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**HERMITE-HADAMARD, JENSEN'S, FEJÉR'S, AND
SIMPSON'S NEW INEQUALITIES FOR A MERGING
BETWEEN TWO CONVEX FUNCTIONS**

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**DOCTOR OF PHILOSOPHY
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Abstrak

Ketaksamaan digunakan secara meluas untuk mencari penyelesaian optimum bagi masalah pengoptimuman, kejuruteraan dan pengaturcaraan linear. Kajian terdahulu, hanya satu fungsi cembung tunggal seperti fungsi s-cembung, h-cembung, dan g-cembung digunakan untuk membina ketaksamaan, yang digunakan sebagai ketaksamaan Hermite-Hadamard, Jensen, Fejér, dan Simpson. Ketaksamaan ini menggunakan sifat fungsi cembung tunggal namun terhad kepada sebilangan masalah Sahaja. Oleh itu, penggabungan fungsi cembung ini dapat menyelesaikan lebih banyak masalah, sekaligus mengembangkan aplikasi dalam pelbagai bidang. Oleh yang demikian, kajian ini bertujuan membina ketaksamaan baharu dengan menggabungkan tiga fungsi cembung tunggal, iaitu fungsi s-cembung, h-cembung, dan g-cembung. Melalui kerangka gabungan kecembungan, kajian ini menyumbang kepada bidang ini secara signifikan dengan menyediakan pelbagai peluasan dalam ketaksamaan Hermite-Hadamard, Jensen, Fejér, dan Simpson bagi fungsi cembung hibrid. Pendekatan menggabungkan dua atau lebih kelas fungsi cembung digunakan untuk membentuk kelas fungsi cembung hibrid bertujuan memperluaskan ketaksamaan bermatematik. Hasil kajian ini membangunkan pelbagai pengembangan ketaksamaan Hermite-Hadamard, Jensen, Fejér, dan Simpson bagi kelas fungsi baharu dengan sifat kecembungan hibrid. Hasil dapatan seterusnya dirumuskan sebagai teorem untuk menerbit anggaran ralat bagi kaedah Simpson, titik tengah, dan petua trapezoid dan juga untuk menentukan nilai purata bagi nombor nyata. Anggaran ralat ini didapati lebih tepat berbanding ketaksamaan sedia ada. Kesimpulannya, kajian ini menambah baik ketaksamaan bermatematik, mengoptimumkan penyelesaian dalam pelbagai bidang di samping mengekalkan sifat kecembungan dan memperluaskan kaedah penghampiran kamiran klasik.

Kata Kunci: Fungsi cembung, ketaksamaan, nilai purata, anggaran ralat.

Abstract

Inequalities are widely used to find the optimal solution for optimisation, engineering and linear programming problems. In previous works, only a single convex function such as s-convex, h-convex and g-convex functions were used to construct the inequalities namely Hermite-Hadamard, Jensen's, Fejér's, and Simpson's inequalities. These inequalities use the properties of a single convex function but can only address a limited number of problems. By combining these convex functions, more problems can be solved, therefore their applicability to various fields. Thus, this study aims to construct new inequalities by combining three single convex functions namely s-convex, h-convex and g-convex functions. The approach of combining two or more classes of convex functions to form hybrid convex function classes aims to extend mathematical inequalities. The findings of the study establish various extensions of the Hermite-Hadamard, Jensen's, Fejér's, and Simpson's inequalities for new classes of functions with hybrid convexity properties. These results are further formulated as theorems to derive error estimations for the Simpson's, midpoint, and trapezoidal rules and also to determine the mean value of real numbers. These error estimations are found more accurate compared to the existing inequalities. Within the framework of convexity combinations, this study contributes significantly to the field by providing various extensions of the Hermite-Hadamard, Jensen's, Fejér's, and Simpson's inequalities for hybrid convex functions. In conclusion, this study enhanced mathematical inequalities, optimizing solutions in various fields while preserving convexity properties and extending classical integral approximation methods.

Keywords: Convex function, inequality, mean value, error estimations.

