i) + f(x_{i+1}) + 4f\left(\frac{x_i + x_{i+1}}{2}\right)\right],$$

and

$$E(f, \Upsilon) = \frac{2(x_i + x_{i+1})^2}{3}\left(\frac{1 + 2^{c+1}}{6(c+1)}\right)^{\frac{1}{c}}$$

$$\times \left\{|f'(x_i)|^d + \left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|^d + \left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|^d + |f'(x_i)|^d\right\}^{\frac{1}{d}}.$$

This is the error estimation formulae for the Simpson's formula using Simpson's inequality.

7.5 Summary

This chapter established some new relationships between newly constructed mathematical inequalities by using new hybrid convex functions from Chapter (3), (4), (5), and (6). Based on these hybrid convex functions H-H, Jensen's, Fejér's, and Simpson's type of inequalities is extended or generalised in Chapter (3), (4), (5), and (6). In section (7.2), the relationship between mathematical inequalities and mathematical mean (average) is created so that this relation can be used in different fields of study. For example, statistics, engineering, linear programming. Furthermore, in section (7.3) error estimations for different numerical formulas such as Trapezoid, mid-point, and Simpson is calculated in the form of theorems. All these applications highlighted the role of hybrid convex functions, and mathematical inequalities in different fields of study.

CHAPTER EIGHT
CONCLUSIONS

The main objective of this study is to construct new mathematical inequalities for Hermite-Hadamard (H-H), Jensen's, Fejér's, and Simpson's by using the approach to construct hybrid convex functions. For the construction of hybrid convex functions s-convex, h-convex, and g-convex functions are considered. Furthermore, harmonically convex, m-convex and similar families of convex functions are considered to extend the performance of convex functions. The performance of the proposed methods with the existing methods are compared in terms of accuracy which shows that all the previously defined results as well as new results can be found using newly constructed inequalities. To demonstrate the application side of the developed inequalities, the relation between mathematical mean with the newly constructed inequalities and the error estimations formulae for Mid-point, Trapezoid and Simpson's formulas are considered. The summary of this study and future research is described in the following sections.

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8.1 Summary

The fundamental concepts of convex function and mathematical inequalities like Hermite-Hadamard (H-H), Jensen's, Fejér's and Simpson's inequalities have been discussed in Chapter One. Some well-known traditional definitions of convex function and mathematical inequalities are also included in this chapter. Moreover, the evolution of convex function is also described. In Chapter Two, a literature review of the previous work on mathematical inequalities was conducted. Based on the literature review, the research gaps have been identified.

Chapter Three focuses on construction of new hybrid convex functions and after that Hermite-Hadamard (H-H) inequality is extended based on these newly constructed hybrid convex functions. This chapter also includes some remarks that show that all the previously defined results can also be found by using newly constructed H-H inequalities.

Chapter Four concentrates on the extension of Jensen's inequality on the bases of newly constructed hybrid convex functions derived in Chapter Three. This chapter also includes some remarks that show that all the previously defined results can also be found by using newly constructed Jensen's type of inequalities.

Chapter Five focuses on the extension of Fejér's inequality based on newly constructed hybrid convex functions found in Chapter Three. This chapter also includes some remarks that show that all the previously defined results can also be found by using newly constructed Fejér's type of inequalities.

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Chapter Six focuses on extension of Simpson's inequality based on newly constructed hybrid convex functions obtained in Chapter three. This chapter also includes some remarks that show that all the previously defined results can also be found by using newly constructed Simpson's type of inequalities.

Chapter seven validates the H-H, Jensen's, Fejér's, and Simpson's inequalities extended in Chapter Three, Four, Five, and Six, respectively. Furthermore, this chapter also discusses about the application of mathematical mean (average) and found error estimation formulae for H-H, Jensen's, Fejér's, and Simpson's inequalities extended in Chapter Three, Four, Five, and Six, respectively. Specific inequalities are chosen from all the chapters and the relationship between mathematical means (averages) with these extensions are established. Moreover, this chapter highlighted the role of extended mathematical inequalities with numerical formulas by calculating the error estimation formulae for Trapezoidal, Mid-point, and Simpson's formulas. Chapter eight highlighted the conclusion with summary of whole thesis and future research.

8.2 Future Research

The outcomes of this study can be used as a building block for future research. This study can also be extended to combine more than three convex functions. Investigating higher-order combinations of functions and establishing general frameworks for these combinations will significantly enrich the field of mathematical inequalities. For valuable insights, future studies should focus on identifying specific classes of functions for which these extended H-H, Jensen's, Fejér's, and Simpson's type of inequalities hold. More inequalities such as Minkowski's, Cauchy–Schwarz, and Chebyshev's types of inequality can also be extended by using newly defined hybrid convex functions. It is also suggested that future researchers can apply newly developed hybrid convex functions and mathematical inequalities in other areas such as optimisation, mathematical inequalities, linear programming, and error estimation.

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