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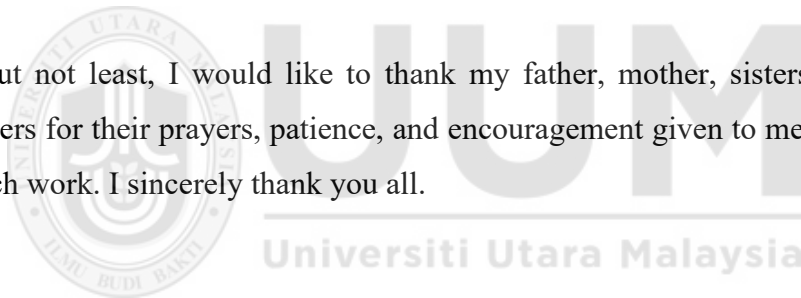


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List of Symbols

$\mathbb{R}$	:	The Set of Real Numbers
$\mathbb{R}_+$	:	The Set of All Positive Real Numbers
$[x, y]$	:	The Line Segment Between x and y
$\mathbb{I}$	:	Real Interval $[p, q]$
$\mathbb{I}^*$	:	Interval in $\mathbb{I}$
$\mathbb{I}^\circ$	:	Interior of $\mathbb{I}$
$\mathbb{J}$	:	Open Interval of 0 and 1
$\mathbb{J}^*$	:	Closed Interval of 0 and 1
$\mathbb{K}$	:	Positive Real Interval = $[0, \infty)$ or $\mathbb{R}_+$
$L[p, q]$	:	Integrable on $[p, q]$
x, p, q, y	:	Random Points in $\mathbb{R}$
k_1^s	:	s -convex Function in the First Sense
k_2^s	:	s -convex Function in the Second Sense
k_3^s	:	s -convex Function in the Third Sense
k_4^s	:	s -convex Function in the Fourth Sense
$A(p, q)$	:	Arithmetic Mean on Random Points in $\mathbb{I}$
$H(p, q)$	:	Harmonic Mean on Random Points in $\mathbb{I}$
$\bar{L}(p, q)$	:	Logarithmic Mean on Random Points in $\mathbb{I}$
$L_n(p, q)$	:	Generalised Log Mean on Random Points in $\mathbb{I}$
$L_m(p, q)$	:	α -logarithmic mean on Random Points in $\mathbb{I}$
Υ	:	Partition of Real Interval $\mathbb{I}$
$\Upsilon(x, y)$	:	Set of All Possible Partitions in $\mathbb{I}$
$T(f, \Upsilon)$	:	Error Estimation Formulae for Trapezoid Rule
$M(f, g, \Upsilon)$	:	Error Estimation Formulae for Midpoint Rule
$S(f, \Upsilon)$	:	Error Estimation Formulae for Simpson's Rule

CHAPTER ONE

INTRODUCTION

1.1 Research Background

Since the nineteenth century, inequality has become increasingly important as a fundamental element of analysis (Fink, 2000). Inequality is defined as the difference between two quantities that are not equal and can be less or greater than one another.

Inequalities are used in numerous branches of mathematics such as in optimisation theory (Rashid, Noor, & Noor, 2020), error estimation (Avci, Kavurmaci, Özdemir, & Emin, 2011), and calculus (Anastassiou, 2011), to name a few. In optimisation theory, inequalities can be employed to find an optimum solution to a given problem (Bertsimas & Popescu, 2005). While in error estimation, the estimated error for a function can be obtained by using inequality theory (Chu, Khan, Adil, & Ali, 2017). Meanwhile, in calculus, inequalities can be applied in diverse areas. Different inequalities, such as trapezoid, midpoint, and Simpson inequalities, can be developed using the inequality concept (Dragomir, Cerone, & Sofo, 1998; Chu, Khan, Adil, & Ali, 2017; Du, Li, & Yang, 2017).

Further, convex function and its derivatives are also used for different purposes. For example, the first-order derivative of a convex function is used for testing the continuity of smooth functions of one variable (Roberts, 1993). The second-order derivative of convex function is used to find maximum error estimation for areas under the curve using midpoint and trapezoidal rule (Alomari, Darus, & Dragomir, 2010; Shen, Wang, Wang, & Wise, 2012; Mehrez & Agarwal, 2018), while the third-order derivative of a convex function is specially used for quasi-convex function (Wu,

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